MAT 307: Combinatorics

Lecture 3: Sperner's lemma and Brouwer's theorem

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1 Sperner's lemma

In 1928, young Emanuel Sperner found a surprisingly simple proof of Brouwer's famous Fixed Point Theorem: *Every continous map of an n-dimensional ball to itself has a fixed point.* At the heart of his proof is the following combinatorial lemma. First, we need to define the notions of simplicial subdivision and proper coloring.

Definition 1. *An n-dimensional simplex is a convex linear combination of $n+1$ points in a general position. I.e., for given vertices $v_1, \ldots, v_{n+1}$, the simplex would be*

$$S = \left\{ \sum_{i=1}^{n+1} \alpha_i v_i : \alpha_i \geq 0, \sum_{i=1}^{n+1} \alpha_i = 1 \right\}.$$

A simplicial subdivision of an n-dimensional simplex S is a partition of S into small simplices ("cells") such that any two cells are either disjoint, or they share a full face of a certain dimension.

Definition 2. *A proper coloring of a simplicial subdivision is an assignment of $n+1$ colors to the vertices of the subdivision, so that the vertices of S receive all different colors, and points on each face of S use only the colors of the vertices defining the respective face of S.*

For example, for $n = 2$ we have a subdivision of a triangle T into triangular cells. A proper coloring of T assigns different colors to the 3 vertices of T, and inside vertices on each edge of T use only the two colors of the respective endpoints. (Note that we do not require that endpoints of an edge receive different colors.)

Lemma 1 (Sperner, 1928)**.** *Every properly colored simplicial subdivision contains a cell whose vertices have all different colors.*

Proof. Let us call a cell of the subdivision a *rainbow cell*, if its vertices receive all different colors. We actually prove a stronger statement, namely that the number of rainbow cells is *odd* for any proper coloring.

Case $n = 1$. First, let us consider the 1-dimensional case. Here, we have a line segment (a, b) subdivided into smaller segments, and we color the vertices of the subdivision with 2 colors. It is required that a and b receive different colors. Thus, going from a to b, we must switch color an odd number of times, so that we get a different color for b. Hence, there is an odd number of small segments that receive two different colors.

Case $n = 2$**.** We have a properly colored simplicial subdivision of a triangle T. Let Q denote the number of cells colored $(1, 1, 2)$ or $(1, 2, 2)$, and R the number of rainbow cells, colored $(1, 2, 3)$. Consider edges in the subdivision whose endpoints receive colors 1 and 2. Let X denote the number of boundary edges colored $(1, 2)$, and Y the number of interior edges colored $(1, 2)$ (inside the triangle T). We count in two different ways:

- Over cells of the subdivision: For each cell of type Q, we get 2 edges colored $(1, 2)$, while for each cell of type R, we get exactly 1 such edge. Note that this way we count internal edges of type $(1, 2)$ twice, whereas boundary edges only once. We conclude that $2Q + R = X + 2Y$.
- Over the boundary of T: Edges colored $(1, 2)$ can be only inside the edge between two vertices of T colored 1 and 2. As we already argued in the 1-dimensional case, between 1 and 2 there must be an odd number of edges colored $(1, 2)$. Hence, X is odd. This implies that R is also odd.

General case. In the general n-dimensional case, we proceed by induction on n. We have a proper coloring of a simplicial subdivision of S using $n + 1$ colors. Let R denote the number of rainbow cells, using all $n + 1$ colors. Let Q denote the number of simplicial cells that get all the colors except $n + 1$, i.e. they are colored using $\{1, 2, \ldots, n\}$ so that exactly one of these colors is used twice and the other colors once. Also, we consider $(n - 1)$-dimensional faces that use exactly the colors $\{1, 2, \ldots, n\}$. Let X denote the number of such faces on the boundary of S, and Y the number of such faces inside S. Again, we count in two different ways.

