VARCHENKO: LECTURE 3

Fix $\kappa \in \mathbb{C}^*$ and $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$ and let

$$\Phi_{k,n}(\mathbf{t}, \mathbf{z}, \mathbf{m}) \;=\; \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{1 \le i \le j \le k} (t_i - t_j)^2 \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_l}$$

$$A_J(\mathbf{t}, \mathbf{z}) \;=\; \frac{1}{j_1! \cdots j_n!} \; \mathrm{Sym}_t \left[\prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_1 + \cdots + j_{l-1} + i} - z_l} \right].$$

In this lecture we address which cycles γ can be chosen in order to make the expressions

$$\phi^\gamma(\mathbf{z}) \;=\; \sum_{|J| = k} \int_\gamma \Phi_{k,n}^{\frac{1}{\kappa}}(\mathbf{t}, \mathbf{z}, \mathbf{m}) A_J(\mathbf{t}, \mathbf{z}) dt_1 \wedge \cdots \wedge dt_k$$

well defined, and thus yield KZ solutions. When $k = 1$ the cycle γ can be chosen to be a closed path whose homotopy class $[\gamma]$ resides in the commutator subgroup of $\pi_1\Big(\mathbb{C} - \{z_1, \ldots, z_n\}\Big)$. For example, a Pochhammer cycle about any points z_i and z_j will serve for γ. In general the appropriate γ will reside in homology groups twisted by coefficients associated with $\Phi_{k,n}^{\frac{1}{\kappa}}$. We start by defining and computing this homology for $k = 1$.

Definition 1. *A twisted k-cell is a pair (Δ^k, s) where Δ^k is a singular k-cell in $\mathbb{C} - \{z_1, \ldots, z_n\}$ and s is a univalent branch of $\Phi_{1,n}^{\frac{1}{\kappa}}$ over Δ^k. A twisted k-chain is a formal linear (over $\mathbb{C}$) combination of k-cells.*

Define the boundary operator d_k on any k-cell by the formula $d_k(\Delta^k, s) = (\partial \Delta^k, s|_{\Delta^k})$. Extend it by linearity to all k-chains.

The kth $\Phi_{1,n}^{\frac{1}{\kappa}}$-twisted homology is defined as

$$H_k\Big(\mathbb{C} - \{z_1, \ldots, z_n\}, \Phi_{1,n}^{\frac{1}{\kappa}}\Big) \;:=\; \frac{\ker d_k}{\mathrm{Image}\, d_{k+1}}.$$

Note that ϕ^γ is a well-defined KZ solution provided $\gamma \in H_1$.

For what follows let $q = e^{\frac{2\pi i}{\kappa}}$ and for any $a \in \mathbb{C}$ set

$$[a] \;=\; \frac{q^{\frac{a}{2}} - q^{-\frac{a}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$

where $q^a = e^{\frac{2a\pi i}{\kappa}}$. Note that if $q \to 1$ (or $\kappa \to \infty$) then $[a] \to a$. We will begin by computing the twisted homology in a toy example; namely $H_*(\mathbb{C} - \{0\}, t^{\frac{-m}{\kappa}})$. Take $p \in \mathbb{C} - \{0\}$ and choose a loop l_1 about 0 passing through p. Fix a value s_0 of $t^{-\frac{m}{\kappa}}$ at p and a branch s of $t^{-\frac{m}{\kappa}}$ over l whose value at l's endpoint is s_0. Computing the boundary map d_1 we obtain:

$$\begin{aligned} d_1(l,s) &= (p, s|_{\text{end point}}) - (p, s|_{\text{start point}}) = (p, s_0) - (p, q^m s_0) \\ &= (1 - q^m)(p, s_0) \end{aligned}$$

Lemma 1. *The chain complex* $0 \longrightarrow \mathbb{C} \cdot (l, s) \xrightarrow{d_1} \mathbb{C} \cdot (p, s_0) \longrightarrow 0$ *computes the homology groups* $H_*(\mathbb{C} - \{0\}, t^{-\frac{m}{\kappa}})$.

