Analytic Fredholm Theory

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The purpose of this note is to prove a version of analytic Fredholm theory, and examine a special case.

Theorem 1.1 (Analytic Fredholm Theory). Let $\Omega$ be a connected open subset of $\mathbb{C}$ and suppose $T(\lambda)$ is an analytic family of Fredholm operators on a Hilbert space $H$. Then either

(i) $T(\lambda)$ is not invertible for any $\lambda \in \mathbb{C}$, or

(ii) There exists a discrete set $S \subseteq \Omega$ such that $T(\lambda)$ is invertible for all $\lambda \notin S$ and furthermore $T^{-1}(\lambda)$ extends to a meromorphic function on all of $\Omega$. Furthermore, every operator appearing as a coefficient of a term of negative order is finite rank.

By analytic, we mean that for every $\lambda_0 \in \Omega$, $T$ is given by a power series

$$T(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n T_n$$

converging in the operator norm, where $T_n : H \to H$ are bounded. Similarly, we say that $T$ is meromorphic if around every $\lambda_0 \in \Omega$ there is a Laurent series

$$T(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_0)^n T_n$$

converging in a punctured neighbourhood of $\lambda_0$.

We will also prove:

Proposition 1.2. Suppose $T : H \to H$ is a self-adjoint, non-negative bounded operator. Suppose that $T - \lambda$ is Fredholm for all $\lambda \in \Omega$, where $\Omega \subset \mathbb{C}$ is a connected open set. Then $(T - \lambda)^{-1}$ has a meromorphic extension to a family with only simple poles, and the residue is $-1$ times the projection onto the kernel of $T - \lambda_0$, where $\lambda_0$ is a pole.

Proof of Analytic Fredholm Theory. Set $\Omega'$ to be a maxima open set containing $\lambda_0$ to which $T^{-1}(\lambda)$ has a meromorphic extension. We divide the proof into several steps.

1. Show that if $T^{-1}(\lambda)$ exists as an operator at a point, it is analytic in a neighbourhood of that point.
2. Show that if $T^{-1}(\lambda)$ has a meromorphic extension near a point, it is analytic in a punctured neighbourhood of that point (this step actually applies to any meromorphic family).

3. Show that $\Omega'$ is open, connected, non-empty.

4. Show that on $\Omega'$, the points for which the meromorphic extension of $T^{-1}(\lambda)$ are analytic are in fact points where the inverse actually exists, and the points where it fails to be analytic form a discrete set.

5. Show that $\Omega' = \Omega$. This last step is the hardest. We will also show the operators appearing as coefficients of negative order are finite rank operators.

Step 4 shows that the points at which $T^{-1}(\lambda)$ fails to exist in $\Omega'$ are discrete, and Step 5 will show that $\Omega' = \Omega$ and complete the proof.

**Step 1.** Fix $\lambda_1 \in \Omega$ for which $T^{-1}(\lambda)$ exists, and write

$$T(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_1)^n T_n.$$  

The Cauchy-Hadamard theorem applies, and in particular there exists $A, B$ such that $\|T_n\| \leq AB^n$. Since $T(\lambda_1) = T_0$ is invertible, we may recursively define operators $S_n$ by $S_0 = T_0^{-1}$ and

$$S_n = -(S_{n-1} T + \cdots S_0 T) T_0^{-1}.$$  

Define $b_0 = \|S_0\|$ and

$$b_n = \|S_0\| (b_{n-1} AB^1 + \cdots b_0 AB^n).$$

It is clear that we have the bound $b_n \leq CD^n$ for some $C, D$, and that $\|S_n\| \leq b_n \leq CD^n$. In particular, the series

$$S(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_1) S_n$$

converges in a small neighbourhood and by construction $S(\lambda) T(\lambda) = 1$ wherever both are defined. Since $T^{-1}(\lambda)$ exists for $\lambda$ near $\lambda_1 (GL(H) \text{ is open})$, $S(\lambda) = T^{-1}(\lambda)$ near $\lambda_1$. In particular, $T^{-1}(\lambda)$ is analytic near $\lambda_1$.

