1 September 7: Intro to Ramsey Numbers

Summary: "Complete disorder is impossible." More explicitly, any very large system contains a subsystem which is well organized.

Proposition 1.1 (Putnam 1952.). *Among any six people, there are three of them any two of whom are friends, or else no two of whom are friends.*

Proof. Let $G = (V, E)$ be a graph with $|V| = 6$.

Case 1.1.1. *There is a vertex of degree* ≥ 3.

The adjacent vertices must either include one edge, forming a triangle, or must have no edges, forming an empty triangle.

Case 1.1.2. *There is not.*

Symmetry with Case 1. □

Definition. *A* clique *is a complete subgraph, an* independent set *is an empty subgraph.*

Definition. $R(s,t)$ *is the minimum number* s *such that any graph on* n *vertices contains a clique of order* s *or an independent set of order* t. *Ex:* $R(3,3) = 6$.

Definition. $R_k(s_1, \ldots, s_k)$ *- here you color the graph in* k *colors and want a monochromatic clique of size* s_i *in color* c_i. *The "normal" Ramsey number is the special case* $k = 2$.

That Ramsey numbers *exist* follows from Ramsey's Theorem.

Theorem 1.2 (Ramsey, Erdős-Szekeres). $R(s,t)$ *exists (and* $R_k(s_1, \ldots, s_k)$ *in general).*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Proof.

Case 1.2.1. *There is a vertex of degree* $\geq R(s-1,t)$.

As before, look at neighbors.

Case 1.2.2. *There is not.*

Symmetry with Case 1. □

$R(2,t) = t$ and $R(s,2) = s$. Using Pascal's Identity,

$$R(s,t) \leq \binom{s+t-2}{s-1}$$

Specifically,

$$R(s,s) \leq \frac{c4^s}{\sqrt{s}}$$

Using the probabilistic method,

$$R(s, s) \geq 2^{s/2}$$

$R(4, 4) = 18$, but already $R(5, 5)$ is not known. It is known that $43 \leq R(5, 5) \leq 49$ and $102 \leq R(6, 6) \leq 165$.

Proof of the existence of the more general Ramsey number $R_k(s_1, \dots, s_k)$.

One way: Assume k even. Then,

$$R_k(s_1, \dots, s_k) \leq R_{k/2}(R(s_1, s_2), R(s_3, s_4), \dots, R_(s_{k-1}, s_k)).$$

Use induction.

Another way:

$$R_k(s_1, \dots, s_k) \leq R_k(s_1 - 1, s_2 \dots, s_k) + R_k(s_1, s_2 - 1, \dots, s_k) + \dots + R_k(s_1, s_2, \dots, s_k - 1)$$

Proof of this along the lines of previous proofs. Now, use induction (on sum of the s_i for instance). □

What is $R_k(3, 3, \dots, 3)$? We have the bounds:

$$2^k \leq R_k(3, 3, \dots, 3) \leq (k + 1)!$$

Open problem: do these Ramsey numbers grow faster than exponential in k?

To get the lower bound, form a complete bipartite graph on 2^k vertices. Color each of the parts the same way. Use induction. No monochromatic triangles.

Also, can get a better lower bound of $(\sqrt{5})^k$ for k even. In general, we have the bound $R_k(3, \dots, 3) - 1 \geq (R_t(3, \dots, 3) - 1)(R_{k-t}(3, \dots, 3) - 1)$ for k a multiple of t.

Theorem 1.3 (Schur)**.** *For any k there exists n such that for any k-coloring of $\{1, \dots, n\}$, there exist x, y, z of the same color such that $x + y = z$.*

Proof. Pick $n = R_k(3, \dots, 3)$. Given a coloring $c : [n] \to [k]$, define an edge-coloring of K_n. The color of edge $\{i, j\}$ is $\chi(\{i, j\}) = c(|j - i|)$. Then, there exists a monochromatic triangle with vertices i, j, k. Assume $i < j < k$. Then,

$$c(j - i) = c(k - j) = c(k - i).$$

Then $x = j - i$, $y = k - j$, and $z = k - i$ is the desired monochromatic solution. □

Can modify proof slightly to make sure x, y are distinct. Another variation: modify to $x^2 + y^2 = z^2$. Answer not known even for two colors.

If we modify to the linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$, the theorem holds iff some non-empty subset of the coefficients sum to 0. This is a special case of Rado's theorem.

Theorem 1.4. *For all $m \geq 1$ there exists p_0 such that for all primes $p > p_0$ the congruence*

$$x^m + y^m \equiv z^m \pmod p$$

has a nontrivial solution.

