Almost sure multifractal spectrum of SLE

Ewain Gwynne
(Joint with Jason Miller and Xin Sun)

Massachusetts Institute of Technology

Conformally invariant scaling limits, University of Cambridge

Outline

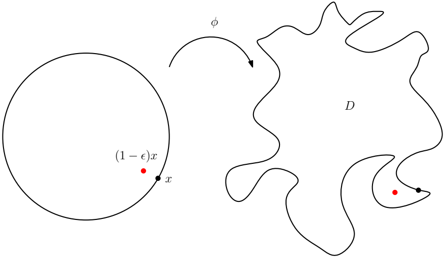
Multifracal spectrum

- Let $D \subset \mathbb{C}$ be a simply connected domain (e.g. a complementary connected component of an SLE_κ curve). Let $\phi : \mathbb{D} \to D$ be a conformal map.
- The multifractal spectrum of D is a means of quantifying the behavior of $|\phi'|$ (resp. $|(\phi^{-1})'|$) near $\partial\mathbb{D}$ (resp. ∂D).

Multifractal spectrum

- Let
$$\widetilde{\Theta}^s(D) := \left\{ x \in \partial\mathbb{D} : \lim_{\epsilon \to 0} \frac{\log|\phi'((1-\epsilon)x)|}{-\log \epsilon} = s \right\}.$$
- Let $\Theta^s(D) := \phi(\widetilde{\Theta}^s(D)) \subset \partial D$.

Multifractal spectrum



Multifractal spectrum

- The *multifractal spectrum* of D is the two functions $s \mapsto \dim_{\mathcal{H}} \widetilde{\Theta}^s(D)$ and $s \mapsto \dim_{\mathcal{H}} \Theta^s(D)$.
- We have $\widetilde{\Theta}^s(D) = \Theta^s(D) = \emptyset$ for $s \notin [-1,1]$, so this is only of interest for $s \in [-1,1]$.
- Related to, e.g., the harmonic measure spectrum of D, the integral means spectrum of D, the Hölder regularity of ϕ, and the Hausdorff dimension of ∂D.

Related results

- Hausdorff dimension computed by Beffara (2008).
- Hölder exponent computed by Lawler and Viklund (2011) building on works by Rohde and Schramm (2005) and Lind (2008).
- Non-rigorous predictions for the multifractal spectrum by Duplantier as early as 2000.
  - Lead Duplantier to conjecture *SLE duality*, the statement that the outer boundary of an SLE_κ for $\kappa > 4$ locally looks like an $\mathrm{SLE}_{16/\kappa}$ (rigorously established in works by Dubedát, Zhan, Miller-Sheffield)
- Lawler and Viklund (2012) computed the *multifractal spectrum at the tip* of SLE.
- Beliaev and Smirnov (2009) computed the *average integral means spectrum* of SLE.
- Alberts, Binder, and Viklund (2015) computed a dimension spectrum for points where SLE hits the boundary.
- More in later talks today.

Multifractal spectrum

Theorem: (Gwynne, Miller, Sun) Let $\kappa > 0$ and let η be an SLE_κ in a smoothly bounded domain $D \subset \mathbb{C}$. Let

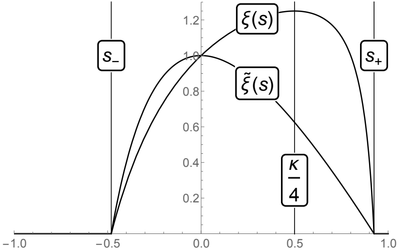
$$s_- = \frac{4\kappa - 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2}, \quad s_+ = \frac{4\kappa + 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2}.$$

Let $s \in [s_-, s_+]$. Almost surely, for each $t > 0$ and each complementary connected V of $\eta([0,t])$, we have

$$\dim_{\mathcal{H}} \widetilde{\Theta}^s(V) = 1 - \frac{(4+\kappa)^2 s^2}{8\kappa(1+s)}$$
$$\dim_{\mathcal{H}} \Theta^s(V) = \frac{8\kappa(1+s-s^2) - 16s^2 - \kappa^2 s^2}{8\kappa(1-s^2)}.$$

For $s \notin [s_-, s_+]$, a.s. $\widetilde{\Theta}^s(V) = \Theta^s(V) = \emptyset$.

