VARCHENKO: LECTURE 2

We begin by sketching the proof of Theorem 2 from Lecture 1. To do so, we modifiy the expressions for ϕ^γ slightly. For any differentail form $f(t_1, \dots, t_k)$ let

$$\mathrm{Ant}_{\mathbf{t}}\, f(t_1, \dots, t_k) \;=\; \sum_{\sigma \in S_k} (-1)^{l(\sigma)} f(t_{\sigma(1)}, \dots, t_{\sigma(k)})$$

where $l(\sigma)$ is the length of the permutation σ. For $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geq 0}$ with $|J| = j_1 + \cdots j_n = k$ set

$$\eta_J \;=\; \frac{1}{j_1! \cdots j_n!} \mathrm{Ant}_{\mathbf{t}} \left[\bigwedge_{l=1}^{n} \bigwedge_{i=1}^{j_l} \frac{d(t_{j_1+\cdots+j_{l-1}+i} - z_l)}{t_{j_1+\cdots+j_{l-1}+i} - z_l} \right].$$

When all z_i specialized to a fixed constant then η_J is $A_J dt_1 \wedge \cdots \wedge dt_k$ as defined in Lecture 1. For $\kappa \in \mathbb{C}^*$ let

$$\phi^\gamma \;=\; \sum_{|J|=k} \int_\gamma \Phi_{k,n}^{\frac{1}{\kappa}} \eta_J f_J v.$$

Claim 1. *The fact that* $\phi^\gamma(\mathbf{z}) \in \mathrm{Sing}\, M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2k\Big]$ *follows from the identities*

$$\left[\kappa\; d\Big(\Phi_{k,n}^{\frac{1}{\kappa}} \eta_J\Big) \;+\; \sum_{l=1}^{n} \Phi_{k,n}^{\frac{1}{\kappa}} \eta_{J+\mathbf{1}_l} (j_l + 1)(m_l - j_l) \right] \Bigg|_{all\ z_i = \ contant} \;=\; 0$$

where $J = (j_1, \dots, j_n)$ *is any multi-index with* $|J| = k - 1$.

The validity of this identity is easily checked by expanding $d(\Phi_{k,n}^{\frac{1}{\kappa}} \eta_J)$. To see that this identity implies $\phi^\gamma \in \mathrm{Sing}\, M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2k\Big]$ integrate the above expression over γ. Since γ is a cycle the integral $\int_\gamma \kappa\; d(\Phi_{k,n}^{\frac{1}{\kappa}} \eta_J)$ vanishes. The remaining part of the integral is exactly the coefficient of $f_J v$ in $e \cdot \phi^\gamma$.

Claim 2. *The fact that* ϕ^γ *is a KZ solution follows from the identity*

$$d\left(\sum_{|J|=k} \Phi_{k,n}^{\frac{1}{\gamma}} \eta_J f_J v \right) \;=\; \frac{1}{\kappa} \sum_{i<j} \Omega^{ij} \left(\frac{d(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \Phi_{k,n}^{\frac{1}{\kappa}} \eta_J f_J v \right).$$

The identity is easily checked by expanding the left hand side with $d = d_{\mathbf{z}} + d_{\mathbf{t}}$. Integrate the above expression over a cycle γ in the coordinate $\mathbf{t}$. From the identity we obtain

$$\int_\gamma d_{\mathbf{t}}\Big[\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\Big] = -\int_\gamma d_{\mathbf{z}}\Big[\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\Big] + \int_\gamma \frac{1}{\kappa}\sum_{i<j}\Omega^{ij}\left(\frac{d(z_i-z_j)}{z_i-z_j}\wedge\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\right).$$

By Stokes' Theorem we know that $\int_\gamma d_{\mathbf{t}}\left[\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\right]$ vanishes so we may conclude that

$$\int_\gamma d_{\mathbf{z}}\Big[\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\Big] = \int_\gamma \frac{1}{\kappa}\sum_{i<j}\Omega^{ij}\left(\frac{d(z_i-z_j)}{z_i-z_j}\wedge\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\right) \text{ or}$$

$$d_{\mathbf{z}}\left[\int_\gamma\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v\right] = \frac{1}{\kappa}\sum_{i<j}\Omega^{ij}\left(\frac{d(z_i-z_j)}{z_i-z_j}\right)\cdot\int_\gamma\sum_{|J|=k}\Phi_{k,n}^{\frac{1}{\kappa}}\eta_J f_J v.$$

By isolating the $\frac{\partial}{\partial z_j}$ component in the right hand side of the above expression we obtain the KZ equation as required.

