MATHEMATICAL IDEAS AND NOTIONS OF QUANTUM FIELD THEORY

1. Generalities on quantum field theory

1.1. **Classical mechanics.** In classical mechanics, we study the motion of a particle. This motion is described by a (vector) function of one variable, $q = q(t)$, representing the position of the particle as a function of time. This function must satisfy the Newton equation of motion,

$$\ddot{q} = -U'(q),$$

where U the potential energy, and the mass of the particle is 1. Another way to express this law of motion is to say that $q(t)$ must be a solution of a certain variational problem. Namely, one introduces *the Lagrangian*

$$\mathcal{L}(q) = \frac{\dot{q}^2}{2} - U(q)$$

(the difference of kinetic and potential energy), and *the action functional*

$$S(q) = \int_a^b \mathcal{L}(q)dt$$

(for some fixed $a < b$). Then the law of motion can be expressed as *the least action principle*: $q(t)$ must be a critical point of S on the space of all functions with given $q(a)$ and $q(b)$. In other words, the Newton equation is the Euler-Lagrange equation for the solution of the variational problem defined by S.

Remark 1. The name "least action principle" comes from the fact that in some cases (for example when $U'' \le 0$) the action is not only extremized but also minimized at the solution $q(t)$. In general, however, it is not the case, and the trajectory of the particle may not be a minimum, but only a saddle point of the action. Therefore, the law of motion is better formulated as the "extremal (or stationary) action principle"; this is the way we will think of it in the future.

Remark 2. Physicists often consider solutions of Newton's equation on the whole line rather than on a fixed interval $[a, b]$. In this case, the naive definition of an extremal does not make sense, since the action integral $S(q) = \int_{\mathbb{R}} \mathcal{L}(q)dt$ is improper and in general diverges. Instead, one makes the following "correct" definition: a function $q(t)$ on $\mathbb{R}$ is an extremal of S if the expression

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} \mathcal{L}(q + s\varepsilon)dt := \int_{\mathbb{R}} (\frac{\partial L}{\partial q}\dot{\varepsilon} + \frac{\partial L}{\partial \dot{q}}\ddot{\varepsilon} + ...),$$

where $\varepsilon(t)$ is any compactly supported perturbation, is identically zero. With this definition, the extremals are exactly the solutions of Newton's equation.

1.2. **Classical field theory.** In classical field theory, the situation is similar. In this case, we should think not of a single particle, but of a "continuum of particles" (e.g. a string, a membrane, a jet of fluid); so the motion is described by a classical field – a (vector) function $\phi(x,t)$ depending on both space and time coordinates ($x \in \mathbb{R}^d$, $t \in \mathbb{R}$). Consequently, the equation of motion is a partial differential equation. For example, for a string or a membrane the equation of motion is the wave equation $\Box\phi = 0$, where $\Box$ is the D'Alambertian $\partial_t^2 - v^2\Delta$ (here Δ is the Laplacian with respect to the space coordinates, and v the velocity of wave propagation).

As in mechanics, in classical field theory there is a Lagrangian $\mathcal{L}(\phi)$ (a differential polynomial in ϕ), whose integral $S(\phi) = \int_D \mathcal{L}(\phi)dxdt$ over a region D in space and time is called the action. The law of motion can be expressed as the condition that the action must be extremized over any closed region D and fixed boundary conditions; so the equations of motion (also called the field equations) are the Euler-Lagrange equations for this variational problem. For example, in the case of string or membrane, the Lagrangian is

$$\mathcal{L}(u) = \frac{1}{2}(\phi_t^2 - v^2(\nabla\phi)^2).$$

Remark. Like in mechanics, solutions of the field equations on the whole space (rather than a closed region D) are extremals of the action in the sense that

$$\frac{d}{ds}|_{s=0}\int_{\mathbb{R}^{d+1}} \mathcal{L}(u + s\varepsilon)dxdt = 0,$$

where ε is a compactly supported perturbation.

1.3. **Brownian motion.** One of the main differences between classical and quantum mechanics is, roughly speaking, that quantum particles do not have to obey the classical equations of motion, but can randomly deviate from their classical trajectories. Therefore, given the position and velocity of the particle at a given time, we cannot determine its position at a later time, but can only determine the density of probability that at this later time the particle will be found at a given point. In this sense quantum particles are similar to random (Brownian) particles. Brownian particles are a bit easier to understand conceptually, so let us begin with them.

The motion of a Brownian particle in $\mathbb{R}^d$ in a potential field $U : \mathbb{R}^d \to \mathbb{R}$ is described by a stochastic process $q = q(t)$, $q = (q_1, \ldots, q_d) \in \mathbb{R}^d$. That is, for each real t we have a random variable $q(t)$ (position of the particle at a time t), such that the dependence of t is regular in some sense. The random dynamics of the particle is "defined" as follows: [1] if $y : [a,b] \to \mathbb{R}^d$ is a continuously differentiable function, then the density of probability that $q(t) = y(t)$ for $t \in [a,b]$ is proportional to $e^{-S(y)/\kappa}$, where $S(y) := \int_a^b (\frac{1}{2}y'^2 - U(y))dt$ is the action for the corresponding classical mechanical system, and κ is the diffusion coefficient. Thus, for given $q(a)$ and $q(b)$, the likeliest $q(t)$ is the one that minimizes S (in particular, solves the classical equations of motion $\ddot{q} = -U'(q)$), while the likelihood of the other paths decays exponentially with the deviation of the action of these paths from the minimal possible.

[1] We put the word "defined" in quotation marks because this definition is obviously heuristic and not rigorous; see below for more explanations

Remark. This discussion assumes that the extremum of S at q is actually a minimum, which we know is not always the case.

All the information we can hope to get about such a process is contained in the correlation functions $< q_{i_1}(t_1) \dots q_{i_n}(t_n) >$, which by definition are the expectation values of the products of random variables $q_{i_1}(t_1) \dots q_{i_n}(t_n)$ (more specifically, by Kolmogorov's theorem the stochastic process $q(t)$ is completely determined by these functions). So such functions should be regarded as the output, or answer, of the theory of the Brownian particle.

So the main question is how to compute the correlation functions. The definition above obviously gives the following answer: given $t_1, \dots, t_n \in [a, b]$, we have

$$< q_{j_1}(t_1) \dots q_{j_n}(t_n) >= \int q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{-S(q)/\kappa} Dq, \tag{1}$$

where integration is carried out over the space of paths $[a, b] \to \mathbb{R}^n$, and Dq is a Lebesgue measure on the space of paths such that $\int e^{-S(q)/\kappa} Dq = 1$. Such an integral is called a path integral, since it is an integral over the space of paths.

It is clear, however, that such definition and answer are a priori not satisfactory from the mathematical viewpoint, since the infinite dimensional integration that we used requires justification. In this particular case, such justification is possible within the framework of Lebesgue measure theory, and the corresponding integration theory is called the theory of Wiener integrals. (To be more precise, one cannot define the measure Dq, but one can define the measure $e^{-S(q)/\kappa} Dq$ for sufficiently nice potentials $U(q)$).

1.4. **Quantum mechanics.** Now let us turn to a quantum particle. Quantum mechanics is notoriosly difficult to visualize, and the randomness of the behavior of a quantum particle is less intuitive and more subtle than that of a brownian particle; nevertheless, it was pointed out by Feynman that the behavior of a quantum particle in a potential field $U(q)$ is correctly described by the same model, with the real positive parameter κ replaced by the imaginary number $i\hbar$, where $\hbar > 0$ is the Planck constant. In other words, the dynamics of a quantum particle can be expressed (we will discuss later how) via the *correlation functions*

$$< q_{j_1}(t_1) \dots q_{j_n}(t_n) >= \int q_{j_1}(t_1) \dots q_{j_n}(t_n) e^{iS(q)/\hbar} Dq, \tag{2}$$

where Dq is normalized so that $\int e^{iS(q)/\hbar} Dq = 1$.

1.5. **Quantum field theory.** The situation is the same in field theory. Namely, a useful theory of quantum fields (used in the study of interactions of elementary particles) is obtained when one considers correlation functions

$$< \phi_{j_1}(x_1, t_1) \dots \phi_{j_n}(x_n, t_n) >= \int \phi_{j_1}(x_1, t_1) \dots \phi_{j_n}(x_n, t_n) e^{iS(\phi)/\hbar} D\phi, \tag{3}$$

where $D\phi$ is normalized so that $\int e^{iS(\phi)/\hbar} D\phi = 1$.

Of course, from the mathematical point of view, this setting is a priori even less satisfactory than the one for the Brownian particle, since it involves integration with respect to the complex valued measure $e^{iS(q)/\hbar} Dq$, which nobody knows how to define. Nevertheless, physicists imagine that certain integrals of this type exist and come to correct and interesting conclusions (both physical and mathematical). Therefore, making sense of such integrals is an interesting problem for mathematicians, and will be one of our main occupation during the course.[2]

2. The steepest descent and stationary phase formulas

Now, let us forget for a moment that the integrals (1,2,3) are infinite dimensional and hence problematic to define, and ask ourselves the following question: why should we expect that when the parameter κ or $\hbar$ goes to zero, we recover the usual classical mechanics or field theory? The answer is that this expectation is based on the *steepest descent* (respectively, *stationary phase*) principle from classical analysis: if $f(x)$ is a function in $\mathbb{R}^d$ then the integrals $\int g(x)e^{-f(x)/\kappa}dx$, $\int g(x)e^{if(x)/\hbar}dx$ "localize" to minima, respectively critical points, of the function f. As this classical fact is of central importance to the whole course, let us now discuss it in some detail.

2.1. The steepest descent formula. Let $f, g : [a,b] \to \mathbb{R}$ be smooth functions.

Theorem 2.1. *(The steepest descent formula) Assume that f attains a global minimum at a unique point $c \in (a,b)$, such that $f''(c) > 0$. Then one has*

$$\int_a^b g(x)e^{-f(x)/\hbar}dx = \hbar^{1/2}e^{-f(c)/\hbar}I(\hbar), \tag{4}$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that $I(0) = \sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}}$.

Proof. Let $I(\hbar)$ be defined by the equation (4).

Let ϵ be a real number, such that $\frac{1}{2} > \epsilon > 0$, and let $I_1(\hbar)$ be defined by the same equation, but with integration over $[c-\hbar^{\frac{1}{2}-\epsilon}, c+\hbar^{\frac{1}{2}-\epsilon}]$. It is clear that $I(\hbar) - I_1(\hbar)$ is "rapidly decaying in $\hbar$" (i.e. it is $O(\hbar^N), \hbar \to 0$ for any N). So it suffices to prove the theorem for $I_1(\hbar)$.

Further, let us make in the integral defining $I_1(\hbar)$ the change of variables $y = (x-c)/\sqrt{\hbar}$. Then we get

$$I_1(\hbar) = \int_{-\hbar^{-\epsilon}}^{\hbar^{-\epsilon}} g(c+y\sqrt{\hbar})e^{(f(c)-f(c+y\sqrt{\hbar}))/\hbar}dy. \tag{5}$$

Now, note that the integrand is a smooth function with respect to $\sqrt{\hbar}$ for $\hbar \geq 0$. Let $I_2(\hbar)$ be the same integral as in (5) but with integrand replaced by its Taylor expansion in $\sqrt{\hbar}$ at 0 modulo $\hbar^N$. Then $|I_1(\hbar) - I_2(\hbar)| \leq C\hbar^{N-\epsilon}$.

Finally, let $I_3(\hbar)$ be defined by the same integral as $I_2(\hbar)$ but with limits from $-\infty$ to ∞. Then $I_2(\hbar) - I_3(\hbar)$ is rapidly decaying in $\hbar$.

Thus, it suffices to show that $I_3(\hbar)$ admits a Taylor expansion in $\hbar^{1/2}$ modulo $\hbar^{N-\epsilon}$, and that the value at zero is as stated. But we know that $I_3(\hbar)$ is a polynomial in $\sqrt{\hbar}$. Also, the integrals giving coefficients of non-integer powers of $\hbar$ are integrals over $\mathbb{R}$ of odd functions, so they are zero. So the first statement (existence of the Taylor expansion) is proved. The value $I_3(0)$ is given by the integral

$$g(c)\int_{-\infty}^{\infty} e^{-\frac{f''(c)y^2}{2}}dy,$$

[2]to be more precise, we will make sense of path integrals as power series in κ or $\hbar$.

so it is computed from the well known Poisson integral:

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}.$$

The theorem is proved. □

2.2. **Stationary phase formula.** This theorem has the following imaginary analog, called the stationary phase formula.

Theorem 2.2. *Assume that f has a unique critical point $c \in (a,b)$, with $f''(c) \neq 0$, and g vanishes with all derivatives at a, b. Then*

$$\int_a^b g(x) e^{if(x)/\hbar} dx = \hbar^{1/2} e^{if(c)/\hbar} I(\hbar),$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that $I(0) = \sqrt{2\pi} e^{\pm \pi i/4} \frac{g(c)}{|\sqrt{f''(c)}|}$, where $\pm$ is the sign of $f''(c)$.

Remark. It is important to assume that g vanishes with all derivatives at the ends of the the integration interval. Otherwise we will get some additional boundary contributions.

Proof. (sketch). The proof is analogous to the real case, but slightly more subtle. The differences are as follows. First of all, the Poisson integral is replaced with the (conditionally convergent) Fresnel integral

$$\int_{-\infty}^{\infty} e^{\frac{iy^2}{2}} dy = \sqrt{2\pi} e^{\pi i/4}.$$

Further, one should partition g in a sum of two smooth functions, one localized around c on an interval of size $2\hbar^{1/2-\epsilon}$, and the other vanishing near c. Next, one needs to show that only the first summand matters, by using Riemann's lemma: if f has no critical points in the support of g then $\int_a^b g(x) e^{if(x)/\hbar}$ is rapidly decaying (prove this!). Finally, for g localized around c, ones makes the change of variable like in the real case. The statement about existence of Taylor expansion is proved as in the real case, and the value at 0 is calculated using Fresnel integral. □

2.3. **Non-analyticity of $I(\hbar)$ and Borel summation.** It is very important to note that the Taylor series for $I(\hbar)$ is usually not convergent and is only an asymptotic expansion, so that the function I is smooth but not analytic at zero. To illustrate this, consider the integral

$$\int_{-\infty}^{\infty} e^{-\frac{x^2+x^4}{2\hbar}} dx = \sqrt{2\pi}\hbar^{1/2} I(\hbar),$$

where

$$I(\hbar) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2+\hbar y^4}{2}} dy.$$

The latter integral expands asymptotically as

$$I(\hbar) = \sum_{n=0}^{\infty} a_n \hbar^n,$$

where

$$a_n = \frac{(-1)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \frac{y^{4n}}{2^n n!} dy = \frac{(-1)^n}{\sqrt{2\pi}} 2^{n+\frac{1}{2}} \Gamma(2n + \frac{1}{2})/n!.$$

It is clear that this sequence has superexponential growth, so the radius of convergence of the series is zero.

Remark. In fact, the non-analyticity of $I(\hbar)$ is related to the fact that the integral defining $I(\hbar)$ is divergent for $\hbar < 0$.

Let us now discuss the question: to what extent does the asymptotic expansion of the function $I(\hbar)$ (which we can find using Feynman diagrams as explained below) actually determines this function? Suppose that $\tilde{I}(\hbar) = \sum_{n\geq 0} a_n \hbar^n$ is a series with zero radius of convergence. In general, we cannot uniquely determine a function I on $[0, \varepsilon)$ whose expansion is given by such a series. However, assume that a_i are such that the series $g(\hbar) = \sum_{n\geq 0} a_n \hbar^n / n!$ is convergent in some neighborhood of 0, analytically continues to $[0, \infty)$, and has at most exponential growth as $\hbar \to \infty$. In this case there is a "canonical" way to construct a smooth function I on $[0, \varepsilon)$ with (asymptotic) Taylor expansion $\tilde{I}$, called *Borel summation* of $\tilde{I}$. Namely, the function I is defined by the formula

$$I(\hbar) = \int_0^\infty g(\hbar u) e^{-u} du$$

The fact that I has the Taylor expansion $\tilde{I}$ follows from the fact that for $t > 0$ one has

$$\int_0^\infty x^n e^{-x} dx = n!.$$

For example, consider the series $\tilde{I} = \sum_{n\geq 0} (-1)^n n! \hbar^n$. Then $g(\hbar) = \sum_{n\geq 0} (-1)^n \hbar^n = \frac{1}{1+\hbar}$. Hence, the Borel summation yields $I(\hbar) = \int_0^\infty \frac{e^{-u}}{1+\hbar u} du$.

Physicists expect that in many situations perturbation expansions in quantum field theory are Borel summable, and the actual answers are obtained from these expansions by Borel summation. The Borel summability of perturbation series has actually been established in a few nontrivial examples of QFT.

2.4. **Application of steepest descent.** Let us give an application of Theorem 2.1. Consider the integral

$$\int_0^\infty t^s e^{-t} dt, s > 0.$$

It is well known that this integral is equal to the Gamma function $\Gamma(s+1)$. By doing a change of variable $t = sx$, we get

$$\frac{\Gamma(s+1)}{s^{s+1}} = \int_0^\infty x^s e^{-sx} dx = \int_0^\infty e^{-s(x - \log x)} dx.$$

Thus, we can apply Theorem 2.1 for $\hbar = 1/s$, $f(x) = x - \log x$, $g(x) = 1$ (of course, the interval $[a, b]$ is now infinite, and the function f blows up on the boundary, but one can easily see that the theorem is still applicable). The function $f(x) = x - \log x$ has a unique critical point on $[0, \infty)$, which is $c = 1$, and we have $f''(c) = 1$. Then we get

$$\Gamma(s+1) = s^s e^{-s} \sqrt{2\pi s}(1 + a_1/s + a_2/s^2 + \cdots).$$

This is the celebrated Stirling's formula.

2.5. **Multidimensional versions of steepest descent and stationary phase.** Theorems 2.1,2.2 have multidimensional analogs. To formulate them, let V be a real vector space of dimension d with a fixed volume element dx, and let f, g be smooth functions in a closed box $\mathbf{B} \subset V$.

Theorem 2.3. *Assume that f has global minimum on $\mathbf{B}$ at a unique interior point c, such that $f''(c) > 0$. Then*

$$\int_{\mathbf{B}} g(x)e^{-f(x)/\hbar}dx = \hbar^{d/2}e^{-f(c)/\hbar}I(\hbar), \tag{6}$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that $I(0) = (2\pi)^{d/2}\frac{g(c)}{\sqrt{\det f''(c)}}$.

Theorem 2.4. *Assume that f has a unique critical point c in $\mathbf{B}$, such that* $\det f''(c) \neq 0$*, and that g vanishes with all derivatives on the boundary of the box. Then*

$$\int_{\mathbf{B}} g(x)e^{if(x)/\hbar}dx = \hbar^{d/2}e^{if(c)/\hbar}I(\hbar), \tag{7}$$

where $I(\hbar)$ extends to a smooth function on $[0,\infty)$ such that $I(0) = (2\pi)^{d/2}e^{\pi i\sigma/4}\frac{g(c)}{\sqrt{|\det f''(c)|}}$, where σ is the signature of the symmetric bilinear form $f''(c)$.

Remark. In presence of a volume element on V, the determinant of a symmetric bilinear form is well defined.

The proofs of these theorems are parallel to the proofs of their one dimensional versions. Namely, the 1-dimensional Poisson and Fresnel integrals are replaced with their multidimensional versions – the Gaussian integrals

$$\int_V e^{-B(x,x)/2}dx = (2\pi)^{d/2}(\det B)^{-1/2},$$

for a symmetric bilinear form $B > 0$, and

$$\int_V e^{iB(x,x)/2}dx = (2\pi)^{d/2}e^{\pi i\sigma(B)/4}|\det B|^{-1/2},$$

for nondegenerate B. These integral formulas are easily deduced from the one-dimensional ones by diagonalizing the bilinear form B.

3. Feynman calculus

3.1. **Wick's theorem.** Let V be a real vector space of dimension d with volume element dx. Let $S(x)$ be a smooth function on a box $\mathbf{B} \subset V$ which attains a minimum at $x = c \in \mathrm{Interior}(B)$, and g be any smooth function on $\mathbf{B}$. In the last section we have shown that the function

$$I(\hbar) = \hbar^{-d/2}e^{S(c)/\hbar}\int_{\mathbf{B}} g(x)e^{-S(x)/\hbar}dx$$

admits an asymptotic power series expansion in $\hbar$:

$$I(\hbar) = A_0 + A_1\hbar + \cdots + A_m\hbar^m + \cdots \tag{8}$$

Our main question now will be: how to compute the coefficients A_i?

It turns out that although the problem of computing $I(\hbar)$ is transcendental, the problem of computing the coefficients A_i is in fact purely algebraic, and involves only differentiation of the functions S and g at the point c. Indeed, recalling the proof of equation 8 (which we gave in the 1-dimensional case), we see that the calculation of A_i reduces to calculation of integrals of the form

$$\int_V P(x)e^{-B(x,x)/2}dx,$$

where P is a polynomial and B is a positive definite bilinear form (in fact, $B(v,u) = (\partial_v \partial_u S)(c)$). But such integrals can be exactly evaluated. Namely, it is sufficient to consider the case when P is a product of linear functions, in which case the answer is given by the following elementary formula, known to physicists as *Wick's theorem*.

For a positive integer k, consider the set $\{1,\ldots,2k\}$. By a *pairing* σ on this set we will mean its partition into k disjoint two-element subsets (pairs). A pairing can be visualized by drawing $2k$ points and connecting two points with an edge if they belong to the same pair (see Fig. 1). This will give k edges, which are not connected to each other.

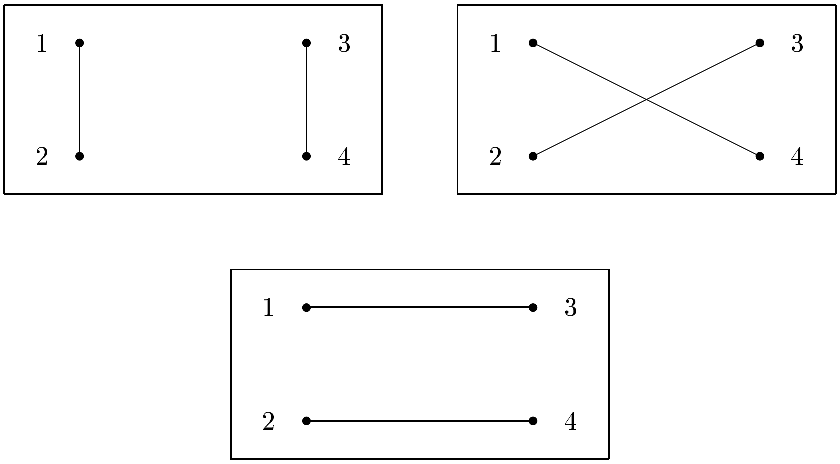


FIGURE 1. Pairings of the set $\{1,2,3,4\}$

Let us denote the set of pairings on $\{1,\ldots,2k\}$ by Π_k. It is clear that $|\Pi_k| = \frac{(2n)!}{2^n \cdot n!}$. For any $\sigma \in \Pi_k$, we can think of σ as a permutation of $\{1,\ldots,2k\}$, such that $\sigma^2 = 1$ and σ has no fixed points. Namely, σ maps any element i to the second element $\sigma(i)$ of the pair containing i.

Theorem 3.1. *Let B^{-1} denote the inverse form on V^*, and let $\ell_1,\ldots,\ell_m \in V^*$. Then, if m is even, we have*

$$\int_V \ell_1(x)\ldots\ell_m(x)e^{-B(x,x)/2}dx = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\sigma\in\Pi_{m/2}} \prod_{i\in\{1,\ldots,m\}/\sigma} B^{-1}(\ell_i,\ell_{\sigma(i)})$$

If m is odd, the integral is zero.

Proof. If m is odd, the statement is obvious, because the integrand is an odd function. So consider the even case. Since both sides of the equation are symmetric polylinear forms in $\ell_1,\ldots,\ell_m$, it suffices to prove the result when $\ell_1 = \cdots = \ell_m = \ell$. Further, it is clear that the formula to be proved is stable under linear changes of variable (check it!), so we can choose a coordinate system in such a way that $B(x,x) = x_1^2 + \cdots + x_d^2$, and $\ell(x) = x_1$. Therefore, it is sufficient to assume that $d = 1$, and $\ell(x) = x$. In this case, the theorem says that

$$\int_{-\infty}^{\infty} x^{2k} e^{-x^2/2} dx = (2\pi)^{1/2} \frac{(2k)!}{2^k k!},$$

which is easily obtained from the definition of the Gamma function by change of variable $y = x^2/2$. □

Examples.

$$\int_V \ell_1(x)\ell_2(x) e^{-B(x,x)/2} dx = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} B^{-1}(\ell_1, \ell_2).$$

$$\int_V \ell_1(x)\ell_2(x)\ell_3(x)\ell_4(x) e^{-B(x,x)/2} dx =$$

$$\frac{(2\pi)^{d/2}}{\sqrt{\det B}} (B^{-1}(\ell_1,\ell_2)B^{-1}(\ell_3,\ell_4) + B^{-1}(\ell_1,\ell_3)B^{-1}(\ell_2,\ell_4) + B^{-1}(\ell_1,\ell_4)B^{-1}(\ell_2,\ell_3)).$$

Wick's theorem shows that the problem of computing A_i is of combinatorial nature. In fact, the central role in this computation is played by certain finite graphs, which are called *Feynman diagrams*. They are the main subject of the remainder of this section.

3.2. **Feynman's diagrams and Feynman's theorem.** We come back to the problem of computing the coefficients A_i. Since each particular A_i depends only on a finite number of derivatives of g at c, it suffices to assume that g is a polynomial, or, more specifically, a product of linear functions: $g = \ell_1 \ldots \ell_N$, $\ell_i \in V^*$. Thus, it suffices to be able to compute the series expansion of the integral

$$< \ell_1 \ldots \ell_N >:= \hbar^{-d/2} e^{S(c)/\hbar} \int_{\mathbf{B}} \ell_1(x) \ldots \ell_N(x) e^{-S(x)/\hbar} dx$$

Without loss of generality we may assume that $c = 0$, and $S(c) = 0$. Then the (asymptotic) Taylor expansion of S at c is $S(x) = \frac{B(x,x)}{2} + \sum_{r\geq 3} \frac{B_r(x,x,\ldots,x)}{r!}$, where $B_r = d^r f(0)$. Therefore, regarding the left hand side as a power series in $\hbar$, and making a change of variable $x \to x/\sqrt{\hbar}$ (like in the last section), we get

$$< \ell_1 \ldots \ell_N >= \hbar^{N/2} \int_V \ell_1(x) \ldots \ell_N(x) e^{-\frac{B(x,x)}{2} - \sum_{r\geq 3} \hbar^{r/2-1} \frac{B_r(x,\ldots,x)}{r!}} dx.$$

(This is an identity of expansions in $\hbar$, as we ignored the rapidly decaying error which comes from replacing the box by the whole space).

The theorem below, due to Feynman, gives the value of this integral in terms of Feynman diagrams. This theorem is easy to prove but is central in quantum field theory, and will be one of the main theorems of this course. Before formulating this theorem, let us introduce some notation.

Let $G_{\geq 3}(N)$ be the set of isomorphism classes of graphs with N 1-valent "external" vertices, labeled by $1, \ldots, N$, and a finite number of unlabeled "internal" vertices, of any valency ≥ 3. Note that here and below graphs are allowed to have multiple edges between two vertices, and loops from a vertex to itself (see Fig. 2).

For each graph $\Gamma \in G_{\geq 3}(N)$, we define the *Feynman amplitude* of Γ as follows.

1. Put the covector ℓ_j at the j-th external vertex.
2. Put the tensor $-B_m$ at each m-valent internal vertex.

3. Take the contraction of the tensors along edges of Γ, using the bilinear form B^{-1}. This will produce a number, called the amplitude of Γ and denoted $F_\Gamma(\ell_1, \dots, \ell_N)$.

Remark. If Γ is not connected, then F_Γ is defined to be the product of numbers obtained from the connected components. Also, the amplitude of the empty diagram is defined to be 1.

Theorem 3.2. *(Feynman) One has*

$$< \ell_1 \dots \ell_N >= \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\Gamma \in G_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $b(\Gamma)$ is is the number of edges minus the number of internal vertices of Γ.

(here by an automorphism of Γ we mean a permutation of vertices AND edges which preserves the graph structure, see Fig. 3; thus there can exist nontrivial automorphisms which act trivially on vertices).

Remark 1. Note that this sum is infinite, but $\hbar$-adically convergent.

Remark 2. We note that Theorem 3.2 is a generalization of Wick's theorem: the latter is obtained if $S(x) = B(x,x)/2$. Indeed, in this case graphs which

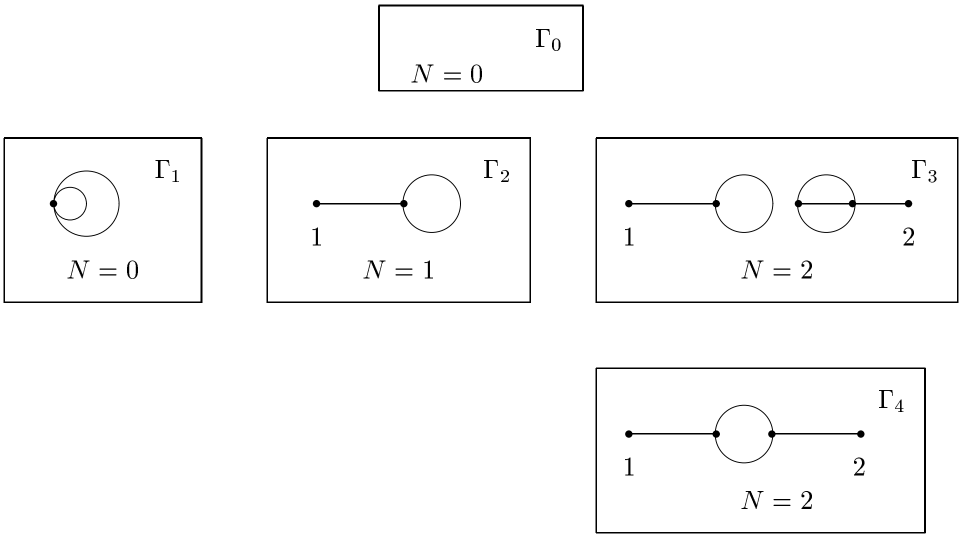


FIGURE 2. Elements of $G_{\geq 3}(N)$

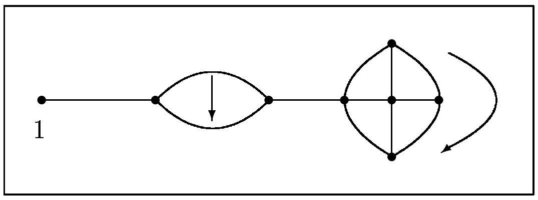


FIGURE 3. An automorphism of a graph

give nonzero amplitudes do not have internal vertices, and thus reduce to graphs corresponding to pairings σ.

Let us now make some comments about the terminology. In quantum field theory, the function $< \ell_1 \dots \ell_N >$ is called the *N-point correlation function*, and graphs Γ are called *Feynman diagrams*. The form B^{-1} which is put on the edges is called *the propagator*. The cubic and higher terms $B_m/m!$ in the expansion of the function S are called *interaction terms*, since such terms (in the action functional) describe interaction between particles. The situation in which S is quadratic (i.e., there is no interaction) is called a *free theory*; i.e. for the free theory the correlation functions are determined by Wick's formula.

