LECTURE 8

During the last lecture we studied the critical points of the function:

$$\Phi_{k,n}(t,z,m) = \prod_{1\le i<j\le k}(t_i - t_j)^2 \prod_{l=1}^{n}\prod_{i=1}^{k}(t_i - z_l)^{-m_l}$$

We studied the action on the set of the critical points of the symmetric group and we discussed the following:

Theorem 0.1.

$$dimSingM^{\otimes m}[|m|-2k] = \binom{k+n-2}{n-2}.$$

and also:

Theorem 0.2. *Let $m_i \in \mathbb{Z}_{>0}$ for $i = 1,...,n$, $k \in \mathbb{Z}_{>0}$ and let $\omega(m = (m_1,...,m_n),k)$ be the multiplicity of $L_{|m|-2k}$ in $L_{m_1} \otimes ... \otimes L_{m_n}$. Then:*
1. If $|m|+1-k > k$, then for generic z all critical points of $\Phi_{k,n}(t)$ are non degenerate and the critical set consists of $\omega(m,k)$ orbits.
2. If $|m|+1-k = k$, then for any z the function $\Phi_{k,n}(t)$ has no critical points.
3. If $0 \le |m|+1-k < k$ then for generic z the function $\Phi_{k,n}(t)$ may have only non isolated critical points. Written in symmetric coordinates $\lambda_1,...,\lambda_k$, the critical set consists of $\omega(m,|m|+1-k)$ straight lines in the space $\mathbb{C}^k_\lambda$.
4. If $|m|+1-k < 0$, then for any z the function $\Phi_{k,n}(t)$ has no critical points.

Today we will discuss the relation of the previous theorem (in particular the statements 2., 3., and 4.) with the theory of the Fuchsian differential equations.

1. Differential Equations with Regular Singular Points.

We will focus to the case of second order equations:

$$u'' + p(z)u' + q(z)u = 0 \tag{1}$$

Let's start with the following:

Definition 1.1. *A point $z_o \in \mathbb{C}$ is called* ordinary *for the differential equation if the functions $p(z)$ and $q(z)$ are holomorphic at a neighborood of this point.*

To study the case $z = \infty$ put $z = \frac{1}{\xi}$:

$$\frac{du}{dz} = \frac{d\xi}{dz}\frac{du}{d\xi} = -\xi^2\frac{du}{d\xi}$$

and

$$\frac{d^2u}{d\xi^2} = \xi^4 \frac{d^2u}{d\xi^2} + 2\xi^3 \frac{du}{d\xi}.$$

With this change of variables the equation turns into:

$$\frac{d^2u}{d\xi^2} + \left[\frac{2}{\xi} - \frac{1}{\xi^2} p\Big(\frac{1}{\xi}\Big)\right] \frac{du}{d\xi} + \frac{1}{\xi^4} q\Big(\frac{1}{\xi}\Big) u. \tag{2}$$

So we have the following:

Definition 1.2. *The point $z = \infty$ is an ordinary point of the equation if $\xi = 0$ is an ordinary point of the transformed equation, i.e if:*

$$\left[\frac{2}{\xi} - \frac{1}{\xi^2} p\Big(\frac{1}{\xi}\Big)\right]$$

and

$$\frac{1}{\xi^4} q\Big(\frac{1}{\xi}\Big)$$

are holomorphic functions at $\xi = 0$.

Definition 1.3. *If z_o is called* singular *point for the equation if it is not ordinary.*

Definition 1.4. *The point z_o is called* regular singular *if the following conditions are verified:*
1) z_o is singular;
2) at z_o the function $p(z)$ at has a pole of order ≤ 1;
3) at z_o the function $q(z)$ has a pole of order ≤ 2.

Remark 1.1. *We have analogous definitions for the point $z = \infty$ and local coordinate $z = \frac{1}{\xi}$.*

In the following two section we will discuss the solutions of the equation at an oridinary point and at a singular point.