Corollary 1. H_0 *and* H_1 *are both* 0 *if* $[m] \neq 0$ *and they are both 1 dimensional if* $[m] = 0$.

Consider $H_*\Big(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{1,n}^{\frac{1}{\kappa}}\Big)$ and assume that $z_1 < \cdots < z_n$ are all real. Fix a point p_0 in the upper half plane and choose loops l_i emanating from p_0 and circling counter-clockwise only around z_i for $i = 1, \dots, n$. Choose branches $s_0, s_1, \dots, s_n$ of $\Phi_{1,n}^{\frac{1}{\kappa}}$ over $p_0, l_1, \dots, l_n$ respectively. Note that if we analytically continue s_i along the loop l_j (for $1 \leq, i, j \leq n$) then s_i at l_j's endpoint is q^{-m_j} times its value at the start point.

Lemma 2. *The complex* $0 \longrightarrow \bigoplus_{j=1}^n \mathbb{C} \cdot (l_j, s_j) \xrightarrow{d_1} \mathbb{C} \cdot (p_0, s_0) \longrightarrow 0$ *computes* $H_*\Big(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{k,n}^{\frac{1}{\kappa}}\Big)$.

Lemma 3. *One can choose* $c_j \in \mathbb{C}$ *for* $j = 0, \dots, n$ *such that*

$$d_1(\omega_j) = [m_j]\Big(q^{\frac{-m_1 - \cdots - m_{j-1} - m_{j+1} + \cdots m_n}{4}}\Big)\omega_0 \quad j = 1, \dots, n$$

where $\omega_0 = c_0(p_0, s_0), \cdots, \omega_n = c_n(l_n, s_n)$.

Corollary 2.

$$H_1 = \Big\{ I_1\omega_1 + \cdots + I_n\omega_n \;\Big|\; \sum_{j=1}^n [m_j]\, q^{\frac{1}{4}\left(-\sum_{i<j} m_i + \sum_{i>j} m_i\right)} I_j = 0 \Big\}.$$

Note that as $q \to 1$ the formula above becomes $\sum_{j=1}^n m_j I_j = 0$ which is precisely the singular vector condition; i.e. the condition to be in $\operatorname{Sing} M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2\Big]$.

Corollary 3. *If* $[m_1] = \cdots = [m_n] = 0$ *then* $\dim H_0 = 1$ *and* $\dim H_1 = n$. *Otherwise* $\dim H_0 = 0$ *and* $\dim H_1 = n - 1$.

Quantum Groups

In what follows set $q^x = e^{\frac{2\pi i}{\kappa}\cdot x}$ for any linear operator x.

Definition 2. *The algebra $U_q(\mathfrak{sl}_2)$ is the algebra, with unit 1, generated by e, f, and q^{xh} (for $x \in \mathbb{C}$) subject to the following relations:*

$$q^{xh}e = eq^{x(h+2)} \qquad q^{xh}f = fq^{x(h-2)}$$

$$[e,f] = \frac{q^{\frac{h}{2}} - q^{-\frac{h}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \qquad q^{xh}q^{x'h} = q^{(x+x')h}$$

We also require $q^{0h} = 1$. The algebra $U_q(\mathfrak{sl}_2)$ is called the quantum group of $\mathfrak{sl}_2(\mathbb{C})$. It is equipped with a Hopf algebra structure with comultipliction Δ given by

$$\Delta(e) = e \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes e$$
$$\Delta(f) = f \otimes q^{\frac{h}{4}} + q^{-\frac{h}{4}} \otimes f$$
$$\Delta(q^{xh}) = q^{xh} \otimes q^{xh}.$$

