Lecture 4

4.1. **Monodromy of KZ equations.** KZ equations are defined over $U = \mathbb{C}^n - D$, where

$$D = \{z \in \mathbb{C}^n \mid \exists i, j \text{ s.t. } z_i = z_j\}.$$

For any $A \in U$, denote Sol_A the space of all solutions of the equations in a neighborhood of A. For any path p from A to B in U the continuation of solutions along p gives an isomorphism

$$\alpha_p : \mathrm{Sol}_A \to \mathrm{Sol}_B \,.$$

If A, B are fixed, then α_p does not depend on continuous deformations of the path. If p is a loop, then the isomorphism becomes an automorphism

$$M(p) : \mathrm{Sol}_A \to \mathrm{Sol}_A$$

and gives the monodromy representation

$$\pi_1(\mathbb{C}^n - D) \to \mathrm{GL}(\mathrm{Sol}_A). \tag{1}$$

4.2. **Topological monodromy.** Consider on $\mathbb{C}^n - D$ the vector bundle whose fiber on $(z_1, \dots, z_n)$ is $H_1(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{1,n}^{1/\kappa})$. There is a flat connection on it: we deform the cycles for close points $(z_1, \dots, z_n) \sim (z_1', \dots, z_n')$ (this is known as the *Gauss–Manin connection*). In particular, if we have a loop in $\mathbb{C}^n - D$ we get a map

$$H_1(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{1,n}^{1/\kappa}) \to H_1(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{1,n}^{1/\kappa}).$$

We get thus another action of the group

$$\pi_1(\mathbb{C}^n - D) \to \mathrm{Aut}(H_1). \tag{2}$$

4.3. R**-matrix.** Recall from the previous lecture that we have an action of the pure braid group

$$PB_n \to \mathrm{Aut}(\mathrm{Sing}\, M^{\otimes m}[|m| - 2]) \tag{3}$$

for $M^{\otimes m} = M_{m_1} \otimes \cdots \otimes M_{m_n}$. Recall also that $PB_n = \pi_1(\mathbb{C}^n - D)$.

4.4. **Relationship.**

Claim 4. (a) Via the correspondence of the space of solutions of KZ with H_1, we get an isomorphism between the two first representations (namely, (1) and (2)).

(b) Via the correspondence of $\mathrm{Sing}\, M^{\otimes m}[|m| - 2]$ with $H_1(\mathbb{C}^n - \{z_1, \dots, z_n\}, \Phi_{1,n}^{1/\kappa})$, we get an isomorphism between the second and third representations (namely, (2) and (3)).

The first part of the claim is clear; the second one is remarkable. It was proved first by Kohno without hypergeometric functions and then by Drinfeld, who developed for this purpose the theory of quasi Hopf algebras.

We give a sketch of a third proof. Consider the KZ equation with values in $\mathrm{Sing}\, M^{\otimes m}[|m| - 2k]$; solutions of it are given, as we know, by

$$\varphi^{(\gamma)}(z_1, \dots, z_n) = \sum_{|J|=k} \int_\gamma \Phi_{k,n}^{1/\kappa}(t, z, m) A_J(t, z)\, dt_1 \wedge \cdots \wedge dt_k$$

$$\Phi_{k,n} = \prod_{i,j} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{i,j} (t_i - t_j)^2 \prod_i \prod_l (t_i - z_l)^{-m_l}$$

For $z \in \mathbb{C}^n$, let $\mathcal{C}_{k,n}(z) = \{t \in \mathbb{C}^k \mid t_i \neq z_j,\ t_i \neq t_j\ \forall i, j\}$. The k-cycles γ are elements of $H_k(\mathcal{C}_{k,n}(z), \Phi_{k,n}^{1/\kappa})$ and each γ of this type defines a solution of KZ. However, not all γ give a nonzero solution. Notice that the differential forms $\eta_J = \Phi_{k,n}^{1/\kappa} A_J\, dt_1 \wedge \cdots \wedge dt_k$ are skew symmetric with respect to permutations of $t_1, \ldots, t_k$:

$$\eta(t_{\sigma(1)}, \ldots, t_{\sigma(k)}) = (-1)^{|\sigma|}\eta(t_1, \ldots, t_k) \quad \forall \sigma \in \mathbb{S}_k.$$

The symmetric group $\mathbb{S}_k$ acts also on the k-cells. This action induces an action of $\mathbb{S}_k$ on the homology spaces. Let $H_k(\mathcal{C}_{k,n}(z), \Phi^{1/\kappa})^-$ be the subspace of skew symmetric elements γ (i.e., $\sigma\gamma = (-1)^{|\sigma|}\gamma$). Let $H_k^\sim$ be the sum of all other isotypical components, $H_k = H_k^- \oplus H_k^\sim$. Then, for any $\gamma \in H_k^\sim$ we have $\varphi^{(\gamma)} \equiv 0$. Thus, solutions of KZ are labeled by $H_k(\mathcal{C}_{k,n}(z), \Phi^{1/\kappa})^-$.