2. Solutions at a neighborood of an ordinary point.

Let z_o an oridinary point, so we can write:

$$p(z) = \sum_{n=0}^{\infty} b_n (z - z_o)^n$$

and

$$q(z) = \sum_{n=0}^{\infty} d_n (z - z_o)^n$$

for $|z - z_o| < R$. We look for solutions of the form:

$$u(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n.$$

Under these hypothesis we have the following:

Theorem 2.1. *For any choice of a_0, a_1 there exist a unique series*

$$u(z) = \sum_{n=0}^{\infty} a_n(z - z_o)$$

with a non zero radius of convergence, satisfying the differential equation:

$$u'' + p(z)u' + q(z)u = 0.$$

Prof. Plugging $u(z) = \sum_{n=0}^{\infty} a_n(z - z_o)$ into the equation $u'' + p(z)u' + q(z)u = 0$ we get:
for $n = 0$, $0a_0 = 0$, for $n = 1$, $0a_1 + 0a_0b_0 = 0$, while for $n \geq 2$,

$$n(n-1)a_n + \sum_{k=0}^{n-1} k a_k b_{n-k-1} + \sum_{k=0}^{n-2} a_k b_{n-k-2}.$$

3. Solutions at a neighborood of a regular singular point.

Let's assume z_o be a regular singular point so that the coefficients of the equation can be written in the following way:

$$p(z) = \sum_{n=0}^{\infty} b_n(z - z_o)^{n-1}$$

and

$$q(z) = \sum_{n=0}^{\infty} d_n(z - z_o)^{n-2},$$

for $|z - z_o| < R$. In this case we look for solutions of the form:

$$u(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^{n+c}$$

where $a_0 \neq 0$ and c is a costant that has to be determined. Plugging the function $u(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^{n+c}$ into the equation, we get the following equation:

$$\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n(z-z_0)^{n+c-2} + \Big[\sum_{n=0}^{\infty} p_n(z-z_0)^{n-1}\Big]\Big[\sum_{n=0}^{\infty}(n+c)a_n(z-z_0)^{n+c-1}\Big] +$$
$$+\Big[\sum_{n=0}^{\infty} q_n(z-z_0)^{n-2}\Big]\Big[\sum_{n=0}^{\infty} a_n(z-z_0)^{n+c}\Big] = 0.$$

From this equation we get the following recurrence relations:
for $n = 0$:

$$[c(c-1) + cp_0 + q_0]a_0 = 0, \tag{3}$$

while for $n \geq 1$:

$$[(n+c)(n+c-1) + (n+c)p_0 + q_0]a_n = -\sum_{k=0}^{n-1} a_k[(k+c)p_{n-k} + q_{n_k}]. \tag{4}$$

Since $a_0 \neq 0$, we have:

$$c^2 + (p_0 - 1) + q_0 = 0. \tag{5}$$

Definition 3.1. *The equation* $c^2 + (p_0 - 1) + q_0 = 0$ *is called* indicial *equation, while the roots* c_1 *and* c_2 *are called the* exponents *of the equation at the point* $z = z_0$*.*

Let's write $F(c) = c^2 + (p_0 - 1) + q_0$ so that we have:

$$F(n+c)a_n = -\sum_{k=0}^{n-1} a_k[(k+c)p_{n-k} + q_{n_k}]$$

for $n \geq 1$, and $F(c) = 0$, for the indicial equation. Let's order the roots c_1, c_2 of the polynomial $F(c)$ in such a way $\Re e c_1 \geq \Re e c_2$ and set $s = c_1 - c_2$. Now if we start with $c = c_1$, we get:

$$F(n+c_1) = n(n + c_1 - c_2) = n(n+s)$$

from which we see that the recurrence equations are satisfied; so we can state the following:

Theorem 3.1. *If* z_0 *is a regular singular point with exponentials* c_1 *and* c_2 *such that* $\Re e c_1 \geq \Re e c_2$*, then the series:*

$$u_1(z) = (z - z_0)^{c_1} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n + c_1} \tag{6}$$

satisfies the equation and it is convergent in a region of the form $0 < |z - z_0| < r$*.*

Question 3.1. *How to find the second solution?*

The answer to this question depends on $s = c_1 - c_2$. Put $c = c_2$, then the equation for $n = 0$ is satisfied; for $n \geq 1$ we get:

$$n(n-s)a_n = \sum_{k=0}^{n-1} q_k \Big[(k + c_2)p_{n-k} + q_{n-k}\Big] :$$

Theorem 3.2. *If* $s \notin \mathbb{Z}$ *then there exists a series:*

$$u_2(z) = (z - z_0)^{c_2} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n + c_1} \tag{7}$$

that satisfies the equation and converges in a region of the form $0 < |z - z_0| < r$*.*

In the case $s = 0$ we don't get any new solution, while if $s \in \mathbb{Z}_{>0}$, for $n = s$, we get:

$$0 = \sum_{k=0}^{s-1} q_k \Big[(k + c_2)p_{s-k} + q_{s-k}\Big].$$

This gives a non trivial constraint: given arbitrary a_0 and a_s we can get:

$$u_1(z) = (z - z_0)^{c_1} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n + c_1}$$

and

$$u_2(z) = (z - z_0)^{c_2} + \sum_{n=1}^{\infty} a_n (z - z_0)^{n + c_2}$$

if a_0 and a_s authomatically satisfy the recurrence relation or:

$$u_2(z) = u_1 \ln(z - z_0) + \sum_{n=0}^{\infty} d_n (z - z_0)^{n+c_2}$$

if not.