Proof. The multiplicative group $\mathbb{Z}_p^\times$ is cyclic, i.e. has a generator g. Each element x can be written as g^{mj+i} for $0 \le i \le m-1$. Color x with $c(x) = i$, so that there are m colors. By Schur's Theorem, there exist $x', y', z' \in \mathbb{Z}_p^\times$ of the same color and with $x' + y' \equiv z'$. We have

$$\begin{aligned} x' &= g^{mj_1+i} \\ y' &= g^{mj_2+i} \\ z' &= g^{mj_3+i} \end{aligned}$$

Divide out by g^i. Now set $x = g^{j_1}$, $y = g^{j_2}$, and $z = g^{j_3}$, and we are done. $\square$

Remark. *Note that we needed the "p sufficiently large" so that we could apply Schur's Theorem.*

2 September 12: Hypergraph Ramsey Numbers I

Definition. *A* hypergraph *is a pair $H = (V, E)$ of a* vertex set *V and an* edge set *E, where elements of the edge set are (distinct) subsets of V. The hypergraph H is* r-uniform *if all the edges have r vertices.*

Definition. *The* hypergraph Ramsey number *$R^r(s,t)$ is the minimum n such that every red-blue coloring of the edges of the complete r-uniform hypergraph K_n^r contains a red s-set or blue t-set.*

Theorem 2.1. *$R^r(s,t)$ exists for all r, s, t and*

$$R^r(s,t) \le R^{r-1}(R^r(s-1,t), R^r(s,t-1)) + 1 = N$$

Proof. We have $R^r(s,r) = s$ and $R^r(r,t) = t$, so proving the inequality proves inductively the existence of the number $R^r(s,t)$.

Consider then a complete two-colored r-uniform hypergraph H on N vertices. Pick a vertex v. We rearrange the remaining vertices into a $(r-1)$-uniform hypergraph H' by coloring every $(r-1)$-set $T \subset V - \{v\}$ with the color of $T \cup \{v\}$. There must be in H' either a red $R^r(s-1,t)$-set or a blue $R^r(s,t-1)$-set. WLOG assume the former and let this red $R^r(s-1,t)$-set be V_0.

We now consider V_0 as a subset of the vertex set of H. All edges of H are red that include $r-1$ vertices from V_0, as well as the vertex v. By the cardinality of V_0, there must be either a red $(s-1)$-set or a blue t-set within it. In the former case, adding v creates a red s-set, and in the latter case, we already have a blue t-set. Therefore, $R^r(s,t) \le N$, as desired. $\square$

Remark. *This bound is awful!!! It goes up by the Ackermann hierarchy...*

Let $N(m)$ be the minimum n such that every n points in general position in $\mathbb{R}^2$ contains m of them in convex position. $N(3) = 3$, $N(4) = 5$, $N(5) = 9$, (computer-aided) $N(6) = 17$. Erdos: "The Happy Ending Problem." Conjecture: $2^{m-2} + 1$. Shown to be a lower bound.

Proposition 2.2. *$N(m)$ exists for each m.*

Proof 1. We here show that $N(m) \le R^4(5,m)$. Color a set of 4 points red if not in convex position and blue otherwise. Now, there can be no red 5-set since $N(4) = 5$, so there must be a blue m-set of points. Since all 4-subsets are convex, the whole set is (by a triangulation argument). $\square$

Proof 2. (By Tarsy, during an exam.) $N(m) \leq R^3(m, m)$. Number the vertices and let the triangle $\{i, j, k\}$ be colored red if i, j, k clockwise. A non-convex polygon cannot have all clockwise (or counterclockwise) triangle subsets. □

Another, better, proof gives the upper bound $\binom{2m-4}{m-2} + 1$. This has since been improved by a factor of roughly 2.

Theorem 2.3 (Erdös-Rado, 1952)**.** *$R^r(s, u) \leq 2^{\binom{t}{r-1}+1} = N$ where $t = R^{r-1}(s-1, u-1) + 1$.*

Proof of case $r = 3$, general case is a similar stepping-down argument. Suppose we have the coloring $c : [N]^3 \to$ {red,blue}.

Objective: Find t vertices $v_1, \ldots, v_t$ such that for all $i < j$ all triples $\{v_i, v_j, v_k\}$ (for $k > j$) have the same color, denoted $\chi(i, j)$. This would solve the problem.

Pick the vertices in rounds. At some point, have $v_1, \ldots, v_m$ and a pool S_m of some of the remaining vertices such that adding any element of S_m to an existing pair v_i, v_j creates a triple with the same color.

In round 1, pick some v_1. Afterwards, suppose we have finished round m. Pick $v_{m+1} \in S_m$ arbitrarily. For each v left in the pool, consider the color vector $(c_1, \ldots, c_m)$ where $c_i = c(\{v_i, v_{m+1}, v\})$, which are the only triples we are worried about. Now, partition $S_m - \{v_{m+1}\}$ into equivalence classes based upon color vector. Let S_{m+1} be the largest of these equivalence classes. Then,

$$|S_{m+1}| \geq 2^{-m}(|S_m| - 1)$$

Combining these inequalities over all m gives the desired bound. □

Remark. *This bound is much better than the Ackermann-type bounds - it is, roughly, a tower of 2's of height r.*

In general, get a tower of 2's one notch higher in upper bound than in lower bound. Improving the gap for 3-uniformity would improve all higher by "stepping up."