Multifractal spectrum



Multifractal spectrum

- Agrees with predictions of Duplantier.
- Invariant under replacing κ with $16/\kappa$ (SLE duality).
- $s \mapsto \xi(s)$ is maximized at $s = \kappa/4$, where is equals $1 + \kappa/8$.
- This yields an alternative proof that $\dim_{\mathcal{H}} \eta = 1 + \kappa/8$ a.s. for $\kappa \in (0, 4]$.

Integral means spectrum

- The *integral means spectrum* of D is the function $\mathrm{IMS}_D : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{IMS}_D(a) = \limsup_{\epsilon \to 0} \frac{\log \int_{\partial B_{1-\epsilon}(0)} |\phi'(z)|^a \, dz}{-\log \epsilon},$$

  where $\phi : \mathbb{D} \to D$ is a conformal map.
- Related to several conjectures in complex analysis.
- Usually hard to compute for deterministic fractals, but can be easier for random fractals.

Integral means spectrum

Average integral means spectrum of SLE computed by Beliaev-Smirnov (2009):

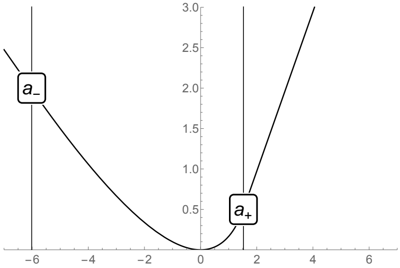
$$\limsup_{r \to 1^+} \frac{\log \int_0^{2\pi} \mathbb{E}|(f_t^{-1})'(re^{i\theta})|^a \, d\theta}{-\log(r-1)}.$$

Integral means spectrum

- We obtain the a.s. bulk integral means spectrum of SLE (which is defined in the same way as the ordinary integral means spectrum, but with small neighborhoods of the tip and starting point of η removed).
- **Corollary:** Let $\kappa > 0$ and let η be an SLE_κ in a smoothly bounded domain $D \subset \mathbb{C}$. Almost surely, for each $t > 0$, each $a \in \mathbb{R}$, and each complementary connected V of $\eta([0,t])$, we have

$$\mathrm{IMS}_V^{\mathrm{bulk}}(a) = \begin{cases} -1 + s_- a, & a < a_- \\ -a + \frac{(4+\kappa)(4+\kappa-\sqrt{(4+\kappa)^2 - 8a\kappa})}{4\kappa}, & a \in [a_-, a_+] \\ -1 + s_+ a, & a > a_+. \end{cases}$$

Integral means spectrum

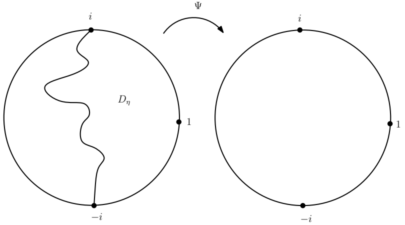


Outline

Setup

- To establish an upper bound for the Hausdorff dimension of the sets $\Theta^s(D)$ and $\widetilde{\Theta}^s(D)$, we need to estimate the probability that a point is contained in these sets.
- By SLE duality it suffices to consider $\kappa \leq 4$.
- By a change of coordinates, we can assume that η is a chordal SLE_κ from $-i$ to i in $\mathbb{D}$. Let D_η be the right complementary connected component of η.
- Also let $\Psi : D_\eta \to \mathbb{D}$ be the conformal map which fixes $-i$, i, and 1.

