LECTURE 11

During the last lecture we introduced the quantized version of the KZ-equations and we studied in detail the rational case. Let's recall the main features of this costruction. Let $V = V_1 \otimes ... \otimes V_n$ a tensor product of sl_2-modules, then the q-KZ equation is given by:

$$\omega(z_1, .., z_m + p, .., z_n) = K_m(z_1, ..., z_n, p)\omega(z_1, ..., z_n), \;\; m = 1, ..., n, \tag{1}$$

where $u = u(z_1, ..., z_n)$ is a V-valued function, p is called the step of the equation, $K_m : V \longrightarrow V, \;\; m = 1, ..., n$ are linear operators that are given in terms of R-matrices, i.e:

$$K_m = R_{V_m,V_{m-1}}(z_m - z_{m-1} + p)...R_{V_m,V_1}(z_m - z_1 + p)e^{\mu h^{(m)}}$$
$$R_{V_m,V_n}(z_m - z_n)...R_{V_m,V_{m+1}}(z_m - z_{m+1}).$$

We also described the geometrical interpretation of this discretized version of the KZ-equation: the q-KZ equation can be interpreted as discrete connection on the trivial vector bundle $V \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Moreover this connection is flat, i.e:

$$K_i(z_1, .., z_j + p, .., z_n)K_j(z_1, ..., z_n) = K_j(z_1, .., z_i + p, .., z_n)K_i(z_1, ..., z_n).$$

Finally we proved that under a suitable limiting procedure, the q-KZ equation turns into the KZ equation.

Remark 0.1.

Lemma 0.1. *Let $\vec{z} = (z_1, ..., z_n)$ and let:*

$$\omega(\vec{z}, p) = e^{\frac{S(\vec{z})}{p}}(f_0(\vec{z}) + pf_1(\vec{z}).....), \tag{2}$$

be an asymptotic solution of 1 , as $p \longrightarrow 0$. Then:

$$K_i(\vec{z}, p = 0)f_0(\vec{z}) = \lambda_i(\vec{z})f_0, \;\; i = 1, ..., n,$$

where $\lambda_i(\vec{z}) = e^{\frac{\partial S(\vec{z})}{\partial z_i}}$.

Prof. Since $\omega = \omega(\vec{z}, p)$ is a solution of 1 we have that:

$$\omega(z_1, .., z_i + p, .., z_n, p) = K_i(\vec{z}, p)\omega(\vec{z}, p).$$

Plugging 2 in the previous equation we get:

$$\begin{aligned} e^{\frac{S(z_1,..,z_i+p,..,z_n)}{p}}(f_0(z_1, .., z_i + p, .., z_n) + pf_1(z_1, .., z_i + p, .., z_n) + ...) \\ = K_i(\vec{z}, p)e^{\frac{S(\vec{z})}{p}}(f_0(\vec{z}) + ...). \end{aligned} \tag{3}$$

expanding with respect to the z_i coordinate:

$$e^{\frac{S(z_1,..,z_i+p,..,z_n)}{p}} = e^{\frac{S(\vec{z})}{p} + \frac{\partial S}{\partial z_i} + ...},$$

Plugging this result in the LHS of 3 and taking the limit for $p \longrightarrow 0$, we get:

$$K_i(z, p = 0)f_0(z) = \lambda_i(z)f_0(\vec{z}).$$

From the previous lemma we deduce that the leading term $f_0(z)$ is an eigenvector of commuting operators $\left\{K_m(z,p=0)\right\}_{m=1,\dots,n}$. These operators are the Hamiltonians of the XXX-quantum spin chain model. So having solutions of the q-KZ equation and computing its quasi-classical asymptotics it is possible to construct eigenvectors for the operators $\left\{K_m(z,p=0)\right\}_{m=1,\dots,n}$. These eigenvectors are the so called Bethe vectors and they are constructed by the Bethe ansatz of the XXX-model.

The main topic of this lecture will be the quantization of the KZ system of differential equation and its geometric interpretations. Let's start recalling the main features of the classical KZ differential equation.