Hyperplane Arrangements

The modern theory of arrangements has some origins in Hilbert's 13th problem. An algebraic function $z = z(x_1,\ldots,x_k)$ is a multi-valued function defined by an equation of the form

$$z^n + P_1(x_1,\ldots,x_k)z^{k-1} + \cdots + P_n(x_1,\ldots,x_k) = 0$$

where the P_i's are polynomials.

Hilbert's 13th Problem: Show that an algebraic function z defined by

$$z^7 + az^2 + bz + c = 0$$

can not be represented as a composition of continuous functions in two variables.

Kolmogorov and Arnold showed that in fact one can find such a two variable decompostion. Despite the Kolmogorov/Arnold result it is believed that the composition is impossible if one restricts to algebraic functions. Arnold wanted to understand why algebraic functions of many variables become more complicated. The idea was to invent invariants of algebraic functions which would detect when compositions are possible. The simplest characteristic of an algebraic function $z(x_1,\ldots,x_k)$ is its discriminant

$$\Delta_P = \left\{\mathbf{x}\in\mathbb{C}^k \;\middle|\; z^n + P_1(\mathbf{x}) + \cdots P_n(\mathbf{x}) = 0 \text{ has multiple roots}\right\}.$$

Any algebraic function is induced from the universal algebraic function $z^n + a_1 z^{n-1} + \cdots + a_n = 0$ by the map $P : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ given by $\mathbf{x} = (x_1,\ldots,x_k) \mapsto \big(P_1(\mathbf{x}),\ldots,P_n(\mathbf{x})\big)$. The discriminant can then be expressed as $\Delta_P = P^{-1}(\Delta)$ where

$$\Delta = \left\{ (a_1, \ldots, a_n) \in \mathbb{C}^n \;\middle|\; z^n + a_1 z^{n-1} + \cdots + a_n = 0 \text{ has multiple roots} \right\}.$$

By passing to cohomology we obtain the map $P^* : H^*(\mathbb{C}^n - \Delta) \longrightarrow H^*(\mathbb{C}^k - \Delta_P)$. For any $\alpha \in H^*(\mathbb{C}^n - \Delta)$ one has a characteristic class $P^*\alpha$ in $H^*(\mathbb{C}^k - \Delta_P)$ for the algebraic function $z(x_1, \ldots, x_k)$. One might hope that the cohomology of the compliment of the discriminant might give obstructions to the representablity of an algebraic function as a composition. As a first step one might try to describe $H^*(\mathbb{C}^n - \Delta)$. There is, however, a simplification. The Vieta map $V : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is defined by $\mathbf{z} = (z_1, \ldots, z_n) \mapsto \big(a_1(\mathbf{z}), \ldots, a_n(\mathbf{z})\big)$ where a_k is $(-1)^k$ times the kth elementary symetric function in the variables $z_1, \ldots, z_n$. Under the Vieta map the set $D = \{\ (z_1, \ldots, z_n) \mid z_i = z_j \ \exists\ i, j\ \}$ maps to Δ. The problem of computing $H^*(\mathbb{C}^n - \Delta)$ thus reduces to the study of $H^*(\mathbb{C}^n - D)$ since $H^*(\mathbb{C}^n - D)$ is the symmetric part of the S_n action on $H^*(\mathbb{C}^n - \Delta)$.

Arnold described $H^*(\mathbb{C}^n - D)$. Consider the 1-forms $\omega_{ij} = \frac{1}{2\pi i}\frac{d(z_i - z_j)}{z_i - z_j}$. Clearly $\omega_{ij} = \omega_{ji}$. An easy exercise of Arnold reveals that for distinct i, j, and k

$$\omega_{ij} \wedge \omega_{jk} \;+\; \omega_{jk} \wedge \omega_{ki} \;+\; \omega_{ki} \wedge \omega_{ij} \;=\; 0.$$

Theorem 1 (Arnold). *Let $\mathcal{A} = \mathbb{C}[\omega_{ij}]$ be the exterior algebra generated by the differentail forms ω_{ij}. The map $\alpha \mapsto [\alpha]$ of $\mathcal{A}$ to $H^*(\mathbb{C}^n - D)$ is an isomorphism.*

Theorem 2 (Arnold). *Let $P_\Delta(t) = \sum_{k=0}^{n} \dim H^k(\mathbb{C}^n - D)t^k$ be the Poincare polynomial, then*

$$P_\Delta(t) \;=\; \Big(1 + t\Big)\Big(1 + 2t\Big) \cdots \Big(1 + (n-1)t\Big).$$