Remark 3. Sometimes it is convenient to consider normalized correlation functions $< \ell_1 \dots \ell_N >_{\mathrm{norm}} = < \ell_1 \dots \ell_N > / < \emptyset >$ (where $< \emptyset >$ denotes the integral without insertions). Feynman's theorem implies that they are given by the formula

$$< \ell_1 \dots \ell_N >_{\mathrm{norm}} = \sum_{\Gamma \in G^*_{\geq 3}(N)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \dots, \ell_N),$$

where $G^*_{\geq 3}(N)$ is the set of all graphs in $G_{\geq 3}(N)$ which have no components without extrenal vertices.

3.3. **Another version of Feynman's theorem.** Before proving Theorem 3.2, we would like to slightly modify and generalize it. Namely, in quantum field theory it is often useful to consider an interacting theory as a deformation of a free theory. This means that $S(x) = B(x,x)/2 + \tilde{S}(x)$, where $\tilde{S}(x)$ is the perturbation $\tilde{S}(x) = \sum_{m \geq 0} g_m B_m(x, x, \dots, x)/m!$, where g_m are (formal) parameters. Consider the *partition function*

$$Z = \hbar^{-d/2} \int_V e^{-S(x)/\hbar} dx$$

as a series in g_i and $\hbar$ (this series involves only positive powers of g_i but arbitrary powers of $\hbar$; however, the coefficient of a given monomial $\prod_i g_i^{n_i}$ is a finite sum, and hence contains only finitely many powers of $\hbar$).

Let $\mathbf{n} = (n_0, n_1, \dots)$ be a sequence of nonnegative integers, almost all zero. Let $G(\mathbf{n})$ denote the set of isomorphism classes of graphs with n_0 0-valent vertices, n_1 1-valent vertices, n_2 2-valent vertices, etc. (thus, now we are considering graphs without external vertices).

Theorem 3.3. *One has*

$$Z = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \sum_{\mathbf{n}} \left(\prod_i g_i^{n_i} \right) \sum_{\Gamma \in G(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$

where F_Γ is the amplitude defined as before, and $b(\Gamma)$ is the number of edges minus the number of vertices of Γ.

We will prove Theorem 3.3 in the next subsection. Meanwhile, let us show that Theorem 3.2 is in fact a special case of Theorem 3.3. Indeed, because of symmetry of the correlation functions with respect to $\ell_1, \dots, \ell_N$, it is sufficient to consider the case $\ell_1 = \dots = \ell_N = \ell$. In this case, denote the correlation function $< \ell^N >$ (expectation value of ℓ^N). Clearly, to compute $< \ell^N >$ for all N, it is sufficient to compute the generating function $< e^{t\ell} > := \sum < \ell^N > \frac{t^N}{N!}$. But this expectation value is exactly the one given by Theorem 3.3 for $g_i = 1$, $i \geq 3$, $g_0 = g_2 = 0$,

$g_1 = -\hbar t$, $B_1 = \ell$, $B_0 = 0$, $B_2 = 0$. Thus, Theorem 3.3 implies Theorem 3.2 (note that the factor $N!$ in the denominator is accounted for by the fact that in Theorem 3.3 we consider unlabeled, rather than labeled, 1-valent vertices – convince yourself of this!).

3.4. **Proof of Feynman's theorem.** Now we will prove Theorem 3.3. Let us make a change of variable $y = x/\sqrt{\hbar}$. Expanding the exponential in a Taylor series, we obtain

$$Z = \sum_{\mathbf{n}} Z_{\mathbf{n}},$$

where

$$Z_{\mathbf{n}} = \int_V e^{-B(y,y)/2} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} (-\hbar^{i/2-1} B_i(y, y, \ldots, y))^{n_i} dy$$

Writing B_i as a sum of products of linear functions, and using Wick's theorem, we find that the value of the integral for each $\mathbf{n}$ can be expressed combinatorially as follows.

1. Attach to each factor $-B_i$ a "flower" — a vertex with i outgoing edges (see Fig. 4).

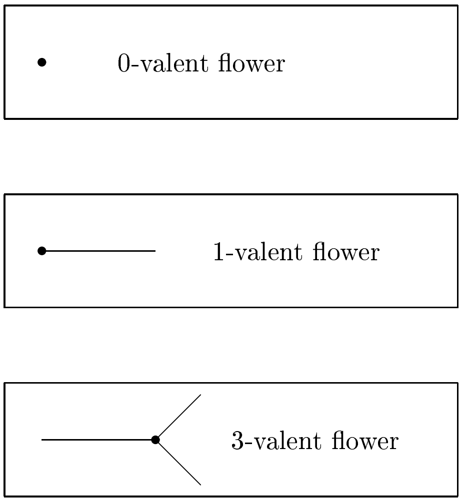


FIGURE 4

2. Consider the set T of ends of these outgoing edges (see Fig. 5), and for any pairing σ of this set, consider the corresponding contraction of tensors $-B_i$ using the form B^{-1}. This will produce a number $F(\sigma)$.

3. The integral $Z_{\mathbf{n}}$ is given by

$$Z_{\mathbf{n}} = \frac{(2\pi)^{d/2}}{\sqrt{\det B}} \prod_i \frac{g_i^{n_i}}{(i!)^{n_i} n_i!} \hbar^{n_i(\frac{i}{2}-1)} \sum_\sigma F_\sigma \tag{9}$$

Now, recall that pairings on a set can be visualized by drawing its elements as points and connecting them with edges. If we do this with the set T, all ends of outgoing edges will become connected with each other in some way, i.e. we will

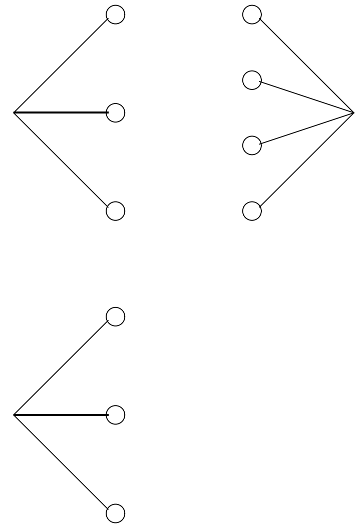


FIGURE 5. The set T for $\vec{n} = (0, 0, 0, 2, 1, 0, 0, \ldots)$ (the set of white circles)

obtain a certain (unoriented) graph $\Gamma = \Gamma_\sigma$ (see Fig. 6). Moreover, it is easy to see that the number $F(\sigma)$ is nothing but the amplitude F_Γ.

It is clear that any graph Γ with n_i i-valent vertices for each i can be obtained in this way. However, the same graph can be obtained many times, so if we want to collect the terms in the sum over σ, and turn it into a sum over Γ, we must find the number of σ which yield a given Γ.

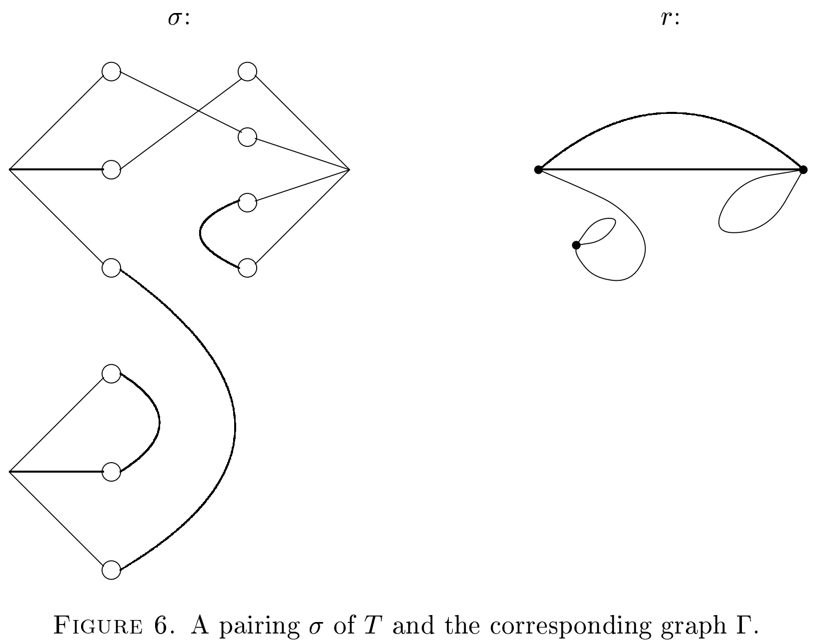


FIGURE 6. A pairing σ of T and the corresponding graph Γ.

For this purpose, we will consider the group G of permutations of T, which preserves "flowers" (i.e. endpoints of any two edges outgoing from the same flower end up again in the same flower). This group involves

1) permutations of "flowers" with a given valency;
2) permutation of the i edges inside each i-valent "flower".

More precisely, the group G is the semidirect product $(\prod_i S_{n_i}) \ltimes (\prod_i S_i^{n_i})$. Note that $|G| = \prod_i (i!)^{n_i} n_i!$, which is the product of the numbers in the denominator of the formula (9).

The group G acts on the set of all pairings σ of T. Moreover, it acts transitively on the set P_Γ of pairings of T which yield a given graph Γ. Moreover, it is easy to see that the stabilizer of a given pairing is $\mathrm{Aut}(\Gamma)$. Thus, the number of pairings giving Γ is

$$\frac{\prod_i (i!)^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|}.$$

Hence,

$$\sum_\sigma F_\sigma = \sum_\Gamma \frac{\prod_i (i!)^{n_i} n_i!}{|\mathrm{Aut}(\Gamma)|} F_\Gamma.$$

Finally, note that the exponent of $\hbar$ in equation (9) is $\sum_i (i/2 - 1)$, which is the number of edges of Γ minus the number of vertices, i.e. $b(\Gamma)$. Substituting this into (9), we get the result.

Example. Let $d = 1$, $V = \mathbb{R}$, $g_i = g$, $B_i = -z^i$ for all i (where z is a formal variable), $\hbar = 1$. Then we find the asymptotic expansion

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + g e^{zx}} = \sum_{n \geq 0} g^n \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|},$$

where $G(n,k)$ is the set of isomorphism classes of graphs with n vertices and k edges. Expanding the left hand side, we get

$$\sum_k \sum_{\Gamma \in G(n,k)} \frac{z^{2k}}{|\mathrm{Aut}(\Gamma)|} = \frac{e^{z^2 n^2/2}}{n!},$$

and hence

$$\sum_{\Gamma \in G(n,k)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{2k}}{2^k k! n!},$$

Exercise. Check this by direct combinatorics.

3.5. **Sum over connected diagrams.** Now we will show that the logarithm of the partition function Z is also given by summation over diagrams, but with only *connected diagrams* taken into account. This significantly simplifies the analysis of Z in the first few orders of perturbation theory, since the number of connected diagrams with a given number of vertices and edges is significantly smaller than the number of all diagrams.

Theorem 3.4. *Let* $Z_0 = \frac{(2\pi)^{d/2}}{\det(B)}$. *Then one has*

$$\ln(Z/Z_0) = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G_c(\mathbf{n})} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$

where $G_c(\mathbf{n})$ *is the set of connected graphs in* $G(\mathbf{n})$.

Remark. We agree that the empty graph is not connected.

Proof. For any graphs Γ_1, Γ_2, let $\Gamma_1\Gamma_2$ stand for the disjoint union of Γ_1 and Γ_2, and for any graph Γ let Γ^n denote the disjoint union of n copies of Γ. Then every graph can be uniquely written as $\Gamma_1^{k_1}\dots\Gamma_l^{k_l}$, where Γ_j are connected non-isomorphic graphs. Moreover, it is clear that $F_{\Gamma_1\Gamma_2} = F_{\Gamma_1}F_{\Gamma_2}$, $b(\Gamma_1\Gamma_2) = b(\Gamma_1) + b(\Gamma_2)$, and $|\mathrm{Aut}(\Gamma_1^{k_1}\dots\Gamma_l^{k_l})| = \prod_j(|(\mathrm{Aut}(\Gamma_j)|^{k_j}k_j!)$. Thus, exponentiating the equation of Theorem 3.4, and using the above facts together with the Taylor series for the function e^x, we arrive at Theorem 3.3. As the Theorem 3.3 has been proved, so is Theorem 3.4 □

3.6. **Loop expansion.** It is very important to note that since summation in Theorem 3.4 is over connected Feynman diagrams, the number $b(\Gamma)$ is the number of loops in Γ minus 1. In particular, the lowest coefficient in $\hbar$ is that of $\hbar^{-1}$, and it is the sum over all trees; the next coefficient is of $\hbar^0$, and it is the sum over all diagrams with one loop (cycle); the next coefficient to $\hbar$ is the sum over two-loop diagrams, and so on. Therefore, physicists refer to the expansion of Theorem 3.4 as *loop expansion.*

Let us study the two most singular terms in this expansion (with respect to $\hbar$), i.e. the terms given by the sum over trees and 1-loop graphs.

Let x_0 be the critical point of the function S. It exists and is unique, since g_i are assumed to be formal parameters. Let $G^{(j)}(\mathbf{n})$ denote the set of classes of graphs in $G_c(\mathbf{n})$ with j loops. Let

$$(\ln(Z/Z_0))_j = \sum_{\mathbf{n}}\prod_i g_i^{n_i}\sum_{\Gamma\in G^{(j)}(\mathbf{n})}\frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|}.$$

Theorem 3.5.

$$(\ln(Z/Z_0))_0 = -S(x_0), \tag{10}$$

and

$$(\ln(Z/Z_0))_1 = \frac{1}{2}\ln\frac{\det(B)}{\det S''(x_0)}. \tag{11}$$

Proof. First note that the statement is purely combinatorial. This means, in particular, that it is sufficient to check that the statement yields the correct asymptotic expansion of the right hand sides of equations (10),(11). in the case when S is a polynomial with real coefficients of the form $B(x,x)/2 + \sum_{i=0}^N g_iB_i(x,x,\dots,x)/i!$. To do this, let $Z = \hbar^{-d/2}\int_{\mathbf{B}} e^{-S(x)/\hbar}$, where $\mathbf{B}$ is a sufficiently small box around 0. For sufficiently small g_i, the function S has a unique global maximum point x_0 in $\mathbf{B}$, which is nondegenerate. Thus, by the steepest descent formula, we have

$$Z/Z_0 = e^{-S(x_0)/\hbar}I(\hbar),$$

where $I(\hbar) = \sqrt{\frac{\det(B)}{\det S''(x_0)}}(1 + a_1\hbar + a_2\hbar^2 + \cdots)$ (asymptotically). Thus,

$$\ln(Z/Z_0) = -S(x_0)\hbar^{-1} + \frac{1}{2}\ln\frac{\det(B)}{\det S''(x_0)} + O(\hbar).$$

This implies the result. □

Physicists call the expression $(\ln(Z/Z_0))_0$ the *classical (or tree) approximation* to the quantum mechanical quantity $\hbar\ln(Z/Z_0)$, and the sum $(\ln(Z/Z_0))_0 + \hbar(\ln(Z/Z_0))_1$ the *one loop approximation.* Similarly one defines higher loop approximations. Note that the classical approximation is obtained by finding the critical point and value of the classical action $S(x)$, which in the mechanics and field theory situation corresponds to solving the classical equations of motion.

3.7. **Nonlinear equations and trees.** As we have noted, Theorem 3.5 does not involve integrals and is purely combinatorial. Therefore, there should exist a purely combinatorial proof of this theorem. Such a proof indeed exists. Here we will give a combinatorial proof of the first statement of the Theorem (formula (10)).

Consider the equation $S'(x) = 0$, defining the critical point x_0. This equation can be written as $x = \beta(x)$, where

$$\beta(x) := -\sum_{i\geq 1} g_i \hat{B}^{-1} B_i(x, x, \ldots, x, ?)/(i-1)!,$$

where $\hat{B}^{-1} : V^* \to V$ is the operator corresponding to the form B^{-1}.

In the sense of power series norm, β is a contracting mapping. Thus, $x_0 = \lim_{N\to\infty} \beta^N(x)$, for any initial vector $x \in V$. In other words, we will obtain x_0 if we keep substituting the series $\beta(x)$ into itself. This leads to summation over trees (explain why!). More precisely, we get the following expression for x_0:

$$x_0 = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\Gamma\in G^{(0)}(\mathbf{n},1)} \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|},$$

where $G^{(0)}(\mathbf{n}, 1)$ is the set of trees with one external vertex and n_i internal vertices of degree i. Now, since $S(x) = B(x, x)/2 + \sum g_i B_i(x, x, \ldots, x)/i!$, the expression $-S(x_0)$ equals the sum of expressions $(\prod g_i^{n_i})\frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|}$ over all trees (without external vertices). Indeed, the term $B(x_0, x_0)/2$ corresponds to gluing two trees with external vertices (identifying the two external vertices, so that they disappear); so it corresponds to summing over trees with a marked edge, i.e. counting each tree as many times as it has edges. On the other hand, the term $g_i B_i(x_0, \ldots, x_0)/i!$ corresponds to gluing i trees with external vertices together at the external vertices (making a tree with a marked vertex). So $\sum g_i B_i(x_0, \ldots, x_0)/i!$ corresponds to summing over trees with a marked vertex, i.e. counting each trees as many times as it has vertices. But the number of vertices of a tree exceeds the number of edges by 1. Thus, the difference $-S(x_0)$ of the above two contributions corresponds to summing over trees, counting each exactly once. This implies formula (10).

3.8. **Counting trees and Cayley's theorem.** In this section we will apply Theorem 3.5 to tree counting problems, in particular will prove a classical theorem due to Cayley that the number of labeled trees with n vertices is n^{n-2}.

We consider essentially the same example as we considered above: $d = 1$, $B_i = -1$, $g_i = g$. Thus, we have $S(x) = \frac{x^2}{2} - ge^x$. By Theorem 3.5, we have

$$\sum_{n\geq 0} g^n \sum_{\Gamma\in T(n)} \frac{1}{|Aut(\Gamma)|} = -S(x_0),$$

where $T(n)$ is the set of isomorphism classes of trees with n vertices, and x_0 is the root of the equation $S'(x) = 0$, i.e. $x = ge^x$.

In other words, let $f(z)$ be the function inverse to xe^{-x} near $x=0$. Then we have $x_0 = f(g)$. Thus, let us find the Taylor expansion of f. This is given by the following classical result.

Proposition 3.6. *One has*

$$f(g) = \sum_{n\geq 1} \frac{n^{n-2}}{(n-1)!} g^n.$$

Proof. Let $f(g) = \sum_{n\geq 1} a_n g^n$. Then

$$a_n = \frac{1}{2\pi i}\oint \frac{f(g)}{g^{n+1}} dg = \frac{1}{2\pi i}\oint \frac{x}{(xe^{-x})^{n+1}} d(xe^{-x}) =$$
$$\frac{1}{2\pi i}\oint e^{nx}\frac{1-x}{x^n}dx = \frac{n^{n-1}}{(n-1)!} - \frac{n^{n-2}}{(n-2)!} = \frac{n^{n-2}}{(n-1)!}.$$

□

Now we find

$$-S(x_0) = -f(g)^2/2 + ge^{f(g)}.$$

Thus

$$-(d/dg)S(x_0) = -f(g)f'(g) + ge^{f(g)}f'(g) + e^{f(g)} = e^{f(g)} = \frac{f(g)}{g}.$$

This means that

$$-S(x_0) = \int_0^g \frac{f(a)}{a} da = \sum_{n\geq 1} \frac{n^{n-2}}{n!} g^n.$$

This shows that

$$\sum_{\Gamma\in T(n)} \frac{1}{|\mathrm{Aut}(\Gamma)|} = \frac{n^{n-2}}{n!}$$

But each isomorphism class of unlabeled trees with n vertices has $\frac{n!}{|\mathrm{Aut}(\Gamma)|}$ nonisomorphic labelings. Thus the latter formula implies

Corollary 3.7. *(A. Cayley) The number of labeled trees with n vertices is n^{n-2}.*

3.9. **Counting trees with conditions.** In a similar way we can count labeled trees with conditions on vertices. For example, let us compute the number of labeled trivalent trees with m vertices (i.e. trees that have only 1-valent and 3-valent vertices). Clearly, $m = 2k$, otherwise there is no such trees. The relevant action functional is $S(x) = \frac{x^2}{2} - g(x + x^3/6)$. Then the critical point x_0 is obtained from the equation $g(x^2/2+1) - x = 0$, which yields $x_0 = \frac{1-\sqrt{1-2g^2}}{g}$. Thus, the tree sum $(\ln(Z/Z_0))_0$ equals

$$(\ln(Z/Z_0))_0 = -S(x_0) = \frac{1-(1-2g^2)^{3/2}}{3g^2} - 1.$$

Expanding this in a Taylor series, we find

$$(\ln(Z/Z_0))_0 = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{(n+2)!} g^{2n+2}$$

Hence, we get

Corollary 3.8. *The number of trivalent labeled trees with $m = 2k$ vertices is* $(2k-3)!!\frac{(2k)!}{(k+1)!}$.

3.10. **Counting oriented trees.** Feynman calculus can be used to count not only non-oriented, but also oriented graphs. For example, suppose we want to count labeled oriented trees, whose vertices are either sources or sinks (see Fig. 7). In this case, it is easy to see (check it!) that the relevant integration problem is in two dimensions, with the action $S = xy - be^x - ae^y$. So the critical point is found from the equations

$$xe^{-y} = a, ye^{-x} = b.$$

Like before, look for a solution $(x, y) = (x_0, y_0)$ in the form

$$x = a + \sum_{p\geq 1, q\geq 1} c_{pq}a^p b^q, y = b + \sum_{p\geq 1, q\geq 1} d_{pq}a^p b^q.$$

A calculation with residues similar to the one we did for unoriented trees yields

$$c_{pq} = \frac{1}{(2\pi i)^2}\oint\oint \frac{x}{a^{p+1}b^{q+1}}da\wedge db = \frac{1}{(2\pi i)^2}\oint\oint \frac{e^{qx+py}}{x^p y^{q+1}}(1-xy)dx\wedge dy = \frac{q^{p-1}p^{q-1}}{(p-1)!q!}.$$

Similarly, $d_{pq} = \frac{q^{p-1}p^{q-1}}{p!(q-1)!}$. Now, similarly to the unoriented case, we find that $-a\partial_a S(x, y) = x$, $-b\partial_b S(x, y) = y$, so

$$-S(x, y) = b + \int_0^a \frac{x}{u}du = a + b + \sum_{p,q\geq 1} \frac{p^{q-1}q^{p-1}}{p!q!}a^p b^q$$

This implies that the number of labeled trees with p sources and q sinks is $p^{q-1}q^{p-1}\frac{(p+q)!}{p!q!}$. In particular, if we specify which vertices are sources and which are sinks, the number of trees is $p^{q-1}q^{p-1}$.

3.11. **1-particle irreducible diagrams and the effective action.** Let $Z = Z_S$ be the partition function corresponding to the action S. In the previous subsections we have seen that the "classical" (or "tree") part $(\ln(Z_S/Z_0))_0$ of the quantity $\ln(Z_S/Z_0)$ is quite elementary to compute – it is just minus the critical value of the action $S(x)$. Thus, if we could find a new "effective" action S_{eff} (a "deformation" of S) such that

$$(\ln(Z_{S_{\text{eff}}}/Z_0))_0 = \ln(Z_S/Z_0)$$

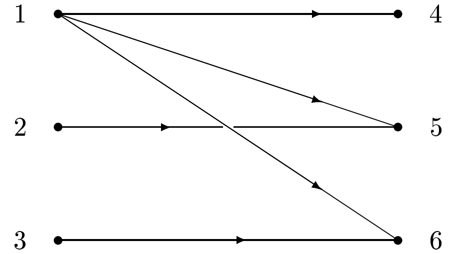


FIGURE 7. A labeled oriented tree with 3 sources and 3 sinks.

(i.e. the classical answer for the effective action is the quantum answer for the original one), then we can regard the quantum theory for the action S as solved. In other words, the problem of solving the quantum theory attached to S (i.e. finding the corresponding integrals) essentially reduces to the problem of computing the effective action S_{eff}.

We will now give a recipe of computing the effective action in terms of amplitudes of Feynman diagrams.

Definition 3.9. An edge e of a connected graph Γ is said to be a bridge, if the graph $\Gamma \setminus e$ is disconnected. A connected graph without bridges is called 1-particle irreducible (1PI).

Remark. This is the physical terminology. The mathematical terminology is "2-connected".

To compute the effective action, we will need to consider graphs with external edges (but having at least one internal vertex). Such a graph Γ (with N external edges) will be called 1-particle irreducible if so is the corresponding "amputated" graph (i.e. the graph obtained from Γ by removal of the external edges). In particular, a graph with one internal vertex is always 1-particle irreducible (see Fig. 8), while a single edge graph without internal vertices is defined *not* to be 1-particle irreducible.

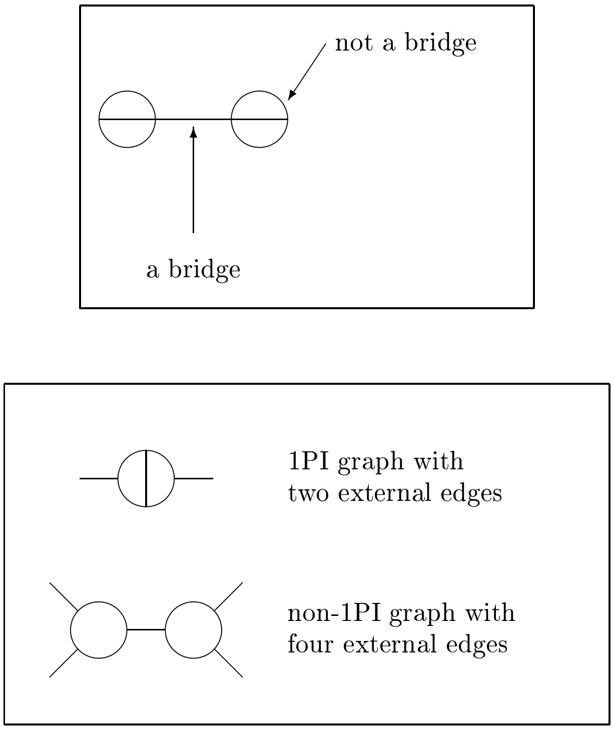


FIGURE 8

Let us denote by $G_{1-\text{irr}}(\mathbf{n}, N)$ the set of isomorphism classes of 1-particle irreducible graphs which N external edges and n_i i-valent internal vertices for each i (where isomorphisms are not allowed to move external edges).

Theorem 3.10. *The effective action S_{eff} is given by the formula*

$$S_{\text{eff}}(x) = \frac{B(x,x)}{2} - \sum_{i \geq 0} \frac{\mathcal{B}_i}{i!},$$

where

$$\mathcal{B}_N(x, x, \ldots, x) = \sum_{\mathbf{n}} (\prod_i g_i^{n_i}) \sum_{\Gamma \in G_{1-\text{irr}}(\mathbf{n}, N)} \frac{\hbar^{b(\Gamma)+1}}{|\text{Aut}(\Gamma)|} F_\Gamma(x_*, x_*, \ldots, x_*),$$

where $x_* \in V^*$ is defined by $x_*(y) := B(x, y)$

Thus, $S_{\text{eff}} = S + \hbar S_1 + \hbar^2 S_2 + ..$ The expressions $\hbar^j S_j$ are called the j-loop corrections to the effective action.

This theorem allows physicists to worry only about 1-particle irreducible diagrams, and is the reason why you will rarely see other diagrams in a QFT textbook. As before, it is very useful in doing low order computations, since the number of 1-particle irreducible diagrams with a given number of loops is much smaller than the number of connected diagrams with the same number of loops.

Proof. The proof is based on the following lemma from graph theory.

Lemma 3.11. *Any connected graph Γ can be uniquely represented as a tree, whose vertices are 1-particle irreducible subgraphs (with external edges), and edges are the bridges of Γ.*

The lemma is obvious. Namely, let us remove all bridges from Γ. Then Γ will turn into a union of 1-particle irreducible graphs, which should be taken to be the vertices of the said tree.

The tree corresponding to the graph Γ is called the skeleton of Γ (see Fig. 9).

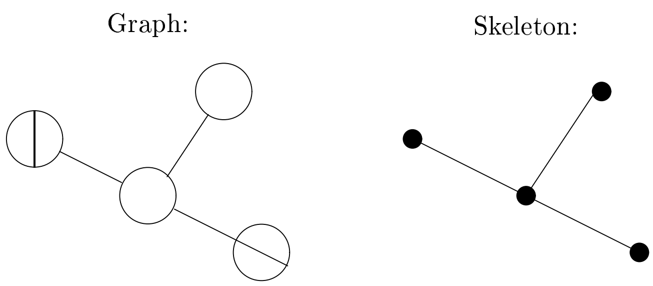


FIGURE 9. The skeleton of a graph.

It is easy to see that the lemma implies the theorem. Indeed, it implies that the sum over all connected graphs occuring in the expression of $\ln(Z_S/Z_0)$ can be written as a sum over skeleton trees, so that the contribution from each tree is (proportional to) the contraction of tensors $\mathcal{B}_i$ put in its vertices, and $\mathcal{B}_i$ is the (weighted) sum of amplitudes of all 1-particle irreducible graphs with i external edges. $\square$

3.12. **1-particle irreducible graphs and the Legendre transform.** Recall the notion of Legendre transform.

Let f be a smooth function on a vector space Y, such that the map $Y \to Y^*$ given by $x \to df(x)$ is a diffeomorphism. Then one can define the Legendre transform of f as follows. For $p \in Y^*$, let $x_0(p)$ be the critical point of the function $(p,x) - f(x)$ (i.e. the unique solution of the equation $df(x) = p$). Then the Legenedre transform of f is the function on Y^* defined by

$$L(f)(p) = (p, x_0) - f(x_0).$$

It is easy to see that the differential of $L(f)$ is also a diffeomorphism $Y^* \to Y$ (in fact, inverse to $df(x)$), and that $L^2(f) = f$.

Example. Let $f(x) = ax^2/2$. Then $px - f = px - x^2/2$ has a critical point at $p = x/a$, and the critical value is $p^2/2a$. $L(ax^2/2) = p^2/2a$. Similarly, if $f(x) = B(x,x)/2$ where B is a nondegenerate symmetric form, then $L(f)(p) = B^{-1}(p,p)/2$.

Now let us consider Theorem 3.10 in the situation of Theorem 3.2. Thus, $S(x) = B(x,x)/2 + O(x^3)$, and we look at

$$Z(J) = \hbar^{-d/2} \int_V e^{\frac{J \cdot x - S(x)}{\hbar}} dx$$

By Theorem 3.10, one has

$$\ln(Z(J)/Z_0) = -S_{\mathrm{eff}}(x_0, J),$$

where the effective action $S_{\mathrm{eff}}(x, J)$ given by summation over 1-particle irreducible graphs.

Now, we must have $S_{\mathrm{eff}}(x, J) = -J \cdot x + S_{\mathrm{eff}}(x)$, since the only 1PI graph which contains 1-valent internal vertices (corresponding to J) is the graph with one edge, connecting an internal vertex with an external one (so it yields the term $-J \cdot x$, and other graphs contain no J-vertices). This shows that $\ln(Z(J)/Z_0)$ is the critical value of $J \cdot x - S_{\mathrm{eff}}(x)$. Thus we have proved the following.

Proposition 3.12. *We have*

$$S_{\mathrm{eff}}(x) = L(\ln(Z(J)/Z_0)), \ \ln(Z(J)/Z_0) = L(S_{\mathrm{eff}}(x)).$$

Physicists formulate this result as follows: the effective action is the Legendre transform of the logarithm of the generating function for quantum correlators (and vice versa).