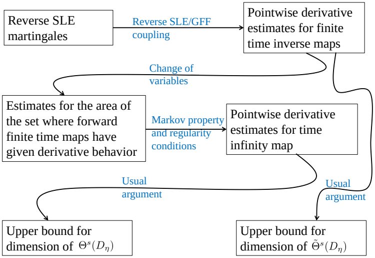
Setup



Setup

- To establish an upper bound for the Hausdorff dimension of the sets $\Theta^s(D)$ and $\widetilde{\Theta}^s(D)$, we need to estimate the probability that a point is contained in these sets.
- By SLE duality it suffices to consider $\kappa \leq 4$.
- By a change of coordinates, we can assume that η is a chordal SLE$_\kappa$ from $-i$ to i in $\mathbb{D}$. Let D_η be the right complementary connected component of η.
- Also let $\Psi : D_\eta \to \mathbb{D}$ be the conformal map which fixes $-i$, i, and 1.
- D_η will be convenient for the two-point estimate because we can grow the curve from both directions.

Upper bound



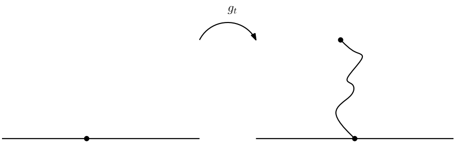
Reverse SLE

- Reverse SLE$_\kappa$ is obtained by solving the reverse Loewner equation

$$\dot{g}_t(z) = -\frac{2}{g_t(z) - U_t}.$$

- The solution is a family of conformal maps $g_t : \mathbb{H} \to \mathbb{H} \setminus K_t$, for (K_t) hulls in $\mathbb{H}$.
- If $U_t = \sqrt{\kappa} B_t$, then $g_t - U_t$ has the same law as the time t centered Loewner map of a forward SLE$_\kappa$.

Reverse SLE



One point estimate for inverse maps

- If $z \in \mathbb{H}$ and $\rho \in \mathbb{R}$, then

$$M_t = |g_t'(z)|^{(8+2\kappa-\rho)\rho/8\kappa} (\operatorname{Im} g_t(z))^{-\rho^2/8\kappa} |g_t(z) - \sqrt{\kappa} B_t|^{\rho/\kappa}$$

  is a martingale.
- Introduced by Lawler (2009).
- Reverse analogue of the Schramm-Wilson martingales for forward SLE.
- Re-weighting by M_t gives a reverse $\mathrm{SLE}_\kappa(\rho)$ with a *force point* at z.
- Call this reweighted law $\mathbb{P}_*^z$.

One point estimate for inverse maps

- If $\operatorname{Im} z = \epsilon$ and z is not too close to 0 or ∞, then

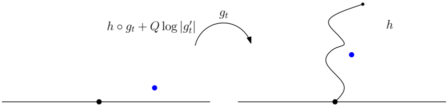
$$\mathbb{P}\left(|g_t'(z)| \approx \epsilon^{-s}\right) \approx \epsilon^{\alpha} \mathbb{P}_*^z \left(|g_t'(z)| \approx \epsilon^{-s}\right)$$

  where $\approx$ means $\epsilon^{-s+u} \leq |g_t'(z)| \leq \epsilon^{-s-u}$ for $u > 0$ small but fixed.
- We claim that if we take $\rho = \frac{(4+\kappa)s}{1+s}$, then $\mathbb{P}_*^z\left(|g_t'(z)| \approx \epsilon^{-s}\right) \asymp 1$.
- Given this we obtain $\mathbb{P}\left(|g_t'(z)| \approx \epsilon^{-s}\right) \approx \epsilon^{\alpha}$.

One point estimate for inverse maps

- To show $\mathbb{P}_*^z(|g_t'(z)| \approx \epsilon^{-s}) \asymp 1$ we use a coupling with a Gaussian free field.
- By a theorem of Sheffield (2011) we can find random distributions h and h_t (GFF's plus harmonic functions) s.t. $h \circ g_t + Q \log |g_t'| \stackrel{d}{=} h_t$ under P_*^z, where $Q = 2/\sqrt{\kappa} + \sqrt{\kappa}/2$.

