LECTURE 7

During the last lecture we proved the following result:
given

$$u = e^{\frac{S(z)}{\kappa}}(f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + ...) \tag{1}$$

asymptotic solution of the KZ equation, $f_0(z)$ is a common eigenvector for the set of commutating operators $H_i = H_i(z)$, $i = 1, ..., n$, where:

$$H_i(z) = \sum_{i \neq j} \frac{\Omega^{(ij)}}{z_i - z_j}$$

with eigenvalues $\frac{\partial S}{\partial z_i}(z)$. We also introduced the Shapovalov form S: given $V = V_m$, generated by v, S is the unique bilinear symmetric form such that:

$$S(f^k v, f^k v) = k!m(m-1)...(m-k+1) \tag{2}$$

and 0 otherwise, and we proved that if $V = V_1 \otimes ... \otimes V_n$, $S = S_1 \otimes ... \otimes S_n$ and $u = e^{\frac{S(z)}{\kappa}}(f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + ...)$ asymptotic solution of the KZ equation, then $S(f_0(z), f_0(z))$ does not depend on z. During this lecture we will discuss how to find explicit asymptotic solutions of KZ equation and hence eigenvectors for the set of hamiltonian operators $H_i(z)$.
Let's consider the space of $SingM^{\otimes m}[|m| - 2k]$. We know that given:

$$\Phi_{k,n}(t, z, m) = \prod_{1 \leq i < j \leq n} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{1 \leq i < j \leq k} (t_i - t_j)^2 \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_l} \tag{3}$$

$$A_J(t, z) = \frac{1}{j_1!...j_n!} Sym_t \Big[\prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_1} + ... + t_{j_{l-1+i}} - z_l} \Big], \tag{4}$$

$$u^\gamma = \Sigma \int_\gamma \Phi_{k,n}^{\frac{1}{\kappa}}(t, z) A_J(t, z) dt_1 \wedge ... \wedge dt_k \tag{5}$$

is a solution of the KZ equation.
We will construct asymptotic solutions taking the limit of u^γ for $\kappa \longrightarrow 0$.

Let's condider the following examples where we introduce the *Steepest Descend Method.*

Example 0.1. *Let's consider the following function:*

$$I(\kappa) = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{\kappa}} a(s) ds. \tag{6}$$

If $a(0) \neq 0$ we can conclude that:

$$I(\kappa) = \int_{-\infty}^{+\infty} e^{-\frac{s^2}{\kappa}} a(s) ds \sim \sqrt{\pi}, \tag{7}$$

since:

$$\int_{-\infty}^{+\infty} e^{-\frac{s^2}{\kappa}} a(s)ds = \sqrt{\pi\kappa}\Big(a(0) + \kappa\frac{a''(0)}{4} + ...\Big) \tag{8}$$

In the same way we could analyze the function:

$$I(\kappa) = \int_{-\infty}^{+\infty} e^{i\frac{s^2}{\kappa}} a(s)ds \tag{9}$$

Definition 0.1. *the function $a(s)$ is called amplitude while the function $\phi(s) = -\frac{s^2}{\kappa}$ is called phase.*

Example 0.2. *Given a phase function of several variables, $S = S(t_1, ..., t_n)$ whose critical points are isolated and non degenerate, we can reduce the analysis of the behaviour of*

$$I(\kappa) = \int_{\mathbb{R}^n} e^{-\frac{S(t_1,...,t_n)}{\kappa}} a(t_1, ..., t_n)dt_1...dt_n \tag{10}$$

to that studied in the previous example. In fact in a neighborood of any of the critical point of S, we can find a new set of coordinates $s_1, ..., s_n$, such that:

$$S(s_1, ..., s_n) = A \pm {s_1}^2 \pm {s_2}^2 ... \pm {s_n}^2 \tag{11}$$

Applying this method to:

$$\int \Phi_{k,n}^{\frac{1}{\kappa}}(t, z)A_J(t, z)dt_1 \wedge ... \wedge dt_k \tag{12}$$

we can conclude that it localizes at the critical points of the function $\Phi_{k,n}^{\frac{1}{\kappa}}(t, z)$.

