Random Matrices: Theory and Applications

Computing with Beta Ensembles and Hypergeometric Functions

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We survey the empirical models for the Hermite, Wishart, Laguerre, Jacobi, and MANOVA beta ensembles. The eigenvalue distributions of these ensembles are expressed in terms of the hypergeometric function of a matrix argument. We present a number of identities for this function that have been instrumental in computing these distributions in practice. These identities include a number of new results.

1. Introduction

Multivariate statistical analysis relies heavily on the classical random matrix ensembles—Hermite, Wishart, Laguerre, Jacobi, and MANOVA—and the distributions of their eigenvalues in order to infer information about multivariate datasets. Muirhead's text [29] provides an in-depth analysis of the real ($\beta = 1$) case. Most of the results have complex ($\beta = 2$) analogues [31] and the extension to quaternions ($\beta = 4$) is straightforward although lacking on applications thus far.

In the past 10–15 years generalizations of these ensembles to any $\beta > 0$ have been introduced. This suggests that most of the classical results (and perhaps most of random matrix theory) also generalize to any $\beta > 0$ [12]. Advances in computational tools have also made these theoretical results very important in practice. The theoretical importance of the distributions of the extreme eigenvalues is also well documented [20,21].

The goal of this paper is to present in one place the empirical models, the eigenvalue distributions, and the identities that have made computing those distributions possible. We include several new results.

This paper is organized as follows. In section 2 we introduce the random matrix ensembles, their empirical models as well as the hypergeometric function of a matrix argument and the various combinatorial objects that we will need later on. In section 3 we present the identities for the hypergeometric function and the Jack function. The known distributions and densities of the extreme eigenvalues of the beta ensembles are in 4. We prove all new results in section 5. The numerical experiments are in section 6. We finish with open problems in section 7.

2. Preliminaries

In this section we survey the classical definitions from combinatorics and multivariate polynomials. We refer to Stanley [34,35] for a more detailed treatment of the material in this section.

For an integer $k \geq 0$ we say that $\kappa = (\kappa_1, \kappa_2, \ldots)$ is a *partition* of k (denoted $\kappa \vdash k$) if $\kappa_1 \geq \kappa_2 \geq \cdots \geq 0$ are integers such that $\kappa_1 + \kappa_2 + \cdots = k$. The quantity $|\kappa| = k$ is called the *size* of κ.

All partitions can be partially ordered: we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i = 1, 2, \ldots$. Then λ/μ is called a *skew shape* and consists of those boxes in the Young diagram of λ that do not belong to μ. Clearly, $|\lambda/\mu| = |\lambda| - |\mu|$.

The skew shape κ/μ is called a *horizontal strip* when $\kappa_1 \geq \mu_1 \geq \kappa_2 \geq \mu_2 \geq \cdots$ [35, p. 339].

The upper and lower *hook lengths* at a point (i, j) in a partition κ (i.e., $i \leq \kappa'_j$, $j \leq \kappa_i$) are, respectively,

$$h^*_\kappa(i,j) \equiv \kappa'_j - i + \frac{2}{\beta}(\kappa_i - j + 1) \quad \text{and} \quad h_*^\kappa(i,j) \equiv \kappa'_j - i + 1 + \frac{2}{\beta}(\kappa_i - j).$$

The products of the upper and lower hook lengths are denoted, respectively, as

$$H^*_\kappa = \prod_{(i,j)\in\kappa} h^*_\kappa(i,j) \qquad \text{and} \qquad H^\kappa_* = \prod_{(i,j)\in\kappa} h^\kappa_*(i,j).$$

Their product is denoted as $j_\kappa \equiv H^*_\kappa H^\kappa_*$.

For a partition $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ and $\beta > 0$, the *Generalized Pochhammer symbol* is defined as

$$\begin{aligned}(a)^{(\beta)}_\kappa &\equiv \prod_{(i,j)\in\kappa}\left(a - \frac{i-1}{2}\beta + j - 1\right) \qquad (2.1)\\ &= \prod_{i=1}^{n}\prod_{j=1}^{\kappa_i}\left(a - \frac{i-1}{2}\beta + j - 1\right)\\ &= \prod_{i=1}^{n}\left(a - \frac{i-1}{2}\beta\right)_{\kappa_i},\end{aligned}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the *rising factorial.*

For a partition $\kappa = (k)$ in only one part $(a)^{(\beta)}_\kappa = (a)_k$ is independent of β.