One point estimate for inverse maps



One point estimate for inverse maps

- To show $\mathbb{P}_*^z(|g_t'(z)| \approx \epsilon^{-s}) \asymp 1$ we use a coupling with a Gaussian free field.
- By a theorem of Sheffield (2011) we can find random distributions h and h_t (GFF's plus harmonic functions) s.t. $h \circ g_t + Q \log |g_t'| \stackrel{d}{=} h_t$ under P_*^z, where $Q = 2/\sqrt{\kappa} + \sqrt{\kappa}/2$.
- By estimating the circle average processes for h and h_t, we get that $|g_t'(z)| \approx \epsilon^{-s}$ with high probability under P_*^z.
- This leads to
$$\mathbb{P}\left(|g_t'(z)| \approx \epsilon^{-s}\right) \approx \epsilon^{\alpha(s)}.$$

One point estimate for inverse maps

- Using stochastic calculus and a symmetry between forward and reverse $\mathrm{SLE}_\kappa(\rho)$ due to Duplantier, Miller, and Sheffield (2014), we can also add extra regularity conditions to our lower bound.
- This is the most technical part of the one-point estimate.
- This is also the main reason why we can't just cite other similar results in the literature (e.g. Rohde-Schramm, Beliaev-Smirnov).

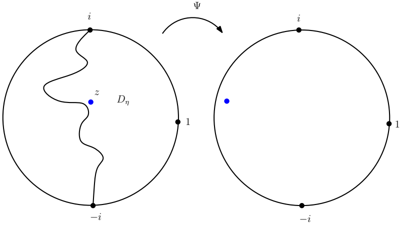
Upper bound

- Our derivative estimates allow us to estimate the expected number of ϵ-balls needed to cover $\widetilde{\Theta}^s(\mathbb{H} \setminus \eta([0,t]))$.
- This gives an upper bound for the Hausdorff dimension of $\widetilde{\Theta}^s(\mathbb{H} \setminus \eta([0,t]))$ and the integral means spectrum of $\mathbb{H} \setminus \eta([0,t])$.
- Basic complex analysis allows us to transfer these upper bounds to D_η.

Area estimate

- By a change of variables and the Koebe quarter theorem ($|f_t'(z)| \operatorname{dist}(z,\eta) \asymp \operatorname{Im} f_t(z)$), we can estimate the area of the set of $z \in \mathbb{H}$ with $|f_t'(z)| \approx \epsilon^s$ and $\operatorname{dist}(z, \eta([0,t])) \approx \epsilon^{1-s}$.
- We then transfer this to an estimate for the area of the set $\mathcal{A}_\epsilon^s$ of $z \in D_\eta$ for which $|\Psi'(z)| \approx \epsilon^s$ and $\operatorname{dist}(z,\eta) \approx \epsilon^{1-s}$.

One point estimate for the forward maps



One point estimate for the forward maps

- Using the Markov property, one can argue that $\mathbb{P}(z \in \mathcal{A}_\epsilon^s)$ does not depend too strongly on z.
- This takes us from area estimates to pointwise estimates:

$$\mathbb{P}\left(|\Psi'(z)| \approx \epsilon^s,\ \operatorname{dist}(z,\eta) \approx \epsilon^{1-s}\right) \approx \epsilon^{\gamma(s)}.$$

- This estimate leads to an upper bound for $\dim_{\mathcal{H}} \Theta^s(D_\eta)$.

Outline

Lower bound

- We prove a lower bound for $\dim_{\mathcal{H}} \Theta^s(D_\eta)$ first.
- Let $q := s/(1-s)$.
- We define nested events $E_n(z)$ for $z \in \mathbb{D}$ such that
  - $\bigcap_{n=1}^{\infty} E_n(z) \subset \{z \in \Theta^s(D_\eta)\}$.
  - $\mathbb{P}\left(E_n(z)\right) \approx e^{-\beta\gamma^*(q)n}$.
  - $\frac{\mathbb{P}(E_n(z)\cap E_n(w))}{\mathbb{P}(E_n(z))\mathbb{P}(E_n(w))} \leq |z-w|^{\gamma^*(q)+o_{|z-w|}(1)}$.