Theorem 5. (a) *Let $z \in \mathbb{C}^n$ be such that $z_1 < \cdots < z_n$. Then there is a natural isomorphism*

$$H_k(\mathcal{C}_{k,n}(z), \Phi^{1/\kappa})^- \simeq \operatorname{Sing} M^{\otimes m,q}[|m| - 2k].$$

Thus, the solutions of KZ are labeled by this space.

(b) *Under this identification, the monodromy of KZ is identified with the R-matrix representation of PB_n in* $\operatorname{Sing} M^{\otimes m,q}[|m| - 2k]$.

The construction of the isomorphism is analogous to the construction of the isomorphism for $k = 1$. The vector $f_J v = f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n}$ is identified with the following cell: take j_i loops around each z_i (see Figure 1). This is a product of k 1-dimensional cells, and then it is a k-dimensional cell. Then, take the antisymmetrization of it.

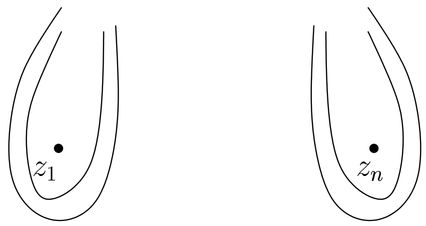

FIGURE 1

This proof was developed later than that of Drinfeld. In it, one realizes KZ equations geometrically, as equations for hypergeometric functions associated with the master functions $\Phi_{k,n}$. In this geometric approach the isomorphism of the KZ monodromy with the R-matrix representation is given explicitly, through the hypergeometric pairing

$$\gamma \otimes \Phi^{1/\kappa} A_J\, dt \mapsto \int_\gamma \Phi^{1/\kappa} A_J\, dt.$$

We address now the following

Problem: Show that the construction we made gives many linearly independent solutions. This will be solved in the next lecture; we develop here some tools.

4.5. **Remark on integration cycles.**

If $\kappa, m_1, \ldots, m_n \in \mathbb{R}$, $m_1, \ldots, m_n < 0$, $\kappa > 0$ and $\kappa \gg 1$, then all exponents of $\Phi_{k,n}$ are big positive integers. Assume $z \in \mathbb{R}^n$. We can consider in $\mathbb{R}^k$ the hyperplanes $\{t_i = z_j\}$ and $\{t_i = t_j\}$. Call $\mathcal{C}_{k,n}(z)(z) = \mathbb{R}^k - \{t \in \mathbb{R}^k \mid t_i = t_j,\ t_i = z_j\}$ complement to these hyperplanes. For $k = 2$ we can picture it as in Figure 2. We can take now γ as any bounded domain in $\mathcal{C}_{k,n}(z)(z)$ (we shaded two of such domains in figure 2).

Claim 6. Under these assumptions, $\varphi^{(\gamma)}(z_1, \ldots, z_n)$ is a solution of the KZ equation with values in $\operatorname{Sing} M^{\otimes m}[|m| - 2k]$.

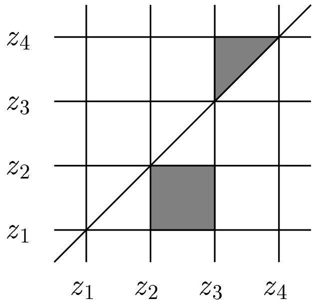

FIGURE 2. Hyperplanes in $\mathbb{R}^k$ ($k = 2$, $n = 4$)

4.6. **Example: the Selberg integral.**
Let $n = 2$; consider KZ with values in

$$\operatorname{Sing} M_{m_1} \otimes M_{m_2}[m_1 + m_2 - 2k] = \{\varphi = \sum_{j_1+j_2=k} I_{(j_1,j_2)} f^{j_1} v_{m_1} \otimes f^{j_2} v_{m_2} \mid e\varphi = 0\}.$$

Calculating explicitly the condition $e\varphi = 0$ we can see that the space is one dimensional and is generated by

$$\omega = \sum_{j_1+j_2=k} \frac{(-1)^{j_1}}{j_1! j_2!} \prod_{i=0}^{j_1-1} \frac{1}{m_1 - i} \prod_{i=0}^{j_2-1} \frac{1}{m_2 - i} f^{j_1} v_{m_1} \otimes f^{j_2} v_{m_2}.$$