4. Matrix integrals

Let $\mathfrak{h}_N$ be the space of Hermitian matrices of size N. The inner product on $\mathfrak{h}_N$ is given by $(A, B) = \mathrm{Tr}(AB)$. In this section we will consider integrals of the form

$$Z_N = \hbar^{-N^2/2} \int_{\mathfrak{h}_N} e^{-S(A)/\hbar} dA,$$

where the Lebesgue measure dA is normalized by the condition $\int e^{-Tr(A^2)/2} dA = 1$, and $S(A) = \mathrm{Tr}(A^2)/2 - \sum_{m \geq 0} g_m \mathrm{Tr}(A^m)/m$ is the action functional.[3] We will be interested the behavior of the coefficients of the expansion of Z_N in g_i for large N. The study of this behavior will lead us to considering not simply Feynman graphs, but actually fat (or ribbon) graphs, which are in fact 2-dimensional surfaces.

[3]Note that we divide by m and not by $m!$. We will see below why such normalization will be more convenient.

Thus, before we proceed further, we need to do some 2-dimensional combinatorial topology.

4.1. **Fat graphs.** Recall from the proof of Feynman's theorem that given a finite collection of flowers and a pairing σ on the set T of endpoints of their edges, we can obtain a graph Γ_σ by connecting (or gluing) the points which fall in the same pair.

Now, given an i-flower, let us inscribe it in a closed disk D (so that the ends of the edges are on the boundary) and take its small tubular neighborhood in D. This produces a region with piecewise smooth boundary. We will equip this region with an orientation, and call it a *fat i-valent flower*. The boundary of a fat i-valent flower has the form $P_1Q_1P_2Q_2\dots P_iQ_iP_1$, where P_i, Q_i are the angle points, the intervals P_jQ_j are arcs on ∂D, and Q_jP_{j+1} are (smooth) arcs lying inside D (see Fig. 10).

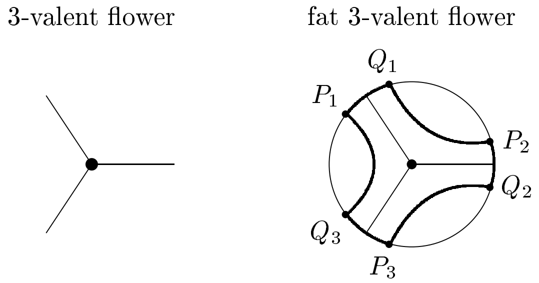


FIGURE 10

Now, given a collection of usual flowers and a pairing σ as above, we can consider the corresponding fat flowers, and glue them (respecting the orientation) along intervals P_jQ_j according to σ. This will produce a compact oriented surface with boundary (the boundary is glued from intervals P_jQ_{j+1}).

We will denote this surface by $\tilde{\Gamma}_\sigma$, and call it *the fattening* of Γ with respect to σ. A fattening of a graph will be called a *fat (or ribbon) graph*. Thus, a fat graph is not just an oriented surface with boundary, but such a surface together with a partition of this surface into fat flowers.

Note that the same graph Γ can have many different fattenings, and in particular the genus g of the fattening is *not* determined by Γ (see Fig. 11).

4.2. **Matrix integrals in large N limit and planar graphs.** Let us now return to the study of the integral Z_N. By the proof of Feynman's theorem,

$$\ln Z_N = \sum_{\mathbf{n}}(\prod \frac{g_i^{n_i}\hbar^{n_i(i/2-1)}}{i^{n_i}n_i!})\sum_\sigma F_\sigma,$$

where the summation is taken over all pairings of $T = T(\mathbf{n})$ that produce a connected graph Γ_σ, and F_σ denotes the contraction of the tensors $\mathrm{Tr}(A^i)$ using σ.

For a surface Σ with boundary, let $\nu(\Sigma)$ denote the number of connected components of the boundary.

Proposition 4.1. $F_\sigma = N^{\nu(\tilde{\Gamma}_\sigma)}$.

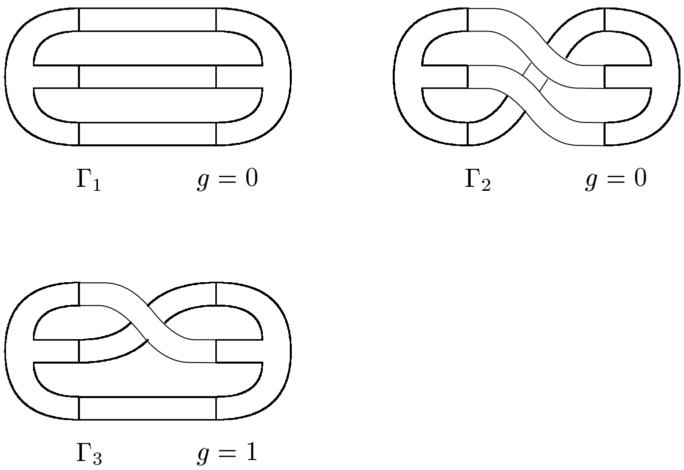
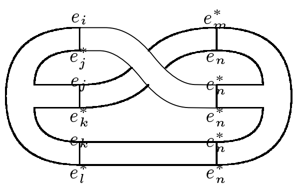


FIGURE 11. Gluing a fat graph from fat flowers

Proof. Let e_i be the standard basis of $\mathbb{C}^N$, and e_i^* the dual basis. Then the tensor $\mathrm{Tr}(A^m)$ can be written as

$$\mathrm{Tr}(A^m) = \sum_{i_1,\ldots,i_m=1}^{N} (e_{i_1} \otimes e_{i_2}^* \otimes e_{i_2} \otimes e_{i_3}^* \otimes \cdots \otimes e_{i_m} \otimes e_{i_1}^*, A^{\otimes m}).$$

One can visualize each monomial in this sum as a labeling of the angle points $P_1, Q_1, \ldots, P_m, Q_m$ on the boundary of a fat m-valent flower by $i_1, i_2, i_2, i_3, \ldots, i_m, i_1$. Now, the contraction using σ of some set of such monomials is nonzero iff the subscript is constant along each boundary component of $\widetilde{\Gamma}_\sigma$ (see Fig. 12). This implies the result. □



Contraction nonzero iff
$i = r,\ j = p,\ j = m,\ k = r,$
$k = p,\ i = m,$
that is
$i = r = k = p = j = m.$

FIGURE 12. Contraction defined by a fat graph.

Let $\widetilde{G}_c(\mathbf{n})$ is the set of isomorphism classes of connected fat graphs with n_i i-valent vertices. For $\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})$, let $b(\widetilde{\Gamma})$ be the number of edges minus the number of vertices of the underlying usual graph Γ.

Corollary 4.2.

$$\ln Z_N = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\widetilde{\Gamma} \in \widetilde{G}_c(\mathbf{n})} \frac{N^{\nu(\widetilde{\Gamma})} \hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Proof. Let $G_{\rm fat}(\mathbf{n}) = \prod S_{n_i} \ltimes (\prod \mathbb{Z}/i\mathbb{Z})^{n_i}$. This group acts on T, so that $\widetilde{\Gamma}_\sigma = \widetilde{\Gamma}_{g\sigma}$, for any $g \in G_{\rm fat}$ (since cyclic permutations of edges of a flower extend to its fattening). Moreover, the group acts transitively on the set of σ giving a fixed fat graph $\widetilde{\Gamma}_\sigma$, and the stabilizer of any σ is $\mathrm{Aut}(\widetilde{\Gamma}_\sigma)$. This implies the result. $\square$

Now for any compact surface Σ with boundary, let $\mathrm{g}(\Sigma)$ be the genus of Σ. Then for a connected fat graph $\widetilde{\Gamma}$, $b(\widetilde{\Gamma}) = 2\mathrm{g}(\widetilde{\Gamma})-2+\nu(\widetilde{\Gamma})$ (minus the Euler characteristic). Thus, defining $\hat{Z}_N(\hbar) = Z_N(\hbar/N)$, we find

Theorem 4.3.

$$\ln \hat{Z}_N = \sum_{\mathbf{n}}(\prod g_i^{n_i}) \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})}\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

This implies the following important result, due to t'Hooft.

Theorem 4.4. *(1) There exists a limit $W_\infty := \lim_{N\to\infty} \frac{\ln \hat{Z}_N}{N^2}$. This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}}(\prod g_i^{n_i}) \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[0]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c(\mathbf{n})[0]$ denotes the set of **planar** *connected fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion $\ln \hat{Z}_N/N^2 = \sum_{\mathrm{g}\ge 0} a_{\mathrm{g}} N^{-2\mathrm{g}}$, where

$$a_{\mathrm{g}} = \sum_{\mathbf{n}}(\prod g_i^{n_i}) \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[\mathrm{g}]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c(\mathbf{n})[\mathrm{g}]$ denotes the set of connected fat graphs which have genus g.

Remark 1. Genus zero fat graphs are said to be planar because the underlying usual graphs can be put on the 2-sphere (and hence on the plane) without self-intersections.

Remark 2. t'Hooft's theorem may be interpreted in terms of the usual Feynman diagram expansion. Namely, it implies that for large N, the leading contribution to $\ln(Z_N(\hbar/N))$ comes from the terms in the Feynman diagram expansion corresponding to planar graphs (i.e. those that admit an embedding into the 2-sphere).

4.3. **Integration over real symmetric matrices.** One may also consider the matrix integral over the space $\mathfrak{s}_N$ of real symmetric matrices of size N. Namely, one puts

$$Z_N = \hbar^{-N(N+1)/4} \int_{\mathfrak{s}_N} e^{-S(A)/\hbar} dA,$$

where S and dA are as above. Let us generalize Theorem 4.4 to this case.

As before, consideration of the large N limit leads to consideration of fat flowers and gluing of them. However, the exact nature of gluing is now somewhat different. Namely, in the Hermitian case we had $(e_i \otimes e_j^*, e_k \otimes e_l^*) = \delta_{il}\delta_{jk}$, which forced us to glue fat flowers preserving orientation. On the other hand, in the real symmetric case $e_i^* = e_i$, and the inner product of the functionals $e_i \otimes e_j$ on the space of symmetric matrices is given by $(e_i \otimes e_j, e_k \otimes e_l) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$. This means that besides the usual (orientation preserving) gluing of fat flowers, we now must allow gluing with a twist of the ribbon by 180^o. Fat graphs thus obtained will be called *twisted fat graphs*. That means, a twisted fat graph is a surface with boundary (possibly not orientable), together with a partition into fat flowers, and orientations on each of them (which may or may not match at the cuts, see Fig.13).

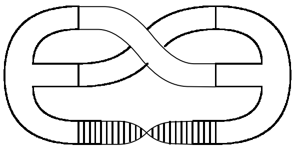


FIGURE 13. Twisted-fat graph

Now one can show analogously to the Hermitian case that the $1/N$ expansion of $\ln \hat{Z}_N$ (where $\hat{Z}_N = Z_N(2\hbar/N)$) is given by the same formula as before, but with summation over the set $\widetilde{G}_c^{\text{tw}}(\mathbf{n})$ of twisted fat graphs:

Theorem 4.5.

$$\ln \hat{Z}_N = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\text{tw}}(\mathbf{n})} \frac{N^{2-2\mathrm{g}(\widetilde{\Gamma})} \hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}.$$

Here the genus g of a (possibly non-orientable) surface is defined by $\mathrm{g} = 1 - \chi/2$, where χ is the Euler characteristic. Thus the genus of $\mathbb{R}P^2$ is $1/2$, the genus of the Klein bottle is 1, and so on.

In particular, we have the analog of t'Hooft's theorem.

Theorem 4.6. *(1) There exists a limit* $W_\infty := \lim_{N\to\infty} \frac{\ln \hat{Z}_N}{N^2}$. *This limit is given by the formula*

$$W_\infty = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\text{tw}}(\mathbf{n})[0]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

where $\widetilde{G}_c^{\text{tw}}(\mathbf{n})[0]$ *denotes the set of* **planar** *connected twisted fat graphs, i.e. those which have genus zero.*

(2) Moreover, there exists an expansion $\ln \hat{Z}_N/N^2 = \sum_{\mathrm{g}\geq 0} a_{\mathrm{g}} N^{-2\mathrm{g}}$, *where*

$$a_{\mathrm{g}} = \sum_{\mathbf{n}} (\prod g_i^{n_i}) \sum_{\widetilde{\Gamma} \in \widetilde{G}_c^{\text{tw}}(\mathbf{n})[\mathrm{g}]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|},$$

and $\widetilde{G}_c^{\text{tw}}(\mathbf{n})[\mathrm{g}]$ *denotes the set of connected twisted fat graphs which have genus* g.

Exercise. Consider the matrix integral over the space$\mathfrak{q}_N$ of quaternionic Hermitian matrices. Show that in this case the results are the same as in the real case, except that each twisted fat graph counts with a sign, equal to $(-1)^m$, where m is the number of twistings (i.e. mismatches of orientation at cuts). In other words, $\ln \hat{Z}_N$ for quaternionic matrices is equal $\ln \hat{Z}_{2N}$ for real matrices with N replaced by $-N$.

Hint: use that the unitary group $U(N, \mathbb{H})$ is a real form of $Sp(2N)$, and $\mathfrak{q}_N$ is a real form of the representation of $\Lambda^2 V$, where V is the standard (vector) representation of $Sp(2N)$. Compare to the case of real symmetric matrices, where the relevant representation is S^2V for $O(N)$, and the case of complex Hermitian matrices, where it is $V \otimes V^*$ for $GL(N)$.

4.4. **Application to a counting problem.** Matrix integrals are so rich that even the simplest possible example reduces to a nontrivial counting problem. Namely, consider the matrix integral Z_N over complex Hermitian matrices in the case $S(A) = \mathrm{Tr}(A^2)/2 - s\mathrm{Tr}(A^{2m})/2m$, where $s^2 = 0$ (i.e. we work over the ring $\mathbb{C}[s]/(s^2)$). In this case we can set $\hbar = 1$. Then from Theorem 4.4 we get

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m}) e^{-\mathrm{Tr}(A^2)/2} dA = P_m(N),$$

where $P_m(N)$ is a polynomial, given by the formula $P_m(N) = \sum_{\mathrm{g} \geq 0} \varepsilon_{\mathrm{g}}(m) N^{m+1-2\mathrm{g}}$, and $\varepsilon_{\mathrm{g}}(m)$ is the number of ways to glue a surface of genus g from a $2m$-gon with labeled sides by gluing sides preserving the orientation. Indeed, in this case we have only one fat flower of valency $2m$, which has to be glued with itself; so a direct application of our Feynman rules leads to counting ways to glue a surface of a given genus from a polygon.

The value of this integral is given by the following non-trivial theorem.

Theorem 4.7. *(Harer-Zagier, 1986)*

$$P_m(x) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} \binom{m}{p} 2^p \frac{x(x-1)\dots(x-p)}{(p+1)!}.$$

The theorem is proved in the next subsections.

Looking at the leading coefficient of P_m, we get.

Corollary 4.8. *The number of ways to glue a sphere of a $2m$-gon is the Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$.*

Corollary 4.8 actually has another (elementary combinatorial) proof, which is as follows. For each pairing σ on the set of sides of the $2m$-gon, let us connect the midpoints of the sides that are paired by straight lines (Fig.14). It is geometrically evident that if these lines don't intersect then the gluing will give a sphere. We claim that the converse is true as well. Indeed, we can assume that the statement is known for the $2m-2$-gon. Let σ be a gluing of the $2m$-gon that gives a sphere. If there is a connection between two adjacent sides, we may glue them and go from a $2m$-gon to a $2m-2$-gon (Fig. 15). Thus, it is sufficient to consider the case when adjacent sides are never connected. Then there exist adjacent sides a and b whose lines (connecting them to some c, d) intersect with each other. Let us now replace σ by another pairing σ', whose only difference from σ is that a is connected to b and c to d (Fig. 16). One sees by inspection (check it!) that this does not decrease the number of boundary components of the resulting surface. Therefore, since σ gives a sphere, so does σ'. But σ' has adjacent sides connected, the case considered before, hence the claim.

Now it remains to count the number of ways to connect midpoints of sides with lines without intersections. Suppose we draw one such line, such that the number of sides on the left of it is $2k$ and on the right is $2l$ (so that $k+l = m-1$). Then we

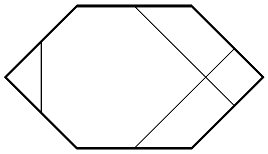


FIGURE 14. Pairing of sides of a 6-gon.

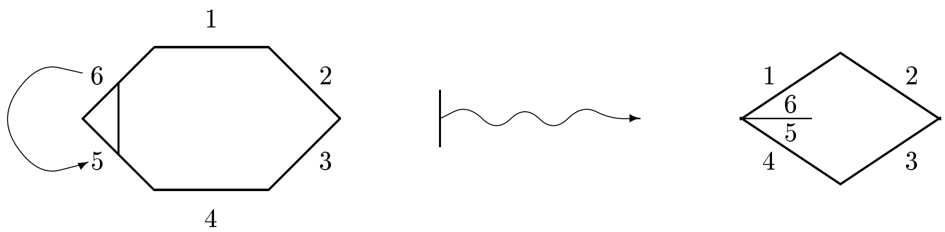


FIGURE 15

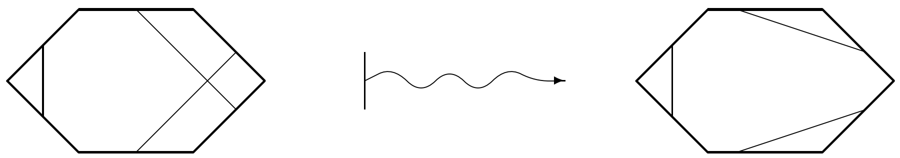


FIGURE 16

face the problem of connecting the two sets of $2k$ and $2l$ sides without intersections. This shows that the number of gluings D_m satisfies the recursion

$$D_m = \sum_{k+l=m-1} D_k D_l$$

In other words, the generating function $\sum D_m x^m = 1+x+\cdots$ satisfies the equation $f-1=xf^2$. This implies that $f=\frac{1-\sqrt{1-4x}}{2x}$, which yields that $D_m = C_m$. We are done.

Corollary 4.8 can be used to derive the the following fundamental result from the theory of random matrices, discovered by Wigner in 1955.

Theorem 4.9. *(Wigner's semicircle law) Let f be a continuous function on $\mathbb{R}$ of at most polynomial growth at infinity. Then*

$$\lim_{N\to\infty} \frac{1}{N}\int_{\mathfrak{h}_N} Trf(A/\sqrt{N})e^{-Tr(A^2)/2}\frac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4-x^2}dx.$$

This theorem is called the semicircle law because it says that the graph of the density of eigenvalues of a large random Hermitian matrix distributed according to 'the "Gaussian unitary ensemble" (i.e. with density $e^{-\mathrm{Tr}(A^2)/2}dA$) is a semicircle.

Proof. By Weierstrass uniform approximation theorem, we may assume that f is a polynomial. (Exercise: Justify this step).

Thus, it suffices to check the result if $f(x) = x^{2m}$. In this case, by Corollary 4.8, the left hand side is C_m. On the other hand, an elementary computation yields $\frac{1}{2\pi}\int_{-2}^{2} x^{2m}\sqrt{4-x^2} = C_m$, which implies the theorem. $\square$

4.5. **Hermite polynomials.** The proof [4] of Theorem 4.7 given below uses Hermite polynomials. So let us recall their properties.

Hermite's polynomials are defined by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

So the leading term of $H_n(x)$ is $(2x)^n$.

We collect the standard properties of $H_n(x)$ in the following theorem.

Theorem 4.10. *(i) The generating function of $H_n(x)$ is $f(x,t) = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!}$ $= e^{2xt-t^2}$.*

(ii) $H_n(x)$ satisfy the differential equation $f'' - 2xf' + 2nf = 0$. In other words, $H_n(x)e^{-x^2/2}$ are eigenfunctions of the operator $L = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2$ (Hamiltonian of the quantum harmonic oscillator) with eigenvalues $n + \frac{1}{2}$.

(iii) $H_n(x)$ are orthogonal:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \delta_{mn}$$

(iv) One has

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2} x^{2m} H_{2k}(x) dx = \frac{(2m)!}{(m-k)!} 2^{2(k-m)}$$

(if $k > m$, the answer is zero).

(v) One has

$$\frac{H_r^2(x)}{2^r r!} = \sum_{k=0}^{r} \frac{r!}{2^k k!^2 (r-k)!} H_{2k}(x).$$

Proof. (sketch)

(i) Follows immediately from the fact that the operator $\sum (-1)^n \frac{t^n}{n!}\frac{d^n}{dx^n}$ maps a function $g(x)$ to $g(x-t)$.

(ii) Follows from (i) and the fact that the function $f(x,t)$ satisfies the PDE $f_{xx} - 2xf_x + 2tf_t = 0$.

(iii) Follows from (i) by direct integration (one should compute $\int_{\mathbb{R}} f(x,t)f(x,u)e^{-x^2}dx$ using a shift of coordinate).

(iv) By (i), one should calculate $\int_{\mathbb{R}} x^{2m} e^{2xt-t^2} e^{-x^2} dx$. This integral equals

$$\int_{\mathbb{R}} x^{2m} e^{-(x-t)^2} dx = \int_{\mathbb{R}} (y+t)^{2m} e^{-y^2} dy == \sqrt{\pi}\sum_p \binom{2m}{2p} \frac{(2m-2p)!}{2^{m-p}(m-p)!} t^{2p}.$$

The result is now obtained by extracting individual coefficients.

(v) By (iii), it suffices to show that

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} H_r^2(x) H_{2k}(x) e^{-x^2} dx = \frac{2^{r+k} r!^2 (2k)!}{k!^2 (r-k)!}$$

[4] I adopted this proof from D.Jackson's notes

To prove this identity, let us integrate the product of three generating functions. By (i), we have

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} f(x,t)f(x,u)f(x,v)e^{-x^2}dx = e^{2(tu+tv+uv)}.$$

Extracting the coefficient of $t^r u^r v^{2k}$, we get the result. □

4.6. **Proof of Theorem 4.7.** We need to compute the integral

$$\int_{\mathfrak{h}_N} \mathrm{Tr}(A^{2m})e^{-\mathrm{Tr}(A^2)/2}dA.$$

To do this, we note that the integrand is invariant with respect to conjugation by unitary matrices. Therefore, the integral can be reduced to an integral over the eigenvalues $\lambda_1,\dots,\lambda_N$ of A.

More precisely, consider the spectrum map $\sigma : \mathfrak{h}_N \to \mathbb{R}^N/S_N$. It is well known (due to H.Weyl) that the direct image $\sigma_* dA$ is given by the formula $\sigma_* dA = Ce^{-\sum_i \lambda_i^2/2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda$, where $C>0$ is a normalization constant that will not be relevant to us. Thus, we have

$$P_m(N) = \frac{\int_{\mathbb{R}^N}(\sum_i \lambda_i^{2m})e^{-\sum\lambda_i^2/2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda}{\int_{\mathbb{R}^N} e^{-\sum\lambda_i^2/2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda}$$

To calculate the integral J_m in the numerator, we will use Hermite's polynomials. Observe that since $H_n(x)$ are polynomials of degree n with highest coefficient 2^n, we have $\prod_{i<j}(\lambda_i-\lambda_j) = 2^{-N(N-1)/2}\det(H_k(\lambda_i))$, where k runs through the set $0,1,\dots,n-1$. Thus, we find

$$\begin{aligned} J_m := \int_{\mathbb{R}^N}(\sum_i \lambda_i^{2m})e^{-\sum\lambda_i^2/2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda = \\ 2^{m+N^2/2}N\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum\lambda_i^2}\prod_{i<j}(\lambda_i-\lambda_j)^2 d\lambda = \\ 2^{m-N(N-2)/2}N\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum\lambda_i^2}\det(H_k(\lambda_j))^2 d\lambda = \\ 2^{m-N(N-2)/2}N\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum\lambda_i^2}(\sum_{\sigma,\tau\in S_N}(-1)^\sigma(-1)^\tau\prod_i H_{\sigma i}(\lambda_i)H_{\tau i}(\lambda_i))d\lambda. \end{aligned} \tag{12}$$

(Here $(-1)^\sigma$ denotes the sign of σ).

Since Hermite polynomials are ortogonal, the only terms which are nonzero are the terms with $\sigma(i)=\tau(i)$ for $i=2,\dots,N$. That is, the nonzero terms have $\sigma=\tau$. Thus, we have

$$\begin{aligned} J_m = 2^{m-N(N-2)/2}N\int_{\mathbb{R}^N}\lambda_1^{2m}e^{-\sum\lambda_i^2}(\sum_{\sigma\in S_N}\prod_i H_{\sigma i}(\lambda_i)^2 d\lambda = \\ 2^{m-N(N-2)/2}N!\gamma_0\dots\gamma_{N-1}\sum_{j=0}^{N-1}\frac{1}{\gamma_j}\int_{-\infty}^{\infty}x^{2m}H_j(x)^2e^{-x^2}dx, \end{aligned} \tag{13}$$

where $\gamma_i = \int_{-\infty}^{\infty} H_i(x)^2 e^{-x^2} dx$ are the squared norms of the Hermite polynomials. Applying this for $m = 0$ and dividing J_m by J_0, we find

$$J_m/J_0 = 2^m \sum_{j=0}^{N-1} \frac{1}{\gamma_j} \int_{-\infty}^{\infty} x^{2m} H_j(x)^2 e^{-x^2} dx$$

Using Theorem 4.10 (iii) and (v), we find: $\gamma_i = 2^i i! \sqrt{\pi}$, and hence

$$J_m/J_0 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^m x^{2m} H_{2k}(x)}{2^k k!^2 (j-k)!} e^{-x^2} dx$$

Now, using (iv), we get

$$J_m/J_0 = \frac{(2m)!}{2^m} \sum_{j=0}^{N-1} \sum_{k=0}^{j} \frac{2^k j!}{(m-k)! k!^2 (j-k)!} =$$

$$\frac{(2m)!}{2^m m!} \sum_{j=0}^{N-1} \sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k}.$$

The sum over k can be represented as a constant term of a polynomial:

$$\sum_{k=0}^{j} 2^k \binom{m}{k} \binom{j}{k} = C.T.((1+z)^m (1+2z^{-1})^k).$$

Therefore, summation over j (using the formula for the sum of the geometric progression) yields

$$J_m/J_0 = \frac{(2m)!}{2^m m!} C.T.((1+z)^m \frac{(1+2z^{-1})^N - 1}{2z^{-1}}) = \frac{(2m)!}{2^m m!} \sum_{p=0}^{m} 2^p \binom{m}{p} \binom{N}{p+1}$$

We are done.

5. The Euler characteristic of the moduli space of curves

Matrix integrals (in particular, computation of the polynomial $P_m(x)$) can be used to calculate the orbifold Euler characteristic of the moduli space of curves. This was done by Harer and Zagier in 1986. Here we will give a review of this result (with some omissions).

5.1. **Euler charactristics of groups.** We start with recalling some basic notions from algebraic topology.

Let Γ be a discrete group, and Y be a contractible finite dimensional CW complex, on which Γ acts cellularly. This means that Γ acts by homeomorphisms of Y that map each cell homeomorphically to another cell. We will assume that the stabilizer of each cell is a finite group (i.e. Y is a proper Γ-complex).

Suppose first that the action of Γ is free (i.e. the stabilizers of cells are trivial). This is equivalent to saying that Γ is torsion free (i.e has no nontrivial finite subgroups), since a finite group cannot act without fixed points on a contractible finite dimensional cell complex (as it has infinite cohomological dimension).

In this case we can define a cell complex Y/Γ (a classifying space for Γ), and we have $H^i(Y/\Gamma, A) = H^i(\Gamma, A)$ for any coefficient group A. In particular, if Y/Γ is finite then Γ has finite cohomological dimension, and the Euler characteristic

$\chi(\Gamma) := \sum(-1)^i \dim H^i(\Gamma, \mathbb{Q})$ is equal to $\sum(-1)^i n_i(Y/\Gamma)$, where $n_i(Y/\Gamma)$ denotes the number of cells in Y/Γ of dimension i.

This setting, however, is very restrictive, since it allows only groups of finite cohomological dimension, and in particular excludes all non-trivial finite groups. So let us consider a more general setting: assume that some finite index subgroup $\Gamma' \subset \Gamma$, rather than Γ itself, satisfies the above conditions. In this case, on may define the Euler characteristic of Γ in the sense of Wall, which is the rational number $[\Gamma : \Gamma']^{-1}\chi(\Gamma')$.

It is easy to check that the Euler characteristic in the sense of Wall can be computed using the following *Quillen's formula*

$$\chi(\Gamma) = \sum_{\sigma \in \mathrm{cells(Y)}/\Gamma} \frac{(-1)^{\dim \sigma}}{|\mathrm{Stab}\sigma|}$$

In particular, this number is independent of Γ' (which is also easy to check directly).

Example 1. If G is a finite group then $\chi(G) = |G|^{-1}$ (one takes the trivial group as the subgroup of finite index).

Example 2. $G = SL_2(\mathbb{Z})$. This group contains a subgroup F of index 12, which is free in two generators (check it!). The group F has Euler characteristic -1, since its classifying space Y/F is figure "eight" (i.e., Y is the universal cover of figure "eight"). Thus, the Euler characteristic of $SL_2(\mathbb{Z})$ is $-1/12$.

The Euler characteristic in the sense of Wall has a geometric interpretation in terms of orbifolds. Namely, suppose that Γ is as above (i.e. $\chi(\Gamma)$ is a well defined rational number), and M is a contractible manifold, on which Γ act freely and properly discontinuously. In this case, stabilizers of points are finite, and thus M/Γ is an orbifold. This means, in particular, that to every point $x \in M/\Gamma$ is attached a finite group $\mathrm{Aut}(x)$, of size $\leq [\Gamma : \Gamma']$. Let X_m be the subset of M/Γ, consisting of points x such that $\mathrm{Aut}(x)$ has order m. It often happens that X_m has the homotopy type of a finite cell complex. In this case, the *orbofold Euler characteristic* of M/Γ is defined to be $\chi_{\mathrm{orb}}(M/\Gamma) = \sum_m \chi(X_m)/m$.

Now, we claim that $\chi_{\mathrm{orb}}(M/\Gamma) = \chi(\Gamma)$. Indeed, looking at the projection $M/\Gamma' \to M/\Gamma$, it is easy to see that $\chi_{\mathrm{orb}}(M/\Gamma) = \frac{1}{[\Gamma:\Gamma']}\chi(M/\Gamma')$. But M/Γ' is a classifying space for Γ', so $\chi(M/\Gamma') = \chi(\Gamma')$, which implies the claim.

Example. Consider the group $\Gamma = SL_2(\mathbb{Z})$ acting on the upper half plane H. Then H/Γ is the moduli space of elliptic curves. So as a topological space it is $\mathbb{C}$, where all points have automorphism group $\mathbb{Z}/2$, except the point i having automorphism group $\mathbb{Z}/4$, and ρ which has automorphism group $\mathbb{Z}/6$. Thus, the orbifold Euler characteristic of H/Γ is $(-1)\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$. This is not surprising since we proved that $\chi_{\mathrm{orb}}(H/\Gamma) = \chi(\Gamma)$, which was computed to be $-1/12$.

5.2. **The mapping class group.** Now let $g \geq 1$ be an integer, and Σ be a closed oriented surface of genus g. Let $p \in \Sigma$, and let Γ_g^1 be the group of isotopy classes of diffeomorphisms of Σ which preserves p. We will recall without proof some standard facts about this group, following the paper of Harer and Zagier, Inv. Math.,v.85, p.457-485.

