Lecture 6

The goal of this lecture is to describe the trigonometric KZ equation and dynamical equation and see their interaction.

6.1. **Dynamical equations.** Let

$$\Phi_{k,n}^{1/\kappa} = \prod_{i,j}(z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \prod_{i,j}(t_i - t_j)^{\frac{2}{\kappa}} \prod_i \prod_l (t_i - z_l)^{\frac{-m_l}{\kappa}}$$

$$\mathrm{I}_J^\gamma(z_1, \dots, z_n) = \int_\gamma \Phi^{1/\kappa} A_j \, dt_1 \wedge \dots \wedge dt_k,$$

$$\varphi^{(\gamma)}(z_1, \dots, z_n) = \sum_{|J|=k} \mathrm{I}_J^\gamma f_J v.$$

be solutions of KZ with values in $\operatorname{Sing} M^{\otimes m}[|m| - 2k]$, $\gamma \in H_k(\mathcal{C}_{k,n}(z), \Phi_{k,n}^{1/\kappa})$. Shift $m_1 \mapsto m_1 + \kappa$. Then the integrand is multiplied by $\frac{\text{const}(z)}{(t_1 - z_1)\cdots(t_k - z_1)}$. This is a rational univalued function; the same cycle γ is good for the new integral. This defines a map in cohomology, given by

$$\{\mathrm{I}_J^\gamma(z, m_1, \dots, m_n)\} \to \{\mathrm{I}_J^\gamma(z, m_1 + \kappa, \dots, m_n)\}.$$

Problem: Describe this map.
The answer will be given in terms of $\mathfrak{sl}_2$-representation theory. Relations of this type will be called *dynamical equations*. Before going over the problem, we make a

Remark 1. Since $\varphi^{(\gamma)} \in \operatorname{Sing} M^{\otimes m}[|m| - 2k]$, there are many linear relations between $\{\mathrm{I}_J^\gamma\}$.

Lemma 2. *If $m_1 \notin \{0, 1, \dots, k-1\}$, then for any J_0, $|J_0| = k$, the integral $\mathrm{I}_{J_0}^\gamma$ is a linear combination of integrals $\{\mathrm{I}_J^\gamma\}$ with J having the form $(0, j_2, \dots, j_n)$, $j_1 + \dots + j_n = k$. The coefficients of this linear combination do not depend on γ.*

For example, for $k = 1$ we have $I_1 = \frac{1}{m_1}\sum_{j=2}^n m_j I_j$. We will study the transformation $m_1 \mapsto m_1 + \kappa$ for "restricted vector valued functions"

$$\psi^{(\gamma)}(z_1, \dots, z_n) = \sum_{\substack{J=(0,j_2,\dots,j_n) \\ |J|=k}} \mathrm{I}_J^\gamma f_J v.$$

Remark 3. Having the restricted function $\psi^{(\gamma)}$ one can reconstruct the initial function $\varphi^{(\gamma)}$. Therefore one can rewrite the KZ differential equation for $\varphi^{(\gamma)}(z)$ as a suitable differential equation for the restricted function. This new differential equation is called *trigonometric KZ equation*

6.2. **Trigonometric r-matrix.** Set

$$\left.\begin{aligned} \Omega^+ &= \tfrac{1}{4} h \otimes h + e \otimes f \\ \Omega^- &= \tfrac{1}{4} h \otimes h + f \otimes e \end{aligned}\right\} \in \mathfrak{sl}_2^{\otimes 2}$$

We have $\Omega^+ + \Omega^- = \Omega$, the Casimir element. Define the *trigonometric r-matrix* $\gamma(z) = \frac{\Omega^+ z + \Omega^-}{z-1}$. A more symmetric form is got when one considers $\gamma(z_i/z_j) = \frac{\Omega^+ z_i + \Omega^- z_j}{z_i - z_j}$. Let $\mathfrak{h} = \mathbb{C}h \subset \mathfrak{sl}_2$ be the Cartan subalgebra. Let $V = V_2 \otimes \cdots \otimes V_n$ be a tensor product of $\mathfrak{sl}_2$ modules, $\kappa \in \mathbb{C}$ and $\vec{\lambda} \in \mathfrak{h}$.

