1 Largest antichains

Suppose we are given a family \( \mathcal{F} \) of subsets of \([n]\). We call \( \mathcal{F} \) an antichain, if there are no two sets \( A, B \in \mathcal{F} \) such that \( A \subset B \). For example, \( \mathcal{F} = \{ S \subseteq [n] : |S| = k \} \) is an antichain of size \( \binom{n}{k} \). How large can an antichain be? The choice of \( k = \lfloor n/2 \rfloor \) gives an antichain of size \( \binom{n}{\lfloor n/2 \rfloor} \). In 1928, Emanuel Sperner proved that this is the largest possible antichain that we can have. In fact, we prove a slightly stronger statement.

**Theorem 1** (Sperner’s theorem). For any antichain \( \mathcal{F} \subset 2^{[n]} \),

\[
\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 1.
\]

Since \( \binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor} \) for any \( A \subseteq [n] \), we conclude that \( |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \).

**Proof.** We present a very short proof due to Lubell. Consider a random permutation \( \pi : [n] \to [n] \). We compute the probability of the event that a prefix of this permutation \( \{\pi_1, \ldots, \pi_k\} \) is in \( \mathcal{F} \) for some \( k \). Note that this can happen only for one value of \( k \), since otherwise \( \mathcal{F} \) would not be an antichain.

For each particular set \( A \in \mathcal{F} \), the probability that \( A = \{\pi_1, \ldots, \pi_{|A|}\} \) is equal to \( k! (n-k)! / n! \), corresponding to all possible orderings of \( A \) and \([n] \setminus A \). By the property of an antichain, these events for different sets \( A \in \mathcal{F} \) are disjoint, and hence

\[
\Pr[\exists A \in \mathcal{F}; A = \{\pi_1, \ldots, \pi_{|A|}\}] = \sum_{A \in \mathcal{F}} \Pr[A = \{\pi_1, \ldots, \pi_{|A|}\}] = \sum_{A \in \mathcal{F}} \frac{|A|! (n-|A|)!}{n!} = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}.
\]

The fact that any probability is at most 1 concludes the proof. \( \square \)

This has the following application. We note that the theorem actually holds for arbitrary vectors and any ball of radius 1, but we stick to the 1-dimensional case for simplicity.

**Theorem 2.** Let \( a_1, a_2, \ldots, a_n \) be real numbers of absolute value \( |a_i| \geq 1 \). Consider the \( 2^n \) linear combinations \( \sum_{i=1}^n \epsilon_i a_i \), \( \epsilon_i \in \{-1, +1\} \). Then the number of sums which are in any interval \((x-1, x+1)\) is at most \( \binom{n}{\lfloor n/2 \rfloor} / 2^n = O(1/\sqrt{n}) \).

An interpretation of this theorem is that for any random walk on the real line, where the \( i \)-th step is either \( +a_i \) or \( -a_i \) at random, the probability that after \( n \) steps we end up in some fixed interval \((x-1, x+1)\) is at most \( \binom{n}{\lfloor n/2 \rfloor} / 2^n = O(1/\sqrt{n}) \).
Therefore, if $I$ is a proper subset of $I'$ then only one of them can correspond to a sum inside $(x-1, x+1)$. Consequently, the sums inside $(x-1, x+1)$ correspond to an antichain and we can have at most $\binom{n}{\lfloor n/2 \rfloor}$ such sums.

**Theorem 3** (Bollobás, 1965). If $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are two sequences of sets such that $A_i \cap B_j = \emptyset$ if and only if $i = j$, then

$$\sum_{i=1}^{m} \left( \frac{|A_i| + |B_i|}{|A_i|} \right)^{-1} \leq 1.$$

Note that if $A_1, \ldots, A_m$ is an antichain on $[n]$ and we set $B_i = [n] \setminus A_i$, we get a system of sets satisfying the conditions above. Therefore this is a generalization of Sperner’s theorem.

**Proof.** Suppose that $A_i, B_i \subseteq [n]$ for some $n$. Again, we consider a random permutation $\pi : [n] \rightarrow [n]$. Here we look at the event that there is some pair $(A_i, B_i)$ such that $\pi(A_i) < \pi(B_i)$, in the sense that $\pi(a) < \pi(b)$ for all $a \in A_i, b \in B_i$. For each particular pair $(A_i, B_i)$, the probability of this event is $\left| A_i \right|! \left| B_i \right|! / (\left| A_i \right| + \left| B_i \right|)!$. On the other hand, suppose that $\pi(A_i) < \pi(B_i)$ and $\pi(A_j) < \pi(B_j)$. Hence, there are points $x_i, x_j$ such that the two pairs are separated by $x_i$ and $x_j$, respectively. Depending on the relative order of $x_i, x_j$, we get either $A_i \cap B_j = \emptyset$ or $A_j \cap B_i = \emptyset$, which contradicts our assumptions. Therefore, the events for different pairs $(A_i, B_i)$ are disjoint. We conclude that

$$\Pr[\exists i : (A_i, B_i) \text{ are separated in } \pi] = \sum_{i=1}^{m} \left( \frac{|A_i|!|B_i|!}{(|A_i| + |B_i|)!} \right) = \sum_{i=1}^{m} \left( \frac{|A_i| + |B_i|}{|A_i|} \right)^{-1} \leq 1.$$

This theorem has an application in the following setting. For a collection of sets $\mathcal{F} \subseteq 2^X$, we call $T \subseteq X$ a transversal of $\mathcal{F}$, if $\forall A \in \mathcal{F}; A \cap T \neq \emptyset$. One question is, what is the smallest transversal for a given collection of sets $\mathcal{F}$. We denote the size of the smallest transversal by $\tau(\mathcal{F})$.