A weaker, probabilistic-method bound:

$$R^r(s, s) \geq 2^{c_r s^{r-1}} = N$$

Proof. Color the r-tuples of an N-set uniformly at random. Each has probability of $2^{1-\binom{s}{r}}$ of being monochromatic. The number of s-sets is $\binom{N}{s}$, so the expected number is

$$2^{1-\binom{s}{r}}\binom{N}{s}$$

We win if this is strictly less than 1, which it is by [calculation stuff]. □

3 September 14: Hypergraph Ramsey Numbers II

PSET 1 will be handed out Monday Sep 19, and will be due Monday Oct 3.

One source for lectures:

- "Ramsey Theory," by Graham, Rothschild, and Spencer. Including Chapters 1, 2 (+new developments), today section 4.7 - the stepping-up lemma (Erdös-Hajnal).

Last class we worked with upper bounds. Today, lower bounds:

$$\begin{aligned} R^2(s,s) &\geq 2^{s/2} \\ R^3(s,s) &\geq 2^{(1+o(1))s^2/6} \\ R^4(s,s) &\geq 2^{2^{cs^2}} \\ R^5(s,s) &\geq 2^{2^{2^{cs^2}}} \\ &\cdots \end{aligned}$$

Definition. *We write $N \to (s)^r_k$ (read "N arrows $(s)^r_k$") if every k-coloring of $[N]^r$ contains a monochromatic s-set. Otherwise, we write $N \not\to (s)^r_k$.*

Lemma 3.1 (Stepping-up Lemma)**.** *If $n \not\to (s)^r$ then $2^n \not\to (2s+r-4)^{r+1}$, for all $r \geq 3$.*

It is not known if $R^3(s,s) \geq 2^{2^{cs}}$. However, $R^3_4(s) = R^3(s,s,s,s) \geq 2^{2^{cs}}$ is known:

Theorem 3.2. *If $n \not\to (s)^2_2$ then $2^n \not\to (s+1)^3_4$.*

4 September 19: Van der Waerden, etc.

PROBLEM SET 1

For a graph H and $n \in N$, let $c(H,n)$ be the minimum fraction, over all 2-edge-colorings of K_n, of the number of (unlabeled or labeled - it gives the same answer) copies of H which must be monochromatic. So, if $|H| = h$, then there are $n(n-1)\cdots(n-h+1)$ labeled copies possible.

Problem 4.1. *Show that*

$$c(H) = \lim_{n\to\infty} c(H,n)$$

exists and is positive.

Problem 4.2. *Show that $c(K_3) = \frac{1}{4}$.*

Problem 4.3. $c(K_{s,t}) = 2^{1-st}$.

Jensen's Inequality and double-counting may be helpful.

Problem 4.4. *Prove that for each positive integer r, there exists $N = N(r)$ such that, for every r-coloring of the edges of the complete graph on the $N \times N$ grid, there are i, j, i', j' such that edges $((i,j),(i',j))$ and $((i,j'),(i',j'))$ have the same color, as do $((i,j),(i,j'))$ and $((i',j),(i',j'))$.*

Conjecture 4.1 (**You get an A+ if you prove this!**)**.** $N(r) = r^{O(1)}$.

Conjecture 4.2 (Implied by Sidorenko's conjecture.)**.** *If H is bipartite with m edges then $c(H) = 2^{1-m}$.*

Theorem 4.3 (Van der Waerden). *For all k, r there exists $w = w(k, r)$ such that every r-coloring of $[w]$ contains a monochromatic k-term arithmetic progression.*

Conjecture 4.4 (Scemerédi). *For all k and $\epsilon > 0$ there exists $N = N(k, \epsilon)$ such that every $S \subset [N]$ with $|S| \geq \epsilon N$ contains a k-term arithmetic progression.*

Roth proved the $k = 3$ case. Gowers gave a better bound. Green and Tao proved that the primes contain arbitrarily long arithmetic progressions.

Theorem 4.5 (Hales-Jewett). *For all k, r there exists n such that every r-coloring of $[k]^n$ contains a monochromatic combinatorial line. A* combinatorial line *is a sequence of points, where each coordinate either increases along them or else is constant. There are thus $(k+1)^n - k^n$ possible combinatorial lines.*

Why does Hales-Jewett imply van der Waerden? Given a coloring of the integers, color the cube based upon the color of the sum of the coordinates, for instance. Other linear functions will do too.

Proof of Hales-Jewett. Can also find beautiful proof (with typos, alas) online, search for "coloring Hales-Jewett."

Use induction on k. The base case of $k = 2$ is easy. Assume holds for $k - 1$ then.

Let $t_1, \ldots, t_r$ be a rapidly increasing (i.e., they get BIG!!!!) sequence of positive integers with $n = t_1 + \ldots + t_r$. Consider an r-coloring of $[k]^n$ Define an induced $r^{k^{n-t_r}}$-coloring on $[k]^{t_r}$, coloring each x according to the function $f : [k]^{n-t_r} \to [r]$ where $f(w)$ is the color of (w, x).