The *multivariate Gamma function* of parameter β is defined as

$$\Gamma^{(\beta)}_m(c) \equiv \pi^{\frac{m(m-1)}{4}\beta}\prod_{i=1}^{m}\Gamma\left(c - \frac{i-1}{2}\beta\right) \quad \text{for } \Re(c) > \frac{m-1}{2}\beta. \qquad (2.2)$$

Definition 2.1 (Jack function).

The Jack function $J^{(\beta)}_\kappa(X) = J^{(\beta)}_\kappa(x_1, x_2, \ldots, x_m)$ is a symmetric, homogeneous polynomial of degree $|\kappa|$ in the eigenvalues $x_1, x_2, \ldots, x_m$ of X. It can be defined recursively [34, Proposition 4.2]:

$$\begin{aligned}&J^{(\beta)}_\kappa(x_1, x_2, \ldots, x_n) = 0, \text{ if } \kappa_{n+1} > 0;\\ &J^{(\beta)}_{(k)}(x_1) = x_1^k\left(1 + \tfrac{2}{\beta}\right)\cdots\left(1 + (k-1)\tfrac{2}{\beta}\right);\\ &J^{(\beta)}_\kappa(x_1, x_2, \ldots, x_n) = \sum_{\mu\subseteq\kappa} J^{(\beta)}_\mu(x_1, x_2, \ldots, x_{n-1})x_n^{|\kappa/\mu|}\beta_{\kappa\mu}, \quad n \geq 2, \qquad (2.3)\end{aligned}$$

where the summation in (2.3) is over all partitions $\mu \subseteq \kappa$ such that κ/μ is a horizontal strip, and

$$\beta_{\kappa\mu} \equiv \frac{\prod_{(i,j)\in\kappa} B^\kappa_{\kappa\mu}(i,j)}{\prod_{(i,j)\in\mu} B^\mu_{\kappa\mu}(i,j)}, \qquad \text{where} \qquad B^\nu_{\kappa\mu}(i,j) \equiv \begin{cases} h^*_\nu(i,j), \text{ if } \kappa'_j = \mu'_j;\\ h^\nu_*(i,j), \text{ otherwise.}\end{cases} \qquad (2.4)$$

Definition 2.2 (Hypergeometric function of matrix arguments). Let $p \geq 0$ and $q \geq 0$ be integers, and let X and Y be $m \times m$ complex symmetric matrices with eigenvalues $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$, respectively. The hypergeometric function of two matrix arguments X and Y, and parameter $\beta > 0$ is defined as

$$\begin{aligned}&{}_pF^{(\beta)}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y)\\ &\quad\equiv \sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{1}{k!}\cdot\frac{(a_1)^{(\beta)}_\kappa\cdots(a_p)^{(\beta)}_\kappa}{(b_1)^{(\beta)}_\kappa\cdots(b_q)^{(\beta)}_\kappa}\cdot\frac{C^{(\beta)}_\kappa(X)C^{(\beta)}_\kappa(Y)}{C^{(\beta)}_\kappa(I)}. \qquad (2.5)\end{aligned}$$