**Step 2.** Suppose that $T^{-1}(\lambda)$ has a meromorphic extension to a point $\lambda_1$. Then

$$T^{-1}(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_1)^n T_n,$$

and for $\lambda_2$ near $\lambda_1$,

$$T^{-1}(\lambda) = \sum_{n=-N}^{\infty} (\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n T_n.$$
If $N \leq 0$ then we may expand $(\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n$ for $n \geq 0$ using the binomial theorem and regroup terms (since the sum converges absolutely by Cauchy-Hadamard) to see that $T^{-1}(\lambda)$ is analytic near $\lambda_2$. If $N > 0$, then we do the same if $n \geq 0$, and if $n < 0$ (and $\lambda$ close enough to $\lambda_2$, we may expand $(\lambda - \lambda_2 + (\lambda_2 - \lambda_1))^n$ as a geometric series and rearrange to see that $T^{-1}(\lambda)$ is analytic near $\lambda_2$. In particular, $T^{-1}(\lambda)$ is analytic (and in particular meromorphic) near $\lambda_2$. This yields two results: the first is that the set of points for which $T^{-1}(\lambda)$ is meromorphic is open. The second is that inside the set for which $T^{-1}(\lambda)$ has a meromorphic extension, the set points for which $T^{-1}(\lambda)$ is not analytic is discrete.

**Step 3.** Let $\{U_\alpha\}$ be the collection of all connected open sets containing $\lambda_0$ for which $T^{-1}(\lambda)$ has a meromorphic extension to $U_\alpha$. By Step 1, this collection is non-empty, since it contains a neighbourhood of $\lambda_0$. Set $\Omega'$ to be the union of all $U_\alpha$. Then $\Omega'$ is connected, since all points can be connected with a path to $\lambda_0$, and is maximal.

**Step 4.**
From Step 2 that $T^{-1}(\lambda)$ has an analytic extension to all but a discrete number of points in $S \subseteq \Omega'$. Since the identities

$$T^{-1}(\lambda)T(\lambda) = 1 = T(\lambda)T^{-1}(\lambda)$$

hold on at least a small open subset $V \subset \Omega'$, and $T^{-1}(\lambda)$ makes sense as an analytic function on a connected open set $\Omega' \setminus \{S\} \subseteq \Omega'$ it follows that the identities persist to all of $\Omega' \setminus \{S\}$. Indeed, for instance,

$$\langle u(T^{-1}(\lambda)T(\lambda) - 1)v \rangle$$

(for $u, v \in H$) is a $C$-valued analytic function which is 0 on an open subset, and thus 0 everywhere. Thus not only does the meromorphic extension of $T^{-1}(\lambda)$ fail to be analytic except on an isolated number of points, but $T^{-1}(\lambda)$ is actually the inverse if $T^{-1}(\lambda)$ is analytic and fails to exist at the same points that it fails to be analytic.

**Step 5.** Let $\lambda_1 \in \partial \Omega'$. We need only show that $\lambda_1 \in \Omega'$, since then $\Omega'$ is open, closed, and non-empty, and thus all of $\Omega$. Without loss of generality we may take $\lambda_1 = 0$. We have to show that $T^{-1}(\lambda)$ extends to a meromorphic function around $\lambda = 0$. $T(\lambda)$ is a continuous family of Fredholm operators which is invertible at some point $\lambda_0$. In particular, the index of $T(\lambda)$ is 0 everywhere. Write $T = T(0)$. Set $V = \ker T^\perp$ and $W = \ker T$, and set $V' = \im T$ and $W' = \im T^\perp$. Since $T$ is Fredholm of index 0, $V, W, V', W'$ are all closed, $V + W = V' + W' = H$ and $T : V \to V'$ is an isomorphism. We denote by $\Pi_X$ the projection onto the subspace $X \subseteq H$.

We divide the rest of Step 5 substeps (i) and (ii).

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1By this we mean the boundary with respect to the subspace topology on $\Omega$, i.e. $\partial \Omega' \cap \Omega$, where here $\partial$ is interpreted as the boundary as a subset of $C$
Substep i) In this substep we find a locally invertible analytic family of operators $R(\lambda)$ such that $R(\lambda)T(\lambda)$ looks like the matrix:

$$
\begin{pmatrix}
A(\lambda) & 0 \\
B(\lambda) & 1
\end{pmatrix},
$$

(1.1)

where the first row and column represent $W$ and the second row and column represent $V$. The point of this is that we can actually write down a nice inverse for this matrix, at least formally.