4. Fuchsian Equations.

Let's start with the following:

Definition 4.1. [Lazarus Fuchs (1833-1902)] *A linear differential equation is called* fuchsian *if all its singular points are regular singular.*

Remark 4.1. *During this section we will focus only on second order differential equations.*

Remark 4.2. *If the equation:*

$$u'' + p(z)u' + q(z)u = 0$$

is fuchsian, then the coefficients $q(z)$ and $p(z)$ are rational functions.

We also have the following:

Proposition 4.1. *A fuchsian equation has at least a singular point.*

In what follows we will give a description of the second order fuchsian differential equations with one two and three singular points. Let's start with the first case. Let's assume that $u'' + p(z)u' + q(z)u = 0$ has only one singular point at $z = z_1$, then:

Proposition 4.2. *The equation has the form:*

$$u'' + \frac{2}{z - z_1} u' = 0. \tag{8}$$

and its solutions are of the form $u(z) = a + \frac{d}{z - z_1}$ for given a and b.

Prof. From the hypothesis we have that:

$$p(z) = \frac{p_1(z)}{z - z_1}$$

and

$$q(z) = \frac{q_1(z)}{(z - z_1)^2}$$

where $p_1(z)$ and $q_1(z)$ are polynomials. Since $z = \infty$ is an ordinary point we have:

$$2z - z^2 p(z) = 2z - \frac{z^2 p_1(z)}{z - z_1} = \frac{2z^2 - z^2 p_1 - 2z_1 z}{z - z_1}$$

and

$$z^4 q(z) = \frac{z^4 q_1(z)}{(z - z_1)^2}$$

are holomorphic at $z=\infty$. From the last equation we deduce that $q_1 \equiv 0$ (otherwise $\deg z^4 q_1(z) \geq 4$) while from the first one follows that $p_1 = 2$. Clearly $u_1(z) = 1$ and $u_2(z) = \frac{1}{z-z_1}$ are solutions.

We also have the following result:

Theorem 4.1. *If $z=\infty$ is the only singular point of a fuchsian differential equation, then the equation is:*

$$u'' = 0. \tag{9}$$

Prof. In this case the coefficients $p(z)$ and $q(z)$ are two polynomials. So we have that the function $2z - z^2 p(z)$ has poles of order ≤ 1 while the function $z^4 q(z)$ has poles of order ≤ 2. From this observation follows that $p(z) = q(z) = 0$.

Now let's discuss the case of fuchsian equations with two singular points z_1 and z_2. In this case we have two different cases:
1) $z_2 = \infty$, or:
2) z_1 and $z_2 \in \mathbb{C}$.
In the first case we have:

Proposition 4.3. *The fuchsian equation can be written as:*

$$u'' + \frac{k_1}{z-z_1}u' + \frac{k_2}{(z-z_1)^2}u = 0, \tag{10}$$

with $k_1, k_2 \in \mathbb{C}$. Moreover the solutions are:

$$u(z) = A(z-z_1)^{c_1} + B(z-z_1)^{c_2} \tag{11}$$

if $c_1 \neq c_2$, or:

$$u(z) = (z-z_1)^c \Big(A + B\ln(z-z_1)\Big) \tag{12}$$

if $c_1 = c_2 = c$.

For the second case, when both z_1 and z_2 belong to $\mathbb{C}$, it is possible to prove the following:

Proposition 4.4. *The fuchsian equation has the form:*

$$u'' + \frac{2z + k_3}{(z-z_1)(z-z_2)}u' + \frac{k_4}{(z-z_1)^2(z-z_2)^2}u = 0. \tag{13}$$

Let's now start to discuss the case of three singular point.

