IDEAL THEORY IN PRÜFER DOMAINS

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KRUll’S THEOREMS AND ARtIN-REES LEMMA

The purpose of this lecture is to prove three important results for Noetherian rings (two of them due to Wolfgang Krull); these are Krull’s Intersection Theorem, Artin-Rees Lemma, and Krull’s Principal Ideal Theorem.

Krull’s Intersection Theorem. The following proposition is a version of Krull Intersection Theorem for Noetherian rings. The proof that we discuss does not use primary decomposition, and was given by H. Perdry in [2].

Proposition 1. Let $R$ be a commutative ring with identity that is Noetherian, and let $I$ be a ideal of $R$. Then there exists $r \in I$ such that $(1 - r) \bigcap_{n \in \mathbb{N}} I^n = (0)$.

Proof. Write $I = (a_1, \ldots, a_\ell)$ and $\bigcap_{n \in \mathbb{N}} I^n = (b_1, \ldots, b_k)$. Now fix $j \in [1, k]$. For every $n \in \mathbb{N}$, the fact that $b_j \in I^n$ guarantees the existence of a homogeneous polynomial $p_n \in R[x_1, \ldots, x_\ell]$ of degree $n$ such that $b_j = p_n(a_1, \ldots, a_\ell)$. For each $n \in \mathbb{N}$, consider the ideal $J_n = (p_1, \ldots, p_n)$ of $R[x_1, \ldots, x_\ell]$. Since the chain of ideals $(J_n)_{n \in \mathbb{N}}$ is ascending and $R[x_1, \ldots, x_\ell]$ is a Noetherian ring by Hilbert Basis Theorem, there is an $n \in \mathbb{N}$ such that $J_{n+1} = J_n$. In particular, $p_{n+1}$ belongs to $J_n$. As a result, we can take polynomials $q_1, \ldots, q_n \in R[x_1, \ldots, x_\ell]$ such that $p_{n+1} = \sum_{i=1}^n q_ip_{n+1-i}$. Observe that there is no loss of generality in assuming that $q_d$ is a homogeneous polynomial of degree $d$ for every $d \in [1, n]$, and we do so. After evaluating both sides of $p_{n+1} = \sum_{i=1}^n q_ip_{n+1-i}$ at $(x_1, \ldots, x_\ell) = (a_1, \ldots, a_\ell)$, we see that

$$b_j = (q_1(a_1, \ldots, a_\ell) + \cdots + q_{n+1}(a_1, \ldots, a_\ell))b_j = r_j b_j$$

for some $r_j \in I$ (here we have used the fact that $q_1, \ldots, q_{n+1}$ are homogeneous polynomials of positive degree). Therefore, for every $j \in [1, k]$, we have found $r_j \in I$ satisfying that $(1 - r_j)b_j = 0$. Then the product $(1 - r_1) \cdots (1 - r_k)$ annihilates $b_j$ for every $j \in [1, k]$. Hence $(1 - r) \bigcap_{n \in \mathbb{N}} I^n = (0)$ when $r = 1 - (1 - r_1) \cdots (1 - r_k)$. □

The previous proposition is specially useful in the context of integral domains and local rings.

Theorem 2 (Krull’s Intersection Theorem). Let $R$ be a Noetherian domain or a Noetherian local ring, and let $I$ be a proper ideal of $R$. Then $\bigcap_{n \in \mathbb{N}} I^n = (0)$. 1
Proof. When $R$ is an integral domain, the statement of the theorem follows immediately from Proposition 1. On the other hand, suppose that $R$ is a local ring with maximal ideal $M$, and set $J = \bigcap_{n \in \mathbb{N}} M^n$. Since $R$ is Noetherian, $J$ is a finitely generated $R$-module. As $MJ = J$, it follows from Nakayama’s Lemma that $J = (0)$. Hence \( \bigcap_{n \in \mathbb{N}} I^n \subseteq \bigcap_{n \in \mathbb{N}} M^n = (0). \) \( \square \)

The conclusion of Krull’s Intersection Theorem does not hold, in general, for Noetherian rings, as the following example indicates.

Example 3. Consider the ring $R = \mathbb{Z}/6\mathbb{Z}$. Since $R$ is finite, it is Noetherian. On the other hand, $R$ is not local (both $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$ are maximal ideals of $R$) and $R$ is not an integral domain ($2 + 6\mathbb{Z}$ and $3 + 6\mathbb{Z}$ are both nonzero zero-divisors). Finally, we observe that $I = (2 + 6\mathbb{Z})$ is an idempotent ideal and, therefore, $2 + 6\mathbb{Z} \in \bigcap_{n \in \mathbb{N}} I^n$.

Artin-Rees Lemma. We proceed to prove the Artin-Rees Lemma, which also deals with ideals in Noetherian rings.