By induction we find a line L_r in $[k]^{t_r}$ such that all points in that line except possibly the point where the variable coordinates is k. We need $t_r \geq HJ(k-1, r^{k^{n-t_r}})$, which works if t_r is REALLY HUGE.

Next step: Pass to $[k]^{n-t_r} \times L_r$, apply the same argument with

$$\left([k]^{n-t_r-t_{r-1}} \times L_r\right) \times [k]^{t_{r-1}}$$

Find that as long as

$$t_{r-1} \geq HJ(k-1, r^{k^{n-t_r-t_{r-1}+1}})$$

can find L_{r-1} a combinatorial line in $[k]^{t_{r-1}}$ such that the color of a point in $[k]^{n-t_r-t_{r-1}} \times L_{r-1} \times L_r$ does not depend on which point in $L_{r-1} \times L_r$ you choose except if you change a k to a non-k (or vice versa).

Proceed thusly until finish. Namely, need

$$t_{r-i} \geq HJ\left(k-1, r^{k^{n-t_{r-i+1}-\cdots-t_r+i}}\right)$$

and find combinatorial lines $L_i \in [k]^{t_i}$ such that the following property holds, which we call *Property P*:

- The color of a point in $L_1 \times L_2 \times \cdots \times L_r$ only depends on which coordinates are k.

This now really boils down to $HJ(2, r) = r$, since only care about k or not-k. Namely, consider the $r + 1$ points $(k-1, \ldots, k-1)$, $(k, k-1, \ldots, k-1)$, $(k, k, k-1, \ldots, k-1)$, $\ldots$, $(k, k, \ldots, k)$. Two of these points will be of the same color - these are the last two points of a combinatorial line. Now, we know the last two are the same color and, by Property P, the first $k-1$ are as well, since they all share the same coordinates of value k. Hence, the entire combinatorial line must be monochromatic. □

5 September 26: Hales-Jewett, etc.

There is a density version of Hales-Jewett, which was originally proven using ergodic theory. Finally, the Polymath Project (online collaboration group) solved it using a *combinatorial proof.*

We will now give an alternate proof of Hales-Jewett. The last one (which was essentially the original proof) gave bad Ackermann-type bounds. This is a much nicer one, improving to Wowzer-type.

Proof by Shelah. We use induction on k. The idea: Where the previous proof used $HJ(k-1)$ at each step and finished with $HJ(2)$, this proof uses $HJ(2)$ at each step and finishes with $HJ(k-1)$. (Intermediate-type proofs are possible too, but they are not as good as this one.)

Let $t_1, \ldots, t_m$ be an increasing sequence of positive integers, where $m = HJ(k-1, r)$. Let $n = t_1 + \ldots + t_m$. Consider an r-coloring of

$$[k]^n = [k]^{t_1} \times [k]^{t_2} \times \cdots \times [k]^{t_m} = [k]^{n-t_m} \times [k]^{t_m}.$$

Define a $r^{k^{n-t_m}}$-coloring of $[k]^{t_m}$ by coloring a point $x \in [k]^{t_m}$ by the function $f : [k]^{n-t_m} \to [r]$ given by $f(w) =$ the color of (w, x) in the original coloring.

By $HJ(2)$ find a combinatorial line L_m in $[k]^{t_m}$ such that the first two points in the line have the same color. To find this, we need that

$$t_m \geq HJ\left(2, r^{k^{n-t_m}}\right) = r^{k^{n-t_m}}.$$

Now, we can pass to the subspace $[k]^{n-t_m} \times L_m$, where changing the last coordinate of a point from a 1 to a 2 (or vice versa) does not change the color of the point.

Here, we run the same argument again by writing

$$[k]^{n-t_m} \times L_m = \left([k]^{n-t_m-t_{m-1}} \times L_m\right) \times [k]^{t_{m-1}}$$

We require

$$t_{m-1} \geq HJ\left(2, r^{k^{n-t_m-t_{m-1}+1}}\right) = r^{k^{n-t_m-t_{m-1}+1}}$$

and get a combinatorial line L_{m-1} in $[k]^{t_{m-1}}$ such that the color of a point in $[k]^{n-t_m-t_{m-1}} \times L_{m-1} \times L_m$ is not changed by switching either of the last two coordinates from a 1 to a 2 (or vice versa).

We continue in this fashion until we obtain a subspace $L_1 \times \cdots \times L_m$, such that changing any coordinate from 1 to 2 (or vice versa) does not change the color of a point. Now, we consider the subspace containing those points having no 1's in any coordinates. Since $m = HJ(k-1, r)$, we then get a combinatorial line L_0 in $L_1 \times \cdots \times L_m$ such that the last $k-1$ points in the line have the same color. Because of the 1-2 changing properties of all the L_i, the first point in line L_0 must have the same color as the other points, and we are done. □

What are the bounds in this proof? We needed t_{i+1} double-exponential in t_i. So $n \approx t_m$ is roughly a tower of 2's of height $2m = 2HJ(k-1, r)$, which gives Wowzer bounds.