Beta ensemble ($m \times m$)	Joint eigenvalue density
Hermite	$\frac{\pi^{\frac{m(m-1)\beta}{4}}}{(2\pi)^{\frac{m}{2}}} \cdot \frac{(\Gamma(1+\beta/2))^m}{\Gamma_m^{(\beta)}(1+\beta/2)} e^{-\frac{\mathrm{tr}(\Lambda^2)}{2}} \mathrm{d}\mu(\Lambda)$
Wishart (n DOF, cov. Σ)	$\frac{\lvert\Sigma\rvert^{-\frac{n}{2}\beta}}{\mathcal{K}_m^{(\beta)}(\frac{n}{2}\beta)} \lvert\Lambda\rvert^{\frac{n}{2}\beta - r} \cdot {}_0F_0^{(\beta)}(-\frac{\beta}{2}\Lambda, \Sigma^{-1}) \mathrm{d}\mu(\Lambda)$
Laguerre (with param. a)	$\frac{1}{\mathcal{K}_m^{(\beta)}(a)} \lvert\Lambda\rvert^{a-r} e^{-\frac{\mathrm{tr}(\Lambda)}{2}\beta} \mathrm{d}\mu(\Lambda)$
Jacobi (with param. a, b)	$\frac{1}{S_m^{(\beta)}(a,b)} \lvert\Lambda\rvert^{a-r} \lvert I - \Lambda\rvert^{b-r} \mathrm{d}\mu(\Lambda)$
MANOVA (param. n, p and covariance Ω)	$\frac{\mathcal{K}_m^{(\beta)}(n+p)}{\mathcal{K}_m^{(\beta)}(n)\mathcal{K}_m^{(\beta)}(p)} \lvert\Omega\rvert^{\frac{p\beta}{2}} \lvert\Lambda\rvert^{\frac{p}{2}\beta - r} \lvert I - \Lambda\rvert^{-\frac{p}{2}\beta - r} \times {}_1F_0^{(\beta)}(\frac{n+p}{2}; \Lambda(\Lambda - I)^{-1}, \Omega) \mathrm{d}\mu(\Lambda)$

Table 1. Joint eigenvalue densities of the beta ensembles (where $r \equiv \frac{m-1}{2}\beta + 1$).

For one matrix argument X, we define

$$ {}_pF_q^{(\beta)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X) \equiv {}_pF_q^{(\beta)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X, I). \tag{2.6} $$

2.1. *A new measure—the Vandermonde determinant*

The Vandermonde determinant to power β is a part of every multivariate integral we consider. To avoid it appearing everywhere, we conveniently incorporate it into a new measure μ

$$ \mathrm{d}\mu(X) = \prod_{i<j} \lvert x_i - x_j\rvert^\beta \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_m, $$

where $X = \mathrm{diag}(x_1, x_2, \ldots, x_m)$.

2.2. *Beta random matrix ensembles*

The Beta random matrices are defined in terms of their joint eigenvalue distributions. A matrix is from a particular beta ensemble if its (unordered) eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ have joint density as in Table 2.2. We denote $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and for any matrix A we denote its determinant as $\lvert A\rvert$.

We require that the parameters above be such that $\beta > 0$ and $a, b > \frac{m-1}{2}\beta$. We define

$$\mathcal{K}_m^{(\beta)}(a) \equiv \frac{m!\left(\frac{2}{\beta}\right)^{ma}}{\pi^{\frac{m(m-1)}{2}\beta}} \cdot \frac{\Gamma_m^{(\beta)}(a)\Gamma_m^{(\beta)}\left(\frac{m}{2}\beta\right)}{\left(\Gamma\left(\frac{\beta}{2}\right)\right)^m}, \tag{2.7}$$

$$S_m^{(\beta)}(a,b) \equiv \frac{m!\Gamma_m^{(\beta)}\left(\frac{m}{2}\beta\right)}{\pi^{\frac{m(m-1)}{2}\beta}\left(\Gamma\left(\frac{\beta}{2}\right)\right)^m} \cdot \frac{\Gamma_m^{(\beta)}(a)\,\Gamma_m^{(\beta)}(b)}{\Gamma_m^{(\beta)}(a+b)}. \tag{2.8}$$

The expression (2.8) is the value of the Selberg Integral [32].

We refer to [10] (Hermite), [8,17] (Wishart), [15] (Laguerre), [11] (Jacobi), and [7] (MANOVA).

2.3. *Empirical models for the classical random matrix ensembles*

The following are the empirical models for the beta–ensembles above which are valid for any $\beta > 0$.

- Hermite [10]:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} N(0,2) & c_{m-1} & & & \\ c_{m-1} & N(0,2) & \ddots & & \\ & \ddots & \ddots & c_2 & \\ & & c_2 & N(0,2) & c_1 \\ & & & c_1 & N(0,2) \end{bmatrix},$$

  where $c_i \sim \chi_{i\beta}$, $i = 1, 2, \ldots, m-1$, are independent and all elements on the diagonal are i.i.d. $N(0,2)$.

