QUANTUM DIMENSION

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Outline for the talk

(1) An example
(2) Semi-simple ribbon categories
(3) Continuation of example
(4) Definition of quantum dimension
(5) Properties thereof
(6) Computations for IPERs of $LSU(2)$

1. Basic example

Let $V = \mathbb{C}^2$ and choose a basis $\{e_2, e_2\}$ with dual basis $\{\epsilon_1, \epsilon_2\}$ for $V^*$. We have an evaluation map

\[ e : V \otimes V^* \to \mathbb{C} \]

defined by eating a vector with a linear functional. There is also an embedding

\[ i : \mathbb{C} \to V \otimes V^* \]

defined by

\[ i(1) = e_1 \otimes \epsilon_2 + e_2 \otimes \epsilon_2. \]

The composition

\[ e \circ i : \mathbb{C} \to V \otimes V^* \to \mathbb{C} \]

yields

\[ e(e_1 \otimes \epsilon_1 + e_1 \otimes \epsilon_2) = 2. \]
2. Semi-simple ribbon categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equipped with the following structure:

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a bilinear functor
- $\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W)$, a functorial isomorphism for all $U, V, W \in \mathcal{C}$
- $I \in \mathcal{C}$ such that $\text{End}(I) = \mathbb{C}$, together with functorial isomorphisms
  $$\lambda_V : I \otimes V \xrightarrow{\sim} V$$
  and
  $$\rho_V : V \otimes I \to V$$
  for all $V \in \mathcal{C}$

Such a structure is called a monoidal structure.

**Example.** $\text{Vec}(\mathbb{C})$, together with the tensor product $\otimes$. \hfill \Box

**Example.** $\text{Rep}(SU(N))$ with tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of the underlying Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2 \in \text{Rep}(SU(N))$ and the action

$$g(x \otimes y) = g(x) \otimes g(y)$$

for $g \in SU(N)$ and $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$. \hfill \Box

**Definition.** A braiding is given by isomorphisms

$$\sigma_{VW} : V \otimes W \xrightarrow{\sim} W \otimes V,$$

which are functorial in $V, W \in \mathcal{C}$, and satisfy some commutative diagrams. A category with a monoidal structure and a braiding is called a braided monoidal category.

**Example.** For the categories $\text{Vec}(\mathbb{C})$ and $\text{Rep}(SU(N))$, the isomorphisms

$$\tau_{VW} : V \times W \to W \times V$$

given by

$$\tau(v, w) = (w, v)$$

give a braiding. \hfill \Box

**Rigidity** is given by homomorphisms

$$e_V : V^* \otimes V \to I$$

and

$$i_V : I \to V \otimes V^*,$$

where $V$ is the right dual of $V$. These maps must make the following diagrams commute:
**Example.** For $\text{Vec}(\mathbb{C})$, we have $V^* = \text{Hom}(V, \mathbb{C})$ and the maps $e_V$ and $i_V$ as before.

**Example.** For $\text{Rep}(G)$, the representation on the dual of $\mathcal{H}$ is given by continuous linear functionals $\mathcal{H}^* = C(\mathcal{H}, \mathbb{C})$ with the representation

$$(gf)(v) = f(g^{-1}v)$$

for $f \in \mathcal{H}^*$, $g \in G$ and $v \in \mathcal{H}$. 

Suppose that we have a map $f : U \to V$. We can construct a dual map $f^*$ by the composition

$$V^* \xrightarrow{\text{id}_V} V^* \otimes U \otimes U^* \xrightarrow{1 \otimes i_U} V^* \otimes V \otimes U^* \xrightarrow{\text{ev} \otimes 1} U^*$$

**Definition.** A ribbon category is a rigid braided tensor category with a functorial isomorphism $\delta_V : V \xrightarrow{\sim} V^{**}$ such that

$$\delta_{V \otimes W} = \delta_V \otimes \delta_W$$

$$\delta_I = 1_I$$

$$\delta_{V^*} = (\delta_V^*)^{-1}$$
3. Quantum dimension

Let $\mathcal{C}$ be a semi-simple ribbon category.

**Definition.** Let $V \in \mathcal{C}$ and $f \in \text{End}(V)$. Define the *trace* of $f$ to be the composition

$$I \rightarrow V \otimes V^* \xrightarrow{f \otimes 1} V \otimes V^* \rightarrow V^{**} \otimes V^* \xrightarrow{ev^*} I$$

One important property of the trace is

$$tr(f \otimes g) = tr(f)tr(g),$$

a fact which needs the ribbon structure.

**Definition.** The *quantum dimension* of $V \in \mathcal{C}$ is defined to by $tr(1_V)$.

If there are finitely many simple objects, the quantum dimension is a real number. For $LSU(N)$, the quantum dimension is, in fact, positive.

4. Calculations

Let

$$N_{fg}^h = \dim(\text{Hom}(V_h, V_f \otimes V_g))$$

**Theorem 4.1** (Wassermann). The Connes fusion satisfies

$$\mathcal{H}_f \boxtimes \mathcal{H}_g = \bigoplus N_{fg}^h \text{sgn}(\sigma_h)\mathcal{H}_{h'}$$

where

$$h' = \sigma(h + \delta) - \delta$$

is a permutation.

**Corollary 4.2.** For $G = SU(2)$, we have

$$\mathcal{H}_l \boxtimes \mathcal{H}_l = \mathcal{H}_0$$

**Theorem 4.3.** For the permutation signature $f$, we have

$$\mathcal{H}_\square \boxtimes \mathcal{H}_f = \bigoplus_{g \equiv f + \square} \mathcal{H}_g$$

**Corollary 4.4.** For $G = SU(2)$ and $1 \leq \lambda \leq l - 1$,

$$\mathcal{H} \boxtimes \mathcal{H}_\lambda = \mathcal{H}_{\lambda-1} \oplus \mathcal{H}_{\lambda+1}$$
Fix a level \( l \in \mathbb{Z} \), consider

\[ \mathcal{H}_0, \ldots, \mathcal{H}_l, \]

and define

\[ d_i = \dim \mathcal{H}_i \]

and

\[ d_l = 1. \]

**Proposition 5.1.** For \( 0 \leq i \leq l \), we have

\[ d_i = d_{l-i}, \]

so that

\[ d_1d_i = d_{i-1} + d_{i+1} \]

and

\[ d_1d_{l-i} = d_{l-i-1} + d_{l-i+1}, \]

and hence

\[ d_{i-1} = d_{l-i+1} \]

for \( 1 \leq i \leq l - 1 \).

We can use this proposition to calculate the quantum dimensions, as in the following picture: