MAT 307 - Spring 2009

Assignment 1

Due: February 25

The solution for each problem should be no longer than one page.

Problem 1. [2 points]
Prove that there is no graph with an odd number of vertices of odd degree.

Problem 2. [4 points]
Let $a_1, a_2, \ldots, a_n$ be n not necessarily distinct integers. Prove that there is a set of consecutive numbers $a_k, a_{k+1}, \ldots, a_\ell$ whose sum is divisible by n.

Problem 3. [4 points]
Prove that every set of $2^n + 1$ vectors in $\mathbb{Z}^n$ (integer coordinates) contains a pair of distinct points whose mean also has integer coordinates.

Problem 4. [4 points]
Prove that for every $k \geq 2$ there exists $n_0 = n_0(k)$ such that every coloring of $1, 2, \ldots, n_0$ in k colors contains three distinct numbers $1 \leq a, b, c \leq n_0$ that have the same color and satisfy $a \cdot b = c$.

Problem 5. [6 points]
Prove that as n tends to infinity, the probability that a random permutation of n elements does not have a 2-cycle tends to $e^{-1/2}$.

Problem 6. [6 points]
A *transitive tournament* is an orientation of a complete graph for which the vertices can be numbered so that (i, j) is a directed edge if and only if $i < j$.

- Show that every orientation of the complete graph K_n contains a transitive tournament on $\lfloor \log_2 n \rfloor$ vertices.
- Show that if $k \geq 2 \log_2 n + 2$, then there is an orientation of K_n with no transitive tournament on k vertices.

Problem 7. [6 points]
Let $g_1(x), \ldots, g_k(x)$ be bounded real functions and $f(x)$ be another real function. Suppose that there are positive constants ϵ and δ such that if $f(x) - f(y) > \epsilon$, then $\max_i((g_i(x) - g_i(y)) > \delta$. Prove that f is also bounded.