LECTURE 10

During the last lecture we introduced the elliptic analoguos of the KZ differential equations. In particular, given $\tau \in \mathbb{H}_+$ and $\lambda \in \mathbb{C}$, and we studied the following equation:

$$2\pi i\kappa \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \lambda^2} + p(p+1)\rho'(\lambda,\tau)u, \tag{1}$$

where $\kappa, p \in \mathbb{Z}_+$ and $\kappa \geq 2p+2$ and ρ is the logaritmic derivative of the first Jacobi theta function:

$$\rho = \frac{\theta'(\lambda,\tau)}{\theta'(\lambda,\tau)}.$$

We also introduced the space of conformal blocks, i.e the space of holomorphic solution of 1 that satisfy the following properties:
1.$u(\lambda+2,\tau) = u(\lambda,\tau)$;
2.$u(\lambda+2,\tau) = e^{-2\pi i\kappa(\lambda+\tau)}u(\lambda,\tau)$;
3.$u(-\lambda,\tau) = (-1)^{p+1}u(\lambda,\tau)$;
4.$u(\lambda,\tau) = \mathcal{O}((\lambda-m-n\tau)^{p+1}))$ as $\lambda \longrightarrow n+m\tau \in \mathbb{Z}+\tau\mathbb{Z}$.
During this lecture we will study the action of the modular group on the space of conformal blocks. To this end we remind the integral formulas introduced at the end of the last lecture. We defined:

$$E(t,\tau) = \frac{\theta(t,\tau)}{\theta'(0,\tau)},$$

$$\sigma_\lambda(t,\tau) = \frac{\theta(\lambda-t,\tau)\theta'(0,\tau)}{\theta(\lambda,\tau)\theta(t,\tau)},$$

and:

$$\Phi(t_1,...,t_p) = \prod_{j=1}^{p} E(t_j,\tau)^{-\frac{2p}{\kappa}} \prod_{1\leq i<j\leq p} E(t_i-t_j,\tau)^{\frac{2}{\kappa}}.$$

If $\Delta_k = \{(t_1,...,t_k\}|0 \leq t_k \leq ... \leq t_1 \leq 1\}$ is the standard k-symplex and $\tilde{\Delta}_k$ is the image of Δ_k under the map:

$$(t_1,...,t_k) \longrightarrow (\tau t_1,...,\tau t_k),$$

we have that:

$$J^{[k]}_{n,\kappa}(\lambda,\tau) = \int_{\Delta_k}\prod_{j=1}^{k}\sigma_\lambda(t_j,\tau)dt_j\int_{\tilde{\Delta}_{p-k}}\prod_{j=k+1}^{p}\sigma_\lambda(t_j,\tau)dt_j\Phi(t_1,...,t_p,\tau)\theta_{\kappa,n}(\lambda+\frac{2}{\kappa}\sum_{j=1}^{p}t_j,\tau), \tag{2}$$

where $\theta_{\kappa,n}$, is a theta function of level κ.
We also defined the functions:

$$u^{[k]}_{n,\kappa}(\lambda,\tau) = J^{[k]}_{n,\kappa}(\lambda,\tau) + (-1)^{p+1}J^{[k]}_{n,\kappa}(-\lambda,\tau)$$

we observed that:

Properties 0.1. *1.* $u^{[k]}_{n,\kappa}(\lambda,\tau) = u^{[k]}_{n,2+\kappa}(\lambda,\tau)$;
2. $u^{[k]}_{n,\kappa}(\lambda,\tau) = -q^{2k(n+p-k)}u^{[k]}_{\kappa,-n-2(p-k)}$

and we stated the following important:

Theorem 0.1 (on Basis)**.** *1. The set* $u^{[p]}_{n,\kappa}(\lambda,\tau)$, $n = p+1, ..., \kappa-p-1$, *is a basis in the space of conformal blocks.*
2. $u^{[p]}_{n,\kappa} \equiv 0$ *if* $0 \leq n \leq p$ *or* $\kappa - p \leq n \leq \kappa$.

In what follows it will be convenient use the following:

Notations 0.1.

$$q = e^{\frac{\pi i}{\kappa}}, \tag{3}$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{4}$$

The following theorem describes some important relations satisfied by the functions $u^{[k]}_{\kappa,n}$ with different k.