The group Γ_g^1 is not torsion free, but it has a torsion free subgroup of finite index. Namely, consider the homomorphism $\Gamma_g^1 \to Sp(2g, \mathbb{Z}/n\mathbb{Z})$ given by the action of Γ_g^1 on $H_1(\Sigma, \mathbb{Z}/n\mathbb{Z})$. Then for large enough n (in fact, $n \geq 3$), the kernel K_n of this map is torsion free.

It turns out that there exists a contractible finite dimensional cell complex Y, to be constructed below, on which Γ_g^1 acts cellularly with finitely many cell orbits. Thus, the Euler characteristic of Γ_g^1 in the sense of Wall is well defined.

Theorem 5.1. *(Harer-Zagier)* $\chi(\Gamma_g^1) = -B_{2g}/2g$, *where* B_n *are the Bernoulli numbers, defined by the generating function* $\sum_{n\geq 0} B_n z^n/n! = \frac{z}{e^z-1}$.

Remark 1. The group Γ_g^1 acts on the Teichmüller space $\mathcal{T}_g^1$, which is, by definition the space of pairs $((R,z),f)$, where (R,z) is a complex Riemann surface with a marked point z, and f is an isotopy class of diffeomorphisms $R \to \Sigma$ that map z to p. One may show that $\mathcal{T}_g^1$ is a contractible manifold of dimension $6g-4$, and that the action of Γ_g^1 on $\mathcal{T}_g^1$ is properly discontinuous. In particular, we may define an orbifold $M_g^1 = \mathcal{T}_g^1/\Gamma_g^1$. This orbifold parametrizes pairs (R,z) as above; therefore, it is called the moduli space of Riemann surfaces (=smooth complex projective algebraic curves) with one marked point. Thus, Theorem 5.1 gives the orbifold Euler characteristic of the moduli space of curves with one marked point.

Remark 2. If $g > 1$, one may define the analogs of the above objects without marked points, namely the mapping class group Γ_g, the Teichmüller space $\mathcal{T}_g$, and the moduli space of curves $M_g = \mathcal{T}_g/\Gamma_g$ (one can do it for $g = 1$ as well, but in this case there is no difference with the case of one marked point, since the translation group allows one to identify any two points on Σ). It is easy to see that we have an exact sequence $1 \to \pi_1(\Sigma) \to \Gamma_g^1 \to \Gamma_g \to 1$, which implies that $\chi(\Gamma_g) = \chi(\Gamma_g^1)/\chi(\Sigma)$. Thus, $\chi(\Gamma_g) = \chi_{orb}(M_g) = B_{2g}/4g(g-1)$

5.3. **Construction of the complex Y.** We begin the proof of Theorem 5.1 with the construction of the complex Y, following the paper of Harer and Zagier. We will first construct a simplicial complex with a Γ action, and then use it to construct Y.

Let $(\alpha_1, ..., \alpha_n)$ be a collection of closed simple unoriented curves on Σ, which begin and end at p, and do not intersect other than at p. Such a collection is called an arc system if two conditions are satisfied:

(A) none of the curves is contractible to a point;

(B) none of the curves is contractible to each other.

Define a simplicial complex A, whose $n-1$-simplices are isotopy classes of arc systems consisting of $n \geq 1$ arcs, and the boundary of a simplex corresponding to $(\alpha_1, ...\alpha_n)$ is the union of simplices corresponding to the arc system $(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_n)$ (α_i is omitted).

It is clear that the group Γ_g^1 acts simplicially on A.

Example. Let $g = 1$, i.e. $\Sigma = S^1 \times S^1$. Then $\Gamma_g^1 = SL_2(\mathbb{Z})$. Up to its action, there are only three arc systems (Fig. 17). Namely, viewing S^1 as the unit circle in the complex plane, and representing arcs parametrically, we may write these three systems as follows:

$$B_0 = \{(e^{i\theta},1)\}; B_1 = \{(e^{i\theta},1),(1,e^{i\theta})\}; B_2 = \{(e^{i\theta},1),(1,e^{i\theta}),(e^{i\theta},e^{i\theta})\}$$

¿From this it is easy to find the simplicial complex A. Namely, let T be the tree with root t_0 connected to three vertices t_1, t_2, t_3, with each t_i connected to two vertices t_{i1}, t_{i2}, each t_{ij} connected to t_{ij1}, t_{ij2}, etc. (Fig.18). Put at every vertex of T a triangle, with sides transversal to the three edges going out of this vertex, and glue triangles along the sides. This yields the complex A, Fig.19 (check it!). The action of $SL_2(\mathbb{Z})$ (or rather $PSL_2(\mathbb{Z})$) on this complex is easy to describe. Namely, recall that $PSL_2(\mathbb{Z})$ is generated by S, U such that $S^2 = U^3 = 1$. The action of S, U on T is defined as follows: S is reflection with flip with respect to a side of the triangle Δ_0 centered at t_0 (Fig.20), and U is rotation by $2\pi/3$ around t_0.

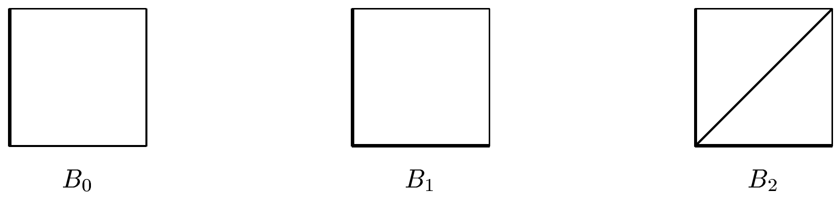


FIGURE 17. Three arc systems.

This example shows that the action of Γ_g^1 on A is not properly discontinuous, and some simplices have infinite stabilizers (in the example, it is the 0-dimensional simplices). Thus, we would like to throw away the "bad" simplices. To do it, let us say that an arc system $(\alpha_1, .., \alpha_n)$ fills up Σ if it cuts Σ into a union of regions diffeomorphic to the open disk. Let A_∞ be the union of the simplices in A corresponding to arc systems that do not fill up Σ. This is a closed subset, since

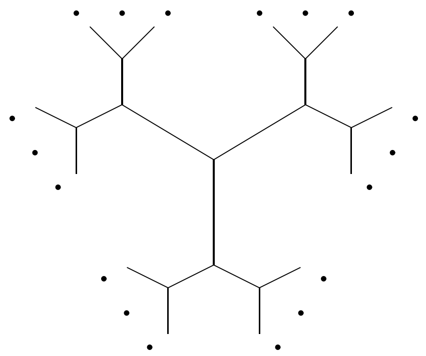


FIGURE 18. Tree T

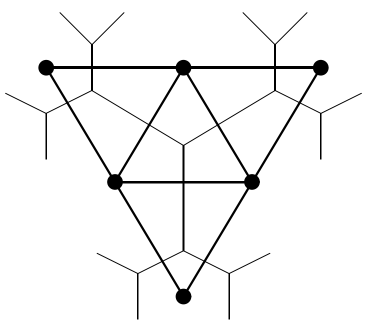


FIGURE 19. Complex A

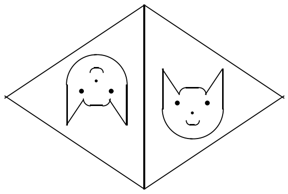


FIGURE 20. Reflection with a flip.

the property of not filling up Σ is obviously stable under taking an arc subsystem. Thus, $A \setminus A_\infty$ is an open subset of A. In the example above, it is the complex A with 0-dimensional simplices removed.

The following theorem shows that $A \setminus A_\infty$ is in fact a combinatorial model for the Teichmüller space $\mathcal{T}_g^1$, with the action of Γ_g^1.

Theorem 5.2. *(Mumford) (a) The action of Γ_g^1 on $A \setminus A_\infty$ is properly discontinuous.*

(b) $A \setminus A_\infty$ is a topologically a manifold, which is Γ_g^1-equivariantly homeomorphic to the Teichmüller space $\mathcal{T}_g^1$; in particular, it is contractible.

Remark 1. Theorem 5.2 exhibits the significance of conditions (A) and (B). Indeed, if either of the conditions were dropped, then one could consider arc systems $(\alpha_1, ..., \alpha_n)$ with arbitrarily large n, while with conditions (A),(B), as seen from Theorem 5.2, the largest value of n is $6g - 3$.

Remark 2. If $g = 1$, Theorem 5.2 is clear from the explicit description of A (convince yourself of this!).

Theorem 5.2 is rather deep, and we will not give its proof, which is beyond the scope of this course. Rather, we will use it to define the "Poincare dual" CW complex Y of $A \setminus A_\infty$. Namely, to each filling arc system $(\alpha_1, ..., \alpha_n)$ we will assign a $6g - 3 - n$-dimensional cell, and the boundary relation is opposite to the one before. The existence of this CW complex follows from the fact that $A \setminus A_\infty$ is a manifold. For instance, in the case $g = 1$ the complex Y is the tree T.

Now, the complex Y is contractible (since so is $A \setminus A_\infty$), and admits a cellular action of Γ_g^1 with finitely many cell orbits and finite stabilizers. This means that the Euler characteristic of Γ_g^1 is given by Quillen's formula.

$$\chi(\Gamma_g^1) = \sum_{\sigma \in \mathrm{cells}(Y)/\Gamma_g^1} (-1)^{\dim \sigma} \frac{1}{|\mathrm{Stab}\sigma|}.$$

Example. In the $g = 1$ case, T has one orbit of 0-cells and one orbit of 1-cells. The stabilizer of a 0-cell in $SL_2(\mathbb{Z})$ is $\mathbb{Z}/6$, and of 1-cell is $\mathbb{Z}/4$. Hence, $\chi(SL_2(\mathbb{Z})) = \frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$, which was already computed before by other methods.

5.4. **Enumeration of cells in Y/Γ_g^1.** Now it remains to count cells in Y/Γ_g^1, i.e. to enumerate arc systems which fill Σ (taking into account signs and stabilizers) To do this, we note that by definition of "filling", any filling arc system S defines a cellular decomposition of Σ. Thus, let S^* be the Poincare dual of this cellular decomposition. Since S has a unique zero cell, S^* has a unique 2-cell. Let n be the number of 1-cells in S (or S^*). Then (Σ, S^*) is obtained by gluing a $2n$-gon (=the unique 2-cell) according to a pairing of its sides preserving orientation. (Note that S can be reconstructed as $(S^*)^*$).

This allows us to link the problem of enumerating filling arc systems with the problem of counting such gluings, which was solved using matrix integrals. Namely, the problem of enumerating filling arc systems is essentially solved modulo one complication: because of conditions (A) and (B) on an arc system, the gluings we will get will be not arbitrary gluings, but gluings which also must satisfy some conditions. Namely, we have

Lemma 5.3. *Let $(\alpha_1, ..., \alpha_n)$ be a system of curves, satisfying the axioms of a filling arc system, except maybe conditions (A) and (B). Then*

(i) $(\alpha_1, ..., \alpha_n)$ satisfies condition (A) iff no edge in the corresponding gluing is glued to a neighboring edge.

(ii) $(\alpha_1, ..., \alpha_n)$ satisfies condition (B) iff no two consequtive edges are glued to another pair of consequtive edges in the opposite order.

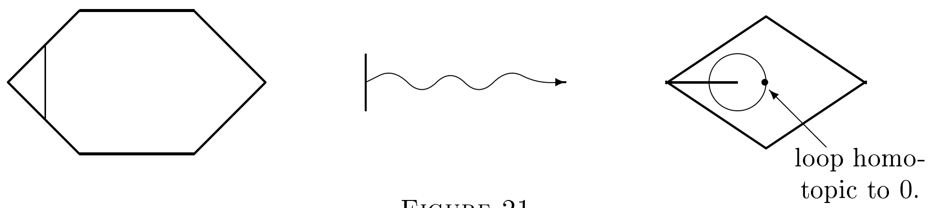


FIGURE 21

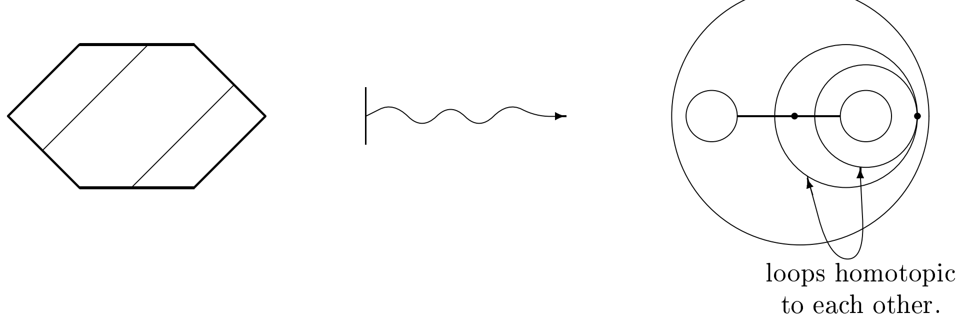


FIGURE 22

The lemma is geometrically evident, and its proof is obtained by drawing a picture (Fig.21 for (i), Fig.22 for (ii)). Motivated by the lemma, we will refer to the conditions on a gluing in (i) and (ii) also as conditions (A) and (B).

Denote by $\varepsilon_g(n), \mu_g(n), \lambda_g(n)$ the numbers of gluings of a (labeled) $2n$-gon into a surface of genus g, with no conditions, condition (A), and conditions (A),(B), respectively (so $\varepsilon_g(n)$ is the quantity we already studied).

Proposition 5.4. *One has*

$$\chi(\Gamma_g^1) = \sum_n (-1)^{n-1} \frac{\lambda_g(n)}{2n}.$$

Proof. Each filling arc system σ arises from $2n/|\mathrm{Stab}(\sigma)|$ gluings (since the labeling of the polygon does not matter for the resulting surface with an arc system). Thus, the result follows from Quillen's formula. $\square$

5.5. **Computation of $\sum_n (-1)^{n-1}\frac{\lambda_g(n)}{2n}$.** Now it remains to compute the sum on the right hand side. To do this, we will need to link $\lambda_g(n)$ with $\varepsilon_g(n)$, which has already been computed. This is accomplished by the following lemma.

Lemma 5.5. *One has*

$$\varepsilon_g(n) = \sum_i \binom{2n}{i} \mu_g(n-i);$$

$$\mu_g(n) = \sum_i \binom{n}{i} \lambda_g(n-i).$$

Proof. Proof of the first equation. Let σ be a gluing of a $2n$-gon Δ with labeled vertices. If there is a pair of consequtive edges that are glued, we can glue them to obtain a $2n-2$-gon. Proceeding like this as long as we can, we will arrive at a $2n-2i$-gon Δ_σ, with a gluing σ' of its sides which satisfies condition (A). Note that Δ_σ and σ' do not depend on the order in which neighboring edges were glued to each other, and Δ_σ has a canonical labeling by $1,...,2n-2i$, in the increasing order of the "old" labels. Now, we claim that each (Δ_σ,σ') is obtained in exactly $\binom{2n}{i}$ ways; this implies the required statement.

Indeed, let us consider vertices of Δ that ended up in the interior of Δ_σ. They have mapped to i points in the interior (each gluing of a pair of edges produces a new point). Let us call these points $w_1,...,w_i$, and let ν_j be the smallest label of a vertex of Δ that goes to w_j. Then $\nu_1,...,\nu_i$ is a subset of $\{1,...,2n\}$. This subset completely determines the gluing σ if (Δ_σ,σ') are given: namely, we should choose ν_j such that $\nu_j+1\neq\nu_k$ for any k, and glue the two edges adjacent to ν_j; then relabel by $1,...,2n-2$ the remaining vertices (in increasing order of "old" labels), and continue the step again, and so on. From this it is also seen that any set of ν_j may arise. This proves the claim.

Proof of the second equation. Let σ be a gluing of Δ (with labeled edges) which satisfies condition (A) but not necessarily (B). If a_1,a_2 are consecutive edges that are glued to consequtive edges b_2,b_1 in the opposite order, then we may unite a_1,a_2 into a single edge a, and b_2,b_1 into b, and obtain a $2n-2$-gon with a gluing. Continuing so as long as we can, we will arrive at a $2n-2i$-gon Δ_σ with a new gluing σ', which satisfies conditions (A) and (B). In Δ_σ, each (j-th) pair of edges is obtained for m_j+1 pairs of edges in Δ. Thus, $\sum_{j=1}^{m-i} m_j = i$. Furthermore, for any (Δ_σ,σ') the collection of numbers $m_1,...,m_{n-i}$ defines (Δ,σ) uniquely, up to deciding which of the m_1+1 edges constituting the first edge of Δ_σ shold be labeled by 1. Thus, each (Δ_σ,σ') arises in the number of ways given by the formula

$$\sum_{m_1,\ldots,m_{n-i}:\sum m_j=i} (m_1+1).$$

It is easy to show (check!) that this number is equal to $\binom{n}{i}$. The second equation is proved. $\square$

The completion of the proof of Theorem 5.1 depends now on the following computational lemma.

Lemma 5.6. *Let $\varepsilon(n), \mu(n), \lambda(n)$, $n \geq 0$, be sequences satisfying the equations*

$$\varepsilon(n) = \sum_i \binom{2n}{i} \mu(n-i);$$

$$\mu(n) = \sum_i \binom{n}{i} \lambda(n-i).$$

Assume also that $\varepsilon(n) = \binom{2n}{n} f(n)$, where f is a polynomial such that $f(0) = 0$. Then $\lambda(0) = 0$, $\lambda(n)$ has finitely many nonzero values, and $\sum(-1)^{n-1}\lambda(n)/2n = f'(0)$.

Proof. Let us first consider any sequences $\varepsilon(n)$, $\mu(n)$, and $\lambda(n)$ linked by the equations of the lemma. Let $E(z)$, $M(z)$, and $L(z)$ be their generating functions (i.e. $E(z) = \sum_{n \geq 0} \varepsilon(n) z^n$ etc.). We claim that

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} L(\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}}).$$

To see this, it suffices to consider the case $\lambda_i = \delta_{ki}$ for some k. In this case,

$$E(z) = \sum_{i,n} \binom{2n}{i} \binom{n-i}{k} z^n = \sum_{p,q \geq 0} \binom{2p+2q}{p} \binom{q}{k} z^{p+q}$$

But the function

$$F_r(z) := \sum_{p \geq 0} \binom{2p+r}{p} z^p$$

equals

$$F_r(z) = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z} \right)^r,$$

as may be easily seen by induction from the recursion $F_r = z^{-1}(F_{r-1} - F_{r-2})$, $r \geq 2$. Substituting this in the formula for $E(z)$, one gets (after trivial simplifications)

$$E(z) = \frac{1+\sqrt{1-4z}}{2(1-4z)} (\frac{1-\sqrt{1-4z}}{2\sqrt{1-4z}})^k,$$

as desired.

Now assume that $\varepsilon(n)$ satisfies the polynomiality condition. This means that $E(z) = P(z\partial)|_{z=0} \frac{1}{\sqrt{1-4z}}$, where P is a polynomial with vanishing constant term. To prove our claim, it suffices to consider the case $P(z) = (1+a)^z - 1$, where a is a formal parameter (so $P'(0) = \ln(1+a)$). In this case we get

$$E(z) = \frac{1}{\sqrt{1-4(1+a)z}} - \frac{1}{\sqrt{1-4z}}$$

Hence,

$$L(u) = \frac{1}{1+u} (\frac{1}{\sqrt{1-4au(1+u)}} - 1)$$

Therefore,

$$\sum_k (-1)^{k-1} \lambda_k / 2k = \frac{1}{2} \int_{-1}^{0} L(u) du/u = \frac{1}{2} \sum_{p \geq 1} \binom{2p}{p} (-1)^{p-1} a^p \int_0^1 x^{p-1}(1-x)^{p-1} dx.$$

But $\int_0^1 x^{p-1}(1-x)^{p-1}dx$ is a beta integral, and it equals $(p-1)!^2/(2p-1)!$. Thus, the above integral equals $\sum_{p\geq 1}(-1)^{p-1}a^p/p = \ln(1+a)$, as desired. $\square$

Now we finish the proof of the theorem. Recall that using matrix integrals we have proved the formula

$$P_n(x) := \sum_g \varepsilon_g(n)x^{n+1-2g} = \frac{(2n)!}{2^n n!}\sum_{p\geq 0}\binom{n}{p}2^p\binom{x}{p+1} \tag{14}$$

Let us set $q = n-p$. Then expression (14) takes the form

$$P_n(x) = \binom{2n}{n}\sum_{q\geq 0}2^{-q}\binom{n}{q}\frac{n!}{(n-q+1)!}x(x-1)...(x-n+q). \tag{15}$$

We claim now that the coefficient of x^{-2g} $(g\geq 1)$ in the polynomial $P_n(x)/x^{n+1}$ are of the form $\binom{2n}{n} f_g(n)$, where f_g is a polynomial. Indeed, contributions to the coefficient of x^{-2g} come from terms with $q\leq 2g$ only, so it suffices to check that each of these contributions is as stated. This reduces to checking that the coefficients of the Laurent polynomial $Q(x,n) = (1-1/x)...(1-n/x)$ are polynomials in n, which vanish at -1 (except, of course, the leading coefficient). To see this, let $Q(x,a) = \frac{\Gamma(x)}{\Gamma(x-a)x^a}$ (this equals to $Q(x,n)$ if $a=n$). This function has an asymptotic Taylor expansion in $1/x$ as $x\to+\infty$, and it is easy to show that the coefficients are polynomials in a. Moreover, $Q(x,-1)=1$, which implies the required statement.

Furthermore, we claim that $f_g(0)=0$: again, this follows from the fact that the non-leading coefficients of the expansion of $Q(x,a)$ vanish at $a=0$. But this is clear, since $Q(x,0)=1$.

Thus, we are in a situation where Lemma 5.6 can be applied. So it remains to compute $\sum_{g\geq 1}f_g'(0)x^{-2g}$. To do this, observe that the terms with $q>1$ do not contribute to $f_g'(0)$, as they are given by polynomials of n that are divisible by n^2. So we only need to consider $q=0$ and $q=1$. For $q=1$, the contribution is the value of $(2x)^{-1}(1-1/x)...(1-n/x)$ at $n=0$, i.e. it is $1/2x$. For $q=0$, the contribution is the derivative at 0 with respect to n of $(1-1/x)...(1-n/x)/(n+1)$, i.e. it is $\frac{d}{da}|_{a=0}\frac{Q(x,a)}{a+1} = -1+\frac{d}{da}|_{a=0}Q(x,a)$. Thus, we have (asymptotically)

$$\sum_{g\geq 1}f_g'(0)x^{-2g} = \frac{1}{2x}+\frac{d}{da}|_{a=0}Q(x,a) = \frac{1}{2x}+\frac{\Gamma'(x)}{\Gamma(x)}-\log x$$

However, the classical asymptotic expansion for Γ'/Γ is:

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - \sum_{g=1}^{\infty}\frac{B_{2g}}{2g}x^{-2g}.$$

This implies that $f_g'(0) = -B_{2g}/2g$. Hence the Harer-Zagier theorem is proved.

6. Matrix integrals and counting planar diagrams

6.1. **The number of planar gluings.** Let us return to the setting of §4. Thus, we have a potential $U(x) = x^2/2 - \sum_{j\geq 0} g_jx^j/j$ (with g_j being formal parameters), and consider the matrix integral

$$Z_N(\hbar) = \hbar^{-N/2}\int_{\mathfrak{h}_N} e^{-TrU(A)}dA.$$

Let $\hat{Z}_N(\hbar) = Z_N(\hbar/N)$. We have seen that

$$\lim_{N\to\infty} \frac{\ln \hat{Z}_N}{N^2} = W_\infty,$$

where W_∞ is given by summation over planar fat graphs:

$$W_\infty = \sum_{\mathbf{n}} \prod_i g_i^{n_i} \sum_{\widetilde{\Gamma}\in \widetilde{G}_c(\mathbf{n})[0]} \frac{\hbar^{b(\widetilde{\Gamma})}}{|\mathrm{Aut}(\widetilde{\Gamma})|}$$

In particular, the coefficient of $\prod g_i^{n_i}$ is (up to a power of $\hbar$) the number of (orientation preserving) gluings of a fat graph of genus zero out of a collection of fat flowers containing n_i i-valent flowers for each i, divided by $\prod i^{n_i} n_i!$.

On the other hand, one can compute W_∞ explicitly as a function of g_i by reducing the matrix integral to an integral over eigenvalues, and then using a fundamental fact from the theory of random matrices: the existence of an asymptotic distribution of eigenvalues as $N \to \infty$. This approach allows one to obtain simple closed formulas for the numbers of planar gluings, which are quite nontrivial and for which direct combinatorial proofs were discovered only very recently.

To illustrate this method, we will restrict ourselves to the case of the potential $U(x) = x^2/2 + gx^4$ (so $g_4 = -4g$ and other $g_i = 0$), and set $\hbar = 1$. Then $W_\infty = \sum_{n\geq 1} c_n(-g)^n/n!$, where c_n is a number of connected planar gluings of a set of n 4-valent flowers. In other words, c_n is the number of ways (up to isotopy) to connect n "crosses" in the 2-sphere so that all crosses are connected with each other, and the connecting lines do not intersect.

Exercise. Check by drawing pictures that $c_1 = 2$, $c_2 = 36$.

Theorem 6.1. *(Brezin, Itzykson, Parisi, Zuber, 1978). One has*

$$c_n = (12)^n(2n-1)!/(n+2)!$$

6.2. **Proof of Theorem 6.1.** Let us prosent the proof of this theorem (with some omissions). We will assume that g is a positive real number, and compute the function $W_\infty(g)$ explicitly. We follow the paper of Brezin, Itzykson, Parisi, and Zuber "Planar diagrams", Comm. Math. Phys. 59, p.35-51, 1978.

The relevant matrix integral has the form

$$\hat{Z}_N = \int_{\mathfrak{h}_N} e^{-N\mathrm{Tr}(A^2/2+gA^4)} dA.$$

Passing to eigenvalues, we get

$$\hat{Z}_N = \frac{J_N(g)}{J_N(0)},$$

where

$$J_N(g) = \int_{\mathbb{R}^N} e^{-N(\sum \lambda_i^2/2 + g\sum \lambda_i^4)} \prod_{i<j}(\lambda_i - \lambda_j)^2 d\lambda.$$

Thus, $W_\infty(g) = E(g) - E(0)$, where $E(g) = \lim_{N\to\infty} N^{-2} \ln J_N(g)$.

Proposition 6.2. *(steepest descent principle) $E(g)$ equals the maximal value of the logarithm of the integrand.*

The proposition says, essentially, that the integrand has a sufficiently sharp maximum, so that the leading behavior of the integral can be computed by the steepest descent formula. We note that we cannot apply the steepest descent formula without explanations, since the integral is over a space whose dimension grows as the perturbation parameter $1/N$ goes to 0. In other words, it is necessary to do some estimates which we will omit. We will just mention that for $g = 0$, this result can be derived from the explicit evaluation of the integral using Hermite polynomials (see §4). For the general case, we refer the reader to the book of P.Deift "Orthogonal polynomials and random matrices: a Riemann-Hilbert approach".

The integrand $K(\lambda_1, ..., \lambda_N) = e^{-N(\sum \lambda_i^2/2+g\sum \lambda_i^4)} \prod_{i<j}(\lambda_i - \lambda_j)^2$ has a unique maximum, because it is logarithmically concave (check it!). The maximum of the integrand is found by equating the partial derivatives to zero. This yields

$$\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = N(\frac{1}{2}\lambda_i + 2g\lambda_i^3). \tag{16}$$

Let $\lambda_1 < \lambda_2 < ... < \lambda_N$ be the unique (up to permutations) solution of this system of equations.

Proposition 6.3. *The normalized counting measures $\frac{1}{N}\sum \delta(x-\lambda_i)$ converge weakly to a measure $\mu(x) = f(x,g)dx$, where $f(x,g)$ is a continuous function, supported on a finite interval $[-2a, 2a]$, and differentiable on this interval.*

For the proof we again refer the reader to the book of P.Deift (p.132 and later). We note that for $g = 0$, by Wigner's semicircular law, $a = 1$ and $f(x,0) = \frac{1}{2\pi}\sqrt{4-x^2}$; so $f(x,g) = \frac{1}{2\pi}\sqrt{4-x^2} + O(g)$.

Now our job will be to find the function $f(x,g)$. Passing to the limit in equation 16 (which requires justification that we will omit), we get

$$\int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx = \frac{1}{2}y + 2gy^3, \ |y| \leq 2a$$

where the integral is understood in the sense of principal value.

This is a linear integral equation on $f(x,g)$, which can be solved in a standard way. Namely, one considers the analytic function $F(y) = \int_{-2a}^{2a} \frac{1}{y-x} f(x,g)dx$ for y in the complex plane but outside of the interval $[-2a, 2a]$. For $y \in [-2a, 2a]$, let $F_+(y)$, $F_-(y)$ denote the limits of $F(y)$ from above and below. Then by the Plemelj formula, the integral equation implies

$$\frac{1}{2}(F_+(y) + F_-(y)) = \frac{1}{2}y + 2gy^3.$$

On the other hand, $F_+(y) = \overline{F_-(y)}$. Hence, $\mathrm{Re}F_+(y) = \mathrm{Re}(F_-(y)) = \frac{1}{2}y + 2gy^3$.

Now set $y = a(z + z^{-1})$. Then, as y runs through the exterior of $[-2a, 2a]$, z runs through the exterior of the unit circle. So the function $G(z) := F(y)$ is analytic on the outside of the unit circle, with decay at infinity, and $\mathrm{Re}G(z) = \frac{1}{2}a(z + z^{-1}) + 2ga^3(z + z^{-1})^3$, $|z| = 1$. This implies that $G(z)$ is twice the sum of all negative degree terms of this Laurent polynomial. In other words, we have

$$G(z) = 4ga^3z^{-3} + (a + 12ga^3)z^{-1}.$$

This yields

$$F(y) = \frac{1}{2}y + 2gy^3 - (\frac{1}{2} + 4ga^2 + 2gy^2)\sqrt{y^2 - 4a^2}.$$

Now $f(y,g)$ is found as the jump of F:

$$f(y,g) = \frac{1}{\pi}(\frac{1}{2} + 4ga^2 + 2gy^2)\sqrt{4a^2 - y^2}.$$

It remains to find a in terms of g. We have $yF(y) \to 1$, $y \to \infty$ (as $\int f(x,g)dx = 1$), hence $zG(z) \to 1/a$, $z \to \infty$. This yields $1/a = a + 12ga^3$, or

$$12ga^4 + a^2 - 1 = 0.$$

This allows one to determine a uniquely:

$$a = \Big(\frac{(1+48g)^{1/2} - 1}{24g}\Big)^{1/2}.$$

Now let us calculate $E(g)$. It follows from the above that

$$E(g) = \int_{-2a}^{2a}\int_{-2a}^{2a} \ln|x-y| f(x,g)f(y,g)dxdy - \int_{-2a}^{2a}(\frac{1}{2}x^2 + gx^4)f(x,g)dx.$$

On the other hand, let us integrate the integral equation defining $f(x,g)$ with respect to y (from 0 to u). Then we get

$$2\int_{-2a}^{2a}(\ln|x-u| - \ln|x|)f(x,g)dx = \frac{1}{2}u^2 + gu^4.$$

Substituting this into the expression for $E(g)$, we get

$$E(g) = \int_{-2a}^{2a}(\ln|u| - \frac{1}{4}u^2 - \frac{1}{2}gu^4)f(u,g)du$$

Since $f(u,g)$ is known, this integral can be computed. In fact, can be expressed via elementary functions, and after calculations we get

$$E(g) - E(0) = \ln a - \frac{1}{24}(a^2-1)(9-a^2).$$

Substituting here the expression for a, after a calculation one finally gets:

$$E(g) - E(0) = \sum_{k=1}^{\infty}(-12g)^k \frac{(2k-1)!}{k!(k-2)!}.$$

This implies the required formula for c_n.

