LECTURE 12

The last part of the last lecture was devoted to discuss the following problem: how to solve the q-KZ equation in terms of q-hypergeometric functions. Let's recall the main points of this construction. We introduced the action of the twisted symmetric group S^k in the space of the function $\mathcal{F}_k = \{\text{functions of k variables}\, t_1, ..., t_k\}$, via the maps:

$$s_j : f(t_1, .., t_k) \longmapsto f(t_1, .., t_{j+1}, t_j, .., t_k)h(t_i - t_j), \tag{1}$$

where $h = h(x)$ is a function that satisfies the condition:

$$h(x) \cdot h(-x) = 1,$$

and we also defined the space:

$$F^k_{m_1,...,m_n}(z_1, ..., z_n) \subset \mathcal{F}_k$$

of the functions of the form:

$$P(t_1, .., t_k) \prod_{l=1}^{n} \prod_{i=1}^{k} \frac{1}{t_i - z_l - m_l} \prod_{1 \le i < j \le k} \frac{t_i - t_j}{t_i - t_j + 1},$$

where $P = P(t_1, .., t_k)$ is a symmetric polynomial (in the standard sense).
We also noticed that the space $F_{m_1,...,m_n}(z_1, ..., z_n) = \bigoplus_{k=0}^{\infty} F^k_{m_1,...,m_n}(z_1, ..., z_n)$ satisfies the following:

Properties 0.1. 1. *dim* $F^k_{m_1,...,m_n}$=*dim* $M_{m_1} \otimes ... \otimes M_{m_n}[|m| - 2k]$;
2. $F^k_{m_1,...,m_n}$ *consists of functions that are symmetric with respect to the action of* S^k *with* $h = \dfrac{x-1}{x+1}$;
3. $F^k_{m_1,...,m_n}(z_1, ..., z_n) = F^k_{m_1,..,m_{j+1},m_j,..,m_n}(z_1, .., z_{j+1}, z_j, .., z_n)$.

Then we defined a tensor product:

$$\begin{array}{ccc} F^{k'}_{m_1,...,m_{n'}}(z_1, ..., z_{n'}) \otimes F^{k''}_{m_{n'+1},...,m_{n'+n''}}(z_{n'+1}, ..., z_{n'+n''}) & \longrightarrow & F^{k'+k''}_{m_1,...,m_{n'+n''}}(z_1, ..., z_{n'+n''}) \\ f \otimes g & \longmapsto & k(t_1, ..., t_{k'+k''}) \end{array}$$

where

$$k(t_1, ..., t_{k'+k''}) = \frac{1}{k'!k''!} Sym_h\Big(f(t_1, .., t_{k'})g(t_{k'+1}, .., t_{k'+k''}) \cdot \prod_{k<j \le k'+k''} \cdot \prod_{1 \le l \le n'} \frac{t_j - z_l + m_l}{t_j - z_l - m_l}\Big)$$

and we state the following:

Claim 0.1. *For generic values of* $z_1, .., z_n$:

$$\bigoplus_{k'+k''=k} F^{k'}_{m_1,...,m_{n'}}(z_1, ..., z_{n'}) \otimes F^{k''}_{m_{n'+1},...,m_{n'+n''}}(z_{n'+1}, ..., z_{n'+n''}) \longrightarrow F^k_{m_1,...,m_{n'+n''}}(z_1, ..., z_{n'+n''})$$

is an isomorphism.

Finally we defined:

$$\begin{array}{ccc} \omega(z):(V_m)^* & \longrightarrow & F_m(z) \\ (f^k v)^* & \longmapsto & \omega_k \end{array}$$

where

$$\omega_k(t_1,..,t_k,z)=\prod_{a<b}\frac{t_a-t_b}{t_a-t_b+1}\prod_{a=1}^{k}\frac{1}{t_a-z-m}, \tag{2}$$

is a generator of the one dimensional vector space $F_m^k(z)$ and from 2 and the tensor product we also defined the map:

$$\omega(z_1,..,z_n):V_{m_1}^*\otimes...\otimes V_{m_n}^*\longrightarrow F_{m_1,...,m_n}(z_1,..,z_n); \tag{3}$$

that has the following property:

Claim 0.2. *For generic $z_1,..,z_n$, the map $\omega=\omega(z_1,...,z_n)$, defined in 3, is an isomorphims.*

Now we will proceed introducing a suitable R-matrix to relate basis in different spaces. Let $z,u\in\mathbb{C}$ such that the map:

$$\omega(z,u):M_a^*\otimes M_b^*\longrightarrow F_{a,b}(z,u) \tag{4}$$

be invertible. Then:

Definition 0.1. *The* rational R-matrix $R_{a,b}(z,u)$ *is the dual map of the composition:*

$$M_a^*\otimes M_b^*\xrightarrow{P}M_b^*\otimes M_a^*\xrightarrow{\omega(u,z)}F_{a,b}(z,u)\xrightarrow{\omega^{-1}(z,u)}M_a^*\otimes M_b^*, \tag{5}$$

Remark 0.1. *Alternatively we can think the map $R_{ab}(z,u)$ defined by 5 as the matrix that express the basis $\omega(u,z)\Big((f^m v_b)^*\otimes((f^l v_a)^*\Big)$ in terms of the basis $\omega(z,u)\Big((f^l v_a)^*\otimes(f^m v_b)^*\Big)$.*

The rational R-matrix just introduced satisfies the following:

Properties 0.2. 1. $R_{ab}(z,u)$ *depends only on the difference $u-z$;*
2. $R_{ab}(z,u)$ *is the standard* R-matrix *associated with the Yangian module structure of the tensor product $M_a(z)\otimes M_b(u)$.*

In the next section we will describe the trigonometric analogous of the previous construction.

1. Trigonometric case.

In what follows with p we will denote the step of the q-KZ equation, $q=e^{\frac{\pi i}{p}}$ and $M_{m_j}^q$, $j=1,..,n$, will be a $\mathcal{U}_q(sl_2)$-Verma module. Let's first introduce the trigonometric analogous of the space

$$F^k_{m_1,...,m_n}(z_1,...,z_n)\subset\mathcal{F}_k$$

Definition 1.1. *Let $\mathcal{F}^k_{m_1,\dots,m_n}(z_1,\dots,z_n) \subset \mathcal{F}_k$ the space of function of the form:*

$$\mathcal{P}(\xi_1,..,\xi_l)\prod_{l=1}^{n}\prod_{i=1}^{k}\frac{\exp(\frac{\pi i(z_l-t_a)}{p})}{\sin(\frac{\pi(t_a-z_l-m_l)}{p})}\prod_{a<b}\frac{\sin(\frac{\pi(t_a-t_b)}{p})}{\sin(\frac{\pi(t_a-t_b+1)}{p})}, \tag{6}$$

where $\xi_a = \exp(\frac{2\pi i t_a}{p})$ and $\mathcal{P} = \mathcal{P}(\xi_1,..,\xi_l)$ is a symmetric polynomial in $\xi_1,\dots,\xi_l$, in the standard sense of degree n in each of its variable.

In analogy to the rational case we have:

Definition 1.2.

$$\mathcal{F}_{m_1,\dots,m_n}(z_1,\dots,z_n) = \bigoplus_{k=0}^{\infty}\mathcal{F}^k_{m_1,\dots,m_n},$$

where, by definition, $\mathcal{F}^0_{m_1,\dots,m_n} = \mathbb{C}$.

The spaces $\mathcal{F}^k_{m_1,\dots,m_n}$ satisfy the following:

Properties 1.1. 1. $\dim\mathcal{F}^k = \binom{n+k-1}{k} = \dim(M^q_{m_1}\otimes..\otimes M^q_{m_n})[|m|-2k]$*;*
2. *$\mathcal{F}^k$ consists of symmetric functions with respect to the action of the group S^k defined by the function:*

$$\tilde{h}(x) = \frac{\sin(\frac{\pi(x-1)}{p})}{\sin(\frac{\pi(x+1)}{p})};$$

3. $\mathcal{F}^k_{m_1,\dots,m_n}(z_1,\dots,z_n) = \mathcal{F}^k_{m_1,..,m_{j+1},m_j,..,m_n}(z_1,..,z_{j+1},z_j,..,z_n)$.