- Wishart [8,17]: The procedure is iterative. Let the eigenvalues of the covariance Σ be $d_1^2, d_2^2, \ldots, d_m^2$. The Wishart model is $W = \frac{1}{\beta}(Z_m^{(\beta)}(n))^T Z_m^{(\beta)}(n)$, where

$$Z_i^{(\beta)}(n) = \begin{bmatrix} \tau_1 & & & \chi_\beta d_i \\ & \ddots & & \vdots \\ & & \tau_{i-1} & \chi_\beta d_i \\ & & & \chi_{(n-i+1)\beta} d_i \end{bmatrix}.$$

  Here $\tau_1, \tau_2, \ldots, \tau_{i-1}$ are the singular values of $Z_{i-1}^{(\beta)}(n)$. The base $(m=1)$ case is $Z_1^{(\beta)}(n) = \chi_{n\beta} d_1$.

  Note than the number of degrees of freedom n need not be integer.

- Laguerre [10,33]: $L \equiv \frac{1}{\beta}BB^T$, where

$$B = \begin{bmatrix} \chi_{2a} & & & \\ \chi_{\beta(m-1)} & \chi_{2a-\beta} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{2a-\beta(m-1)} \end{bmatrix}, \; a > \frac{\beta}{2}(m-1).$$

- Jacobi [14] (see also [23,26]):

$$J \equiv Z^T Z, \quad \text{where} \quad Z \equiv \begin{bmatrix} c_m & -s_m c'_{m-1} & & \\ & c_{m-1} s'_{m-1} & \ddots & \\ & & \ddots & -s_2 c'_1 \\ & & & c_1 s'_1 \end{bmatrix},$$

with

$$\begin{aligned} c_k &\sim \sqrt{\mathrm{Beta}\big(a - \tfrac{m-k}{2}\beta, b - \tfrac{m-k}{2}\beta\big)}, \quad s_k = \sqrt{1 - c_k^2}; \\ c'_k &\sim \sqrt{\mathrm{Beta}\big(\tfrac{k}{2}\beta, a + b - \tfrac{2m-k-1}{2}\beta\big)}, \quad s'_k = \sqrt{1 - c_k'^2}. \end{aligned}$$

- MANOVA [7] Let Λ be $m \times m$ beta–Wishart with n degrees of freedom and covariance Ω and let M be $m \times m$ Wishart with p degree of freedom and covariance Λ^{-1}. Then

$$(M^{-1} + I)^{-1}$$

is $m \times m$ beta–MANOVA with parameters n, p and covariance Ω.

3. Identities involving ${}_pF_q^{(\beta)}(a_{1:p}; b_{1:q}; X, Y)$

Let

$$\begin{aligned} r &\equiv \tfrac{m-1}{2}\beta + 1; \\ X &\equiv \mathrm{diag}(x_1, x_2, \ldots, x_m); \\ Y &\equiv \mathrm{diag}(y_1, y_2, \ldots, y_m); \\ \bar{X} &\equiv \mathrm{diag}(x_2 - x_1, x_3 - x_1, \ldots, x_m - x_1); \\ \bar{Y} &\equiv \mathrm{diag}(y_2 - y_1, y_3 - y_1, \ldots, y_m - y_1); \\ \Delta(X) &\equiv \prod_{i<j} (x_i - x_j); \\ \Delta(Y) &\equiv \prod_{i<j} (y_i - y_j). \end{aligned}$$

