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This work is dedicate to my lovely girlfriend (hopefully wife soon) Shiran ♥
1 Abstract

The present thesis deals with questions concerning an existence of almost precipitous ideals raised in [5]. It is shown that every successor of a regular cardinal can carry an almost precipitous ideal in a generic extension of $L$. In $L[\mu]$ every regular cardinal which is less than the measurable carries an almost precipitous non-precipitous ideal. Also, results of [4] are generalized- thus assumptions on precipitousness are replaced by those on $\infty$-semi precipitousness.
2 Introduction

In [7] Jech and Prikry introduced a class of ideals called precipitous and in [8] Jech observed that such ideals can be characterized in terms of infinite games, due to F. Galvin. The existence of a precipitous ideal $I$ on a regular $\kappa$ implies that $\kappa$ is measurable in an inner model of ZFC so it’s natural to ask whether weaker forms of precipitousness are good enough to get ultrapowers which are "enough" well founded and yet, avoid assuming large cardinals assumptions as strong as the existence of a measurable cardinal in an inner model of ZFC. In the present paper we study ideals called almost − precipitous and answer some questions raised in [5]. It was shown in [5] that $\aleph_1$ is almost precipitous once there is an $\aleph_1$-Erdős cardinal and we would like to show that $\aleph_1$ and even bigger cardinals can be almost − precipitous even if there is no measurable cardinal in an inner model. Also, we would like to find a relation between semi − precipitous ideals (due to Donder-Levinski [1]) and almost − precipitous ideals and then we present some forcing notions which preserve the property of $\kappa$ being semi − precipitous.

We begin with several elementary definitions.

Let $\kappa$ be a regular cardinal and let $I$ be an ideal over $\kappa$.

We denote $I^+ = \{X \in P(\kappa) \mid X \in P(\kappa) - I\}$ and call the elements of $I^+$ sets of positive − measure.

$I$ is called normal iff for every positive measure set $X \in I^+$ and for every function $f$ ,if $f(\alpha) < \alpha$ for every $\alpha \in X$ then $f$ is constant on some positive measure set $Y \subseteq X$.

Let $\lambda$ be a cardinal, then $I$ is $\lambda$ − saturated iff whenever $\{X_\alpha \mid \alpha < \delta\}$ is a collection of $I$ positive measure sets so that $\alpha \neq \beta$ implies $X_\alpha \cap X_\beta \in I$ then $\delta < \lambda$.
3 On semi precipitous and almost precipitous ideals

3.1 Definitions

Definition 3.1 (Jech-Prikry) Let $\kappa$ be a regular uncountable cardinal, an Ideal $I$ above $\kappa$ is precipitous iff for every generic $G \subseteq P(\kappa)/I$, the ultrapower $V^\kappa/G$ is well founded.

In [4] and [5] the following ideals were considered:

Definition 3.2 Let $\kappa$ be a regular uncountable cardinal, $\tau$ a ordinal and $I$ a $\kappa$-complete ideal over $\kappa$. We call $I$ $\tau$-almost precipitous iff every generic ultrapower of $I$ is wellfounded up to the image of $\tau$.

Clearly, any such $I$ is $\tau$-almost precipitous for each $\tau < \kappa$. Also, note that if $\tau \geq (2^\kappa)^+$ and $I$ is $\tau$-almost precipitous, then $I$ is precipitous.

Definition 3.3 Let $\kappa$ be a regular uncountable cardinal. We call $\kappa$ almost precipitous iff for each $\tau < (2^\kappa)^+$ there is $\tau$-almost precipitous ideal over $\kappa$.

It was shown in [5] that $\aleph_1$ is almost precipitous once there is an $\aleph_1$-Erdős cardinal. The following questions were raised in [5]:
1. Is $\aleph_1$-Erdős cardinal needed?
2. Can cardinals above $\aleph_1$ be almost precipitous without a measurable cardinal in an inner model?

We will construct two generic extensions of $L$ such $\aleph_1$ will be almost precipitous in the first and $\aleph_2$ in the second.

Some of the ideas of Donder and Levinski [1] will be crucial here.

Definition 3.4 (Donder-Levinski [1]) Let $\kappa$ be a cardinal and $\tau$ be a limit ordinal of cofinality above $\kappa$ or $\tau =$ On. $\kappa$ is called $\tau$-semi-precipitous iff there exists a forcing notion $P$ such the following is forced by the weakest condition:

there exists an elementary embedding $j : V_\tau \rightarrow M$ such that

1. $\text{crit}(j) = \kappa$
2. $M$ is transitive.

$\kappa$ is called $< \lambda$-semi-precipitous iff it is $\tau$-semi-precipitous for every $\tau < \lambda$.
$\kappa$ is called a semi-precipitous iff it is $\tau$-semi-precipitous for every $\tau$. 
κ is called $\infty$-semi-precipitous iff it is $On$-semi-precipitous for every $\tau$.

Note if $\kappa$ is a semi-precipitous, then it is not necessarily $\infty$-semi-precipitous, since by Donder and Levinski [1] semi-precipitous cardinals are compatible with $V = L$, and $\infty$-semi-precipitous cardinals imply an inner model with a measurable.

For a cardinal $\kappa$ which is $\tau$-semi precipitous, let us call

$$ F = \{ X \subseteq \kappa \mid 0_p \parallel \kappa \in j(X) \} $$

a $\tau$-semi-precipitous filter. Note that such $F$ is a normal filter over $\kappa$.

**Lemma 3.5** Let $F$ be a $\tau$-almost precipitous normal filter over $\kappa$ for some ordinal $\tau$ above $\kappa$. Then $F$ is $\tau$-semi-precipitous.

**Proof.** Force with $F^+$. Let $i : V \rightarrow N = V \cap \mathcal{V}/G$ be the corresponding generic embedding. Set $j = i \upharpoonright \tau$. Then $j : V_\tau \rightarrow (V_{i(\tau)})^N$. Set $M = (V_{i(\tau)})^N$. We claim that $M$ is well founded. Suppose otherwise. Then there is a sequence $\langle g_n \mid n < \omega \rangle$ of functions such that

1. $g_n \in V$
2. $g_n : \kappa \rightarrow V_\tau$
3. $\{ \alpha < \kappa \mid g_{n+1}(\alpha) \in g_n(\alpha) \} \in G$

Replace each $g_n$ by a function $f_n : \kappa \rightarrow \tau$. Thus, set $f_n(\alpha) = \text{rank}(g_n(\alpha))$. Clearly, still we have

$$ \{ \alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha) \} \in G. $$

But this means that $N$ is not well-founded below the image of $\tau$. Contradiction. $\square$

Note that the opposite direction does not necessary hold. Thus for $\tau \geq (2^\kappa)^+$, $\tau$-almost precipitousness implies precipitousness and hence a measurable cardinal in an inner model. By Donder and Levinski [1], it is possible to have semi-precipitous cardinals in $L$.

The following is an analog of a game that was used in [5] with connection to almost precipitous ideals.

**Definition 3.6** (The game $G_\tau(F)$)

Let $F$ be a normal filter on $\kappa$ and let $\tau > \kappa$ be an ordinal.

The game $G_\tau(F)$ is defined as follows:
Player I starts by picking a set $A_0$ in $F^+$. Player II chooses a function $f_1 : A_0 \to \tau$ and either a partition $\langle B_i \mid i < \xi < \kappa \rangle$ of $A_0$ into less than $\kappa$ many pieces or a sequence $\langle B_\alpha \mid \alpha < \kappa \rangle$ of disjoint subsets of $\kappa$ so that

$$\nabla_{\alpha < \kappa} B_\alpha \supseteq A_0.$$ 

The first player then supposed to respond by picking an ordinal $\alpha_2$ and a set $A_2 \in F^+$ which is a subset of $A_0$ and of one of $B_i$’s or $B_\alpha$’s.

At the next stage the second player supplies again a function $f_3 : A_2 \to \tau$ and either a partition $\langle B_i \mid i < \xi < \kappa \rangle$ of $A_2$ into less than $\kappa$ many pieces or a sequence $\langle B_\alpha \mid \alpha < \kappa \rangle$ of disjoint subsets of $\kappa$ so that

$$\nabla_{\alpha < \kappa} B_\alpha \supseteq A_2.$$ 

The first player then supposed to respond by picking a stationary set $A_4$ which is a subset of $A_2$ and of one of $B_i$’s or $B_\alpha$’s on which everywhere $f_1$ is either above $f_3$ or equal $f_3$ or below $f_3$. In addition he picks an ordinal $\alpha_4$ such that

$$\alpha_2, \alpha_4 \text{ respect the order of } f_1 \restriction A_4 \text{ and } f_3 \restriction A_4.$$ 

Intuitively, $\alpha_2 n$ pretends to represent $f_{2n-1}$ in a generic ultrapower.

Continue further in the same fashion.

Player I wins if the game continues infinitely many moves. Otherwise Player II wins.

Clearly it is a determined game.

The following lemma is analogous to [5] (Lemma 3).

**Lemma 3.7** Suppose that $\lambda$ is a $\kappa$-Erdős cardinal. Then for each ordinal $\tau < \lambda$ Player II does not have a winning strategy in the game $G_\tau(Cub_\kappa)$.

**Proof.** Suppose otherwise. Let $\sigma$ be a strategy of two. Find a set $X \subset \lambda$ of cardinality $\kappa$ such that $\sigma$ does not depend on ordinals picked by Player I from $X$. In order to get such $X$ let us consider a structure

$$\mathfrak{A} = \langle H(\lambda), \in, \lambda, \kappa, P(\kappa), F, G_\tau(F), \sigma \rangle.$$ 

Let $X$ be a set of $\kappa$ indiscernibles for $\mathfrak{A}$.

Pick now an elementary submodel $M$ of $H(\chi)$ for $\chi > \lambda$ big enough of cardinality less than $\kappa$, with $\sigma, X \in M$ and such that $M \cap \kappa \in On$. Let $\alpha = M \cap \kappa$. Let us produce an infinite play in which the second player uses $\sigma$. This will give us the desired contradiction. Consider the set $S = \{ f(\alpha) \mid f \in M, f \text{ is a partial function from } \kappa \text{ to } \tau \}$. Obviously, $S$ is
countable. Hence we can fix an order preserving function $\pi : S \to X$.
Let one start with $A_0 = \kappa$. Consider $\sigma(A_0)$. Clearly, $\sigma(A_0) \in M$. It consists of a function $f_1 : A_0 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \kappa \rangle$ of disjoint subsets of $\kappa$ so that

$$\bigcup_{\xi < \kappa} B_\xi \supseteq A_0.$$ 

Now, $\alpha \in A_0$, hence there is $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_0 \cap B_{\xi^*} \in M$ and $\alpha \in A_0 \cap B_{\xi^*}$. Let $A_2 = A_0 \cap B_{\xi^*}$. Note that $A_2 \cap C \neq \emptyset$, for every closed unbounded subset $C$ of $\kappa$ which belongs to $M$, since $\alpha$ is in both $A_2$ and $C$.

