

1 Bounds on Ramsey numbers

Ramsey number of particular interest are the *diagonal Ramsey numbers* $R(s, s)$. The bound we have proved gives

$$R(s, s) \leq \binom{2s-2}{s-1} \leq \frac{4^s}{\sqrt{s}}.$$

This bound has not been improved significantly for over 50 years! All we know currently is that exponential growth is the right order of magnitude, but the base of the exponential is not known. The following is an old lower bound of Erdős. Note that to get a lower bound, we need to show that there is a large graph without cliques and independent sets of a certain size. Equivalently, we need to prove there is a 2-coloring such that there is no monochromatic clique of a certain size in either color. This is quite difficult to achieve by an explicit construction. (The early lower bounds on $R(s, s)$ were polynomial in s .)

The interesting feature of Erdős's proof is that he never presents a specific coloring. He simply proves that choosing a coloring at random almost always works! This was one of the first occurrences of the *probabilistic method* in combinatorics. The probabilistic method has been used in combinatorics ever since with phenomenal success, using much more sophisticated tools; we will return to this later.

Theorem 1. For $s \geq 3$,

$$R(s, s) > 2^{s/2}.$$

Proof. Let $n = 2^{s/2}$. Consider a random coloring of K_n where each edge is colored independently red or blue with probability $1/2$. For any particular s -tuple of vertices S , the probability that the clique on S has all edges of the same color is $2/2^{\binom{s}{2}}$. The number of s -tuples of vertices is $\binom{n}{s}$ and therefore the probability that there is at least one monochromatic s -clique is at most

$$\binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < \frac{2n^s}{s!2^{s(s-1)/2}} = \frac{2^{1+s/2}}{s!} < 1$$

for $s \geq 3$. Therefore, with non-zero probability, there is no monochromatic clique of size s and such a coloring certifies that $R(s, s) > 2^{s/2}$. \square

Determining Ramsey numbers exactly, even for small values of s , is a notoriously difficult task. The currently known diagonal values are: $R(2, 2) = 2$, $R(3, 3) = 6$, $R(4, 4) = 18$. $R(5, 5)$ is known to be somewhere between 43 – 49, and $R(6, 6)$ between 102 – 165. ¹

¹A famous quote from Paul Erdős goes as follows: “Imagine an alien force, vastly more powerful than us, demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. Then we should attempt to destroy the aliens.”

2 Ramsey's theorem for hypergraphs

Next, we will talk about “generalized graphs” which are called *hypergraphs*. While a graph is given by a collection of pairs (edges) on a set of vertices V , a hypergraph can contain “hyperedges” of arbitrary size. Thus a hypergraph in full generality is any collection of subsets of V . Hypergraphs of particular importance are *r-uniform hypergraphs*, where all hyperedges have size r . Thus graphs can be viewed as 2-uniform hypergraphs.

Definition 1. An *r-uniform hypergraph* is a pair $H = (V, E)$ where V is a finite set of vertices and $E \subseteq \binom{V}{r}$ is a set of hyperedges.

An *empty hypergraph* is a hypergraph with no hyperedges.

A *complete r-uniform hypergraph* is $K_n^{(r)} = (V, \binom{V}{r})$ where $|V| = n$.

A *subhypergraph of H induced by a set of vertices S* is $H[S] = (S, E \cap \binom{S}{r})$.

An *independent set* is a set of vertices S that induces an empty hypergraph.

A *clique* is a set of vertices T that induces a complete hypergraph.

We define Ramsey numbers for hypergraphs in a way similar to the previous lecture.

Definition 2. The *hypergraph Ramsey number* $R^{(r)}(s, t)$ is the minimum number n such that any *r-uniform hypergraph on n vertices* contains an independent set of size s or a clique of size t .

The *Ramsey number* $R_k^{(r)}(s_1, s_2, \dots, s_k)$ is the minimum number n such that any coloring of the edges of the complete hypergraph $K_n^{(r)}$ with k colors contains a clique of size s_i whose edges all have color i , for some i .

Theorem 2 (Ramsey for hypergraphs). For any $s, t \geq r \geq 1$, the Ramsey number $R^{(r)}(s, t)$ is finite and satisfies

$$R^{(r)}(s, t) \leq R^{(r-1)}\left(R^{(r)}(s-1, t), R^{(r)}(s, t-1)\right) + 1.$$

Similarly, it can be shown that the hypergraph Ramsey numbers for k colors are finite.

Proof. We know from the previous lecture that the Ramsey numbers $R^{(2)}(s, t) = R(s, t)$ are finite. It's also easy to see that $R^{(r)}(s, r) = s$ (because any hypergraph on s vertices is either empty or it contains some edge on r vertices). Similarly $R^{(r)}(r, t) = t$.