Brieskorn proved Theorem 1 for any hyperplane arrangement and Orlik and Solomon combinatorially described the algebra of logarithmic differential forms. Realize each hyperplane H in the arrangement $\mathcal{H}$ as the vanishing of an equation f_H. Associate to f_H the closed 1-form $\omega_H = \frac{1}{2\pi i}\frac{df_H}{f_H}$; note that ω_H does not depend on the choice of f_H. Consider the exterior algebra $\mathcal{A} = \mathbb{C}[\omega_H]_{H \in \mathcal{H}}$. The relations in $\mathcal{A}$ are given as follows. A collection of hyperplanes $H_1, \ldots, H_j$ is said to intersect transversally if codim $\big(H_1 \cap \cdots \cap H_j\big) = j$. To obtain the relations for $\mathcal{A}$, take any collection of hyperplanes $H_1, \ldots, H_{j+1}$ such that $H_1 \cap \cdots \cap H_{j+1} \neq \phi$ and such that $H_1, \ldots, H_{j+1}$ do **not** intersect transversally. Associate to this collection the relation

$$\sum_{i=1}^{j+1} (-1)^j \; \omega_{H_1} \wedge \cdots \wedge \hat{\omega_{H_i}} \wedge \cdots \wedge \omega_{H_{j+1}} \;=\; 0.$$

Theorem 3 (Orlik & Solomon). *These are the defining relations for $\mathcal{A} = \mathbb{C}[\omega_H]$.*

Theorem 2 is generalized in the following manner. Let G be a Coxeter group and let $\mathcal{H}$ be the collection of reflecting hyperplanes for G. Set $\mathcal{H}_{\mathbb{C}}$ to be $\mathcal{H}$'s complexification.

Theorem 4 (Terao)**.** *Let $P(t)$ be the Poincare polynomial of $\mathbb{C}^k - \bigcup_{H\in\mathcal{H}} H$ then,*

$$P(t) \;=\; \prod_j \Big(1 + (d_j - 1)t\Big)$$

where the d_j's are the degrees of the basis set of homogeneous polynomials invariant under the action of G.

Classical Hypergeometric Series

For a complex number z define the nth rising power $(z)_n = z(z+1)\cdots(z+n-1)$ with the provision that $(z)_0 = 1$. The classical hypergeometric series is defined as:

$$F(a,b,c;z) \;=\; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

The rising power can be expressed via the Gamma function as $(z)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ where

$$\Gamma(z) \;=\; \int_0^{\infty} e^{-t}t^{z-1}dt.$$

Note that the Gamma function is defined and holomorphic for $z \in \mathbb{C} - \mathbb{Z}_{<0}$.

Properties of the Hypergeometric series: Gauss 1812

1. The series $F(a,b,c;z)$ is convergent for $|z| < 1$ and it converges for $|z| = 1$ provided $\Re(c-a-b) > 0$.

2. Continguous Recurrence Relations:

$$(c-a-b)F \;+\; a(1-z)F(a+) \;-\; (c-b)F(b-) \;=\; 0$$

   where $F(a\pm) = F(a\pm 1, b, c; z)$, $F(b\pm 1) = F(a, b\pm 1, c; z)$, and $F(c\pm 1) = F(a,b,c\pm 1; z)$ are the associated continguous functions. Gauss showed there is a linear relation between F and any two continguous functions with coefficients which are linear in z.

3. $F(a,b,c;1) \;=\; \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$.

Theorem 5 (Euler)**.** *F satisfies the Gauss Hypergeometric Equation*

$$z(1-z)\frac{d^2u}{dz^2} \;+\; \Big(c-(a+b+1)z\Big)\frac{du}{dz} \;-\; abu \;=\; 0.$$

Theorem 6 (Euler)**.**

$$F(a,b,c,z) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt.$$

If the specialization $z = 1$ is made in the above expression formula #3 is recovered.

$$\text{Set } J \;=\; \int_a^b (s-a)^\alpha (b-s)^\beta (z-s)^\gamma ds.$$

If we make the change of variables $t = \frac{s-a}{b-a}$ in the integral J we obtain

$$J \;=\; \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}(b-a)^{\alpha+\beta+1}(z-a)^\gamma F\left(-\gamma, \alpha+1, \alpha+\beta+2, \frac{b-a}{z-a}\right).$$