Lower bound

This will allow us to construct a *Frostman measure* on a self-similar subset of $\Theta^s(D_\eta)$ (the "perfect points"), i.e. a positive finite measure satisfying

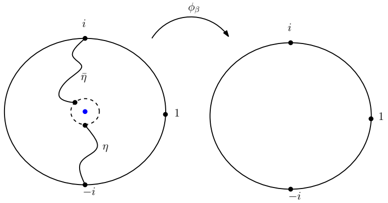
$$\int\int \frac{1}{|z-w|^\alpha} \, d\nu(z) d\nu(w) < \infty$$

for given $\alpha < \xi(s)$.

Lower bound

- Let $\overline{\eta}$ be the time reversal of η, which is an SLE_κ from $-i$ to i.
- We will look at the behavior of η and $\overline{\eta}$ at the first time they hit $B_{e^{-\beta}}(z)$.

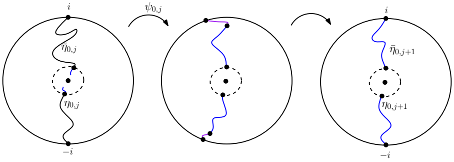
Lower bound



Lower bound

- Now we iterate this.
- Let $\eta_{0,1} = \eta$. Inductively let $\eta_{0,j+1}$ be the curve obtained by running $\eta_{0,j}$ and $\overline{\eta}_{0,j}$ up to the first time they $B_{e^{-\beta}}(0)$, then applying the map $\psi_{0,j}$ which takes the complement of the two sides of the curve to $\mathbb{D}$, with 0 fixed.
- Then we have $\eta_{0,j} \overset{d}{=} \eta$, modulo perturbations of the endpoints (which we can deal with by growing out a little more of the curve).

Lower bound



Lower bound

- Let $\phi_{0,j}$ be defined in the same manner as ϕ_β above, but with $\eta_{0,j}$ in place of η.
- Let $E_{0,j}$ be the event that $|\phi_j'(0)| \approx e^{-\beta q}$ (plus a bunch of regularity conditions).
- Let $E_n(0) = \bigcap_{j=1}^n E_{0,j}$.
- Define $E_n(z)$ for $z \in \mathbb{D}$ by first mapping z to 0.
- Define the *perfect points* to be the (approximately) the set of $z \in \mathbb{D}$ for which $E_n(z)$ occurs.

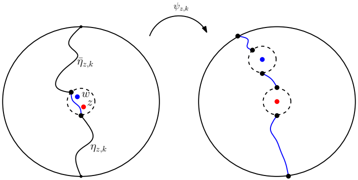
Lower bound

- Our events are set up so that $\bigcap_{n=1}^\infty E_n(z) \subset \Theta^s(D_\eta)$.
- The probability that $|\phi_\beta'(0)| \approx e^{-\beta q}$ is of the same order as the probability that $|\Psi'(0)| \approx e^{-\beta q}$ and $\mathrm{dist}(z,\eta) \approx e^{-\beta}$.
- We know the latter probability is $e^{-\beta\gamma^*(q)}$ by the one-point estimate (with $\epsilon = e^{-\beta/(1-s)}$).
- Hence $\mathbb{P}(E_n(z)) \approx e^{-n\beta\gamma^*(q)}$.

Lower bound

- Consider two points z and w with $|z-w| \approx e^{-k\beta}$.
- We need to estimate $\mathbb{P}(E_n(z) \cap E_n(w))$ for $n \geq k$.
- The points $0 = \psi_{z,k}(z)$ and $\psi_{z,k}(w)$ are at constant order distance apart.