Now, a new type of problem:

Question: How many edges can a graph G on n vertices have with no cycle?
Answer: $n-1$ (forests)

What about no *odd* cycle? Answer: $\lfloor \frac{n^2}{4} \rfloor$. Because...

Proposition 5.1. *A graph is bipartite iff it has no odd cycle.*

Proof. If it is bipartite, it clearly has no odd cycle (try to 2-color the vertices of the cycle and you get a problem). The other way: If it has no odd cycle, pick a vertex to start a 2-coloring at and greedily continue, adding neighbors in the opposite color and so on, with no contradictions since there is no odd cycle. □

(The maximal number of edges in an n-vertex bipartite graph is, by AM-GM, achieved when the parts are equal.)

Theorem 5.2 (Mantel 1907)**.** *A triangle-free graph $G = (V, E)$ has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.*

Proof. Let d_x be the degree of vertex x. If $(x, y) \in E$ then $d_x + d_y \le n$, since the graph is triangle-free. Then, if $m = |E|$, we have

$$mn \ge \sum_{(x,y)\in E} (d_x + d_y) = \sum_{x \in V} d_x^2.$$

Now, by Cauchy-Schwarz,

$$n \sum_{x\in V} d_x^2 \ge \left(\sum_{x \in V} d_x \right)^2 = (2m)^2.$$

Combining the two inequalities, we get $mn \ge (2m)^2/n$, and so $m \le n^2/4$. □

The *Turán graph* $T_{n,r}$ is the complete r-partite graph on n vertices with parts of "equal" size (i.e., differing in size by at most 1).

Theorem 5.3 (Turán 1941)**.** *$T_{n,r}$ is the (unique) K_{r+1}-free graph on n vertices with the maximal number of edges. Then,*

$$|E| \le \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Definition. *Let* $\mathrm{ex}(n, H)$ *be the maximum number of edges possible in an H-free graph on n vertices.*

Then, Turán's Theorem implies that

$$\mathrm{ex}(n, K_{r+1}) \approx \left(1 - \frac{1}{r}\right) \frac{n^2}{2}.$$

Theorem 5.4 (Erdös-Stone-Simonovits)**.** *If H has chromatic number $\chi(H) = r + 1$ then*

$$ex(n, H) = \left(1 - \frac{1}{r} + o(1)\right) \frac{n^2}{2}.$$

This is not very informative for H bipartite.

Proof of Turán's Theorem. Use induction.

Let G be an n-vertex K_{r+1}-free graph with the maximum possible number of edges. Pick a vertex v of maximum degree d. Let S be the set of neighbors of v and T be the complement of S (so that $v \in T$). Now, we add to G all edges across T and S and delete all edges in T.

We get G', with $e(G') \geq e(G)$ and such that G' is K_{r+1}-free. Using the inductive hypothesis, we can show S is a Turán graph. Now, we apply to AM-GM to show that the maximal number of edges in a multipartite graph is achieved when the sizes of the parts are the same. □

6 September 28, 2011: Turán's Theorem, etc.

MIT Combinatorics Seminar every Wednesday (and Friday, but at Microsoft) 4:15-5:15 in 2-135.

Alternative proof of Turán's Theorem. Let G be a K_{k+1}-avoiding graph. Consider a probability distribution on the vertices $p_1, \dots, p_n$, with $\sum p_i = 1$. We wish to maximize

$$P = \sum_{i,j|(v_i,v_j)\in E(G)} p_i p_j$$

If we pick $p_i = i/n$, then $P = 2|E|/n^2$. Claim: if $p_i, p_j > 0$ and (i, j) is not an edge, then you can increase P by changing one of p_i, p_j to 0 and the other to $p_i + p_j$.

Proof of claim: Set $s_i = \sum_{k|\{i,k\}\in E} p_k$ and $s_j = \sum_{k|\{j,k\}\in E} p_k$. If $s_i \geq s_j$, set p_i to $p_i + p_j$ and p_j to 0. Otherwise, set p_j to $p_i + p_j$ and set p_i to 0. This increases P by $p_j(s_i - s_j)$ or $p_i(s_j - s_i)$ in the two cases, respectively.

Repeat the operation until impossible; when you stop, you must have a clique Q formed from the positive-probability vertices. In this case, $P = 1 - \sum_{i\in Q} p_i^2$. Then, P is maximized, by Cauchy-Schwarz, when every p_i equals $1/|Q|$. Hence,

$$1 - \frac{1}{|Q|} = P \geq \frac{2|E|}{n^2}$$

Hence, since Q is of size at most k (because we are avoiding K_{k+1}),

$$1 - \frac{1}{k} \leq \frac{2|E|}{n^2}$$

which gives the Turán bound. If one is careful with the proof, one can show equality exists only in the Turán graph. □

Proof of Erdös-Stone-Simonowitz (statement in previous section). Since H is a subgraph of a Turán graph, we obtain that $\mathrm{ex}(n, H) \geq (1 - 1/k)n^2/2$ by Turán's Theorem. Now we do the other way

Conceptual idea: There must be at least one K_{k+1} clique in the big graph. In fact, there must be lots - turn the graph into a $(k+1)$-uniform hypergraph with an edge if the vertices form a clique. If H has t vertices and $\chi(H) = k+1$, then H is a subgraph of the Turán graph $K_{\underbrace{t,t,\dots,t}_{k+1}}$. Show that this must exist because of the number of edges in the hypergraph, applying Jensen's Inequality.