3.1. Identities for ${}_pF_q^{(\beta)}$

$$
\begin{aligned}
{}_0F_0^{(\beta)}(X) &= e^{\mathrm{tr}(X)}; && (3.1)\\
{}_0F_0^{(\beta)}(X,Y) &= e^{\mathrm{tr}(X)}\,{}_0F_0^{(\beta)}(X, Y-I) && (3.2)\\
&= e^{y_1\mathrm{tr}(X)+x_1\mathrm{tr}(Y)-mx_1y_1}\,{}_1F_1^{(\beta)}(\tfrac{m-1}{2}\beta; \tfrac{m}{2}\beta; \bar{X}, \bar{Y});\\
{}_1F_0^{(\beta)}(a;X) &= |I-X|^{-a}, \qquad (X<I); && (3.3)\\
{}_1F_0^{(\beta)}(a;X,Y) &= |I-X|^{-a}\cdot {}_1F_0^{(\beta)}(a; X(I-X)^{-1}, Y-I); && (3.4)\\
{}_1F_0^{(\beta)}(\tfrac{m}{2}\beta; X, Y) &= \textstyle\prod_{i,j}(1-x_iy_j)^{-\frac{\beta}{2}}; && (3.5)\\
{}_1F_1^{(\beta)}(a;c;X) &= e^{\mathrm{tr}(X)}\cdot {}_1F_1^{(\beta)}(c-a;c;-X); && (3.6)\\
{}_1F_1^{(\beta)}(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X) &= {}_1F_1^{(\beta)}(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X-tI)\cdot e^t; && (3.7)\\
{}_2F_1^{(\beta)}(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; X) &= (1-t)^n\cdot {}_2F_1^{(\beta)}(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; (1-t)X+tI),\ (t\neq 1); && (3.8)\\
{}_2F_1^{(\beta)}(a,b;c;I) &= \frac{\Gamma_m^{(\beta)}(c)\Gamma_m^{(\beta)}(c-a-b)}{\Gamma_m^{(\beta)}(c-a)\Gamma_m^{(\beta)}(c-b)}; && (3.9)\\
{}_2F_1^{(1)}(a,b;c;xI) &= \mathrm{Pf}(A) \quad \text{(see [19] for the definition of } A); && (3.10)\\
{}_2F_1^{(\beta)}(a,b;c;X) &= {}_2F_1^{(\beta)}\left(c-a, b; c; -X(I-X)^{-1}\right)\cdot |I-X|^{-b} && (3.11)\\
&= {}_2F_1^{(\beta)}\left(c-a, c-b; c; X\right)\cdot |I-X|^{c-a-b};\\
{}_2F_1^{(\beta)}(a,b;c;X) &= {}_2F_1^{(\beta)}\big(a,b;a+b+r-c; I-X\big)\\
&\quad\times {}_2F_1^{(\beta)}(a,b;c;I), \quad (a \text{ or } b\in\mathbb{Z}_{\le 0}); && (3.12)\\
{}_pF_q^{(\beta)}(\tfrac{\beta}{2}, a_{2:p}; b_{1:q}; xI) &= {}_pF_q(\tfrac{m}{2}\beta, a_{2:p}; b_{1:q}; x); && (3.13)\\
{}_pF_q^{(2)}(a_{1:p}; b_{1:q}; X) &= \frac{\left|\left(x_i^{m-j}\,{}_pF_q(a_{1:p}-j+1; b_{1:q}-j+1; x_i)\right)_{i,j=1}^m\right|}{\Delta(X)}; && (3.14)\\
{}_pF_q^{(2)}(a_{1:p}; b_{1:q}; X, Y) &= \prod_{i=1}^m \frac{(i-1)!\prod_{j=1}^q (b_j-m+1)_{m-i}}{\prod_{j=1}^p (a_j-m+1)_{m-i}}\\
&\quad\times\frac{\left|\left({}_pF_q(a_{1:p}-m+1; b_{1:q}-m+1; x_iy_j)\right)_{i,j=1}^m\right|}{\Delta(X)\Delta(Y)}; && (3.15)\\
{}_2F_0^{(\beta)}(r,-t;-X) &= \frac{\Gamma_m^{(\beta)}(r+t)}{\Gamma_m^{(\beta)}(r)}|X|^t\sum_{k=0}^{mt}\sum_{\kappa\vdash k,\,\kappa_1\le r}\frac{C_\kappa^{(\beta)}(X^{-1})}{k!}, \quad (t\in\mathbb{Z}_{\ge 0}); && (3.16)\\
{}_pF_q^{(\beta)}(a_{1:p}; b_{1:q}; X) &= \lim_{c\to\infty} {}_{p+1}F_q^{(\beta)}(a_{1:p}, c; b_{1:q}; \tfrac{1}{c}X)\\
&= \lim_{c\to\infty} {}_pF_{q+1}^{(\beta)}(a_{1:p}; b_{1:q}, c; cX). && (3.17)
\end{aligned}
$$