1. Classical KZ-system.

Given $m_1, \dots m_n, \kappa \in \mathbb{C}$ consider the following function:

$$\Phi = \prod_{l=1}^{n} (t - z_l)^{\frac{m_l}{\kappa}}.$$

and let $\gamma_m = [z_m, z_{m+1}]$ be the oriented interval. Let's define the following function:

$$I^\gamma(z_1, \dots, z_n) = \Big(\int_\gamma \Phi \frac{dt}{t - z_1}, \dots, \int_\gamma \Phi \frac{dt}{t - z_n} \Big), \tag{4}$$

where $\gamma = \gamma_1, \dots, \gamma_{n-1}$, then:

$$\frac{\partial I^\gamma}{\partial z_i} = \frac{1}{\kappa} \sum_{i \neq j} \frac{\Omega_{i,j}}{z_i - z_j} I^\gamma. \tag{5}$$

Remark 1.1. *In 5 Ω depends only on $m_1, \dots, m_n$.*

Remark 1.2. *The system 5 is an example of KZ equation.*

1.1. Differential Forms.

For $z_1, \dots, z_n$ fixed consider the complex:

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \longrightarrow 0, \tag{6}$$

where by definition:

$$\Omega^0 = \{ f(t)\Phi(t,z) | f \text{ is a rational function on } \mathbb{C} \backslash \{z_1, \dots, z_n\} \}, \tag{7}$$

$$\Omega^1 = \{ \omega = f(t)\Phi(t,z)dt | f \text{ is a rational function on } \mathbb{C} \backslash \{z_1, \dots, z_n\} \} \tag{8}$$

and d is the usual Cartan differential. Under these hypothesis we have the following:

Theorem 1.1. *For generic $m_1, \dots, m_n$ and κ, $H^0 = 0$ and dim $H^1 = n - 1$. Moreover the differential forms $\omega_l = \Phi \frac{dt}{t - z_l}$, $l = 1, \dots, n - 1$, form a basis in H^1.*

Remark 1.3. *From the previous theorem follows that I^γ for $\gamma = \gamma_1, \dots, \gamma_{n-1}$, consists of integrals of closed forms.*

Let $H_1 = (H^1)^*$ be the dual space of the first cohomology group. Each interval γ_m, $m = 1, ..., n-1$, defines a linear functional on Ω^1:

$$[\gamma_m] : f\Phi dt \longmapsto \int_{\gamma_m} f\Phi dt.$$

By the Stokes theorem we have that $\int_{\gamma_m} d(f\Phi) = 0$, hence $[\gamma_m]$ defines an element of H_1. We have that:

Theorem 1.2. *The elements* $[\gamma_m]$, $m = 1, ..., n-1$, *form a basis in* H_1.

Prof. The prof of the theorem is based on the following formula:

$$\det_{1\leq l,m\leq n-1}\Big(\frac{m_l}{\kappa}\int_{\gamma_m}\Phi\frac{dt}{t-z_l}\Big)\frac{\Gamma(\frac{m_1}{\kappa}+1)\cdot ..\cdot\Gamma(\frac{m_n}{\kappa}+1)}{\Gamma(\frac{m_1+...+m_n}{\kappa}+1)}\prod_{i\neq j}(z_i-z_j)^{\frac{m_j}{\kappa}}. \tag{9}$$

Example 1.1. *Let's suppose* $n = 2$ *and* $(z_1, z_2) = (0, 1)$. *Then the formula 9 becomes:*

$$b\int_0^1 t^a(1-t)^{b-1}dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}. \tag{10}$$

Remark 1.4. *In the following table we summarize the main relations between conformal field theory and the geometry of the classical KZ equation:*

Conformal Field Theory	*Geometry of hypergeometric functions*
Space of Conformal Blocks	*Cohomology space* H^1 *associated to the function* $\Phi(t,z)$
KZ equation	*Gauss-Manin connection*
Solutions to KZ equation	*Cycles in* H_1

In what follows we proceed to the quantization of the geometry associated to the KZ equation.

2. Geometric Construction of Quantized KZ Equation.

We know that KZ can be realized as differential equation for integrals of the basic closed differential one forms over cycles. It turns out that there is a quantization of all those geometric object leading to a geometric construction of the q-KZ equation. Now we want introduce the p-analogous of the main ingredients of the classical picture. Let's start to define the p-analogous of the function $\Phi(t,z) = \prod_{l=1}^n (t-z_l)^{\frac{m_l}{\kappa}}$.

