MAT 307 - Spring 2009

Assignment 4

Due: April 15

The solution for each problem should be no longer than one page.

Problem 1. [4 points]
Let G_1 and G_2 be two graphs on the same vertex set V. Prove that the chromatic number of their union $G_1 \cup G_2$ (we take the union of the edge sets of both graphs) satisfies

$$\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2).$$

Use this to show that if $H_1, \ldots, H_t$ are bipartite graphs whose union is a complete graph on n vertices, then $t \geq \log_2 n$.

Problem 2. [6 points]
Let k be a positive integer. Suppose T is a tournament with at least k vertices such that every subset U of k vertices from T is dominated by some vertex $v \notin U$, i.e., $v \longrightarrow u$ for all $u \in U$. We have seen in class that there are such tournaments with $O(k^2 2^k)$ vertices. Prove that there is no such tournament with 2^k vertices.

Problem 3. [6 points]
Let $A_1, \ldots, A_m$ be subsets of an n-element sets such that $|A_i|$ is not divisible by 6 for every i, but the sizes of all pairwise intersections $|A_i \cap A_j|$ are divisible by 6. Prove that $m \leq 2n$.

Problem 4. [6 points]
Prove that every three-uniform hypergraph with n vertices and $m \geq n/3$ hyperedges contains an independent set of size at least $\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}$.

Problem 5. [8 points]
Let F be a finite collection of binary strings of finite length and assume no member of F is a prefix of another one. Let N_i denote the number of strings of length i in F. Prove that

$$\sum_i \frac{N_i}{2^i} \leq 1.$$