Lower bound



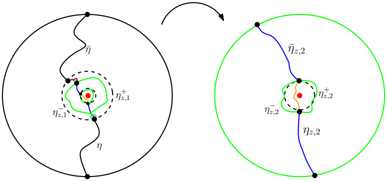
Lower bound

- We would like to say that the behaviors of the curve near the two points are approximately independent.
- However, we are interested in the derivative of a certain conformal map, which may depend on the whole curve.
- To get around this we need to localize.

Flow lines

- We can couple η with a GFF h on $\mathbb{D}$ in such a way that η is the "flow line" of h started from $-i$ (in the sense of Miller and Sheffield's *Imaginary Geometry* papers).
- At each stage in the construction of $E_n(z)$, we add auxiliary flow lines $\eta^{\pm}_{z,j}$ for h started from the tip of the part of η we have grown so far.

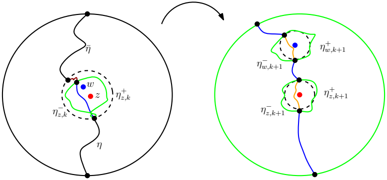
Flow lines



Flow lines

- The auxiliary flow lines form "pockets" with the property that the intersection of η with each pocket is conditionally independent of what happens outside the pocket, given the pocket.
- We re-define the curves $\eta_{z,j}$ so that they only depend on the part of the curve inside the $j-1$th pocket around z.

Flow lines



Lower bound

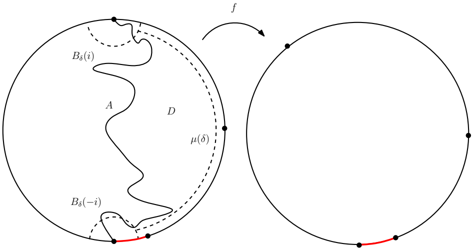
- This leads to an estimate for $\frac{\mathbb{P}(E_n(z)\cap E_n(w))}{\mathbb{P}(E_n(z))\mathbb{P}(E_n(w))}$ in terms of $|z-w|$.
- Once we have such an estimate, we get a lower bound for $\dim_{\mathcal{H}} \Theta^s(D_\eta)$ via the usual (Frostman measure) argument.
- Can use the same estimates (and some relatively minor tricks) to get a lower bound for $\dim_{\mathcal{H}} \widetilde{\Theta}^s(D_\eta)$ and $\mathrm{IMS}^{\mathrm{bulk}}(D_\eta)$.

Outline

Reverse continuity conditions

- Throughout this talk, all of our events have involved "regularity conditions". The most important (but by no means the only) such regularity conditions are the following.
- Let $A \subset \overline{\mathbb{D}}$ be a closed set. Let D be a connected component of $\mathbb{D} \setminus A$ and let $f : D \to \mathbb{D}$ be a conformal map. Let $\mu : (0,\infty) \to (0,\infty)$ be an increasing function.
- $\mathcal{G}(f,\mu)$: for each $\delta > 0$ and each $x, y \in \partial D \cap \partial \mathbb{D}$ with $|x - y| \geq \delta$, we have $|f(x) - f(y)| \geq \mu(\delta)$.
- $\mathcal{G}'(A,\mu)$: for each $\delta > 0$, A lies at distance at least $\mu(\delta)$ from $\partial \mathbb{D} \setminus (A \cap \partial \mathbb{D})$.

Reverse continuity conditions



Reverse continuity conditions

- $\mathcal{G}(f,\mu)$ and $\mathcal{G}'(A,\mu)$ are "equivalent" in the sense that for each μ, there exists μ' (depending only on μ) such that $\mathcal{G}(f,\mu) \Rightarrow \mathcal{G}'(A,\mu')$ and vice versa.