Pick $\alpha_2 = \pi(f_1(\alpha))$.
Consider now the answer of two which plays according to $\sigma$. It does not depend on $\alpha_2$, hence it is in $M$. Let it be a function $f_3 : A_2 \to \tau$ and, say a sequence $\langle B_\xi | \xi < \kappa \rangle$ of disjoint subsets of $\kappa$ so that

$$\bigcup_{\xi < \kappa} B_\xi \supseteq A_2.$$ 

As above find $\xi^* < \alpha$ such that $\alpha \in B_{\xi^*}$. Then $B_{\alpha^*} \in M$, as $M \supseteq \alpha$. Hence, $A_2 \cap B_{\xi^*} \in M$ and $\alpha \in A_2 \cap B_{\xi^*}$. Let $A'_2 = A_2 \cap B_{\xi^*}$. Split it into three sets $C_<, C_=', C_>$ such that

$$C_\xi = \{ \nu \in A'_2 | f_3(\nu) < f_1(\nu) \},$$

$$C_\xi = \{ \nu \in A'_2 | f_3(\nu) = f_1(\nu) \},$$

$$C_\xi = \{ \nu \in A'_2 | f_3(\nu) > f_1(\nu) \}.$$ 

Clearly, $\alpha$ belongs to only one of them, say to $C_\xi$. Set then $A_4 = C_\xi$. Then, clearly, $A_4 \in M$, it is stationary and $f_3(\alpha) < f_1(\alpha)$. Set $\alpha_4 = \pi(f_3(\alpha))$.

Continue further in the same fashion.

□

It follows that the first player has a winning strategy.

The next game was introduced by Donder and Levinski in [1].

**Definition 3.8** A set $R$ is called $\kappa$-plain iff

1. $R \neq \emptyset$,
2. $R$ consists of normal filters over $\kappa$,
3. for all $F \in R$ and $A \in F^+$, $F + A \in R$. 


Definition 3.9 (The game $H_R(F, \tau)$)

Let $R$ be a $\kappa$-plain, $F \in R$ be a normal filter on $\kappa$ and let $\tau > \kappa$ be an ordinal. The game $H_R(F, \tau)$ is defined as follows. Set $F_0 = F$. Let $1 \leq i < \omega$. Player I plays at stage $i$ a pair $(A_i, f_i)$, where $A_i \subseteq \kappa$ and $f_i : \kappa \to \tau$. Player II answers by a pair $(F_i, \gamma_i)$, where $F_i \in R$ and $\gamma_i$ is an ordinal. The rules are as follows:

1. For $0 \leq i < \omega, A_{i+1} \in (F_i)^+$
2. For $0 \leq i < \omega, F_{i+1} \supseteq F_i[A_{i+1}]

Player II wins iff for all $1 \leq i, k \leq n < \omega : (f_i <_{F_n} f_k) \to (\gamma_i < \gamma_k)$

Donder and Levinski [1] showed that an existence of a winning strategy for Player II in the game $H_R(F, \lambda)$ for some $R, F$ is equivalent to $\kappa$ being $\tau$-semi precipitous by proving the following theorem:

Theorem 3.10 Let $\lambda$ be a cardinal such that $cf(\lambda) > \kappa$, the following are equivalent:

1. $\kappa$ is $\lambda$-semi precipitous
2. There exists a $\kappa$ plain $R$ so that for every $F \in R$ player II has a winning strategy in the game $H_R(F, \lambda)$
3. There exists a normal filter $F$ above $\kappa$ so that player II has a winning strategy in the game $H(F, \lambda)$

3.2 Theorems and Remarks

In the proof of 3.10 ( (3) implies (1) ), Donder and Levinski showed that $P = Col(\omega, \lambda^\kappa)$ satisfies the definition of $\kappa$ being a $\lambda$-semi precipitous so it’s natural to ask whether we can find a ”better” forcing notion which witnessing the same. Gitik showed that if the forcing notion $P$ is $\aleph_2$-closed then $0^#$ exists.

Lemma 3.11 Let $\lambda$ be a cardinal such that $cf(\lambda) > \kappa$, suppose that $\kappa \geq \aleph_2$, $\kappa$ is $\lambda$-semi precipitous and let $P$ be a forcing notion which witnessing it. If $P$ is $\aleph_2$-closed then $0^#$ exists.

Proof. Suppose otherwise. Now, in $V[G]$, let $j : L_\lambda \to L_\theta$ be a $P$-generic elementary embedding with $crit(j) = \kappa$. Let us define (in $V[G]$) $F := \{X \in P(\kappa) \mid X \in L \land \kappa \in j(X)\}$
and we would like to prove that $F$ is a $V[G]$ $\sigma-$closed filter (which yields an elementary embedding $j : L \rightarrow L$, contradiction). Let $\{X_n \mid n < \omega\} \in V[G]$ be a sequence of $F$-members. By the covering lemma of Jensen we can find a sequence $\{Y_\alpha \mid \alpha < \aleph_1\} \in L$ which covers $\{X_n \mid n < \omega\}$. Notice that because $P$ is $\aleph_2-$closed we get that $j\{Y_\alpha \mid \alpha < \aleph_1\} = \{j(Y_\alpha) \mid \alpha < \aleph_1\}$. Also notice that the set $\{Y_\alpha \mid \alpha < \aleph_1, \kappa \in j(Y_\alpha)\}$ is in $L$. We get that $\{X_n \mid n < \omega\} \subseteq \{Y_\alpha \mid \alpha < \aleph_1, \kappa \in j(Y_\alpha)\}$ and because $\kappa \in j(\cap\{Y_\alpha \mid \alpha < \aleph_1, \kappa \in j(Y_\alpha)\})$ we get that $\cap_{n<\omega}X_n \in F$.

□

We don’t know whether the existence of a $\aleph_1$ closed notion of forcing witnessing semi precipitousness of $\kappa$ is a stronger assumption then in the definition of Donder-Levinski.

Next two lemmas deal with connections between winning strategies for the games $G_\tau(F)$ and $H_R(F, \tau)$.

**Lemma 3.12** Suppose that Player $\Pi$ has a winning strategy in the game $H_R(F, \tau)$, for some $\kappa$-plain $R$, a normal filter $F \in R$ over $\kappa$ and an ordinal $\tau$. Then Player $\Pi$ has a winning strategy in the game $G_\tau(F)$.

**Proof.** Let $\sigma$ be a winning strategy of Player $\Pi$ in $H_R(F, \tau)$. Define a winning strategy $\delta$ for Player $\Pi$ in the game $G_\tau(F)$. Let the first move according to $\delta$ be $\kappa$. Suppose that Player $\Pi$ responds by a function $f_1 : \kappa \rightarrow \tau$ and a partition $B_1$ of $\kappa$ to less then $\kappa$ many subsets or a sequence $B_1 = \{B_\alpha \mid \alpha < \kappa\}$ of $\kappa$ many subsets such that $\nabla_{\alpha<\kappa}B_\alpha \supseteq \kappa$. Turn to the strategy $\sigma$. Let $\sigma(\kappa, f_1) = (F_1, \gamma_1)$, for some $F_1 \supseteq F, F_1 \in R$ and an ordinal $\gamma_1$. Now we let Player $\Pi$ pick $A_1 \in (F_1)^+$ such that there is a set $B \in B_1$ with $A_1 \subseteq B$ (he can always choose such an $A_1$ because $F_1$ is normal and $\nabla_{\alpha<\kappa}B_\alpha \supseteq A_1$. Back in $H_R(F, \tau)$, we consider the answer according $\sigma$ of Player $\Pi$ to $(A_1, f_2)$, i.e. $\sigma((\kappa, f_1), (A_1, f_2)) = (F_2, \gamma_2)$. Choose $A_2 \in (F_2)^+$ such that there is a set $B \in B_2$ with $A_2 \subseteq B$ (it is always possible to find such $A_2$ because $F_2$ is normal and $\nabla_{\alpha<\kappa}B_\alpha \subset (F_2)^+$) on which either $f_1 < f_2$ or $f_1 > f_2$ or $f_1 = f_2$. Let the respond according to $\delta$ be $(A_2, \gamma_2)$. Continue in a similar fashion. The play will continue infinitely many moves. Hence Player $\Pi$ will always win once using the strategy $\delta$.

□
Lemma 3.13 Suppose that Player \( I \) has a winning strategy in the game \( G_\tau(F) \), for a normal filter \( F \) over \( \kappa \) and an ordinal \( \tau \). Then Player \( II \) has a winning strategy in the game \( H_R(D, \tau) \) for some \( \kappa \)-plain \( R \) and \( D \in R \).

Proof. Let \( \sigma \) be a winning strategy of Player \( I \) in \( G_\tau(F) \). Set
\[
J = \{ X \subseteq \kappa \mid X \text{ and any of its subsets are never used by } \sigma \},
\]
and for every finite play \( t = \langle t_1, \ldots, t_{2n} \rangle \)
\[
J_t = \{ X \subseteq \kappa \mid X \text{ and any of its subsets are never used by } \sigma \text{ in the continuation of } t \}.
\]
It is not hard to see that such \( J \) and \( J_t \)'s are normal ideals over \( \kappa \). Denote by \( D \) and \( D_t \) the corresponding dual filters.

Pick \( R \) to be a \( \kappa \)-plain which includes \( D \) and all \( D_t \)'s.

Define a winning strategy \( \delta \) for Player \( II \) in the game \( H_R(D, \tau) \). Let \( (A_1, g_1) \) be the first move in \( H_R(D, \tau) \). Then \( A_1 \in D^+ \). Hence \( \sigma \) picks \( A_1 \) in a certain play \( t \) as a move of Player \( I \) in the game \( G_\tau(F) \). Continue this play, and let Player \( II \) responde by a trivial partition of \( A_1 \) consisting of \( A_1 \) itself and by function \( g_1 \) restricted to \( A_1 \). Let \( (B_1, \gamma_1) \) be the respond of Player \( I \) according to \( \sigma \). Set \( t_1 = t^{\frown} (\{A_1\}, g_1) \). Then \( B_1 \in D_{t_1} \). Now we set the respond of Player \( II \) according to \( \delta \) to be \( (D_{t_1}, \gamma_1) \).

Continue in similar fashion.

\( \Box \)

Theorem 3.14 Suppose that \( \lambda \) is a \( \kappa \)-Erdős cardinal, then \( \kappa \) is \( \tau \)-semi precipitous for every \( \tau < \lambda \).


\( \Box \)

Combining the above with Theorem 17 of [5], we obtain the following:

Theorem 3.15 Assume that \( 2^{\aleph_1} = \aleph_2 \) and \( \|f\| = \omega_2 \), for some \( f : \omega_1 \to \omega_1 \). Let \( \tau < \aleph_3 \). If there is a \( \tau \)-semi-precipitous filter over \( \aleph_1 \), then there is a normal \( \tau \)-almost precipitous filter over \( \aleph_1 \) as well.

