

1 What is combinatorics?

Defining *combinatorics* within the larger field of mathematics is not an easy task. Typically, combinatorics deals with finite structures such as graphs, hypergraphs, partitions or partially ordered sets. However, rather than the object of study, what characterizes combinatorics are its *methods*: counting arguments, induction, inclusion-exclusion, the probabilistic method - in general, surprising applications of relatively elementary tools, rather than gradual development of a sophisticated machinery. That is what makes combinatorics very elegant and accessible, and why combinatorial methods should be in the toolbox of any mainstream mathematician.

Let's start with a few examples where combinatorial ideas play a key role.

1. **Ramsey theory.** In the 1950's, a Hungarian sociologist S. Szalai studied friendship relationships between children. He observed that in any group of around 20 children, he was able to find four children who were mutual friends, or four children such that no two of them were friends. Before drawing any sociological conclusions, Szalai consulted three eminent mathematicians in Hungary at that time: Erdős, Turán and Sós. A brief discussion revealed that indeed this is a mathematical phenomenon rather than a sociological one. For any symmetric relation R on at least 18 elements, there is a subset S of 4 elements such that R contains either all pairs in S or none of them. This fact is a special case of Ramsey's theorem proved in 1930, the foundation of *Ramsey theory* which developed later into a rich area of combinatorics.
2. **Tournament paradox.** Suppose that n basketball teams compete in a tournament where each pair of teams plays exactly one game. The organizers want to award the k best teams. However, whichever k teams they pick, there is always another team that beats them all! Is this possible? It can be proved using a random construction that for any $k > 0$ there is $n > k$ such that this can indeed happen.
3. **Brouwer's Theorem.** In 1911, Luitzen Brouwer published his famous Fixed Point Theorem:
Every continuous map $f : B^n \rightarrow B^n$ (where B^n is an n -dimensional ball) has a fixed point, $f(x) = x$.
The special case of $n = 1$ follows easily from the intermediate value theorem. For higher dimensions, however, the original proof was complicated. In 1928, Emanuel Sperner found a simple combinatorial result which implies Brouwer's fixed point theorem in an elegant way. The proof of Sperner's lemma is equally elegant, by double counting.
4. **Borsuk's conjecture.** In 1933, Karol Borsuk published a paper which contained a proof of a conjecture stated by Stanislaw Ulam:
Every continuous map $f : S^n \rightarrow R^n$ (where S^n is an n -dimensional sphere) maps two antipodal points to the same value, $f(x) = f(-x)$.

Borsuk also asked whether the following conjecture is true:

Every set $S \subset \mathbb{R}^n$ of finite diameter can be partitioned into $n + 1$ sets of strictly smaller diameter.

The example of an n -dimensional regular simplex shows that $n + 1$ parts are necessary, since we need to separate the $n + 1$ vertices into different parts. Borsuk's conjecture was proved for smooth bodies, using the Borsuk-Ulam theorem above. However, the general case was open until 1993 when Kahn and Kalai disproved it in a dramatic way. They constructed discrete sets of points that cannot be partitioned into fewer than $1.2\sqrt{n}$ parts.

5. Littlewood and Offord studied the following problem in 1943:

Given complex numbers z_1, z_2, \dots, z_n of absolute value $|z_i| \geq 1$, what is the maximum number of distinct sums $\sum_{i=1}^n \pm z_i$ that lie inside some unit disk? Kleitman and Katona proved that the maximum number is $\binom{n}{n/2}$, using the methods of extremal combinatorics.

2 Graph theory

Let us begin with an area of combinatorics called *graph theory*. What we mean by a *graph* here is not the graph of a function, but a structure consisting of *vertices* some of which are connected by *edges*.

Definition 1. A graph is a pair $G = (V, E)$ where V is a finite set whose elements we call vertices and $E \subseteq \binom{V}{2}$ is a collection of pairs of vertices that we call edges.

Some basic examples of graphs are:

- K_n : the complete graph on n vertices, where every pair of vertices forms an edge.
- $K_{s,t}$: the complete bipartite graph, where $V = S \cup T$, $|S| = s$, $|T| = t$, and every pair in $S \times T$ forms an edge.
- C_ℓ : the cycle of length ℓ , where $V = \{v_0, v_1, \dots, v_{\ell-1}\}$ and $\{v_i, v_j\} \in E$ if $j = i + 1 \pmod{\ell}$.

A famous story which stands at the beginning of graph theory is the problem of the bridges of Königsberg. Königsberg had 7 bridges connecting different parts of the town. Local inhabitants were wondering whether it was possible to walk across each bridge exactly once and return to the same point. In the language of graph theory, the bridges are edges connecting different vertices, i.e. parts of town. The bridges of Königsberg looked like the following graph (or rather, “multigraph”, since here multiple edges connect the same pair of vertices):

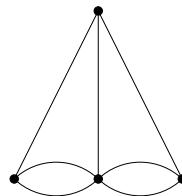


Figure 1: The graph of Königsberg.

The question is whether it is possible to walk around the graph so that we traverse each edge exactly once and come back to the same vertex. Such a walk is called an *Eulerian circuit* and graphs where this is possible are called *Eulerian* (after a paper by Euler where this question was considered in 1736). The following result shows that the graph of Königsberg is *not* Eulerian. First, we need some definitions.

Definition 2. We say that an edge e is incident with a vertex v , if $v \in e$. Two vertices are adjacent, if they form an edge. The degree $d(v)$ of a vertex is the number of edges incident with v .

Definition 3. A walk is a sequence of vertices (v_1, v_2, \dots, v_k) (with possible repetition) such that each successive pair (v_i, v_{i+1}) forms an edge.

An Eulerian circuit is a walk where $v_k = v_1$ and each edge appears exactly once.

A path is a walk (v_1, v_2, \dots, v_k) without repetition of vertices.

A graph G is called connected if there is a path between any pair of vertices.

Theorem 1. A (multi)graph G is Eulerian if and only if it is connected and the degree of every vertex is even.

Proof. Suppose G has an Eulerian circuit. Draw arrows along the edges as they are traversed by the circuit. Then the circuit induces a connecting path for any pair of vertices. Also, for every vertex the number of incoming edges equals the number of outgoing edges. Hence the degree of each vertex is even.

Conversely, assume that G is connected and all degrees are even. First, we prove the following: the set of edges E can be decomposed into a union of disjoint cycles C_1, \dots, C_n . This is true, because for any graph with even degrees, we can find a cycle by walking arbitrarily until we hit our own path. Then we remove the cycle, all degrees are still even, and we continue as long as there are any edges in the graph.

Now we construct the Eulerian circuit L inductively as follows. Start with $L := C_1$. We incorporate the cycles into L one by one. At any point, there must be an unused cycle C_j that intersects L , otherwise the graph is disconnected. Then, we add C_j to the circuit L by walking along L up to the first point where we hit C_j , then walk around C_j and continue along L . We repeat this procedure until all cycles are exhausted. Since the cycles cover each edge exactly once, our circuit also covers each edge exactly once at the end. \square