- Each cell of type R contributes exactly one face colored $\{1, 2, \ldots, n\}$. Each cell of type Q contributes exactly two faces colored $\{1, 2, \ldots, n\}$. However, inside faces appear in two cells while boundary faces appear in one cell. Hence, we get the equation $2Q + R = X + 2Y$.
- On the boundary, the only $(n - 1)$-dimensional faces colored $\{1, 2, \ldots, n\}$ can be on the face $F \subset S$ whose vertices are colored $\{1, 2, \ldots, n\}$. Here, we use the inductive hypothesis for F, which forms a properly colored $(n - 1)$-dimensional subdivision. By the hypothesis, F contains an odd number of rainbow $(n - 1)$-dimensional cells, i.e. X is odd. We conclude that R is odd as well.

□

2 Brower's Fixed Point Theorem

Theorem 1 (Brouwer, 1911)**.** *Let B^n denote an n-dimensional ball. For any continuous map $f : B^n \to B^n$, there is a point $x \in B^n$ such that $f(x) = x$.*

We show how this theorem follows from Sperner's lemma. It will be convenient to work with a simplex instead of a ball (which is equivalent by a homeomorphism). Specifically, let S be a simplex embedded in R^{n+1} so that the vertices of S are $v_1 = (1, 0, \ldots, 0)$, $v_2 = (0, 1, \ldots, 0)$, ..., $v_{n+1} = (0, 0, \ldots, 1)$. Let $f : S \to S$ be a continuous map and assume that it has no fixed point.

We construct a sequence of subdivisions of S that we denote by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$. Each $\mathcal{S}_j$ is a subdivision of $\mathcal{S}_{j-1}$, so that the size of each cell in $\mathcal{S}_j$ tends to zero as $j \to \infty$.

Now we define a coloring of $\mathcal{S}_j$. For each vertex $x \in \mathcal{S}_j$, we assign a color $c(x) \in [n+1]$ such that $(f(x))_{c(x)} < x_{c(x)}$. To see that this is possible, note that for each point $x \in S$, $\sum x_i = 1$, and also $\sum f(x)_i = 1$. Unless $f(x) = x$, there are coordinates such that $(f(x))_i < x_i$ and also $(f(x))_{i'} > x_{i'}$. In case there are multiple coordinates such that $(f(x))_i < x_i$, we pick the smallest i.

Let us check that this is a proper coloring in the sense of Sperner's lemma. For vertices of S, $v_i = (0, \ldots, 1, \ldots, 0)$, we have $c(x) = i$ because i is the only coordinate where $(f(x))_i < x_i$ is possible. Similarly, for vertices on a certain faces of S, e.g. $x = \text{conv}\{v_i : i \in A\}$, the only coordinates where $(f(x))_i < x_i$ is possible are those where $i \in A$, and hence $c(x) \in A$.

Sperner's lemma implies that there is a rainbow cell with vertices $x^{(j,1)}, \ldots, x^{(j,n+1)} \in \mathcal{S}_j$. In other words, $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for each $i \in [n+1]$. Since this is true for each $\mathcal{S}_j$, we get a sequence of points $\{x^{(j,1)}\}$ inside a compact set S which has a convergent subsequence. Let us throw away all the elements outside of this subsequence - we can assume that $\{x^{(j,1)}\}$ itself is convergent. Since the size of the cells in $\mathcal{S}_j$ tends to zero, the limits $\lim_{j \to \infty} x^{(j,i)}$ are the same in fact for all $i \in [n+1]$ - let's call this common limit point $x^* = \lim_{j \to \infty} x^{(j,i)}$.

We assumed that there is no fixed point, therefore $f(x^*) \neq x^*$. This means that $(f(x^*))_i > x_i^*$ for some coordinate i. But we know that $(f(x^{(j,i)}))_i < x_i^{(j,i)}$ for all j and $\lim_{j \to \infty} x^{(j,i)} = x^*$, which implies $(f(x^*))_i \leq x_i^*$ by continuity. This contradicts the assumption that there is no fixed point.