MAT 307 - Spring 2009

Assignment 2

Due: March 23

The solution for each problem should be no longer than one page.

Problem 1. [4 points]
Let G be a graph with n vertices and m edges. Prove that the number of triangles in G is at least

$$\frac{4m}{3n}\left(m - \frac{n^2}{4}\right).$$

Show that this bound is tight when $m = n^2/3$.

Problem 2. [4 points]
Show that any 2-coloring of the edges of K_n with $n \geq 6$ contains at least $\frac{1}{20}\binom{n}{3}$ monochromatic triangles.
Bonus: Show that, as n tends to infinity, the fraction of triangles which must be monochromatic tends to $1/4$.

Problem 3. [6 points]
Let X be a set of n points in a circle of radius 1. Prove that there are at most $\lfloor n^2/4 \rfloor$ pairs of distinct points in X of distance more than $\sqrt{3}$.
Show that this bound is tight.

Problem 4. [6 points]
Let X be a set of n distinct points in the plane. Prove that the number of pairs of points in X with distance one is at most $cn^{3/2}$, for some absolute constant c.

Problem 5. [6 points]
Let D be a directed graph on n vertices such that the outdegree of each vertex is more than $\log_2 n$. Prove that D contains an even directed cycle.
Hint: Prove that D has a bipartite subgraph such that every vertex has outdegree at least one.