Note that when $q \to 1$ these relations degenerate into the defining relations for the classical enveloping algebra $U(\mathfrak{sl}_2)$. By analogy with $\mathfrak{sl}_2(\mathbb{C})$, we define the quantum Verma modules. For $\lambda \in \mathbb{C}$, the Verma module M_λ^q is the infinite dimensional $U_q(\mathfrak{sl}_2)$-module generated by one vector v_λ satisfying $e \cdot v_\lambda = 0$ and $q^{xh} \cdot v_\lambda = q^{x\lambda}v_\lambda$. It has a basis $v_\lambda, fv_\lambda, f^2v_\lambda, \ldots$ subject to

$$f \cdot f^k v_\lambda = f^{k+1}v_\lambda \qquad q^{xH} \cdot f^k v_\lambda = q^{x(\lambda-2k)}f^k v_\lambda \qquad e \cdot f^k v_\lambda = [k][\lambda - k + 1]f^{k-1}v_\lambda.$$

If $\lambda \in \mathbb{Z}_{\geq 0}$ then M_λ^q possesses a non-trivial submodule spanned by the vectors $f^{\lambda+1}v_\lambda, f^{\lambda+2}v_\lambda, \ldots$. The quotient L_λ^q is irreducible. Again, as $q \to 1$ the representations L_λ^q degenerate to the classical finite dimensional irreducible representations L_λ of $\mathfrak{sl}_2(\mathbb{C})$. If no confusion arises we shall omit the superscript q in the notation and write only M_λ and L_λ.

Unlike the classical theory, the linear isomorphism $v \otimes w \mapsto w \otimes v$ between $V \otimes W$ and $W \otimes V$ of two $U_q(\mathfrak{sl}_2)$-modules V and W is **not** an isomorphism of representations. Indeed, the $U_q(\mathfrak{sl}_2)$-module isomorphism exists when e's action on V an W is locally nilpotent. It is constructed using the R matrix:

$$R \;=\; q^{\frac{h\otimes h}{4}} \sum_{k \geq 0} q^{\frac{-k(k+1)}{4}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k}{[k]!}\; q^{\frac{kh}{4}}e^k \otimes q^{-\frac{kh}{4}}f^k \;\in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$$

The above expression is in fact a finite sum when applied to any particular vector in $V \otimes W$ since e is locally nilpotent. The $U_q(\mathfrak{sl}_2)$-isomorphism between $V \otimes W$ and $W \otimes V$ is $P \cdot R$ where $P(v \otimes w) = w \otimes v$.

As an example, consider the case of $L_m \otimes L_l$. Its basis is given by vectors of the form $v_m \otimes v_l, fv_m \otimes v_l, v_m \otimes fv_l, \ldots$. Assume q is not a root of unity (i.e. κ is not rational). We will evaluate $P \cdot R$ on the first three basis vectors. Since the highest power of f occuring in any of these vectors is 1, it is enough to truncate R as

$$q^{\frac{h\otimes h}{4}} \;+\; q^{\frac{h\otimes h}{4}} q^{-\frac{1}{2}} q^{\frac{h}{4}} e \otimes q^{-\frac{h}{4}} f.$$

The isomorphism $P \cdot R : L_m \otimes L_l \longrightarrow L_l \otimes L_m$ evaluates on $v_m \otimes v_l$, $fv_m \otimes v_l$, and $v_m \otimes fv_l$ as

$$\begin{array}{lcl} v_m \otimes v_l & \longmapsto & q^{\frac{ml}{4}} v_l \otimes v_m \\ v_m \otimes fv_l & \longmapsto & q^{\frac{m(l-2)}{4}} v_l \otimes fv_m \\ fv_m \otimes v_l & \longmapsto & q^{\frac{l(m-2)}{4}} v_l \otimes fv_m + [m]\, q^{\frac{ml-m-l}{4}} fv_l \otimes v_m \end{array}$$