Real proof: Suppose $\epsilon > 0$, k, and t are fixed, n sufficiently large. Consider G on n vertices and with at least $(1 - 1/k + \epsilon)(n^2/2)$ edges. We use induction on k. The base case of $k = 1$ follows from Jensen's Inequality. (See below.) We may assume that every vertex has degree at least $(1 - 1/k + \epsilon/2)n$. (If not, then delete, progressively, the vertices for which this fails. Since all these vertices had small degree, you will be left with some edges and hence will not have deleted all the vertices. One can then consider the resulting smaller graph.)

By the inductive hypothesis, there exists $K_{\underbrace{s,\ldots,s}_{k}}$ with $s = \lceil t/\epsilon \rceil$. Let $B_1, \ldots, B_k$ be the parts of this multipartite graph, so that $|B_i| = s$ for every i.

Let $U = V \backslash (B_1 \cup \cdots \cup B_k)$. Let $W \subset U$ be the set of those vertices with at least t neighbors in each B_i. If $|W| > \binom{s}{t}^k (t-1)$, then we are done, by a Pigeonhole argument. Namely, there must be a set A_{k+1} of some t vertices such that each of them has, for each i, the same set A_i of neighbors in B_i. Then, $\bigcup_{i=1}^{k+1} A_i$ is a graph $K_{\underbrace{t,t,\ldots,t}_{k+1}}$.

Hence, we now merely need to show that $|W| > \binom{s}{t}^k (t-1)$. Let $\tilde{m}$ be the number of missing edges between U and $B_1 \cup \cdots B_k$. We have

$$\tilde{m} \leq \left(\frac{1}{k} - \frac{\epsilon}{2}\right) nks$$

and also that

$$\tilde{m} \geq |U \backslash W|(s-t) \geq (n - ks - |W|)(1-\epsilon)s.$$

Combining the two inequalities, moving things around, and using $k \geq 1$, we obtain the desired inequality.

Now, we must prove the base case. Statement: Assume ϵ, t fixed and n sufficiently large. If G has n vertices and at least ϵn^2 edges. Then, G contains $K_{t,t}$.

Proof: Count the number of pairs (v, T), where v is a vertex and T is a vertex subset of order t, with v adjacent to all vertices in T. The number of such pairs is

$$\sum_{v \in V} \binom{\deg(v)}{t},$$

which, by Jensen's Inequality is minimized when all degrees are the same, and hence is at least $n\binom{2m/n}{t}$.

By Pigeonhole, we are done if there are more than $(t-1)\binom{n}{t}$ such pairs. The inequality

$$n\binom{2m/n}{t} > (t-1)\binom{n}{t}$$

is routine to verify. □

7 October 3, 2011: More Extremal Graph Theory

It is true that $\mathrm{ex}(n, H) = o(n^2) \leq c_{|H|} n^{2-1/|H|}$. (Exact constant given by Erdös-Sós Conjecture.) If H is a tree, $\mathrm{ex}(n, H) = O(n)$.

If one is very careful with the approach in the last section, one can get $\mathrm{ex}(n, K_{r,s}) \leq cs^{1/r}n^{2-1/r} + O(n)$ for $s \geq r$. It is conjectured that $\mathrm{ex}(n, K_{r,r}) \geq cn^{2-1/r}$ (true for $r = 2, 3$, open for 4).

Proof for $K_{2,2} = C_4$ (Erdös et al). Let the graph G have vertices (x, y) with x, y distinct residues modulo p a prime. (So there are $p(p-1)$ vertices.) Let (a, b) and (x, y) form an edge iff $ax + by \equiv 1 \pmod p$. The number of edges, since the graph is p-regular, is

$$\frac{1}{2}p \cdot p(p-1) \approx \frac{1}{2}n^{3/2}.$$

Having a copy of C_4 would require 2 solutions to 2 independent linear equations - a contradiction. □

Proof for $K_{3,3}$ (Brown). Vertices are triples (x, y, z) which are distinct residues modulo p. There are then $p(p-1)(p-2)$ vertices. Let (a, b, c) and (x, y, z) be adjacent iff

$$(a-x)^2 + (b-y)^2 + (c-z)^2 = 1 \pmod p$$

It turns out the graph is regular and has $(1 + o(1))n^{5/3}$ edges. It turns out also that since three general spheres intersect in two points, there can be no $K_{3,3}$. □

This method of proof does not, however, generalize, due to a topological change that comes about at dimension 4.

It is true that $\mathrm{ex}(n, C_{2t}) \leq ctn^{1+1/t}$. We prove now that $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2t}\}) \leq cn^{1+1/t}$.