Example 1.1. *As in the rational case we have an explicit description of the generators for the case $n=1$. In fact in this hypothesis we have that the space $\mathcal{F}^k_m(z)$ is generated by:*

$$W_k(t_1,\dots,t_k,z) = \prod_{a=1}^{k}\frac{\exp(\frac{\pi i(z-t_a)}{p})}{\sin(\frac{\pi(t_a-z-m)}{p})}\prod_{a<b}\frac{\sin(\frac{\pi(t_a-t_b)}{p})}{\sin(\frac{\pi(t_a-t_b+1)}{p})}. \tag{7}$$

Let's introduce the tensor product:

$$\begin{array}{ccc}\mathcal{F}^{k'}_{m_1,\dots,m_{n'}}(z_1,\dots,z_{n'})\otimes\mathcal{F}^{k''}_{m_{n'+1},\dots,m_{n'+n''}}(z_{n'+1},\dots,z_{n'+n''}) & \longrightarrow & \mathcal{F}^{k'+k''}_{m_1,\dots,m_{n'+n''}}(z_1,\dots,z_{n'+n''}) \\ f\otimes g & \longmapsto & u(t_1,\dots,t_{k'+k''})\end{array}$$

where:

$$u(t_1,\dots,t_{k'+k''}) = \frac{1}{k'!k''!}Sym_{\tilde{h}}\Big(f(t_1,..,t_{k'})g(t_{k'+1},..,t_{k'+k''})\prod_{k'<j\le k'+k''}\prod_{1\le l\le n'}\frac{\sin(\frac{t_j-z_l+m_l}{p})}{\sin(\frac{t_j-z_l-m_l}{p})}\Big)$$

Also in the this case we have that:

Claim 1.1. *For generic values of $z_1,..,z_n$:*

$$\bigoplus_{k'+k''=k}\mathcal{F}^{k'}_{m_1,\dots,m_{n'}}(z_1,\dots,z_{n'})\otimes\mathcal{F}^{k''}_{m_{n'+1},\dots,m_{n'+n''}}(z_{n'+1},\dots,z_{n'+n''}) \longrightarrow \mathcal{F}^k_{m_1,\dots,m_{n'+n''}}(z_1,\dots,z_{n'+n''})$$

is an isomorphism.

Now let's define:

Definition 1.3.
$$\begin{aligned} W(z) : (M_m)^q &\longrightarrow F_m(z) \\ (f^k v) &\longmapsto C_k W_k(t_1, .., t_n, z), \end{aligned} \tag{8}$$

where:
$$C_k = \prod_{s=0}^{k-1} \frac{\sin(\frac{\pi(2m-s)}{p})}{\sin(\frac{\pi}{p})} = \prod_{s=0}^{k-1} \frac{q^{2(m-s)} - q^{-2(m-s)}}{q - q^{-1}}.$$

The map 8 induces a map:
$$W(z_1, .., z_n) : M_{m_1} \otimes .. \otimes M_{m_n} \longrightarrow \mathcal{F}_{m_1,..,m_n}(z_1, .., z_n). \tag{9}$$

We have:

Claim 1.2. *For generic $z_1, .., z_n$ the map 9 is non degenerate.*

Let's now introduce the trigonometric version of the R-matrix 4. Let $z, u \in \mathbb{C}$ such that
$$W(u, z) : V_b \otimes V_a \longrightarrow \mathcal{F}_{a,b}(z, u)$$
is non degenerate. Then:

Definition 1.4. *The* trigonometric R-matrix $R^{trig}_{a,b}$ *is given by following composition:*
$$M_a \otimes M_b \xrightarrow{W(z,u)} \mathcal{F}_{a,b}(z, u) \xrightarrow{W^{-1}(u,z)} M_b \otimes M_a \xrightarrow{p} M_a \otimes M_b, \tag{10}$$

The trigonometric R-matrix defined by 10 has the following:

Properties 1.2. 1. $R^{trig}_{a,b}$ *is a map between $M^q_a \otimes M^q_b$ and itself;*
2. $R^{trig}_{a,b}$ *depends only on the difference $z - u$, is p-periodic and coincides with the trigonometric R-matrix $R_{M^q_a,M^q_b}\Big(\exp(\frac{2\pi i(z-u)}{p})\Big)$.*

Remark 1.1. *For any highest weight $\mathcal{U}_q(sl_2)$-modules V^q_1 and V^q_2, there is $R^q_{V^q_1,V^q_2}(z) \in End(V^q_1 \otimes V^q_2)$ satisfying the QYE:*
$$R^q_{V^q_1,V^q_2}(\frac{z_1}{z_2}) R^q_{V^q_1,V^q_3}(\frac{z_1}{z_3}) R^q_{V^q_2,V^q_3}(\frac{z_2}{z_3}) = R^q_{V^q_2,V^q_3}(\frac{z_2}{z_3}) R^q_{V^q_1,V^q_3}(\frac{z_1}{z_3}) R^q_{V^q_1,V^q_2}(\frac{z_1}{z_2}).$$
$R^q_{V^q_1,V^q_2}(z)$ is called trigonometric R-matrix, it is constucted using $\mathcal{U}_q(\widehat{sl_2})$ and normalized requiring that:
$$R^q_{V^q_1,V^q_2}(z)(v_1 \otimes v_2) = v_1 \otimes v_2$$
where $v_i \in V^q_i$ are highest weight vectors.