By Donder and Levinski [1], \( 0^\# \) implies that the first indiscernible \( c_0 \) for \( L \) is in \( L \) \( \tau \)-semi-precipitous for each \( \tau \). They showed [1](Theorem 7) that the property "\( \kappa \) is \( \tau \)-semi-precipitous" relativizes down to \( L \). Also it is preserved under \( \kappa \)-c.c. forcings of cardinality \( \leq \kappa \) ([1](Theorem 8)).
Now combine this with 3.15. We obtain the following:

**Theorem 3.16** Suppose that \( \kappa < \kappa^{++} \)-semi-precipitous cardinal in \( L \). Let \( G \) be a generic subset of the Levy Collapse \( \text{Col}(\omega, < \kappa) \). Then for each \( \tau < \kappa^{++} \), \( \kappa \) carries a \( \tau \)-almost precipitous normal ideal in \( L[G] \).

**Proof.** In order to apply 3.15, we need to check that there is \( f : \omega_1 \to \omega_1 \) with \( \| f \| = \omega_2 \). Suppose otherwise. Then by Donder and Koepke [2] (Theorem 5.1) we will have \( w\text{CC}(\omega_1) \) (the weak Chang Conjecture for \( \omega_1 \)). Again by Donder and Koepke [2] (Theorem D), then \( (\aleph_2)^{L[G]} \) will be almost \( < (\aleph_1)^L \)-Erdös in \( L \). But note that \( (\aleph_2)^{L[G]} = (\kappa^+)^L \) and in \( L \), \( 2^\kappa = \kappa^+ \). Hence, in \( L \), we must have \( 2^\kappa \rightarrow (\omega)^2_\kappa \), as a particular case of \( 2^\kappa \) being almost \( < \aleph_1 \)-Erdös. But \( 2^\kappa \not\rightarrow (3)_{\kappa^+}^2 \). Contradiction.

\( \square \)

**Corollary 3.17** The following are equivalent:

1. \( \text{Con}(\text{there exists an almost precipitous cardinal}) \),
2. \( \text{Con}(\text{there exists an almost precipitous cardinal with normal ideals witnessing its almost precipitousness}) \),
3. \( \text{Con}(\text{there exists} \ < \kappa^{++} \text{-semi-precipitous cardinal} \kappa) \).

In particular the strength of existence of an almost precipitous cardinal is below \( 0^\# \).

## 4 An almost precipitous ideal on \( \omega_2 \)

In this section we will construct a model with \( \aleph_2 \) being almost precipitous.

### 4.1 The model

The initial assumption will be an existence of a Mahlo cardinal \( \kappa \) which carries a \( (2^\kappa)^+ \)-semi precipitous normal filter \( F \) with \( \{ \tau < \kappa \mid \tau \text{ is a regular cardinal} \} \in F \).

Again by Donder and Levinski [1] this assumption is compatible with \( L \). Thus, under \( 0^\dagger \), the first indiscernible will be like this in \( L \).

Assume \( V = L \).

Let \( \langle P_i, Q_j \mid i \leq \kappa, j < \kappa \rangle \) be Revised Countable Support iteration (see [11]) so that for each \( \alpha < \kappa \), if \( \alpha \) is an inaccessible cardinal (in \( V \)), then \( Q_\alpha \) is \( \text{Col}(\omega_1, \alpha) \) which turns it to \( \aleph_1 \).
and $Q_{\alpha+1}$ will be the Namba forcing which changes the cofinality of $\alpha^+$ (which is now $\aleph_2$) to $\omega$. In all other cases let $Q_\alpha$ be the trivial forcing.

By [11](Chapter 9), the forcing $P_\kappa$ turns $\kappa$ into $\aleph_2$, preserves $\aleph_1$, does not add reals and satisfies the $\kappa$-c.c. Let $G$ be a generic subset of $P_\kappa$.

By Donder and Levinski ([1]), a $\kappa$-c.c. forcing preserves semi precipitousness of $\kappa$. Hence $\kappa$ is $\kappa^{++} = \aleph_4$-semi precipitous in $L[G]$ and we are able to continue the generic elementary embedding by extending $F$ to $F^*$. In addition,
\[
\{ \tau < \kappa | \text{cof}(\tau) = \omega_1 \} \in F^*
\]
and
\[
\{ \tau < \kappa | \text{cof}((\tau^+)^V) = \omega \} \in F^*.
\]
For our convenience we shall denote $F^*$ by $F$.

Now, there is a forcing $Q$ in $L[G]$ so that in $L[G]^Q$ we have a generic embedding
\[ j : L_{\kappa^{++}}[G] \to M \]
such that $M$ is a transitive and $\kappa \in j(A)$ for every $A \in F$. By elementarity, then $M$ is of the form $L_{\lambda}[G^*]$, for some $\lambda > \kappa^{++}$, and $G^* \subseteq j(P_\kappa)$ which is $L_\lambda$-generic.
Note that $Q_\kappa$ collapses $\kappa$ to $(\aleph_1)^M$ because it was an inaccessible cardinal, and at the very next stage its successor changes the cofinality to $\omega$. That means that there is a function $H \in L_{\kappa^{++}}[G]$ such that $j(H)(\kappa) : \omega \to (\aleph_3)^{L[G]}$ is an increasing and unbounded in $(\kappa^+)^L = (\aleph_3)^{L[G]}$ function.
We will use such $H$ as a replacement of the corresponding function of [4]. Together with the fact that in the model $L[G]$ we have a filter on $\aleph_2$ which is $\aleph_4$ semi precipitous this will allow us to construct $\tau$- almost precipitous filter on $\aleph_2$, for every $\tau < \aleph_4$.

### 4.2 The construction

Fix $\tau < \kappa^{++}$.

By [1], we can assume that $Q = Col(\omega, \tau^\kappa)$ Denote by $B$ the complete Boolean algebra $RO(Q)$. Further by $\leq$ we will mean the order of $B$.

For each $p \in B$ set
\[ F_p = \{ X \subseteq \kappa | p \forces \kappa \in j(X) \} \]

We will use the following easy lemma:

**Lemma 4.1** 1. $p \leq q \Rightarrow F_p \supseteq F_q$
2. $X \in (F_p)^+$ iff there is a $q \leq p, q \Vdash \kappa \in j(X)$

3. Let $X \in (F_p)^+$, then for some $q \leq p$, $F_q = F_p + X$

Proof. (1) and (2) are trivial. Let us prove (3).

Suppose that $X \in (F_p)^+$. Set $q = |\kappa \in j(X)|_B \wedge p$. We claim that $F_q = F_p + X$. The inclusion $F_q \supseteq F_p + X$ is trivial. Let us show that $F_p + X \supseteq F_q$. Suppose not, then there are $Y \in (F_p)^+, Y \subseteq X$ and $Z \in F_q$ such that $Y \cap Z = \emptyset$. But $Y \in (F_p)^+$, so we can find $s \leq p$ such that $s \Vdash \kappa \in j(Y)$. Now, $s \leq p$ and $s \Vdash \kappa \in j(X)$, since $Y \subseteq X$. Hence, $s \leq q$. But then

$s \Vdash \kappa \in j(Y), \kappa \in j(Z), j(Z \cap Y) = \emptyset$.

Contradiction.

□

Define $\{A_{n\alpha} \mid \alpha < \kappa^+, n < \omega\}$ as in [4]:

$$A_{n\alpha} = \{\eta > \kappa \mid \exists p \in B \quad p \Vdash H(\eta)(n) = h_\alpha(\eta)\},$$

where $\langle h_\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of $\kappa^+$ canonical functions from $\kappa$ to $\kappa$ (in $V^B$). Note that here $H$ is only cofinal and not onto, as in [4].

The following lemmas were proved in [4] and hold without changes in the present context:

Lemma 4.2 For every $n < \omega$ there is an ordinal $\alpha < \kappa^+$ so that $A_{n\alpha} \in (F_1^B)^+$

Lemma 4.3 For every $\alpha < \kappa^+$ and $p \in B$ there is $n < \omega$ and $\alpha < \beta < \kappa^+$ so that $A_{n\beta} \in (F_p)^+$

Lemma 4.4 Let $n < \omega$ and $p \in B$. Then the set:

$$\{A_{n\alpha} \mid \alpha < \kappa^+ \text{ and } A_{n\alpha} \in (F_p)^+\}$$

is a maximal antichain in $(F_p)^+$.

The following is an analog of a lemma due Assaf Rinot in [4], 3.5.

Lemma 4.5 Let $D$ be a family of $\kappa^+$ dense subsets of $B$, there exists a sequence $\langle p_\alpha \mid \alpha < \kappa^+ \rangle$ such that for all $Z \in (F_1^B)^+, p' \in Q$ and $n < \omega$ if

$$Z_{n,p'} = \{\alpha < \kappa^+ \mid A_{n\alpha} \cap Z \in (F_{p'})^+\}$$

has cardinality $\kappa^+$ then:
1. For any \( p \in \mathcal{B} \) there exists \( \alpha \in Z_{n,p'} \) with \( p \geq p_{\alpha} \).

2. For any \( D \in \mathcal{D} \) there exists \( \alpha \in Z_{n,p'} \) with \( p_{\alpha} \models \kappa \in j(A_{n\alpha} \cap Z), p_{\alpha} \leq p' \) and \( p_{\alpha} \in D \).

Proof. Let \( \{ S_{i} \mid i < \kappa^+ \} \subseteq [\kappa^+]^{\kappa^+} \) be some partition of \( \kappa^+ \), \( \{ D_{\alpha} \mid \alpha < \kappa^+ \} \) an enumeration of \( \mathcal{D}, \{ q_{\alpha} \mid \alpha < \kappa^+ \} \) an enumeration of \( Q \) and let \( \triangleleft \) be a well ordering of \( \kappa^+ \cup \kappa^+ \times \kappa^+ \) of order type \( \kappa^+ \). Now, fix a surjective function \( \varphi : \kappa^+ \to \{ (Z,n,p) \in ((F_{18})^+, \omega, Q) \mid |Z_{n,p}| = \kappa^+ \} \).

We would like to define a function \( \psi : \kappa^+ \to \kappa^+ \cup \kappa^+ \times \kappa^+ \) and the sequence \( (p_{\alpha} \mid \alpha < \kappa^+) \).

For that, we now define two sequences of ordinals \( \{ L_{\alpha} \mid \alpha < \kappa^+ \}, \{ R_{\alpha} \mid \alpha < \kappa^+ \} \) and the values of \( \psi \) and the sequence on the intervals \( [L_{\alpha}, R_{\alpha}] \) by recursion on \( \alpha < \kappa^+ \). For \( \alpha = 0 \) we set \( L_{0} = R_{0} = 0, \psi(0) = 0 \) and \( p_{0} = q_{0} \).

