Lecture 5: Permutation Inversions and q-Binomials

In this lecture, we introduce \( q \)-analogs of \( n! \) and \( \binom{n}{k} \), which are corresponding combinatorial expressions depending on a variable \( q \) that when we evaluate at \( q = 1 \) we recover \( n! \) and \( \binom{n}{k} \), respectively. Most importantly, in the same way \( n! \) and \( \binom{n}{k} \) count linear arrangements and \( k \)-subsets of a given set of size \( n \), their corresponding \( q \)-analogs count chain of subspaces and \( k \)-dimensional subspaces of a given vector space of dimension \( n \) (over a field of \( q \) elements).

Counting Inversions. Let \( S_n \) denote the set consisting of all permutations of \([n]\). The inversion table of a permutation \( w \in S_n \) is an \( n \)-tuple \( I(w) := (a_1, \ldots, a_n) \), where \( a_i \) denotes the number of elements \( j \) in \( w \) to the left of \( i \) with \( j > i \). Observe that \( 0 \leq a_i \leq n - i \) for every \( i \in [n] \).

**Proposition 1.** For each \( n \in \mathbb{N} \), the map \( I: S_n \to \mathbb{N}^n \), where \( I(w) = (a_1, \ldots, a_n) \) is the inversion table of \( w \), is a bijection.

**Proof.** Set \( T_n := [0, n-1] \times [0, n-2] \times \cdots \times [0, 0] \). Since \( |S_n| = |T_n| = n! \), it suffices to show that the function \( I \) is surjective. Take \((a_1, \ldots, a_n) \in T_n \) and let us construct \( w \in S_n \) as follows. Consider the element \( n \) as an initial linear arrangement of length 1. Then suppose that we have inserted the elements \( n-1, n-2, \ldots, n-i+1 \) (in this order) into the initial length-1 linear arrangement. In the \( i \)-th step, insert \( n-i \) in the current length-\( i \) linear arrangement so that there are exactly \( a_{n-i} \) elements to the left of \( n-i \). After inserting 1, we obtain a length-\( n \) linear arrangement of \([n]\), that is, a permutation \( w \in S_n \). Observe that in our construction of \( w \), right after we inserted \( n-i \) there were precisely \( a_{n-i} \) elements \( j \) in the linear arrangement to the left of \( n-i \) such that \( j > n-i \), and this number was unchanged during the remaining steps as only elements less than \( n-i \) were inserted then. Hence \( I(w) = (a_1, \ldots, a_n) \), and we can conclude that \( I \) is a surjective. \( \square \)

An inversion of \( w := w_1w_2 \cdots w_n \in S_n \) is a pair \((w_i, w_j)\) such that \( i < j \) but \( w_i > w_j \). The number of inversions of a permutation \( w \) is denoted by \( \text{inv}(w) \). Observe that if \( I(w) = (a_1, \ldots, a_n) \) is the inversion table of \( w \), then \( \text{inv}(w) = a_1 + \cdots + a_n \).
Proposition 2. The identity
\[
\sum_{w \in S_n} q^{\text{inv}(w)} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1})
\]
holds for every \( n \in \mathbb{N} \).

Proof. Set \( T_n := [0, n-1] \times [0, n-2] \times \cdots \times [0, 0] \). Since the assignment \( w \mapsto I(w) \) induces a bijection \( S_n \rightarrow T_n \), and \( \text{inv}(w) = a_1 + \cdots + a_n \) for every \( w \in S_n \) with \( I(w) = (a_1, \ldots, a_n) \), it follows that
\[
\sum_{w \in S_n} q^{\text{inv}(w)} = \sum_{(a_1, \ldots, a_n) \in T_n} q^{a_1+\cdots+a_n} = \prod_{k=0}^{n-1} (1 + q + \cdots + q^k),
\]
which is the desired identity. \( \square \)

Motivated by the previous proposition, for every \( n \in \mathbb{N} \) we define the following \( q \)-analogs
\[
(n)_q := 1 + q + \cdots + q^{n-1} \quad \text{and} \quad (n)_q! := (1)_q(2)_q \cdots (n)_q,
\]
of \( n \) and \( n! \), respectively. By convention, we set \((0)_q = 0\), and so \((0)_q! = 1\). We call \((n)_q!\) the \( q \)-factorial of \( n \). In general, and roughly speaking, a \( q \)-analog of a mathematical object, is another mathematical object depending on a variable \( q \) that specializes to the former object when \( q = 1 \). One can see that \((n)_1 = n \) and \((n)_1! = n! \). Observe that we can also write \((n)_q\) and \((n)_q!\) as follows:
\[
(n)_q = \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1} \quad \text{and} \quad (n)_q! = \prod_{k=1}^{n} \frac{q^k - 1}{q - 1}.
\]