Write $T(\lambda) = T(\lambda)\Pi_V + T(\lambda)\Pi_W$.

Then both summands are analytic and we have

$$
T(\lambda) = \sum_{n=0}^{\infty} \lambda^n T_n \Pi_V + \sum_{n=0}^{\infty} \lambda^n T_n \Pi_W.
$$

By assumption $T(0)\Pi_V = T\Pi_V$ is invertible as an operator from $V$ onto $V'$. Thus the analytic family $V \to V'$ given by

$$
T(\lambda)\Pi_V = \sum_{n=0}^{\infty} \lambda^n T_n \Pi_V
$$

is invertible for small $\lambda$. Call this inverse $Q(\lambda) : V \to V'$ (which is analytic by an argument like in Step 1. We have of course generalized to the case where the domain and codomain are not the same; however they are still isomorphic). We extend $Q(\lambda)$ to an analytic family of operators $H \to H$ by multiplying on the right by $\Pi_V'$. The new operator, $Q_{\text{new}}(\lambda) = Q(\lambda)\Pi_V$, we will henceforth call $Q(\lambda)$, too, in order to reduce unnecessary notation.

Since $T$ has index 0, $\dim W = \dim W' < \infty$, and we may pick an isomorphism $P$ of $W'$ onto $W$. Of course $PT\Pi_V = 0$. We show that we can extend this to small $\lambda$, i.e. find an analytic function $P(\lambda)$ with $P(0) = P$ for which $P(\lambda)T(\lambda)\Pi_V = 0$. Define recursively $P_0 = P$ and

$$
P_{k+1} = (\Pi_W - (P_{n-1}T_1 + \cdots P_0 T_n))Q_0.
$$

Similiar to Step 1, setting

$$
P(\lambda) = \sum_{n=0}^{\infty} \lambda^n P_n
$$

defines an analytic family. Since $Q_0T_0\Pi_V = \Pi_V$ and $\Pi_W\Pi_V = 0$, $P(\lambda)T(\lambda)\Pi_V = 0$.

Now set $R(\lambda) = Q(\lambda) + P(\lambda)$. Since $R(0)$ is invertible, $R(\lambda)$ is locally invertible with analytic inverse (by Step 1, for instance).

Set $A(\lambda) = \Pi_W R(\lambda) T(\lambda) \Pi_W$. Then $A(\lambda)$ can be interpreted as an analytic family of operators from $W \to W$, i.e. $A(\lambda)$ is an analytic family of square matrices. Similarly, set $B(\lambda) = \Pi_V R(\lambda) T(\lambda) \Pi_W$. Observe that

$$
\Pi_W R(\lambda) T(\lambda) \Pi_V = \Pi_W (Q(\lambda) T(\lambda) + P(\lambda) T(\lambda)) \Pi_V = \Pi_W \Pi_V + 0 = 0,
$$
and similarly
\[ \Pi_V R(\lambda)T(\lambda)\Pi_V = \Pi_V \Pi_V = \Pi_V. \]

Writing
\[ R(\lambda)T(\lambda) = (\Pi_V + \Pi_W)R(\lambda)T(\lambda)(\Pi_V + \Pi_W) \]

arrives at the matrix representation (1.1).

Substep ii) From the matrix representation, it is easy to see that formally an inverse should be given by the operator corresponding to
\[
\begin{pmatrix}
A^{-1}(\lambda) & 0 \\
-B(\lambda)A^{-1}(\lambda) & 1
\end{pmatrix}.
\]

(1.2)

In the rest of this substep, we show that this is actually a well-defined meromorphic extension of \((R(\lambda)T(\lambda))^{-1}\) and show that this gives a well-defined meromorphic extension of \(T(\lambda)^{-1}\).

