Prüfer Domains I

Throughout this lecture, \( R \) is an integral domain. Recall that \( \text{qf}(R) \) denotes the quotient field of \( R \).

**Definition and Examples.** Prüfer domains, which are natural generalizations of valuation domains, play a fundamental role in multiplicative ideal theory. In this lecture, we start our discussion of Prüfer domains.

A fractional ideal \( I \) of \( R \) is invertible if there is a fractional ideal \( J \) such that \( IJ = R \), in which case \( J = (R : I) = \{ r \in \text{qf}(R) : rI \subseteq R \} \). It is clear that the set of all invertible fractional ideals of \( R \) is an abelian group with identity \( R \). Observe that such a group contains the set of all nonzero principal fractional ideals as a subgroup.

**Definition 1.** An integral domain \( R \) is a Prüfer domain if every nonzero finitely generated ideal of \( R \) is invertible.

Fields and PIDs are clearly Prüfer domains. Recall that a Bezout domain is an integral domain where every finitely generated ideal is principal. Since nonzero principal ideals are invertible, every Bezout domain is a Prüfer domain. In particular, every valuation domain is a Prüfer domain. Let us briefly exhibit two further examples of Prüfer domains.

**Example 2.** The set \( \text{Int}(R) := \{ p(x) \in \mathbb{Q}[x] : p(\mathbb{Z}) \subseteq \mathbb{Z} \} \) is a subring of \( \mathbb{Q}[x] \) called the ring of integer-valued polynomials. We shall prove soon enough that \( \text{Int}(R) \) is a non-Noetherian Prüfer domain of Krull dimension 2.

**Example 3.** The ring consisting of all the entire function on the complex plane is a Bezout domain of infinite Krull dimension. In particular, it is a Prüfer domain.

Although every PID is Prüfer, this is not the case for UFDs. The following example sheds some light upon this observation.

**Example 4.** For a field \( F \), consider the ring of polynomials \( R := F[x, y] \) and the ideal \( I = Rx + Ry \) of \( R \). If \( f \in \text{qf}(R) \) belongs to \( J := (R : I) \), then \( Rx f + Ry f \subseteq R \), and so \( xf \in R \) and \( yf \in R \). Therefore \( f \in x^{-1}R \cap y^{-1}R = R \). Then \( J \subseteq R \) (indeed, \( J = R \)), and we see that \( IJ \subseteq I \). Thus, \( I \) is not an invertible ideal even though it is finitely generated, and this allows us to conclude that \( R \) is not a Prüfer domain. Note, however, that \( R \) is a UFD.
Characterizations. We will discuss various of the many characterizations of Prüfer domains. Let us start by the following.

**Proposition 5.** For an integral domain $R$, the following statements are equivalent.

(a) $R$ is a Prüfer domain.

(b) Every two-generated ideal of $R$ is invertible.

**Proof.** (a) $\Rightarrow$ (b): This is obvious.

(b) $\Rightarrow$ (a): We will show that every nonzero finitely generated ideal of $R$ is invertible by using induction on the minimum number $n$ of generators of such an ideal. It is clear when $n = 1$, and it follows from part (b) when $n = 2$. Suppose, therefore, that $I$ can be generated by $n$ elements, where $n > 2$, and assume that every nonzero ideal of $R$ that can be generated by less than $n$ elements is invertible. Now write $I = Rc_1 + \cdots + Rc_n$ for some nonzero elements $c_1, \ldots, c_n \in R$. Set $I_1 := Rc_1 + \cdots + Rc_{n-1}$, $I_2 := Rc_2 + \cdots + Rc_n$, and $I_3 := Rc_1 + Rc_n$. By induction, $I_1$, $I_2$, and $I_3$ are invertible.

Then $J := c_1I_1^{-1}I_3^{-1} + c_nI_2^{-1}I_3^{-1}$ is a fractional ideal of $R$. We claim that $J$ is the inverse of $I$. To show this, first observe that

$$IJ = (I_1 + Rc_n)c_1I_1^{-1}I_3^{-1} + (Rc_1 + I_2)c_nI_2^{-1}I_3^{-1}$$
$$= c_1I_3^{-1} + c_1c_nI_1^{-1}I_3^{-1} + c_1c_nI_2^{-1}I_3^{-1} + c_nI_3^{-1}$$
$$= c_1I_3^{-1}(R + c_nI_2^{-1}) + c_nI_3^{-1}(R + c_1I_1^{-1}).$$

As $I_1$ and $I_2$ are invertible ideals and $c_1 \in I_1$ and $c_n \in I_2$, it follows that $c_1I_1^{-1} \subseteq R$ and $c_nI_2^{-1} \subseteq R$. This, along with the previous chain of equalities, guarantees that $IJ = c_1I_3^{-1} + c_nI_3^{-1} = I_3I_3^{-1} = R$. Hence $I$ is an invertible ideal. 

We proceed to characterize Prüfer domains in terms of valuation domains.

**Proposition 6.** For an integral domain $R$, the following statements are equivalent.

(a) $R$ is a Prüfer domain.

