LECTURE 9

During this lecture we will study the elliptic generalization of the hypergeometric functions. Motivations come from CFT (Conformal Field Theory) where to any Riemann surfaces Σ with marked points decorated with finite dimensional representations of a simple Lie algebra is assigned the (finite dimensional) vector space of *conformal blocks*. The set of all conformal blocks forms a vector bundle over the moduli space of the marked Riemann surfaces, this vector bundle carries a projectively flat connection. We will discuss in detail the case of genus 1:

$$\Sigma = T = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, \quad Im\tau > 0.$$

Notations: we will consider the Lie algebra $\mathfrak{g}$=sl_2, the Cartan subalgebra $\mathfrak{h}$=$\mathbb{C}h$, the module: $V = V_1 \otimes ... \otimes V_n$, and we define: $V[0] = \{v \in V | hv = 0\}$. The conformal block will be given by the function:

$$u = u(z_1, ..., z_n, \lambda, \tau) \in V[0], \quad \lambda, z_i \in \mathbb{C}, \;\; i = 1, .., n \;\; \tau \in \mathbb{H},$$

i.e will be a function of λ and will depend on the parameter $z_1, ..., z_n, \tau$.
We will discuss the following equations:

$$1. \;\; \kappa\frac{\partial u}{\partial z_i} = \sum_{j\neq i} r^{(i,j)}(z_i - z_j, \lambda, \tau)u - h^{(i)}\frac{\partial u}{\partial\lambda}, \;\; i = 1, ..., n; \tag{1}$$

$$2. \;\; 4\pi i\kappa\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial\lambda^2} + \sum_{i,j} H^{(i,j)}(z_i - z_j, \lambda, \tau)u, \;\; i = 1, ...n. \tag{2}$$

Remark 0.1. *The equation 2 is called KZB heat equation.*

Example 0.1. *For $n = 1$ we have $V = V_1 = L_{2p}$ ($dim\, L_{2p} = 2p + 1$) and $V[0] = L_{2p}[0] = \mathbb{C}f^p v_{2p}$. In this case the function $u = u(\lambda, z, \tau)$ is a scalar valued function and we have that:*

$$1. \qquad \frac{\partial u}{\partial z_1} = 0, \tag{3}$$

$$2. \;\; 2\pi i\kappa\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial\lambda^2} + p(p+1)\rho'(\lambda, \tau)u, \tag{4}$$

where:

$$\rho(\lambda, \tau) = \frac{\theta'(\lambda, \tau)}{\theta(\lambda, \tau)}$$

and

$$\theta(\lambda, \tau) = -\sum_{j\in\mathbb{Z}} e^{\pi i(j+\frac{1}{2})^2\tau + 2\pi i(j+\frac{1}{2})(\lambda+\frac{1}{2})}$$

Remark 0.2. *The derivative in the equation 4 is taken w.r.t λ, i.e ($' = \frac{\partial}{\partial\lambda}$).*

Remark 0.3. *The function θ is the first Jacobi theta function.*

Remark 0.4. *We have that:*

$$\rho'(\lambda,\tau) = -\wp(\lambda,\tau) + c(\tau)$$

where:

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{n,m\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2}\right)$$

is the Weierstrass $\wp$-function.

Example 0.2. *For $g = sl_N$, $n = 1$, $V = S^{pN}\mathbb{C}^N$ and $h = \mathbb{C}/\mathbb{C}(1,...,1)$, the KZB equation become:*

$$4\pi i\kappa \frac{\partial u}{\partial \tau} = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial \lambda_i^2} + 2p(p+1)\sum_{i<j} \rho'(\lambda_i - \lambda_j, \tau)u. \tag{5}$$

The RHS of the previous equation is the hamiltonian operator of the quantum elliptic Calogero-Moser N-body system.

Remark 0.5. *A remarkable fact about all form of KZ equation is that them can be realized geometrically. Their solutions can be expressed in terms of hypergeometric integrals depending on parameters. The KZ equation(s) can also be realized as Gauss-Manin connection.*

Example 0.3. *Let $n = 1$, $p = 1$ so $V = L_2$ ($dim\ L_2 = 3$). In this case the solution of the KZB equation can be written in the following form:*

$$u(\lambda,\tau) = u_g(\lambda,\tau) = \int_0^1 \left(\frac{\theta(t,\tau)}{\theta'(0,\tau)}\right)^{-\frac{2}{\kappa}} \frac{\theta(\lambda-t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)} g(\lambda - \frac{2}{\kappa}t,\tau)dt,$$

where $g = g(\lambda,\tau)$ is any holomorphic solution of:

$$2\pi i\kappa \frac{\partial g}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2}.$$