Let's define $S = \frac{\log \Phi_{k,n}(t,z)}{\kappa}$, let's suppose that $z^o \in \mathbb{C}^n$ has distinct coordinates and let $t^o \in \mathbb{C}^k$ be a non-degenerate critical point of $\Phi_{k,n}$ w.r.t t, i.e

$$\frac{\partial \Phi_{k,n}}{\partial t_i}(t^o, z^o) = 0 \tag{13}$$

and

$$\det \frac{\partial^2 \Phi_{k,n}}{\partial t_i \partial t_j}(t^o, z^o) \neq 0. \tag{14}$$

Theorem 0.1. (Implicit Function.) *There exist a unique holomorphic $\mathbb{C}^k$ valued function $t = t(z)$, such that:*

1) $\frac{\partial \Phi_{k,n}}{\partial t_i}(t(z), z) = 0 \;\; i = 1, ..., k;$

2) $t^o = t(z^o)$.

Under these hypotesis we have the following:

Theorem 0.2. *The KZ equation with values in $SingM^{\otimes m}[|m|-2k]$ has an asymptotic solution, u, in a neighborood of z^o given by the following expression:*

$$u(z) = \frac{\Phi_{k,n}(t(z),z)^{\frac{1}{\kappa}}}{\sqrt{\det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z)}} \Big(\sum_{|J|=k} \mathcal{A}_J(t(z),z) f_J v + o(\kappa) \Big). \tag{15}$$

Corollary 0.1. *If $t^o \in \mathbb{C}^k$ is a non degenerate critical point of $\Phi_{k,n} = \Phi_{k,n}(t, z^o)$ then the vector:*

$$\omega(t^o, z^o) = \sum_{|J|=k} \mathcal{A}_J(t(z), z) f_J v \tag{16}$$

is an eigenvector of the set of commutating hamiltonians $H_i = H_i(z)$, $i = 1, ..., n$.

Definition 0.2. *$\omega(t^o, z^o)$ is called the Bethe vector.*

We also have the following:

Corollary 0.2.

$$\frac{S(\omega((t(z),z),\omega(t(z),z))}{\det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z)} = cost$$

Question 0.1. *What is this constant?*

Remark 0.1. *Even if this question makes sense for arbitrary Kac-Moody algebras the answer is known only for $sl(2)$; in this case we have:*

Theorem 0.3.

$$S(\omega(t(z),z),\omega(t(z),z)) = \det \frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_{k,n}(t(z),z)$$

i.e. the constant is 1.

Remark 0.2. *The set of critical points $t^o = (t_1, ..., t_k)$ of $\Phi_{k,n} = \Phi_{k,n}(t, z^o)$ is invariant w.r.t the permutation of the coordinates, moreover the Bethe vector corresponding to the point belonging to the same orbit are equal.*

we will address the following:

Question 0.2. *Is it true that:*
1) the number of the orbits of critical points of $\Phi_{k,n}$ is equal to the dimension of $SingM^{\otimes m}[|m|-2]$?
2) the Bethe vectors form a basis?

To answer these question we introduce the following general result; given an arrengement of hyperplanes $\mathcal{A} = \{\mathcal{H}\}$ in $\mathbb{R}^k$, defined by the equations $f_H = 0$, set:

$$\Phi_{\mathcal{A}}^{\lambda} := \prod_{H \in \mathcal{A}} f_H^{\lambda_H}, \lambda \in \mathbb{C}. \tag{17}$$

Question 0.3. *What is the number of critical points of the function $\Phi_{\mathcal{A}}^{\lambda}$ for generic $\{\lambda_H\}$?*

The equation that gives us the critical points is $\frac{d\Phi}{\Phi} = 0$ or in other words:

$$\sum \lambda_H \frac{\frac{\partial f_H}{\partial t_i}}{f_H} = 0, \quad i = 1, ..., k.$$

Remark 0.3. *Considering a real arrangement in $\mathbb{R}^k$, and positive λ_H, then each bounded component has a critical point. It is also possible to show that each of these critical points is non degenerate.*

Let $\mathcal{A}_R \in \mathbb{R}^k$ be a real arrangement, $\mathcal{A} \subset \mathbb{C}^k$ its complexification and let's define $M = \mathbb{C}^k - \bigcup_{H \in \mathcal{A}} H$. Then for positive $\{\lambda_H\}$ it is possible to prove the following:

Theorem 0.4. $\sharp\Big($ *of critical points of* $\Phi^\lambda_\mathcal{A}$ *in* $M\Big) = \sharp\Big($ *bounded component of* $\mathbb{R}^k - \bigcup_{H \in \mathcal{A}} H\Big) = |\chi(M)|$

where $\chi(M)$ is the Euler charateristic of the open complex manifols M.

Remark 0.4. *It was conjectured by Varchenko that for generic $\{\lambda_k\}$ this result holds for any complex arrangement $\mathcal{A} \subset \mathbb{C}^k$. This conjecture was proved by Orlic and Terao.*

From the theorem follows:

Corollary 0.3. *If the highest weights $m_1, ..., m_n$ are negative, i.e. the modules M_{m_j} are all irreducible*

$\sharp\Big($ *orbits of critical points of* $\Phi_{k,n}\Big)$=*dim* $SingM^{\otimes m}[|m| - 2k]$.

