VARCHENKO: LECTURE 1

Given $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{z} = (z_1, \ldots, z_n)$ fixed vectors in $\mathbb{C}^n$ and given fixed $\kappa \in \mathbb{C}$ we define the following multi-valued function and associated set of 1-forms:

$$\Phi \;=\; \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \; \prod_{l=1}^{n} (t - z_l)^{\frac{-m_l}{\kappa}}$$

$$\eta_j \;=\; \Phi \, \frac{dt}{t - z_j} \quad j = 1, \ldots, n$$

The 1-forms η_j are closed and, moreover, are cohomologically dependent as they satisfy the relation

$$m_1 \eta_1 + \cdots + m_n \eta_n \;=\; -\kappa d\phi.$$

Let $G = \pi_1(\mathbb{C} - \{z_1, \ldots, z_n\}, *)$ and choose any contour γ emanating from the base point * with the property that $[\gamma]$ resides in the commutator subgroup $[G, G]$. Since $[\gamma] \in [G, G]$ one can pull back the multi-valued 1-form η_j to a single valued 1-form on S^1 - after chosing a branch of η_j about the base point *. In this manner the contour integral $\int_\gamma \eta_j$ becomes well defined and its value will only depend on γ's homotopy class. Define I^γ to be the following holomorphic vector valued function:

$$I^\gamma \;=\; (I_1, \ldots, I_n) \;=\; \Big(\int_\gamma \eta_1, \ldots, \int_\gamma \eta_n \Big)$$

Theorem 1. *I^γ satisfies the following system of differential equations:*

$$\frac{\partial I}{\partial z_i} \;=\; \frac{1}{\kappa} \sum_{j \ne i} \frac{\Omega_{ij}}{z_i - z_j} I \quad \textit{for } i = 1, \ldots, n \textit{ where}$$

$$\Omega_{ij} \;=\; \begin{array}{c} \\ \\ i \cdots \\ \\ j \cdots \\ \\ \end{array} \begin{pmatrix} & i & & j & \\ & \vdots & & \vdots & \\ & \frac{(m_i - 2) m_j}{2} & \cdots & m_j & \cdots \\ & \vdots & & \vdots & \\ & m_i & \cdots & \frac{m_i (m_j - 2)}{2} & \cdots \\ & \vdots & & \vdots & \end{pmatrix}$$

All other diagonal entries are $\frac{m_i m_j}{2}$ and the remaining off-diagonal entries are all zero.

Proof. For example, in the case of $i = 1$ and the component $\frac{\partial I_2^\gamma}{\partial z_1}$ of $\frac{\partial I^\gamma}{\partial z_1}$, we have

$$\frac{\partial I_2^\gamma}{\partial z_1} = \int_\gamma \frac{\partial \eta_2}{\partial z_1} dt = \int_\gamma \Big(\sum_{j\neq 1}^{n} \frac{m_i m_j}{2\kappa} \frac{1}{z_1 - z_j} + \frac{m_1}{\kappa} \frac{1}{t - z_1} \Big) \Phi \frac{dt}{t - z_2}.$$

Using the identity $\frac{1}{t-z_1}\frac{1}{t-z^2} = \frac{1}{z_1-z_2}\left(\frac{1}{t-z_1} - \frac{1}{t-z_2}\right)$ we obtain the desired result

$$\frac{\partial I_2^\gamma}{\partial z_1} = \int_\gamma \left(\frac{m_1(m_2-2)}{2\kappa} \frac{1}{z_1 - z_2} \eta_2 + \sum_{j\notin\{1,2\}} \frac{m_1 m_j}{2\kappa} \frac{1}{z_1 - z_j} \eta_2 + \frac{m_1}{\kappa} \frac{1}{z_1 - z_2} \eta_1 \right).$$

□

We will now describe a generalization of the above system of differential equations which arrises in the context $\mathfrak{sl}_2(\mathbb{C})$-representations.

Recall that $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra with generators

$$\left\langle e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

satisfying the familier commutation relations

$$[h, e] = 2e \qquad [e, f] = h \qquad [h, f] = -2f.$$

The element $\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \in \mathfrak{g} \otimes \mathfrak{g}$ is called the Casimir element and for all $x \in \mathfrak{sl}_2(\mathbb{C})$ it satisfies the commuation relation $[x \otimes 1 + 1 \otimes, \Omega] = 0$ inside $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \otimes \mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$.