For instance we can take the function:

$$g = e^{\lambda\mu + \frac{\mu^2}{2\pi i\kappa}\tau} \quad \mu \in \mathbb{C}.$$

Remark 0.6. *The simplest KZ equations have the classical Gauss hypergeometric function as their solution. So it is natural to consider solutions of all KZ type equations as generalized hypergeometric functions. The Gauss hypergeometric function is:*

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b,c;z) = \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}dt$$

We can see that the function:

$$\left(\frac{\theta(t,\tau)}{\theta'(0,\tau)}\right)^{\alpha}$$

is the analogue of the function t^α, while:

$$\frac{\theta(\lambda-t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)}$$

is the analogue of the function $\frac{1}{t}$.

1. Asymptotic Solutions.

Let's consider the asymptotic behaviour of the function $u = u(\lambda, \tau)$, i.e let's study the behaviour of this function for $\kappa \longrightarrow 0$.

Remark 1.1. *Under this hypothesis we are dealing with the following eigenvalue problem:*

$$\frac{d^2}{d\lambda^2}v(\lambda) - 2\wp(\lambda,\tau)v(\lambda) = Ev(\lambda)$$

where E is a number. The equation:

$$\frac{d^2}{d\lambda^2}v(\lambda) - p(p+1)\wp(\lambda,\tau)v(\lambda) = Ev(\lambda) \tag{6}$$

is called Lame' *equation and its solutions are called Lame' functions.*

We are looking for solutions of the form:

$$u(\lambda,\tau) = e^{\frac{S(\tau)}{\kappa}}(f_0(\lambda,\tau) + \kappa f_1(\lambda,\tau) + ...).$$

Plugging this expression into the equation:

$$2\pi i\kappa\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} - p(p+1)\rho'(\lambda,\tau)u \tag{7}$$

we get recurrence relations for the functions $f_i(\lambda,\tau) \quad n = 0, 1, ...$

Definition 1.1. *We say that the function $u = u(\lambda,\tau)$ is a formal asymptotic solution if all the recurrence relations are satisfied.*

Corollary 1.1. *If $u = u(\lambda,\tau)$ is an asymptotic solution then for every fixed τ the function $f_0 = f_0(\lambda,\tau)$ is an eigenvector of the operator:*

$$\frac{\partial^2}{\partial\lambda^2} + p(p+1)\rho'$$

with eigenvalue $\frac{\partial S}{\partial \tau}$.

Applying the stationary phase method to the integral:

$$u(\lambda,\tau) = \int_0^1 \left(\frac{\theta(t,\tau)}{\theta'(0,\tau)}\right)^{-\frac{2}{\kappa}} \frac{\theta(\lambda - t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)} e^{\lambda\mu + \frac{\mu^2}{2\pi i\kappa}\tau} dt,$$

we get the following:

Theorem 1.1. *For any $\mu \in \mathbb{C}$, the function:*

$$v(\lambda) = e^{\lambda\mu}\frac{\theta(\lambda - t_0,\tau)}{\theta(\lambda,\tau)}$$

is an eigenfunction of:

$$\frac{\partial^2}{\partial\lambda^2} - 2\wp(\lambda,\tau)$$

if t_0 is a critical point, w.r.t t, of the function:

$$\theta(t,\tau)e^{-\mu t}.$$

Remark 1.2. *This is an example of Bethe ansazt. Hermite developed this method for the Lame' equation (1872).*

Remark 1.3. *Similar formulas are available for the eigenfunctions of the Many-Body Elliptic Calogero-Moser hamiltonian.*

2. Special Solutions and Conformal Blocks.

In what follows we will consider the differential equation:

$$2\pi i\kappa\frac{\partial u}{\partial\tau}=\frac{\partial^2 u}{\partial\lambda^2}+p(p+1)\rho'(\lambda,\tau)u. \tag{8}$$

Let's fix $\kappa\in\mathbb{Z}_{>0}$ and $p\in\mathbb{Z}_{\geq 0}$ and assume that $\kappa\geq 2p+2$. We will consider holomorphic solutions $u=u(\lambda,\tau)$ of the KZB equation with the following properties:
1. $u(\lambda+2,\tau)=u(\lambda,\tau)$;
2. $u(\lambda+2\tau,\tau)=e^{-2\pi i\kappa(\lambda+\tau)}u(\lambda,\tau)$;
3. $u(-\lambda,\tau)=(-1)^{p+1}u(\lambda,\tau)$;
4. $u(\lambda,\tau)=\mathcal{O}((\lambda-m-n\tau)^{p+1})$, as $\lambda\longrightarrow m+n\tau,\ \ m,n\in\mathbb{Z}$.