7. Quantum mechanics

So far we have considered quantum field theory with 0-dimensional spacetime (to make a joke, one may say that the dimension of the space is -1). In this section, we will move closer to actual physics: we will consider 1-dimensional spacetime, i.e. the dimension of the space is 0. This does not mean that we will study motion in a 0-dimensional space (which would be really a pity) but just means that we will consider only point-like quantum objects (particles) and not extended quantum objects (fields). In other words, we will be in the realm of quantum mechanics.

7.1. **The path integral in quantum mechanics.** Let $U(q)$ be a smooth function on the real line (the potential). We will assume that $U(0) = 0$, $U'(0) = 0$, and $U''(0) = m^2$, where $m > 0$.

Remark. In quantum field theory the parameter m in the potential is called the *mass* parameter. To be more precise, in classical mechanics it has the meaning of frequency ω of oscillations. However, in quantum theory thanks to Einstein frequency is identified with energy ($E = \hbar\omega/2\pi$), while in relativisitic theory energy is identified with mass (again thanks to Einstein, $E = mc^2$).

We want to construct the theory of a quantum particle moving in the potential field $U(q)$. According to what we discussed before, this means that we want to give sense to and to evaluate the normalized correlation functions

$$< q(t_1)\dots q(t_n) >:= \frac{\int q(t_1)\dots q(t_n)e^{iS(q)/\hbar}Dq}{\int e^{iS(q)/\hbar}Dq},$$

where $S(q) = \int \mathcal{L}(q)dt$, and $\mathcal{L}(q) = \dot{q}^2/2 - U(q)$.

As we discussed, such integrals cannot be handled rigorously by means of measure theory if $\hbar$ is a positive number; so we will only define these path integrals "in perturbation theory", i.e. as formal series in $\hbar$.

Before giving this (fully rigorous) definition, we will explain the motivation behind it. We warn the reader that this explanation is heuristic and involves steps which are mathematically non-rigorous (or "formal" in the language of physicists).

7.2. **Wick rotation.** In §1 we discussed path integrals with imaginary exponential (quantum mechanics), as well as real exponential (Brownian motion). If $\hbar$ is a number, then the integrals with imaginary exonential cannot be defined measure theoretically. Therefore, people study integrals with real exponential (which can be rigorously defined), and then perform a special analytic continuation procedure called the Wick rotation.

In our formal setting ($\hbar$ is a formal parameter), one can actually define the integrals in both the real and the imaginary case. Still, the real case is a bit easier, and thus the Wick rotation is still useful. Besides, the Wick rotation is very important conceptually. Therefore, while it is not technically necessary, we start with introducing the Wick rotation here.

Namely, let us denote $< q(t_1)...q(t_n) >$ by $\mathcal{G}_n^M(t_1,...,t_n)$, and "formally" make a change of variable $\tau = it$ in the formula for $\mathcal{G}_n^M(t_1,...,t_n)$. Let $q(t) = q_*(\tau)$. Then, taking into account that $d\tau = idt$, $dq/dt = idq_*/d\tau$ we get

$$\mathcal{G}_n^M(t_1,...,t_n) = \frac{\int q_*(\tau_1)\dots q_*(\tau_n)e^{-\int[\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*)]/\hbar}Dq_*}{\int e^{-\int[\frac{1}{2}(\frac{dq_*}{d\tau})^2+U(q_*)]/\hbar}Dq_*}.$$

This shows that

$$\mathcal{G}_n^M(t_1,...,t_n) = \mathcal{G}_n^E(it_1,...,it_n),$$

where

$$\mathcal{G}_n^E(t_1,...,t_n) := \frac{\int q(t_1)\dots q(t_n)e^{-S_E(q)/\hbar}Dq}{\int e^{-S_E(q)/\hbar}Dq}.$$

where $S_E(q) = \int \mathcal{L}_E(q)dt$, and $\mathcal{L}_E(q) = \dot{q}^2/2 + U(q)$ (i.e. $\mathcal{L}_E$ is obtained from $\mathcal{L}$ by replacing U with $-U$).

This manipulation certainly does not make rigorous sense, but it motivates the following definition.

Definition 7.1. The function $\mathcal{G}_n^M(t_1, ..., t_n)$ ($t_i \in \mathbb{R}$) is the analytic continuation of the function $\mathcal{G}_n^E(s_1, ..., s_n)$ from the point $(t_1, ..., t_n)$ to the point $(it_1, ..., it_n)$ along the path $e^{i\theta}(t_1, ..., t_n)$, $0 \le \theta \le \pi/2$.

Of course, this definition will only make sense if we define the function $\mathcal{G}_n^E(t_1, ..., t_n)$ and show that it admits the required analytic continuation. This will be done below.

Terminological remark. The function $\mathcal{G}_n^M(t_1, ..., t_n)$ is called the Minkowskian (time ordered) correlation function, while $\mathcal{G}_n^E(t_1, ..., t_n)$ is called the Euclidean correlation function (hence the notation). This terminology will be explained later, when we consider relativistic field theory.

From now on, we will mostly deal with Euclidean correlation functions, and therefore will omit the superscript E when there is no danger of confusion.

7.3. **Definition of Euclidean correlation functions.** Now our job is to define the Euclidean correlation functions $\mathcal{G}_n(t_1, ..., t_n)$. Our strategy (which will also be used in field theory) will be as follows. Recall that if our integrals were finite dimensional then by Feynman's theorem the expansion of the correlation functions in $\hbar$ would be given by a sum of amplitudes of Feynman diagrams. So, in the infinite dimensional case, we will use the sum over Feynman diagrams as a *definition* of correlation functions.

More specifically, because of conditions on U we have an action functional without constant and linear terms in q, so that the correlation function $\mathcal{G}_n(t_1, ..., t_n)$ should be given by the sum

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\Gamma \in G^*_{\ge 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma(\ell_1, \ldots, \ell_n), \tag{17}$$

Thus, we should make sense of (=define) the amplitudes F_Γ in our situation. For this purpose, we need to define the following objects.

1. The space V.
2. The form B on V which defines B^{-1} on V^*.
3. The tensors corresponding to non-quadratic terms in the action.
4. The covectors ℓ_i.

It is clear how to define these objects naturally. Namely, V should be a space of functions on $\mathbb{R}$ with some decay conditions. There are many choices for V, which do not affect the final result; for instance, a good choice (which we will make) is the space $C_0^\infty(\mathbb{R})$ of compactly supported smooth functions on $\mathbb{R}$, Thus V^* is the space of generalized functions on $\mathbb{R}$. Note that V is equipped with the inner product $(f, g) = \int_\mathbb{R} f(x)g(x)dx$.

The form B, by analogy with the finite dimensional case, should be twice the quadratic part of the action. In other words, $B(q, q) = \int(\dot{q}^2 + m^2q^2)dt = (Aq, q)$, where A is the operator $-d^2/dt^2 + m^2$. This means that $B^{-1}(f, f) = (A^{-1}f, f)$

The operator A^{-1} is an integral operator, with kernel $K(x, y) = G(x - y)$, where $G(x)$ is the Green's function of A, i.e. the fundamental (decaying at infinity) solution of the differential equation $(AG)(x) = \delta(x)$. It is straightforward to find that

$$G(x) = e^{-m|x|}/2m.$$

(thus B^{-1} is actually defined not on the whole V^* but on a dense subspace of V^*).

Remark. Here we see the usefulness of the Wick rotation. Namely, the spectrum of A in L^2 is $[m^2, +\infty)$, so it is invertible and the inverse is bounded. However, if we did not make a Wick rotation, we would deal with the operator $A' = -d^2/dt^2 - m^2$, whose spectrum is $[-m^2, +\infty)$, i.e. contains 0, so that the operator is not invertible in the naive sense.

To make sense of the cubic and higher terms in the action as tensors, consider the decomposition of U in the (asymptotic) Taylor series at $x = 0$: $U(x) = m^2x^2/2 + \sum_{n\geq 3} a_n x^n/n!$. This shows that cubic and higher terms in the action have the form

$$B_r(q, q, , ..., q) = \int q^r(t)dt$$

Thus $B_r(q_1, ..., q_r)$ is an element of $(S^rV)^*$ given by the generalized function $\delta_{t_1=...=t_r}$ (the delta function of the diagonal).

Finally, the functionals ℓ_i are given by $\ell_i(q) = q(t_i)$, so $\ell_i = \delta(t - t_i)$.

This leads to the following Feynman rules of defining the amplitude of a diagram Γ.

1. To the i-th external vertex of Γ assign the number t_i.
2. To each internal vertex j of Γ, assign a variable s_j.
3. For each internal edge connecting vertices j and j', write $G(s_j - s_{j'})$.
4. For each external edge connecting i and j write $G(t_i - s_j)$.
5. For each external edge connecting i and i' write $G(t_i - t_{i'})$.
6. Let $G_\Gamma(\mathbf{t}, \mathbf{s})$ be the product of all these functions.
7. Let $F_\Gamma(\ell_1, ...\ell_n) = \prod_j (-a_{v(j)}) \int G_\Gamma(\mathbf{t}, \mathbf{s})d\mathbf{s}$, where $v(j)$ is the valency of j.

We are finally able to give the following definition.

Definition 7.2. The function $\mathcal{G}_n(t_1, ..., t_n)$ is defined by the formula 17.

Remark. Note that the integrals defining F_Γ are convergent since the integrand always decays exponentially at infinity. It is, however, crucial that we consider only graphs without components having no external vertices; for example, if Γ has a single 4-valent vertex connected to itself by two loops (Fig.23) then the amplitude integral involves $\int_{\mathbb{R}} G(0)^2 ds$, which is obviously divergent.

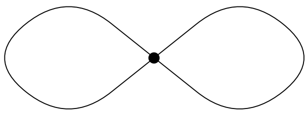


FIGURE 23

With this definition, the function $\mathcal{G}_n(t_1, ..., t_n)$ is a Laurent series in $\hbar$, whose coefficients are symmetric functions of $t_1, ..., t_n$, given by linear combinations of explicit (and convergent) finite dimensional integrals. Furthermore, it is easy to see that these integrals are in fact computable in elementary functions, i.e. are (in the region $t_1 \geq ... \geq t_n$) linear combinations of products of functions of the form $t_i^r e^{at_i}$. This implies the existence of the analytic continuation required in the Wick rotation procedure.

Remark. An alternative setting for making this definition is to assume that a_i are formal parameters. In this case, $\hbar$ can be given a numerical value, e.g. $\hbar = 1$, and the function $\mathcal{G}_n$ will be a well defined power series in $a_3, a_4, ...$.

Example 1. The free theory: $U(q) = m^2q^2/2$. In this case, there is no internal vertices, and hence we have

Proposition 7.3. *(Wick's theorem) One has* $\mathcal{G}_n(t_1, ..., t_n) = 0$ *if* n *is odd, and*

$$\mathcal{G}_{2k}(t_1, ..., t_{2k}) = \hbar^k \sum_{\sigma \in \Pi_k} \prod_{i \in \{1,...,2k\}/\sigma} G(t_i - t_{\sigma(i)}).$$

In particular, $\mathcal{G}_2(t_1, t_2) = \hbar G(t_1 - t_2)$. In other words, $\mathcal{G}_2(t_1, t_2)$ is (proportional to) the Green's function. Motivated by this, physicists often refer to correlation functions of a quantum field theory as *Green's functions*.

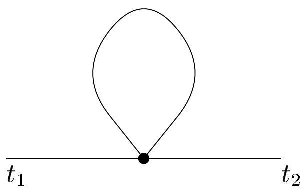


FIGURE 24

Example 2. Consider the potential $U(q) = m^2q^2/2 + gq^4/24$, and set $\hbar = 1$. In this case, let us calculate the 2-point correlation function modulo g^2. In other words, we have to compute the coefficient of g in this function. Thus we have to consider Feynman diagrams with two external edges and one internal vertex. Such a diagram Γ is unique: it consists of one edge with a loop attached in the middle (Fig. 24). This diagram has automorphism group $\mathbb{Z}/2$. The amplitude of this diagram is

$$F_\Gamma = -g \int_{\mathbb{R}} G(s, t_1)G(s, t_2)G(s, s)ds = \frac{g}{8m^3} \int_{\mathbb{R}} e^{-m(|s-t_1|+|s-t_2|)} ds$$

Because of symmetry in t_1 and t_2, we may assume that $t_1 \geq t_2$. Splitting the integral in a sum of three integrals, over $(-\infty, t_2], [t_2, t_1]$, and $[t_1, \infty)$, respectively we get:

$$\mathcal{G}_2(t_1, t_2) = \widetilde{G}(t_1 - t_2),$$

where

$$\widetilde{G}(t) = \frac{1}{2m} e^{-m|t|}(1 - \frac{g}{8m^2}(\frac{1}{m} + |t|)) + O(g^2).$$

This expression is called the 1-loop approximation to the 2-point function, because it comes from 0-loop and 1-loop Feynman diagrams.

Remark. Here we are considering quantum mechanics of a single 1-dimensional particle. However, everything generalizes without difficulty to the case of an n-dimensional particle or system of particles (i.e. to path integrals over the space of vector valued, rather than scalar, functions of one variable). Indeed, if q takes values in a Euclidean space V then the quadratic part of the Lagrangian is of the form $\frac{1}{2}(\dot{q}^2 - M(q))$, where M is a positive definite quadratic form on V. Reducing M to principal axes, we may assume that the quadratic part of the Lagrangian looks like $\frac{1}{2}\sum_i(\dot{q_i}^2 - m_i^2q_i^2)$, which corresponds to a system of independent harmonic oscillators. Thus in quantum theory the propagator will be the diagonal matrix with diagonal entries $e^{-m_i|t-s|}/2m_i$, and the correlation functions can be defined by the usual Feynman diagram procedure.

7.4. **Connected Green's functions.** Let $\mathcal{G}_n^c(t_1, ..., t_n)$ be the connected Green's functions, defined by the sum of the same amplitudes as $\mathcal{G}_n(t_1, ..., t_n)$ but taken over connected Feynman diagrams only. It is clear that

$$\mathcal{G}_n(t_1, ..., t_n) = \sum_{\{1,...,n\}=S_1\sqcup...\sqcup S_k} \prod \mathcal{G}^c_{|S_i|}(t_j; j \in S_i).$$

For example, $\mathcal{G}_2(t_1, t_2) = \mathcal{G}_2^c(t_1, t_2) + \mathcal{G}_1^c(t_1)\mathcal{G}_1^c(t_2)$, etc. Thus, to know the correlation functions, it is sufficient to know the connected correlation functions.

Example 1. In a free theory ($U = m^2q^2/2$), all connected Green's functions except $\mathcal{G}_2$ vanish.

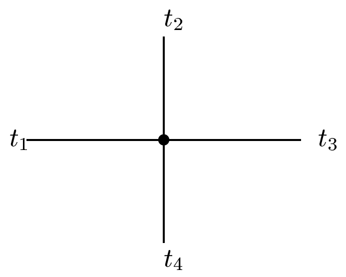


FIGURE 25

Example 2. Let us compute the connected 4-point function in the theory associated to quartic potential U as above, modulo g^2. This means, we should compute the contribution of connected Feynman diagrams with one internal vertex and 4 external edges. Such diagram Γ is unique –it is the cross (with one internal vertex), Fig.25. This diagram no nontrivial automorphisms. Thus,

$$\mathcal{G}_4^c(t_1, t_2, t_3, t_4) = -g \int_{\mathbb{R}} G(t_1 - s)G(t_2 - s)G(t_3 - s)G(t_4 - s)ds + O(g^2).$$

It is elementary to compute this integral; we leave it as an exercise.

7.5. **The clustering property.** Note that the Green's function $G(t)$ goes to zero at infinity. This implies the following *clustering property* of the correlation functions of the free theory:

$$\lim_{z\to\infty} \mathcal{G}_n(t_1, ..., t_r, t_{r+1} + z, ..., t_n + z) = \mathcal{G}_r(t_1, ..., t_r)\mathcal{G}_{n-r}(t_{r+1}...t_n).$$

Moreover, it is easy to show that the same is true in the interacting theory (i.e. with potential) in each degree with respect to $\hbar$ (check it!). The clustering property can be more simply expressed by the equation

$$\lim_{z\to\infty} \mathcal{G}_n^c(t_1, ..., t_r, t_{r+1} + z, ..., t_n + z) = 0.$$

This property has a physical interpretation: processes distant from each other are almost statistically independent. Thus it can be viewed as a necessary condition of a quantum field theory to be "physically meaningful".

Remark. Nevertheless, there exist theories (e.g. so called topological quantum field theories) which do not satisfy the clustering property but are interesting both form a physical and mathematical point of view.

7.6. **The partition function.** Let $J(t)dt$ be a compactly supported measure on the real line. Consider the "partition function with external current J", which is the formal expression

$$Z(J) = \int e^{\frac{-S_E(q)+(J,q)}{\hbar}} Dq.$$

Then we have a formal equality

$$\frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int_{\mathbb{R}^n} \mathcal{G}_n(t_1,...,t_n) J(t_1)...J(t_n) dt_1...dt_n,$$

which, as before, we will use as definition of $Z(J)/Z(0)$. So the knowledge of $Z(J)/Z(0)$ is equivalent to the knowledge of all the Green's functions (in other words, $Z(J)/Z(0)$ is their generating function). Furthermore, as in the finite dimensional case, we have

Proposition 7.4. *One has*

$$W(J) := \ln \frac{Z(J)}{Z(0)} = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^c(t_1,...,t_n) J(t_1)...J(t_n) dt_1...dt_n$$

(i.e. W is the generating function of connected Green's functions)

The proof of this proposition is the same as in the finite dimensional case.

Remark. The statement of the proposition is equivalent to the relation between usual and connected Green's functions, given in the previous subsection.

Remark. The fact that we can only define amplitudes of graphs whose all components have at least one 1-valent vertex (see above) means that we actually cannot define either $Z(0)$ or $Z(J)$ but can only define their ratio $Z(J)/Z(0)$.

Like in the finite dimensional case, we have an expansion

$$W(J) = \hbar^{-1} W_0(J) + W_1(J) + \hbar W_2(J) + ...,$$

where W_j are the j-loop contributions (in particular, W_0 is given by a sum over trees). Furthermore, we have explicit formulas for W_0 and W_1, analogously to the finite dimensional case.

Proposition 7.5. *One has*

$$W_0(J) = -S_E(q_J) + (q_J, J),$$

where q_J is the extremal of the functional $S_E^J(q) := S_E(q) - (q, J)$ which decays at infinity. Furthermore,

$$W_1(J) = -\frac{1}{2} \ln \det L_J,$$

where L_J is the linear operator on V such that $d^2 S_E^J(q_J)(f_1, f_2) = d^2 S_E^0(0)(L_J f_1, f_2)$.

The proof of this proposition, in particular, involves showing that q_J is well defined and that $\det L_J$ exists. It is analogous to the proof of the same result in the finite dimensional case which is given in §3.7 (to be precise, we gave a proof only in the 0-loop case; but in the 1-loop case, the proof is similar). Therefore we will not give this proof; rather, we will illustrate the statement by an example.

Example. Let U be the quartic potential and $J(t) = a\delta(t)$. In this case, $S_E^J(q) = \int (\dot{q}^2/2 + U(q))dt - aq(0)$. The Euler-Lagrange equation has the form

$$\ddot{q} = m^2 q + g q^3/6 - a\delta(t).$$

Thus, the function q_J is continuously glued from two solutions q_+, q_- of the non-linear differential equation $\ddot{q} = m^2 q + gq^3/6$ on $(-\infty, 0]$ and $[0, \infty)$, with jump of derivative at 0 equal to $-a$.

The solutions q_+, q_- are required to decay at infinity, so they must be solutions of zero energy ($E = \dot{q_\pm}^2/2 - U(q_\pm) = 0$). Thus, by the standard formula for solutions of Newton's equation, they are defined by the equality

$$t - t_\pm = \int \frac{dq}{\sqrt{2(E + U(q))}} = \int \frac{dq}{mq\sqrt{1 + \frac{gq^2}{12m^2}}} = \frac{1}{2m} \ln \frac{\sqrt{1 + \frac{gq^2}{12m^2}} - 1}{\sqrt{1 + \frac{gq^2}{12m^2}} + 1}.$$

After a calculation one gets

$$q_J(t) = \frac{\sqrt{\frac{12m^2}{g}}}{C^{-1}e^{m|t|} - Ce^{-m|t|}},$$

where C is the solution of the equation

$$\frac{C + C^{-1}}{(C - C^{-1})^2} = \sqrt{\frac{g}{12m^2}} \frac{a}{2m}$$

which is given by a power series in a with zero constant term. ¿From this it is elementary (but somewhat lengthy) to compute $W_0 = -S_E^J(q_J)$.

Now, the operator L_J is given by the formula $L_J = 1 + gA^{-1}q_J(t)^2/2$, where $A = -d^2/dt^2 + m^2$. Thus $\det L_J$ makes sense. Indeed, the operator $A^{-1}q_J(t)^2$ is given by the kernel $\frac{e^{-m|x-y|}q_J(y)^2}{2m}$, which decays exponentially at infinity; hence this operator is trace class and therefore, $1 + gA^{-1}q_J(t)^2/2$ is determinant class.

7.7. **1-particle irreducible Green's functions.** Let $\mathcal{G}_n^{1PI}(t_1, ..., t_n)$ denote 1-particle irreducible Green's functions, i.e. those defined by the sum of the same amplitudes as the usual Green's functions, but taken only over 1-particle irreducible Feynman graphs. Define also the amputated 1-particle irreducible Green's function: $\mathcal{G}_n^{1PIa} = A^{\otimes n}\mathcal{G}_n^{1PI}$ (it is defined by the same sum of amplitudes, except that instead of $G(t_i - s_j)$ for external edges, we write $\delta(t_i - s_j)$).

Let $S_{\text{eff}}(q)$ be the generating function of $\mathcal{G}_n^{1PIa}$ i.e.,

$$S_{\text{eff}}(q) = \sum_n \frac{\hbar^{-n}}{n!} \int \mathcal{G}_n^{1PIa}(t_1, ..., t_n) q(t_1)...q(t_n) dt_1 ... dt_n,$$

Proposition 7.6. *The function $W(J) = \ln(Z(J)/Z(0))$ is the Legendre transform of $S_{\text{eff}}(q)$, i.e. it equals $-S_{\text{eff}}(\tilde{q}_J) + (J, \tilde{q}_J)$, where $\tilde{q}_J$ is the extremal of $-S_{\text{eff}}(q) + (J, q)$ (decaying at infinity).*

The proof of this proposition is the same as in the finite dimensional case. The proposition shows that in order to know the Green's functions, it "suffices" to know amputated 1-particle irreducible Green's functions (the generating function of usual Green's functions can be reconstructed from that for 1PI Green's functions by taking the Legendre transform and exponentiation). Which is a good news, since there are a lot fewer 1PI diagrams than general connected diagrams.

7.8. **Momentum space integration.** We saw that the amplitude of a Feynman diagram is given by an integral over the space of dimension equal to the number of internal vertices. This is sometimes inconvenient, since even for tree diagrams such integrals can be rather complicated. However, it turns out that if one passes to Fourier transforms then Feynman integrals simplify and in particular the number of integrations for a connected diagram becomes equal to the number of loops (so for tree diagrams we have no integrations at all).

Namely, we will proceed as follows. Instead of the time variable t we will consider the dual energy variable E. A function $q(t)$ with compact support will be replaced by its Fourier transform $\hat{q}(E)$. Then, by Plancherel formula, for real functions q_1, q_2, we have

$$(q_1, q_2) = \int_{\mathbb{R}} \hat{q}_1(E)\overline{\hat{q}_2(E)}dE = \int_{\mathbb{R}} \hat{q}_1(E)\hat{q}_2(-E)dE.$$

This implies that the propagator is given by

$$B^{-1}(f,f) = \int_{\mathbb{R}} \frac{1}{E^2+m^2}\hat{f}(E)\hat{f}(-E)dE$$

The vertex tensors standing at k-valent vertices were $\delta_{s_1=...=s_k}$, so they will be replaced by $\delta_{Q_1+...+Q_k=0}$, where Q_i are dual variables to s_i.

Terminological remark. Physicists refer to the time variables t_i, s_j as position variables, and to energy variables E_i, Q_k as momentum variables, since in relativistic mechanics (which is the setting we will deal with when we study field theory) there is no distinction between time and position and between energy and momentum (due to the action of the Lorenz group).

This shows that the Feynman rules "in momentum space" for a given connected Feynman diagram Γ with n external vertices are as follows.

1. Orient the diagram Γ, so that all external edges are oriented inwards.

2. Assign variables E_i to external edges, and variables Q_j to internal ones. These variables are subject to the linear equations of "the first Kirchhof law": at every internal vertex, the sum of the variables corresponding to the incoming edges equals the sum of those corresponding to the outgoing edges. Let $Y(\mathbf{E})$ be the space of solutions $\mathbf{Q}$ of these equations (it depends on Γ, but we will not write the dependence explicitly). It is easy to show that this space is nonempty only if $\sum E_i = 0$, and in that case $\dim Y(\mathbf{E})$ equals the number of loops of Γ (show this!).

3. For each external edge, write $\frac{1}{E_i^2+m^2}$, and for each internal edge, write $\frac{1}{Q_k^2+m^2}$. Let $\phi_\Gamma(\mathbf{E},\mathbf{Q})$ be the product of all these functions.

4. Define the *momentum space amplitude* of Γ to be the distribution $\hat{F}_\Gamma(\mathbf{E})$ on the hyperplane $\sum E_i = 0$ defined by the formula

$$\hat{F}_\Gamma(E_1,...,E_n) = \prod_j (-a_{v(j)}) \int_{Y(\mathbf{E})} \phi_\Gamma(\mathbf{E},\mathbf{Q})d\mathbf{Q}.$$

We will regard it as a distribution on the space of all n-tuples $E_1,...,E_n$, extending it by zero. It is clear that this distribution is independent on the orientation of Γ.

Remark. Here we must specify the normalization of the Lebesgue measure on the space $\sum E_i = 0$ and the space $Y(\mathbf{E})$. The first one is just $dE_1...dE_{n-1}$. To define the second one, let $Y_{\mathbb{Z}}(0)$ be the set of integer elements in $Y(0)$. Then the measure on $Y(\mathbf{E})$ is defined in such a way that the volume of $Y(\mathbf{E})/Y_{\mathbf{Z}}(0)$ is 1.

Now we have

Proposition 7.7. *The Fourier transform of the function* $F_\Gamma(\delta_{t_1}, ..., \delta_{t_n})$ *is* $\hat{F}_\Gamma(E_1, ..., E_n)$. *Hence, the Fourier transform of the connected Green's functions is*

$$\hat{\mathcal{G}}_n^c(E_1, ..., E_n) = \sum_{\Gamma \in G^*_{\geq 3}(n)} \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \hat{F}_\Gamma(E_1, \ldots, E_n). \tag{18}$$

The proof of the proposition is straightforward.

To illustrate the proposition, consider an example.

Example 1. The connected 4-point function for the quartic potential, modulo g^2, in momentum space, looks like:

$$\hat{\mathcal{G}}_n^c(E_1, E_2, E_3, E_4) = -g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta(\sum E_i) + O(g^2).$$

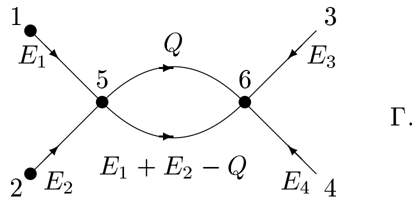


FIGURE 26

Example 2. Let us compute the 1PI 4-point function in the same problem, modulo g^3. Thus, in addition to the above, we need to compute the g^2 coefficient, which comes from 1-loop diagrams. There are three such diagrams, differing by permutation of external edges. One of these diagrams is as follows: it has external vertices $1, 2, 3, 4$ and internal ones $5, 6$ such that $1, 2$ are connected to 5, $3, 4$ to 6, and 5 and 6 are connected by two edges (Fig.26). This diagram has the symmetry group $\mathbb{Z}/2$, so its contribution is

$$\frac{g^2}{2} \left(\int_{\mathbb{R}} \frac{dQ}{(Q^2 + m^2)((E_1 + E_2 - Q)^2 + m^2)} \right) \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \delta(\sum E_i).$$

The integral inside is easy to compute for example by residues. This yields

$$\hat{\mathcal{G}}_n^c(E_1, E_2, E_3, E_4) =$$

$$-g \prod_{i=1}^{4} \frac{1}{E_i^2 + m^2} \left(1 - \frac{\pi g}{m} \sum_{i=2}^{4} \frac{1}{(E_1 + E_i)^2 + 4m^2}\right) \delta(\sum E_i) + O(g^3).$$

7.9. **The Wick rotation in momentum space.** To obtain the correlation functions of quantum mechanics, we should, after computing them in the Euclidean setting, Wick rotate them back to the Minkowski setting. Let us do it at the level of Feynman integrals in momentum space. (We could do it in position space as well, but it is instructive for the future to do it in momentum space, since in higher dimensional field theory which we will discuss later, the momentum space representation is more convenient).

Consider the Euclidean propagator

$$\frac{1}{E^2 + m^2} = \int G(t) e^{iEt} dt,$$

where G is the Green's function. When we do analytic continuation back to the Minkowski setting, we must replace in the correlation functions the time variable t with $e^{i\theta}t$, where θ varies from 0 and $\pi/2$. In particular, the Green's function $G(t)$ must be replaced by $G(e^{i\theta}t)$. So we must consider

$$\int G(e^{i\theta}t)e^{iEt}dt = e^{-i\theta}\int G(t)e^{ie^{-i\theta}Et}dt = \frac{e^{-i\theta}}{e^{-2i\theta}E^2+m^2}.$$

As $\theta \to \pi/2$, this function tends (as a distribution) to the function $\lim_{\varepsilon\to 0+}\frac{i}{E^2-m^2+i\varepsilon}$. For brevity the limit sign is usually dropped and this distribution is written as $\frac{i}{E^2-m^2+i\varepsilon}$.

We see that in order to compute the correlation functions in momentum space in the Minkowski setting, we should use the same Feynman rules as in the Euclidean setting except that the propagator put on the edges should be

$$\frac{i}{E^2-m^2+i\varepsilon}.$$

For instance, the contribution of the diagram in Fig.26 is

$$-\frac{g^2}{2}\Big(\int_{\mathbb{R}}\frac{dQ}{(Q^2-m^2+i\varepsilon)((E_1+E_2-Q)^2-m^2+i\varepsilon)}\Big)\prod_{j=1}^{4}\frac{1}{E_j^2-m^2+i\varepsilon}\delta(\sum E_j).$$

7.10. **Quantum mechanics on the circle.** It is reasonable (at least mathematically) to consider Euclidean quantum mechanical path integrals in the case when the time axis has been replaced with a circle of length L, i.e. $t\in\mathbb{R}/L\mathbb{Z}$. In this case, the theory is the same, except the Green's function $G(t)$ is replaced by the periodic solution $G_L(t)$ of the equation $(-d^2/dt^2+m^2)f=\delta(t)$ on the circle. This solution has the form

$$G_L(t)=\sum_{k\in\mathbb{Z}}G(t-kL)=\frac{e^{-m(t-L/2)}+e^{-m(L/2-t)}}{2m(e^{mL/2}-e^{-mL/2})},\ 0\le t\le L.$$

We note that in the case of a circle, there is no problem with graphs without external edges (as integral over the circle of a constant function is convergent), and hence one may define not only correlation functions (i.e. $Z(J)/Z(0)$), but also $Z(0)$ itself. Namely, let $U(q)=m^2q^2/2+\sum_{n\ge 3}a_nq^n/n!$, and let $m^2=m_0^2+a_2$ (where a_i are formal parameters). Then we can make sense of the ratio $Z_{m_0,\mathbf{a},L}(0)/Z_{m_0,0,L}(0)$ (where $Z_{m,\mathbf{a},L}(0)$ denotes the partition function for the specified values of parameters; from now on the argument 0 will be dropped). Indeed, this ratio is defined by the formula

$$\frac{Z_{m_0,\mathbf{a},L}}{Z_{m_0,0,L}}=\sum_{\Gamma\in G_{\ge 2}(0)}\frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}F_\Gamma,$$

which is a well defined expression.