The references for the above identities are:

(3.1) [16, (13.3), p. 593] (also [29, p. 262] for $\beta = 1$ and [31, p. 444] for $\beta = 2$);
(3.2) [2, section 6] (also [27] for $\beta = 1$);
(3.3) [16, p. 593, (13.4)] (also [29, p. 262] for $\beta = 1$ and [31, p. 444] for $\beta = 2$);
(3.4) [27, (6.29)];
(3.5) [27, (6.10)];

(3.6) [16, p. 596, (13.16)] (also [29, (6), p. 265] for the case $\beta = 1$);
(3.7) New result, see Theorem 5.2. For $m = 3$, $\beta = 1$ this is due to Bingham [3, Lemma 2.1];
(3.8) New result, see Theorem 5.2;
(3.9) [16, (13.14), p. 594] (also [5, p. 24, (51)] for the case $\beta = 1$);
(3.10) [19];
(3.11) [16, Proposition 13.1.6, p. 595]; see also [29, p. 265, (7)] for the case $\beta = 1$;
(3.12) [16, Proposition 13.1.7, p. 596]. The condition a or $b \in \mathbb{Z}_{\leq 0}$ implies that the series expansion for ${}_2F_1^{(\beta)}$ terminates.
(3.13) New result, see Theorem 5.3. See also [18, (5.13)] for $\beta = 2$;
(3.14) [30, (33), p. 281];
(3.15) [18, Theorem 4.2] (there is a typo in [18, (4.8)]: β_n^{-1} should be β_n), see also [30, (34), p. 281];
(3.16) [13];
(3.17) [16, (13.5)]. Trivially implied by $\lim_{x\to\infty}(x)_\kappa^{(\beta)} \cdot x^{-|\kappa|} = 1$.

3.2. *Integral identities*

Define

$$c_m^{(\beta)} \equiv \pi^{-\frac{m(m-1)}{4}\beta} m! \prod_{i=1}^{m} \frac{\Gamma(\frac{i}{2}\beta)}{\Gamma(\frac{\beta}{2})}$$

and recall that $r = \frac{m-1}{2}\beta + 1$. Then

$$\begin{aligned} &{}_{p+1}F_q^{(\beta)}(a_{1:p}, a; b_{1:q}; Y) \\ &\qquad = \frac{1}{c_m^{(\beta)}\Gamma_m^{(\beta)}(a)} \int_{\mathbb{R}_+^m} e^{-\mathrm{tr}(X)} {}_pF_q^{(\beta)}(a_{1:p}; b_{1:q}; X, Y)|X|^{a-r} \mathrm{d}\mu(X); \end{aligned} \tag{3.18}$$

$$\begin{aligned} &{}_{p+1}F_{q+1}^{(\beta)}(a_{1:p}, a; b_{1:q}, b; Y) \\ &= \frac{1}{S_m^{(\beta)}(a, b-a)} \int_{[0,1]^m} {}_pF_q^{(\beta)}(a_{1:p}; b_{1:q}; X, Y)|X|^{a-r}|I - X|^{b-a-r} \mathrm{d}\mu(X); \end{aligned} \tag{3.19}$$

$$C_\kappa^{(\beta)}(Y^{-1}) = \frac{|Y|^a}{c_m^{(\beta)}\Gamma_m^{(\beta)}(a) \cdot (a)_\kappa^{(\beta)}} \int_{\mathbb{R}_m^+} {}_0F_0^{(\beta)}(-X, Y)|X|^{a-r} C_\kappa^{(\beta)}(X) \mathrm{d}\mu(X). \tag{3.20}$$

The references for the above identities are:

(3.18) [27, (6.20)] (see also [13]);
(3.19) [27, (6.21)] (see also [13]);
(3.20) This is a conjecture of Macdonald [27, Conjecture C, section 8], proven by Baker and Forrester [2, section 6], with Dubbs and Edelman [7] correcting a typo in [2].