Example 1.2. *Let $V^q_1 = V^q_2 = \mathbb{C}^2$, then:*
$$R^{trig}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x-1}{xq-q^{-1}} & \frac{q-q^{-1}}{xq-q^{-1}} & 0 \\ 0 & \frac{x(q-q^{-1})}{xq-q^{-1}} & \frac{x-1}{xq-q^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

Remark 1.2. *There is an analogous construction for the* elliptic q-KZ equation.

2. Hypergeometric Pairing.

In what follows we will assume that $p \in \mathbb{R}$ $(p < 0)$. Let's introduce the following map:

Definition 2.1.

$$J(z_1, ..., z_n) : \mathcal{F}_{m_1,..,m_n}(z_1, .., .z_n) \otimes F_{m_1,..,m_n}(z_1, ..., z_n) \longrightarrow \mathbb{C}$$

such that:

$$J(z_1, ..., z_n)(f \otimes g) = \int_\gamma \Phi_{m_1,..,m_n}(t_1, .., t_k, z_1, .., z_n) f(t_1, .., t_k) g(t_1, .., t_k) dt_1 \cdot .. \cdot dt_k, \tag{12}$$

where $\Phi_{m_1,..,m_n}(t_1, .., t_k, z_1, .., z_n)$ is a q-deformation of the Master function, defined by:

$$\Phi_{m_1,..,m_n}(t_1, .., t_k, z_1, .., z_n) = \exp(\mu \sum_{a=1}^{k} t_a/p) \prod_{l=1}^{n} \prod_{a=1}^{k} \frac{\Gamma((t_a - z_l + m_l)/p)}{\Gamma((t_a - z_l - m_l)/p)} \prod_{a<b} \frac{\Gamma((t_a - t_b - 1)/p)}{\Gamma((t_a - t_b + 1)/p)},$$

and the cicle γ is defined by:

$$\gamma = \{(t_1, .., t_k) \in \mathbb{C}^k \,|\, \Re e t_i = 0, \, \forall i = 1, .., k\}.$$

Remark 2.1. *If $m_1, .., m_n \in \mathbb{R}$ and $<< 0$ and $z_1, .., z_n \in i\mathbb{R}$ the integral 12 is convergent. If the previous conditions are not satisfied we define 12 by analytic continuation.*

The hypergeometric pairing has the following property:

Proposition 2.1. 1. *For generic $z_1, ..., z_n$ and generic $m_1, ..., m_n$ 12 is non degenerate.*

Prof. The prof follows from the calculation of suitable determinants of hypergeometric integrals.

Example 2.1 (q-Selberg integral)**.** *Let $n = 1$. Then:*

$$\int_{-i\infty}^{i\infty} .. \int_{-i\infty}^{i\infty} \prod_{i=1}^{k} \left((u^{2t_i} \Gamma(a + t_i)\Gamma(a - t_i) \right) \prod_{i<j} \frac{\Gamma(t_i - t_j + x)\Gamma(t_j - t_i + x)}{\Gamma(t_i - t_j)\Gamma(t_j - t_i)} dt_1 \cdot \cdot \cdot dt_k =$$

$$(2\pi\sqrt{-1})^k (u + u^{-1})^{-k(2a+(k-1)x)} \prod_{i=1}^{k} \frac{\Gamma(1 + ix)\Gamma(2a + (i - 1)x)}{\Gamma(1 + x)}$$

Remark 2.2. *The q-Selberg integral defined in the previous example gives the multidimensional generalization of the Barnes' formula.*