Lemma 7.1. *Every graph with average degree d has a subgraph with minimum degree at least $d/2$.*

Proof of Lemma. Delete, one by one, the vertices of degree less than $d/2$. Calculate the number of deleted edges and stuff happens... □

Proof sketch of $\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2t}\}) \leq cn^{1+1/t}$. Assume WLOG that each vertex has degree at least $n^{1/t}+1$. Work out the size of the neighborhood of a vertex, then of the distance 2 neighborhood, distance 3, etc. No overlap is possible since no cycles. Contradiction. □

Unit distance problem: How many unit distances can there be among n points in the plane. Can easily get $n-1$ by putting in a line. Can do better by seeking frequent distances in a grid - which involves how many times a number can be written as the sum of two squares... Calculation gives you lower bound of $ne^{c\sqrt{\log n}}$.

Trivial upper bound: $\binom{n}{2}$. Erdös: upper bound of $\mathrm{ex}(n, K_{2,3}) \leq cn^{3/2}$. Proof: make a graph where edges between points unit distance apart - must be $K_{2,3}$ by geometric observation (because two unit circles cannot intersect in more than 2 points). Can also get upper bound of $cn^{4/3}$ by a different argument.

Similar problem: How many distinct distances are possible? The grid has about $\frac{n}{\sqrt{\log n}}$. Katz and Guth recently proved that there must be at least $\frac{cn}{\log n}$ distances.

Representing squares: Let A be a set of integers, and set $A + A = \{a + a' \mid a, a' \in A\}$. Suppose $A + A$ contains the first n squares. How small can $|A|$ be?

Theorem 7.2 (Erdös-Newman)**.** $|A| \geq n^{2/3-o(1)}$.

Proof. For every $1 \leq x \leq n$ connect some pair (a, a') such that $a + a' = x^2$. If $|A| = m = n^{2/3} - \epsilon$ then this graph has m vertices, $n \geq m^{3/2+\delta}$ edges. This implies that there exists K_{2,n^δ} in this graph, so a, a' exist with at least n^δ common neighbors. Then $a - a'$ can be written as a difference of squares in at least n^δ ways. Can do some figuring to determine any integer $\leq n$ must have $n^{o(1)}$ common divisors. Thus, we obtain a contradiction. □

Theorem 7.3 (Katona)**.** *Let X_1, X_2 be independently chosen identical distribution random vectors in $\mathbb{R}^d$ Then,*

$$Prob[|X_1 + X_2| \geq 1] \geq 1/2Prob^2[|X_1| \geq 1].$$

Preparatory observation: Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^d$ with length at least 1. Then, pairs v_i, v_j with $|v_i + v_j| < 1$ cannot be part of a triangle (simple calculation). Which means, by Mantel's Theorem that at least $n(n-2)/2$ pairs i, j must have $|v_i + v_j| \geq 1$.

Proof of Katona. Sample some vectors $X_1, \ldots, X_m$ from the distribution. Let $a = \text{Prob}[|X_1| \geq 1]$ and $b = \text{Prob}[|X_1 + X_2| \geq 1]$, so roughly am vectors will have length at least 1. The number of i, j with $|X_i + X_j| \geq 1$ is, then, both approximately $bm(m-1)$ and also at least $am(m-2)/2$ by the preparatory observation. From

$$bm(m-1) \geq \frac{am(m-2)}{2}$$

we obtain $b \geq a^2/2$, as desired. □

8 October 5: Szemerédi's Regularity Lemma

The presentation used here for the Lemma is from Alon and Spencer. Another good one is Komlos and Simonovits.

Let $G = (V, E)$ be a graph. For $A, B \subset V$, let $e(A, B)$ be the number of pairs in $A \times B$ which are edges, and let $d(A, B)$ be the fraction of pairs which are edges, so that $d(A, B) = e(A, B)/(|A| \cdot |B|)$. We say that (A, B) is *ϵ-regular* if for all $X \subset A$ with $|X| \geq \epsilon|A|$ and $Y \subset B$ with $|Y| \geq \epsilon|B|$, we have

$$d(X, Y) - d(A, B)| \leq \epsilon$$

If P and P' are partitions, we say that P' is a refinement of P if every part of P is a union of parts of P'. We say that a partition of V into $V_0, V_1, \ldots, V_k$ is an *equipartition* if $|V_1| = \ldots = |V_k|$ (with V_0 an *exceptional set*).

We say that an equipartition P is ϵ-regular if $|V_0| \leq \epsilon|V|$ and all pairs (V_i, V_j) with $1 \leq i, j \leq k$ are ϵ-regular, except possibly ϵk^2 pairs.

Theorem 8.1 (Szemerédi's Regularity Lemma)**.** *For every $\epsilon > 0$ and t there exists $T = T(\epsilon, t)$ such that every graph (with at least T vertices) has an ϵ-regular partition $P : V = V_0 \cup V_1 \cup \cdots \cup V_k$ with $t \leq k \leq T$.*

(Alas, T is very big: This proof shows that it is at most a tower of 2's of height $O(\epsilon^{-5})$.)