Counting Subspaces. Using the previous \( q \)-analog of \( n! \), we can naturally define \( q \)-analogs for the binomial coefficients:
\[
\binom{n}{k}_q := \frac{(n)_q!}{(k)_q!(n-k)_q!}
\]
for every \( n \in \mathbb{N}_0 \) and \( k \in [0, n] \). It is clear that \( \binom{n}{k}_q \) is a \( q \)-analog of \( \binom{n}{k} \), and the former is called a \( q \)-binomial coefficient. As the next proposition indicates, \( \binom{n}{k}_q \) counts the set of \( k \)-dimensional subspaces of an \( n \)-dimensional vector space over a finite field of size \( q \). Let \( \mathbb{F}_q \) denote a finite field such that \( |\mathbb{F}_q| = q \). As for vector spaces over \( \mathbb{R} \), a vector space over \( \mathbb{F}_q \) of dimension \( d \) can be treated as (i.e., is isomorphic to) \( \mathbb{F}_q^d \).

\(^1\)There exists a finite field of size \( q \) precisely when \( q \) is a positive power of a prime, in which case there is exactly one field of size \( q \) (up to isomorphism).
Proposition 3. For all \( n \in \mathbb{N}_0 \) and \( k \in [0, n] \), the number of \( k \)-dimensional subspaces of the vector space \( \mathbb{F}_q^n \) is \( \binom{n}{k}_q \).

Proof. Let \( A(n, k) \) denote the number of \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \), and let \( L(n, k) \) denote the number of sequences consisting of \( k \) linearly independent vectors in \( \mathbb{F}_q^n \). We proceed to count the number \( L(n, k) \) of \( k \)-sequences \( v_1, \ldots, v_k \) of linearly independent vectors in \( \mathbb{F}_q^n \) in two different ways. Choose \( v_1 \) to be any nonzero vector of \( \mathbb{F}_q^n \), which can be done in \( q^n - 1 \) different ways. Then choose \( v_2 \) in \( \mathbb{F}_q^n \) so that \( v_2 \) is not a multiple (i.e., a linear combination) of \( v_1 \); this can be done in \( q^n - q \) ways. In the \( i \)-th step, choose \( v_i \) in \( \mathbb{F}_q^n \) so that it is not a linear combination of \( v_1, \ldots, v_{i-1} \), which can be done in \( q^n - q^{i-1} \) different ways. Therefore

\[
L(n, k) = (q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}).
\]

We can also obtain \( L(n, k) \) as follows. First, we choose a \( k \)-dimensional subspace \( W \) of \( \mathbb{F}_q^n \) in \( A(n, k) \) possible ways, and then we choose a linearly independent sequence of \( k \) vectors in \( W \cong \mathbb{F}_q^k \), which can be done in \( (q^k - 1)(q^n - q) \cdots (q^k - q^{k-1}) \) by mimicking the way we just described above to choose a sequence of \( k \) linearly independent vectors of \( \mathbb{F}_q^n \). As a result, \( L(n, k) = A(n, k)(q^k - 1)(q^n - q) \cdots (q^k - q^{k-1}) \), and taking into account the equality (0.1) we obtain

\[
A(n, k) = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \frac{\prod_{j=1}^{n} \frac{q^j - 1}{q^j - 1}}{\prod_{j=1}^{k} \frac{q^j - 1}{q^j - 1} \prod_{j=1}^{n-k} \frac{q^j - 1}{q^j - 1}} = \frac{(n)_q!}{(k)_q! (n-k)_q!} = \binom{n}{k}_q.
\]

\[\square\]

Practice Exercises

Exercise 1. For \( n \in \mathbb{N} \), argue that there are \( (n)_q! \) ordered sequences \( V_1, \ldots, V_n \) of subspaces of \( \mathbb{F}_q^n \) with \( \dim V_i = i \) for every \( i \in [n] \) such that \( V_1 \subset V_2 \subset \cdots \subset V_n \).

Exercise 2. For any \( n \in \mathbb{N}_0 \) and \( k \in [0, n] \), prove that

\[
\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^{n-k} \binom{n-1}{k-1}_q.
\]