**MAT 307: Combinatorics**

# Lecture 14: Topological methods

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## 1 The Borsuk-Ulam theorem

We have seen how combinatorics borrows from probability theory. Another area which has been very beneficial to combinatorics, perhaps even more surprisingly, is topology. We have already seen Brouwer's fixed point theorem and its combinatorial proof.

**Theorem 1** (Brouwer)**.** *For any continuous function $f : B^n \to B^n$, there is a point $x \in B^n$ such that $f(x) = x$.*

A more powerful topological tool which seems to stand at the root of most combinatorial applications is a somewhat related result which can be stated as follows. Here, $S^n$ denotes the $n$-dimensional sphere, i.e. the surface of the $(n+1)$-dimensional ball $B^{n+1}$.

**Theorem 2** (Borsuk-Ulam)**.** *For any continuous function $f : S^n \to R^n$, there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

There are many different proofs of this theorem, some of them elementary and some of them using a certain amount of the machinery of algebraic topology. All the proofs are, however, more involved than the proof of Brouwer's theorem. We will not give the proof here.

In the following, we use a corollary (in fact an equivalent re-statement of the Borsuk-Ulam theorem).

**Theorem 3.** *For any covering of $S^n$ by $n+1$ open or closed sets $A_0, \ldots, A_n$, there is a set $A_i$ which contains two antipodal points $x, -x$.*

Let's just give some intuition how this is related to Theorem 2. For now, let us assume that all the sets $A_i$ are closed. (The extension to open sets is a technicality but the idea is the same.) We define a continuous function $f : S^n \to R^n$,

$$f(x) = (dist(x, A_1), dist(x, A_2), \ldots, dist(x, A_n))$$

where $dist(x, A) = \inf_{y \in A} ||x - y||$ is the distance of $x$ from $A$. By Theorem 2, there is a point $x \in S^n$ such that $f(x) = f(-x)$. This means that $dist(x, A_i) = dist(-x, A_i)$ for $1 \le i \le n$. If $dist(x, A_i) = 0$ for some $i$, then we are done. If $dist(x, A_i) = dist(-x, A_i) \ne 0$ for all $i \in \{1, \ldots, n\}$, it means that $x, -x \notin A_1 \cup \ldots \cup A_n$. But then $x, -x \in A_0$.

## 2 Kneser graphs

Similarly to the previous sections, Kneser graphs are derived from the intersection pattern of a collection of sets. More precisely, the vertex set of a Kneser graph consists of all $k$-sets on a given ground set, and two $k$-sets form an edge if they are *disjoint*.

**Definition 1.** *The Kneser graph on a ground set* $[n]$ *is*

$$KG_{n,k} = \left( \binom{[n]}{k}, \{(A,B) : |A| = |B| = k, A \cap B = \emptyset\} \right).$$

Thus, the maximum independent set in $KG_{n,k}$ is equivalent to the maximum intersecting family of $k$-sets - by the Erős-Ko-Rado theorem, $\alpha(KG_{n,k}) = \binom{n-1}{k-1} = \frac{k}{n}|V|$ for $k \leq n/2$. The maximum clique in $KG_{n,k}$ is equivalent to the maximum number of disjoint $k$-sets, i.e. $\omega(KG_{n,k}) = \lfloor n/k \rfloor$.

Another natural question is, what is the chromatic number of $KG_{n,k}$? Note that for $n = 3k-1$, the Kneser graph does not have any triangle, and also $\alpha(KG_{3k-1,k}) \approx \frac{1}{3}|V|$. Yet, we will show that the chromatic number $\chi(KG_{n,k})$ grows with $n$. Therefore, these graphs give another example of a triangle-free graph of high chromatic number.

**Theorem 4** (Lovász-Kneser)**.** *For all* $k > 0$ *and* $n \geq 2k-1$, $\chi(KG_{n,k}) = n - 2k + 2$.

*Proof.* First, we show that $KG_{n,k}$ can be colored using $n - 2k + 2$ colors. This means assigning colors to $k$-sets, so that all $k$-sets of the same color intersect. This is easy to achieve: color each $k$-set with all elements in $[2k-1]$ with one color, and every other $k$-set by their largest element. We have $n - 2k + 2$ colors and all $k$-sets of a given color intersect.