Definition 2.1. *Holomorphic solutions of 8 satisfying the conditions 1., 2., 3., and 4. are called* conformal blocks *associated with the elliptic curve $T=\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ and the point $z=0$.*

Remark 2.1. *It known that under this hypopthesis the space of conformal blocks has dimension equal to $\kappa-2p-1$.*

3. Theta functions of level κ.

Before describing the space of conformal blocks it is useful discuss some elementary properties of theta functions.

Definition 3.1. *Let $\kappa\in\mathbb{Z}_{>0}$. A holomorphic function $u=u(\lambda,\tau)$ is called a theta function of level κ w.r.t the lattice $2\mathbb{Z}+2\tau\mathbb{Z}$ if:*
1. $u(\lambda+2,\tau)=u(\lambda,\tau)$;
2. $u(\lambda+2\tau,\tau)=e^{-2\pi i\kappa(\lambda+\tau)}u(\lambda,\tau)$.

Lemma 3.1. *The space of theta functions of level κ is 2κ dimensional. A basis is given by:*

$$\theta_{n,\kappa}(\lambda,\tau)=\sum_{j\in\mathbb{Z}}e^{2\pi i\kappa(j+\frac{n}{2\kappa})^2+2\pi i\kappa(j+\frac{n}{2\kappa})\lambda}$$

$n=0,...,2\kappa-1$

Prof. Follows from the definition and from the Fourier expansion.

These functions have the following remarkable:

Properties 3.1. *Let* $q = e^{\frac{\pi i}{\kappa}}$*:*

1. $\theta_{n,\kappa}(\lambda,\tau)$ *is a solution of:*

$$2\pi i\kappa \frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial \lambda^2};$$

2. $\theta_{n,\kappa}(\lambda,\tau) = \theta_{n+2\kappa,\kappa}(\lambda,\tau)$*;*

3. $\theta_{n,\kappa}(-\lambda,\tau) = \theta_{-n,\kappa}(\lambda,\tau)$*;*

4. $\theta_{n,\kappa}(\lambda + \frac{2}{\kappa},\tau) = q^{2n}\theta_{n,\kappa}$*;*

5. $\theta_{n,\kappa}(\lambda + \frac{2t}{\kappa},\tau)e^{-2\pi i t - \frac{2\pi i \tau}{\kappa}}\theta_{n+2,\kappa}(\lambda,\tau)$*.*

3.1. **Modular Transformations.** Let's start with:

Definition 3.2. *The* Modular Group *is* $M = SL(2,\mathbb{Z})/(\pm Id)$*. The generators are:*

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and they satisfy the relations:

$$S^2 = 1, \quad (ST)^3 = 1.$$

M acts on $\mathbb{H}$ by $T : \tau \longrightarrow \tau + 1 \quad S : \tau \longrightarrow -\frac{1}{\tau}$.

Remark 3.1. *Elliptic curves corresponding to parameters* $\tau, \tau+1, -\frac{1}{\tau}$ *are isomorphic so it natural to expect relations among conformal blocks associated to these curves.*

Let's introduce the following maps:

Definition 3.3.

$$(Tu)(\lambda,\tau) = u(\lambda,\tau+1) \tag{9}$$

and

$$(Su)(\lambda,\tau) = e^{-\pi i\kappa\frac{\lambda^2}{2\tau}}\tau^{-1/2}u(\lambda/\tau,-1/\tau). \tag{10}$$

Definition 3.4. Θ_κ *will denote the space of theta functions of level* κ*.*

Lemma 3.2. *The maps* T *and* S *preserve the space of* Θ_κ*-functions:*

$$1. \qquad \theta_{n,\kappa}(\lambda,\tau+1) = q^{\frac{n^2}{2}}\theta_{n,\kappa}(\lambda,\tau); \tag{11}$$

$$2.\ \theta_{n,\kappa}\Big(\frac{\lambda}{\tau},-\frac{1}{\tau}\Big) = \sqrt{-\frac{i\tau}{2\kappa}}e^{\pi i\kappa\frac{\lambda^2}{2\tau}}\sum_{m=0}^{2\kappa-1} q^{-mn}\theta_{m,\kappa}(\lambda,\tau). \tag{12}$$

Remark 3.2. *The maps* T *and* S *define a projective representation of the Modular group in the space of* Θ_κ*-functions.*

3.2. **Symmetric Theta Functions.**

Definition 3.5. *Let's define:*

$$\theta^S_{n,\kappa}(\lambda,\tau) = \theta_{n,\kappa}(\lambda,\tau) + \theta_{n,\kappa}(-\lambda,\tau) = \theta_{n,\kappa}(\lambda,\tau) + \theta_{-n,\kappa}(\lambda,\tau). \tag{13}$$

$\theta^S_{n,\kappa}$ *is called symmetric theta function of level* κ.