Let $V_1, \ldots, V_n$ be $\mathfrak{sl}_2(\mathbb{C})$-modules and set $V = V_1 \otimes \cdots \otimes V_n$. An element $x \in \mathfrak{sl}_2(\mathbb{C})$ acts on V by $x \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes x$. For indices $1 \leq i < j \leq n$ let $\Omega^{ij} : V \longrightarrow V$ be the operator which acts by Ω on the ith and jth positions and as the idenity on all others. For example:

$$\Omega^{12}(v_1 \otimes \cdots \otimes v_n) \ = \ \Omega(v_1 \otimes v_2) \otimes v_3 \cdots \otimes v_n$$

Definition 1. *the KZ equation for a V-valued function $\phi(z_1, \ldots, z_n)$ is*

$$\frac{\partial \phi}{\partial z_i} = \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega^{ij}}{z_i - z_j} \ \phi \quad \textit{for } i = 1 \ldots n.$$

This equation is defined on $U = \left\{ \ (z_1, \ldots, z_n) \in \mathbb{C}^n \ \middle| \ z_i \neq z_j \ \right\}$.

Mention connections with CFT and decorated Riemann surfaces.

A solution ϕ of the KZ equation one can be translated by an element $x \in \mathfrak{sl}_2(\mathbb{C})$ to obtain a new V-valued funtion $x \cdot \phi$. It follows from the commutation relation $[\Omega, x \otimes 1 + 1 \otimes x] = 0$ that $x \cdot \phi$ is also a solution of the KZ equation. From this we can conclude that

Corollary 1. *For any $x \in \mathfrak{sl}_2(\mathbb{C})$ the KZ equation preserves the eigenspaces of the action of x on V. In other words, if ϕ is a KZ solution such that $x \cdot \phi(\mathbf{z}_0) = \lambda\phi(\mathbf{z}_0)$ for some $\mathbf{z}_0 \in \mathbb{C}^n$ then $x \cdot \phi(\mathbf{z}) = \lambda\phi(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^n$.*

For $m \in \mathbb{C}$, the Verma module M_m is the infinite dimensional $\mathfrak{sl}_2(\mathbb{C})$-represention with heighest weight m generated by a single vector v_m where $hv_m = mv_m$ and $ev_m = 0$. The vectors $f^k v_m$ for $k = 0, 1, \ldots$ form a basis for M_m. The generators e, f, and h act on this basis as indicated below:

$$f \cdot f^k v_m \;=\; f^{k+1} v_m \qquad h \cdot f^k v_m \;=\; (m-2k) f^k v_m \qquad e \cdot f^k v_m \;=\; k(m-k+1) f^{k-1} v_m$$

If $m \notin \mathbb{Z}_{\geq 0}$ then M_m is irriducible, otherwise $< f^{m+1} v_m >= \mathfrak{sl}_2(\mathbb{C}) \cdot f^{m+1} v_m$ will be an invariant subspace. Let $L_m = M_m / < f^{m+1} v_m >$. This quotient is irriducible with basis $v, fv, \ldots, f^m v$.

It is a familiar exercise in theory of $\mathfrak{sl}_2(\mathbb{C})$- representations to check the following tensor decompostion:

$$L_m \otimes L_l \;=\; L_{m+l} \oplus L_{m+l-2} \oplus \cdots \oplus L_{m-l}$$

Let $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$ and set $|\mathbf{m}| = m_1 + \cdots + m_n$. Let $M^{\otimes \mathbf{m}} = M_{m_1} \otimes \cdots \otimes M_{m_n}$. For $J = (j_1, \ldots, j_n) \in \mathbb{Z}_{\geq 0}$ let $f_J v = f^{j_1} v_{m_1} \otimes \cdots \otimes f^{j_n} v_{m_n}$ where v_{m_i} is the primitive vector in M_{m_i}. The vectors $f_J v$ form a basis for the module $M^{\otimes \mathbf{m}}$.

Now $h \cdot f_J v = \Big(|\mathbf{m}| - 2|J|\Big) f_J v$. Define $M^{\otimes \mathbf{m}}[\lambda] = \Big\{\ v \in M^{\otimes \mathbf{m}}\ \Big|\ h \cdot v = \lambda v\ \Big\}$ and $\mathrm{Sing}\, M^{\otimes \mathbf{m}}[\lambda] = \Big\{\ v \in M^{\otimes \mathbf{m}}[\lambda]\ \Big|\ e \cdot v = 0\ \Big\}$. The singular eigenspaces $\mathrm{Sing}\, M^{\otimes \mathbf{m}}[\lambda]$ generate the entire module $M^{\otimes \mathbf{m}}$ and, in view of Corollary 1, it follows that it is enough to produce KZ solutions for the singular eigenspaces only.