The proof that $n - 2k + 1$ colors are not enough is more interesting. Let $d = n - 2k + 1$ and assume that $KG_{n,k}$ is colored using $d$ colors. Let $X$ be a set of $n$ points on $S^d$ in a general position (there are no $d+1$ points lying on a $d$-dimensional hyperplane through the origin). Each subset $A \in \binom{X}{k}$ corresponds to a vertex of $KG_{n,k}$ which is colored with one of $d$ colors. Let $\mathcal{A}_i$ be the collection of $k$-sets corresponding to color $i$.

We define sets $U_1, \ldots, U_d \subseteq S^d$ as follows: $x \in U_i$, if there exists $A \in \mathcal{A}_i$ such that $\forall y \in A; x \cdot y > 0$. In other words, $x \in U_i$ if some $k$-set of color $i$ lies in the open hemisphere whose pole is $x$. Finally, we define $U_0 = S^d \setminus (U_1 \cup U_2 \cup \ldots \cup U_d)$. It's easy to see that the sets $U_1, \ldots, U_d$ are open and $U_0$ is closed. By Theorem 3, there is a set $U_i$ and two antipodes $x, -x \in U_i$.

If this happens for $i = 0$, then we have two antipodes $x, -x$ which are not contained in any $U_i, i > 0$. This means that both hemispheres contain fewer than $k$ points, but then $n - 2(k-1) = d+1$ points must be contained in the "equator" between the two hemispheres, contradicting the general position of $X$. Therefore, $x, -x \in U_i$ for some $i > 0$, which means we have two $k$-sets of color $i$ lying in opposite hemispheres. This means that they are disjoint and hence forming an edge in $KG_{n,k}$, which is a contradiction. □

# 3 Dolnikov's theorem

The Kneser graph can be defined naturally for any set system $\mathcal{F}$: two sets form an edge if they are disjoint. We denote this graph by $KG(\mathcal{F})$:

$$KG(\mathcal{F}) = \{\mathcal{F}, \{(A,B) : A, B \in \mathcal{F}, A \cap B = \emptyset\}\}.$$

We derive a bound on the chromatic number of $KG(\mathcal{F})$ which generalizes Theorem 4. For this purpose, we need the notion of a *2-colorability defect.*

**Definition 2.** *For a hypergraph (or set system)* $\mathcal{F}$*, the 2-colorability defect* $cd_2(\mathcal{F})$ *is the smallest number of vertices, such that removing them and all incident hyperedges from* $\mathcal{F}$ *produces a 2-colorable hypergraph.*

For example, if $\mathcal{H}$ is the hypergraph of all $k$-sets on $n$ vertices, we need to remove $n-2k+2$ vertices and then the remaining hypergraph of $k$-sets on $2k-2$ vertices is 2-colorable, i.e. $cd_2(\mathcal{H}) = n-2k+2$. Coincidentally, this is also the chromatic number of the corresponding Kneser graph. We prove the following.

**Theorem 5** (Dolnikov)**.** *For any hypergraph (or set system) $\mathcal{F}$,*

$$\chi(KG(\mathcal{F})) \geq cd_2(\mathcal{F}).$$

We remark that equality does not always hold, and also $cd_2(\mathcal{F})$ is not easy to determine for a given hypergraph. The connection between two very different coloring concepts is quite surprising, though. Our first proof follows the lines of the Kneser-Lovász theorem.

*Proof.* Let $d = \chi(KG(\mathcal{F}))$ and consider a coloring of $\mathcal{F}$ by $d$ colors. Again, we identify the ground set of $\mathcal{F}$ with a set of points $X \subset S^d$ in general position, with no $d+1$ points on the same hyperplane through the origin. We define $U_i \subseteq S^d$ by $x \in U_i$ iff some set $F \in \mathcal{F}$ of color $i$ is contained in $H(x) = \{y \in S^d : x \cdot y > 0\}$. Also, we set $A_0 = S^d \setminus (A_1 \cup \ldots \cup A_d)$.

By Theorem 3, there is a set $A_i$ containing two antipodal points $x, -x$. This cannot happen for $i \geq 1$, because then there would be two sets $F, F' \in \mathcal{F}$ of color $i$ such that $F \subset H(x)$ and $F' \subset H(-x)$. This would imply $F \cap F' = \emptyset$, contradicting the coloring property of the Kneser graph $KG(\mathcal{F})$.