It is possible to prove the following:

Lemma 3.3. *The space of symmetric theta functions of level* κ *has dimension equal to* $\kappa+1$ *and it is generated by* $\theta^S_{0,\kappa}, ..., \theta^S_{\kappa,\kappa}$.

Let's go back to the general theory of theta function and discuss the following:

Example 3.1 (First Jacobi theta function)**.** *The first Jacobi theta function is definided by the following formula:*

$$\theta(\lambda,\tau) = -\sum_{j\in\mathbb{Z}} e^{\pi i(j+\frac{1}{2})^2\tau + 2\pi i(j+\frac{1}{2})(\lambda+\frac{1}{2})}. \tag{14}$$

This function satisfies the following product formula*:*

$$\theta(\lambda,\tau) = ie^{\pi i(\frac{\tau}{4}-\lambda)}(\kappa,\tilde{q})(\frac{\tilde{q}}{\kappa},\tilde{q})(\tilde{q},\tilde{q}) \tag{15}$$

where $\tilde{q} = e^{2\pi i\tau}$, $\kappa = e^{2\pi i\lambda}$ *and* $(\kappa,\tilde{q}) = \prod_{n=0}^{\infty}(1-\kappa\tilde{q}^n)$. *Moreover this function has the following:*

Properties 3.2. *1.* $\theta(-\lambda,\tau) = -\theta(\lambda,\tau)$;
2. $\theta(\lambda+1,\tau) = -\theta(\lambda,\tau)$;
3. $\theta(\lambda+\tau,\tau) = e^{-\pi i\tau - 2\pi i\lambda}\theta(\lambda,\tau)$;
4. $theta(\lambda,\tau)$ *has simple zeros at* $\mathbb{Z}+\tau\mathbb{Z}$*;*
5. $theta(\lambda,\tau)$ *is a theta function of level 2;*
6. $theta(\lambda,\tau)$ *is a solution of the heat equation:*

$$2\pi i\kappa\frac{\partial g}{\partial\tau} = \frac{\partial^2 g}{\partial\lambda^2};$$

7.(Modular property) *The function* $\theta(\lambda,\tau)$ *satisfies the following two equations:*

$$\theta(\lambda,\tau+1) = e^{\frac{\pi i}{4}}\theta(\lambda,\tau); \tag{16}$$

$$\theta\Big(\frac{\lambda}{\tau},-\frac{1}{\tau}\Big)\frac{\sqrt{-i\tau}}{\tau}e^{\pi i\lambda^2}\tau\theta(\lambda,\tau) \tag{17}$$

From the property we see that θ *generates a one dimensional subspace, invariant w.r.t the action of the modular group.*

Let's go back to the general theory and let's start to investigate the relation between conformal blocks and theta functions.

Lemma 3.4. *For fixed τ the space of conformal blocks has dimension $\kappa - 2p + 1$ and consists of linear combinations of functions of the form:*

$$\theta^{p+1}(\lambda, \tau)\theta^S_{n,\kappa-2p}(\lambda, \tau), \quad n = 0, ..., \kappa - 2p.$$

Example 3.2. *Let's suppose that $\kappa = 2p+2$; then the space of conformal blocks is one dimensional. This implies that the holomorphic solutions of the KZB equation must be of the form:*

$$u(\lambda, \tau) = A(\tau)\theta^{p+1}(\lambda, \tau). \tag{18}$$

Lemma 3.5. *The function $u(\lambda, \tau) = \theta^{p+1}(\lambda, \tau)$ is a solution of the KZB heat equation for $\kappa = 2p + 2$.*

4. Integral Formulas.

In this section we will introduce the integral fomulas for the conformal blocks.