The function $\Phi_p(t,z)$: first we observe that the monomial $T^{\frac{m}{\kappa}}$ satisfies the following differential equation:

$$\frac{dY}{dT} = \frac{\frac{m}{\kappa}}{T}Y(T). \tag{11}$$

The analogous of 11 is:

$$y(t+p) = \frac{t+m}{t-m}y(t). \tag{12}$$

In fact if we pose $t = ST$ and and we define $Y(T) = y(ST)$, then:

$$Y\Big(T+\frac{p}{S}\Big) = \Big(1+\frac{2m}{S}\frac{1}{T-\frac{m}{S}}\Big) \tag{13}$$

and as $S \longrightarrow \infty$ we get that 13 turns into the equation:

$$\frac{dY}{dT} = \frac{\frac{2m}{p}}{T} Y(T). \tag{14}$$

p-Solutions: the solution of the equation 12 is given by the following:

$$y(t,m) = \Gamma(\frac{t+m}{p})\Gamma(1-\frac{t-m}{p})e^{-\frac{\pi i t}{p}},$$

so the p-analogous of the Master function will be given by:

$$\Phi_{(p)} = \prod_{l=1}^{n} y(t-z_l, m_l) = \prod_{l=1}^{n} \Gamma(\frac{t-z_l+m_l}{p})\Gamma(1-\frac{t-z_l+m_l}{p})e^{-\frac{\pi i t}{p}}.$$

Singularities of the function $\Phi_p(t,z)$: these correspond to the points:

$$t = z_l - m_l - Np$$

$$t = z_l + m_l + (N+1)p$$

where $l = 1, ..., n$, $N = 0, 1, 2, ...$. We will also introduce the following:

Notations 2.1. $\widetilde{Sing}$=$\{t \in \mathbb{C} |\ t = z_l - m_l + (N+1)p,\ t = z_l + m_l - Np,\ l = 1, ..., n,\ N = 0, 1, 2, ...\}$

The complex: now we define the p-analogous of the complex given in 6. Let's introduce the space of the 0 and 1 p-form:

$\Omega^0_{(p)} =$

$\Omega^1_{(p)} = \{\Phi_{(p)} f(t) |\ f$ is a rational function regular on $\mathbb{C} \backslash \widetilde{Sing}$ an with first order poles on it$\}$,

and let the p differential $\mathcal{D}_p : \Omega^0_{(p)} \longrightarrow \Omega^1_{(p)}$ defined by the following:

$$(\mathcal{D}_p h)(t) = h(t+p) - h(t). \tag{15}$$

Starting from these data we have the following:

Definition 2.1. *the p analogous of the complex 6 is:*

$$0 \longrightarrow \Omega^0_p \xrightarrow{\mathcal{D}_p} \Omega^1_p \longrightarrow 0. \tag{16}$$

The first cohomology group is given by:

$$H^1_{(p)} = \Omega^1_{(p)} \backslash \mathcal{D}_p \Omega^{(0)}_{(p)},$$

and the first homology group will be:

$$H^{(p)}_1 = \left(H^1_{(p)}\right)^*.$$

Theorem 2.1. 1. *For generic $z_1, ..., z_n, p, m_l$, $H^1_{(p)}$ is (n-1)-dimensional and the following 1 forms form a basis:*

$$\omega_l = \frac{1}{t-z_l-m_l} \cdot \frac{t-z_{l-1}+m_{l-1}}{t-z_{l-1}-m_{l-1}} \cdot .. \cdot \frac{t-z_1+m_1}{t-z_1-m_1} dt.$$

2. *Let $t = ST$, $z_j = SZ_j$, where $S >> 1$ and T, Z_j are new variables. As $S \longrightarrow \infty$ the p-complex turns into the de Rham complex of $\prod_{l=1}^{n}(T - Z_l)^{\frac{2m_l}{p}}$ and $\{\omega_l\}_{l=1,..,n}$ turn into $\left\{\frac{dT}{T-Z_l}\right\}_{l=1,..,n}$ which give a basis in H^1 of the de Rham complex.*

2.1. **p-Homology Theory.** In this subsection we will construct linear functionals on the space $\Omega^1_{(p)} \backslash \mathcal{D}_p$, i.e linear functionals:

$$l : \Omega^1_{(p)} \longrightarrow \mathbb{C}$$

such that:

$$l(g(t+p) - g(t)) = 0, \ \forall g \in \Omega^1_{(p)}.$$

Naive approch: let $\xi \in \mathbb{C}$ and $h(t) = \Phi_{(p)}(t)f(t)$.