It is instructive to compute this expression in the case $a_2=a$, $a_3=a_4=...=0$. In this case, we have only 2-valent vertices, so the only connected Feynman diagrams are N-gons, which are 1-loop. Hence,

$$\ln\frac{Z_{m_0,\mathbf{a},L}}{Z_{m_0,0,L}}=W_1=-\frac{1}{2}\ln\det M,$$

where $M = 1 + a(-d^2/dt^2 + m_0^2)^{-1}$. This determinant may be computed by looking at the eigenvalues. Namely, the eigenfunctions of $-d^2/dt^2 + m_0^2$ in the space $C^\infty(\mathbb{R}/L\mathbb{Z})$ are $e^{2\pi int/L}$, with eigenvalues $\frac{4\pi^2n^2}{L^2} + m_0^2$. So,

$$\det M = \prod_{n\in\mathbb{Z}} (1 + \frac{a}{\frac{4\pi^2n^2}{L^2} + m_0^2}).$$

Hence, using the Euler product formula for $\sinh(z)$, we get

$$\frac{Z_{m_0,\mathbf{a},L}}{Z_{m_0,0,L}} = \frac{\sinh(m_0L/2)}{\sinh(mL/2)}$$

Remark. More informally speaking, we see that the partition function Z for the theory with $U = m^2q^2/2$ has the form $\frac{C}{\sinh(mL/2)}$, where C is a constant of our choice. Our choice from now on will be $C = 1/2$; we will see later why such a choice is convenient.

7.11. **The massless case.** Consider now the massless case, $m = 0$. In this case the propagator should be obtained by inverting the operator $-\frac{d^2}{dt^2}$, i.e. it should be a the integral operator with kernel $G(t - s)$, where $G(t)$ is an even function satisfying the differential equation $-G''(t) = \delta(t)$. There is a 1-parameter family of such solutions, $G(t) = -\frac{1}{2}|t| + C$. Using them (for any choice of C), one may define the correlation functions of the theory by the Wick formula.

Note that the function G does not decay at infinity. Therefore, this theory will not satisfy the clustering property (i.e. is not "physically meaningful").

We will also have difficulties in defining the corresponding interacting theory (i.e. one with a non-quadratic potential), as the integrals defining the amplitudes of Feynman diagrams will diverge. Such divergences are called *infrared divergences*, since they are caused by the failure of the integrand to decay at large times (or, in momentum space, its failure to be regular at low frequencies).

7.12. **Circle valued quantum mechanics.** Consider now the theory with the same Lagrangian in which $q(t)$ takes values in the circle of radius r, $\mathbb{R}/2\pi r\mathbb{Z}$ (the "sigma-model"). We can do this at least classically, since the Lagrangian $\dot{q}^2/2$ makes sense in this case.

Let us define the corresponding quantum theory. The main difference from the line-valued case is that since $q(t)$ is circle valued, we should consider not the usual correlators $< q(t_1)...q(t_n) >$, but rather correlation functions exponentials $< e^{ip_1q(t_1)/r}...e^{ip_nq(t_k)/r} >$, where p_i are integers. They should be defined by the path integral

$$\int e^{ip_1q(t_1)/r}...e^{ip_nq(t_n)/r}e^{-S(q)/\hbar}Dq, \tag{19}$$

$S(q) = \frac{1}{2}\int \dot{q}^2dt$ where $\int e^{-S(q)/\hbar}Dq$ is agreed to be 1. Note that we should only consider the case $\sum p_i = 0$, otherwise the group of translations along the circle acts nontrivially on the integrand, and hence under any reasonable definition the integral should be zero.

Now let us define the integral (19). Since the integral is invariant under shifts along the target circle, we may as well imagine that we are integrating over $q : \mathbb{R} \to \mathbb{R}$ with $q(0) = 0$. Now, let us use the finite dimensional analogy. Following this analogy, by completing the square we would get

$$e^{-\frac{\hbar}{2r^2}B^{-1}(\sum p_j q(t_j),\sum p_j q(t_j))}=e^{-\hbar\sum p_lp_jG(t_l-t_j)/2r^2}=e^{\hbar\sum_{l<j}p_lp_j|t_l-t_j|/2r^2}$$

where $B(q,q)=\int \dot q^2 dt$. Thus, it is natural to define the correlators by the formula

$$<e^{ip_1q(t_1)/r}...e^{ip_nq(t_k)/r}>=e^{\hbar\sum_{l<j}p_lp_j|t_l-t_j|/2r^2}.$$

We note that this theory, unlike the line-valued one, does satisfy the clustering property. Indeed, if $\sum p_j=0$ (as we assumed), then (assuming $t_1\ge t_2\ge ...\ge t_n$), we have

$$\sum_{l<j}p_lp_j(t_l-t_j)=\sum_{j=1}^{n-1}(t_j-t_{j+1})(p_{j+1}+...+p_n)(p_1+...+p_j)=-\sum_j(t_j-t_{j+1})(p_1+...+p_j)^2,$$

so the clustering property follows from the fact that $(p_1+...+p_j)^2\ge 0$.

7.13. **Massless quantum mechanics on the circle.** Consider now the theory with Lagrangian $\dot q^2/2$, where q is a function on the circle of length L. In this case, according to the Feynman yoga, we must invert the operator $-d^2/dt^2$ on the circle $\mathbb{R}/L\mathbb{Z}$, or equivalently solve the differential equation $-G''(t)=\delta(t)$. Here we run into trouble: the operator $-d^2/dt^2$ is not invertible, since it has eigenfunction 1 with eigenvalue 0; correspondingly, the differential equation in question has no solutions, as $\int G''dt$ must be zero, so $-G''(t)$ cannot equal $\delta(t)$ (one may say that the quadratic form in the exponential is degenerate, and therefore the Gaussian integral turns out to be meaningless). This problem can be resolved by the following technique of "killing the zero mode". Namely, let us invert the operator $-d^2/dt^2$ on the space $\{q\in C^\infty(\mathbb{R}/L\mathbb{Z}):\int qdt=0\}$ (this may be interpreted as integration over this codimension one subspace, on which the quadratic form is nondegenerate). This means that we must find the solution of the differential equation $-G''(t)=\delta(t)-\frac{1}{L}$, such that $\int Gdt=0$. Such solution is indeed unique, and it equals $G(t)=\frac{(t-L/2)^2}{2L}-\frac{L}{24}$, $t\in[0,L]$. Thus, for example $<q(0)^2>=L/12$.

Higher correlation functions are defined in the usual way. Moreover, one can define the theory with an arbitrary potential using the standard procedure with Feynman diagrams.

Finally, let us consider the circle valued version of the same theory. Thus, our integration variable is a map $q:\mathbb{R}/L\mathbb{Z}\to\mathbb{R}/2\pi r\mathbb{Z}$. Let us first consider integration over degree zero maps. Then we should argue in the same way as in the case $t\in\mathbb{R}$, and make the definition

$$<e^{ip_1q(t_1)/r}...e^{ip_nq(t_n)/r}>_0=e^{-\hbar\sum_{l,j}p_lp_jG(t_l-t_j)/2r^2},$$

where $\sum p_j=0$. (Here subscript 0 stands for degree zero maps). Assuming that $1\ge t_1,...,t_n\ge 0$, we find after a short calculation

$$<e^{ip_1q(t_1)/r}...e^{ip_nq(t_n)/r}>_0=e^{\frac{\hbar}{2r^2}(\sum_{l<j}p_lp_j|t_l-t_j|+(\sum p_jt_j)^2/L)}.$$

(the second summand disappears as $L\to\infty$, and we recover the answer on the line).

It is, however, more natural (as we will see later) to integrate over all (and not only degree zero) maps q. Namely, let N be an integer. Then all maps of degree N have the form $q(t)+2\pi rNt/L$, where q is a map of degree zero. Thus, if we want to integrate over maps of degree N, we should compute the same integral as in degree zero, but with shift $q\to q+2\pi rNt/L$. But it is easy to see that this shift results simply in rescaling of the integrand by the factor $e^{2\pi i \sum p_j t_j N/L - 2\pi^2 r^2 N^2/\hbar L}$. Thus, the integral over all maps should be defined by the formula

$$< e^{ip_1 q(t_1)/r} ... e^{ip_n q(t_n)/r} >=$$

$$e^{\frac{\hbar}{2r^2}(\sum_{l<j} p_l p_j |t_l - t_j| + (\sum p_j t_j)^2/L)} \cdot \frac{\sum_N e^{2\pi i(\sum p_j t_j)N/L - 2\pi^2 r^2 N^2/\hbar L}}{\sum_N e^{-2\pi^2 r^2 N^2/\hbar L}}.$$

Introduce the elliptic theta-function

$$\theta(u, T) = \sum_{N \in \mathbb{Z}} e^{2\pi i u N - T N^2/2}$$

Then the last formula (for $t_1 \geq ... \geq t_n$) can be rewritten in the form
(20)

$$< e^{ip_1 q(t_1)/r} ... e^{ip_n q(t_n)/r} >= e^{\frac{\hbar}{2r^2}(\sum_j (t_{j+1} - t_j)(p_1 + ... + p_j)^2 + (\sum p_j t_j)^2/L)} \frac{\theta(\frac{\sum p_j t_j}{L}, \frac{4\pi^2 r^2}{\hbar L})}{\theta(0, \frac{4\pi^2 r^2}{\hbar L})}.$$

8. Operator approach to quantum mechanics

In mechanics and field theory (both classical and quantum), there are two main languages – Lagrangian and Hamiltonian. In the classical setting, the Lagrangian language is the language of variational calculus (i.e. one studies extremals of the action functional), while the Hamiltonian language is that of symplectic geometry and Hamilton equations. Correspondingly, in the quantum setting, the Lagrangian language is the language of path integrals, while the Hamiltonian language is the language of operators and Schrödinger equation. We have now studied the first one (at least in perturbation expansion) and are passing to the second.

8.1. **Hamilton's equations in classical mechanics.** We start with recalling the Lagrangian formalism of classical mechanics. For more details, we refer the reader to the excellent book of Arnold "Mathematical methods of classical mechanics".

Consider the motion of a classical particle (or system of particles). The position of a particle is described by a point q of the configuration space X, which we will assume to be a manifold. The Lagrangian of the system is a (smooth) function $\mathcal{L} : TX \to \mathbb{R}$ on the total space of the tangent bundle of X. Then the action functional is $S(q) = \int \mathcal{L}(q, \dot{q})dt$. The trajectories of the particle are the extremals of S. The condition for $q(t)$ to be an extremal of S is equivalent to the Euler-Lagrange equation (=the equation of motion), which in local coordinates has the form

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = \frac{\partial L}{\partial q}.$$

For example, if X is a Riemannian manifold, and $\mathcal{L}(q, v) = v^2/2 - U(q)$, where $U : X \to \mathbb{R}$ is a potential function, then the Euler-Lagrange equation is the Newton equation

$$\ddot{q} = -\mathrm{grad} U(q),$$

where $\ddot{q} = \nabla_{\dot{q}} \dot{q}$ is the covariant derivative with respect to the Levi-Civita connection.

Consider now a system with Lagrangian $\mathcal{L}(q, v)$, whose differential with respect to v (for fixed q) is a diffeomorphism $T_q X \to T_q^* X$. This is definitely true in the above special case of Riemannian X.

Definition 8.1. The *Hamiltonian (or energy function)* of the system with Lagrangian $\mathcal{L}$ is the function $H : T^*X \to \mathbb{R}$, which is the Legendre transform of $\mathcal{L}$ along fibers; that is, $H(q,p) = pv_0 - \mathcal{L}(q, v_0)$, where v_0 is the (unique) critical point of $pv - \mathcal{L}(q,v)$. The manifold T^*X is called the *phase space (or space of states)*. The variable p is called the momentum variable.

For example, if $\mathcal{L} = v^2/2 - U(q)$, then $H(q,p) = p^2/2 + U(q)$.

Remark. Since Legendre transform is involutive, we also have that the Lagrangian is the fiberwise Legendre transform of the Hamiltonian.

Let q_i be local coordinates on X. This coordinate system defines a coordinate system (q_i, p_i) on T^*X.

Proposition 8.2. *The equations of motion are equivalent to the Hamilton equations*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

in the sense that they are obtained from Hamilton's equations by elimination of p_i.

It is useful to write Hamilton's equations in terms of Poisson brackets. Recall that the manifold T^*X has a canonical symplectic structure ω. In fact, $\omega = d\alpha$, where α is a canonical 1-form on T^*M constructed as follows: for any $z \in T_{(q,p)}(T^*X)$, $\alpha(z) = (p, d\pi(q,p)z)$, where $\pi : T^*X \to X$ is the projection. In local coordinates, we have $\alpha = \sum p_i dq_i$, and $\omega = \sum dp_i \wedge dq_i$.

Now let (M, ω) be a symplectic manifold (in our case $M = T^*X$). Since ω is nondegenerate, one can define the Poisson bivector ω^{-1}, which is a section of the bundle $\wedge^2 TM$. Now, given any two smooth functions f, g on M, one can define a third function – their *Poisson bracket*

$$\{f, g\} = (df \otimes dg, \omega^{-1})$$

This operation is skew-symmetric and satisfies Jacobi identity, i.e. it is a Lie bracket on $C^\infty(M)$. For $M = T^*X$, in local coordinates we have

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

This shows that Hamilton's equations can be written in the following manner in terms of Poisson brackets:

$$\frac{d}{dt} f(q(t), p(t)) = \{f, H\}(q(t), p(t)). \tag{21}$$

for any smooth function ("classical observable") $f \in C^\infty(T^*X)$. In other words, Hamilton's equations say that the rate of change of the observed value of f equals the observed value of $\{f, H\}$.

Note that for a given Lagrangian, the unique function H (up to adding a constant) for which equations (21) are equivalent to the equations of motion is the Hamiltonian. This provides another definition of the Hamiltonian, which does not use the notion of the Legendre transform.

8.2. **Hamiltonians in quantum mechanics.** The yoga of quantization says that to quantize classical mechanics on a manifold X we need to replace the classical space of states T^*X by the quantum space of states – the Hilbert space $\mathcal{H} = L^2(X)$ on square integrable complex half-densities on X (or, more precisely, the corresponding projective space). Further, we need to replace classical observables, i.e.

real functions $f \in C^\infty(T^*X)$, by quantum observables $\hat{f}$, which are (unbounded) self-adjoint operators on $\mathcal{H}$ (not commuting with each other, in general). Then the (expected) value of an observable A at a state $\psi \in H$ of unit norm is by definition $(\psi, A\psi)$.

The operators $\hat{f}$ should linearly depend on f. More importantly, they should depend on a positive real parameter $\hbar$ called the Planck constant, and satisfy the following relation:

$$[\hat{f}, \hat{g}] = i\hbar\widehat{\{f,g\}} + O(\hbar^2),\ \hbar \to 0.$$

Since the role of Poisson brackets of functions is played in quantum mechanics by commutators of operators, this relation expresses the condition that classical mechanics should be the limit of quantum mechanics as $\hbar \to 0$.

We must immediately disappoint the reader by confessing that there is no canonical choice of the quantization map $f \to \hat{f}$. Nevertheless, there are some standard choices of $\hat{f}$ for particular f, which we will now discuss.

Let us restrict ourselves to the situation $X = \mathbb{R}$, so on the phase space we have coordinates q (position) and p (momentum). In this case there are the following standard conventions.

1. $\hat{f} = f(q)$ (multiplication operator by $f(q)$) when f is independent of p.
2. $\widehat{p^m} \to (-i\hbar\frac{d}{dq})^m$.

(Note that these conventions satisfy our condition, since $[\hat{q}, \hat{p}] = i\hbar$, while $\{q, p\} = 1$.)

Example. For the classical Hamiltonian $H = p^2/2 + U(q)$ considered above, the quantization will be $\hat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q)$.

Remark. The extension of these conventions to other functions is not unique. However, such an extension will not be used, so we will not specify it.

Now let us see what the quantum analog of Hamilton's equations should be. In accordance with the outlined quantization yoga, Poisson brackets should be replaced in quantum theory by commutators (with coefficient $(i\hbar)^{-1} = -i/\hbar$). Thus, the Hamilton's equation should be replaced by the equation

$$\frac{d}{dt}(\psi(t), A\psi(t)) = (\psi(t), \frac{[A, \hat{H}]}{i\hbar}\psi(t)) = -\frac{i}{\hbar}(\psi(t), [A, \hat{H}]\psi(t)),$$

where $(,)$ is the Hermitian form on $\mathcal{H}$ (antilinear on the first factor) and $\hat{H}$ is some quantization of the classical Hamiltonian H. Since this equation must hold for any A, it is equivalent to the Schrödinger equation

$$\dot{\psi} = -\frac{i}{\hbar}\hat{H}\psi.$$

Thus, the quantum analog of the Hamilton equation is the Schrödinger equation.

Remark. This "derivation" of the Schrödinger equation is definitely not a mathematical argument. It is merely a reasoning aimed to motivate a definition.

The general solution of the Schrödinger equation has the form

$$\psi(t) = e^{-it\hat{H}/\hbar}\psi(0).$$

Therefore, for any quantum observable A it is reasonable to define a new observable $A(t) = e^{it\hat{H}/\hbar}A(0)e^{-it\hat{H}/\hbar}$ (such that to observe $A(t)$ is the same as to evolve for time t and then observe A). The observable $A(t)$ satisfies the equation

$$A'(t) = -i[A(t), \hat{H}]/\hbar,$$

and we have

$$(\psi(t), A\psi(t)) = (\psi(0), A(t)\psi(0)).$$

The two sides of this equation represent two pictures of quantum mechanics: Heisenberg's (observables change, states don't), and Schrödinger's (states change, observables don't). The equation expresses the equivalence of the two pictures.

8.3. **Feynman-Kac formula.** Let us consider a 1-duimensional particle with potential $U(q) = m^2q^2 + \sum_{j\geq 3} g_j q^j/j!$. Let us assume that $U \geq 0$ and $U(q) \to \infty$ as $|q| \to \infty$. In this case, the operator $\hat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2} + U(q)$ is positive definite, and its spectrum is discrete. In particular, we have unique lowest eigenvector Ω, which is given by a positive function with norm 1. The correlation functions in the Hamiltonian setting are defined by the formula

$$\mathcal{G}_n^{\mathrm{Ham}}(t_1, ..., t_n) := (\Omega, q(t_1)...q(t_n)\Omega).$$

Remark 1. The vecor Ω is called the ground, or vacuum state, since it has lowest energy, and physicists often shift the Hamiltonian by a constant, so that the energy of this state is zero (i.e. there is no matter).

Remark 2. Physicists usually write the inner product (v, Aw) as $< v|A|w >$. In particular, Ω is written as $< 0|$ or $|0 >$.

Theorem 8.3. *(Feynman-Kac formula) If $t_1 \geq ... \geq t_n$ then the function $\mathcal{G}_n^{\mathrm{Ham}}$ admits an asymptotic expansion in $\hbar$ (near $\hbar = 0$), which coincides with the path integral correlation function $\mathcal{G}_n^M$ constructed above. Equivalently, the Wick rotated function $\mathcal{G}_n^{\mathrm{Ham}}(-it_1, ..., -it_n)$ equals $\mathcal{G}_n^E$.*

This theorem plays a central role in quantum mechanics, and we will prove it below. Before we do so, let us formulate an analog of this theorem for "quantum mechanics on the circle".

Let $\mathcal{G}_{n,L}(t_1, ..., t_n)$ denote the correlation function on the circle of length L (for $0 \leq t_n \leq ... \leq t_1 \leq L$), and let Z_L be the partition function on the circle of length L, defined from path integrals. Also, let

$$Z_L^{\mathrm{Ham}} = \mathrm{Tr}(e^{-L\hat{H}/\hbar}),$$

and

$$\mathcal{G}_{n,L}(t_1, ..., t_n) = \frac{\mathrm{Tr}(q(-it_n)...q(-it_1)e^{-L\hat{H}/\hbar})}{\mathrm{Tr}(e^{-L\hat{H}/\hbar})}$$

Theorem 8.4. *(Feynman-Kac formula on the circle) The functions Z_L^{Ham}, $\mathcal{G}_{n,L}^{\mathrm{Ham}}$ admit asymptotic expansions in $\hbar$, which coincide with the functions Z_L and $\mathcal{G}_{n,L}$ computed from path integrals.*

Note that Theorem 8.3 is obtained from Theorem 8.4 by sending L to infinity. Thus, it is sufficient to prove Theorem 8.4.

Remark. As we mentioned before, the function $\mathcal{G}_n^E$ can be defined by means of Wiener integral, and the equality $\mathcal{G}_n^{\mathrm{Ham}}(-it_1, ..., -it_n) = \mathcal{G}_n^E(t_1, ..., t_n)$ actually holds for numerical values of $\hbar$, and not only in the sense of power series expansions. The same applies to the equalities $Z_L^{\mathrm{Ham}} = Z_L$, $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$. However, these results is technically more complicated and are beyond the scope of these notes.

Example. Consider the case of the quadratic potential. By renormalizing variables, we can assume that $\hbar = m = 1$, so $U = q^2/2$. In this case we know that

$Z_L = \frac{1}{2\sinh(L/2)}$. On the other hand, $\hat{H}$ is the Hamiltonian of the quantum harmonic oscillator:

$$\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}.$$

The eigenvectors of this operator are well known: they are $H_k(x)e^{-x^2/2}$, where H_k are the Hermite polynomials ($k \geq 0$), and the eigenvalues are $k+1/2$ (see Theorem 4.10). Hence,

$$Z_L^{\mathrm{Ham}} = e^{-L/2} + e^{-3L/2} + ... = \frac{1}{e^{L/2} - e^{-L/2}} = Z_L,$$

as expected from the Feynman-Kac formula. (This shows the significance of the choice $C = 1/2$ in the normalization of Z_L).

8.4. **Proof of the Feynman-Kac formula in the free case.** Consider again the quadratic Hamiltonian $\hat{H} = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{q^2}{2}$. Note that it can be written in the form

$$\hat{H} = a^\dagger a + 1/2,$$

where $a = \frac{1}{\sqrt{2}}(\frac{d}{dq}+q)$, $a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dq}+q)$. The operators $a, a^\dagger$ define a representation of the Heisenberg algebra on $\mathcal{H}$:

$$[a, a^\dagger] = 1.$$

Thus the eigenvectors of $\hat{H}$ are $(a^\dagger)^n\Omega$ (where $\Omega = e^{-q^2/2}$) is the lowest eigenvector), and the eigenvalues $n + \frac{1}{2}$ (as we already saw before in Theorem 4.10).

Remark. The operators a and $a^\dagger$ are called the annihilation and creation operators, since $a\Omega = 0$, while all eigenvectors of $\hat{H}$ can be "created" from Ω by action of powers of $a^\dagger$.

Now, we have

$$q(0) = q = \frac{1}{\sqrt{2}}(a + a^\dagger).$$

Since $[a^\dagger a, a] = -a$, $[a^\dagger a, a^\dagger] = a^\dagger$, we have

$$q(t) = \frac{1}{\sqrt{2}}e^{ita^\dagger a}(a + a^\dagger)e^{-ita^\dagger a} = \frac{1}{\sqrt{2}}(e^{-it}a + e^{it}a^\dagger)$$

This shows that

$$\mathcal{G}_{n,L}^{\mathrm{Ham}}(-it_1, ..., -it_n) = 2^{-n/2}\frac{\mathrm{Tr}(\prod_{j=1}^n (e^{t_j}a^\dagger + e^{-t_j}a)e^{-L(a^\dagger a + \frac{1}{2})})}{\mathrm{Tr}(e^{-L(a^\dagger a+\frac{1}{2})})}.$$

Now we can easily prove Theorem 8.4. Indeed, let us move the terms $e^{t_1}a^\dagger$ and $e^{-t_1}a$ around the trace (using the cyclic property of the trace). This will yield, after a short calculation,

$$\mathcal{G}_{n,L}^{\mathrm{Ham}}(t_1, ..., t_n) = \sum_{j=2}^n \frac{1}{2}\mathcal{G}_{n-2,L}(t_2, ..., t_{j-1}, t_{j+1}, ..., t_n)(\frac{e^{t_1-t_j}}{e^L - 1} - \frac{e^{t_j-t_1}}{e^{-L}-1}) =$$

$$\sum_{j=2}^n \mathcal{G}_{n-2,L}(t_2, ..., t_{j-1}, t_{j+1}, ..., t_n)G_L(t_1 - t_j).$$

This implies the theorem by induction.

Note that in the quadratic case there is no formal expansions and the Feynman-Kac formula holds as an equality between usual functions.

8.5. **Proof of the Feynman-Kac formula (general case).** Now we consider an arbitrary potential $U = m^2q^2/2 + V(q)$, where $V(q) = \sum_{k\geq 3} g_k q^k/k!$. For simplicity we will assume that $\hbar = 1$ and coefficients g_j as formal parameters (this does not cause a loss of generality, as this situation can be achieved by rescaling). Let us first consider the case of partition function. We have $Z_L^{\mathrm{Ham}} = \mathrm{Tr}(e^{-L\hat{H}}) = \mathrm{Tr}(e^{-L(\hat{H}_0+V)})$, where $\hat{H}_0 = -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{2}m^2q^2$ is the free (=quadratic) part of the Hamiltonian. Since g_j are formal parameters, we have a series expansion

(22)
$$e^{-L(\hat{H}_0+V)} = e^{-L\hat{H}_0} + \sum_{N\geq 1}(-1)^N \int_{L\geq s_1\geq \ldots\geq s_N\geq 0} e^{-(L-s_1)\hat{H}_0}Ve^{-(s_1-s_2)\hat{H}_0}V \ldots e^{-(s_{n-1}-s_n)\hat{H}_0}Ve^{-s_n\hat{H}_0}d\mathbf{s}$$

This follows from the general fact that in the (completed) free algebra with generators A, B, one has

$$e^{A+B} = e^A + \sum_{N\geq 1}\int_{1\geq s_1\geq \ldots\geq s_N\geq 0} e^{(1-s_1)A}Be^{(s_1-s_2)A}B \ldots e^{(s_{N-1}-s_N)A}Be^{s_NA}d\mathbf{s} \tag{23}$$

(check this identity!).

Equation 22 shows that

$$Z_L^{\mathrm{Ham}} = \sum_{N\geq 0}(-1)^N \sum_{j_1,\ldots,j_N=3}^{\infty} \frac{g_{j_1}\cdots g_{j_N}}{j_1!\ldots j_N!}\mathrm{Tr}(q_0(-is_1)^{j_1}\ldots q_0(-is_N)^{j_N}e^{-L\hat{H}_0}),$$

where $q_0(t)$ is the operator $q(t)$ in the free theory, associated to the potential $m^2q^2/2$.

Since the Feynman-Kac formula for the free theory has been proved, the trace on the right hand side can be evaluated as a sum over pairings. To see what exactly is obtained, let us collect the terms corresponding to all permutations of $j_1, .., j_N$ together. This means that the summation variables will be the numbers $i_3, i_4, ..$ of occurences of $3, 4, ..$ among $j_1, ..., j_N$. Further, to every factor $q_0(-is)^j$ will be assigned a j-valent vertex, with a variable s attached to it, and it is easy to see that Z_L^{Ham} equals the sum over all ways of connecting the vertices (i.e. Feynman diagrams Γ) of integrals

$$\int_{0\leq s_1,\ldots,s_N\leq L} \prod_{\mathrm{edges\, v-w}} G_L(s_v - s_w)d\mathbf{s},$$

multiplied by the coefficients $\frac{\prod(-g_k)^{i_k}}{|\mathrm{Aut}\Gamma|}$. Thus, $Z_L^{\mathrm{Ham}} = Z_L$, as desired.

Now let us consider correlation functions. Thus we have to compute

$$\mathrm{Tr}(e^{-(L-t_1)\hat{H}}qe^{-(t_1-t_2)\hat{H}}q \ldots qe^{-t_n\hat{H}}).$$

Explanding each exponential inside the trace as above, we will clearly get the same Feynman diagram sum, except that the Feynman diagrams will contain n external vertices marked by variables $t_1, ..., t_n$. This implies that $\mathcal{G}_{n,L}^{\mathrm{Ham}} = \mathcal{G}_{n,L}$, and we are done.

8.6. **The massless case.** Consider now the massless case, $m = 0$, in the Hamiltonian setting. For maps $q : \mathbb{R} \to \mathbb{R}$, we have $\mathcal{H} = L^2(\mathbb{R})$, and $\hat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dq^2}$. This operator has continuous spectrum, and there is no lowest eigenvector Ω (more precisely, there is a lowest eigenvector $\Omega = 1$, but it is not in L^2), which means that we cannot define the correlation functions in the usual way, i.e. as $< \Omega, q(t_1) \ldots q(t_n)\Omega >$.

(This is the reflection, in the Hamiltonian setting, of the difficulties related to the growth of the Green's function at infinity, which we encountered in the Lagrangian setting).

Consider now the case $q : \mathbb{R} \to S^1 = \mathbb{R}/2\pi r\mathbb{Z}$. In this case, we have the same Hamiltonian but acting in the space $\mathcal{H} := L^2(S^1)$. The eigenvectors of this operator are $e^{iNq/r}$, with eigenvalues $\hbar^2 N^2/2r^2$. In particular, the lowest eigenvector is $\Omega = 1$. Thus the Hamiltonian correlation functions (in the Euclidean setting, for $t_1 \geq ... \geq t_n$) are

$$(\Omega, e^{t_1\hat{H}/\hbar} e^{ip_1 q/r} e^{(t_2-t_1)\hat{H}/\hbar} ... e^{ip_n q/r} e^{-t_n \hat{H}/\hbar}\Omega) =$$

$$e^{\frac{\hbar}{2r^2}\sum (t_j - t_{j+1})(p_1 + ... p_j)^2},$$

which is equal to the correlation function in the Lagrangian setting. Thus the Feynman-Kac formula holds.

Now we pass to the case of quantum mechanics on the circle. First consider circle valued maps q. In this case, we have $\mathrm{Tr}(e^{-L\hat{H}/\hbar}) = \sum_N e^{-N^2 L\hbar/2r^2}$, and

$$\mathrm{Tr}(e^{t_1\hat{H}/\hbar} e^{ip_1 q/r} e^{(t_2-t_1)\hat{H}/\hbar} ... e^{ip_n q/r} e^{(L-t_n)\hat{H}/\hbar}) = \sum_N e^{\frac{\hbar}{2r^2}\sum_{j=0}^n (t_j - t_{j+1})(N - p_1 - ... - p_j)^2},$$

where $t_{n+1} := L$, $t_0 := 0$. Simplifying this expression, we obtain

$$e^{\frac{\hbar}{2r^2}\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2} \sum_N e^{-\frac{\hbar}{2r^2}(LN^2 + 2\sum_{j=1}^n p_j t_j N)} =$$

$$e^{\frac{\hbar}{2r^2}\sum_j (t_j - t_{j+1})(p_1 + ... + p_j)^2} \theta(\frac{\hbar}{2\pi i r^2}\sum p_j t_j, \frac{L\hbar}{r^2}).$$

Comparing with (20), we see that the Feynman-Kac formula holds, and follows from the modular invariance of the theta-function:

$$\theta(u, T) = e^{-2\pi^2 u^2/T}\theta(\frac{2\pi u}{iT}, \frac{4\pi^2}{T}).$$

(which follows from the Poisson summation formula).