Now, suppose that \( \{ L_{\beta}, R_{\beta} \mid \beta < \alpha \} \) and \( \psi \upharpoonright \bigcup_{\beta < \alpha} [L_{\beta}, R_{\beta}] \) were defined. Take \( i \) to be the unique index such that \( \alpha \in S_{i} \). Let \( (Z,n,p) = \varphi(i) \) and set \( L_{\alpha} = \min(\kappa^+ \cup \bigcup_{\beta < \alpha} [L_{\beta}, R_{\beta}]) \), \( R_{\alpha} = \min(Z_{n,p} \setminus L_{\alpha}) \).

Now, for each \( \beta \in [L_{\alpha}, R_{\alpha}] \) we set \( \psi(\beta) = t \), where:

\[
 t = \min\alpha(\kappa^+ \cup \{ i \} \times \kappa^+) \setminus \psi''(Z_{n,p} \cap L_{\alpha}).
\]

If \( t \in \kappa^+ \) then we set \( p_{\beta} = q_{t} \) for each \( \beta \in [L_{\alpha}, R_{\alpha}] \). Otherwise, \( t = (i, \delta) \) for some \( \delta < \kappa^+ \) and because \( A_{nR_{\alpha}} \cap Z \subseteq F_{p}^+ \) and \( D_{\delta} \) is dense we can find some \( q \in D_{\delta}, q \leq p, q \models \kappa \in j(A_{nR_{\alpha}} \cap Z) \) and set \( p_{\beta} = q \) for each \( \beta \in [L_{\alpha}, R_{\alpha}] \). This completes the construction.

Now, we would like to check that the construction works. Fix \( Z \subseteq F_{18}^+, p \in Q \) and \( n < \omega \) so that \( |Z_{n,p}| = \kappa^+ \). Let \( i < \kappa^+ \) be such that \( \varphi(i) = Z_{n,p} \) and notice that the construction insures that \( \psi''Z_{n} = \kappa^+ \cup \{ i \} \times \kappa^+ \).

(1) Let \( p' \in \mathcal{B} \): There exists a \( t < \kappa^+ \) so that \( q_{t} \leq p' \). Let \( \alpha \in Z_{n} \) be such that \( \psi(\alpha) = t \), so \( p_{\alpha} = q_{t} \leq p' \).

(2) Let \( D \in \mathcal{D} \). There exist \( \delta < \kappa^+ \) and \( \alpha \in Z_{n,p} \) such that \( D_{\delta} = D \) and \( \psi(\alpha) = (i, \delta) \). Then, by the construction we have that \( p_{\alpha} \in D_{\delta}, p_{\alpha} \models \kappa \in j(A_{nR_{\alpha}} \cap Z) \) and \( p_{\alpha} \leq p' \).

Define \( \mathcal{D} = \{ D_{f} \mid f \in (\tau^{n})^{V} \} \), where

\[
 D_{f} = \{ p \in \mathcal{B} \mid \exists \gamma \in On \quad p \models j(\tilde{f})(\kappa) = \gamma \}
\]

and let \( (p_{\alpha} \mid \alpha < \kappa^+) \) be as in lemma 4.5.

We turn now to the construction of filters which will be similar to those of [4]).

Start with \( n = 0 \). Let \( \alpha < \kappa^+ \). Consider three cases:
Case I: If \(|\{\xi < \kappa^+ \mid A_0\xi \in (F_{1B})^+\}\}| = \kappa^+ and \(p_\alpha \models \kappa \in j(A_{0a})\) then we define \(q_{<\alpha>} = p_\alpha and extend \(F_{1B}\) to \(F_{q_{\alpha<\xi}>}\).

Case II: If \(A_{0a} \in (F_{1B})^+\) then we define \(q_{<\alpha>} = \|\kappa \in j(A_{0a})\|_B\) and extend our filter to \(F_{q_{\alpha<\xi}>}\).

Case III: If \(A_{0a} \in \bar{F}_{1B}\) (the dual ideal of \(F_{1B}\)) then \(q_{<\alpha>}\) is not defined.

Notice that by Lemma 4.2, there exists some \(\alpha < \kappa^+\) with \(A_{0a} \in (F_{1B})^+\), thus \(\{\alpha < \kappa^+ \mid F_{q_{\alpha<\xi}>}\) is defined \} is non-empty.

**Definition 4.6** Set \(F_0 = \bigcap\{ F_{q_{\alpha<\xi>}\mid \alpha < \kappa^+, F_{q_{\alpha<\xi>}}\) is defined \}, and denote the corresponding dual ideals by \(I_{q_{\alpha<\xi>}}\) and \(I_0\).

Clearly, \(I_0 = \bigcap\{ I_{q_{\alpha<\xi>}}\mid \alpha < \kappa^+, I_{q_{\alpha<\xi>}}\) is defined \}. Also, \(F_0 \supseteq F_{1B}\) and \(I_0 \supseteq \bar{F}_{1B}\), since each \(F_{q_{\alpha<\xi>}}\supseteq F_{1B}\) and \(I_{q_{\alpha<\xi>}}\supseteq \bar{F}_{1B}\). Note that \(F_0\) is a \(\kappa\) complete, normal and proper filter since it is an intersection of such filters and also \(I_0\) is.

We now describe the successor step of the construction, i.e., \(m = n + 1\).

Let \(\sigma : m \rightarrow \kappa^+\) be a function with \(F_{p_\alpha}\) defined and \(\alpha < \kappa^+\). There are three cases:

Case I: If \(|\{\xi < \kappa^+ \mid A_{m\xi} \in (F_{p_\alpha})^+\}\}| = \kappa^+ , p_\alpha \leq p_\sigma and p_\alpha \models \kappa \in j(A_{ma})\), then we define \(q_{\sigma - \alpha} = p_\alpha and extend \(F_{p_\alpha}\) to \(F_{q_{\sigma - \alpha}}\).

Case II: If Case I fails, but \(A_{ma} \in (F_{\sigma})^+\), then let \(q_{\sigma - \alpha} = \|\kappa \in j(A_{ma})\|_B \wedge q_\sigma\) and extend \(F_{q_\sigma}\) to \(F_{q_{\sigma - \alpha}}\).

Case III: If \(A_{ma} \in I_{p_\sigma}\), then \(q_{\sigma - \alpha}\) and \(F_{q_{\sigma - \alpha}}\) would not be defined.

This completes the construction.

**Definition 4.7** Let \(F_{n+1} = \bigcap\{ F_{\sigma\mid \sigma : n + 2 \rightarrow \kappa^+, F_{p_\sigma}\) is defined \}, and define the corresponding dual ideals \(I_{n+1}, I_{p_\sigma}\).

Notice that all \(F_n\)s and \(I_n\)s are \(\kappa\) complete, proper and normal as an intersection of such filters and ideals respectively.

**Definition 4.8** Let \(F_\omega\) be the closure under \(\omega\) intersections of \(\bigcup_{n<\omega} F_n\).

Let \(I_\omega\) = the closure under \(\omega\) unions of \(\bigcup_{n<\omega} I_n\).

**Lemma 4.9** \(F_{1B} \subseteq F_0 \subseteq \ldots \subseteq F_n \subseteq \ldots \subseteq F_\omega\) and \(\bar{F}_{1B} \subseteq I_0 \subseteq \ldots \subseteq I_n \subseteq \ldots \subseteq I_\omega\), and \(I_\omega\) is the dual ideal to \(F_\omega\).
Lemma 4.10  Let \( s : m \to \kappa^+ \) with \( F_p s \) defined; then:

1. \( \{ \alpha < \kappa^+ \mid F_{s_{\sim \alpha}} \text{ is defined} \} = \{ \xi < \kappa^+ \mid A_{m\xi} \in F^+_p \} \);

2. There exists an extension \( \sigma \supseteq s \) such that \( F_p \sigma \) is defined and:

\[ |\{ \xi < \kappa^+ \mid A_{\text{dom}(\sigma)\xi} \in F^+_\sigma \}| = \kappa^+. \]

Proof. 1) is clear from the construction above. For 2), let us assume that for every extension \( \sigma \supseteq s \) such that \( F_p \sigma \) is defined:

\[ |\{ \xi < \kappa^+ \mid A_{\text{dom}(\sigma)\xi} \in F^+_\sigma \}| \leq \kappa. \]

That means that \( \Sigma = \{ \sigma : n \to \kappa^+ | n \geq m \text{ and } \sigma \supseteq s \} \) is of cardinality less or equal \( \kappa \), so \( \nu = \bigcup_{\sigma \in \Sigma} \text{ran}(\sigma) \) is less then \( \kappa^+ \) and \( p_s \) or some extension of it will force that \( j(H)(\kappa) \) is bounded, contradiction.

\[ \square \]

The rest of the proof is exactly as in [4]( Theorem 2.5) and in the next section we shall give a similar proof so we omit it. We just mention that the key lemma for the almost well foundedness is the following:

Lemma 4.11  \( F^+_\omega = \bigcup \{ F_s \mid s \in \kappa^+, F_s \text{ is defined} \} \)

\[ \square \]

5  Constructing of almost precipitous ideals from semi-precipitous

5.1 The main theorem

Suppose \( \kappa \) is a \( \lambda \) semi-precipitous cardinal for some ordinal \( \lambda \) which is a successor ordinal \( > \kappa \) or a limit one with \( \text{cof}(\lambda) > \kappa \). Let \( P \) be a forcing notion witnessing this. Then, for each generic \( G \subseteq P \), in \( V[G] \) we have an elementary embedding \( j : V_\lambda \to M \) with \( cp(j) = \kappa \) and \( M \) is transitive. Consider

\[ U = \{ X \subseteq \kappa \mid X \in V, \kappa \in j(X) \}. \]

Then \( U \) is a \( V \)-normal ultrafilter over \( \kappa \). Let \( i_U : V \to V \cap \kappa V/U \) be the corresponding elementary embedding. Note that \( V \cap \kappa V/U \) need not be well founded, but it is well founded
up to the image of $\lambda$. Thus, denote $V \cap ^*V/U$ by $N$. Define $k : (V_{i(\lambda)})^N \to M$ in a standard fashion by setting

$$k([f]_U) = j(f)(\kappa),$$

for each $f : \kappa \to V_{i(\lambda)}$, $f \in V$. Then $k$ will be elementary embedding, and so $(V_{i(\lambda)})^N$ is well founded.

For every $p \in P$ set

$$F_p = \{ X \subset \kappa \mid p \forces X \in j(X) \}.$$

Clearly, if $G$ is a generic subset of $P$ with $p \in G$ and $U_G$ is the corresponding $V$-ultrafilter, then $F_p \subseteq U_G$.