Definition 2.2 (Jackson's Integral)**.** *The Jackson integral of the function $h = h(t)$ over $[\xi]_p$ is given by the following formula:*

$$\int_{[\xi]_p} h(t)dt = p \sum_{l=-\infty}^{\infty} h(\xi + lp). \tag{17}$$

Properties 2.1. 1. *If $h(t) = g(t+p) - g(t)$ then:*
$\displaystyle\int_{[\xi]_p} h(t)dt = p \sum_{l=\infty}^{\infty} \Big(g(\xi + (l+1)p) - g(\xi + lp)\Big) = 0;$
2. $\displaystyle\lim_{p \to 0} \int_{[\xi]_p} h(t)dt = \int_{\xi + line\ \mathbb{R}p} h(t)dt.$

In our case this definition does not work since the integral 17 is divergent.
Right approch: let's choose our data $z_1, ..., z_n, m_1, ..., m_n, p$ such that the points z_i, for $i = 1, ..., n$, lie on the immaginary line and $m_1, ..., m_n, p \in \mathbb{R}$, with $m_1, ..., m_n, p < 0$. For a given $m \in \{1, ..., n-1\}$ let's introduce the following function:

$$\begin{array}{rcl} \mathbb{G}_m : \mathbb{C} & \longrightarrow & \mathbb{C} \\ t & \longmapsto & exp(\frac{2\pi i m t}{p}) \end{array}$$

Remark 2.1. *The $\mathbb{G}_m$ are p-periodic functions.*

Now for every m let's cosider the following functionals:

$$\begin{array}{rcl} [\mathbb{G}_m] : \Omega^1_{(p)} & \longrightarrow & \mathbb{C} \\ \Phi_{(p)}f & \longmapsto & \int_{i\mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t,z)dt \end{array}$$

Remark 2.2. *The functionals $[\mathbb{G}_m]$ are the p analogous of the intervals $[\gamma_m]$.*

In what follows we will describe some of the properties of the functionals $[\mathbb{G}_m]$ just introduced.

Properties 2.2. 1. *The functionals* $[\mathbb{G}_m]$ *are well defined if* $m \in \{1, ..., n-1\}$ *and are not defined if* $m \notin \{1, ..., n-1\}$.
2. $[\mathbb{G}_m]|_{\mathcal{D}_p\Omega^0_{(p)}} \equiv 0$.
From the previous property we get the following:

Corollary 2.1. *The functionals* $[\mathbb{G}_m]$, $m \in \{1, .., n-1\}$, *define elements in* $H_1^{(p)}$.

3. *If* $z_l = SZ_l$, *then:*

$$\lim_{S \to +\infty} [\mathbb{G}_m] = [z_m, z_{m+1}]$$

in the following sense:

$$\int_{i\mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t,z)\omega_l dt \longrightarrow C_m(S,p)\Big(\int_{Z_m}^{Z_{m+1}} \prod_{j=1}^{n} (T - Z_j)^{\frac{2m_j}{p}} \frac{dT}{T - Z_l} + \mathcal{O}(S^{-1})\Big), \forall m.$$

Here $C_m = C_m(S,p)$ *is some explicitely known function.*
4. *Let* $u_m = u_m(z_1, ..., z_n)$, *for* $m = 1, .., n-1$, *be the following function:*

$$u_m(z_1, ..., z_n) = \Big(\int_{i\mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t,z)\omega_1 dt, ..., \int_{i\mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t,z)\omega_{n-1} dt\Big). \tag{18}$$

Then for every $m \in \{1, .., n-1\}$, *18 is a solution of the q-KZ equation with values in* $M_{m_1}\otimes, .., \otimes M_{m_n}[|m| - 2]$ *with* $\mu = 0$. *The functions* $u_m = u_m(z_1, ..., z_n)$ *define by 18 are called* q-hypergeometric *functions associated with* $\Phi_{(p)}$.
5. *The functionals* $[\mathbb{G}_m]$, $m \in \{1, .., n-1\}$, *form a basis in the homology space* $H_1^{(p)}$.

Prof.
1. Follows applying the Stirling formula to the definition of the p-master function $\Phi_{(p)}$:

$$\Phi_{(p)} = \prod_{l=1}^{n} y(t - z_l, m_l) = \prod_{l=1}^{n} \Gamma\Big(\frac{t - z_l + m_l}{p}\Big)\Gamma\Big(1 - \frac{t - z_l + m_l}{p}\Big)e^{-\frac{\pi i t}{p}}.$$