Remark 0.1. *We recall that k represents the number of horizontal integrations performed in 2.*

Theorem 0.2 (Stokes Theorem)**.** *For $0 \leq k \leq p-1$ and any n, we have:*

$$[p-k]\Big(q^{m+p-k} - q^{-m-p+k}\Big)u^{[k]}_{\kappa,m} = q^{-m-k-1}[k+1]\Big(q^{-2(k+1)}u^{[k+1]}_{\kappa,m+2} - u^{[k+1]}_{\kappa,m}\Big). \tag{5}$$

Prof. We will give the idea of the prof of the theorem. Let $p = 1$ and $k = 0$. In these hypothesis the integrals are 1 dimensional:

$$\int_{interval} \Phi(t,\tau)\sigma_\lambda(t,\tau)\theta_{\kappa,m}(\lambda + \frac{2}{\kappa}t,\tau)dt. \tag{6}$$

Now will perform the integration on the foundamental domain $(1,\tau)$. Since the integrand is holomorphic we get that:

$$I_1 + I_2 + I_3 + I_4 = 0$$

and from the periodicity properties of the θ-functions, we get $I_1 = c_1 u^{[1]}_m$, $I_4 = c_4 u^{[0]}_m$, $I_2 = c_2 u^{[0]}_m$, $I_3 = c_1 u^{[1]}_{m+2}$.

Remark 0.2. *The Stokes theorem allows to express an arbitrary function $u^{[k]}_{\kappa,m}$ in terms of basic ones.*

1. Modular Transformations of Conformal Blocks.

Let's introduce the following two transformations:

$$(Tu)(\lambda,\tau) = u(\lambda,\tau+1), \tag{7}$$

$$(Su)(\lambda,\tau) = e^{-\pi i\kappa\frac{\lambda^2}{2\tau}}\tau^{-\frac{1}{2}}u(\frac{\lambda}{\tau},-\frac{1}{\tau}). \tag{8}$$

We have the following:

Theorem 1.1. *If $u = u(\lambda,\tau)$ is a solution of the KZB heat equation the Tu and Su a also solutions of the same equation. Moreover the maps T and S preserve the space of conformal blocks.*

We also have the following:

Proposition 1.1. *The restrictions of the maps T and S to space of conformal blocks satisfy the following identities:*

$$S^2 = (-1)^p i q^{-p(p+1)} Id, \tag{9}$$

$$(ST)^3 = (-1)^p i q^{-p(p+1)} Id, \tag{10}$$

i.e they define a projective representation of the modular group on the space of conformal blocks.

Now we want to discuss the following problem: find an explicit description of the operators S and T.

The solution of this problem is contained in the following two lemmas:

Lemma 1.1.

$$Tu^{[p]}_{\kappa,n}(\lambda,\tau) = q^{\frac{n^2}{2}}u^{[p]}_{\kappa,n},$$

and

Lemma 1.2.

$$Su^{[p]}_{\kappa,n}(\lambda,\tau) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2\kappa}}\sum_{m=0}^{2\kappa-1} q^{-mn}u^{[0]}_{\kappa,m}(\lambda,\tau). \tag{11}$$

Remark 1.1. *The proofs of the previous lemmas are straigthforward. We observe that in 11 the RHS is given in terms of functions obtained taking p-vertical integrations while the LHS is a function obtained taking p-horizontal integrations. To get the formula 11 we need to apply p-times the Stokes theorem to get the form:*

$$Su^{[p]}_{\kappa,n} = \sum_m S_{m,n}u^{[p]}_{\kappa,m}. \tag{12}$$

To describe the coefficients $S_{n,m}$ in 12 of the previous remark we need the following:

Theorem 1.2.

$$S_{n,m} = \sqrt{\frac{-i}{2\kappa}}q^{p(n-p)-\frac{p(p+1)}{2}}(q^{-m}-q^m)\prod_{j=1}^{p}(q^{-n+j}-q^{n-j})P^{(p+1)}_{n-p-1}(m). \tag{13}$$

Definition 1.1. *The function $P^{(p+1)}_{n-p-1}(x)$ is the Macdonald polynomial of level $p+1$ and degree $n-p-1$ associated with the Lie algebra sl_2.*