Note that, if for some $p \in P$ the filter $F_p$ is $\kappa^+$-saturated, then each $U_G$ with $p \in G$ will be generic over $V$ for the forcing with $F_p$-positive sets. Thus, every maximal antichain in $F_p^+$ consists of at most $\kappa$ many sets. Let $\langle A_\nu \mid \nu < \kappa \rangle \in V$ be such maximal antichain. Without loss of generality we can assume that $\min(A_\nu) > \nu$, for each $\nu < \kappa$. Then there is $\nu^* < \kappa$ with $\kappa \in j(A_{\nu^*})$. Hence $A_{\nu^*} \in U_G$ and we are done.

It follows that in such a case $N$ which is the ultrapower by $U_G$ is fully well founded.

Note that in general if some forcing $P$ produces a well founded $N$, then $\kappa$ is $\omega$-semi precipitous. Just $i$ and $N$ will witness this.

Our aim will be to prove the following:

**Theorem 5.1** Assume that $2^\kappa = \kappa^+$ and $\kappa$ carries a $\lambda$-semi-precipitous filter for some limit ordinal $\lambda$ with $\text{cof}(\lambda) > \kappa$. Suppose in addition that there is a forcing notion $P$ witnessing $\lambda$-semi-precipitous with corresponding $N$ ill founded. Then

1. if $\lambda < \kappa^{++}$, then $\kappa$ is $\lambda$-almost precipitous witnessed by a normal filter,
2. if $\lambda \geq \kappa^{++}$, then $\kappa$ is an almost precipitous witnessed by a normal filters.

**5.2 The proof**

*Proof.* The proof will be based on an extension of the method of constructing normal filters of [4] which replaces restrictions to positive sets by restrictions to filters. An additional idea will be to use a witness of a non-well-foundedness in the construction in order to limit it to $\omega$ many steps.
Let $\kappa, \tau, P$ be as in the statement of the theorem. Preserve the notation that we introduced above. Then

$$0_P \Vdash (V_{i(\lambda)})^N$$

is well founded and $N$ is ill founded.

Fix a sequence $\langle g_n \mid n < \omega \rangle$ of names of functions witnessing an ill foundedness of $N$, i.e.

$$0_P \Vdash [g_n] > [g_{n+1}],$$

for every $n < \omega$. Note that, as was observed above, for every $p \in P$, the filter $F_p$ is not $\kappa^+$-saturated.

Fix some $\tau < \kappa^{++}, \tau \leq \lambda$. We should construct a normal $\tau$-almost precipitous filter over $\kappa$.

For each $p \in P$ choose a maximal antichain $\{A_{p\beta} \mid \beta < \kappa^+\}$ in $F_p^+$.

Let $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ enumerate all the functions from $\kappa$ to $\tau$. Fix an enumeration $\langle X_\alpha \mid \alpha < \kappa^+ \rangle$ of $F_p^+$.

Start now an inductive process of extending of $F_0$.

Let $n = 0$. Assume for simplicity that there is a function $g_0 : \kappa \to On \in V$ so that

$$0_P \Vdash \bar{g}_0 = g_0.$$ 

We construct inductively a sequence of ordinals $\langle \xi_{0\beta} \mid \beta < \kappa^+ \rangle$ and a sequence of conditions $\langle p_{0\beta} \mid \beta < \kappa^+ \rangle$. Let $\alpha < \kappa^+$.

Case I. There is a $\xi < \kappa^+$ so that $\xi \neq \xi_{0\beta}$, for every $\beta < \alpha$ and $X_\alpha \cap A_{0\beta} \in F_0^+$.

Then let $\xi_{0\alpha}$ be the least such $\xi$. We would like to attach an ordinal to $f_{\xi_{0\alpha}}$. Let us pick $p \in P$, such that $p \Vdash \kappa \in j(X_\alpha \cap A_{0\beta})$ and for some $\gamma$ such that $p \Vdash j(f_{\xi_{0\alpha}})(\kappa) = \gamma$.

Now, set $p_{0\alpha} = p$ and extend $F_0$ to $F_{p_{0\alpha}}$.

Case II. Not Case I.

Then we will not define $F_{p_{0\alpha}}$. Set $\xi_{0\alpha} = 0$ and $p_{0\alpha} = 0_P$.

Note that if Case I fails then we have $X_\alpha \subseteq \nabla_{\beta < \kappa} A_{1\xi_{0\beta}} \mod F_{0_p}$ for a surjective $\tau : \kappa \to \alpha$.

Set $F_0 = \bigcap\{F_{p_{0\alpha}} \mid \alpha < \kappa^+ \text{ and } F_{p_{0\alpha}} \text{ is defined} \}$, and denote the corresponding dual ideals by $I_{p_{0\alpha}}$ and $I_0$.

Clearly, $I_0 = \bigcap\{I_{p_{0\alpha}} \mid \alpha < \kappa^+\}$. Also, $F_0 \supseteq F_{0p}$ and $I_0 \supseteq I_{0p}$, since each $F_{p_{0\alpha}} \supseteq F_{0p}$ and $I_{p_{0\alpha}} \supseteq I_{0p}$. Note that $F_0$ is a $\kappa$ complete, normal and proper filter since it is an intersection of such filters and also $I_0$ is.
We now describe the successor step of the construction, i.e., \( n = m + 1 \).

Let \( \sigma : m \to \kappa^+ \). Find some \( p \in P, p \geq p_\sigma \) and a function \( g_m : \kappa \to On \in V \) such that \( p \vDash g_m = g_m, p \vDash g_m < g_{m-1} \). Denote \( S_\sigma = \{ \nu \mid g_m(\nu) < g_{m-1}(\nu) \} \). We extend \( F_{p_\sigma} \) to \( F_p + S_\sigma \). By 4.1, there is \( q_\sigma \in P, q_\sigma \geq p \) and \( F_{q_\sigma} \supseteq F_p + S_\sigma \).

We construct now by induction a sequence of ordinals \( \langle \xi_{\sigma \beta} \mid \beta < \kappa^+ \rangle \) and a sequence of conditions \( \langle p_{\sigma \beta} \mid \beta < \kappa^+ \rangle \). Let \( \alpha < \kappa^+ \):

**Case I.** There is \( \xi < \kappa^+ \) so that \( \xi \neq \xi_{\sigma \beta} \) for every \( \beta < \alpha \) and \( X_\alpha \cap A_{q_\sigma \xi} \in F_{q_\sigma}^+ \).

Then let \( \xi_{\sigma \alpha} \) be the least such \( \xi \). We would like to attach an ordinal to \( f_{\xi_{\sigma \alpha}} \). Let us pick \( p \in P \) so that \( p \geq q_\sigma, p \vDash \kappa \in j(X_\alpha \cap A_{q_\sigma \xi_{\sigma \alpha}}) \) and there is an ordinal \( \gamma \) such that \( p \vDash (f_{\xi_{\sigma \alpha}})(\kappa) = \gamma \). Now, set \( p_{\sigma \alpha} = p \) and extend \( F_{q_\sigma} \) to \( F_{p_{\sigma \alpha}} \).

**Case II.** Case I fails.

Then we will not define \( F_{p_{\sigma \alpha}} \). Set \( \xi_{\sigma \alpha} = 0 \) and \( p_{\sigma \alpha} = 0_p \).

This completes the construction.

Set \( F_n = \bigcap \{ F_{p_{\sigma \alpha}} \mid \sigma : m \to \kappa^+, \alpha < \kappa^+ \text{ and } F_{p_{\sigma \alpha}} \text{ is defined} \} \), and denote the corresponding dual ideals by \( I_{p_{\sigma \alpha}} \) and \( I_n \).

will use the following:

**Definition 5.2** Let \( F_\omega \) be the closure under \( \omega \) intersections of \( \bigcup_{n < \omega} F_n \).

Let \( I_\omega \) = the closure under \( \omega \) unions of \( \bigcup_{n < \omega} I_n \).

**Lemma 5.3** \( F_0 \subseteq F_0 \subseteq \ldots \subseteq F_n \subseteq \ldots \subseteq F_\omega \) and \( \tilde{F}_n \subseteq I_0 \subseteq \ldots \subseteq I_n \subseteq \ldots \subseteq I_\omega \), and \( I_\omega \) is the dual ideal to \( F_\omega \).

Our purpose now will be to show that we cannot continue the construction further beyond \( \omega \) and then we would be able to show that \( F_\omega \) is a \( \tau \)-almost precipitous filter.

**Lemma 5.4** \( F_\omega^+ \subseteq \bigcup \{ F_{p_\sigma} \mid \sigma \in {}^{<\omega}\kappa^+ \} \).

**Proof.** Let \( X \in (F_\omega)^+ \) and assume that \( X \notin F_{p_\sigma} \) for each \( \sigma \in [\kappa^+]^{<\omega} \) so that \( F_{p_\sigma} \) is defined. Let us show that then there are at most \( \kappa \) many \( \sigma \)'s so that \( X \in F_{p_\sigma} \). Thus, for \( n=0, \{ \alpha < \kappa^+ \mid X \cap A_{0\alpha} \in F_{0\alpha}^+ \} \) is of cardinality less or equal \( \kappa \). Suppose otherwise. Let \( \nu < \kappa^+ \) be such that \( X = X_\nu \). Then \( F_{p_\nu} \) is defined according to Case I and \( X \in F_{p_\nu} \). Contradiction.

For every \( \nu < \kappa^+ \) with \( X \in F_{p_\nu} \), the set \( \{ \alpha < \kappa^+ \mid X \cap A_{q(0,\nu)\alpha} \in F_{q(0,\nu)}^+ \} \) is of cardinality less or equal \( \kappa \). Otherwise, we must have that for \( \xi < \kappa^+ \) with \( X = X_\xi \) the filter \( F_{p_{(0,\nu)\xi}} \) is defined according to Case I and \( X \in F_{p_{(0,\nu)\xi}} \). We continue in a similar fashion and obtain
that the set \( T = \{ \sigma \in [\kappa^+]^{<\omega} \mid F_{p_{\sigma}} \text{ is defined} , X \in F_{p_{\sigma}}^+ \} \) is of cardinality at most \( \kappa \). Also note, that for every \( \sigma \in T \) the set
\[
B_{\sigma} = \{ \beta < \kappa^+ \mid A_{q_\sigma \beta} \cap X \in F_{q_\sigma}^+ \}
\]
is of cardinality at most \( \kappa \). Otherwise, we can always find \( \xi, \alpha < \kappa^+ \) so that \( X = X_\alpha \) , \( X_\alpha \cap A_{q_\xi \alpha} \in F_{q_\alpha}^+ \) and \( \xi \neq \xi_\alpha \beta \), for every \( \beta < \alpha \). Then, according to Case 1, \( X_\alpha \in F_{q_\xi \alpha} \).

For every \( \sigma \in T \) , fix \( \psi_\sigma : \kappa \leftarrow B_{\sigma} \). Note that

\[
X \setminus \nabla_\beta < \kappa^+ A_{q_\sigma \psi_\sigma (\beta)}
\]
is in the ideal \( I_{q_\sigma} \).