Remark 0.5. *Varchenko and Reshetikhin checked that for generic points $z_1, ..., z_n$ and negative $m_1, ..., m_n$ the Bethe vectors form a basis in $SingM^{\otimes m}[|m| - 2k]$. In this case we have:*

$$dimSingM^{\otimes m}[|m| - 2k] = \binom{k+n-2}{n-2} = \frac{\mathcal{B}}{k!}$$

where $\mathcal{B}$ in the number of the bounded components of the arrangement.

Let's now consider the following:

Example 0.3. *Let's consider the Selberg integrand:*

$$\tilde{\Phi}(t; \alpha, \beta, \gamma) = \prod_{j=1}^{k} t_j^\alpha (1 - t_j)^\beta \prod_{1 \leq i < j \leq k} (t_i - t_k)^{2\gamma}$$

where $\alpha, \beta, \gamma > 0$. This function has $k!$ critical points, one on each bounded domain. The symmetric group acts on the set of critical points and we have exactly one orbit.

Let's consider the elementary symmetric functions $\lambda_1 = \sum_{i=1}^k t_i$, $\lambda_2 = \sum_{i<j} t_i t_j$,...,$\lambda_k = \prod_{i=1}^k t_i$. We have the following: The critical points of the Selberg integrand satisfy the following equations:

$$\frac{\alpha}{t_i} + \frac{\beta}{t_i - 1} + \sum_{j \neq i} \frac{2\gamma}{t_i - t_j} = 0, \tag{18}$$

for $i = 1, ..., k$. *If* $(t_1, ..., t_k)$ *is a critical point then:*

Theorem 0.5.

$$\lambda_j = \binom{k}{j} \prod_{i=1}^{j} \frac{\alpha(k-i)\gamma}{\alpha+\beta+(2k-i-1)\gamma}, \tag{19}$$

for $j = 1, ..., k$.

Prof. The equation for the critical point can be rearranged in the following way:

$$(j+1)(\alpha+j\gamma)\lambda_{i+1} = (k-j)(\alpha-\beta+(k+j-1)\gamma)\lambda_j.$$

Now let $t \in \mathbb{C}^k$ be a singular point of the function $\tilde{\Phi}(t; -m_1, -m_2, 1)$ and let's consider the Bethe vector:

$$\omega = \sum_{p=0}^{k} f^{k-p} v_{m_1} \otimes f^p v_{m_2} A_{k-p,p}(t). \tag{20}$$

where:

$$A_{k-p,p}(t) = \sum_{1 \le i_1 < ... < i_j \le k} \frac{1}{(t_{i_1}-1)...(t_{i_p}-1)} \prod_{i \notin (i_1,...,i_p)} \frac{1}{t_i}$$

This vector belongs to $SingM_{m_1} \otimes M_{m_2}[m_1+m_2-2k]$ and w.r.t. the standard generator

$$\omega_{m_1+m_2-2k} = \sum_{p=0}^{k} \frac{f^{k-p} v_{m_1} \otimes f^p v_{m_2}}{(k-p)! m_1(m_1-1)...(m_1-k+p+1) m_2(m_2-1)...(m_2-k+p+1)}$$

of $SingM_{m_1} \otimes M_{m_2}[m_1+m_2-2k]$ we have the following:

Theorem 0.6.

$$\omega_{Bethe} = k! \prod_{j=1}^{k} (m_1+m_2-(2k-j-1))\omega_{m_1+m_2-2k}.$$

Theorem 0.7. *Let* $S = S_{m_1} \otimes S_{m_2}$ *the Shapovalov form on* $M_{m_1} \otimes M_{m_2}$ *and* t^o *a critical point of the function* $\tilde{\Phi}(t; -m_1, -m_2, 1)$. *Then:*

$$S(\omega_{Bethe}, \omega_{Bethe}) = \det\Big(\frac{\partial^2}{\partial t_i \partial t_j} \ln \tilde{\Phi}(t)\Big)$$

The theorem follows from the following lemmas:

Lemma 0.1.

$$S(\omega_{Bethe}, \omega_{Bethe}) = k! \prod_{l=0}^{k-1} \frac{(m_1+m_2-2k+l+2)^3}{(m_1-l)(m_2-l)}$$

Prof. The result is based on:

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Lemma 0.2.

$$\det\Big(\frac{\partial^2}{\partial t_i \partial t_j}\ln\tilde{\Phi}(t^o,-m_1,-m_2,1)\Big)=k!\prod_{l=0}^{k-1}\frac{(m_1+m_2-2k+l+2)^3}{(m_1-l)(m_2-l)}.$$

Prof. This follows comparing the asymptotic expansion of the Selberg integral:

$$I(\kappa)=\int_{0\leq t_k\leq\ldots\leq t1\leq 1}\tilde{\Phi}^{\frac{1}{\kappa}}dt_1\wedge\ldots\wedge dt_k$$

with its expression in term of Γ-functions.