Note that the Feynman-Kac formula would be false if in the Lagrangian setting we had ignored the topologically nontrivial maps. Thus we may say that the Feynman-Kac formula "sees topology". This ability of the Feynman-Kac formula to "see topology" (in much more complex situations) lies at the foundation of many interrelations between geometry and quantum field theory.

Remark. It should be noted that the contributions of topologically nontrivial maps from the source circle to the target circle are, strictly speaking, beyond our usual setting of perturbation theory, since they are exponentially small in $\hbar$. To be specific, the contribution from maps of degree N mostly comes from those maps which are close to the minimal action map $q_N(t) = 2\pi t N r/L$, so it is of the order $e^{-2\pi^2 N^2 r^2/L\hbar}$. The maps $q_N(t)$ are the simplest examples of "instantons" – nonconstant solutions of the classical equations of motion, which have finite action (and are nontrivial in the topological sense). Exponentially small contributions to the path integral coming from integration over neighborhoods of instantons are called "instanton corrections to the perturbation series".

9. Fermionic integrals

9.1. **Bosons and fermions.** In physics there exist two kinds of particles – bosons and fermions. So far we have dealt with bosons only, but many important particles are fermions: e.g., electron, proton, etc. Thus it is important to adapt our techniques to the fermionic case.

In quantum theory, the difference between bosons and fermions is as follows: if the space of states of a single particle is $\mathcal{H}$ then the space of states of the system of k such particles is $S^k\mathcal{H}$ for bosons and $\Lambda^k\mathcal{H}$ for fermions. In classical theory, this means that the space of states of a bosonic particle is a usual real vector space (or more generally a manifold), while for a fermionic particle it is an *odd* vector space. Mathematically "odd" means that the ring of smooth functions on this space (i.e. the ring of classical observables) is an *exterior* algebra (unlike the case of a usual, *even* space, for which the ring of polynomial functions is a *symmeteric* algebra).

More generally, one may consider systems of classical particles or fields some of which are bosonic and some fermionic. In this case, the space of states will be a supervector space, i.e. the direct sum of an even and an odd space (or, more generally, a supermanifold – a notion we will define below).

When such a theory is quantized using the path integral approach, one has to integrate functions over supermanifolds. Thus, we should learn to integrate over supermanifolds and then generalize to this case our Feynman diagram techniques. This is what we do in this section.

9.2. **Supervector spaces.** Let k be a field of characteristic zero. A supervector space (or shortly, superspace) over k is just a $\mathbb{Z}/2$-graded vector space: $V = V_0 \oplus V_1$. If $V_0 = k^n$ and $V_1 = k^m$ then V is denoted by $k^{n|m}$. The notions of a homomorphism, direct sum, tensor product, dual space for supervector spaces are defined in the same way as for $\mathbb{Z}/2$-graded vector spaces. In other words, the tensor category of supervector spaces is the same as that of $\mathbb{Z}/2$-graded vector spaces.

However, the notions of a supervector space and a $\mathbb{Z}/2$-graded vector space are *not* the same. The difference is as follows. The category of vector (and hence $\mathbb{Z}/2$-graded vector) spaces has an additional *symmetry* structure, which is the standard isomorphism $V \otimes W \to W \otimes V$ (given by $v \otimes w \to w \otimes v$). This isomorphism allows one to define symmetric powers S^mV, exterior powers Λ^mV, etc. For supervector spaces, there is also a symmetry $V \otimes W \to W \otimes V$, but it is defined differently. Namely, $v \otimes w$ goes to $(-1)^{mn} w \otimes v$, $v \in V_m, w \in V_n$ $(m, n \in \{0, 1\})$. In other words, it is the same as usual except that if v, w are odd then $v \otimes w \to -w \otimes v$. As a result, we can define the superspaces S^mV and Λ^mV for a superspace V, but they are not the same as the symmetric and exterior powers in the usual sense. For example, if V is purely odd $(V = V_1)$, then S^mV is the exterior m-th power of V, and Λ^mV is the m-th symmetric power of V (purely even for even m and purely odd for odd m).

For a superspace V, let ΠV be the same space with opposite parity, i.e. $(\Pi V)_i = V_{1-i}$, $i = 0, 1$. With this notation, the equalities explained in the previous paragraph can be written as: $S^mV = \Pi^m(\Lambda^m \Pi V)$, $\Lambda^mV = \Pi^m(S^m\Pi V)$.

Let $V = V_0 \oplus V_1$ be a finite dimensional superspace. Define the algebra of polynomial functions on V, $\mathcal{O}(V)$, to be the algebra SV^* (where symmetric powers are taken in the super sense). Thus, $\mathcal{O}(V) = SV_0^* \otimes \Lambda V_1^*$, where V_0 and V_1 are regarded as usual spaces. More explicitly, if $x_1, ..., x_n$ are linear coordinates on V_0, and $\xi_1, ..., \xi_m$ are linear coordinates on V_1, then $\mathcal{O}(V) = k[x_1, ..., x_n, \xi_1, ..., \xi_m]$, with defining relations

$$x_i x_j = x_j x_i, x_i \xi_r = \xi_r x_i, \xi_r \xi_s = -\xi_s \xi_r$$

Note that this algebra is itself a (generally, infinite dimensional) supervector space, and is commutative in the supersense. Also, if V, W are two superspaces, then $\mathcal{O}(V \oplus W) = \mathcal{O}(V) \otimes \mathcal{O}(W)$, where the tensor product of algebras is understood in the supersense (i.e. $(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd)$, where $p(x)$ is the parity of x).

9.3. **Supermanifolds.** Now assume that $k = \mathbb{R}$. Then by analogy with the above for any supervector space V we can define the algebra of smooth functions, $C^\infty(V) := C^\infty(V_0) \otimes \Lambda V_1^*$. In fact, this is a special case of the following more general setting.

Definition 9.1. A supermanifold M is a usual manifold M_0 with a sheaf C_M^∞ of $\mathbb{Z}/2\mathbb{Z}$ graded algebras (called the structure sheaf), which is locally isomorphic to $C_{M_0}^\infty \otimes \Lambda(\xi_1, ..., \xi_m)$.

The manifold M_0 is called the reduced manifold of M. The dimension of M is the pair of integers $\dim M_0 | m$.

For example, a supervector space V is a supermanifold of dimension $\dim V_0 | \dim V_1$. Another (more general) example of a supermanifold is a superdomain $U := U_0 \times V_1$, i.e. a domain $U_0 \subset V_0$ together with the sheaf $C_U^\infty \otimes \Lambda V_1^*$. Moreover, the definition of a supermanifold implies that any supermanifold is "locally isomorphic" to a superdomain.

Let M be a supermanifold. An open set U in M is the supermanifold $(U_0, C_M^\infty|_{U_0})$, where U_0 is an open subset in M_0.

By the definition, supermanifolds form a category $\mathcal{SMAN}$. Let us describe explicitly morphisms in this category, i.e. maps $F : M \to N$ between supermanifolds M and N. By the definition, it suffices to assume that M, N are superdomains, with global coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$, and $y_1, ..., y_p, \eta_1, ..., \eta_q$, respectively (here x_i, y_i are even variables, and ξ_i, η_i are odd variables). Then the map F is defined by the formulas:

$$y_i = f_{0,i}(x_1, ..., x_n) + f_{2,i}^{j_1 j_2}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2} + ...,$$

$$\eta_i = a_{1,i}^j(x_1, ..., x_n)\xi_j + a_{3,i}^{j_1 j_2 j_3}(x_1, ..., x_n)\xi_{j_1}\xi_{j_2}\xi_{j_3} + ...,$$

where $f_{0,i}, f_{2,i}^{j_1 j_2}, a_{1,i}^j, a_{3,i}^{j_1, j_2, j_3}, ...$ are usual smooth functions, and we assume summation over repeated indices. These formulas, determine F completely, since for any $g \in C^\infty(N)$ one can find $g \circ F \in C^\infty(M)$ by Taylor's formula. For example, if $M = N = \mathbb{R}^{1|2}$ and $F(x, \xi_1, \xi_2) = (x + \xi_1\xi_2, \xi_1, \xi_2)$, and if $g = g(x)$, then $g \circ F(x, \xi_1, \xi_2) = g(x + \xi_1\xi_2) = g(x) + g'(x)\xi_1\xi_2$.

Remark. For this reason, one may consider only C^∞ (and not C^r) functions on supermanifolds. Indeed, if for example $g(x)$ is a C^r function of one variable which is not differentiable $r+1$ times, then the expression $g(x + \sum_{i=1}^{r+1} \xi_{2i-1}\xi_{2i})$ will not be defined, because the coefficient of $\xi_1 ... \xi_{2r+2}$ in this expression should be $g^{(r+1)}(x)$, but this derivative does not exist.

9.4. **Supermanifolds and vector bundles.** Let M_0 be a manifold, and E be a vector bundle on M_0. Then we can define the supermanifold $M := \mathrm{Tot}(\Pi E)$, the total space of E with changed parity. Namely, the reduced manifold of M is M_0, and the structure sheaf C_M^∞ is the sheaf of sections of ΛE^*. This defines a functor $S : \mathcal{BUN} \to \mathcal{SMAN}$, from the category of manifolds with vector bundles to the category of supermanifolds. We also have a functor S_* in the opposite direction: namely, $S_*(M)$ is the manifold M_0 with the vector bundle $(R/R^2)^*$, where R is the nilpotent radical of C_M^∞.

The following proposition (whose proof we leave as an exercise) gives a classification of supermanifolds.

Proposition 9.2. *$S_*S = Id$, and $SS_* = Id$ on isomorphism classes of objects.*

The usefulness of this proposition is limited by the fact that, as one can see from the above description of maps between supermanifolds, SS_* is *not* the identity on morphisms (e.g. it maps the automorphism $x \to x + \xi_1\xi_2$ of $\mathbb{R}^{1|2}$ to Id), and hence, S is not an equivalence of categories. In fact, the category of supermanifolds is not equivalent to the category of manifolds with vector bundles (namely, the category of supermanifolds "has more morphisms").

Remark. The relationship between these two categories is quite similar to the relationship between the categories of (finite dimensional) filtered and graded vector spaces, respectively (namely, for them we also have functors S, S_* with the same properties – check it!). Therefore in supergeometry, it is better to avoid using realizations of supermanifolds as $S(M_0, E)$, similarly to how in linear algebra it is better to avoid choosing a grading on a filtered space.

9.5. **Integration on superdomains.** We would now like to develop integration theory on supermanifolds. Before doing so, let us recall how it is done for usual manifolds. In this case, one proceeds as follows.

1. Define integration of compactly supported (say, smooth) functions on a domain in $\mathbb{R}^n$.

2. Find the transformation formula for the integral under change of coordinates (i.e. discover the factor $|J|$, where J is the Jacobian).

3. Define a density on a manifold to be a quantity which is locally the same as a function, but multiplies by $|J|$ under coordinate change (unlike true functions, which don't multiply by anything). Then define integral of compactly supported functions on the manifold using partitions of unity. The independence of the integral on the choices is guaranteed by the change of variable formula and the definition of a density.

We will now realize this program for supermanifolds. We start with defining integration over superdomains.

Let $V = V_0 \oplus V_1$ be a supervector space. The *Berezinian* of V is the line $\Lambda^{\mathrm{top}} V_0 \otimes \Lambda^{\mathrm{top}} V_1^*$. Suppose that V is equipped with a nonzero element dv of the Berezinian (called a supervolume element).

Let U_0 be an open set in V_0, and $f \in C^\infty(U) \otimes \Lambda V_1^*$ be a compactly supported smooth function on the superdomain $U := U_0 \times V_1$ (i.e. $f = \sum f_i \otimes \omega_i$, $f_i \in C^\infty(U)$, $\omega_i \in \Lambda V_1^*$, and f_i are compactly supported). Let dv_0, dv_1 be volume forms on V_0, V_1 such that $dv = dv_0/dv_1$.

Definition 9.3. The integral $\int_U f(v)dv$ is $\int_{U_0}(f(v), (dv_1)^{-1})dv_0$.

It is clear that this quantity depends only on dv and not on dv_0 and dv_1 separately.

Thus, $\int f(v)dv$ is defined as the integral of the suitably normalized top coefficient of f (expanded with respect to some homogeneous basis of ΛV_1^*). To write it in coordinates, let $\xi_1,...,\xi_m$ be a linear system of coordinates on V such that $dv = \frac{dx_1...dx_n}{d\xi_1...d\xi_m}$ (such coordinate systems will be called unimodular with respect to dv). Then $\int f(v)dv$ equals $\int f_{\text{top}}(x_1,...,x_n)dx_1...dx_n$, where f_{top} is the coefficient of $\xi_1...\xi_n$ in the expansion of f.

9.6. **The Berezinian of a matrix.** Now we generalize to the supercase the definition of determinant (since we need to generalize Jacobian, which is a determinant).

Let R be a supercommutative ring. Fixt two nonnegative integers m, n. Let A be a $n+m$ by $n+m$ matrix over R. Split A in the blocks $A_{11}, A_{12}, A_{21}, A_{22}$ so that A_{11} is n by n, and A_{22} is m by m. Assume that the matrices A_{11}, A_{22} have even elements, while A_{21} and A_{12} have odd elements. Assume also that A_{22} is invertible.

Definition 9.4. The Berezinian of A is the element

$$\mathrm{Ber}(A) := \frac{\det(A_{11} - A_{12}A_{22}^{-1}A_{21})}{\det(A_{22})} \in R$$

(where the determinant of the empty matrix is agreed to be 1; so for $m = 0$ one has $\mathrm{Ber}A = \det A$, and for $n = 0$ one has $\mathrm{Ber}A = (\det A)^{-1}$).

Remark. Recall for comparison that if A is purely even then

$$\det(A) := \det(A_{11} - A_{12}A_{22}^{-1}A_{21})\det(A_{22}).$$

The Berezinian has the following simpler description. Any matrix A as above admits a unique factorization $A = A_+A_0A_-$, where A_+, A_0, A_- are as above, and in addition A_+, A_- are block upper (respectively, lower) triangular with 1 on the diagonal, while A_0 is block diagonal. Then $\mathrm{Ber}(A) = \det((A_0)_{11})/\det((A_0)_{22})$.

Proposition 9.5. *If A, B be matrices as above, then* $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{Ber}(B)$.

Proof. From the definition using triangular factorization, it is clear that it suffices to consider the case $A = A_-$, $B = B_+$. Let $X = (A_-)_{21}$, $Y = (B_+)_{12}$ (matrices with odd elements). Then the required identity is

$$\det(1 - Y(1 + XY)^{-1}X) = \det(1 + XY).$$

To prove this relation, let us take the logarithm of both sides and expand using Taylor's formula. Then the left hand side gives

$$-\sum_{k\geq 1}\mathrm{Tr}(Y(1+XY)^k(XY)^{k-1}X)/k$$

Using the cyclic property of the trace, we transform this to

$$\sum_{k\geq 1}\mathrm{Tr}((1+XY)^k(XY)^k)/k$$

(the minus disappears since X, Y have odd elements). Summing the series, we find that the last expression equals

$$-\mathrm{Tr}\ln(1 - (1+XY)^{-1}XY) = \mathrm{Tr}\ln(1+XY),$$

as desired. ∎

The additive analog of Berezinian is supertrace. Namely, for A as above, $\mathrm{sTr}A = \mathrm{Tr}A_{11} - \mathrm{Tr}A_{22}$. It is the correct superanalogue of the usual trace, as it satisfies the equation $\mathrm{sTr}(AB) = \mathrm{sTr}(BA)$ (while the usual trace does not). The connection between the supertrace and the Berezinian is given by the formula

$$\mathrm{Ber}(e^A) = e^{\mathrm{sTr(A)}}.$$

Exercise. Prove this formula.

9.7. **Berezin's change of variable formula.** Let V be a vector space, $f \in \Lambda V^*$, $v \in V$. Denote by $\frac{\partial f}{\partial v}$ the result of contraction of f with v.

Let U, U' be superdomains, and $F : U \to U'$ be a morphism. As explained above, given linear coordinates $x_1, ..., x_n, \xi_1, ..., \xi_m$ on U and $y_1, ..., y_p, \eta_1, ..., \eta_q$ on U', we can describe f by expressing y_i and η_i as functions of x_j and ξ_j. Define the Berezin matrix of F, $A := DF(x, \xi)$ by the formulas:

$$A_{11} = (\frac{\partial y_i}{\partial x_j}),\ A_{12} = (\frac{\partial y_i}{\partial \xi_j}),\ A_{21} = (\frac{\partial \eta_i}{\partial x_j}),\ A_{22} = (\frac{\partial \eta_i}{\partial \xi_j}).$$

Clearly, this is a superanalog of the Jacobi matrix.

The main theorem of supercalculus is the following theorem.

Theorem 9.6. *(Berezin) Let g be a smooth function with compact support on U', and $F : U \to U'$ be an isomorphism. Let dv, dv' be supervolume elements on U, U'. Then*

$$\int_{U'} g(v')dv' = \int_U g(F(v))|\mathrm{Ber}DF(v)|dv,$$

where the Berezinian is computed with respect to unimodular coordinate systems.

Remark. If $f(\xi) = a+$terms containing ξ_j then by definition $|f(\xi)| := f(\xi)$ is $a > 0$ and $-f(\xi)$ if $a < 0$.

Proof. The chain rule of the usual calculus extends verbatim to supercalculus. Also, we have shown that $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\mathrm{BerB}$. Therefore, if we know the statement of the theorem for two isomorphisms $F_1 : U_2 \to U_1$ and $F_2 : U_3 \to U_2$, then we know it for the composition $F_1 \circ F_2$.

Let $F(x_1, ..., x_n, \xi_1, ..., \xi_m) = (x'_1, ..., x'_n, \xi'_1, ..., \xi'_m)$. From what we just explained it follows that it suffices to consider the following cases.

1. x'_i depend only on x_j, $j = 1, ..., n$, and $\xi'_i = \xi_i$.
2. $x'_i = x_i + z_i$, where z_i lie in the ideal generated by ξ_j, and $\xi'_i = \xi_i$.
3. $x'_i = x_i$.

Indeed, it is clear that any isomorphism F is a composition of isomorphisms of the type $1, 2, 3$.

In case 1, the statement of the theorem follows from the usual change of variable formula. Thus it suffices to consider cases 2 and 3.

In case 2, it is sufficient to consider the case when only one coordinate is changed by F, i.e. $x'_1 = x_1 + z$, and $x'_i = x_i$ for $i \geq 2$. In this case we have to show that the integral of

$$g(x_1 + z, x_2, ..., x_n, \xi)(1 + \frac{\partial z}{\partial x_1}) - g(x, \xi)$$

is zero. But this follows easily upon expansion in powers of z, since all the terms are manifestly total derivatives with respect to x_1.

In case 3, we can also assume $\xi'_i = \xi_i$, $i \geq 2$, and a similar (actually, even simpler) argument proves the result. $\square$

9.8. **Integration on supermanifolds.** Now we will define densities on supermanifolds. Let M be a supermanifold, and $\{U_\alpha\}$ be an open cover of M together with isomorphisms $f_\alpha : U_\alpha \to U'_\alpha$, where U'_α is a superdomain in $\mathbb{R}^{n|m}$. Let $g_{\alpha\beta} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ be the transition map $f_\alpha f_\beta^{-1}$. Then a density s on M is a choice of an element $s_\alpha \in C_M^\infty(U_\alpha)$ for each α, such that on $U_\alpha \cap U_\beta$ one has $s_\beta(z) = s_\alpha(z)|\mathrm{Ber}(g_{\alpha\beta})(f_\beta(z))|$.

Remark. It is clear that a density on M is a global section of a certain sheaf on M, called the sheaf of densities.

Now, for any (compactly supported) density ω on M, the integral $\int_M \omega$ is well defined. Namely, it is defined as is usual calculus: one uses partition of unity ϕ_α such that $\mathrm{Supp}\phi_\alpha \subset (U_\alpha)_0$ are compact subsets, and sets $\int_M \omega := \sum_\alpha \int_M \phi_\alpha \omega$ (where the summands can be defined using f_α). Berezin's theorem guarantees then that the final answer will be independent on the choices made.

9.9. **Gaussian integrals in an odd space.** Now let us generalize to the odd case the theory of Gaussian integrals, which was, in the even case, the basis for the path integral approach to quantum mechanics and field theory.

Recall first the notion of Pfaffian. Let A be a skew-symmetric matrix of even size. Then the determinant of A is the square of a polynomial in the entries of A. This polynomial is determined by this condition up to sign. The sign is usually fixed by requiring that the polynomial should be 1 for the direct sum of matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With this convention, this polynomial is called the Pfaffian of A and denoted PfA. The Pfaffian obviously has the property $\mathrm{Pf}(X^TAX) = \mathrm{Pf}(A)\det(X)$ for any matrix X.

Let now V be an $2m$-dimensional vector space with a volume element dv, and B a skew-symmetric bilinear form on V. We define the Pfaffian PfB of B to be the Pfaffian of the matrix of B in any unimodular basis by the above transfoamtion formula, it does not depend on the choice of the basis). It is easy to see (by reducing B to canonical form) that

$$\frac{\wedge^m B}{m!} = \mathrm{Pf}(B)dv.$$

In terms of matrices, this translates into the following (well known) formula for the Pfaffian of a skew symmetric matrix of size $2m$:

$$\mathrm{Pf}(A) = \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1,...,2m\}, i<\sigma(i)} a_{i\sigma(i)},$$

where Π_m is the set of pairings of $\{1,...,2m\}$, and ε_σ is the sign of the permutation sending $1,...,2m$ to $i_1, \sigma(i_1), ..., i_m, \sigma(i_m)$ (where $i_r < \sigma(i_r)$ for all r). For example, for $m = 2$ (i.e. a 4 by 4 matrix),

$$\mathrm{Pf}(A) = a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24}.$$

Now consider an odd vector space V of dimension $2m$ with a volume element $d\xi$. Let B be a symmetric bilinear form on V (i.e. a skewsymmetric form on ΠV). Let $\xi_1, ..., \xi_{2m}$ be unimodular linear coordinates on V (i.e. $d\xi = d\xi_1 \wedge ... \wedge d\xi_m$). Then if $\xi = (\xi_1, ..., \xi_n)$ then $B(\xi, \xi) = \sum_{i,j} b_{ij}\xi_i\xi_j$, where b_{ij} is a skewsymmetric matrix.

Proposition 9.7.

$$\int_V e^{\frac{1}{2}B(\xi,\xi)}(d\xi)^{-1} = \mathrm{Pf}(B).$$

Proof. The integral equals $\frac{\wedge^m B}{m! d\xi}$, which is $\mathrm{Pf}(B)$. $\square$

Example. Let V is a finite dimensional odd vector space, and $Y = V \oplus V^*$. The space Y has a canonical volume element $dvdv^*$, defined as follows: if $e_1, ..., e_m$ be a basis of V and $e_1^*, ..., e_n^*$ is the dual basis of V^* then $dvdv^* = e_1 \wedge e_1^* ... \wedge e_n \wedge e_n^*$. Let $dy = (dvdv^*)^{-1}$ be the corresponding supervolume element.

Let $A : V \to V$ be a linear operator. Then we can define an even smooth function S on the odd space Y as follows: $S(v, v^*) = (Av, v^*)$. More explicitly, if ξ_i be coordinates on V corresponding to the basis e_i, and η_i the dual system of coordinates on V^*, then

$$S(\xi_1, ..., \xi_m, \eta_1, ..., \eta_m) = \sum a_{ij} \xi_j \eta_i,$$

where (a_{ij}) is the matrix of A in the basis e_i.

Proposition 9.8.

$$\int_Y e^S dy = \det A$$

Proof. We have $S(v, v_*) = \frac{1}{2} B((v, v_*), (v, v_*))$, where B is the skew form on ΠY, which is given by the formula $B((v, v^*), (w, w^*)) = (Av, w^*) - (Aw, v^*)$. It is easy to see that $\mathrm{Pf}(B) = \det(A)$, so Proposition 9.8 follows from Proposition 9.7.

Another proof can be obtained by direct evaluation of the top coefficient. $\square$

9.10. **The Wick formula in the odd case.** Let V be a $2m$-dimensional odd space with a volume form $d\xi$, and $B \in S^2 V$ a nondegenerate form (symmetric in the supersense and antisymmetric in the usual sense). Let $\lambda_1, ..., \lambda_n$ be linear functions on V (regarded as the usual space). Then $\lambda_1, ..., \lambda_n$ can be regarded as odd smooth functions on the superspace V.

Theorem 9.9.

$$\int_V \lambda_1(\xi) ... \lambda_n(\xi) e^{-\frac{1}{2} B(\xi, \xi)} (d\xi)^{-1} = \mathrm{Pf}(-B) \mathrm{Pf}(B^{-1}(\lambda_i, \lambda_j)).$$

(By definition, this is zero if n is odd). In other words, we have:

$$\int_V \lambda_1(\xi) ... \lambda_n(\xi) e^{-\frac{1}{2} B(\xi, \xi)} (d\xi)^{-1} = \mathrm{Pf}(-B) \sum_{\sigma \in \Pi_m} \varepsilon_\sigma \prod_{i \in \{1, ..., 2m\}, i < \sigma(i)} (B^{-1}(\lambda_i, \lambda_{\sigma(i)})).$$

Proof. We prove the second formula. Choose a basis e_i of V with respect to which the form B is standard: $B(e_j, e_l) = 1$ if $j = 2i - 1, l = 2i$, and $B(e_j, e_l) = 0$ for other pairs $j < l$. Since both sides of the formula are polylinear with respect to $\lambda_1, ..., \lambda_n$, it suffices to check it if $\lambda_1 = e_{i_1}^*, ..., \lambda_n = e_{i_n}^*$. This is easily done by direct computation (in the sum on the right hand side, only one term may be nonzero). $\square$

10. Quantum mechanics for fermions

10.1. **Feynman calculus in the supercase.** Wick's theorem allows us to extend Feynman calculus to the supercase. Namely, let $V = V_0 \oplus V_1$ be a finite dimensional real superspace with a supervolume element dv, equipped with a symmetric nondegenerate form $B = B_0 \oplus B_1$ ($B_0 > 0$). Let $S(v) = \frac{1}{2} B(v, v) + \sum_{r \geq 3} \frac{B_r(v, v, ..., v)}{r!}$ be an even function on V (the action). Note that B_r, $r \geq 3$ can contain mixed terms involving both odd and even variables, e.g. $x\xi_1\xi_2$ (the so called "Yukawa term"). We will consider the integral

$$I(\hbar) = \int_V \ell_1(v_0)...\ell_n(v_0)\lambda_1(v_1)...\lambda_p(v_1)e^{-S(v)/\hbar}dv.$$

(where v_0, v_1 are the even and odd components of v). Then this integral has an expansion in $\hbar$ written in terms of Feynman diagrams. Since v has both odd and even part, these diagrams will contain "odd" and "even" edges (which could be depicted by straight and wiggly lines respectively). More precisely, let us write

$$B_r(v,v,...,v) = \sum_{s=0}^{r} \binom{r}{s} B_{s,r-s}(v_1,...,v_1,v_0,...,v_0),$$

where $B_{s,r-s}$ has homogeneity degree s with respect to v_1 and $r-s$ with respect to v_0 (i.e. it will be nonzero only for even s). Then to each term $B_{s,r-s}$ we assign an $(s,r-s)$-valent flower, i.e. a flower with s odd and $r-s$ even outgoing edges, and for the set of odd outgoing edges, it has been specified which orderings are even. Then, given an arrangement of flowers, for every pairing σ of outgoing edges, we can define an amplitude F_σ by contracting the tensors $-B_{s,r-s}$ (and being careful with the signs). It is easy to check that all pairings giving the same graph will contribute to $I(\hbar)$ with the same sign, and this we have almost the same formula as in the bosonic case:

$$I(\hbar) = (2\pi)^{\dim V_0/2}\hbar^{\frac{\dim V_0 - \dim V_1}{2}} \frac{\mathrm{Pf}(-B_1)}{\sqrt{\det B_0}} \sum_\Gamma \frac{\hbar^{b(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} F_\Gamma,$$

where the summation is taken over graphs with n even and p odd outgoing edges.

Remark. More precisely, we can define the sign ε_σ of a pairing σ as follows: label outgoing edges by $1,...,$ starting from the fisrt flower, then second, etc., so that the labeling is even on each flower. Then write the labels in a sequence, enumerating (in any order) the pairs defined by σ (the element with the smaller of the two labels goes first). The sign ε_σ is by definition the sign of this ordering (as a permutation of $1,...$). Then F_Γ is F_σ for any pairing σ yielding Γ which is *positive*, i.e. such that $\varepsilon_\sigma = 1$. For negative pairing, $F_\Gamma = -F_\sigma$.

In most (but not all) situations considered in physics, the action is quadratic in the fermionic variables, i.e. $S(v) = S_b(v_0) - S_f(v_0)(v_1, v_1)$, where $S_f(v_0)$ is a skew-symmetric bilinear form on ΠV_1. In this case, using fermionic Wick's theorem, we can perform exact integration with respect to v_1, and reduce $I(\hbar)$ to a purely bosonic integral. For example, if we have only ℓ_i and no λ_i, we have

$$I(\hbar) = \int_{V_0} \ell_1(v_0)...\ell_n(v_0)e^{-S_b(v_0)/\hbar}\mathrm{Pf}(S_f(v_0))dv_0.$$

In this situation, all vertices which have odd outgoing edges, will have only two of them, and therefore in any Feynman diagram with even outgoing edges, odd lines form nonintersecting simple curves, called fermionic loops (in fact, the last formula is nothing but the result of regarding these lopps as a new kind of vertices – convince yourself of this). In this case, there is the following simple way of assigning signs to Feynman diagrams. For each vertex with two odd outgoing edges, we orient the first edge inward and the second one outward. We allow only connections (pairings) that preserve orientations (so the fermionic loops become oriented). Then the sign is

$(-1)^r$, where r is the number of fermionic loops (i.e. each fermionic loop contributes a minus sign).

10.2. **Fermionic quantum mechanics.** Let us now pass from finite dimensional fermionic integrals to quantum mechanics, i.e. integrals over fermionic functions of one (even) real variable t.

Let us first discuss fermionic classical mechanics, in the Lagrangian setting. Its difference with the bosonic case is that the "trajectory" of the particle is described by an *odd*, rather than even, function of one varible, i.e. $\psi : \mathbb{R} \to \Pi V$, where V is a vector space. Mathematically this means that the space of fields (=trajectories) is an odd vector space $\Pi C^\infty(\mathbb{R}, V)$. A Lagrangian $\mathcal{L}(\psi)$ is a local expression in such a field (i.e. a polynomial in $\psi, \dot{\psi}, ...$), and an action is the integral $S = \int_{\mathbb{R}} \mathcal{L}dt$. This means that the action is an element of the space $\Lambda C_0^\infty(\mathbb{R}, V)^*$.

Consider for example the theory of a single scalar-valued free fermion $\psi(t)$. By definition, the Lagrangian for such a theory is $\mathcal{L} = \frac{1}{2}\psi\dot{\psi}$, i.e. the action is $S = \frac{1}{2}\int \psi\dot{\psi}dt$.

This Lagrangian is the odd analog of the Lagrangian of a free particle, $\dot{q}^2/2$.