Definition 4. The *trigonometric KZ equations* on a V-valued function $u(z_2, \ldots, z_n, \vec{\lambda}) \in V$ are the equations

$$\kappa z_i \frac{\partial}{\partial z_i} u = \sum_{j \neq i} \frac{(\Omega^+ z_i + \Omega^- z_j)^{(i,j)}}{z_i - z_j} u + \vec{\lambda}^{(i)} u, \quad i = 2, \ldots, n,$$

where $\vec{\lambda}^{(i)} = 1 \otimes \cdots \otimes 1 \otimes \vec{\lambda} \otimes 1 \otimes \cdots \otimes 1$ in the i-th factor. $\vec{\lambda}$ and κ are parameters of the equation.

We introduce the *trigonometric KZ operators*

$$\nabla_i(\kappa, \vec{\lambda}) = \kappa z_i \frac{\partial}{\partial z_i} - \sum_{j \neq i} r^{(i,j)}(z_i/z_j) - \vec{\lambda}^{(i)}, \quad i = 2, \ldots, n.$$

Then the trigonometric KZ equations have the form

$$\nabla_i(\kappa, \vec{\lambda}) u = 0, \quad i = 2, \ldots, n.$$

6.3. **Hypergeometric solutions of trigonometric equations.** We consider solutions with values in $V = M_{m_2} \otimes \cdots \otimes M_{m_n}[\sum_{j=2}^n m_j - 2k]$. This is so because the trigonometric KZ equation does not preserve singular spaces, though it does preserve weight spaces (there is a correspondence between singular spaces in $V_{m_1} \otimes \cdots \otimes V_{m_n}$ and weight spaces of $V_{m_2} \otimes \cdots \otimes V_{m_n}$). Take $\vec{\lambda} = \lambda \frac{h}{2}$, $\lambda \in \mathbb{C}$. Set $\nu = \sum_{j=2}^n m_j - 2k$ and put

$$\Phi(t_1, \ldots, t_k, z_2, \ldots, z_n) = \prod_{2 \le i < j \le n} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{i=2}^n z_i^{\frac{1}{2} m_i(\lambda + (m_i - \nu)/2)}$$
$$\times \prod_{i=1}^k \prod_{j=2}^n (t_i - z_j)^{-m_j} \prod_{1 \le i < j \le k} (t_i - t_j)^2 \prod_{i=1}^k t_i^{-(\lambda - 1 - \nu/2)}$$

For any $J = (j_2, \ldots, j_n)$, $|J| = k$, introduce the function A_J by the same formulas as before. Set $f_J v = f^{j_2} v_{m_2} \otimes \cdots \otimes f^{j_n} v_{m_n}$.

Remark 5. The complement to the singularities of Φ with respect to $t_1, \ldots, t_k$ is the space $\mathcal{C}_{k,n} \subset \mathbb{C}^k$, which is also the complement to the singularities of $\Phi_{k,n}$ with $z = (0, z_2, \ldots, z_n)$.

Theorem 6. *For any $\gamma(z) \in H_k(\mathcal{C}_{k,n}(z), \Phi^{1/\kappa})$, the function*

$$\varphi^{(\gamma)}(z_1, \ldots, z_n) = \sum_{|J|=k} \int_{\gamma(z)} \Phi^{1/\kappa} A_J \, dt f_J v$$

is a solution of the trigonometric KZ equation with parameter $\vec{\lambda} = \lambda \frac{h}{2}$.

Let us describe then the Dynamical Equations $\vec{\lambda} \mapsto \vec{\lambda} + \kappa\frac{h}{2}$ (this corresponds to $\lambda \mapsto \lambda + \kappa$). For $\lambda \in \mathbb{C}$, set

$$P(\lambda) = \sum_{i=0}^{\infty} f^i e^i \frac{1}{i!} \prod_{j=1}^{i} \frac{1}{\lambda - \frac{h}{2} - j}.$$

Remark 7. If V is a f.d. $\mathfrak{sl}_2$ module or a Verma module, then $P(\lambda) : V \to V$ is well defined. Similarly, $P(\lambda)$ is well defined on a tensor product of such modules.

Definition 8. The *Dynamical Equation* for $u(z_2, \ldots, z_n) \in V_2 \otimes \cdots \otimes V_n$ is

$$u(z_2, \ldots, z_n, \lambda + \kappa) = \prod_{i=2}^{n} (z_i^{\frac{h}{2}})^{(i)} P(\lambda) u(z_2, \ldots, z_n, \lambda).$$

Denote $K(z_2, \ldots, z_n, \lambda) = \prod_{i=2}^{n} (z_i^{\frac{h}{2}})^{(i)} P(\lambda)$. It is clear that $K(z, \lambda)$ preserves the weight decomposition of V.