2. Macdonald Polynomials.

The Macdonald polynomials of level k associated with sl_2 are x-even polynomials of q^{mx}, $m \in \mathbb{Z}$ defined by the following rules:
1. $P^{(k)}_0(x) \equiv 1$ and $P^{(k)}_n(x) = q^{nx} + q^{-nx}$+lower order terms;
2. $\langle P^{(k)}_m(x), P^{(k)}_n(x)\rangle = 0$ if $m \neq n$, where:

$$\langle f(x), g(x)\rangle = costant\ term\ \Big(f(x)g(x)\prod_{j=0}^{k-1}(1-q^{2(j+x)})(1-q^{2(j-x)}\Big).$$

Example 2.1. *For $n > 0$ we have that:*

$$P^{(0)}_n(x) = q^{nx} + q^{-nx}.$$

In the following we will define an operator that play an importan role in the theory of Macdonald polynomials:

Definition 2.1 (Shift Operator)**.** *The shift operator D, in the sl_2-case, is defined by:*

$$Df(x) = \frac{f(x-1)+f(x+1)}{q^x + q^{-x}}. \tag{14}$$

where $f = f(x)$ is a given function.

Given the shift operator it is possible to obtain Macdonald polynomials of higher level knowing polynomials of lower level. In fact we have the following:

Theorem 2.1 (Askey-Ismail)**.**

$$(DP^{(k)}_n)(x) = (q^{-n} - q^n)P^{(k+1)}_{n-1}(x). \tag{15}$$

Example 2.2. *For $n > 0$ we have that:*

$$P^1_{n-1}(x) = \frac{q^{nx} + q^{-nx}}{q^x + q^{-x}}. \tag{16}$$

This follows applying the definition of the shift operator and previous theorem to $P^{(0)}_n(x) = q^{nx} + q^{-nx}$. From 16 we get that $P^{(2)}_0(x) = 1$, $P^{(2)}_1(x) = q^x + q^{-x}$ and that

$$P^{(2)}_2(x) = q^{2x} + \frac{1+2q^2+q^4}{1+q^2+q^4} + q^{-2x}.$$

The following theorem describes the relation between the shift operator and the modular transformations of the basic elliptic hypergeometric functions:

Theorem 2.2. *For $0 \le k \le p$ we have:*

$$Su^{[p]}_{\kappa,n}(\lambda,\tau) = \sqrt{\frac{-i}{2\kappa}}\sum_{m=-p+2k+1}^{\kappa-p-1} q^{pn-km-\frac{k(k+1)}{2}}\begin{bmatrix} p \\ k \end{bmatrix}_q^{-1}(q^{m-p+k} - q^{m+p-k})$$

$$\prod_{j=1}^{k}(q^{-n+j}-q^{n-j})P^{(k+1)}_{n-k+1}(m+p-k)u^{[k]}_m(\lambda,\tau).$$

Remark 2.1. *Comparing this formula and the Stokes theorem we see that the Stokes formula gives the shift operator relation 15.*

3. Trace of Intertwining Operators.

In what follows $q=e^{\frac{\pi i}{\kappa}}$. The quantum group $U_q(sl_2)$ has generators E, F and q^{ch}, where $c\in\mathbb{C}$, with relations:
1. $q^{ch}q^{c'h}=q^{(c+c')h}$;
2. $q^{ch}Eq^{-ch}=q^{2ch}E$;
3. $q^{ch}Fq^{-ch}=q^{-2ch}F$;
4. $EF-FE=\dfrac{q^h-q^{-h}}{q-q^{-1}}$. The comultiplication is defined by the following:
$\Delta(E)=E\otimes q^h+1\otimes E$, $\Delta(F)=F\otimes 1+q^{-h}\otimes F$, $\Delta(q^{ch})=q^{ch}\otimes q^{ch}$. Let's identify weights for $U_q(sl_2)$ with complex number as follows and let's say that a vector v in a $U_q(sl_2)$-module has weight $\nu\in\mathbb{C}$ if $q^hv=q^\nu v$. Let M_μ the Verma module over $U_q(sl_2)$ with highest weight μ and let v_μ be its highest vector. Let k be a non negative integer such that $\kappa\geq 2k+2$, U an irreducible finite dimensional representation of $U_q(sl_2)$ of weight $2k$ and let's denote with $U[0]$ the zero weight subspace of U. Let $u\in U[0]$. For generic μ let:

$$\Phi^\nu_\mu: M_\mu\longrightarrow M_\mu\otimes U \tag{17}$$

be the intertwining operator defined by:

$$\Phi^\nu_\mu v_\mu = v_\mu\otimes u+\frac{Fv_\mu}{[-\mu]}\otimes Eu+...+\frac{F^jv_\mu}{[j]!(-\mu,q)_j}\otimes E^ju+... \tag{18}$$

where $(-\mu,q)_j=[n][n+1]...[n+j-1]$.
Introduce now and End(U[0])-valued function $\psi^{(k)}(q,\nu,\mu)$ defined by:

$$\psi^{(k)}(q,\nu,\mu)u=Tr|_{M_\mu}(\Phi^\nu_\mu q^{\nu h}). \tag{19}$$

Since $U[0]$ is 1 dimensional, this function is scalar. It is is possible give an explicit description of this function:

Theorem 3.1 (Etingof-Varchenko)**.** *The function $\psi^k(q,\nu,\mu)$ is given by the formula:*

$$\psi^{(k)}(q,\nu,\mu)=q^{\nu\mu}\sum_{j=0}^{k}(-1)^jq^{\frac{j(j-3)}{2}}(q-q^{-1})^{-j-1}\frac{[k+j]!}{[j]![k-j]!}\frac{q^{-j\mu-(j-1)\nu}}{\prod_{l=0}^{j-1}[\mu-l]\prod_{l=0}^{j}[\nu-l]}.$$

Let's introduce now the renormalized trace functions:

Definition 3.1.

$$\Psi^{(k)}(q,\nu,\mu)=\prod_{j=1}^{k}\left(\frac{q^{\mu+1-j}-q^{-\mu-1+j}}{q^{\nu+j}-q^{-\nu-j}}\right).$$

Remark 3.1. *The functions $\Psi^{(k)}$ are holomorphic functions of μ.*

The relation between Macdonald polynomials and the trace functions is given by the following:

Theorem 3.2 (Etingof-Styrkas)**.**

$$\Psi^{(k)}(q^{-1},-m-p+k,n-1)-\Psi^{(k)}(q^{-1},-m-p+k,-n-1)=P^{(k+1)}_{n-k-1}(x)\prod_{j=1}^{k}(q^{-n+2j}-q^{n}).$$

Now we will explain how the trace functions enter in the picture. Our starting point was the formula 11:

$$Su^{[p]}_{\kappa,n}(\lambda,\tau)=\frac{e^{-\frac{\pi i}{4}}}{\sqrt{2\kappa}}\sum_{m=0}^{2\kappa-1}q^{-mn}u^{[0]}_{\kappa,m}(\lambda,\tau).$$

Then using the Stokes theorem:

$$[p-k]\Big(q^{m+p-k}-q^{-m-p+k}\Big)u^{[k]}_{\kappa,m}=q^{-m-k-1}[k+1]\Big(q^{-2(k+1)}u^{[k+1]}_{\kappa,m+2}-u^{[k+1]}_{\kappa,m}\Big),$$

we obtained for any $0\le k\le p$ an expression of the form:

$$Su^{[p]}_{\kappa,n}(\lambda,\tau)=\sqrt{\frac{-i}{2\kappa}}\Big[\sum_{m=-p+2k+1}^{k-p-1}f^{(k)}_{m,n}u^{[k]}_{\kappa,m}(\lambda,\tau)+\sum_{m=\kappa-p+2k+1}^{2\kappa-p-1}f^{(k)}_{m,n}u^{[k]}_{\kappa,m}(\lambda,\tau)\Big]$$

for suitable numbers $f^{(k)}_{m,n}$ given by the following:

Theorem 3.3.

$$f^{(k)}_{m,n}=q^{pn-km-k(k+1)}(q^{m+p-k}-q^{-m-p+k})\begin{bmatrix}p\\k\end{bmatrix}_q^{-1}\Psi^{(k)}(q^{-1},-m-p+k,n-1). \tag{20}$$

4. q-KZ Equations.