Now, let \( n = 0 \). Turn the family \( \{ A_{0_{p_\psi (\gamma)}} \mid \gamma < \kappa \} \) into a family of disjoint sets as follows:

\[
A_{0_{p_\psi (0)}}' := A_{0_{p_\psi (0)}} - \{ 0 \}
\]

and for each \( \gamma < \kappa \) let

\[
A_{0_{p_\psi (\gamma)}}' := A_{0_{p_\psi (\gamma)}} - (\bigcup_{\beta < \gamma} A_{0_{p_\psi (\beta)}} \cup (\gamma + 1))
\]

Note that

\[
\nabla_{\beta < \kappa^+} A_{0_{p_\psi (\beta)}} = \{ \nu < \kappa \mid \exists \beta < \nu \text{ so that } \nu \in A_{0_{p_\psi (\beta)}} \}
\]

and, because \( \nu \in A_{1_{\psi_0 (\beta)}} \rightarrow \nu > \beta \), we get that the right hand side is equal to

\[
\bigcup \{ A_{0_{p_\psi (\gamma)}}' \mid \gamma < \kappa \}.
\]

Also note that

\[
\nabla_{\beta < \kappa^+} A_{0_{p_\psi (\beta)}}' = \nabla_{\beta < \kappa^+} A_{0_{p_\psi (\beta)}}
\]

So \( \{ X \cap A_{0_{p_\psi (\gamma)}}' \mid \gamma < \kappa \} \) is still a maximal antichain in \( F_{0_{p_{\psi (0)}}}^+ \) below \( X \) and \( X \subseteq \nabla_{\beta < \kappa^+} A_{0_{p_\psi (\beta)}} \) mod \( F_{0_{p_\psi (0)}} \).

Set \( R_0 := X \setminus \bigcup_{\beta < \kappa} A_{0_{p_\psi (\beta)}} \). Then \( R_0 \in I_{0_{p_\psi (0)}} \).

Now, for each \( \beta < \kappa \) with \( F_{p_{\sigma \beta}} = F_{p_{\psi (\beta)}} \) defined, let us turn the family \( \{ A_{q_{\sigma \beta} \psi_{\sigma \beta} (\gamma)} \mid \gamma < \kappa \} \) into a disjoint one \( \{ A_{q_{\sigma \beta} \psi_{\sigma \beta} (\gamma)}' \mid \gamma < \kappa \} \) as described above. Then

\[
R_{\sigma \beta} := X \cap A_{0_{p_\psi (\beta)}}' \setminus \bigcup_{\gamma < \kappa} (A_{q_{\sigma \beta} \psi_{\sigma \beta} (\gamma)}' \cap S_{\sigma \beta}) \in I_{\sigma \beta},
\]

where \( S_{\sigma \beta} \) was defined during the construction above. Set \( R_1 = \bigcup \{ R_{\sigma \beta} \mid \sigma \beta \in T \} \).

**Claim 1** \( R_1 \in I_0 \).
Proof. Suppose otherwise. Then $R_1 \in (F_0)^+$. Note that $R_1 \subseteq \{X \cap A'_{0,\psi(\beta)} : \langle \psi(\beta) \rangle \in T\}$ and that the right hand side is a disjoint union. Maximality of $\{X \cap A'_{0,\psi(\beta)} : \beta < \kappa\}$ implies that $R_1 \cap A'_{0,\psi(\alpha)} \in F_{p\sigma}^+$, for some $\alpha < \kappa$. But $R_1 \cap A'_{0,\psi(\alpha)} = R_{\sigma_n}$ and $R_{\sigma_n} \in I_{\sigma_n}$, contradiction.

$\Box$ of the claim.

Continue similar for each $n < \omega$. We will have $R_n \in I_{n-1}$. Set

$$R_\omega := \bigcup_{n<\omega} R_n.$$ 

Then $R_\omega \in I_\omega$ and $X - R_\omega \in (F_\omega)^+$. Now, let $\alpha \in X - R_\omega$. We can find a non decreasing sequence $\langle p_n \mid n < \omega \rangle$ and $\langle \beta_n \mid n < \omega \rangle$ so that

$$\alpha \in \bigcap_{n<\omega} (A'_{p_n,\beta_n} \cap S_{p_n}).$$

Recall that $g_{n+1}(\nu) < g_n(\nu)$, for each $n < \omega$ and $\nu \in \bigcap_{k \leq n+1} (A'_{p_k,\beta_k} \cap S_{p_k})$. So the intersection $\bigcap_{n<\omega} (A'_{p_n,\beta_n} \cap S_{p_n})$ must be empty, but on the other hand, $\alpha$ is a member of this intersection. Contradiction.

$\Box$

**Lemma 5.5** Generic ultapower by $F_\omega$ is well founded up to the image of $\tau$

Proof. Suppose that $\langle h_n \mid n < \omega \rangle$ is a sequence of $(F_\omega)^+$-names of old (in $V$) functions from $\kappa$ to $\tau$. Let $G \subseteq (F_\omega)^+$ be a generic ultrafilter. Choose $X_0 \in G$ and a function $h_0 : \kappa \rightarrow \tau, h_0 \in V$ so that $X_0 \Vdash_{F_\omega^+} h_0 = h_0$. Let $\alpha_0 < \kappa^+$ be so that $f_{\alpha_0} = h_0$. By Lemma 5.4, we can find $\sigma_0 \in [\kappa^+]^{\omega}$ such that $F_{p_{\sigma_0}}$ is defined and $X_0 \in F_{p_{\sigma_0}}$. Note that at the next stage of the construction there will be $\beta$ with $A_{p_{\sigma_0} \alpha_0} \in F_{p_{\sigma_0}}$, and so the value of $j(f_{\alpha_0})(\kappa)$ will be decided. Denote this value by $\gamma_0$. Assume for simplicity that $A_{p_{\sigma_0} \alpha_0} \cap X_0$ is in $G$ (otherwise we could replace $X_0$ by another positive set using density). Continue below $A_{p_{\sigma_0} \alpha_0} \cap X_0$ and pick $X_1 \in G$ and a function $h_1 : \kappa \rightarrow \tau, h_1 \in V$ so that $X_1 \Vdash_{F_\omega^+} h_1 = h_1$. Let $\alpha_1 < \kappa^+$ be so that $f_{\alpha_1} = h_1$. By Lemma 5.4, we can find $\sigma_1 \in [\kappa^+]^{\omega}$ such that $F_{p_{\sigma_1}}$ is defined, $\sigma_1 \supseteq \sigma_0$ and $X_1 \in F_{p_{\sigma_1}}$. Again, note that at the next stage of the construction there will be $\beta$ with $A_{p_{\sigma_1} \alpha_1} \in F_{p_{\sigma_1}}$, and so the value of $j(f_{\alpha_1})(\kappa)$ will be decided. Denote this value by $\gamma_1$. Continue the process for every $n < \omega$. There must be $k < m < \omega$ such that $\gamma_k \leq \gamma_m$ and $X_m \cap A_{\sigma_m \alpha_m} \in G$. So the sequence $\langle [h_n]_G \mid n < \omega \rangle$ is not strictly decreasing.

$\Box$
5.3 Conclusions

Let us deduce now some conclusions concerning an existence of almost precipitous filters.
The following answers a question raised in [5].

**Corollary 5.6** Assume $0^\sharp$. Then every cardinal can be an almost precipitous witnessed by normal filters in a generic extension of $L$.

*Proof.* By Donder, Levinski [1], every cardinal can be semi-precipitous in a generic extension of $L$. Now apply 5.1. Clearly, there is no saturated ideals in $L[0^\sharp]$.

□

**Corollary 5.7** Assume there are class many Ramsey cardinals. Then every cardinal is an almost precipitous witnessed by normal filters.

*Proof.* It follows from 3.7 and 5.1.

□

**Corollary 5.8** Assume $V = L[U]$ with $U$ a normal ultrafilter over $\kappa$. Then

1. every regular cardinal less than $\kappa$ is an almost precipitous witnessed by normal filters and non precipitous,

2. for each $\tau \leq \kappa^+$, $\kappa$ carries a normal $\tau$-almost precipitous non precipitous filter.

*Proof.* Let $\eta$ be a regular cardinal less than $\kappa$. By 3.14, $\eta$ is $< \kappa$-semi-precipitous. Note that no cardinal less than $\kappa$ can be $\infty$-semi precipitous. Hence, $\eta$ is an almost precipitous witnessed by a normal filter, by 5.1. This proves (1).

Now,

$$A = \{ \eta < \kappa \mid \eta \text{ is an almost precipitous witnessed by a normal filter and non precipitous } \}$$

is in $U$. Hence, in $M \simeq {}^\ast V/U$, for each $\tau < (\kappa^{++})^M$ there is a normal $\tau$-almost precipitous non precipitous filter $F_\tau$ over $\kappa$. Then $F_\tau$ remains such also in $V$, since ${}^\ast M \subseteq M$.

□

We do not know if (2) remains valid once we replace $\tau \leq \kappa^+$ by $\tau < \kappa^{++}$.

Let us turn to the case of $\infty$-semi precipitous cardinals which was not covered by Theorem 5.1

Combining constructions of [4] with the present ones (mainly, replacing restrictions to sets by restrictions to filters) we obtain the following.
Theorem 5.9 Assume that $\aleph_1$ is $\infty$-semi precipitous and $2^{\aleph_1} = \aleph_2$. Suppose that for some witnessing this forcing $P$

$$0_P\Vdash_{P} \dot{\mathcal{A}}(\aleph_1) > (\aleph_1^+)^V.$$

Then $\aleph_1$ is almost precipitous witnessed by normal filters.

Theorem 5.10 Assume that $\kappa$ is $\infty$-semi precipitous, $2^{\kappa} = \kappa^+$ and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where $\kappa^-$ denotes the immediate predecessor of $\kappa$. Suppose that for some witnessing this forcing $P$

1. $0_P\Vdash_{P} \dot{\mathcal{A}}(\kappa) > (\kappa^+)^V$

2. $0_P\Vdash_{P} \kappa \in \{\nu < \dot{\mathcal{A}}(\kappa) \mid \cof(\nu) = \kappa^-\}$.

Then $\kappa$ is almost precipitous witnessed by normal filters.

Theorem 5.11 Suppose that there is no inner model satisfying ($\exists \alpha \ o(\alpha) = \alpha^{++}$). Assume that $\aleph_1$ is $\infty$-semi precipitous and $2^{\aleph_1} = \aleph_2$. If $\aleph_3$ is not a limit of measurable cardinals of the core model, then there exists a normal precipitous ideal on $\aleph_1$.