The important example for applications is the case of the tensor product of finite dimensional representation and not for generic Verma modules. Given $m_1,\ldots,m_n\in\mathbb{Z}_{>0}$ let's consider the function:

$$\Phi_{k,n}(t,z,m)=\prod_{1\leq i<j\leq k}(t_i-t_j)^2\prod_{l=1}^{n}\prod_{i=1}^{k}(t_i-z_l)^{-m_l}$$

General problem 0.1. *study the critical set of this function.*

Remark 0.6. *It's natural to conjecture that the number of orbits of the critical points must be related to the dimension of Sing $M^{\otimes m}[|m|-2k]$.*

Remark 0.7. *Let $m_1,\ldots,m_n\in\mathbb{Z}_{>0}$ and let's consider*

$$L^{\otimes m}=L_{m_1}\otimes\ldots\otimes L_{m_n}=\oplus L_a$$

where each a has the form:
1. $a=|m|-2k$;
2. $a\geq 0$.
Let's denote $\omega(m,k)$ the multiplicity of $L_{|m|-2k}$ in $L^{\otimes m}$.

Example 0.4. *Let take $m=(1,1,1)$, $|m|=3$, so:*

$$L_1\otimes L_1\otimes L_1=L_3\oplus L_1\oplus L_1.$$

In this case we have $\omega(m,0)=1$ and $\omega(m,1)=2$.

Note that dim $M^{\otimes m}[|m|-2k]=\omega(m,k)$ and each direct summand L_a contains one singular vector of weight a.

Now we are ready to describe the critical set of the function

$$\Phi_{k,n}(t,z,m)=\prod_{1\leq i<j\leq k}(t_i-t_j)^2\prod_{l=1}^{n}\prod_{i=1}^{k}(t_i-z_l)^{-m_l}$$

for $m_1,\ldots,m_n\in\mathbb{Z}_{>0}$. Let $\lambda_1=\sum_{i=1}^{k}t_i$, $\lambda_2=\sum_{i<j1}t_it_j$,....,$\lambda_k=\prod_{i=1}^{k}t_i$ the standard symmetric functions of $t_1,\ldots,t_k$ and denote $\mathbb{C}^k_\lambda$ the space with coordinates $\lambda_1,\ldots,\lambda_k$.

Theorem 0.8. *Let m_i, $i+1,\ldots,n$, as before and $k\in\mathbb{Z}_{>0}$.*
1. If $|m|+1-k>k$, then for generic z all critical points of $\Phi_{k,n}(t)$ are non degenerate and the critical set consists of $\omega(m,k)$ orbits.

2. If $|m|+1-k=k$*, then for any* z *the function* $\Phi_{k,n}(t)$ *has no critical points.*
3. If $0 \leq |m|+1-k < k$ *then for generic* z *the function* $\Phi_{k,n}(t)$ *may have only non isolated critical points. Written in symmetric coordinates* $\lambda_1, ..., \lambda_k$*, the critical set consists of* $\omega(m, |m|+1-k)$ *straight lines in the space* $\mathbb{C}^k_\lambda$*.*
4. If $|m|+1-k < 0$*, then for any* z *the function* $\Phi_{k,n}(t)$ *has no critical points.*

Example 0.5. *Let* $m=(1,1,1)$*,* $|m|=3$*. In this case we have:*

$$\prod_{1\leq i<j\leq k}(t_i-t_j)^2\prod_{l=1}^{3}\prod_{i=1}^{k}(t_i-z_l)^{-1}.$$

If:
1. $k=1$*,* $|m|+1-k=3$ *so* $\Phi_{1,3}(t)$ *has two non degenerate critical points;*
2. $k=2$*,* $\Phi_{2,3}(t)$ *has no critical points;*
3. $k=3$*,* $\Phi_{3,3}(t)$ *has two straight lines of critical points;*
4. $k=4$*,* $\Phi_{4,3}(t)$ *has one straight line of critical points;*
5. $k\geq 5$*,* $\Phi_{k,3}(t)$ *has no critical points.*