Solutions

Case 1: $\mathrm{Sing}\, M^{\otimes \mathbf{m}}[|\mathbf{m}|]$

$\mathrm{Sing}\, M^{\otimes \mathbf{m}}[|\mathbf{m}|]$ is spanned by the single vector $v_{m_1} \otimes \cdots \otimes v_{m_n}$. If ϕ is a KZ solution residing in the $|\mathbf{m}|$-eigenspace then it is of the form $\phi = I_0(z_1, \ldots, z_n) v_{m_1} \otimes \cdots \otimes v_{m_n}$ where I_0 is a scalar valued function. In an effort to compute the right hand side of the KZ equation we obtain:

$$\Omega^{ij}\Big(v_{m_1} \otimes \cdots \otimes v_{m_n}\Big) \;=\; \frac{m_i m_j}{2}\, v_{m_1} \otimes \cdots \otimes v_{m_n}$$

In this case the KZ equation reads as:

$$\frac{\partial I_0}{\partial z_i} = \sum_{j\neq i} \frac{m_i m_j}{2\kappa} \frac{I_0}{z_i - z_j} \quad \text{for } i = 1 \dots n$$

with solution given by

$$I_0 = \prod_{1\le i<j\le n} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}}$$

In view of Corollary 1 we can conclude that $f^k \cdot I_0 v_{m_1} \otimes \cdots \otimes v_{m_n} = I_0 f^k \cdot v_{m_1} \otimes \cdots \otimes v_{m_n}$ is also a KZ solution for any $k \in \mathbb{Z}_{\ge 0}$.

Case 2: Sing $M^{\otimes \mathbf{m}}[|\mathbf{m}| - 2]$

This eigenspace is

$$\text{Sing } M^{\otimes \mathbf{m}}[|\mathbf{m}| - 2] = \left\{ \; w = \sum_{l=1}^{m} I_l v_{m_1} \otimes \cdots \otimes f \cdot v_{m_l} \otimes \cdots \otimes v_{m_n} \;\middle|\; e \cdot w = 0 \;\right\}$$

where the I_l's are scalars. In this case the condition $e \cdot w = 0$ can be reformulated as $m_1 I_1 + \cdots + m_n I_n = 0$. In order to compute the right hand side of the KZ equation we compute the action of Ω^{ij} on the basis vectors $v_{m_1} \otimes \cdots \otimes f \cdot v_{m_l} \otimes \cdots \otimes v_{m_n}$. Assume $i < j$. If the basis vector is of the form $\mathbf{v} = \cdots \otimes v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots$ then $\Omega^{ij}(\mathbf{v}) = \frac{m_i m_j}{2} \mathbf{v}$. If $\mathbf{v}$ is of the form $\cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots$ then

$$\Omega^{ij}\mathbf{v} = m_i \Big(\cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \Big) + \frac{(m_i - 2) m_j}{2} \Big(\cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots \Big).$$

If the basis vector $\mathbf{v}$ is of the form $\cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots$ then

$$\Omega^{ij}\mathbf{v} = m_j \Big(\cdots \otimes f \cdot v_{m_i} \otimes \cdots \otimes v_{m_j} \otimes \cdots \Big) + \frac{(m_j - 2) m_i}{2} \Big(\cdots \otimes v_{m_i} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \Big).$$

From these computations we may surmise that the matrix representing Ω^{ij} is of the form

$$\Omega_{ij} = \begin{array}{c} \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{pmatrix} & \vdots & & \vdots & \\ i \cdots & \frac{(m_i-2)m_j}{2} & \cdots & m_j & \cdots \\ & \vdots & & \vdots & \\ j \cdots & m_i & \cdots & \frac{m_i(m_j-2)}{2} & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

where all other diagonal entries are $\frac{m_i m_j}{2}$ and the remaining off-diagonal entries are all zero. Of course this is the same matrix we encountered in the beginning of the lecture. We know explicit solutions namely,

$$\phi^\gamma(z_1, \dots, z_n) = \sum_{j=1}^{n} \int_\gamma \Phi \frac{dt}{t - z_j} v_{m_1} \otimes \cdots \otimes f \cdot v_{m_j} \otimes \cdots \otimes v_{m_n} \quad \text{where}$$

$$\Phi \;=\; \prod_{i<j} (z_i - z_j)^{\frac{m_i m_j}{2\kappa}} \; \prod_{l=1}^{n} (t - z_l)^{\frac{-m_l}{\kappa}}.$$