A set system $\mathcal{F}$ is called $\tau$-critical, if removing any member of $\mathcal{F}$ decreases $\tau(\mathcal{F})$. An example of a $\tau$-critical system is the collection $\mathcal{F} = \binom{k+\ell}{k}$ of all subsets of size $k$ out of $k + \ell$ elements. The smallest transversal has size $\ell + 1$, because any set of size $\ell + 1$ intersects every member of $\mathcal{F}$, whereas no set of size $\ell$ is a transversal, since its complement is a member of $\mathcal{F}$. Moreover, removing any set $A \in \mathcal{F}$ decreases $\tau(\mathcal{F})$ to $\ell$, because then $\bar{A}$ is a transversal of $\mathcal{F} \setminus \{A\}$. This is an example of a $\tau$-critical system of size $\binom{k+\ell}{k}$, where $\tau(\mathcal{F}) = \ell + 1$ and $\forall A \in \mathcal{F}; |A| = k$.

Observe that if $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ is $\tau$-critical and $\tau(\mathcal{F}) = \ell + 1$, then there is a transversal $B_i, |B_i| = \ell$ for each $i$, which intersects each $A_j, j \neq i$. However, $B_i$ does not intersect $A_i$, otherwise it would also be a transversal of $\mathcal{F}$. Therefore, Theorem 3 implies the following.

**Theorem 4.** Suppose $\mathcal{F}$ is a $\tau$-critical system, where $\tau(\mathcal{F}) = \ell + 1$ and each $A \in \mathcal{F}$ has size $k$. Then

$$|\mathcal{F}| \leq \binom{k+\ell}{k}.$$
2 Intersecting families

Here we consider a different type of family of subsets. We call $F \subseteq 2^{[n]}$ intersecting, if $A \cap B \neq \emptyset$ for any $A, B \in F$. The question what is the largest such family is quite easy: For any set $A$, we can take only one of $A$ and $[n] \setminus A$. Conversely, we can take exactly one set from each pair like this - for example all the sets containing element 1. Hence, the largest intersecting family of subsets of $[n]$ has size exactly $2^{n-1}$.

A more interesting question is, how large can be an intersecting family of sets of size $k$? We assume $k \leq n/2$, otherwise we can take all $k$-sets.

**Theorem 5** (Erdős-Ko-Rado). For any $k \leq n/2$, the largest size of an intersecting family of subsets of $[n]$ of size $k$ is $\binom{n-1}{k-1}$.

Observe that an intersecting family of size $\binom{n-1}{k-1}$ can be constructed by taking all $k$-sets containing element 1. To prove the upper bound, we use an elegant argument of Katona. First, we prove the following lemma.

**Lemma 1.** Consider a circle divided into $n$ intervals by $n$ points. Let $k \leq n/2$. Suppose we have “arcs” $A_1, \ldots, A_t$, each $A_i$ containing $k$ successive intervals around the circle, and each pair of arcs overlapping in at least one interval. Then $t \leq k$.

**Proof.** No point $x$ can be the endpoint of two arcs - then they are either the same arc, or two arcs starting from $x$ in opposite directions, but then they do not share any interval.

Now fix an arc $A_1$. Every other arc must intersect $A_1$, hence it must start at one of the $k-1$ points inside $A_1$. Each such endpoint can have at most one arc. $\square$

Now we proceed with the proof of Erdős-Ko-Rado theorem.

**Proof.** Let $F$ be an intersecting family of sets of size $k$. Consider a random permutation $\pi : [n] \to [n]$. We consider each set $A \in F$ mapped onto the circle as above, by associating $\pi(A)$ with the respective set of intervals on the circle. Let $X$ be the number of sets $A \in F$ which are mapped onto contiguous arcs $\pi(A)$ on the circle. For each set $A \in F$, the probability that $\pi(A)$ is a contiguous arc is $nk!(n-k)!/n! = n/\binom{n}{k}$. Therefore,

$$E[X] = \sum_{A \in F} \Pr[\pi(A) \text{ is contiguous}] = \frac{n}{(n/k)} |F|.$$ 

On the other hand, we know by our lemma that $\pi(A)$ can be contiguous for at most $k$ sets at the same time, because $F$ is an intersecting family. Therefore,

$$E[X] \leq k.$$ 

From these two bounds, we obtain

$$|F| \leq \frac{k}{n} \binom{n}{k} = \binom{n-1}{k-1}.$$ 

$\square$