Theorem 9. *The Dynamical Equation is compatible with the trigonometric KZ equation. Explicitly,*

$$\nabla_i(\kappa, \vec{\lambda} + \kappa\frac{h}{2}) K(z, \lambda) = K(z, \lambda) \nabla_i(\kappa, \vec{\lambda}).$$

Corollary 10. *If $u(z_2, \ldots, z_n, \lambda)$ is a solution of the trigonometric KZ equation with parameter $\vec{\lambda} = \lambda\frac{h}{2}$, then $K(z, \lambda)u(z, \lambda)$ is a solution of the trigonometric KZ equation with parameter $\vec{\lambda} + \kappa\frac{h}{2}$.*

For hypergeometric solutions we can say more: consider a hypergeometric solution of the trigonometric KZ equation with values in $M_{m_2} \otimes \cdots \otimes M_{m_n}[\sum_{i=2}^{n} m_i - 2k]$,

$$\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda) = \sum_{\substack{J=(j_2,\ldots,j_n) \\ |J|=k}} \int_{\gamma} \Phi^{1/\kappa} A_J \, dt_1 \wedge \cdots \wedge dt_k.$$

Theorem 11. *The function $\varphi^{(\gamma)}$ satisfies also the Dynamic Equation*

$$\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda + \kappa) = K(z, \lambda) \varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda).$$

Remark 12. The function $\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda + \kappa)$ is well defined since the corresponding integrand differs from the integrand of $\varphi^{(\gamma)}(z_2, \ldots, z_n, \lambda)$ by a univalued factor.

Example 13. Take $V = M_m$ and consider a weight subspace $V[m - 2k] = \mathbb{C} f^k v_m$. Let $\vec{u}(z, \lambda) \in V[m - 2k]$, $\vec{u}(z, \lambda) = u(z, \lambda) f^k v_m$ for a scalar $u(z, \lambda)$. We describe trigonometric KZ, Dyn. Eqn. and its hypergeometric solutions (here $\lambda, z \in \mathbb{C}$).
Trigonometric KZ: $\kappa z \frac{\partial}{\partial z} \vec{u} = \vec{\lambda}\vec{u}$, $\vec{\lambda} = \lambda\frac{h}{2}$. Then the equation reads as

$$\kappa z \frac{\partial}{\partial z} u = \lambda \frac{m - 2k}{2} u.$$

Since $K(z, \lambda) f^k v_m = z^{h/2} P(\lambda) f^k v_m$, we must compute $P(\lambda) f^k v_m$.

Theorem 14. *We have*

$$P(\lambda)f^k v_m = \frac{(\lambda+\frac{m}{2})(\lambda+\frac{m}{2}-1)\cdots(\lambda+\frac{m}{2}-k+1)}{(\lambda-\frac{m}{2})(\lambda-\frac{m}{2}+1)\cdots(\lambda-\frac{m}{2}+k-1)} f^k v_m.$$

Proof.

$$\begin{aligned} P(\lambda)f^k v_m &= \sum_{j\geq 0} \frac{k(k-1)\cdots(k-j+1)(m-k+1)\cdots(m-k+j)}{j!(\lambda-\frac{m-2k}{2}-1)\cdots(\lambda-\frac{m-2k}{2}-j)} f^k v_m \\ &= F(-k, m-k+1, -\lambda+\frac{m-2k}{2}+1, z=1) f^k v_m, \end{aligned}$$

and by Gauss' formula we have $F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$. □

Corollary 15. *The Dynamical Equation is*

$$u(z, \lambda+\kappa) = z^{\frac{m-2k}{2}} \prod_{j=0}^{k-1} \frac{(\lambda+\frac{m}{2}-j)}{(\lambda-\frac{m}{2}+j)} u(z,\lambda).$$