(b) $R_P$ is a valuation domain for every prime ideal $P$.

(c) $R_M$ is a valuation domain for every maximal ideal $M$.

**Proof.** (a) $\Rightarrow$ (b): Assume that $R$ is a Prüfer domain, and let $P$ be a prime ideal of $R$. Since $R_P$ is a local ring, it is enough to prove that it is a Bezout domain. Let $\frac{a_1}{s_1}R_P + \cdots + \frac{a_k}{s_k}R_P$ be a nonzero finitely generated ideal of $R_P$, where $a_1, \ldots, a_k \in I$ and $s_1, \ldots, s_k \in R \setminus P$. Then $I := Ra_1 + \cdots + Ra_k$ satisfies that $I_P = \frac{a_1}{s_1}R_P + \cdots + \frac{a_k}{s_k}R_P$.

Since $R$ is a Prüfer domain, $I$ is invertible. Let $J$ be a fractional ideal such that $JI = R$, then $(J|_R)I_P = R_P$, and so $I_P$ is invertible in $R_P$. Since $R_P$ is local, $I_P$ is a principal ideal. Hence $R_P$ is a valuation domain.

(b) $\Rightarrow$ (c): This is clear.
(c) \(\Rightarrow\) (a): Assume, by way of contradiction, that there is a nonzero finitely generated ideal \(I\) of \(R\) that is not invertible. Write \(I = Ra_1 + \cdots + Ra_n\) for \(a_1, \ldots, a_n \in R\). Since \(I\) is not invertible, \(IJ \subseteq R\), where \(J := (R : I)\). So there is a maximal ideal \(M\) of \(R\) such that \(IJ \subseteq M\). Because the extension \(I_M\) of \(I\) is a finitely generated ideal of the Bezout domain \(R_M\), there is an \(a \in I\) satisfying \(I_M = aR_M\). For each \(i \in [1, n]\), we can now take \(s_i \in R \setminus M\) with \(s_ia_i \in aR\). After setting \(s = s_1 \cdots s_n\), we see that \(sa^{-1}a_i \in R\) for every \(i \in [1, n]\), and so \(sa^{-1}I \subseteq R\). This implies that \(sa^{-1} \in J\) and, therefore, \(s = a(sa^{-1}) \in IJ \subseteq M\), which is a contradiction. \(\square\)

**Corollary 7.** Let \(R\) be a Prüfer domain, and let \(P\) be a prime ideal of \(R\). Then the set of all \(P\)-primary ideals of \(R\) is totally ordered, and the intersection \(P'\) of all such primary ideals is a prime ideal satisfying that there is no prime ideal strictly between \(P'\) and \(P\).

**Proof.** It follows from Proposition 6 that \(R_P\) is a valuation domain. Now the corollary follows from the correspondence between the \(P\)-primary ideals of \(R\) and the \(P_P\)-primary ideals of \(R_P\), as we have seen before that the statement of the corollary holds for valuation domains. \(\square\)

Prüfer domains can also be characterized using cancellation of finitely generated ideals.

**Proposition 8.** For an integral domain \(R\), the following statements are equivalent.

(a) \(R\) is a Prüfer domain.

(b) For every nonzero finitely generated ideal \(I\) of \(R\), whenever \(IB = IC\) for ideals \(B\) and \(C\) the equality \(B = C\) must hold.

(c) For every finitely generated ideal \(I\) of \(R\), whenever an ideal \(J\) is contained in \(I\), there is an ideal \(K\) such that \(J = IK\).

**Proof.** (a) \(\Rightarrow\) (b): Let \(I\) be a finitely generated nonzero ideal of \(R\), and let \(J\) and \(K\) be ideals of \(R\) such that \(IJ = IK\). Since \(R\) is a Prüfer domain, \(I\) is invertible and so \(J = I^{-1}IJ = I^{-1}IK = K\).

(b) \(\Rightarrow\) (a): Suppose, on the other hand, that every finitely generated nonzero ideal of \(R\) is cancellative. We start by observing that if \(I\) is a nonzero finitely generated ideal of \(R\) and \(IJ \subseteq IK\) for ideals \(J\) and \(K\) of \(R\), then \(J \subseteq K\). Indeed, \(IK = IJ + IK = I(J + K)\) implies that \(K = J + K\), which means that \(J \subseteq K\).

To prove that \(R\) is Prüfer it suffices to argue that the localization of \(R\) at any prime ideal is a valuation domain. Let \(P\) be a prime ideal of \(R\). Take \(a, b \in R\), and let us show that either \(aR_P \subseteq bR_P\) or \(bR_P \subseteq aR_P\). The assertion clearly holds when \(a = 0\) or \(b = 0\). So we assume that \(ab \neq 0\). Note that \(Rab(Ra + Rb) \subseteq (Ra^2 + Rh^2)(Ra + Rb)\), and so \(Rab \subseteq Ra^2 + Rh^2\). Take \(x, y \in R\) such that \(ab = xa^2 + yh^2\), and observe that \(Rhb(Ra + Rh) \subseteq Ra(Ra + Rh)\). Therefore \(yb = ra\) for some \(r \in R\), and we can write
ab = x^2a^2 + rab, that is, xa = b(1 - r). If r \notin P, then a = b(y/r) \in bR_P and so aR_P \subseteq bR_P. On the other hand, if r \in P, then 1 - r \notin P and so b = ax/(1-r) \in aR_P, which implies that bR_P \subseteq aR_P. Hence R_P is a valuation domain for every prime ideal P.