Remark 4.3. *We have the following general:*

Theorem 4.2. *A second order fuchsian equation with $n+1$ singular points $z_1, ..., z_n, \infty$ has the form:*

$$u'' + \frac{T_{n-1}}{(z-z_1)...(z-z_n)}u' + \frac{T_{2n-2}}{(z-z_1)^2...(z-z_n)^2}u = 0, \tag{14}$$

where T_{n-1} and T_{2n-2} are polynomials of degree less or equal to $n-1$ and $2n-2$.

If we call $\alpha_{1,k}$ and $\alpha_{2,k}$ the exponents of the equation at the point $z = z_k$ and $\alpha_{1,\infty}$ and $\alpha_{2,\infty}$ the exponents at the point $z = \infty$, then:

Theorem 4.3. [Fuchsian Invariants.]

$$\alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{n}(\alpha_{1,k} + \alpha_{2,k}) = n - 1. \tag{15}$$

Let's focus on the case of three singular points z_1, z_2 and ∞. In this case the exponents are: $(\alpha_{1,\infty}, \alpha_{2,\infty})$ $(\alpha_{1,1}, \alpha_{2,1})$ $(\alpha_{1,2}, \alpha_{2,2})$ and we have:

$$\alpha_{1,\infty} + \alpha_{2,\infty} + \sum_{k=1}^{2}(\alpha_{1,k} + \alpha_{2,k}) = 1. \tag{16}$$

Claim 4.1. *the fuchsian equation can be written in the following way:*

$$u'' + \left[\frac{1-\alpha_{1,1}-\alpha_{2,1}}{z-z_1} + \frac{1-\alpha_{1,2}-\alpha_{2,2}}{z-z_2}\right]u' +$$
$$\left[\frac{\alpha_{1,1}\alpha_{2,1}}{(z-z_1)^2} + \frac{\alpha_{2,1}\alpha_{2,2}}{(z-z_2)^2} + \frac{\alpha_{1,\infty}\alpha_{2,\infty} - \alpha_{1,1}\alpha_{2,1} - \alpha_{1,2}\alpha_{2,2}}{(z-z_1)(z-z_2)}\right]u = 0$$

Remark 4.4. *The previous equation is the form of the general Fuchsian equation of the second order and with three regular singular points written in terms of the exponents.*

Using the notation $z = a, b, c$ for the singular points (all of them different by ∞) and (a', a''), (b', b''), (c', c'') for the correspondents exponents, we have:

Theorem 4.4. *The previous equation can be written as:*

$$u'' + \left[\frac{1-a'-a''}{z-a} + \frac{1-b'-b''}{z-b} + \frac{1-c'-c''}{z-c}\right]u' +$$
$$\left[\frac{a'a''(a-b)(a-c)}{z-c} + \frac{b'b''(b-a)(b-c)}{z-b} + \frac{c'c''(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)} = 0.$$

Remark 4.5. *This equation is called* Riemann-Papperitz equation.

Definition 4.2. [Riemann P-symbol.]

Let's write:

$$u = P\begin{pmatrix} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{pmatrix} \tag{17}$$

This is called the Riemann P-symbols. *This denotes a solution of a Fuchsian differntial equation with singular points at a, b, c (distinct but can be ∞) and exponentials (a', a''), (b', b''), (c', c'').*

Remark 4.6. *Under these hypothesis we have:*

$$a' + a'' + b' + b'' + c' + c'' = 1.$$

Remark 4.7. *The informations given in the P-symbol determines the equation uniquely.*

Theorem 4.5. *Let's suppose $a, c \in \mathbb{C}$ and:*

$$u = P\left(\begin{array}{cccc} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{array}\right). \tag{18}$$

If

$$u = \left(\frac{z-a}{z-c}\right)^k w$$

then:

$$u = P\left(\begin{array}{cccc} a & b & c & \\ a'-k & b' & c'+k & z \\ a''-k & b'' & c''+k & \end{array}\right). \tag{19}$$

Theorem 4.6. *If:*

$$u = P\left(\begin{array}{cccc} a & b & \infty & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{array}\right) \tag{20}$$

and $u = (z-a)^k$ then:

$$u = P\left(\begin{array}{cccc} a & b & \infty & \\ a'-k & b' & c'+k & z \\ a''-k & b'' & c''+k & \end{array}\right). \tag{21}$$