Now let's consider:

$$J(z_1, ..., z_n) : \mathcal{F}_{m_1,..,m_n}(z_1, .., .z_n) \otimes F_{m_1,..,m_n}(z_1, ..., z_n) \longrightarrow \mathbb{C}$$

and let ν, $\tau \in S^n$, where S^n is the permutation group. Then J defines the following paring:

$$(M^q_{m_{\tau_1}} \otimes ... \otimes M^q_{m_{\tau_n}}) \otimes (M_{m_{\nu_1}} \otimes ... \otimes M_{m_{\nu_n}})^* \longrightarrow \mathbb{C} \tag{13}$$

or, equivalently, the following map:

$$I_{\tau,\nu}(z_1,..,z_n) : M^q_{m_{\tau_1}} \otimes ... \otimes M^q_{m_{\tau_n}} \longrightarrow M_{m_{\nu_1}} \otimes ... \otimes M_{m_{\nu_n}}. \tag{14}$$

Let's now consider 14 for $\nu = id$:

$$I_{\tau,id}(z_1,..,z_n) = I_\tau(z_1,..,z_n) : M^q_{m_{\tau_1}} \otimes ... \otimes M^q_{m_{\tau_n}} \longrightarrow M_{m_1} \otimes ... \otimes M_{m_n}. \tag{15}$$

We can now state the following:

Theorem 2.1. *For any $\tau \in S^n$, I_τ defined in 15 satisfied the* rational q-KZ equation. *with values in $M_{m_1} \otimes ... \otimes M_{m_n}$.*

Remark 2.3. *We can rephrase the content of the previous theorem saying that the solution of the rational q-KZ equation, with values in $M_{m_1} \otimes ... \otimes M_{m_n}$, product of sl_2-modules, are labeled elements of the product of $\mathcal{U}_q(sl_2)$ corresponding modules.*

We observe that for $\tau \in S^n$, the product $M^q_{m_{\tau_1}} \otimes ... \otimes M^q_{m_{\tau_n}}$ has a natural basis given by:

$$f_J v_\tau = f^{j_1} v^q_{m_{\tau_1}} \otimes \cdot .. \cdot \otimes f^{j_n} v^q_{m_{\tau_n}}, \tag{16}$$

where $J = (j_1, .., j_n)$ and $v^q_{m_i} \in M^q_{m_i}$ is the generating vector. The basis $\left\{ f_J v_J \right\}_J$ given in 16 corresponds to a solution of the q-KZ equation and for any element $\tau \in S^n$ we have a basis of solutions. It turns out that these basis have remarkable asymptotic properties.

3. Quantization of the Drinfeld-Kohno theorem.

In this section we will discuss the discrete analogue of the Drinfeld-Kohno theorem. Let's start describing the monodromy of a difference equation.

Example 3.1. *Let's consider the following scalar difference equation:*

$$\psi(z+p) = K(z)\psi(z),\ z \in \mathbb{C}\ and\, p < 0. \tag{17}$$

We have that if $\psi = \psi(z)$ is a non zero solution then any other solution $\tilde{\psi}$ of this equation has the form:

$$\tilde{\psi}(z) = P(z)\psi(z),$$

where $P = P(z)$ is p-periodic, i.e:

$$P(z+p) = P(z)\,\forall z.$$

Let's now let's suppose that the function $K = K(z)$ satisfies the following properties:

$$K(z) = e^{az}(1 + \alpha/z + \mathcal{O}(1/z)),\ z \to +\infty, \tag{18}$$

$$K(z) = e^{bz}(1 + \beta/z + \mathcal{O}(1/z)),\ z \to -\infty, \tag{19}$$

where a, b, α and β are some numbers. Then we have that there are solutions f_+ and f_- of 17 such that:

$$f_+(z) \sim e^{az/p} z^{\alpha/p}(1 + \mathcal{O}(1/z)) \text{ as } z \to +\infty$$

and similarly

$$f_-(z) \sim e^{bz/p} z^{\beta/p}(1 + \mathcal{O}(1/z)) \text{ as } z \to -\infty.$$

The function:

$$P_{+,-} = P_{+,-}(z) = \frac{f_+(z)}{f_-(z)} \tag{20}$$

is called the scattering *or* transition *matrix associated to the solutions 18 and 19.*

Remark 3.1. *The scattering martrix replaces the concept of monodromy in the case of difference equations.*

We want to apply a procedure similar to the one described in the previous example to the q-KZ equation.

Remark 3.2. *We observe that the q-KZ equation has $n!$ asymptotic zones:*

$$A_\tau, \ \tau \in S^n :$$

$$A_\tau : \ \Re e z_{\tau_1} << \Re e z_{\tau_2} << ....\Re e z_{\tau_n},$$

i.e. we can approach ∞ in $n!$ different ways.