Since \(W, W'\) are fixed finite dimensional spaces, \(A^{-1}(\lambda)\) can be written formally in the form \(A^{-1}(\lambda) = p(\lambda)^{-1}C(\lambda)\) where \(p(\lambda) = \text{det} A(\lambda)\). This can be seen using Cramer’s rule, for instance. Since \(A(\lambda)\) is certainly analytic, \(p(\lambda)\) is a \(C\)-valued analytic function, and \(C(\lambda)\) is analytic. Now \(p(\lambda)\) is not identically 0. Indeed, if it were then \(A(\lambda)\) would not be invertible anywhere. This means, using (1.1), that neither is \(R(\lambda)T(\lambda)\), which contradicts the fact \(R(\lambda)\) is invertible and \(T(\lambda)\) is invertible everywhere but a discrète number of points. So \(p(\lambda)\) is holomorphic and not identically 0, and so \(p(\lambda)^{-1}\) is meromorphic, and thus \(A^{-1}(\lambda)\) is meromorphic. Moreover, since \(A\) is a square matrix, all coefficients of terms of negative order in the Laurent expansion of \(A(\lambda)\) are finite-rank operators.

In particular, the operator family
\[ S(\lambda) = A^{-1}(\lambda) - B(\lambda)A^{-1}(\lambda) + \Pi_V, \]
corresponding to the inverse matrix written above, is also meromorphic near \(\lambda_1\) (we remark that some care needs to be taken when interpreting this formula; one needs to extend \(A^{-1}(\lambda)\) to a function on all of \(H\) by setting it to be 0 on \(V'\); the extended \(A^{-1}(\lambda)\) is still meromorphic). We clearly still have that all coefficients of terms of negative order in \(S(\lambda)\) are finite rank. \(S(\lambda)\) is actually an inverse to \(R(\lambda)T(\lambda)\) wherever \(p(\lambda) \neq 0\). Thus \(S(\lambda)R(\lambda) = T^{-1}(\lambda)\) in the same area. But \(S(\lambda)R(\lambda)\) is meromorphic near \(\lambda_1\), and thus we conclude that \(T^{-1}(\lambda)\) has a meromorphic extension to \(\lambda_1\), i.e. \(\lambda_1 \in \Omega'\). Lastly, since the finite-rank operators form an ideal, all coefficients of terms of negative order in the Laurent expansion of \(S(\lambda)R(\lambda) = T^{-1}(\lambda)\) are finite rank operators.

Remark 1.3. If one knew as a black box that a meromorphic extension existed, one could deduce that the coefficients of terms of negative order have finite rank without looking at the proof. We sketch here how to do it. Suppose without loss of generality that \(T^{-1}(\lambda)\) does not exist as \(\lambda = 0\). Write
\[ T(\lambda) = \sum_{n=0}^{\infty} T_n \lambda^n \]
and
\[ T^{-1}(\lambda) = \sum_{n=-N}^{\infty} R_n \lambda^n, \]
for \( N \geq 1 \). We need to show that each \( R_{-k} \), \( k \geq 1 \) has finite rank. Since \( T(\lambda) \) is Fredholm for all \( \lambda \), in particular \( T(0) = T_0 \) is Fredholm, and so has finite-dimensional kernel. Since \( T(\lambda)T^{-1}(\lambda) = 1 \), the coefficient of \( \lambda^{-k} \) in the product is 0 for \( k \geq 1 \). This means that the following equations are valid for \( 1 \leq k \leq N \):
\[ 0 = \sum_{j=0}^{N-k} T_j R_{-k-j}. \]
Using this we inductively show that \( R_{-k} \) has finite rank. The equation for \( k = N \) simply reads \( T_0 R_{-N} = 0 \). Since \( \dim \ker T_0 < \infty \), this means that \( R_{-N} \) has finite rank. For \( k < N \), we may rearrange the equation to get
\[ T_0 R_{-k} = -\sum_{j=1}^{N-k} T_j R_{-k-j}. \]
By induction, the right-hand side is finite-rank, and therefore so is the left. By the rank-nullity theorem
\[ \dim \ker R_{-k} = \dim \ker T_0 \cap \ker R_{-k} + \dim \ker T_0 R_{-k} < \infty, \]
i.e. \( R_{-k} \) has finite rank.

We now turn to the proof of the proposition:

Proof. By the spectral theorem for self-adjoint operators, \( T - \lambda \) is invertible (and hence analytic) off of \([0, \infty)\). Since \( \Omega \) is open, it necessarily intersects points not in this set. Thus analytic Fredholm theory applies.