Notations 4.1. *Let's define:*

$$\sigma_\lambda(t, \tau) = \frac{\theta(\lambda - t, \tau)\theta'(0, \tau)}{\theta(\lambda, \tau)\theta(t, \tau)}; \tag{19}$$

$$E(t, \tau) = \frac{\theta(t, \tau)}{\theta'(0, \tau)}. \tag{20}$$

Let's also define the Master function:

$$\Phi(t_1, ..., t_p) = \prod_{j=1}^{p} E(t_j, \tau)^{-\frac{2p}{\kappa}} \prod_{1\le i<j\le p} E(t_i - t_j, \tau)^{\frac{2}{\kappa}}. \tag{21}$$

Remark 4.1. *21 is the elliptic analogous of the Selberg integral.*

We also introduce the integration cycles:

$$\Delta_k \subset \mathbb{R}^k \subset \mathbb{C}^k$$

where

$$\Delta_k = \{(t_1, ..., t_k | 0 \le t_k \le ... \le t_1 \le 1\}.$$

We define $\tilde{\Delta_k}$ the image of the standard real symplex under the map:

$$(t_1, ..., t_k) \longrightarrow (\tau t_1, ..., \tau t_k). \tag{22}$$

Remark 4.2. *22 define a rotation of the standard symplex.*

For $0 \le k \le p$ let's define:

$$J^{[k]}_{n,\kappa}(\lambda, \tau) = \int_{\Delta_k} \prod_{j=1}^{k} \sigma_\lambda(t_j, \tau)dt_j \int_{\tilde{\Delta}_{p-k}} \prod_{j=k+1}^{p} \sigma_\lambda(t_j, \tau)dt_j \Phi(t_1, ..., t_p, \tau)\theta_{\kappa,n}(\lambda + \frac{2}{\kappa}\sum_{j=1}^{p} t_j, \tau).$$

Definition 4.1.

$$u^{[k]}_{n,\kappa}(\lambda,\tau) = J^{[k]}_{n,\kappa}(\lambda,\tau) + (-1)^{p+1} J^{[k]}_{n,\kappa}(-\lambda,\tau). \tag{23}$$

Theorem 4.1. *For any $0 \le k \le p$ and any n*

$$u^{[k]}_{n,\kappa}(\lambda,\tau)$$

is a conformal block.

We have the following:

Definition 4.2. *The*

$$u^{[k]}_{n,\kappa}(\lambda,\tau)$$

are called elliptic hypergeometric functions.

Now we state some elementary properties of the elliptic hypergeometric functions:

Properties 4.1. *1.* $u^{[k]}_{n,\kappa}(\lambda,\tau) = u^{[k]}_{n,2+\kappa}(\lambda,\tau)$*;*
2. $u^{[k]}_{n,\kappa}(\lambda,\tau) = -q^{2k(n+p-k)} u^{[k]}_{\kappa,-n-2(p-k)}$.

Theorem 4.2 (on Basis)**.** *1. The set* $u^{[p]}_{n,\kappa}(\lambda,\tau)$, $n = p+1, ..., \kappa-p-1$, *is a basis in the space of conformal blocks.*
2. $u^{[p]}_{n,\kappa} \equiv 0$ *if* $0 \le n \le p$ *or* $\kappa - p \le n \le \kappa$.

Example 4.1. *Let $\kappa = 2p+2$ so the dimension of the space of conformal blocks is equal to 1. In this case we have that:*

$$u^{[p]}_{n,\kappa}(\lambda,\tau) \equiv 0 \ \ if \ \ n \ne p+1 \ \ mod \ \ \kappa.$$

In this case the p-dimensional integral can be calculate explicitely. This integral can be called the elliptic Selberg integral. Recall that:

$$B_p(\alpha,\beta,\gamma) = \int_{\Delta_p} \prod_{j=1}^{p} t_j^{\alpha-1}(1-t_j)^{\beta-1} \prod_{1\le i<j\le p} (t_j - t_k)^{2\gamma} dt_1 \wedge ... \wedge d_k =$$
$$\frac{1}{p!} \prod_{j=0}^{p-1} \frac{\Gamma(1+\gamma(1+j))}{\Gamma(1+\gamma)} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(\alpha+\beta+(p+j-1)\gamma)}.$$

Now we can state the following:

Theorem 4.3.

$$u^{[p]}_{2p+2}(\lambda,\tau) = C_p B_p\Big(\frac{1}{2} + \frac{1}{2(p+1)}, -1 + \frac{1}{p+1}, \frac{1}{2(p+1)}\Big)\theta(\lambda,\tau)^{p+1},$$

where:

$$C_p = (2\pi)^{\frac{p}{2}} e^{-\pi i \left[\frac{p(3p-1)}{4(p+1)} + \frac{p+1}{2}\right]} \prod_{j=1}^{p} \left(1 - e^{-2\pi i \frac{j}{2p+2}}\right)$$