Remark. Note that $\psi\dot{\psi} \neq \frac{d}{dt}(\psi^2/2) = 0$, since $\psi\dot{\psi} = -\dot{\psi}\psi$, so this Lagrangian is "reasonable". On the other hand, the same Lagrangian would be unreasonable in the bosonic case, as it would be a total derivative, and hence the action would be zero. Finally, note that it would be equally unreasonable to use in the fermionic case the usual bosonic Lagrangian $\frac{1}{2}(\dot{q}^2 - m^2q^2)$; it would identically vanish if q were odd-valued.

The Lagrangian $\mathcal{L}$ is invariant under the group of reparametrizations $\mathrm{Diff}_+(\mathbb{R})$, and the Euler-Lagrange equation for this Lagrangian is $\dot{\psi} = 0$ (i.e. no dynamics). Theories with such properties are called *topological field theories.*

Let us now turn to quantum theory in the Lagrangian setting, i.e. the theory given by the Feynman integral $\int \psi(t_1)...\psi(t_n)e^{iS(\psi)}D\psi$. In the bosonic case, we "integrated" such expressions over the space $C_0^\infty(\mathbb{R})$. This integration did not make immediate sense because of failure of measure theory in infinite dimensions. So we had to make sense of this integration in terms of $\hbar$-expansion, using Wick's formula and Feynman diagrams. In the fermionic case, the situation is analogous. Namely, now we must integrate functions over $\Pi C_0^\infty(\mathbb{R})$, which are elements of $\Lambda\mathcal{D}(\mathbb{R})$, where $\mathcal{D}(\mathbb{R})$ is the space of distributions on $\mathbb{R}$. Although in the fermionic case we don't need measure theory (as integration is completely algebraic), we still have trouble defining the integral: recall that by definition the integral should the top coefficient of the integrand as the element of $\Lambda\mathcal{D}(\mathbb{R})$, which makes no sense since in the exterior algebra of an infinite dimensional space there is no top component. Thus we have to use the same strategy as in the bosonic case, i.e. Feynman diagrams.

Let us, for instance, define the quantum theory for a free scalar valued fermion, i.e one described by the Lagrangian $\mathcal{L} = \frac{1}{2}\psi\dot{\psi}$. According to the yoga we used in the bosonic case, the two point function of this theory $< \psi(t_1)\psi(t_2) >$ should be the function $G(t_1 - t_2)$, where G is the solution of the differential equation $-i\frac{dG}{dt} = \delta(t)$.

The general solution of this equation has the form $G(t) = -\frac{1}{2i}\mathrm{sign}(t) + C$. Because of the fermionic nature of the field $\psi(t)$, it is natural to impose the requirement that $G(-t) = -G(t)$, i.e that the correlation functions are antisymmetric; this singles out the solution $G(t) = -\frac{1}{2i}\mathrm{sign}(t)$ (we also see from this condition that we should set $G(0) = 0$). As usual, the $2n$-point correlation functions are defined by the Wick formula. That is, for distinct t_i,

$$< \psi(t_1)...\psi(t_{2n}) >= (2n-1)!!(i/2)^n \mathrm{sign}(\sigma),$$

where σ is the permutation that orders t_i in the decreasing order. If at least two points coincide, the correlation function is zero.

Thus we see that the correlation functions are invariant under $\mathrm{Diff}_+(\mathbb{R})$. In other words, using physical terminology, we have a *topological quantum field theory.*

Note that the correlation functions in the Euclidian setting for this model are the same as in the Minkowski setting, since they are (piecewise) constant in t_i. In particular, they don't decay at infinity, and hence our theory does not have the clustering property.

We have considered the theory of a massless fermionic field. Consider now the massive case. This means, we want to add to the Lagrangian a quadratic term in ψ which does not contain derivatives. If we have only one field ψ, the only choice for such term is ψ^2, which is zero. So in the massive case we must have at least two fields. Let us therefore consider the theory of two fermionic fields ψ_1, ψ_2 with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\psi_1\dot{\psi}_1 + \psi_2\dot{\psi}_2 - m\psi_1\psi_2),$$

where $m > 0$ is a mass parameter. The Green's function for this model satisfies the differential equation

$$G' - MG = i\delta,$$

where $M = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}$ (and G is a 2 by 2 matrix valued function). The general solution of this equation that satisfies the antisymmetry condition $G^T(-t) = -G(t)$ (which we will impose as in the massless case) has the form

$$G(t) = -(\frac{1}{2i}\mathrm{sign}(t)I + aM)e^{Mt},$$

where I is the identity matrix, and a is a number. Furthermore, it is natural to require that the theory at hand satisfies the clustering property (after being Wick-rotated). This means, $G(-it)$ decays at infinity for real t. It is easy to compute that this condition is satisfied for only one value of a, namely $a = 1/2m$. For this value of a, the solution has the form $G(t) = \mp iP_\pm e^{-iMt}$ for $\mp t > 0$, where $P_\pm$ is the projector to the eigenspace of M with eigenvalue $\pm im$ (and $G(0) = 0$).

Remark. It is easy to generalize this analysis to the case when ψ takes values in a positive definite inner product space V, and $M : V \to V$ is a skewsymmetric operator, since such a situation is a direct sum of the situations considered above.

In the case when M is nondegenerate, one can define the corresponding theory with interactions, i.e. with higher than quadratic terms in ψ. Namely, one defines the correlators as sums of amplitudes of appropriate Feynman diagrams. We leave it to the reader to work out this definition, by analogy with the finite dimensional case which we have discussed above.

10.3. **Super Hilbert spaces.** The space of states of a quantum system is a Hilbert space. As we plan to do Hamiltonian quantum mechanics for fermions, we must define a superanalog of this notion.

Recall that a sesquilinear form on a complex vector space is a form $(,)$ which is additive in each variable, and satisfies the conditions $(ax, y) = \bar{a}(x, y)$, $(x, ay) = a(x, y)$ for $a \in \mathbb{C}$.

Now suppose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a $\mathbb{Z}/2$-graded complex vector space.

Definition 10.1. (i) A Hermitian form on $\mathcal{H}$ is an even sesquilinear form $(,)$, such that $(x, y) = \overline{(y, x)}$ for even x, y, and $(x, y) = -\overline{(y, x)}$ for odd x, y.

(ii) A Hermitian form is positive definite if $(x, x) > 0$ for even $x \neq 0$, and $-i(x, x) > 0$ for odd $x \neq 0$. A super Hilbert space is a superspace with a positive definite Hermitian form $(,)$, which is complete in the corresponding norm.

(iii) Let $\mathcal{H}$ be a super Hilbert space, and $T : \mathcal{H}_0 \oplus \Pi\mathcal{H}_1 \to \mathcal{H}_0 \oplus \Pi\mathcal{H}_1$ be a linear operator between the underlying purely even spaces. The Hermitian adjoint operator T^* is defined by the equation $(x, Ty) = (-1)^{p(x)p(T)}(T^*x, y)$, where p denotes the parity.

10.4. **The Hamiltonian setting for fermionic quantum mechanics.** Let us now discuss what should be the Hamiltonian picture for the theory of a free fermion. More precisely, let V be a positive definite finite dimensional real inner product space, and consider the Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \dot{\psi}) - (\psi, M\psi))$, where $\psi : \mathbb{R} \to \Pi V$, and $M : V \to V$ is a skew-symmetric operator.

To understand what the Hamiltonian picture should be, let us compare with the bosonic case. Namely, consider the Lagrangian $\mathcal{L}_b = \frac{1}{2}(\dot{q}^2 - m^2q^2)$, where $q : \mathbb{R} \to V$. In this case, the classical space of states is $Y := T^*V = V \oplus V^*$. The equations of motion are Newton's equations $\ddot{q} = -m^2q$, which can be reduced to Hamilton's equations $\dot{q} = p$, $\dot{p} = -m^2q$. The algebra of classical observables is $C^\infty(Y)$, with Poisson bracket defined by $\{a, b\} = (a, b)$, $a, b \in Y^*$, where $(,)$ is the form on Y^* inverse to the natural symplectic form on Y. The hamiltonian H is determined (up to adding a constant) by the condition that the equations of motion are $\dot{f} = \{f, H\}$; in this case it is $H = (p^2 + m^2q^2)/2$.

The situation in the fermionic case is analogous, with some important differences which we will explain below. Namely, it is easy to compute that the equation of motion (i.e. the Euler-Lagrange equation) is $\dot{\psi} = M\psi$. The main difference with the bosonic case is that this equation is of first and not of second order, so the space of classical states is just ΠV (no momentum or velocity variables are introduced). Hence the algebra of classical observables is $C^\infty(\Pi V) = \Lambda V^*$. To define a Poisson bracket on this algebra, recall that ΠV has a natural "symplectic structure", defined by the *symmetric* form $(,)$ on V. Thus we can define a Poisson bracket on ΛV^* by the same formula as above: $\{a, b\} = (a, b)$. More precisely, $\{,\}$ is a unique skew symmetric (in the supersense) bilinear operation on ΛV^* which restricts to (a, b) for $a, b \in V^*$, and is a derivation with respect to each variable:

$$\{a, bc\} = \{a, b\}c + (-1)^{p(a)p(b)}b\{a, c\},$$

where $p(a)$ denotes the parity of a.

Now it is easy to see what should play the role of the Hamiltonian. More precisely, the definition with Legendre transform is not valid in our situation, since the Legendre transform was done with respect to the velocity variables, which we don't have in the fermionic case. On the other hand, as we discussed in §8, in the bosonic case the equation of motion $\dot{f} = \{f, H\}$ determines H uniquely, up to a constant. The situation is the same in the fermionic case. Namely, by looking at the equation of motion $\dot{\psi} = M\psi$, it is easy to see that the Hamiltonian equals $H = \frac{1}{2}(\psi, M\psi)$. In particular, if $M = 0$ (massless case), the Hamiltonian is zero (a characteristic property of topological field theories).

Now let us turn to quantum theory. In the bosonic case the algebra of quantum observables is a noncommutative deformation of the algebra $C^\infty(Y)$ in which the relation $\{a, b\} = (a, b)$ is replaced with its quantum analog $ab - ba = i(a, b)$ (up to the Planck constant factor which we will now ignore). In particular, the subalgebra of polynomial observables is the Weyl algebra $W(Y)$, generated by Y^* with this defining relation. By analogy with this, we must define the algebra of quantum observables in the fermionic case to be generated by V^* with the relation $ab+ba = i(a, b)$ (it deforms the relation $ab + ba = 0$ which defines ΛV^*). So we recall the following definition.

Definition 10.2. Let V be a vector space over a field k with a symmetric bilinear form Q. The Clifford algebra $Cl(V, Q)$ is generated by V with defining relations $ab + ba = Q(a, b)$, $a, b \in V$.

We see that the algebra of quantum observables should be $Cl(V_{\mathbb{C}}^*, i(,))$. Note that like in the classical case, this algebra is naturally $\mathbb{Z}/2$ graded, so that we have even and odd quantum observables.

Now let us see what should be the Hilbert space of quantum states. In the bosonic case it was $L^2(V)$, which is, by the well known Stone-von Neumann theorem, the unique irreducible unitary representation of $W(Y)$. By analogy with this, in the fermionic case the Hilbert space of states should be an irreducible even unitary representation of $Cl(V)$ on a supervector space $\mathcal{H}$.

The structure of the Clifford algebra $Cl(V^*)$ is well known. Namely, consider separately the cases when $\dim V$ is odd and even.

In the even case, $\dim V = 2d$, $Cl(V^*)$ is simple, and has a unique irreducible representation $\mathcal{H}$, of dimension 2^d. It is constructed as follows: choose a decomposition $V_{\mathbb{C}} = L \oplus L^*$, where L, L^* are Lagrangian subspaces. Then $\mathcal{H} = \Lambda L$, where $L \subset V_{\mathbb{C}}^*$ acts by multiplication and L^* by differentiation (multiplied by $-i$). The structure of the superspace on $\mathcal{H}$ is the standard one on the exterior algebra.

In the odd case, $\dim V = 2d+1$, choose a decomposition $V_{\mathbb{C}} = L \oplus L^* \oplus K$, where L, L^* are maximal isotropic, and K is a nondegenerate 1-dimensional subspace orthogonal to L and L^*. Let $\mathcal{H} = \Lambda(L \oplus K)$, where L, K act by multiplication and L^* by ($-i$ times) differentiation. This is a representation of $Cl(V^*)$ with a $\mathbb{Z}/2$ grading. This representation is not irreducible, and decomposes in a direct sum of two non-isomorphic irreducible representations $\mathcal{H}_+ \oplus \mathcal{H}_-$ (this is related to the fact that the Clifford algebra for odd $\dim V$ is not simple but is a sum of two simple algebras). However, this decomposition is not consistent with the $\mathbb{Z}/2$-grading, and therefore as superrepresentation, $\mathcal{H}$ is irreducible.

Now, it is easy to show that both in the odd and in the even case the space $\mathcal{H}$ carries a unique up to scaling Hermitian form, such that $V^* \subset V_{\mathbb{C}}^*$ acts by selfadjoint operators. This form is positive definite. So the situation is similar to the bosonic case for any $\dim V$.

Let us now see which operator on $\mathcal{H}$ should play the role of the Hamiltonian of the system. The most natural choice is to define the quantum Hamiltonian to be the obvious quantization of the classical Hamiltonian $H = \frac{1}{2}(\psi, M\psi)$. Namely, if ε_i is a basis of V^* and a_{ij} is the matrix of M in this basis, then one sets $\hat{H} = \frac{1}{2}\sum_{i,j} a_{ij}\varepsilon_i\varepsilon_j$. To compute this operator more explicitly, we will assume (without loss of generality) that the decomposition of $V_{\mathbb{C}}$ that we chose is stable under M. Let ξ_j be an eigenbasis of M in L (with eigenvalues im_j), and ∂_j be differentiations along the vectors of this basis. Then

$$\hat{H} = \sum_j m_j(\xi_j \partial_j - \partial_j \xi_j) = \sum_j m_j(2\xi_j\partial_j - 1).$$

This shows that if $\dim V$ is even then the partition function on the circle of length L for our theory is

$$Z = \mathrm{sTr}(e^{-L\hat{H}}) = \prod_j (e^{m_j L} - e^{-m_j L})$$

If the dimension of V is odd then the partition function is zero.

Now we would like to consider the fermionic analog of the Feynman-Kac formula. For simplicity consider the fully massive case, when $\dim V$ is even and $m_j \neq 0$ (i.e. M is nondegenerate). In this case, we have a unique up to scaling lowest eigenvector of $\hat{H}$, namely $\Omega = 1$.

Let $\psi(0) \in V \otimes \mathrm{End}(\mathcal{H})$ be the element corresponding to the action map $V^* \to \mathrm{End}(\mathcal{H})$, and $\psi(t) = e^{it\hat{H}}\psi(0)e^{-it\hat{H}}$. Also, let $< \psi(t_1)...\psi(t_n) >$, $t_1 \geq ... \geq t_n$, be the correlation function for the free theory in the Lagrangian setting, taking values in $V^{\otimes n}$, so in this expression $\psi(t_i)$ is a formal symbol and not an operator.

Theorem 10.3. *(i) For the free theory on the line we have*

$$< \psi(t_1)...\psi(t_n) >= (\Omega, \psi(t_1)...\psi(t_n)\Omega).$$

(ii) For the free theory on the circle of length L we have

$$< \psi(t_1)...\psi(t_n) >= \frac{\mathrm{sTr}(\psi(t_1)...\psi(t_n)e^{-L\hat{H}})}{\mathrm{sTr}(e^{-L\hat{H}})}$$

Exercise. Prove this theorem. (The proof is analogous to Theorem 8.3 in the free case).

It should now be straightforward for the reader to formulate and prove the Feynman-Kac formula for an interacting theory which includes both bosonic and Fermionic massive fields. We leave this as an instructive exercise.

11. Free field theories in higher dimensions

11.1. **Minkowski and Euclidean space.** Now we pass from quantum mechanics to quantum field theory in dimensions $d \geq 1$. As we explained above, we have two main settings.

1. Minkowski space. Fields are functions on a spacetime V_M, which is a real inner product space of signature $(1, d-1)$. This is where physical processes actually "take place". The symmetry group of V, $G = SO(1, d-1)$, is called the *Lorenz group*; it is the group of transformation of spacetime in special relativity. Therefore, field theories in Minkowski space which are in an approariate sense "compatible" with the action of G are called *relativistic.*

Recall some standard facts and definitions. *The light cone* in V is the cone described by the equation $|\mathbf{v}|^2 = 0$, where $|\mathbf{v}|^2 := (\mathbf{v}, \mathbf{v})$. Vectors belonging to the light cone are called lightlike. The light cone divides the space V into spacelike vectors $|\mathbf{v}|^2 < 0$ (the outside of the cone), and timelike vectors $|\mathbf{v}|^2 > 0$ (inside the cone). We will choose one of the two components of the interior of the cone and call it positive; it will be denoted by V_+. The opposite (negative) component is denoted by V_-. The group of $g \in SO(V) = SO(1, d-1)$ which preserve V_+

is denoted by $SO_+(1, d-1)$; it is the connected component of the identity of the group $SO(1, d-1)$ (which has two connected components).

Often (e.g. when doing Hamiltonian field theory) it is necessary to split V in an orthogonal direct sum $V = V_s \oplus \mathbb{R}$ of space and time. In this decomposition, the space V_s is required to be spacelike (i.e. negative definite), which implies that the time axis $\mathbb{R}$ has to be timelike (positive definite). Note that such a splitting is not unique, and that fixing it breaks the Lorenz symmetry $SO_+(1, d-1)$ down to the usual rotation group $SO(d-1)$. To do explicit calculations, one further chooses Cartesian coordinates $x_1, ..., x_{d-1}$ on V_s and t on the time axis $\mathbb{R}$.

2. Euclidean space. Fields are functions on a spacetime V_E, which is a positive definite inner product space. It plays an auxiliary role and has no direct physical meaning, although path integrals computed in this space are similar to expectation values in statistical mechanics.

The two settings are related by the "Wick rotation". Namely the Euclidean space V_E corresponding to the Minkowski space V_M is the real subspace in $(V_M)\mathbb{C}$ consisting of vectors $(it, x_1, ..., x_{d-1}$, where t and x_j are real. In other words, to pass to the Euclidean space one needs to make a change of variable $t \to it$. Note that under this change, the standard metric on the Minkowski space, $dt^2 - \sum dx_i^2$, goes into a negative definite metric $-dt^2 - \sum dx_i^2$. However, the minus sign is traditionally dropped and one considers instead the positive metric $dt^2 + \sum dx_i^2$ on V_E.

11.2. **Free scalar bosons.** Consider the theory of a free scalar bosonic field ϕ of mass m. The procedure of quantization of this theory in the Lagrangian setting is a straightforward generalization from the case of quantum mechanics. Namely, the Lagrangian for this theory in Minkowski space is $\mathcal{L} = \frac{1}{2}((d\phi)^2 - m^2\phi^2)$, and the Euler-Lagrange equation is $\Box\phi = -m^2\phi$, where $\Box$ is the D'Alambertian, $\Box = \frac{\partial^2}{\partial t^2} - \sum_i \frac{\partial^2}{\partial x_i^2}$. Thus to define the corresponding quantum theory, we should invert the operator $\Box + m^2$. As in the quantum mechanics case, this operator is not invertible (0 is in the spectrum), so we should proceed using the Wick rotation.

After the Wick rotation (i.e. transformation $t \to it$), we arrive at the following Euclidean Lagrangian: $\mathcal{L}_E = \frac{1}{2}((d\phi)^2 + m^2\phi^2)$, and the Euler-Lagrange equation is $\Delta\phi = m^2\phi$. So to define the quantum theory, i.e. the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

(where $S = \int \mathcal{L}$), we now need to invert the operator $A = -\Delta + m^2$. The operator A^{-1} is an integral operator, whose kernel is $G(x-y)$, where $G(x)$ is the Green's function, i.e. the fundamental solution of the differential equation

$$-\Delta G + m^2 G = \delta$$

To solve this equation, note that the solution is rotationally invariant. Therefore, outside of the origin, $G(x) = g(|x|)$, where g is a function on $(0, \infty)$ such that

$$-g'' - \frac{d-1}{r}g' + m^2 g = 0$$

(where the left hand side is the radial part of the operator A). This is a version of the Bessel equation. If $m > 0$, the basis of solutions is $r^{\frac{2-d}{2}} J_{\pm\frac{2-d}{2}}(imr)$. (Actually, these functions are elementary for odd d). There exists a unique up to scaling solution, which decays at infinity, namely,

$$g = Cr^{\frac{2-d}{2}}(J_{\frac{2-d}{2}}(imr) + i^d J_{-\frac{2-d}{2}}(imr)), (d \neq 2).$$

For $d = 2$, this expression is zero, and one should instead take the limit of the right hand side divided by $d - 2$ as $d \to 2$ (which will generate a logarithmic factor $\ln r$). The normalizing constant C can be found from the condition that $AG = \delta$.

Remark. It is easy to check that for $d = 1$ this function equals the familiar Green's function for quantum mechanics, $e^{-mr}/2m$.

If $m = 0$ (massless case), the basis of solutions is: $1, r$ for $d = 1$, $1, \ln r$ for $d = 2$, and $1, r^{2-d}$ for $d > 2$. Thus, if $d \leq 2$, we don't have a decaying solution and thus the corresponding quantum theory will be deficient: it will not satisfy the clustering property. On the other hand, for $d > 2$ we have a unique up to scaling decaying solution $g = Cr^{1-d}$. The constant C is found as in the massive case.

The higher correlation founctions are found from the 2-point function via the Wick formula, as usual.

We should now note a fundamental difference between quantum mechanics and quantum field theory in $d > 1$ dimensions. This difference comes from the fact that while for $d = 1$, the Green's function $G(x)$ is continuous at $x = 0$, for $d > 1$ it is singular at $x = 0$. Namely, $G(x)$ behaves like $C|x|^{2-d}$ as $x \to 0$ for $d > 2$, and as $C \ln |x|$ as $d = 2$. Thus for $d > 1$, unlike the case $d = 1$, the path integral

$$\int \phi(x_1)...\phi(x_n)e^{-S(\phi)}D\phi$$

(as defined above) makes sense only if $x_i \neq x_j$. In other words, this path integral should be regarded not as a function but rather as a distribution. Luckily, there is a canonical way to do it, since the Green's function $G(x)$ is locally L^1.

Now we can Wick rotate this theory back into the Minkowski space. It is clear that the Green's function will then turn into $G_M(x) = g(\sqrt{|x|^2 + i\varepsilon})$, The higher correlation functions, as before, are determined from this by the Wick formula.

Actually, it is more convenient to describe this theory "in momentum space", where the Green's function can be written more explicitly. Namely, the Fourier transform $\hat{G}(p)$ of the distribution $G(x)$ is a solution of the equation

$$p^2\hat{G} + m^2\hat{G} = 1,$$

obtained by Fourier transforming the differential equation for G. Thus,

$$\hat{G}(p) = \frac{1}{p^2 + m^2},$$

as in the quantum mechanics case. Therefore, like in quantum mechanics, the Wick rotation produces the distribution

$$\hat{G}_M(p) = \frac{i}{p^2 - m^2 + i\varepsilon},$$

which is the Fourier transform of $G_M(x)$.

11.3. **Spinors.** To consider field theory for fermions, we must generalize to the case of $d > 1$ the basic fermionic Lagrangian $\frac{1}{2}\psi\frac{d\psi}{dt}$. To do this, we must replace $\frac{d}{dt}$ by some differential operator on V. This operator should be of first order, since in fermionic quantum mechanics it was important that the equations of motion are first order equations. Clearly, it is impossible to define such an operator so that the Lagrangian is $SO_+(V)$-invariant, if ψ is a scalar valued (odd) function on V. Thus, a fermionic field in field theory of dimension $d > 1$ cannot be scalar valued, but rather must take values in a real representation S of $SO_+(V)$, such that there exists a nonzero intertwining operator $V \to \mathrm{Sym}^2 S^*$. This property is satisfied by spinor representations. They are indeed basic in fermionic field theory, and we will now briefly discuss them (for more detail see "Spinors" by P.Deligne, in "QFT and string theory: a course for mathematicians").

First consider the complex case. Let V be a complex inner product space of dimension $d > 1$. Let $Cl(V)$ be the Clifford algebra of V, defined by the relation $\xi\eta + \eta\xi = 2(\xi, \eta)$, $\xi, \eta \in V$. As we discussed, for even d it is simple and has a unique irreducible representation S of dimension $2^{d/2}$, while for odd d it has two such representations S', S'' of dimension $2^{(d-1)/2}$. It is easy to show that the space $CL_2(V)$ of quadratic elements of $CL(V)$ (i.e. the subspace spanned elements of the form $\xi\eta - \eta\xi, \xi, \eta \in V$) is closed under bracket, and coonstitutes the Lie algebra $\mathfrak{o}(V)$. Thus $\mathfrak{o}(V)$ acts on S (respectively, S', S''). This action does not integrate to an action of $SO(V)$, but integrates to an action of its double cover $\mathrm{Spin}(V)$.

If d is even, the representation S of $\mathrm{Spin}(V)$ is not irreducible. Namely, recall that S is the exterior algebra of a Lagrangian subspace of V. Thus is splits in a direct sum $S = S_+ \oplus S_-$ (odd and even elements). The subspaces S_+, S_- are subrepresentations of S, which are irreducible. They are called half-spin representations.

If d is odd, the representations S' and S'' of $\mathrm{Spin}(V)$ are irreducible and isomorphic. Any of them will be denoted by S and called the spinor representation. Thus, we have the spinor representation S for both odd and even d, but for even d it is reducible.

An important structure attached to the spinor representation S is the intertwining operator $\Gamma : V \to \mathrm{End}S$, given by the action of $V \subset Cl(V)$ in S. This intertwiner allows us to define the Dirac operator

$$\mathbf{D} = \sum \Gamma_i \frac{\partial}{\partial x_i}$$

where x_i are coordinates on V associated to an orthornormal basis e_i, and $\Gamma_i = \Gamma(e_i)$. This operator acts on functions form V to S, and $\mathbf{D}^2 = \Delta$, so $\mathbf{D}$ is a square root of the Laplacian. The matrices Γ_i are called Γ matrices.

Note that for even d, one has $\Gamma(v) : S_\pm \to S_\mp$, so $\mathbf{D}$ acts from functions with values in $S_\pm$ to functions with values in $S_\mp$.

By a spinor representation of $\mathrm{Spin}(V)$ we will mean any linear combination of S_+, S_- for even d, and any multiple of S for odd d. Thus, for every spinor representation we have $\Gamma(v) : Y \to Y'$, where $S'_+ := S_-$ and $S'_- := S_+$, and $S' := S$.

Now assume that V is a real inner product space with Minkowski metric. In this case we can define the group $\mathrm{Spin}_+(V)$ to be the preimage of $SO_+(V)$ under the map $\mathrm{Spin}(V_{\mathbb{C}}) \to SO(V_{\mathbb{C}})$. It a double cover of $SO_+(V)$ (if $d = 2$, this double cover is disconnenced and actually a direct product by $\mathbb{Z}/2$).

By a real spinor representation of $SO_+(V)$ we will mean a real representation Y such that $Y_{\mathbb{C}}$ is a spinor representation of $\mathrm{Spin}(V_{\mathbb{C}})$.

11.4. **Fermionic Lagrangians.** Now let us consider Lagrangians for a spinor field ψ with values in a spinor representation Y. As the Lagrangian is supposed to be real in the Minkowski setting, we will require in that case that Y is real. First of all, let us see what we need in order to write the "kinetic term" $(\psi, \mathbf{D}\psi)$. Clearly, to define such a term (so that it is nonzero), we need an invariant nondegenerate pairing $(,)$ between Y and Y' such that the for any $v \in V$, the bilinear form $(x, \Gamma(v)y)$ on Y is symmetric.

Let us find for which Y this is possible (for complex V). The behavior of Spin groups depends on d modulo 8. Thus we will list the answers labeling them by d mod 8 (they are easily extracted from the tables given in Deligne's text).

0. $n(S_+ \oplus S_-)$.
1. nS.
2. $nS_+ \oplus kS_-$.
3. nS.
4. $n(S_+ \oplus S_-)$.
5. $2nS$.
6. $2nS_+ \oplus 2kS_-$.
7. $2S$.

Let us now find when we can also add a mass term. Recall that the mass term has the form $(\psi, M\psi)$, so it corresponds to an invariant skew-symmetric operator $M : Y \to Y^*$. Let us list those Y from the above list, for which such nondegenerate operator exists.

0. $2n(S_+ \oplus S_-)$.
1. $2nS$.
2. $n(S_+ \oplus S_-)$.
3. nS.
4. $n(S_+ \oplus S_-)$.
5. $2nS$.
6. $2n(S_+ \oplus S_-)$.
7. $2S$.

To pass to the real Minkowski space (in both massless and massive case), one should put the additional requirement that Y should be a real representation.

We note that upon Wick rotation to Minkowski space, it may turn out that a real spinor representation Y will turn into a complex representation which has no real structure. Namely, this happens for massless spinors that take values in $S_\pm$ if $d = 2$ mod 8. These representations have a real structure for Minkowskian V (i.e. for $SO_+(1, d-1)$), but no real structure for Euclidean V (i.e. for $SO(d)$). This is quite obvious, for example, when $d = 2$ (check!).

One may think that this causes a problem in quantum field theory, where we would be puzzled what to integrate over – real or complex space. However, the problem in fact does not arise, since we have to integrate over fermions, and integration over fermions (say, in the finite dimensional case) is purely algebraic and does not make a distinction between real and complex.

11.5. **Free fermions.** Let us now consider a free theory for a spinor $\psi : V \to \Pi Y$, where Y is a spinor representation, defined by a Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \mathbf{D}\psi) - (\psi, M\psi))$, where M is allowed to be degenerate (we assume that Y is such that this expression makes sense). The equation of motion in Minkowski space is $\mathbf{D}\psi = M\psi$. where we have identified Y' and Y^* using the pairing $Y \times Y' \to \mathbb{R}$ used to define the kinetic term. Thus, to define the corresponding quantum theory, we need to invert the operator $\mathbf{D} - M$. As usual, this cannot be done because of a singularity, and we have to consider the Wick rotation.

The Wick rotation produces the Euclidean Lagrangian $\mathcal{L} = \frac{1}{2}((\psi, \mathbf{D}_E\psi) + (\psi, M\psi))$ (note that the i in the kinetic term is hidden in the definition of the Euclidean Dirac operator). We invert $\mathbf{D}_E + M$, to obtain the Euclidean Green's function. To do this, it is convenient to go to momentum space, i.e. perform a Fourier transform. Namely, after Fourier transform $\mathbf{D}_E$ turns into the operator $i\mathbf{p}$, where $\mathbf{p} = \sum p_j \Gamma_j$, and p_j are the operators of multiplication by the coordinates p_j. Thus, the Green's function (i.e. the 2-point function) $G(x) \in \mathrm{Hom}(Y', Y)$ is the Fourier transform of the matrix-valued function $\frac{1}{i\mathbf{p}+M}$. Let $M^\dagger : Y' \to Y$ be the operator such that $\Gamma_i M = M^\dagger \Gamma_i$. Then, $(-i\mathbf{p} + M^\dagger)(i\mathbf{p} + M) = p^2 + M^\dagger M$, so that

$$\hat{G}(p) = (p^2 + M^\dagger M)^{-1}(-i\mathbf{p} + M^\dagger).$$

This shows that $G(x)$ is expressed through the Green's function in the bosonic case by differentiations (how?). After Wick rotation back to the Minkowski space, we get

$$\hat{G}_M(p) = (p^2 - M^\dagger M + i\varepsilon)^{-1}(\mathbf{p} + iM^\dagger).$$

Finally, the higher correlation functions, as usual, are found from the Wick formula.