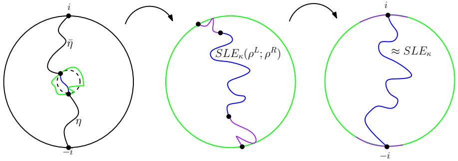
Reverse continuity conditions

- $\mathcal{G}(f, \mu)$ is useful because many of our maps are normalized so that they fix $-i$, i, and 1. In order to achieve such a normalization, we sometimes need to apply a Möbius transformation. The condition $\mathcal{G}(f, \mu)$ ensures that such a transformation does not distort distances too much.
- The condition $\mathcal{G}'(A, \mu)$ (typically with $A = \eta$ or some part of η) is useful because many of our estimates degenerate near the boundary.
- $\mathcal{G}(f, \mu)$ is well-behaved under compositions of maps. $\mathcal{G}'(A, \mu)$ is easy to deal with geometrically.

Strict mutual absolute continuity

- The parts of η inside the "pockets" used in the proof of the lower bound are $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$'s, not ordinary SLE_κ's.
- However, if we grow out a little bit of the curves, then map back, we get curves whose laws are *strictly mutually absolutely continuous* with respect to the law of ordinary SLE_κ curve, meaning that their laws are absolutely continuous, with Radon-Nikodym derivative bounded above and below by deterministic constants.
- We can do this at the same time we move the endpoints to $-i$ and i.

Strict mutual absolute continuity



Strict mutual absolute continuity

- Growing the initial (purple) segments of the curve means that the starting point for the auxiliary flow lines is not a stopping time for η.
- One has to be careful to make sure that the results of the imaginary geometry papers are applicable (this actually involves growing a second pair of auxiliary flow lines).

"o(1)" errors

- In our events, we require that the derivative of the conformal map is "$\approx \epsilon$", i.e. between ϵ^{s-u} and ϵ^{s+u}, where u is a small parameter.
- In order to ensure that the perfect points are actually contained in the set $\Theta^s(D_\eta)$, we need to shrink u a little bit at each stage.
- To counteract the increasing constants in the estimates, we also need to increase β a little bit at each stage.
- So, the nth event in the definition of the perfect points is defined using the ball of radius $e^{-\beta_n}$, rather than $e^{-\beta}$, where $\beta_n \to \infty$ (at approximately a logarithmic rate) as $n \to \infty$.

"o(1)" errors

- The diameter of the nth pocket is $e^{-\overline{\beta}_n(1+o_n(1))}$, where $\overline{\beta}_n = \sum_{j=1}^n \beta_j$.
- To make sure the pockets surrounding z and w are disjoint, we need to skip $no_n(1)$ scales when we do the two-point estimate.
- This is okay, as it just leads to an $o_n(1)$ error in the exponent.

Outline

Future directions

- Winding spectrum of SLE—asymptotics of arg ϕ' rather than $|\phi'|$
  - Can also consider mixed spectrum.
  - Predictions by Duplantier and Duplantier/Binder
  - Upper bound for winding proven by Aru (2014).
  - Probably lower bound can be done in a similar manner as for the multifractal spectrum.
- Multifractal spectrum for $\text{SLE}_\kappa(\underline{\rho})$ near where it intersects the boundary.
  - Same as for ordinary SLE away from the boundary by absolute continuity.
  - Maybe could be done using techniques similar to those of our paper and/or or those of Alberts-Binder-Viklund.

References

- E. Gwynne, J. Miller, and X. Sun. *Almost sure multifractal spectrum of SLE.* ArXiv.
- B Duplantier. *Conformally invariant fractals and potential theory.* Physical Review Letters.
- G. Lawler and F. J. Viklund. *Almost sure multifractal spectrum for the tip of an SLE curve.* Acta Math.
- D. Beliaev and S. Smirnov. *Harmonic measure and SLE.* Comm. Math. Phys.
- S. Sheffield. *Conformal weldings of random surfaces: SLE and the quantum gravity zipper.* ArXiv.
- J. Miller and S. Sheffield. *Imaginary Geometry I-IV.* ArXiv.