Theorem 1. *Let V_1, V_2, and V_3 be $U_q(\mathfrak{sl}_2)$-modules then the following diagram is commutative*

$$\begin{array}{ccccc} V_1 \otimes V_2 \otimes V_3 & \xrightarrow{(23)} & V_1 \otimes V_3 \otimes V_2 & \xrightarrow{(12)} & V_3 \otimes V_1 \otimes V_2 \\ {\scriptstyle (12)}\big\downarrow & & & & \big\downarrow{\scriptstyle (23)} \\ V_2 \otimes V_1 \otimes V_3 & \xrightarrow{(23)} & V_2 \otimes V_3 \otimes V_1 & \xrightarrow{(12)} & V_3 \otimes V_2 \otimes V_2 \end{array}$$

where $(12) = PR \otimes 1$ *and* $(23) = 1 \otimes PR$.

Commutativity of the above diagram implies that the R-matrix satisfies the Yang-Baxter equation.

Braid Groups

Definition 3. *The Braid group B_n on n-strings is the group with generators $b_1, \ldots, b_{n-1}$ with defining relations*

- $b_i b_j = b_j b_i$ *if* $|i - j| > 1$
- $b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ *for* $1 \le i \le n-2$.

The Pure Braid group PB_n is defined as the kernel of the homomorphism $\sigma : B_n \longrightarrow S_n$ given by $b_i \mapsto (i, i+1)$.

An element of the Braid group is visualized as a family of n non-intersecting paths in space joining n points in a plane to the "same" n points in a parallel plane. Multiplication in B_n is interpreted as concatenation of paths. The Pure Braid group PB_n is visualized as families of paths which start and end at the "same" point - where the second plane and the points are identified with the first.

Recall that $\Delta = \left\{ (a_1, \ldots, a_n) \in \mathbb{C}^n \;\middle|\; z^n + a_1 z^{n-1} + \cdots + a_n \text{ has multiple roots} \right\}$. Every element in the complement $\mathbb{C}^n - \Delta$ can be identified with a unique monic nth order polynomial with distinct roots; namely associate the point $(a_1, \ldots, a_n)$ with the polynomial $(z - a_1) \cdots (z - a_n)$. Given a braid - which we interpret as a family of non-intersecting paths joining the distinct points $(a_1, 0)\ldots,(a_n,0)\in \mathbb{C}\times\{0\}\subset\mathbb{R}^3=\mathbb{C}\times\mathbb{R}$ and the points $(a_1,1),\ldots,(a_n,1)\in\mathbb{C}\times\{1\}\subset\mathbb{R}^3$- we may associate a loop in $\mathbb{C}^n-\Delta$ emanating from $(a_1,\ldots,a_n)$. The position of the loop at time $t\in[0,1]$ is the point $(b_1,\ldots,b_n)$ where each (b_i,t) is the intersection point of a strand of the braid with the plane $\mathbb{C}\times\{t\}$. Using this construction one obtains the following theorem.

Theorem 2. $\pi_1\big(\mathbb{C}^n-\Delta\big)=B_n$.

Using induction and the long exact sequence of homotopy groups for fibrations one can prove that $\pi_k\big(\mathbb{C}^n-\Delta\big)=0$ for $k>1$. Taken together with Theorem 2 this implies that $\mathbb{C}^n-\Delta$ is the classifying space for B_n.

Let $D=\left\{\ (z_1,\ldots,z_n)\in\mathbb{C}^n\ \middle|\ \exists\, i\neq j \text{ such that } z_i=z_j\ \right\}$. By a similar construction using pure braids one obtains the following theorem.

Theorem 3. *The classifying space for the Pure Braid group PB_n is $\mathbb{C}^n-D$.*

Let $V_1,\ldots,V_n$ be $U_q(\mathfrak{sl}_2)$-modules and let $R_i^{\vee}: V_1\otimes\cdots\otimes V_i\otimes V_{i+1}\otimes\cdots\otimes V_n\longrightarrow V_1\otimes\cdots\otimes V_{i+1}\otimes V_i\otimes\cdots\otimes V_n$ be the map $P\cdot R$ in the $i,i+1$ positions and identity in all others.