Definition 3.1. *We will say that a basis of solutions $\psi_1, \psi_2, ...$ of the q-KZ equation, forms an asymptotic solution in a given zone A, if:*

$$\psi_j = \exp(\sum_{m=1}^{n} a_{m_j} z_m/p) \prod_{m<l} (z_l - z_m)^{b_{jlm}} (v_j + \mathcal{O}(1)) \, \forall j, \tag{21}$$

where a_{m_j}, b_{jlm} are numbers, $v_1, v_2, ...$ are vectors forming a basis in V and $\mathcal{O}(1)$ tends to zero as $z \to \infty$ in A.

Remark 3.3. *We recall that $K_m = K_m(z_1, .., z_n, p)$ can be written in terms of R-matrices as:*

$$K_m = R_{V_m,V_{m-1}}(z_m - z_{m-1} + p)...R_{V_m,V_1}(z_m - z_1 + p)e^{\mu h^{(m)}}$$

$$R_{V_m,V_n}(z_m - z_n)...R_{V_m,V_{m+1}}(z_m - z_{m+1}).$$

In every asymptotic zone A_τ we have:

$$K_m(z_1, .., z_n) \sim e^{\mu h^{(m)}},$$

i.e at ∞ the $K_m(z_1, .., z_n)$ operators are diagonal and the vectors $v_1, v_2, ...$ introduced in the previous definition are common eigenvectors. These vectors can be easily described, namely consider the vectors $\left\{ f^{j_1} v_{m_1} \otimes .. \otimes f^{j_n} v_{m_n} \right\}$: they form an eigen basis for $h^{(m)}$. This basis will be called the monomial basis in $M_{m_1} \otimes .. \otimes M_{m_n}$.

Now we can state the main result:

Theorem 3.1. *For every asymptotic zone A_τ the basis in the space of solutions corresponding to the basis:*

$$\Big\{f_J v_\tau\Big\} \subset M_{m_{\tau_1}} \otimes .. \otimes M_{m_{\tau_n}}$$

is asymptotic in the zone A_τ.

We also have the following description of the scattering matrices:

Corollary 3.1. *The transition functions between the asymptotic solutions corresponding to neighboring zones are given in terms of the corresponding trigonometric R-matrices. Namely $\forall \tau \in S^n$, $m \in \{1,..,n\}$, let*

$$T_{\tau,m}(z_1,..,z_n) : M^q_{\tau_1} \otimes .. \otimes M^q_{\tau_{m+1}} \otimes M^q_{\tau_m} \otimes .. \otimes M^q_{\tau_n} \longrightarrow M^q_{\tau_1} \otimes .. \otimes M^q_{\tau_n} \tag{22}$$

be the composition of $R_{M^q_{\tau_{m+1}}, M^q_{\tau_m}}(e^{2\pi i(z_{\tau_{m+1}} - z_{\tau_m})/p})$ and:

$$P : M^q_{\tau_{m+1}} \otimes M^q_{\tau_m} \longrightarrow M^q_{\tau_m} \otimes M^q_{\tau_{m+1}}.$$

Then the transition function between the solutions associated to the zones A_τ and $A_{\sigma_m \tau}$ is identified with $T_{\tau,m}(z_1,..,z_n)$.

Example 3.2. *Let $a, b, c, d, \theta \in \mathbb{C}$ with $\Re e\theta > 0$.*
Set $\lambda = \sqrt{a^2 - bc}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A(u) = \dfrac{1}{d+u}\begin{pmatrix} a+u & b \\ c & a-u \end{pmatrix}$.
We have that:

$$A(u,\theta) = \theta^{uh} A(u) \theta^{-uh}.$$

We have the following:

Theorem 3.2.

$$\lim_{s\to\infty} \Big(s^{-ah} h^s \Big(\prod_{r=-s}^{s} A(u+r;\theta) \Big) h^s s^{ah} \Big) = A^q(u),$$

where:

$$A^q(u) = \frac{1}{\sin(\pi(d+u))} \begin{pmatrix} \sin(\pi(a+u)) & \frac{\pi b(\theta+\theta^{-1})^{2a}}{\Gamma(1+a+\lambda)\Gamma(1+a-\lambda)} \\ \frac{\pi c(\theta+\theta^{-1})^{-2a}}{\Gamma(1-a+\lambda)\Gamma(1-a-\lambda)} & \sin(\pi(a-u)) \end{pmatrix}$$