Theorem 4.7. *If*

$$u = P\left(\begin{array}{cccc} a & b & c & \\ a' & b' & c' & z \\ a'' & b'' & c'' & \end{array}\right) \tag{22}$$

the fractional transformation:

$$v = \frac{Az+B}{Cz+D}$$

with $AD - CB \neq 0$ maps $z = a, b, c$ into $v = a_1, b_1, c_1$ and the solution u can be witten in terms of v in the following way:

$$u = P\left(\begin{array}{cccc} a_1 & b_1 & c_1 & \\ a' & b' & c' & v \\ a'' & b'' & c'' & \end{array}\right) \tag{23}$$

We also have the following general:

Lemma 4.1. *Any linear fractional transformation:*

$$v = \frac{Az+B}{Cz+D}$$

with $AD - BC \neq 0$ is the composition of elementary transformations:
$v = \gamma z$, $v = \gamma + z$ and $v = \frac{1}{z}$.

We have the following important:

Theorem 4.8. *Any P-symbols can be reduced to a P-symbols of the form:*

$$y = P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & x \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix} \tag{24}$$

Remark 4.8. *The corrisponding differential equation is the following:*

$$\frac{d^2y}{dx^2} + \left[\frac{1-(1-\gamma)}{x} + \frac{1-(\gamma-\alpha-\beta)}{x-1}\right]\frac{dy}{dx} + \frac{\alpha\beta}{x(x-1)}y = 0. \tag{25}$$

Multiplying both sides by $x(x-1)$ we get:

$$x(x-1)y''(x) + [\gamma - (\alpha+\beta+1)x]y' - \alpha\beta y = 0, \tag{26}$$

i.e. the Gauss *differential equation.*

Let's start the discussion the connectione of the theory of the Fuchsian differential equations with the critical points of the master function. Let's consider the following:

$$F(x)u''(x) + G(x)u'(x) + H(x)u(x) = 0 \tag{27}$$

where F, G and H are polynomials of degree n, $n-1$ and $n-2$ respectively.

Claim 4.2. *If the polynomial F has no multiple roots then the previous equation is Fuchsian.*

In this case let's write:

$$F(x) = \prod_{j=1}^{n}(x-z_j)$$

$$\frac{G(x)}{F(x)} = \sum_{j=1}^{n} -\frac{m_j}{x-z_j}$$

for suitable complex numbers m_j and z_j.

Claim 4.3. *It is possible to prove that* 0 *and* m_j+1 *are exponents of the above equation and if $-k$ is one of the exponents at $z=\infty$ then the other is $k-l(m)-1$.*

Problem 4.1. *Given polynomials $F(x)$ and $G(x)$ as above:*
1. find a polynomial $H(x)$ of degree at most $n-2$ such that the equation has a polynomial solution of a preassigned degree k;
2. find the number of solution of 1.

we have the following classical results (reference: Szego G., "Orthogonal Polynomials"):

Theorem 4.9. *i. Let $u(x)$ a polynomial solution of degree k of the equation, with roots $t_1^0, ..., t_k^0$ of multiplicity one. Then $t^0 = (t_1^0, ..., t_k^0)$ is a critical point of the function $\Phi_{k,n}(t; z, m)$, where $z = (z_1, ..., z_n)$ and $m = (m_1, ..., m_n)$.*

ii. Let t^0 be a critical point of the function $\Phi_{k,n}(t;z,m)$, then the polynomial $u(x) = (x-t_1^0)...(x-t_k^0)$ of degree k is a solution of the equation with

$$H(x) = \frac{\Big(-F(x)u''(x) - G(x)u'(x)\Big)}{u(x)}$$

being a polynomial of degree at most $n-2$.

Remark 4.9. *Recall that the critical points of the function $\Phi_{k,n}(t;z,m)$ are given by:*

$$\sum_{j=1}^{n} \frac{-m_j}{t_i - z_j} + \sum_{j \neq i} \frac{2}{t_i - t_j}, i = 1, ..., k. \tag{28}$$

Remark 4.10. *This can be seen as an equivalent version of the Bethe ansazt problem.*

Prof.(of part i.). Let $u(x) = (x-t_1^0)...(x-t_k^0)$ be a solution of the equation:

$$Fu'' + Gu' + Hu = 0,$$

and set $x = t_i^0$; then:

$$F(t_i^0)u''(t_i^0) + G(t_i^0)u'(t_i^0) + H(t_i^0)u(t_i^0) = 0.$$

From this we can deduce:

$$\frac{u''(t_i^0)}{u'(t_i^0)} + \frac{G(t_i^0)}{F(t_i^0)} = 0,$$

but by definition we have:

$$\frac{G(t_i^0)}{F(t_i^0)} = \sum \frac{-m_i}{t_i^0 - z_i}.$$

To conclude we need the following:

Lemma 4.2. *If $u(x) = (x-t_1^0)...(x-t_k^0)$, then:*

$$\frac{u''(t_i^0)}{u'(t_i^0)} = \sum_{j \neq i} \frac{2}{t_i^0 - t_j^0}. \tag{29}$$

Example 4.1. *If:*

$$u(x) = (x-t_1^0)(x-t_2^0)$$

then:

$$u(x) = (x-t_1^0) + (x-t_2^0),$$

$$u'' = 2$$

so:

$$\frac{u''}{u'}(t_1^0) = \frac{2}{t_1^0 - t_2^0}$$

This give us part i. of the previous theorem.

Remark 4.11. *The critical points label solutions of the problem 2., and a critical point defines a differential equation and a polynomial solution of that equation.*

Remark 4.12. *In 19-th century Heine and Stieltjes showed that for fixed real $z_1, ..., z_n$ and negative $m_1, ..., m_n$ the problem 2. has:*

$$\binom{k+n-2}{n-2}$$

solutions (reference: Szego G., "Orthogonal Polynomials"*).*

Let $m_1, ..., m_n \in \mathbb{Z}_{>0}$ and $z \in \mathbb{Z}_{>0}$, such that $|m| - 2k \geq 0$, then $k < |m| - k + 1$. Assume that t^0 is a critical point of $\Phi_{k,n}(t; z, m)$, then it is possible to construct a line of critical points of the function $\Phi_{|m|-k+1,n}(t; z, m)$.

Example 4.2. *Let $m = (1, 1, 1)$ and*

$$\Phi_{1,3}(t; z, m) = (t - z_1)^{-1}(t - z_2)^{-1}(t - z_3)^{-1}.$$

it is possible to construct a line of critical points of the function: Given the two critical points of the function $\Phi_{1,3}(t; z, m)$ we get two lines of critical points of the functon.

$$\Phi_{3,3}(t; z, m) = \prod_{i=1}^{3} \prod_{l=1}^{3} (t_i - z_l)^{-1} \prod_{1 \leq i < j \leq 3} (t_i - t_j)^2.$$

Moreover it is possible to show that for generic z_1, z_2, z_3 the points on those two lines are the only critical points.

5. Enumerative Algebraic Geometry.

It is possible to reformulate the main theorem in terms of Enumerative Algebraic Geometry. Let V a two dimensional space of polynomials of one variable with coefficients in $\mathbb{C}$. Let k_1 the degree of a typical polynomial and k_2 the degree of a special polynomial ($k_1 < k_2$)

Definition 5.1. *Given two functions $f(x)$ and $g(x)$ let's denote with:*

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \tag{30}$$

their Wronskian.

Remark 5.1. *If (f, g) and $(\tilde{f}, \tilde{g})$ are two basis of V then there exits a costant c such that:*

$$W(f, g) = cW(\tilde{f}, \tilde{g}).$$

Lemma 5.1. *If $deg f = k_1$ and $deg g = k_2$ then $deg W(f, g) = k_1 + k_2 - 1$.*

Prof. This follows from direct calculation:

$$W(f, g) = \begin{vmatrix} x^{k_1} + .. & x^{k_2} + .. \\ k_1 x^{k_1 - 1} + .. & k_2 x^{k_2 - 1} + .. \end{vmatrix} = (k_2 - k_1) x^{k_1 + k_2 - 1} + ... \tag{31}$$

Let $W_V = x^{k_1+k_2-1} + ...$ the Wronskian of V and let's suppose that $W_V(x) = \prod_{l=1}^{n}(x - z_l)^{m_l}$, where $m_l \geq 1$.

Definition 5.2. *We will say that V is not singular if $\forall x_0 \in \mathbb{C}$ there is $f \in V$ such that $f(x_0) \neq 0$.*

Now we can state the following:

Problem 5.1. *Assume that $k_1 < k_2$ and $W_V(x) = \prod_{l=1}^{n}(x - z_l)^{m_l}$ are fixed. What is the number of non singular vector spaces V of polynomials with such data?*

It is possible to get an answer to question in terms of representation of $sl(2)$:

Theorem 5.1. *For generic $z_1, ..., z_n$ the numeber of non singular vector spaces with such data is equal to:*
$multL_{k_1-k_1-1} \subset L_{m_1} \otimes ... \otimes L_{m_n}$.