Theorem 4 (Artin-Rees Lemma). Let $R$ be a Noetherian ring, and let $I, J,$ and $K$ be ideals of $R$. Then there exist $m \in \mathbb{N}$ such that
\[
(0.1) \quad I^n J \cap K = I^{n-m}(I^m J \cap K)
\]
for every $n \in \mathbb{N}$ with $n \geq m$.

Proof. Write $I = (a_1, \ldots, a_k)$. For each $n \in \mathbb{N}_0$, let $H_n$ be the set consisting of homogeneous polynomials $f \in R[x_1, \ldots, x_n]$ of degree $n$ with $f(a_1, \ldots, a_k) \in I^n J \cap K$. Now let $I'$ be the homogeneous ideal generated by the set $H := \bigcup_{n \in \mathbb{N}_0} H_n$. In light of Hilbert Basis Theorem, we can write $I' = (f_1, \ldots, f_t)$ for some $f_1, \ldots, f_t \in R[x_1, \ldots, x_k]$. Since $I'$ is a homogeneous ideal, we can assume that $f_1, \ldots, f_t$ are homogeneous polynomials. For each $i \in [1, t]$, set $d_i := \deg f_i$, and then set $m = \max\{d_i : i \in [1, t]\}$ and fix $n \in \mathbb{N}$ with $n \geq m$.

To argue the inclusion $I^n J \cap K = I^{n-m}(I^m J \cap K)$, take $a \in I^n J \cap K$. As $a \in I^n$, we can pick a polynomial $f \in H_n$ such that $a = f(a_1, \ldots, a_k)$. Now write $f = \sum_{i=1}^t g_i f_i$, for some $g_1, \ldots, g_t \in R[x_1, \ldots, x_t]$. Since $f$ is homogeneous of degree $n$, there is no loss of generality in assuming that $g_i$ is homogeneous of degree $n - d_i$ for every $i \in [1, t]$. Then the fact that
\[
a = f(a_1, \ldots, a_k) = \sum_{i=1}^t g_i(a_1, \ldots, a_k) f_i(a_1, \ldots, a_k) \in \sum_{i=1}^t I^{n-d_i}(I^{d_i} J \cap K),
\]
along with
\[
\sum_{i=1}^t I^{n-d_i}(I^{d_i} J \cap K) \subseteq I^{n-m} \sum_{i=1}^t (I^m J) \cap I^{m-d_i} K) \subseteq I^{n-m}(I^m J \cap K),
\]

F. GOTTI
allows us to conclude that \( a \in I^{n-m}(F^m J \cap K) \). Hence the direct inclusion of (0.1) holds. The reverse inclusion follows easily: \( I^{n-m}(F^m J \cap K) \subseteq I^n J \cap I^{n-m} K \subseteq I^n J \cap K \). Hence (0.1) holds for every \( n \geq m \).

Krull’s Principal Ideal Theorem. Our next goal is to prove Krull’s Principal Ideal Theorem (Krull’s Hauptidealsatz), which states that, in a Noetherian ring, every minimal prime ideal over a principal ideal has height at most one.

Let \( R \) be a commutative ring with identity. The height of a prime ideal \( P \) of \( R \), which is denoted by \( \text{ht}(P) \), is the maximum \( h \in \mathbb{N}_0 \cup \{\infty\} \) such that there is a chain

\[ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h = P, \]

where \( P_0, \ldots, P_h \) are prime ideals of \( R \). Given an ideal \( I \) of \( R \), recall that a minimal prime ideal over \( I \) is a prime ideal \( P \) containing \( I \) such that for every prime ideal \( Q \) with \( I \subset Q \subset P \) the equality \( Q = P \) holds. Finally, we need the following lemma.

Lemma 5. Let \( R \) be a Noetherian ring, and let \( P \) be a prime ideal of \( R \). For every \( n \in \mathbb{N} \), set \( P^{(n)} := P^n R_P \cap R \). Then \( P^{(n)} R_P = P^n R_P \).

Proof. Exercise. \( \square \)

The ideal \( P^{(n)} \) in the previous lemma is called the \( n \)-th symbolic power of \( P \). We are now in a position to prove Krull’s Principal Ideal Theorem.

Theorem 6 (Krull’s Principal Ideal Theorem). Let \( R \) be a Noetherian domain, and let \( I \) be a proper principal ideal of \( R \). Then each minimal prime ideal over \( I \) has height at most one.

Proof. Let \( P \) be a minimal prime ideal over \( I \). After localizing \( R \) at \( P \) if necessary, all the relevant data is preserved and we can further assume that \( R \) is a local ring with maximal ideal \( P \). Suppose, by way of contradiction, that \( \text{ht}(P) \geq 2 \). Let \( Q_0 \) and \( Q \) be prime ideals in \( R \) such that \( Q_0 \subsetneq Q \subsetneq P \). Observe that if we replace \( R \) by \( R/Q_0 \), then we can assume that \( R \) is a Noetherian domain that is local with maximal ideal \( P \) satisfying that \( P \) is a minimal prime over \( I \) and \((0) \subsetneq Q \subsetneq P \).