Applying our translation argument again we may conclude that $f^k \cdot \phi^\gamma$ is a KZ solution with values in $M^{\otimes \mathbf{m}}\big[|\mathbf{m}| - 2 - 2k\big]$.

Case 3: $\operatorname{Sing} M^{\otimes \mathbf{m}}\big[|\mathbf{m}| - 2k\big]$ where $k \in \mathbb{Z}_{\geq 0}$

Define $\Phi_{k,n}(\mathbf{t}, \mathbf{z}, \mathbf{m}) = \Phi_{k,n}(t_1, \ldots, t_k, z_1, \ldots, z_n, m_1, \ldots, m_n)$ to be the following expression:

$$\prod_{i<j} (z_i - z_j)^{\frac{m_i m_j}{2}} \prod_{1 \leq i \leq j \leq k} (t_i - t_j)^2 \prod_{l=1}^{n} \prod_{i=1}^{k} (t_i - z_l)^{-m_l}.$$

Comments about discriminantal arrangement of hyperplanes associated to the singularities $t_i = z_l$ and vanishings $t_i = t_j$.

Recall that $M^{\otimes \mathbf{m}}\big[|\mathbf{m}| - 2k\big]$ has a basis of vectors $f_J v$ where $J = (j_1, \ldots, j_n)$ and $|J| = k$. Associate to each basis vector $f_J v$ the following rational function:

$$A_J(\mathbf{t}, \mathbf{z}) \;=\; \frac{1}{j_1! \cdots j_n!} \operatorname{Sym}_t \left[\prod_{l=1}^{n} \prod_{i=1}^{j_l} \frac{1}{t_{j_1 + \cdots + j_{l-1} + i} - z_l} \right] \quad \text{where}$$

$$\operatorname{Sym}_t f(t_1, \ldots, t_k) := \sum_{\sigma \in S_k} f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

For example: $A(1, 0, \ldots, 0) = \frac{1}{t_1 - z_1}$ and $A(2, 0, \ldots, 0) = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_1}$ and $A(1, 1, 0, \ldots, 0) = \frac{1}{t_1 - z_1} \frac{1}{t_2 - z_2} + \frac{1}{t_2 - z_1} \frac{1}{t_1 - z_2}$.

We define the following $M^{\otimes \mathbf{m}}[|\mathbf{m}| - 2k]$-valued functions:

$$\phi^\gamma(z_1, \ldots, z_n) \;=\; \sum_{|J| = k} \left(\int_\gamma \Phi_{k,n}(\mathbf{t}, \mathbf{z}, \mathbf{m}))^{\frac{1}{\kappa}} A_J(\mathbf{t}, \mathbf{z}) \; dt_1 \wedge \cdots \wedge dt_k \right) \cdot f_J v.$$

In the next lecture we shall prove the following theorem:

Theorem 2. *$\phi^\gamma(\mathbf{z}) \in \operatorname{Sing} M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2k\Big]$ and it is a KZ solution.*

Definition 2. *The functions $I_J^\gamma := \int_\gamma \Phi_{k,n}^{\frac{1}{\kappa}} A_J \; dt_1 \wedge \cdots \wedge dt_k$ are called the hypergeometric functions associated with $\operatorname{Sing} M^{\otimes \mathbf{m}}\Big[|\mathbf{m}| - 2k\Big]$.*

Claim 1. *Let* $J+\mathbf{1}_l=(j_1,\dots,j_l+1,\dots,j_n)$*. If* $|J|=k-1$ *then*

$$\sum_{l=1}^{n} I_{J+\mathbf{1}_l}(\mathbf{z})(j_l+1)(m_l-j_l) \;=\; 0.$$

This statement is equivalent to the claim that $\phi^\gamma(\mathbf{z})\in \mathrm{Sing}\, M^{\otimes\mathbf{m}}\Big[|\mathbf{m}|-2k\Big]$.