So it remains to prove the inductive step. Fix r, s, t and assume that $R^{(r)}(s-1, t)$, $R^{(r)}(s, t-1)$ and $R^{(r-1)}(u, v)$ for all u, v are finite. Let $n = R^{(r-1)}\left(R^{(r)}(s-1, t), R^{(r)}(s, t-1)\right) + 1$ and consider any *r-uniform hypergraph H* on n vertices. Fix a vertex $v \in V$ and define an $(r-1)$ -uniform hypergraph $L(v)$ on the remaining vertices: an $(r-1)$ -tuple R' is in $L(v)$, if and only if $R' \cup \{v\}$ is a hyperedge of H .²

Since $L(v)$ is an $(r-1)$ -uniform hypergraph on $R^{(r-1)}\left(R^{(r)}(s-1, t), R^{(r)}(s, t-1)\right)$ vertices, it has either an independent set of size $R^{(r)}(s-1, t)$ or a clique of size $R^{(r)}(s, t-1)$.

- If $L(v)$ has an independent set S of size $R^{(r)}(s-1, t)$, then we know that for any $(r-1)$ -tuple $R' \in \binom{S}{r-1}$, $\{v\} \cup R'$ is *not* a hyperedge in H . Also, by induction we can apply the Ramsey property to the induced hypergraph $H[S]$: it contains either an independent set S_1 of size $s-1$ or a clique T_1 of size t . In the first case, $S_2 \cup \{v\}$ is an independent set of size s in H , and in the second case, T_1 is a clique of size t in H .

² $L(v)$ is sometimes called the *link of v*. It generalizes the notion of a neighborhood in a graph.

- If $L(v)$ has a clique T of size $R^{(r)}(s, t-1)$, then we know that for any $(r-1)$ -tuple $R' \in \binom{T}{r-1}$, $T \cup \{v\}$ is a hyperedge of H . Again, by induction on $H[T]$, we obtain that $H[T]$ has either an independent set S_1 of size s or a clique T_1 of size $t-1$. In the first case, we are done and in the second case, $T_1 \cup \{x\}$ is a clique of size t in H .

□

The proof that the Ramsey numbers $R_k^{(r)}(s_1, s_2, \dots, s_k)$ are finite is more complicated but in the same spirit. Let us just state a corollary of this statement, Ramsey's theorem for coloring sets.

Theorem 3. *For any $t \geq r \geq 2$ and $k \geq 2$, there is n such that for any coloring $\chi : \binom{[n]}{r} \rightarrow [k]$, there is a subset $T \subset [n]$ of size $|T| = t$ such that all subsets $R \in \binom{T}{r}$ have the same color $\chi(R)$.*

Note that this corresponds to a special case where we color a complete r -uniform hypergraph with k colors and we want to find a monochromatic clique of size t in some color. Therefore, n can be chosen as $n = R_k^{(r)}(t, t, \dots, t)$.

3 Convex polygons among points in the plane

Ramsey theory has many applications. The following is a geometric statement which follows quite surprisingly from Ramsey's theorem for 4-uniform hypergraphs.

Theorem 4 (Erdős-Szekeres, 1935). *For any $m \geq 4$, there is n , such that given any configuration of n points in the plane, no three on the same line, there are m points forming a convex polygon.*

To deduce this theorem from Ramsey's theorem, we need only the following two geometric facts. We say that points are in a *convex position* if they form a convex polygon.

Fact 1. Among five points in the plane, no three on the same line, there are always four points in a convex position.

Fact 2. If out of m points, any four are in a convex position, then they are all in a convex position.

The first fact can be seen by checking three cases, depending on whether the convex hull of the 5 points consists of 3, 4 or 5 points. Note that this proves the statement of the theorem for $m = 4$.

The second fact holds because if one of the m points is inside the convex hull of others, then it is also inside a triangle formed by 3 other points and then these 4 are not in a convex position. Now we are ready to prove the theorem.

Proof. Given m , let $n = R^{(4)}(5, m)$ and consider any set of n points in the plane, no three on the same line. We define a 4-uniform hypergraph H on this set of n points. Let a 4-tuple of points u, v, x, y form a hyperedge if they are in a convex position. By Ramsey's theorem, H must contain either an independent set of size 5 or a clique of size m . However, an independent set of size 5 would mean that there are 5 points without any 4-tuple in a convex position. This would contradict Fact 1. Therefore, there must be a clique of size m which means m points where all 4-tuples are in a convex position. Then Fact 2 implies that these m points form a convex polygon. □