Take \( a \in R \) such that \( I = Ra \) and, for each \( n \in \mathbb{N} \), set \( Q^{(n)} = Q^n R_Q \cap R \). Observe that \( Q^{(n)} \) is a \( Q \)-primary ideal for every \( n \in \mathbb{N} \). The quotient ring \( R/Ra \) has only one prime ideal, namely, \( P/Ra \). Therefore it is a zero-dimensional Noetherian (local) ring, and so it is also an Artinian ring. As a result, the chain of ideals \( ((Q^{(n)} + Ra)/Ra)_{n \in \mathbb{N}} \) of \( R/Ra \) eventually stabilizes, and so there is an \( N \in \mathbb{N} \) such that \( Q^{(n)} + Ra = Q^{(n+1)} + Ra \) for every \( n \geq N \).

Fix \( n \geq N \), and then take \( q_n \in Q^{(n)} \). Since \( Q^{(n)} \subseteq Q^{(n+1)} + Ra \), we can write \( q_n = q_{n+1} + ra \) for some \( q_{n+1} \in Q^{(n+1)} \) and \( r \in R \). Note that \( ra = q_n - q_{n+1} \in Q^{(n)} \). In addition, \( a \notin Q \) because \( P \) is a minimal prime over \( Ra \) in \( R \). This, along with the fact that \( Q^{(n)} \) is \( Q \)-primary, ensures that \( r \in Q^{(n)} \). As a consequence, \( Q^{(n)} \subseteq Q^{(n+1)} + Q^{(n)} a, \)}
which implies that \( Q^{(n)} = Q^{(n+1)} + Q^{(n)}a \). Therefore the \( R \)-module \( M = Q^{(n)}/Q^{(n+1)} \) satisfies that \( M = aM \). So it follows from Nakayama’s Lemma that \( M = \{0\} \), whence \( Q^{(n)}/Q^{(n+1)} = \{0\} \).

Thus, for each \( n \geq N \) the equality \( Q^{(n)} = Q^{(N)} \) holds, and so \( Q^nR_Q = Q^N R_Q \) by virtue of Lemma 5. Take a nonzero \( q \in Q \). As \( R \) is an integral domain, \( q^N \) is a nonzero element of \( Q^nR_Q \) for every \( n \in \mathbb{N} \). Now since \( R_Q \) is a Noetherian local ring with maximal ideal \( QR_Q \), the fact that \( q^N \in \bigcap_{n \in \mathbb{N}} Q^n R_Q \) generates a contradiction with Krull’s Intersection Theorem, which completes the proof. \( \square \)

The following related statement follows as a consequence of Krull’s Principal Ideal Theorem.

**Corollary 7.** Let \( R \) be a Noetherian ring, and suppose that \( a \in R \) is not a zero-divisor. Prove that \( \text{ht}(P) = 1 \) for every minimal prime ideal over \( Ra \).

**Proof.** Exercise. \( \square \)

**Exercises**

**Exercises 1** (Nagata’s Idealization Trick). Let \( R \) be any commutative ring identity, and let \( M \) be a module over \( R \). For the abelian group \( S := R \times M \), prove the following statements.

1. \( S \) is a commutative ring with identity under the multiplication operation
   \[(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1) .\]
2. \( I := \{0\} \times M \) is an ideal of \( S \) satisfying that \( S/I \cong R \) and \( I^2 = (0) \).
3. Every prime ideal of \( S \) has the form \( P \times M \) for some prime ideal \( P \) of \( R \).
4. \( S \) is a local ring provided that \( R \) is a local ring.
5. \( S \) is Noetherian provided that both \( R \) and \( M \) are Noetherian.

**Exercises 2** (Krull’s Intersection Theorem for Modules). Let \( R \) be a Noetherian local ring with maximal ideal \( P \), and let \( M \) be a finitely generated module over \( R \). Prove that \( \bigcap_{n \in \mathbb{N}} P^n M = 0 \). [Hint: Use Nagata’s Idealization Trick.]

**Exercises 3.** Let \( R \) be a Noetherian ring, and let \( P \) be a prime ideal of \( R \). Prove that \( P^{(n)}P = P^nP \) for every \( n \in \mathbb{N} \).

**Exercises 4.** Let \( R \) be a Noetherian ring, and suppose that \( a \in R \) is not a zero-divisor. Prove that \( \text{ht}(P) = 1 \) for every minimal prime ideal over \( Ra \). [Hint: Argue that in a Noetherian ring every minimal prime ideal consists of zero-divisors.]
REFERENCES


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