Fix \( \lambda_0 \in [0, \infty) \cap \Omega \) for which \( T \) is not invertible, and set \( S = T - \lambda_0 \). Since \( S \) is self-adjoint and Fredholm, it is invertible: \( \im S = \ker S^\perp \rightarrow \im S \), and \( S \equiv 0 \) as a map \( \ker S \rightarrow \ker S = \im S^\perp \). We can thus picture \( S - \mu \) as the matrix
\[ \begin{pmatrix} -\mu & 0 \\ 0 & S - \mu \end{pmatrix}, \]
where the first row and column represent \( \ker S \) and the second row and column represent \( \im S \). In particular, \( S - \mu \) is invertible for small \( \mu \neq 0 \). Notice that \( S \) is invertible on \( \im S \). This means that for \( v \in \im S \),
\[ ||(S - \mu)^{-1}v|| \leq (||S|_{\im S}^{-1} - \mu)^{-1}||v||; \quad (1.3) \]
which is uniformly bounded as \( \mu \rightarrow 0 \).
Write $H \ni u = v + w$, where $v \in \text{im} \, S$, and $w \in \ker \, S$. Then, for $\mu \neq 0$ small,

$$
\|(S - \mu)^{-1}(u)\|^2 = \|(S - \mu)^{-1}v\|^2 + \|\mu^{-1}w\|^2.
$$

(1.4)

For arbitrary $\|u\| = 1$, we can use (1.3) to bound (1.4) by

$$
(\|S\|^{-1}_{\text{im} \, S} - \mu)^{-2} + \mu^{-2} \lesssim \mu^{-2},
$$

(1.5)

since the second term is uniformly bounded. If $\|u\| = 1$ and $u \in \ker \, S$, then using (1.4),

$$
\|(S - \mu)^{-1}u\| \geq \mu^{-1}.
$$

(1.6)

We conclude from (1.5) and (1.6) that

$$
\|(S - \mu)\| \sim \mu^{-1}
$$

as $\mu \to 0$.

It follows immediately that

$$
\|(T - \lambda)^{-1}\| \sim |\lambda - \lambda_0|^{-1},
$$

as $\lambda \to \lambda_0$. But it is also clear that $\|(T - \lambda)^{-1}\|$ blows up like $|\lambda - \lambda_0|^{-m}$, where $m$ is the order of the pole at $\lambda_0$. Indeed, we may write

$$
\|(T - \lambda)^{-1}\| = |\lambda - \lambda_0|^{-m}R_{-n} + (\lambda - \lambda_0)R_{-n+1} + \cdots,
$$

and the series in the second factor converges to a continuous function of $\lambda$ for $\lambda$ near $\lambda_0$ (by Cauchy-Hadamard, for instance). Thus the second factor converges to $\|R_{-n}\|$ and we have the desired blow up.

It follows that $m = 1$, and so the pole is simple.

Next, write

$$(T - \lambda)^{-1} = \sum_{n=-1}^{\infty} (\lambda - \lambda_0)^n R_n$$

around a pole $\lambda_0$. We have that

$$(T - \lambda)^{-1}(T - \lambda) = (T - \lambda)(T - \lambda)^{-1} = 1$$

and

$$
T - \lambda = (T - \lambda_0) - (\lambda - \lambda_0)
$$

are both valid around $\lambda_0$ (but of course not at it). Expand the first identity out as a series, and looking at the $n = -1, 0$ terms yields $R_{-1}(T - \lambda_0) = (T - \lambda_0)R_{-1} = 0$ and $R_0(T - \lambda_0) = (T - \lambda_0)R_0 = 1 + R_{-1}$. The first of the two implies that $R_{-1}$ takes values in $\ker(T - \lambda_0)$ and acts trivially on $\text{im}(T - \lambda_0)$. The second identity implies that

$$
0 = R_0(T - \lambda_0)w = w + R_{-1}w
$$

for $w \in \ker(T - \lambda_0)$, i.e. $R_{-1}$ acts as $-1$ times the identity on $\ker(T - \lambda_0)$. Putting both these things together we prove the proposition. 

$\square$