(a) \Rightarrow (c): Let I be a finitely generated ideal of R, and let J be an ideal of R contained in I. If I is the zero ideal so is J, and we can take K = R (or any ideal of R). If I is nonzero, then it is invertible and so we can take K to be I^{-1}J.

(c) \Rightarrow (a): Finally, suppose that the statement (c) holds. We will show that every localization of R at a prime ideal is a valuation. To do so, take a prime ideal P of R. Take a, b \in R and let us verify that the principal ideals aR_P and bR_P are comparable. Since Ra \subseteq Ra + Rb, there is an ideal I such that Ra = (Ra + Rb)I. After writing a = xa + yb for some x, y \in I, we see that yb = a(1-x) \in aR. If x \in P, then 1-x \notin P, and from a = by/(1-x) \in bR_P we obtain that aR_P \subseteq bR_P. On the other hand, if x \notin P, then bx \in bI \subseteq (Ra + Rb)I = Ra ensures that b \in aR_P, that is, bR_P \subseteq aR_P. Hence R_P is a valuation domain for every prime ideal P.

Finally, we characterize Prüfer domains by using certain distributivity laws.

**Proposition 9.** For an integral domain R, the following statements are equivalent.

(a) R is a Prüfer domain.
(b) A(B \cap C) = AB \cap AC for all ideals A, B, and C of R.
(c) A \cap (B + C) = A \cap B + A \cap C for all ideals A, B, and C of R.

**Proof.** (a) \Rightarrow (b): Suppose that R is a Prüfer domain, and let A, B, and C be ideals of R. Let P be a maximal ideal of R. Since R_P is a valuation domain by Proposition 6, the ideals BR_P and CR_P of R_P are comparable and, therefore,

\[ A(B \cap C)R_P = AR_P(BR_P \cap CR_P) = (AR_P)(BR_P) \cap (AR_P)(CR_P) = (AB \cap AC)R_P. \]

Since the maximal ideal P was arbitrarily taken, the equality A(B \cap C) = AB \cap AC must hold.

(b) \Rightarrow (a): Take C = Rc_1 + Rc_2 for some c_1, c_2 \in R, and let us check that C is an invertible ideal. This is clear if C is principal. Therefore suppose that c_1 \neq 0 and c_2 \neq 0. Set A = Rc_1 and B := Rc_2, and observe that AB \subseteq CA \cap CB = C(A \cap B), and so AB = (A \cap B)C. As both A and B are invertible ideals,

\[ C(A \cap B)B^{-1}A^{-1} = ABB^{-1}A^{-1} = R, \]

which implies that C is also an invertible ideal. Since every two-generated ideal of R is invertible, it follows from Proposition 5 that R is a Prüfer domain.

(a) \Rightarrow (c): This follows the same argument used to establish (a) \Rightarrow (b).

(c) \Rightarrow (a): Fix a prime ideal P, and let us verify that R_P is a valuation domain. To do so, take a, b \in R and observe that, in light of the distributive law in (c),

\[ Ra = Ra \cap (Rb + R(a - b)) = (Ra \cap Rb) + (Ra \cap R(a - b)). \]
As a consequence, one can pick \( t \in Ra \cap Rb \) and \( r \in R \) with \( r(a - b) \in Ra \) such that \( a = t + r(a - b) \). Then we see that \( rb \in Ra \) and \( (1 - r)a \in Rb \). Thus, if \( r \in P \), then \( 1 - r \notin P \), which implies that \( a \in bR_P \). On the other hand, if \( r \notin P \), then \( a - b \in aR_P \) and so \( b \in aR_P \). Therefore the ideals \( aR_P \) and \( bR_P \) are comparable. Because any two principal ideals of \( R_P \) are comparable, \( R_P \) is a valuation domain, and it follows from Proposition 6 that \( R \) is a Prüfer domain.

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**Exercises**

**Exercise 1.** Let \( R \) be a Prüfer domain, and let \( P \) be a prime ideal of \( R \). Prove that \( R/P \) is also a Prüfer domain.

**Exercise 2.** Let \( R \) be an integral domain. Prove that the following statements are equivalent.

1. \( R \) is a Prüfer domain.
2. \((J + K) : I = (J : I) + (K : I)\) for all ideals \( I, J, \) and \( K \) of \( R \) with \( I \) finitely generated.
3. \( I : (J \cap K) = (I : J) + (I : K)\) for all ideals \( I, J, \) and \( K \) of \( R \) with \( J \) and \( K \) finitely generated.

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