Theorem 5.12 Suppose that there is no inner model satisfying ($\exists \alpha \ o(\alpha) = \alpha^{++}$). Assume that $\kappa$ is $\infty$-semi precipitous, $2^{\kappa} = \kappa^+$ and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where $\kappa^-$ denotes the immediate predecessor of $\kappa$. Suppose that for some witnessing this forcing $P$

$$0_P\Vdash_{P} \kappa \in \{\nu < \dot{\mathcal{A}}(\kappa) \mid \cof(\nu) = \kappa^-\}.$$

If $\kappa^{++}$ is not a limit of measurable cardinals of the core model, then there exists a normal precipitous ideal on $\kappa$.

Theorem 5.13 Assume that $\aleph_1$ is $\infty$-semi precipitous. Let $P$ be a witnessing this forcing such that

$$0_P\Vdash_{P} \dot{\mathcal{A}}(\aleph_1) > (\aleph_1^+)^V.$$

Then, after forcing with $\text{Col}(\aleph_2, |P|)$, there will be a normal precipitous filter on $\aleph_1$.

Theorem 5.14 Assume that $\kappa$ is $\infty$-semi precipitous and $(\kappa^-)^{<\kappa^-} = \kappa^-$, where $\kappa^-$ denotes the immediate predecessor of $\kappa$. Let $P$ be a witnessing this forcing such that

1. $0_P\Vdash_{P} \dot{\mathcal{A}}(\kappa) > (\kappa^+)^V$

2. $0_P\Vdash_{P} \kappa \in \{\nu < \dot{\mathcal{A}}(\kappa) \mid \cof(\nu) = \kappa^-\}.$
Then, after forcing with $\text{Col}(\kappa^+, |P|)$, there will be a normal precipitous filter on $\kappa$.

**Sketch of the proof of 5.13.** Let $P$ be a forcing notion witnessing $\infty$-semi precipitousness such that

$$0_P \Vdash \dot{\mathcal{N}}_1 > (\kappa^+_1)^V.$$  

Fix a function $H$ such that for some $p \in P$

$$p \Vdash \dot{H}(\kappa) : \omega \rightarrow^{\text{onto}} (\kappa^+)^V,$$

where here and further $\kappa$ will stand for $\aleph_1$. Assume for simplicity that $p = 0_P$. Let $\langle h_\alpha | \alpha < \kappa \rangle$ be a sequence of the canonical functions from $\kappa$ to $\kappa$. For every $\alpha < \kappa^+$ and $n < \omega$ set

$$A_{n\alpha} = \{ \nu | H(\nu)(n) = h_\alpha(\nu) \}.$$  

Then, the following hold:

**Lemma 5.15** For every $\alpha < \kappa^+$ and $p \in P$ there is $n < \omega$ so that $A_{n\alpha} \in F_0^+.$

**Lemma 5.16** Let $n < \omega$ and $p \in P$. Then the set

$$\{ A_{n\alpha} | \alpha < \kappa^+ \text{ and } A_{n\alpha} \in F_0^+ \}$$

is a maximal antichain in $F_0^+.$

Denote by

$$\text{Col}(\aleph_2, P) = \{ t | t \text{ is a partial function of cardinality at most } \aleph_1 \text{ from } \aleph_2 \text{ to } P \}.$$  

Let $G \subseteq \text{Col}(\aleph_2, P)$ be a generic and $C = \bigcup G.$

We extend $F_0$ now as follows.

Start with $n = 0$. If $| \{ \alpha | A_{0\alpha} \in F_0^+ \} | < \kappa^+$, then set $F_0 = F_{00}.$

Suppose otherwise. Let $\alpha < \kappa^+$. If $A_{0\alpha}$ in the ideal dual to $F_0$, then set $F_{0\alpha} = F_{00}$. If $A_{0\alpha} \in F_0^+$, then we consider $F_{C(\alpha)}$. If $A_{0\alpha} \notin F_0^+$, then pick some $p(0\alpha) \in P$ forcing $\kappa \in \dot{\mathcal{N}}(A_{0\alpha})$ and set $F_{0\alpha} = F_{00}(p(0\alpha))$. If $A_{0\alpha} \in F_0^+$, then pick some $p(0\alpha) \in P$, $p(0\alpha) \geq C(\alpha)$ forcing $\kappa \in \dot{\mathcal{N}}(A_{0\alpha})$ and set $F_{0\alpha} = F_{00}(p(0\alpha))$.

Set $F_0 = \bigcap \{ F_{0\alpha} | \alpha < \kappa^+ \}.$

Let now $n = 1$. Fix some $\gamma < \kappa^+$ with $F_{0\gamma}$ defined. If $| \{ \alpha | A_{1\alpha} \in F_{0\gamma}^+ \} | < \kappa^+$, then we do nothing. Suppose that it is not the case. Let $\alpha < \kappa^+$. We define $F_{(\gamma, 1\alpha)}$ as follows:

- if $A_{1\alpha} \notin F_{0\gamma}^+$, then set $F_{(\gamma, 1\alpha)} = F_{0\gamma}.$
• if \( A_1 \alpha \in F_{0,1} \), then consider \( F_{C(\alpha)} \). If there is no \( p \) stronger than both \( C(\alpha), p(0,1) \) and forcing \( \kappa \in \dot{j}(A_1 \alpha) \), then pick some \( p(0,1) \geq p(0) \alpha \) which forces \( \kappa \in \dot{j}(A_1 \alpha) \) and set \( F_{p(0,1)} = F_{p(0)} \alpha \). Otherwise, pick some \( p(0,1) \geq C(\alpha), p(0) \alpha \) which forces \( \kappa \in \dot{j}(A_1 \alpha) \) and set \( F_{p(0,1)} = F_{p(0)} \alpha \).

Set \( F_1 = \bigcap \{ F_{0,1} \alpha \mid \alpha, \gamma < \kappa^+ \} \).

Continue by induction and define similar filters \( F_s, F_n \) and conditions \( p(s) \) for each \( n < \omega, s \in [\omega \times \kappa^+] \langle \omega \rangle \). Finally set

\[ F_\omega = \text{the closure under } \omega \text{ intersections of } \bigcup_{n<\omega} F_n. \]

The arguments like those of 5.1 transfer directly to the present context. We refer to [4] which contains more details.

Let us prove the following crucial lemma.

**Lemma 5.17** \( F_\omega \) is a precipitous filter.

**Proof.** Suppose that \( \langle g, n < \omega \rangle \) is a sequence of \( F_\omega^+ \)-names of old (in \( V \)) functions from \( \kappa \to \On \).

Let \( G \subseteq F_\omega^+ \) be a generic ultrafilter. Pick a set \( X_0 \in G \) and a function

\[ g_0 : \kappa \to \On \]

in \( V \) such that

\[ X_0 \|_{F_\omega^+} \bar{g}_0 = \bar{g}_0. \]

Pick some \( t_0 \in \text{Col}(\aleph_2, P), t \subseteq C \) such that

\[ \langle t_0, X_0 \rangle \|_{\text{Col}(\aleph_2, P) * F_\omega^+} \bar{g}_0 = \bar{g}_0 \]

and for some \( s_0 = \langle \xi_0, ..., \xi_n \rangle \in [\omega \times \kappa^+] \langle \omega \rangle \)

\[ t_0 \| X_0 \in F_{s_0}, \]

moreover, for each \( i \leq n, \xi_i \in \text{dom}(t_0) \) and \( t_0(\xi_n) = p(s_0) \).

**Claim 2** For each \( \langle t, Y \rangle \in \text{Col}(\aleph_2, P) * F_\omega^+ \) with \( \langle t, Y \rangle \geq \langle t_0, X_0 \rangle \) there are \( \langle g_0, Z_0 \rangle \geq \langle t, Y \rangle, \rho_0 \in \On \) and \( s'_0 \) extending \( s_0 \) such that

\[ \langle t, Y \rangle \|_{F_\omega^+} \bar{g}_0 = \bar{g}_0. \]
1. \( q(s'_0(|s'_0|)) \leq p(s'_0) \),
2. \( q \|_{Col(\aleph_2, P)} Z_0 \in F_{s'_0} \),
3. \( p(s'_0) \|_{P_\gk(g_0)(\kappa)} = \rho_0 \).

**Proof.** Suppose for simplicity that \( \langle t, Y \rangle = \langle t_0, X_0 \rangle \). We know that \( t_0 \) decides \( F_{s_0} \), \( t_0(s_0(|s_0|)) = p(s_0) \) and \( X_0 \in F_{s_0} \). Find \( s \) extending \( s_0 \) of the smallest possible length such that the set \( B = \{ \alpha \mid A_{|s|s} \in F_{s_0}^+ \} \) has cardinality \( \kappa^+ \). Remember that we do not split \( F_{s_0} \) before getting to such \( s \). Pick some \( \alpha \in B \setminus \text{dom}(t_0) \). \( A_{|s|\alpha} \in F_{s_0}^+ \), hence there is some \( p' \in P, p' \geq p(s_0) \) which forces \( \kappa \in \dot{\jmath}(A_{|s|\alpha}) \). Find some \( p \in P, p \geq p' \) and \( \rho_0 \) such that

\[
p \|_{P_\gk(g_0)(\kappa)} = \rho_0.
\]

Extend now \( t_0 \) to \( t \) by adding to it \( \langle \alpha, p \rangle \). Let \( s'_0 = s \upharpoonright \alpha \) and \( Z_0 = X_0 \cap A_{|s|\alpha} \).

\( \Box \) of the claim.

By the genericity we can find \( \langle g_0, Z_0 \rangle \) as above in \( C * G \). Back in \( V[C, G] \), find \( X_1 \subseteq Z_0 \) in \( G \) and a function

\[
g_1 : \kappa \rightarrow \text{On}
\]

in \( V \) such that

\[
X_1 \|_{F_{\gk}\dot{g}_1} = \dot{g}_1.
\]

Proceed as above only replacing \( X_0 \) by \( X_1 \). This will define \( q_1, Z_1 \) and \( \rho_1 \) for \( g_1 \) as in the claim.

Continue the process for each \( n < \omega \). The ordinals \( \rho_n \) will witness the well foundness of the sequence \( \langle [g_n]_C | n < \omega \rangle \)

\( \Box \)

\( \Box \)

Note that if there is a precipitous ideal (not a normal one) over \( \kappa \), then we can use its positive sets as \( P \) of Theorems 5.13, 5.14. The cardinality of this forcing is \( 2^\kappa \). So adding a Cohen subset to \( \kappa \) will suffice.

Embeddings witnessing \( \infty \)-semi precipitousness may have a various sources. Thus for example they may come from strong, supercompact, huge cardinals etc or their generic relatives. An additional source of examples is Woodin Stationary Tower forcings, see Larson [6].