2. Follows from the periodicity of the functions $\mathbb{G}_m$, in fact since these functions are p periodic we have:

$$\int_{i\mathbb{R}} \mathbb{G}_m(t)\Big[\Phi_{(p)}(t+p,z)f(t+p) - \Phi_{(p)}(t,z)f(t)\Big]dt = \int_{i\mathbb{R}+p} \mathbb{G}_m(t)\Phi_{(p)}(t,z)f(t)dt$$

$$- \int_{i\mathbb{R}} \mathbb{G}_m(t)\Phi_{(p)}(t,z)f(t) = 0.$$

3. Follows from the Stirling formula.
5. This is a consequence of the following formula:

$$\det_{1 \le l,m \le n-1} \Big(\frac{2m_l}{p} \int_{\mathbb{R}} \mathbb{G}_m \Phi_{(p)}\omega_l dt\Big) =$$

$$(2\pi i)^{\frac{n(n-1)}{2}} \frac{\prod_{l=1}^{n-1} \Gamma\Big(1 + \frac{2m_l}{p}\Big)}{\Gamma\Big(1 + \frac{2}{p}\sum_{j=1}^{n-1} m_j\Big)} \prod_{j<l} \Gamma\Big(\frac{z_j + m_j - z_l + m_l}{p}\Big)\Gamma\Big(1 + \frac{z_l + m_l - z_j + m_j}{p}\Big).$$

Example 2.1. *For $n = 2$ from the previous determinant we get the following classical formula:*

$$\int_{-i\infty}^{i\infty} \Gamma(a+t)\Gamma(a+t)\Gamma(c-t)\Gamma(d-t)dt = 2\pi i \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}. \tag{19}$$

Remark 2.3. *19 is the q-analogous of the β-function and it is called is called* Barnes' *formula.*

In the next section we will describe explicit solutions for the q-KZ equation in terms of q-hypergeometric functions.

3. q-KZ Equation: sl_2 case.

Let's start with some preliminary remarks about the action of the twisted symmetric group S^k on space $\mathcal{F}_k = \{\text{function of k-variables } t_1, .., t_k\}$.
Given $h = h(x)$ a function such that

$$h(x)h(-x) = 1. \tag{20}$$

let's define:

$$s_j : f(t_1, .., t_k) \longmapsto f(t_1, .., t_{j+1}, t_j, .., t_k)h(t_i - t_j). \tag{21}$$

Definition 3.1.

$$S^k = \{s_j : \mathcal{F}_k \longrightarrow \mathcal{F}_k \text{ such that 21 is satisfied}\}.$$

Lemma 3.1. *The maps s_j, $j = 1, ..., k-1$, given by 21, define an action of S^k on the space $\mathcal{F}_k$.*

Prof. It suffices check that $s_j^2 = 1$ and $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ for every
$j \in \{1, .., k-1\}$.

Example 3.1. *The following are examples of functions that satisfy the condition 20.*

$$h(x) = \frac{x-1}{x+1}, \tag{22}$$

$$h(x) = \frac{\sin(\frac{\pi(x-1)}{p})}{\sin(\frac{\pi(x+1)}{p})}, \tag{23}$$

$$h(x) = \frac{\theta(x-a,\tau)}{\theta(x+a,\tau)}. \tag{24}$$

Remark 3.1. *In the formula 24 $\theta = \theta(x, \tau)$ is the first Jacobi theta-function.*

Remark 3.2. *The functions just described in the previous example enter in the decription of the solutions of the q-KZ equation. In particular the functions 22 and 23 are used in the* rational *case. For the* trigonometric *case we need the functions 23 and 24, while for the elliptic one we need only 24.*

We need the following:

Definition 3.2. *Let $F^k_{m_1,\dots,m_n}(z_1,\dots,z_n) \subset \mathcal{F}_k$ be the $\mathbb{C}$ vector space of the functions of the form:*

$$P(t_1,..,t_k)\prod_{l=1}^{n}\prod_{i=1}^{k}\frac{1}{t_i-z_l-m_l}\prod_{1\le i<j\le k}\frac{t_i-t_j}{t_i-t_j+1},$$

where $P=P(t_1,..,t_k)$ is a symmetric polynomial (in the standard sense), of degree less than n in each of the k variables.

Example 3.2. *For $k=1$ we have that the function in $F^k_{m_1,\dots,m_n}(z_1,\dots,z_n)$ are of the form:*

$$\frac{P(t)}{(t-z_1-m_1)\cdot..\cdot(t-z_n-m_n)},$$

with $\deg P<n$.

Definition 3.3. *Let's define:*

$$F_{m_1,\dots,m_n}(z_1,\dots,z_n)=\bigoplus_{k=0}^{\infty}F^k_{m_1,\dots,m_n}(z_1,\dots,z_n),$$

where by definition, $F^0_{m_1,\dots,m_n}(z_1,\dots,z_n)=\mathbb{C}$.

