The goal of this lecture is to see how many independent solutions one can get by taking cycles on the bounded domains in the case $\kappa > 0$, $m_i < 0\ \forall i$.

5.1. **Determinant formulas.**
Consider the KZ equation with values in $\operatorname{Sing} M^{\otimes m}[|m|-2]$:

$$\Phi_{1,n} = \prod_{1\le i<j\le n} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{l=1}^{n} (t - z_l)^{-m_l},$$

$$\varphi^{(\gamma)}(z_1,\ldots,z_n) = \sum_{j=1}^{n} \int_\gamma \Phi_{1,n}^{1/\kappa} \frac{dt}{t-z_j}\, v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n}$$

and call $\mathrm{I}_j^\gamma(z) = \int_\gamma \Phi_{1,n}^{1/\kappa} \frac{dt}{t-z_j}$. We have $\mathrm{I}_1^\gamma(z) = -\frac{1}{m_1}\sum_{j=2}^n m_j\, \mathrm{I}_j^\gamma(z)$, which is the condition for the vector $\sum_j \mathrm{I}_j^\gamma\, v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n}$ to be a singular vector.

Corollary 1. *If $m_1 \neq 0$, then* $\dim \operatorname{Sing} M^{\otimes m}[|m|-2] = n-1$.

Let $z_1 < \ldots < z_n$ be real. Let $\Delta_i(z) = [z_{i-1}, z_i]$, $i = 2,\ldots,n$. Let $\kappa, m_1,\ldots,m_n \in \mathbb{R}$, $\kappa > 0$, $m_1,\ldots,m_n < 0$. Then $\varphi^{(\Delta_i)}(z_1,\ldots,z_n)$, $i = 2,\ldots,n$ are solutions of KZ.

Theorem 2. *These solutions are linearly independent for generic $m_1,\ldots,m_n,\kappa$.*

The theorem follows from the following lemma

Lemma 3 (Determinant Formula)**.** *Let $\varphi^1,\ldots,\varphi^{n-1}$ be a set of solutions. Let $w^1,\ldots,w^{n-1}$ be a basis of* $\operatorname{Sing} M^{\otimes m}[|m|-2]$, *and write $\varphi^i = \sum_j \varphi^i_j w^j$. Then*

$$\det(\varphi^i_j) = (\text{const}) \prod_{i<j} (z_i - z_j)^{(n-1)\frac{m_i m_j}{2\kappa} - \frac{m_i}{\kappa} - \frac{m_j}{\kappa}}$$

Proof. Notice that if one has a differential equation $\frac{dy}{dx} = A(x)y$, where A, y are $n\times n$ matrices, then $\det' y = \operatorname{tr} A(x) \det y$ (since $\det(I+\epsilon A) = 1 + \epsilon \operatorname{tr} A + o(\epsilon^2)$). Our system is

$$\frac{\partial y}{\partial z_i} = \frac{1}{\kappa} \sum_{j\neq i} \frac{\Omega^{(ij)}}{z_i - z_j} y,$$

and then $\frac{\partial \det y}{\partial z_i} = \frac{1}{\kappa}\sum_{j\neq i} \frac{\operatorname{tr}\Omega^{(ij)}}{z_i - z_j} \det y$, whence $\det y = (\text{const}) \prod_{i<j} (z_i - z_j)^{\frac{\operatorname{tr}\Omega^{(ij)}}{\kappa}}$.

We show now that $\operatorname{tr}\Omega^{(ij)} = \operatorname{tr}\Omega^{(ij)}|_{\operatorname{Sing} M^{\otimes m}[|m|-2]} = (n-1)\frac{m_i m_j}{2\kappa} - \frac{m_i}{\kappa} - \frac{m_j}{\kappa}$. We prove it for $\Omega^{(12)}$, the general case is analogous. We can use the basis

$$w^1 = \frac{1}{m_1} f v \otimes v \otimes \cdots \otimes v - \frac{1}{m_2} v \otimes f v \otimes \cdots \otimes v,$$

$$w^i = \frac{1}{m_1 + m_2}(f v \otimes \cdots \otimes v + v \otimes f v \otimes \cdots \otimes v) - \frac{1}{m_i} v \otimes \cdots \otimes f v \otimes \cdots \otimes v$$

and then

$$\begin{aligned}
\Omega^{(12)} w^1 &= ((\frac{1}{2} h \otimes h + e \otimes f + f \otimes e) \otimes 1 \otimes \cdots \otimes 1)(w^1) \\
&= (\frac{1}{2} m_1 m_2 - m_1 - m_2) w^1 \\
\Omega^{(12)} w^i &= \frac{1}{2} m_1 m_2 w^i .
\end{aligned}$$

□

We compute now the constant in the determinant. Let $\bar{\Phi} = \prod_{l=1}^{n} (t - z_l)^{\alpha_l}$, and the differential forms $\eta_j = \frac{d(t-z_j)}{t-z_j}$, $j = 1, \ldots, n$. Let $z_1 < \ldots < z_n$ as before. We fix then the arguments in each interval $\Delta_i(z)$, and then the integrals $\int_{z_{i-1}}^{z_i} \bar{\Phi} \eta_j$ are well defined.

Theorem 4.

$$\det_{2 \le i,j \le n} (\alpha_j \int_{z_{i-1}}^{z_i} \bar{\Phi} \eta_j) = \frac{\Gamma(\alpha_1 + 1) \cdots \Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \cdots + \alpha_n + 1)} \prod_{i<j} (z_i - z_j)^{\alpha_i + \alpha_j} .$$

Before giving the proof we notice that for $n = 2$, $z_1 = 0$ and $z_2 = 1$ the theorem says

$$\alpha_2 \int_0^1 t^{\alpha_1} (1 - t)^{\alpha_2 - 1} dt = \alpha_2 \frac{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)}{\Gamma(\alpha_1 + \alpha_2 + 1)} ,$$

and this is Euler's formula.

Proof. By induction. For $n = 2$, we saw that this is Euler's formula. Consider then $z_1, \ldots, z_{n-1}$ fixed and let $z_n \to \infty$; we will see that the ratio RHS/LHS tends to 1.

We remark that

(1) For $i, j \le n - 1$,

$$\int_{z_{i-1}}^{z_i} \alpha_j \bar{\Phi} \frac{dt}{t - z_j} = (-z_n)^{\alpha_n} \left[\int_{z_{i-1}}^{z_i} \alpha_j \prod_{l=1}^{n-1} (t - z_l)^{\alpha_l} \frac{dt}{t - z_j} + \mathcal{O}(\frac{1}{z_n}) \right]$$

(2) $\int_{z_{i-1}}^{z_i} \alpha_n \bar{\Phi} \frac{dt}{t-z_n} = \mathcal{O}((-z_n)^{\alpha_n - 1})$

(3) $\int_{z_{n-1}}^{z_n} \alpha_n \bar{\Phi} \frac{dt}{t-z_i} = \mathcal{O}((-z_n)^{\alpha_1 + \cdots + \alpha_n})$

(4) $\int_{z_{n-1}}^{z_n} \alpha_n \bar{\Phi} \frac{dt}{t-z_n} = (-z_n)^{\alpha_1 + \cdots + \alpha_n} \int_0^1 \alpha_n t^{\alpha_1 + \cdots + \alpha_{n-1}} (1 - t)^{\alpha_n - 1} \, dt + \mathcal{O}(z_n^{\alpha_n})$.

We have then a matrix

	$\bar{\Phi} \frac{dt}{t-z_1}$	$\cdots\cdots$	$\bar{\Phi} \frac{dt}{t-z_{n-1}}$	$\bar{\Phi} \frac{dt}{t-z_n}$
Δ_1				$\mathcal{O}((-z_n)^{\alpha_n - 1})$
				$\vdots$
Δ_{n-1}				$\mathcal{O}((-z_n)^{\alpha_n - 1})$
Δ_n	$\mathcal{O}((-z_n)^{\alpha_1 + \cdots + \alpha_n})$	$\cdots\cdots$	$\mathcal{O}((-z_n)^{\alpha_1 + \cdots + \alpha_n})$	

Thus, dividing the i-th row ($i < n$) by $(-z_n)^{\alpha_n}$ and the n-th row by $(-z_n)^{\alpha_1+\cdots+\alpha_n}$, we get

$$\det_{(n)}(z_1,\ldots,z_n) = (-z_n)^{(n-1)\alpha_n} \det_{(n-1)}(z_1,\ldots,z_{n-1}) \frac{\Gamma(\alpha_1+\cdots+\alpha_{n-1}+1)\Gamma(\alpha_n+1)}{\Gamma(\alpha_1+\cdots+\alpha_n+1)}.$$

Now, it is easy to proof that RHS/LHS is holomorphic with respect to z_n. Since its limit at ∞ is 1, we finished. □

Remark 5. We proved that we constructed all solutions for generic $m_1,\ldots,m_n,\kappa$. The case of degenerate parameters is very explicit: it happens in the poles of $\Gamma(\alpha_1+\cdots+\alpha_n+1)$.

Remark 6. Considering solutions with values in $\operatorname{Sing} M^{\otimes m}[|m|-2k]$, one can similarly show that the set of hypergeometric solutions generate all solutions for generic $m_1,\ldots,m_n,\kappa$. For $n=2$, one has to replace Euler's formula by the Selberg integral.

There are remarkable general determinant formulas by Terao and Donai. Let A be a finite collection of hyperplanes in $\mathbb{R}^k$ and chose for each $H \in A$ a number λ_H and a linear function f_H. Then

$$\det\left(\int_\Delta \prod_{H\in A} f_H^{\lambda_H}\omega\right) = (\text{Alternating product of } \Gamma \text{ functions}) \prod \left(\text{values of } f_H^{\lambda_H} \text{ at vertices}\right),$$

where Δ runs through bounded domains and ω through a suitable basis of logaritmic differential forms. In our case there is an action of the symmetric group and one must consider isotypical components.

Another determinant formula: let A be again an arrangement of hyperplanes in $\mathbb{R}^k$, now in generic position. It can be proved that the number of bounded domains is $\binom{n-1}{k}$. On the other hand, the number of monomials $t^a = t_1^{a_1}\cdots t_k^{a_k}$ such that $a_1+\cdots+a_k \le n-k-1$ is also $\binom{n-1}{k}$. One can consider $\det(\int_\Delta t^a\, dt_1\cdots dt_k)$, where Δ runs on the bounded domains and t^a runs on the preceding monomials. We have

$$\det = \pm[(n-1)!]^{\binom{n-2}{k-1}} \prod_{S\subset\mathbb{R}^k}(k!\operatorname{vol}(S)),$$

where S runs on the k-simplices with faces in A.

Yet another example: let $z_1 < \cdots < z_n \in \mathbb{R}$. Consider $\det(\int_{z_i}^{z_{i+1}} t^j\, dt)$. This is a Vandermonde determinant, and we have

$$\det(\int_{z_i}^{z_{i+1}} t^j\, dt) = \text{const}\prod_{j<i}(z_i - z_j).$$

5.2. **Resonances.** Consider the case $k=1$. Then

$$\det(\text{KZ Solutions}) = \frac{\prod_j \Gamma(-\frac{m_j}{\kappa}+1)}{\Gamma(-\frac{\sum_i m_i}{\kappa}+1)}\prod(z_i-z_j)^{\boxed{*}}.$$

When $-\frac{\sum_i m_i}{\kappa}+1 = 0$ then we have a pole in the denominator and then we get linearly dependent solutions. In this case, the bundle $\operatorname{Sing} M^{\otimes m}[|m|-2]$ has a nontrivial subbundle which is invariant by the KZ equation. We want to describe this subbundle. It turns out that in CFT the appearence of such subspace was predicted, and it was described in terms of representation theory.

For a general k, let $m_1, \dots, m_n \in \mathbb{Z}_{>0}$ and consider $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$ the tensor product of irreducible f.d. representations. Assume that $\kappa \in \mathbb{Z}$, $\kappa > 2$, $0 \le m_1, \dots, m_n$ and $|m|-2 \le \kappa - 2$. Consider the KZ equation with values in

$$\operatorname{Sing} L^{\otimes m}[|m|-2k] = \{v \in L^{\otimes m} \mid hv = (|m|-2k)v,\ ev = 0\}.$$

Set $p = \kappa - 1 - |m| + 2k$, this is a positive integer. Take $W(z_1, \dots, z_n) \subset \operatorname{Sing} L^{\otimes m}[|m|-2k]$ given by

$$W(z) = \{w \in L^{\otimes m} \mid hw = (|m|-2k)w, \quad ew = 0, \quad (ze)^p w = 0\},$$

where $ze : L^{\otimes m} \to L^{\otimes m}$ is the operator $w_1 \otimes \dots \otimes w_n \mapsto \sum_i z_i w_1 \otimes \dots \otimes e w_i \otimes \dots \otimes w_n$.

Theorem 7. *The subspaces $W(z) \subset \operatorname{Sing} L^{\otimes m}[|m|-2k]$ are preserved by KZ.*

Definition 8. These subspaces are called *spaces of conformal blocks.*

Theorem 9. *All the hypergeometric solutions $\varphi^{(\gamma)}(z)$ belong to the bundle of conformal blocks.*

This means that hypergeometric functions "know" about CFT and its conformal blocks.

Consider the case $k = 1$. We have

$$\operatorname{Sing} L^{\otimes m}[|m|-2] = \{w = \sum_{j=1}^{n} I_j v_{m_1} \otimes \dots \otimes f v_{m_j} \otimes \dots \otimes v_{m_n} \mid \sum_{j=1}^{n} I_j m_j = 0\}.$$

Here $p = \kappa + 1 - |m| \in \mathbb{Z}_{>0}$. If $p > 1$ then $W(z) = \operatorname{Sing} L^{\otimes m}[|m|-2]$ (by a degree argument $(ze)^2 = 0$). If $p = 1$ then

$$W(z) = \{w = \sum I_j v_{m_1} \otimes \dots \otimes f v_{m_j} \otimes \dots \otimes v_{m_n} \mid \sum m_j I_j = 0, \quad \sum z_j m_j I_j = 0\}.$$

Notice that $p = 1 \iff \kappa = \sum m_j$. Then

Theorem 10. *For any hypergeometric solution $\varphi^{(\gamma)}(z)$ we have $\sum z_j m_j I_j = 0$.*

Proof. It is similar to the proof that $\sum m_j I_j = 0$. In that case, we have

$$-\kappa d_t(\Phi^{1/\kappa}) = \sum m_j \Phi^{1/\kappa} \frac{dt}{t - z_j},$$

and integrating over a cycle γ we get the identity. Now, we have $\kappa d_t(t\Phi^{1/\kappa}) = \sum z_j m_j \Phi^{1/\kappa} \frac{dt}{t-z_j}$, i.e.,

$$\begin{aligned} d_t(t \prod (z_i - z_j)^* \prod (t - z_l)^{-m_l/\kappa}) &= \Phi^{1/\kappa} dt + \Phi^{1/\kappa} \sum -\frac{m_l}{\kappa} \frac{(t - z_l) + z_l}{t - z_l} \\ &= (1 - \sum_{l=1}^{n} \frac{m_l}{\kappa}) \Phi^{1/\kappa} dt - \sum z_l \frac{m_l}{\kappa} \Phi^{1/\kappa} \frac{dt}{t - z_l}, \end{aligned}$$

and, again, integrating over a cycle γ we get the identity. ∎

5.3. **Dynamical equations.** Let $\Phi_{k,n}^{1/\kappa} = \prod_{i,j}(z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \prod_{i,j}(t_i - t_j)^{\frac{2}{\kappa}} \prod_i \prod_l (t_i - z_l)^{\frac{-m_l}{\kappa}}$. Let

$$\varphi^{(\gamma)}(z_1, \ldots, z_n) = \sum_{|J|=k} \mathrm{I}_J^\gamma f_J v,$$

$$\mathrm{I}_J^\gamma(z_1, \ldots, z_n) = \int_\gamma \Phi^{1/\kappa} A_j \, dt_1 \wedge \cdots \wedge dt_k.$$

be solutions of KZ with values in $\operatorname{Sing} M^{\otimes m}[|m| - 2k]$, $\gamma \in H_k(\mathcal{C}_{k,n}(z), \Phi_{k,n}^{1/\kappa})$. Shift $m_1 \mapsto m_1 + \kappa$. Then the integrand is multiplied by $\frac{\mathrm{const}(z)}{(t_1 - z_1)\cdots(t_k - z_1)}$. This is a rational univalued function; the same cycle γ is good for the new integral. This defines a map

$$\{\mathrm{I}_J^\gamma(z, m_1, \ldots, m_n)\} \to \{\mathrm{I}_J^\gamma(z, m_1 + \kappa, \ldots, m_n)\}$$

in $H^1(\mathcal{C}_{k,n}(z), \Phi_{k,n}^{1/\kappa})$.

Problem: Describe this map.

Again, the answer will be given in terms of $\mathfrak{sl}_2$-representation theory. Relations of this type will be called *dynamical equations*. Before going over the problem, we make a

Remark 11. Since $\varphi^{(\gamma)} \in \operatorname{Sing} M^{\otimes m}[|m| - 2k]$, there are many linear relations between $\{\mathrm{I}_J^\gamma\}$.

Lemma 12. *If $m_1 \notin \{0, 1, \ldots, k-1\}$, then for any J_0, $|J_0| = k$, the integral $\mathrm{I}_{J_0}^\gamma$ is a linear combination of integrals $\{\mathrm{I}_J^\gamma\}$ with J having the form $(0, j_2, \ldots, j_n)$, $j_1 + \cdots + j_n = k$. The coefficients of this linear combination do not depend on γ.*

For example, for $k = 1$ we have $I_1 = \frac{1}{m_1} \sum_{j=2}^n m_j I_j$. We will study the transformation $m_1 \mapsto m_1 + \kappa$ for "restricted vector valued functions"

$$\psi^{(\gamma)}(z_1, \ldots, z_n) = \sum_{\substack{J=(0,j_2,\ldots,j_n)\\ |J|=k}} \mathrm{I}_J^\gamma f_J v.$$

Remark 13. Having the restricted function $\psi^{(\gamma)}$ one can reconstruct the initial function $\varphi^{(\gamma)}$. Therefore one can rewrite the KZ differential equation for $\varphi^{(\gamma)}(z)$ as a suitable differential equation for the restricted function. This new differential equation is called *trigonometric KZ equation*