Therefore, there are two antipodal points $x, -x \in A_0$. This implies that there is no set $F \in \mathcal{F}$ in either hemisphere $H(x)$ or $H(-x)$. By removing the points on the equator between $H(x)$ and $H(-x)$, whose number is at most $d = \chi(KG(\mathcal{F}))$, and also removing all the sets in $\mathcal{F}$ containing them, we obtain a hypergraph $\mathcal{F}'$ such that all the sets $F \in \mathcal{F}'$ touch both hemispheres $H(x), H(-x)$. This hypergraph can be colored by 2 colors corresponding to the two hemispheres. □

Next, we present Dolnikov's original proof, which is longer but perhaps more intuitive. It relies on the following geometric lemma, which follows from the Borsuk-Ulam theorem.

**Lemma 1.** *Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_d$ be families of convex bounded sets in $R^d$. Suppose that each family $\mathcal{C}_i$ is* intersecting*, i.e. $\forall C, C' \in \mathcal{C}_i; C \cap C' \neq \emptyset$. Then there is a hyperplane intersecting all the sets in $\bigcup_{i=1}^d \mathcal{C}_i$.*

*Proof.* Let's consider a vector $v \in S^{d-1}$, which defines a line in $R^d$, $L_v = \{\alpha v : \alpha \in R\}$. For each family $\mathcal{C}_i$, we consider its projection on $L_v$. Formally, for each $C \in \mathcal{C}_i$ we consider

$$P(C, v) = \{x \cdot v : x \in C\}.$$

Since each $C$ is a convex bounded set, $P(C, v)$ is a bounded interval. $C \cap C' \neq \emptyset$ for all $C, C' \in \mathcal{C}_i$, and therefore all the intervals $P(C, v)$ are pairwise intersecting as well. Hence, the intersection of all these intervals, $\bigcap_{C \in \mathcal{C}_i} P(C, v)$, is a nonempty bounded interval as well. Let $f_i(v)$ denote the midpoint of $\bigcap_{C \in \mathcal{C}_i} P(C, v)$. This means that the hyperplane

$$H(v, \lambda) = \{x \in R^d : x \cdot v = \lambda\}$$

for $\lambda = f_i(v)$ intersects all the sets in $\mathcal{C}_i$.

For each $1 \le i \le d-1$, define $g_i(v) = f_i(v) - f_d(v)$. Observe that $P(C, -v) = -P(C, v)$ and hence $f_i(-v) = -f_i(v)$, and also $g_i(-v) = -g_i(v)$. By Theorem 2, there is a point $v \in S^{d-1}$ such that for all $1 \le i \le d-1$, $g_i(v) = g_i(-v)$. Since $g_i(-v) = -g_i(v)$, this implies that in fact $g_i(v) = 0$. In other words, $f_i(v) = f_d(v) = \lambda$ for all $1 \le i \le d-1$. This means that the hyperplane $H(v, \lambda)$ intersects all the sets in $\mathcal{C}_i$, for each $1 \le i \le d$. $\square$

Now we can give the second proof of Dolnikov's theorem.

*Proof.* We consider a coloring of the Kneser graph $KG(\mathcal{F})$ by $d$ colors. Denote by $\mathcal{F}_i$ the collection of sets in $\mathcal{F}$ corresponding to vertices of color $i$.

We represent the ground set of $\mathcal{F}$ by a set of points $X \subset R^d$ in general position. (Observe that in the first proof, we placed the points in $S^d \subset R^{d+1}$.) Again, we assume that there are no $d+1$ points on the same hyperplane. We define $d$ families of convex sets: for every $i \in [d]$,

$$\mathcal{C}_i = \{conv(F) : F \in \mathcal{F}_i\}.$$

In other words, these are polytopes corresponding to sets of color $i$. In each family, all polytopes are pairwise intersecting, by the coloring property of $KG(\mathcal{F})$. Therefore by Lemma 1, there is a hyperplane $H$ intersecting all these polytopes in each $\mathcal{C}_i$. Let $Y = H \cap X$ be the set of points exactly on the hyperplane. Let's remove $Y$ and all the sets containing some point in $Y$, and denote the remaining sets by $\mathcal{F}'$. Each set $F' \in \mathcal{F}'$ must contain vertices on both sides of $H$, otherwise $conv(F')$ would not be intersected by $H$. Therefore, coloring the open halfspaces on the two sides of $H$ by 2 colors, we obtain a valid 2-coloring of $\mathcal{F}'$. $\square$