The spaces $F^k_{m_1,\dots,m_n}(z_1,\dots,z_n)$ have the following:

Properties 3.1. 1. *$\dim F^k_{m_1,\dots,m_n}=\dim M_{m_1}\otimes\dots\otimes M_{m_n}[|m|-2k]$;*
2. *$F^k_{m_1,\dots,m_n}$ consists of functions that are symmetric with respect to the action of S^k with h given by 22.*

Example 3.3. *If*

$$g=\frac{1}{t_1-z_1-m_1}\cdot\frac{1}{t_2-z_1-m_1}\cdot\frac{t_1-t_2}{t_1-t_2+1},$$

then:

$$s_1(g)=\frac{1}{t_1-z_1-m_1}\cdot\frac{1}{t_2-z_1-m_1}\cdot\frac{t_2-t_1}{t_2-t_1+1}\cdot\frac{t_1-t_2-1}{t_1-t_2+1}=$$
$$\frac{1}{t_1-z_1-m_1}\cdot\frac{1}{t_2-z_1-m_1}\cdot\frac{t_2-t_1}{t_2-t_1+1}=g.$$

3. $F^k_{m_1,\dots,m_n}(z_1,\dots,z_n)=F^k_{m_1,..,m_{j+1},m_j,..,m_n}(z_1,..,z_{j+1},j,..,z_n)$.

Example 3.4. *For $n=1$ we have that the space $F^k_m(z)$ is one dimensional and it is spanned by:*

$$\omega_k(t_1,..,t_k,z)=\prod_{a<b}\frac{t_a-t_b}{t_a-t_b+1}\prod_{a=1}^{k}\frac{1}{t_a-z-m}. \tag{25}$$

Let's introduce the following map:

$$\begin{array}{ccc} F^{k'}_{m_1,...,m'_n}(z_1,...,z'_n)\otimes F^{k''}_{m_{n'+1},...,m_{n'+n''}}(z_{n'+1},...,z_{n'+n''}) & \longrightarrow & F^{k'+k''}_{m_1,...,m_{n'+n''}}(z_1,...,z_{n'+n''}) \\ f\otimes g & \longmapsto & k(t_1,...,t_{k'+k''}) \end{array}$$

where

$$k(t_1,...,t_{k'+k''})=\frac{1}{k'!k''!}Sym_h\Big(f(t_1,..,t'_k)g(t_{k'+1},..,t_{k'+k''})\cdot\prod_{k<j\le k'+k''}\cdot\prod_{1\le l\le n'}\frac{t_j-z_l+m_l}{t_j-z_l-m_l}\Big)$$

Example 3.5. *For $k'=0$ and $k''=1$ we have:*

$$\begin{array}{ccc} F^0_{m_1}(z_1)\otimes F^1_{m_2}(z_2) & \longrightarrow & F^1_{m_1,m_2}(z_1,z_2) \\ 1\otimes\frac{1}{t-z_2-m_2} & \longmapsto & \frac{t-z_1+m_1}{t-z_1-m_1}\frac{1}{t-z_2-m_2} \end{array}$$

and

$$\begin{array}{ccc} F^1_{m_1}(z_1)\otimes F^0_{m_2}(z_2) & \longrightarrow & F^1_{m_1,m_2}(z_1,z_2) \\ \frac{1}{t-z_1-m_1}\otimes 1 & \longmapsto & \frac{1}{t-z_1-m_1} \end{array}$$

Claim 3.1. *For generic values of $z_1,..,z_m$:*

$$\bigoplus_{k'+k''=k}F^{k'}_{m_1,...,m'_n}(z_1,...,z'_n)\otimes F^{k''}_{m_{n'+1},...,m_{n'+n''}}(z_{n'+1},...,z_{n'+n''})\longrightarrow F^{k}_{m_1,...,m_{n'+n''}}(z_1,...,z_{n'+n''})$$

is an isomorphism.

Now we can define:

Definition 3.4.

$$\begin{array}{ccc} \omega(z):(V_m)^* & \longrightarrow & F_m(z) \\ (f^kv)^* & \longmapsto & \omega_k \end{array}$$

The map ω_z induces a map:

$$\omega(z_1,..,z_n):V^*_{m_1}\otimes...\otimes V^*_{m_n}\longrightarrow F_{m_1,...,m_n}(z_1,..,z_n). \qquad (26)$$

Claim 3.2. *For generic $z_1,..,z_n$, the map $\omega=\omega(z_1,...,z_n)$, 26 is an isomorphims.*