The hypergeometric solution is given by

$$\begin{aligned} u(z,\lambda) &= \int_{0\leq t_k\leq\cdots\leq t_1\leq z} \Phi^{1/\kappa} A\, dt_1\wedge\cdots\wedge dt_k, \quad \textit{where} \\ \Phi^{1/\kappa} &= z^{\frac{1}{2\kappa}(\lambda+k)m} \prod_{i=1}^{k} t_k^{-\frac{1}{\kappa}(\lambda-1-\frac{m-2k}{2})} (t_i-z)^{-m/\kappa} \prod_{i,j} (t_i-t_j)^{2/\kappa} \\ A &= \frac{1}{(t_1-z)\cdots(t_k-z)}. \end{aligned}$$

Changing variables $t_i = zs_i$ we get

$$u(z,\lambda) = z^{\lambda\frac{m-2k}{2\kappa}} \int_k (-\frac{1}{\kappa}(\lambda-1-\frac{m-2k}{2})+1, -\frac{m}{\kappa}, \frac{1}{\kappa}),$$

where as before $\int_k(a,b,c) = \prod_{j=0}^{k-1} \frac{\Gamma(1+(j+1)c)}{\Gamma(1+c)} \frac{\Gamma(a+jc)\Gamma(b+jc)}{\Gamma(a+b+(k+j-1)c)}$ is the Selberg integral, and we have the relation

$$\int_k (a+1,b,c) = \prod \frac{a+jc}{a+b+(k+j-1)c} \int_k (a,b,c),$$

from where we can see that $u(z,\lambda)$ satisfies both equations.

6.4. **Quasiclassical asymptotics of hypergeometric functions and Bethe ansatz.** Let $V = V_1\otimes\cdots\otimes V_n$ be a tensor product of $\mathfrak{sl}_2$ modules. Set $H_i(z) = \sum_{j\neq i} \frac{\Omega^{(ij)}}{z_i-z_j}$ as operators on V.

Theorem 16. *For any z, the operators $H_i(z)$ commute.*

Proof. Follows from the identity $[\Omega, x\otimes 1 + 1\otimes x] = 0\ \forall x\in\mathfrak{sl}_2$. □

The operators $\{H_i(z)\}$ are called *the Hamiltonians* of the Gaudin model of an "inhomogeneous magnetic chain". In integrable models of quantum mechanics, usually one has a vector space $V = V_1\otimes\cdots\otimes V_n$ called the *space of states.* One may think about n particles at the points

$z_1, \ldots, z_n \in \mathbb{C}$, and the spaces $V_1, \ldots, V_n$ are their spin spaces. Usually one has a family of commutative linear operators on V called Hamiltonians. The main problem is to diagonalize them and to find their eigenvalues. The next problem is to find the limit when $n \to \infty$. The Bethe ansatz is a method for diagonalizing the Hamiltonians. One looks for eigenvectors of the form

$$v(t_1, \ldots, t_k) \in V,$$

where $v(t_1, \ldots, t_k)$ is some suitable function of parameters $\vec{t}$. One then tries to find the values of the parameters such that $v(t_1, \ldots, t_k)$ is an eigenvector of the Hamiltonians. One proves that if $t_1, \ldots, t_k$ satisfy some explicit system of equations

$$F_j(t_1, \ldots, t_k) = 0, \quad j = 1, \ldots, k, \tag{17}$$

then $v(t_1, \ldots, t_k) \in V$ is an eigenvector.

The equations (17) are called *the Bethe equations*, $v(t)$ is called *the Bethe vector*. The method is called *the Bethe ansatz method.*

The standard conjectures are

(1) The number of solutions properly counted = dim of the space of states.
(2) Bethe vectors from a basis

Important observation: In all known examples there exists a single (master) function

$$\Phi(t_1, \ldots, t_k)$$

such that $\frac{\partial \Phi}{\partial t_j} = F_j$, $j = 1, \ldots, n$. Thus, with a given integrable model of quantum mechanics there is associated a (master) function $\Phi(t_1, \ldots, t_k)$ such that

Conjectures (reformulated)

(1) The number of critical points of Φ = dim of the space of states,
(2) the Bethe vectors $\{v(t_0)\}|_{t_0 \in \text{crit. pts.}}$ form a basis,
(3) the Hessian $\det(\frac{\partial^2 \ln \Phi}{\partial t_i \partial t_j})(t_0)$ "=" square of the length of the Bethe vector $v(t_0)$

Remark 18. For f.d. irreducible representations of $\mathfrak{sl}_2$ this is recently known to be true.

