THE SYMMETRIC MONOIDAL 3-CATEGORY OF
CONFORMAL NETS

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The goal of this talk is to give a definition of symmetric monoidal 3-categories, show that conformal nets form such a category and relate the subject to Chern-Simons theory. (This is work with Bartles, Douglas and Henriques.)

1. Motivation: Topological QFT

Let \( \text{Bord}^{n-1}_{n-1} \) be the category of bordisms of \( n \)-manifolds. This category is a symmetric monoidal category with operation given by disjoint unions and identity object the empty set \( \emptyset \).

**Definition.** An \( n \)-dimensional topological quantum field theory or TQFT is a symmetric monoidal functor

\[
Z : \text{Bord}^{n-1}_{n-1} \rightarrow (\text{Hilbert}, \otimes)
\]

\( \Omega \text{Bord}^{n-1}_{n-1} = \text{End}(\emptyset) = \text{closed } n\text{-manifolds/diffeomorphisms} \)

\( \Omega \text{Hilb} = \text{End}(\mathbb{C}) = \mathbb{C} \)

So \( Z \) assigns \( \mathbb{C} \)-valued diffeomorphism invariant to closed \( n \)-manifolds.

\( \text{Bord}^k_n \) is the symmetric \( (n-k) \)-category which we think of as manifolds with bordisms of bordisms of ..., We have

\[
\Omega \text{Bord}^k_n \cong \text{Bord}^n_{k+1}
\]

**Definition.** Let \( \mathcal{C} \) be a symmetric monoidal \( n \)-category with \( \Omega^{n-1} \mathcal{C} \cong \text{Hilb}_\mathbb{C} \). A \( \mathbb{C} \)-valued local TQFT is a symmetric monoidal functor \( \text{Bord}^n_0 \rightarrow \mathcal{C} \)

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Available online at [http://math.mit.edu/~eep/CFTworkshop](http://math.mit.edu/~eep/CFTworkshop). Please email eep@math.mit.edu with corrections and improvements!
Theorem 1.1 (BDH). There exists a symmetric monoidal 3-category $CN$ whose objects are conformal nets and $\Omega^2 CN \cong \text{Hilb}$.

Theorem 1.2 (Cobordism hypothesis of Baez-Dolan, 99% certainty proof by Hopkins-Lurie). Framed local $C$-valued n-dimensional TQFTs are in one-to-one correspondence with dualizable objects in $C$.

Theorem 1.3. $A \in CN$ is dualizable if and only if it is the direct sum of irreducible conformal nets with finite $\mu$-index.

2. Warm-up: algebras and bimodules

We assign to a 0-dimensional manifold an algebra $A$. Given a 1-morphism between points with algebras $A$ and $B$, we assign an $(A, B)$-bimodule $V$. We compose these bimodules using the tensor product. To a 2-morphism, we assign a bi-module homomorphism between the corresponding bi-modules. There are two ways to compose these morphisms (horizontal and vertical) and in this case we ask that they agree.

Remark. We can thing of a symmetric monoidal category as a bi-category with a 1-object, so a symmetric monoidal category is something at least 4-categorical in nature.
There is a symmetric monoidal category of algebras with \( \otimes \). There is also a symmetric monoidal category of bimodules with \( \otimes \):

\[
(A V_B) \otimes (A' V'_B') = A \otimes A'(V \otimes V')_{B \otimes B'}.
\]

In this case the functors \( s \) and \( t \) with take an arrow to its source and target are symmetric monoidal functors \( Bimod \rightarrow Alg \) with

\[
\boxtimes : Bimod \times \rightarrow Bimod \rightarrow Bimod
\]

The upshot is that \((Alg, Bimod)\) is a category object in the 2-category \( SMC \) of symmetric monoidal categories. Let us, then, think of conformal nets as a bicategory object in \( SMC \).

3. Conformal nets revisited

**Definition.** We call \( Int \) the (topological) category whose objects are oriented intervals and whose morphisms are smooth embeddings, which are not necessarily orientation preserving. The topology on the hom-sets is given by point-wise convergence.

Notice that we are allowing more information than just the transformation given by Mobiüs transformations.

**Definition.** A conformal net is a continuous functor

\[
\mathcal{A} : Int \rightarrow vN-alg
\]

from intervals to von Neumann-algebras satisfying the usual axioms, as well as

- For \( \phi : I \rightarrow I \), such that \( \phi \) is the identity in a neighbourhood of \( \partial I \), then \( \mathcal{A}(\phi) : \mathcal{A}(I) \rightarrow \mathcal{A}(I) \) is inner
- If \( \phi : I \rightarrow J \) is orientation preserving (resp. reversing) then \( \mathcal{A}(\phi) \) is a homomorphism (resp. anti-homomorphism).

4. Conformal nets and 2-algebras

Given a conformal net \( \mathcal{A} \), let \( A = \mathcal{A}([0, 1]) \). Define the standard inclusions

\[
i, j : [0, 1] \rightarrow [0, 2]
\]

where \( i \) is the inclusion and \( j \) is inclusion plus translation one unit to the right. Also pick an isomorphism

\[
s : [0, 2] \rightarrow [0, 1]
\]
which has derivative 1 in a neighborhood of $\partial[0, 2]$. We define

$$\mu : A \times A \to A$$

by

$$\mu(x, y) = s_*(i_*(x)j_*(y)) = (si_*)(x)(sj_*)(y) = \begin{pmatrix} x \\ y \end{pmatrix}$$

### Claim.

There exists $v \in A$ such that

1. $$v \begin{pmatrix} x \\ y \\ z \end{pmatrix} v^{-1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2. $$v^2 = \begin{pmatrix} 1 \\ v \end{pmatrix} v \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We get identities like
The morphisms in the category are:

- 1-morphisms: defects
- 2-morphisms: sectors
- 3-morphisms: homomorphisms of sectors

**Definition.** A bicolored interval is an interval $I$ with two subintervals $I_w$ and $I_b$ (the white and black intervals) with $I = I_w \cup I_b$ and such that either

1. $I_w = \emptyset$,
2. $I_b = \emptyset$, or
3. $I = I_w \cup I_b$

together with a coordinate function $c : nbd(I_w \cap I_b) \to \mathbb{R}$.

We can define a category $Int_{bc}$ of bicolored intervals.

**Definition.** A defect $D : A \to B$ for $A, B \in CN$ is a cosheaf $D : Int_{bc} \to vN-alg$ such that

$$D|_{\text{white int.}} = A$$

and

$$D|_{\text{black int.}} = B$$

5. **Sectors**

Consider intervals $I$ in $S^1$ such that either $i \notin I$ or $-i \notin I$. We can bicolor such intervals: call the pieces to the left of $\pm i$ black and those to the right of $\pm i$ white.
We have bimodules with defects and a composition using Connes fusion:

There is a natural isomorphism between the different ways of fusing, which uses the machinery we’ve been discussing this week.