The hypergeometric solutions have the form

$$\varphi^{(\gamma)}(z_1, z_2) = \sum_{j_1+j_2=k} \int_{\Delta(z_1,z_2)} \Phi_{k,2}^{\frac{1}{\kappa}} A_{(j_1,j_2)} \, dt_1 \wedge \cdots \wedge dt_k \; f^{j_1} v_{m_1} \otimes f^{j_2} v_{m_2},$$

where $\Delta(z_1, z_2) = \{t \in \mathbb{R}^k \mid z_1 \le t_k \le \cdots \le t_1 \le z_2\}$. Then $\varphi(z_1, z_2)$ is proportional to ω. To see that $\varphi \not\equiv 0$, we consider the $(k, 0)$ coordinate of the solution (i.e., $(j_1, j_2) = (k, 0)$):

$$(z_1 - z_2)^{\frac{m_1 m_2}{2\kappa}} \int_{\Delta(z_1,z_2)} \prod_{i=1}^{k} (t_i - z_1)^{-\frac{m_1}{\kappa} - 1} (t_i - z_2)^{-\frac{m_2}{\kappa}} \prod (t_i - t_j)^{\frac{2}{\kappa}} \, dt_1 \wedge \cdots \wedge dt_k.$$

Using new variables $t_i = (z_2 - z_1)s_i + z_1$, we can write this as

$$(z_1 - z_2)^{\frac{\Delta(m_1+m_2-2k)}{\kappa} - \frac{\Delta(m_1)}{\kappa} - \frac{\Delta(m_2)}{\kappa}} \frac{1}{k!} \int_k \left(-\frac{m_1}{\kappa}, -\frac{m_2}{\kappa} + 1, \frac{1}{\kappa} \right),$$

where $\Delta(a) = \frac{a(a+2)}{4}$. Here $\int_k (a, b, c)$ is the Selberg integral:

$$\int_k (a, b, c) = \int_{0 \le s_k < \cdots < s_1 \le 1} \prod_{i=1}^{k} s_i^{a-1} (1 - s_i)^{b-1} \prod_{1 \le i < j \le k} (s_i - s_j)^{2c} \, ds_1 \wedge \cdots \wedge ds_k.$$

In 1944 Selberg showed that

$$\int_k (a, b, c) = \prod_{j=0}^{k-1} \frac{\Gamma(1 + c + jc)}{\Gamma(1 + c)} \frac{\Gamma(a + jc)\Gamma(b + jc)}{\Gamma(a + b + (k + j - 1)c)}.$$

For $n = 1$, we get the beta integral $\int_1 (a, b, c) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. The Selberg integral is one of the most remarkable hypergeometric functions. In particular, we see that our hypergeometric solution is not zero for generic m_1, m_2, κ.

Taking a suitable limit of the Selberg integral we get

$$\int_{\mathbb{R}^k} e^{-a\sum_{i=1}^k s_i^2} \prod_{i<j} (s_i - s_j)^{2c}\, ds_1 \wedge \cdots \wedge ds_k = (2\pi)^{\frac{k}{2}} (2a)^{-k\frac{c(k-1)+1}{2}} \prod_{j=1}^{k} \frac{\Gamma(1+jc)}{\Gamma(1+c)}.$$

4.7. **Connection with finite reflection groups.**
Macdonald (1982) observed that $s_i - s_j = 0$, $1 \le i < j \le k$ are equations of the reflecting hyperplanes of the finite group A_{k-1}. There are other finite groups generated by reflections. In the k-dimensional Euclidean space $\mathbb{R}^k$ consider a certain number of hyperplanes, all passing through the origin. If the angles between the hyperplanes are properly chosen, then the group generated by reflections corresponding to them is finite. These groups were enumerated and classified by Coxeter as

$$A_k\ (k \ge 1),\ B_k\ (k \ge 2),\ D_k\ (k \ge 4)\ E_6,\ E_7,\ E_8,\ F_4,\ G_2,\ H_3,\ H_4,\ I_2(p)\ (p > 5).$$

Let G be one of these reflecting groups. Let $P(s)$ be the product of distances of the point $(s_1, \ldots, s_k) = s$ from all reflecting hyperplanes belonging to G. Let N be the number of hyperplanes. Not knowing Selberg integrals, Macdonald conjectured that

$$\int_{\mathbb{R}^k} e^{-\sum s_i^2/2} |P(s)|^{2c}\, ds_1 \wedge \cdots \wedge ds_k = 2^{-Nc} (2\pi)^{k/2} \prod_{j=1}^{k} \frac{\Gamma(1+cd_j)}{\Gamma(1+c)},$$

where d_j are the degrees of a basis set of the space of homogeneous polynomials which are invariant with respect to G.

The conjecture was proved by Opdam. The q-analogs were proved by Cherednik.