In what follows we will discuss the theory of the *q-hypergeometric* functions and the related KZ-equations. These are *difference equations* that can be thought as a quantized version of the previous ones. In the "classical" case we had that given $V=V_1\otimes...\otimes V_n$, tensor product of sl_2 modules, we could define the following system of differential KZ-equations:

$$\kappa\frac{\partial u}{\partial z_i}=\sum_{j\neq i}\frac{\Omega^{(i,j)}}{z_i-z_j}u$$

where $u=u(z_1,...,z_n)$ is a V-valued function and $\Omega=e\otimes f+f\otimes e+\frac{1}{2}h\otimes h$ is the Casimir operator. These equations define a flat connection on the flat vector bundle $V\times\mathbb{C}^n\longrightarrow\mathbb{C}^n$, that has regular singularities along the diagonals. The monodromy of this connection can be described in terms of the universal R-matrix associated to the quantum group $U_q(sl_2)$, where $q=e^{\frac{2\pi i}{\kappa}}$, acting on the tensor product $V^q=V_1^q\otimes...\otimes V_n^q$, where the V_i^q are $U_q(sl_2)$-modules associated to the sl_2 modules V_i for $i=1,...,n$. Moreover the KZ equations can be also realized geometrically:
if we consider the map $p:\mathbb{C}^{k+n}\longrightarrow\mathbb{C}^n$ and the Master function $\Phi_{k,n}:\mathbb{C}^{k+n}\longrightarrow\mathbb{C}$,

then the KZ-equations can be identified with the Gauss-Manin connection of the vector bundle $\mathcal{H}$ whose whose fibers are the homology groups $H_k(p^{-1}(z), \Phi)$. Now we want to study the discrete version of all these objects. There are three versions of the classical KZ-equation:
1. rational;
2. trigonometric and;
3. elliptic.
In what follows we will describe the discretization of the rational KZ-system. Let's consider the tensor product of sl_2 modules $V = V_1 \otimes ... \otimes V_n$.

Definition 4.1. *The rational q-KZ equation on a V-valued function $\omega = \omega(z_1, ..., z_n)$ has the form:*

$$\omega(z_1, .., z_m + p, .., z_n) = K_m(z_1, ..., z_n, p)\omega(z_1, ..., z_n), \ \ m = 1, ..., n. \tag{21}$$

p is called the step of the equations and $K_m : V \longrightarrow V$ are linear operators.

Now we have the following data: the step map, a trivial fibration $\pi : V \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and the set of linear maps $\Big\{K_m(z, p)\Big\}_{m=1,...,n}$. These operators define and identification of the neighboring fibers w.r.t the lattice generated by $p\mathbb{Z}^n \subset \mathbb{C}^n$:

Definition 4.2. *We'll call this structure a* discrete connection *and we will say that such connection is flat if the translation along the sides of any elementary square gives the identity operator, i.e:*

$$K_i(z_1, .., z_j + p, .., z_n)K_j(z_1, ..., z_n)K_j(z_1, .., z_i + p, .., z_n)K_i(z_1, ..., z_n) \tag{22}$$

Remark 4.1. *The q-KZ discrete connection is flat and the q-KZ equations are the equations for flat sections of the discrete connection.*

In the next section we will discuss the explicit formulas for the q-KZ operators K_m.

5. Yangian $Y(gl_2)$ and Rational R-matrices.

Definition 5.1. *The Yangian $Y(gl_2)$ is an associative algebra with unit and generators $T^{(k)}_{i,j}$, $i, j = 1, 2$ $k = 1, 2, 3, ...$*

To write the relations let's introduce the generating series:

$$T_{i,j}(u) = \delta_{i,j} + \sum_{k=1}^{\infty} T^{(k)}_{i,j} u^{-k}. \tag{23}$$

The relations are given by the following:

$$(u - v)\Big[T_{i,j}(u), T_{k,l}(v)\Big] = T_{j,k}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v). \tag{24}$$

The Yangian is a Hopf algebra whose coproduct $\Delta : Y(gl_2) \longrightarrow Y(gl_2) \otimes Y(gl_2)$ is given by:

$$(\Delta T_{i,j})(u) = \sum_{k=1}^{2} T_{i,k}(u) \otimes T_{k,j}(u). \tag{25}$$

On $Y(gl_2)$ are also defined:
1. a family of homomorphims $\rho_z : Y(gl_2) \longrightarrow Y(gl_2)$, where:

$$(\rho_z T_{i,j})(u) = T_{i,j}(u-z), \tag{26}$$
$$\rho_z\Big(\frac{1}{u^k}\Big) = \frac{1}{u^k}\frac{1}{\Big(1-\frac{z}{u}\Big)^k}, \tag{27}$$

and:
2. the evaluation homomorphism $\epsilon : Y(gl_2) \longrightarrow U(sl_2)$, such that:

$$T^k_{1,1} = \delta_{1,k} h \qquad T^k_{1,2} = \delta_{1,k} f, \tag{28}$$

$$T^k_{2,1} = \delta_{1,k} e \qquad T^k_{2,2} = -\delta_{1,k} h. \tag{29}$$

If V is any sl_2-module:

Definition 5.2. *let's denote $V(z)$ the Yangian module defined via the homomorphism:*

$$\epsilon \circ \rho_z : Y(gl_2) \longrightarrow U(sl_2). \tag{30}$$

Now let V_1 and V_2 two highest weight sl_2 modules with highest weight vectors $v_1 \in V_1$ and $v_2 \in V_2$. For generic complex numbers $x, y \in \mathbb{C}$ the Yangian modules $V_1(x) \otimes V_2(y)$ and $V_2(y) \otimes V_1(x)$ are isomorphic.

Definition 5.3. *The rational R-matrix $R_{V_1,V_2}(x-y)$ can be defined as the unique element of End$(V_1 \otimes V_2)$, such that:*

1. $\qquad R_{V_1,V_2} : v_1 \otimes v_2 \longrightarrow v_1 \otimes v_2$
(identity operator on the tensor product of the highest weight vectors),

2. $\qquad V_1(x) \otimes V_2(y) \xrightarrow{R_{V_1,V_2}} V_1(x) \otimes V_2(y) \xrightarrow{P} V_2(y) \otimes V_1(x)$
(the composition $P \circ R_{V_1,V_2}$ is an isomorphism of Yangian modules).

Remark 5.1. *P is the permutation operator.*

Example 5.1. *Let $V_1 = V_2 = \mathbb{C}^2$, v_+, v_- a basis for $\mathbb{C}^2$ where v_+ is highest weight vector. In $V_1 \otimes V_2$ we have a basis given by $v_+ \otimes v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+$ and $v_- \otimes v_-$.*

Under these hypotesis we have that the R-matrix with respect to the ordered basis written above is:

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x}{x+1} & \frac{1}{x+1} & 0 \\ 0 & \frac{1}{x+1} & \frac{x}{x+1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{31}$$

Remark 5.2. *The R-matrices satisfy the QYE:*

$$R^{(12)}_{V_1,V_2}(x_1-x_2)R^{(13)}_{V_1,V_3}(x_1-x_3)R^{(23)}_{V_2,V_3}(x_2-x_3) = R^{(23)}_{V_2,V_3}(x_2-x_3)R^{(13)}_{V_1,V_3}(x_1-x_3)R^{(12)}_{V_1,V_2}(x_1-x_2)$$

Now let $V = V_1 \otimes ... \otimes V_n$ the tensor product of highest weight sl_2-modules. Then:

Definition 5.4. *The q-KZ operators K_m are defined by the following:*

$$K_m = R_{V_m,V_{m-1}}(z_m - z_{m-1} + p)...R_{V_m,V_1}(z_m - z_1 + p)e^{\mu h^{(m)}}$$
$$R_{V_m,V_n}(z_m - z_n)...R_{V_m,V_{m+1}}(z_m - z_{m+1})$$

Esercise 5.1. *Check that the q-KZ discrete connection defined by the operators K_m given in the previous definition is flat.*

Hint. *Use the fact that the R-matrices satisfy the QYE.*

Remark 5.3. *The rational q-KZ difference equations turn into the KZ differential equations under the following limiting procedure.*

Let $z_m = SZ_m$, $m = 1, ..., n$, with $S >> 1$ a number and Z_m a new variable and $\mu = \frac{\nu}{S}$, where ν is a new parameter. Introduce $u(Z) = \omega(Sz)$. Then the q-KZ equation is:

$$u(Z_1, ..., Z_m + \frac{p}{S}, ...Z_n) = K_m(SZ, p)u(Z), \;\; m = 1, ..., n \tag{32}$$

where:

$$K_m(SZ, p) = 1 + \frac{1}{S}\sum_{j\neq m} \frac{\Omega^{(m,j)} + c_{m,j}}{Z_m - Z_j} + \frac{\nu h^{(m)}}{S}, \tag{33}$$

for suitable numbers $c_{m,j}$. As $S \to \infty$ the difference equation 32 turns into a differential equation:

$$p\frac{\partial u}{\partial z_m} = \sum_{j\neq m} \frac{\Omega^{(m,j)} + c_{m,j}}{z_m - z_j} + \nu h^{(m)}, \;\; m = 1, ..., n. \tag{34}$$