**Corollary 5.18** Suppose that \( \delta \) is a Woodin cardinal and there is \( f : \omega_1 \rightarrow \omega_1 \) with \( \|f\| \geq \omega_2 \). Then in \( V^{Col(\aleph_2, \delta)} \) there is a normal precipitous ideal over \( \aleph_1 \).
Remark. Woodin following Foreman, Magidor and Shelah [3] showed that $Col(\aleph_1, \delta)$ turns $NS_{\aleph_1}$ into a presaturated ideal. On the other hand Schimmerling and Velickovic [10] showed that there is no precipitous ideals on $\aleph_1$ in $L[E]$ up to at least a Woodin limit of Woodins. Also by [10], there is $f : \omega_1 \to \omega_1$ with $\|f\| \geq \omega_2$ in $L[E]$ up to at least a Woodin limit of Woodins.

Proof. Let $\delta$ be a Woodin cardinal. Force with $P_{<\delta}$, (refer to the Larson book [6] for the definitions) above a stationary subset of $\omega_1$. This will produce a generic embedding $i : V \to N$ with a critical point $\omega_1$, $N$ is transitive and $i(\omega_1) > (\omega_2)^V$. The cardinality of $P_{<\delta}$ is $\delta$. So 5.13 applies.

□

Similar, using 5.14, one can obtain the following:

Corollary 5.19 Suppose that $\delta$ is a Woodin cardinal, $\kappa < \delta$ is the immediate successor of $\kappa^-$, $(\kappa^-)^{<\kappa^-} = \kappa^-$ and there is $f : \kappa \to \kappa$ with $\|f\| \geq \kappa^+$. Then in $V^{Col(\kappa^+ , \delta)}$ there is a normal precipitous ideal over $\kappa$.

6 Extensions of generic elementary embeddings

In this section we would like to introduce some forcing notions which allow us to extend generic elementary embeddings hence preserve semi-precipitousness of a cardinal $\kappa$.

The following lemma was proved in [1] and shows that ”small” $\kappa$-c.c forcing notions allow us to extend generic embeddings.

Lemma 6.1 Let $\kappa$ be a $\lambda$-semi precipitous cardinal and let $P$ be a $\kappa$-c.c notion of forcing. If $|P| < \lambda$ then $V^P \models \kappa \text{ is } \lambda \text{ semi precipitous }$.

Let us show that $\kappa^+$-distributive forcings preserve semi-precipitousness of a cardinal $\kappa$, as well.

Lemma 6.2 Let $\kappa$ be a semi-precipitous cardinal and let $P$ be a $\kappa^+$-distributive forcing. Then, $V^P \models \kappa \text{ is semi-precipitous }$.

Proof. Fix a cardinal $\lambda$ so that $P \in V_\lambda$. Let us show that $\kappa$ remains a $\lambda$-semi-precipitous in $V^P$. It is enough for every $p \in P$ to find a generic subset $G$ of $P$ with $p \in G$, such that $\kappa$ is
a λ-semi-precipitous in $V[G]$. Fix some $p_0 \in P$.

In $V$, $\kappa$ is λ-semi-precipitous so the forcing $Q = Col(\omega, \mu)$, with $\mu \geq \lambda$ big enough, produces an elementary embedding $j : V_\lambda \to M \models (V_\lambda)^\kappa/U$, with $M$ transitive and $U$ a normal $V$-ultrafilter over $\kappa$ (in $V^Q$).

Note that $|P| = \aleph_0$ in $V^Q$. So there is a set $G \in V^Q$ which is a $V$-generic subset of $P$ with $p_0 \in G$. Set

$$G^* = \{ p \in \bar{P} \mid \text{there is a } q \in P, p \geq j(q) \}.$$ 

Clearly, $G^*$ is directed and we would like to show that it meets every open dense subset of $j(P)$ which belongs to $M$. Let $D$ be such a subset. There is a function $f \in V_\lambda, f : \kappa \to V_\lambda$ so that $[f]_U = D$. We can assume that for each $\alpha < \kappa$ $f(\alpha)$ is an open dense subset of $P$. $P$ is $\kappa^+$-distributive, hence $\bigcap\{f(\alpha) \mid \alpha < \kappa\} = D'$ is a dense subset of $P$. So $G \cap D' \neq \emptyset$. Let $q \in G \cap D'$. Then $j(q) \in G^*$ which implies that $G^* \cap D \neq \emptyset$.

Now it is easy to extend $j$ to $j^* : V_\lambda[G] \to M[G^*]$.

So, in $V^Q$, we found a $V$-generic subset $G$ of $P$ with $p_0 \in G$ and an elementary embedding of $V_\lambda[G]$ into a transitive model. Note that this actually implies λ-semi-precipitousness of $\kappa$ in $V[G]$. Thus, force with $Q/G$ over $V[G]$. Clearly, $V[G]^Q/G = V^Q$. Hence the forcing $Q/G$ produces the desired elementary embedding.

□

We can use the previous lemma in order to show the following:

**Theorem 6.3** Suppose that $\kappa$ is a λ-semi-precipitous, for some $\lambda > (2^\kappa)^+$. Then $\kappa$ will be an almost precipitous after adding of a Cohen subset to $\kappa^+$.

**Proof.** First note that if $\kappa$ caries a precipitous filter, then this filter will remain precipitous in the extension. By Lemma 6.2, $\kappa$ caries a $\lambda$-semi-precipitous filter in $V^{Cohen(\kappa^+)}$. If there is a precipitous filter over $\kappa$, then we are done. Suppose that it is not the case. Note that in the generic extension we have $2^\kappa = \kappa^+$, so the results of Section 3 apply and give the desired conclusion.

□

The following lemma shows that small forcing notions cannot destroy semi precipitousness:

**Lemma 6.4** Let $\kappa$ be a $\lambda$ semi-precipitous cardinal and let $P$ be a forcing notion so that $P \in V_\kappa$. Then, $V^P \models " \kappa$ is $\lambda$ semi-precipitous ".

Proof. As before, \( Q = \text{Col}(\omega, \mu) \), with \( \mu \geq \lambda \) big enough, produces an elementary embedding
\( j: V_\lambda \rightarrow M \cong (V_\lambda)^\kappa / U \), with \( M \) transitive and \( U \) a normal \( V \)-ultrafilter over \( \kappa \) (in \( V^Q \)).
Note that \( P \in V_\kappa \) implies that \( j(P) = P \), hence every \( P \) generic \( G \) over \( V \) is also \( j(P) = P \)
generic over \( M \) and we allow to extend the generic embedding. □

Remark. Note that if we try to violate GCH on \( \kappa \) which is semi-precipitous we might encounter the same problems as doing the same on a measurable cardinal. If we assume some stronger assumptions on the closeness of the generic embedding yields by \( \kappa \) being a semi precipitous cardinal we can just apply the same known arguments of Silver, Woodin or Gitik to violate GCH on \( \kappa \). For example, if we assume \( \kappa \) acts similar to a super-compact cardinal \( (j(\kappa) > \kappa^{++} \text{ and } \kappa^{++} \subseteq M) \) then the same proof of Silver for violating GCH on a measurable cardinal holds.

The following questions raised and we don’t know the answer yet:
a) How strong is the previous assumption?
b) Can we do the same under the assumption ”\( \kappa \) is \( \lambda \)-semi precipitous” where \( \lambda \) is big enough?
c) Is it possible under the assumption that \( \kappa \) is ”only” \( \kappa^+ \) or \( \kappa^{++} \) semi precipitous (which according to [1] if we begin with \( V=L \) then the first semi precipitous cardinal is such that).
d) Can we decide under ”\( V=L \)” whether the first semi-precipitous cardinal \( \kappa \) is \( \kappa^+ \) semi precipitous or \( \kappa^{++} \) semi precipitous?
It seems that for b)+c) we shall try to build another generic \( j \) instead of extending the old one or to prove it in terms of winning strategy in Donder-Levinski ’s game or the game defined in 3.6.

7 A remark on pseudo-precipitous ideals

Pseudo-precipitous ideals were introduced by T. Jech in [9]. The original definition was based on a game. We will use an equivalent definition, also due to T. Jech [9].
Let \( I \) be a normal ideal over \( \kappa \). Consider the forcing notion \( Q_I \) which consists of normal ideals \( J \) extending \( I \). We say that \( J_1 \) is stronger than \( J_2 \), if \( J_1 \supseteq J_2 \).
Let \( G \) be a generic subset of \( Q_I \). Then \( \bigcup G \) is a prime ideal with respect to \( V \). Let \( F_G \) denotes its dual \( V \)-ultrafilter.

**Definition 7.1** (Jech [9]) An ideal \( I \) is called a pseudo-precipitous iff \( I \) forces in \( Q_I \) that \( ^{\kappa}V \cap V / F_G \) is well founded.
T. Jech [9] asked how strong is the consistency of "there is a pseudo-precipitous ideal on $\aleph_1$"?

Not that if $U$ is a normal ultrafilter over $\kappa$ then Player Two trivially has a winning strategy in $G_6(U)$ and so $U$ is pseudo-precipitous.

Let us address the consistency strength of existence of a pseudo-precipitous ideal over a successor cardinal.

**Theorem 7.2** If there is a pseudo-precipitous ideal over a successor cardinal then there is an inner model with a strong cardinal. In particular, an existence of precipitous ideal does not necessary imply an existence of a pseudo-precipitous one.

**Remark 7.3** By Jech [9], any normal saturated ideal is pseudo-saturated. S. Shelah showed that starting with a Woodin cardinal it is possible to construct a model with a saturated ideal on $\aleph_1$. So the strength of existence of a pseudo-precipitous ideal requires at least a strong but not more than a Woodin cardinal.

**Proof.** Suppose that $I$ is a pseudo-precipitous ideal over $\lambda = \kappa^+$. Assume

$I \models_{Q_1} j(\lambda) > (\lambda^+)^V$,

just otherwise we will have large cardinals. This is basically due to Mitchell, see Lemmas 2.31, 2.32 of [4].

Find $J \supseteq I$ and a function $H$ such that

$J \models_{Q_j} j(H) : \kappa \rightarrow^{ont} (\lambda^+)^V$.

Fix $\langle h_\nu \mid \nu < \lambda^+ \rangle$ canonical functions. Now there is $\xi < \kappa$ such that for $\lambda^+$ ordinals $\nu < \lambda^+$, we have

$A_\nu := \{ \alpha < \lambda \mid H(\alpha)(\xi) > h_\nu(\alpha) \} \in J^+$.\n
Extend $J$ to $J'$ by adding to it all the compliments of $A_\nu$'s and their subsets. Then $J'$ will be a normal ideal extending $J$. Now extend $J'$ to $J''$ deciding $j(H)(\lambda)(\xi)$. Let $\eta$ be the decided value. Then for each $\nu < \lambda^+$ we have $\eta > \nu$. But

$J \models_{Q_j} \text{ran}(j(H)(\lambda)) = (\lambda^+)^V$.

Contradiction.

□

The following natural questions remain open:

Question: Suppose that $I$ is a pseudo-precipitous. Is $I$ a precipitous?
References