6.5. **Asymptotic solutions of KZ and eigenvectors of $H_i(z) = \sum_{j \neq i} \frac{\Omega^{(ij)}}{z_i - z_j}$.**
Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of $\mathfrak{sl}_2$ modules. The KZ equation for $u(z_1, \ldots, z_n) \in V$ is

$$x \frac{\partial}{\partial z_i} u = H_i(z) u, \quad i = 1, \ldots, n. \tag{19}$$

Definition 20. Let

$$u = e^{\frac{S(z)}{\kappa}} (f_0(z) + \kappa f_1(z) + \kappa^2 f_2(z) + \cdots) \tag{21}$$

for certain functions $S, f_0, \ldots$. We say that u is an *asymptotic solution of KZ* if the subsitution of u in (19) gives 0 in the expansion as a formal power series in κ.

Theorem 22. *Let $u(z)$ be an asymptotic solution of KZ. Then $\forall i$ the vector $f_0(z)$ is an eigenvector of $H_i(z)$ with eigenvalue $\frac{\partial S}{\partial z_i}(z)$.*

Proof.

$$\kappa\Big[\frac{1}{\kappa}\frac{\partial S}{\partial z_i}e^{\frac{S(z)}{\kappa}}(f_0+\kappa f_1+\cdots)+e^{\frac{S(z)}{\kappa}}(\frac{\partial f_0}{\partial z_i}+\kappa\frac{\partial f_1}{\partial z_i}+\cdots)\Big]=H_ie^{\frac{S(z)}{\kappa}}(f_0+\kappa f_1+\cdots),$$

and then

$$\frac{\partial S}{\partial z_i}f_0=H_if_0,\quad \frac{\partial S}{\partial z_i}f_1+\frac{\partial f_0}{\partial z_i}=H_if_1,\ldots$$

$\square$

Remark 23. With the following powers we can in general recover f_i from f_0 $\forall i>0$.

6.6. **Shapovalov form.** Let $V=V_m$ generated by v. There is a unique bilinear symmetric form S on V such that $S(v,v)=1$, $S(hx,y)=S(x,hy)$, $S(ex,y)=S(x,fy)$ $\forall x,y\in V$. For example $S(fv,fv)=S(v,efv)=S(v,hv)=m$, and in general we get

$$S(f^kv,f^kv)=k!m(m-1)\cdots(m-k+1).$$

The form S is called *Shapovalov form.*

Consider now $V_1\otimes V_2$ and $S=S_1\otimes S_2$ the tensor product of the Shapovalov forms on V_1 and V_2, i.e., $S(x_1\otimes x_2,y_1\otimes y_2)=S(x_1,y_1)S(x_2,y_2)$. We have

Lemma 24. $S(\Omega(x_1\otimes x_2),y_1\otimes y_2)=S(x_1\otimes x_2,\Omega(y_1\otimes y_2))$.

Corollary 25. *For $V=V_1\otimes\cdots\otimes V_n$, $S=S_1\otimes\cdots\otimes S_n$, we have $\forall x,y\in V$*

$$S(H_i(z)x,y)=S(x,H_i(z)y).$$

Lemma 26. *Let $u_1(z)$ be a solution of KZ with parameter κ. Let $u_2(z)$ be a solution with parameter $-\kappa$. Then $S(u_1(z),u_2(z))$ does not depend on z.*

Proof.

$$\frac{\partial}{\partial z_i}S(u_1,u_2)=S(\frac{\partial}{\partial z_i}u_1,u_2)+S(u_1,\frac{\partial}{\partial z_i}u_2)=S(\frac{1}{\kappa}H_iu_1,u_2)+S(u_1,-\frac{1}{\kappa}H_iu_2)=0.$$

$\square$

Lemma 27. *Let $u(z,\kappa)=e^{\frac{S(z)}{\kappa}}(f_0(z)+\kappa f_1(z)+\cdots)$ be an asymptotic solution of KZ with parameter κ. Then $S(f_0(z),f_0(z))$ does not depend on z.*

Proof. Let $u_2(z,\kappa)=u(z,-\kappa)$. It is an asymptotic solution to KZ with parameter $-\kappa$, and then $S(u,u_2)$ does not depend on z. But

$$S(e^{\frac{S(z)}{\kappa}}(f_0(z)+\kappa f_1(z)+\cdots),e^{-\frac{S(z)}{\kappa}}(f_0(z)-\kappa f_1(z)+\cdots))=S(f_0(z),f_0(z))+\mathcal{O}(\kappa).$$

$\square$