MAT 307: Combinatorics

Lecture 5: Ramsey Theory

Instructor: Jacob Fox

1 Ramsey's theorem for graphs

The metastatement of Ramsey theory is that "complete disorder is impossible". In other words, in a large system, however complicated, there is always a smaller subsystem which exhibits some sort of special structure. Perhaps the oldest statement of this type is the following.

Proposition 1. *Among any six people, there are three any two of whom are friends, or there are three such that no two of them are friends.*

This is not a sociological claim, but a very simple graph-theoretic statement: in other words, in any graph on 6 vertices, there is a triangle or three vertices with no edges between them.

Proof. Let $G = (V, E)$ be a graph and $|V| = 6$. Fix a vertex $v \in V$. We consider two cases.

- If the degree of v is at least 3, then consider three neighbors of v, call them x, y, z. If any two among $\{x, y, z\}$ are friends, we are done because they form a triangle together with v. If not, no two of $\{x, y, z\}$ are friends and we are done as well.
- If the degree of v is at most 2, then there are at least three other vertices which are not neighbors of v, call them x, y, z. In this case, the argument is complementary to the previous one. Either $\{x, y, z\}$ are mutual friends, in which case we are done. Or there are two among $\{x, y, z\}$ who are not friends, for example x and y, and then no two of $\{v, x, y\}$ are friends.

□

More generally, we consider the following setting. We color the edges of K_n (a complete graph on n vertices) with a certain number of colors and we ask whether there is a complete subgraph (a *clique*) of a certain size such that all its edges have the same color. We shall see that this is always true for a sufficiently large n. Note that the question about frienships corresponds to a coloring of K_6 with 2 colors, "friendly" and "unfriendly". Equivalently, we start with an arbitrary graph and we want to find either a clique or the complement of a clique, which is called an *independent set*. This leads to the definition of *Ramsey numbers*.

Definition 1. *A clique of size t is a set of t vertices such that all pairs among them are edges.*
An independent set of size s is a set of s vertices such that there is no edge between them.

Ramsey's theorem states that for any large enough graph, there is an independent set of size s or a clique of size t. The smallest number of vertices required to achieve this is called a *Ramsey number*.

Definition 2. *The Ramsey number $R(s, t)$ is the minimum number n such that any graph on n vertices contains either an independent set of size s or a clique of size t.*
The Ramsey number $R_k(s_1, s_2, \ldots, s_k)$ is the minimum number n such that any coloring of the edges of K_n with k colors contains a clique of size s_i in color i, for some i.

Note that it is not clear a priori that Ramsey numbers are finite! Indeed, it could be the case that there is no finite number satisfying the conditions of $R(s,t)$ for some choice of s,t. However, the following theorem proves that this is not the case and gives an explicit bound on $R(s,t)$.

Theorem 1 (Ramsey's theorem)**.** *For any $s,t \geq 1$, there is $R(s,t) < \infty$ such that any graph on $R(s,t)$ vertices contains either an independent set of size s or a clique of size t. In particular,*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

We remark that the bound given here is stronger than Ramsey's original bound.

Proof. We show that $R(s,t) \leq R(s-1,t) + R(s,t-1)$. To see this, let $n = R(s-1,t) + R(s,t-1)$ and consider any graph G on n vertices. Fix a vertex $v \in V$. We consider two cases:

- There are at least $R(s,t-1)$ edges incident with v. Then we apply induction on the neighbors of v, which implies that either they contain an independent set of size s, or a clique of size $t-1$. In the second case, we can extend the clique by adding v, and hence G contains either an independent set of size s or a clique of size t.

- There are at least $R(s-1,t)$ non-neighbors of v. Then we apply induction to the non-neighbors of v and we get either an independent set of size $s-1$, or a clique of size t. Again, the independent set can be extended by adding v and hence we are done.

Given that $R(s,t) \leq R(s-1,t) + R(s,t-1)$, it follows by induction that these Ramsey numbers are finite. Moreover, we get an explicit bound. First, $R(s,t) \leq \binom{s+t-2}{s-1}$ holds for the base cases where $s = 1$ or $t = 1$ since every graph contains a clique or an independent set of size 1. The inductive step is as follows:

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$$

by a standard identity for binomial coefficients. □

For a larger number of colors, we get a similar statement.

Theorem 2. *For any $s_1, \ldots, s_k \geq 1$, there is $R_k(s_1, \ldots, s_k) < \infty$ such that for any k-coloring of the edges of $K_n, n \geq R_k(s_1, \ldots, s_k)$, there is a clique of size s_i in some color i.*

We only sketch the proof here. Let us assume for simplicity that $k \geq 4$ is even. We show that

$$R_k(s_1, s_2, \ldots, s_k) \leq R_{k/2}(R(s_1, s_2), R(s_3, s_4), \ldots, R(s_{k-1}, s_k)).$$

To prove this, let $n = R_{k/2}(R(s_1, s_2), R(s_3, s_4), \ldots, R(s_{k-1}, s_k))$ and consider any k-coloring of the edges of K_n. We pair up the colors: $\{1,2\}, \{3,4\}, \{5,6\}$, etc. By the definition of n, there exists a subset S of $R(s_{2i-1}, s_{2i})$ vertices such that all edges on S use only colors $2i-1$ and $2i$. By applying Ramsey's theorem once again to S, there is either a clique of size s_{2i-1} in color $2i-1$, or a clique of size s_{2i} in color $2i$.

2 Schur's theorem

Ramsey theory for integers is about finding monochromatic subsets with a certain arithmetic structure. It starts with the following theorem of Schur (1916), which turns out to be an easy application of Ramsey's theorem for graphs.

Theorem 3. *For any $k \geq 2$, there is $n > 3$ such that for any k-coloring of $\{1, 2, \ldots, n\}$, there are three integers x, y, z of the same color such that $x + y = z$.*

Proof. We choose $n = R_k(3, 3, \ldots, 3)$, i.e. the Ramsey number such that any k-coloring of K_n contains a monochromatic triangle. Given a coloring $c : [n] \to [k]$, we define an edge coloring of K_n: the color of edge $\{i, j\}$ will be $\chi(\{i, j\}) = c(|j - i|)$. By the Ramsey theorem for graphs, there is a monochromatic triangle $\{i, j, k\}$; assume $i < j < k$. Then we set $x = j - i$, $y = k - j$ and $z = k - i$. We have $c(x) = c(y) = c(z)$ and $x + y = z$. $\square$

Schur used this in his work related to Fermat's Last Theorem. More specifically, he proved that Fermat's Last Theorem is false in the finite field Z_p for any sufficiently large prime p.

Theorem 4. *For every $m \geq 1$, there is p_0 such that for any prime $p \geq p_0$, the congruence*

$$x^m + y^m = z^m \pmod{p}$$

has a solution.

Proof. The multiplicative group Z_p^* is known to be cyclic and hence it has a generator g. Each element of Z_p^* can be written as $x = g^{mj+i}$ where $0 \leq i < m$. We color the elements of Z_p^* by m colors, where $c(x) = i$ if $x = g^{mj+i}$. By Schur's theorem, for p sufficiently large, there are elements $x, y, z \in Z_p^*$ such that $x' + y' = z'$ and $c(x') = c(y') = c(z')$. Therefore, $x' = g^{mj_x+i}$, $y' = g^{mj_y+i}$ and $z' = g^{mj_z+i}$ and

$$g^{mj_x+i} + g^{mj_y+i} = g^{mj_z+i}.$$

Setting $x = g^{j_x}$, $y = g^{j_y}$ and $z = g^{j_z}$, we get a solution of $x^m + y^m = z^m$ in Z_p. $\square$