18.735: Topics in algebra
TR 9.30-11, Rm. 12-102
Introduction to derived categories in algebra and geometry.

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Prerequisites: 1st year algebra and topology; basic algebraic geometry.

Derived categories have become a standard tool in algebra, algebraic geometry, topology, representation theory etc.

[Roughly speaking, for an object of a "classical" theory given by a system of equations the corresponding "derived" object carries (in a very well-packaged form) some information about how overdetermined this system of equations is (i.e. it remembers relations between equations etc.)]

Though the notion itself is about 40 years old, both applications and foundations of the subject were being actively developed in the recent years.

The course is intended for beginners, its aim is to give an introduction to the subject, touching upon some applications to algebraic geometry.

Approximate plan of the course

(1) Brief recollection of the background: complexes, 5-lemma, Euler characteristic, lack of exactness of standard functors (Hom, tensor product for modules, global sections for sheaves), Ext^1, Tor_1.
(2) (Universal) ∂ functors of left (or right) exact functors for modules over a ring.
(3) Resolutions of modules. Derived category of modules as a homotopy category of projectives/injectives. Examples of ∂ functors: Ext and Tor. Hochschild (co)homology.
(4) Abelian and exact categories. Full embedding theorems. Above constructions in this setting.
(5) DG categories. Triangulated axioms. Homotopy category is triangulated.
(6) Quotient of a triangulated category by a thick subcategory. Standard definition of derived categories. Discussion of available DG models.
(7) Derived functors: universal property, existence when there are enough adjusted objects, or target has limits.
(8) Sheaves and derived global sections.

    Examples of triangulated equivalences: Koszul duality; tilting (in algebra); Beilinson's description of $D^b(Coh(\mathbb{P}^n))$, and Kapranov's for Grassmannian (time permitting); Fourier-Mukai transform.
(9) Application: Grothendieck duality, $f^!$. Serre functor.
(10) "Higher" intersection of subschemes: DGA. Triangulated category of DG modules. Quasi-isomorphism induces quasi-equivalence.
(11) t-structures. Relation to derived equivalences.
(12) Gluing of t-structures.