Let the *mean square density*

$$q(P) = \sum_{i,j} d^2(V_i, V_j) p_i p_j,$$

where $p_i = |V_i|/|V|$. Properties:

1. $0 < q(P) \leq 1$.
2. If P' is a refinement of P, then $q(P') \geq q(P)$. (This claim will be proved later.)

Proof idea: Start with an arbitrary equipartition P_1 into t parts. Step i: we have P_i with k parts. If P_i is ϵ-regular, we are done. If not, find a refinement P_{i+1} of P_i with at most $k \cdot 4^k$ parts and with $q(P_{i+1}) \geq q(P_i) + \epsilon^5/2$. This process must stop at some point, because $q(P)$ is bounded above by 1.

For $U, W \subset V$, let $q(U,W) = d^2(U,W)\frac{|U|\cdot|W|}{|V|^2}$. For partitions P_U of U and P_W of W, let

$$q(P_U, P_W) = \sum_{U' \in P_U, W' \in P_W} q(U', W').$$

Lemma 8.2. *$q(P_U, P_W) \geq q(U,W)$.*

Proof of Lemma. PIck a random $u \in U$ and $w \in W$. Let $W' \in P_W$ be such that $w \in W'$ and let $U' \in P_U$ be such that $u \in U'$. Let $Z = d(U', W')$. Now, we relate expected values: By Cauchy-Schwarz

$$\mathbb{E}[Z^2] \geq \mathbb{E}[Z]^2. \tag{1}$$

We have that

$$\begin{aligned} \mathbb{E}[Z] &= \sum_{U' \in P_U, W' \in P_W} \frac{|U'| \cdot |W'|}{|U| \cdot |W|} \cdot d(U', W') \\ &= \frac{1}{|U| \cdot |W|} \sum e(U', W') \\ &= d(U,W), \end{aligned}$$

from which we conclude

$$\mathbb{E}[Z]^2 = \frac{|V|^2}{|U| \cdot |W|} \cdot q(U,W) \tag{2}$$

Now, we know,

$$\mathbb{E}[Z^2] = \sum \frac{|U'|}{|W'|} \cdot \frac{|W'|}{|W|} \cdot d^2(U', W') = \frac{|V|^2}{|U| \cdot |W|} \cdot q(P_U, P_W) \tag{3}$$

Combining (1), (2), and (3) yields the desired result. □

This also proves Property 2 above.

Lemma 8.3. *Suppose that $\epsilon > 0$ and $U, W \subset V$ with (U,W) not ϵ-regular. Then, there exist $P_U : U = U_1 \cup U_2$ and $P_W : W = W_1 \cup W_2$ such that*

$$q(P_U, P_W) > q(U,W) + \frac{\epsilon^4 |U| \cdot |W|}{|V|^2}$$

Proof of lemma. Let $U_1 \subset U$ with $|U_1| \geq \epsilon |U|$ and $W_1 \subset W$ with $|W_1| \geq \epsilon |W|$ such that ϵ-regularity fails - i.e.,

$$|d(U_1, W_1) - d(U, W)| > \epsilon$$

Let $U_2 = U \backslash U_1$ and $W_2 = W \backslash W_1$. Define Z as in the previous lemma's proof, and compute the variance of Z: it must, by the calculations in that proof, be

$$\frac{|V|^2}{|U| \cdot |W|}(q(P_U, P_W) - q(U, W))$$

We note that $\mathbb{E}[Z] = d(U, W)$ and there is a probability of at least ϵ^2 that Z deviates from $\mathbb{E}[Z]$. Hence, the variance of Z is also at least $\epsilon^2 \cdot \epsilon^2 = \epsilon^4$, from which the result follows. $\square$

Proof of the regularity lemma. Take P which is not ϵ-regular. Let $P_{i,j}$ be a partition of V_i into one or two parts, with two parts if (V_i, V_j) is not ϵ-regular. Let P_i be a common refinement of all P_{ij}. Then, P_i has at most 2^{k-1} parts. Let Q be the partition of V which has as parts the parts of the P_i. Then, $|Q| \leq k2^{k-1}$. Refine each part of Q into parts of size $|V|/(k4^k)$, to obtain the partition P'. The number of remaining vertices (which we put in V_0) is no more than

$$\frac{|V|}{k4^k} \cdot k2^{k-1} = |V|/2^k,$$

which is small, so we need not worry about having enlarged V_0 too much.

We want to show that $q(P') \geq q(P) + \epsilon^5/2$. Now, we gain, by Lemma 8.3, at least $|V_i| \cdot |V_j| \cdot \epsilon^4/|V|^2$ in mean square density, for each irregular pair (V_i, V_j). Now, we had $P : V = V_0 \cup V_1 \cup \cdots \cup V_k$, so

$$|V_i| \approx \frac{|V|}{k} \geq \frac{(1-\epsilon)|V|}{k},$$

implying that we gain (in total) at least

$$\left(\frac{(1-\epsilon)|V|}{k}\right)^2 \cdot \frac{\epsilon^4}{|V|^2} \cdot \epsilon k^2 \geq \frac{\epsilon^5}{2}$$

as desired. $\square$