Theorem 4. *For $1\le i\le n-2$ we have the Yang-Baxter identity $R_i^{\vee}R_{i+1}^{\vee}R_i^{\vee}=R_{i+1}^{\vee}R_i^{\vee}R_{i+1}^{\vee}$.*

Let V be a $U_q(\mathfrak{sl}_2)$-module. In view of Theorem 4 the map $B_n\longrightarrow \mathrm{GL}(V^{\otimes n})$ given by $b_i\mapsto R_i^{\vee}$ is a representation of B_n. If V is taken to be $V_1\oplus\cdots\oplus V_n$ then the representation of B_n on $V^{\otimes n}$ restricts to a representation of PB_n on the component $V_1\otimes\cdots\otimes V_n\subset V^{\otimes n}$.

Quantum Singular Vectors

For $\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{C}^n$ let $M^{\otimes\mathbf{m}}$ be the tensor product of $U_q(\mathfrak{sl}_2)$ Verma modules $M_{m_1}\otimes\cdots\otimes M_{m_n}$. Let $\operatorname{Sing} M^{\otimes\mathbf{m}}\Big[|\mathbf{m}|-2k\Big]=\left\{\ w\in M^{\otimes\mathbf{m}}\ \middle|\ e\cdot w=0 \text{ and } q^{\lambda h}\cdot w=q^{\lambda(|\mathbf{m}|-2k)}w\ \right\}$. Note that e acts as $e\otimes q^{\frac{h}{4}}\otimes\cdots\otimes q^{\frac{h}{4}}\ +\ \cdots\ +\ q^{-\frac{h}{4}}\otimes\cdots\otimes q^{\frac{-h}{4}}\otimes e$. As in the classical case, $\operatorname{Sing} M^{\otimes\mathbf{m}}\Big[|\mathbf{m}|\Big]$ is spanned by the vector $v_{m_1}\otimes\cdots\otimes v_{m_n}$. When $k=1$ we see that $\operatorname{Sing} M^{\otimes\mathbf{m}}\Big[|\mathbf{m}|-2\Big]$ is given by

$$\left\{\ w=\sum_{j=1}^{n} I_j v_{m_1}\otimes\cdots\otimes f v_{m_j}\otimes\cdots\otimes v_{m_n}\ \middle|\ \text{eigenspace condition and}\right.$$
$$\left.\sum_{j=1}^{n} I_j[m_j]q^{\frac{1}{4}\left(\sum_{i>j}m_i-\sum_{i<j}m_i\right)}=0\ \right\}.$$

Corollary 4. *The map* $v_{m_1} \otimes \cdots \otimes f v_{m_j} \otimes \cdots \otimes v_{m_n} \stackrel{\psi}{\longmapsto} \omega_j = c_j(l_j, s_j)$ *and* $v_{m_1} \otimes \cdots \otimes v_{m_n} \stackrel{\psi}{\longmapsto} \omega_0 = c_0(p_0, s_0)$ *makes the following diagram commute:*

$$\begin{array}{ccccccc} 0 \longrightarrow & M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2\Big] & \stackrel{e}{\longrightarrow} & M^{\otimes \mathbf{m}}\Big[|\mathbf{m}|\Big] & \longrightarrow 0 \\ & \psi \Big\downarrow & & \psi \Big\downarrow & \\ 0 \longrightarrow & \bigoplus_{j=1}^{n} \mathbb{C} \cdot \omega_j & \stackrel{d_1}{\longrightarrow} & \mathbb{C} \cdot \omega_0 & \longrightarrow 0 \end{array}$$

As an immediate consequence of this corollary we obtain

$$\psi : \operatorname{Sing} M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2\Big] \;\cong\; H_1\Big(\mathbb{C} - \{z_1, \dots, z_n\}, \Phi_{1,n}^{\frac{1}{\kappa}}\Big)$$

for $z_1 < \cdots < z_n$ real.