Define $T \equiv \operatorname{diag}(t_1, t_2, \ldots, t_n)$. The following identities are due to Kaneko [22]

$$\int_{[0,1]^m} \prod_{i,k=1}^{m,n} (x_i - t_k) \prod_{i=1}^{m} x_i^a (1-x_i)^b \mathrm{d}\mu(X) = S_m^{(\beta)}(a+r, b+r) \times {}_2F_1^{(\frac{4}{\beta})}\left(-m, \tfrac{2}{\beta}(a+b+n+1)+m-1; \tfrac{2}{\beta}(a+n); T\right); \quad (3.21)$$

$$\int_{[0,1]^m} \prod_{i,k=1}^{m,n} (1 - x_i t_k)^{-\frac{\beta}{2}} \prod_{i=1}^{m} x_i^a (1-x_i)^b \mathrm{d}\mu(X) = S_m^{(\beta)}(a+n+r, b+r) \times {}_2F_1^{(\frac{4}{\beta})}\left(-\tfrac{m\beta}{2}, \tfrac{\beta}{2}(m-1)+a+1; \beta(m-1)+a+b+2; T\right), \quad (3.22)$$

where $S_m^{(\beta)}$ is the value of the Selberg integral (2.8).

3.3. *Identities involving the Jack function*

We will predominantly use the "C" normalization of the Jack function, but there are other normalizations, $J_\kappa^{(\beta)}$, $P_\kappa^{(\beta)}$, and $Q_\kappa^{(\beta)}$, related as in (3.25). The properties of these normalizations are that $J_\kappa^{(\beta)}(X)$ is such that for $|\kappa| = n$ the coefficient of $x_1 x_2 \cdots x_n$ in $J_\kappa^{(\beta)}(X)$ is $n!$ [34, Theorem 1.1]. The zonal polynomial is $C_\kappa^{(1)}(X)$ and $P_\kappa^{(2)}(X) = Q_\kappa^{(2)}(X) = s_\lambda(X)$ is the Schur function.

$$\sum_{\kappa \vdash k} C_\kappa^{(\beta)}(X) = (\mathrm{tr} X)^k; \quad (3.23)$$

$$J_\kappa(xI_m) = (x\tfrac{2}{\beta})^{|\kappa|} \left(\tfrac{m}{2}\beta\right)_\kappa^{(\beta)} \quad (3.24)$$

$$J_\kappa^{(\beta)}(X) = \frac{j_\kappa}{(\frac{2}{\beta})^{|\kappa|}|\kappa|!} C_\kappa^{(\beta)}(X) = H_\kappa^* Q_\kappa^{(\beta)}(X) = H_*^\kappa P_\kappa^{(\beta)}(X); \quad (3.25)$$

$$J_\kappa^{(\beta)}(X) = |X| \cdot J_{(\kappa_1-1,\kappa_2-1,\ldots,\kappa_m-1)}^{(\beta)}(X) \prod_{i=1}^{m} (m-i+1+\tfrac{2}{\beta}(\kappa_i - 1)), \quad (3.26)$$

where $\kappa_m > 0$;

$$\frac{C_\kappa^{(\beta)}(I+X)}{C_\kappa^{(\beta)}(I)} = \sum_{\sigma \subseteq \kappa} \binom{\kappa}{\sigma} \frac{C_\sigma^{(\beta)}(X)}{C_\sigma^{(\beta)}(I)}; \quad (3.27)$$

$$P_{\hat{\mu}}^{(\beta)}(X) = |X|^N P_\mu^{(\beta)}(X^{-1}), \quad (3.28)$$

where $\mu_1 \le N$ and $\hat{\mu}_i = N - \mu_{m+1-i}$, $i = 1, 2, \ldots, m$

(i.e., the partition $\hat{\mu}$ is the complement of μ in (N^m));

For partitions $\kappa = (k)$ in only one part,

$$C_k^{(\beta)}(I) = \left(\tfrac{m}{2}\beta\right)_k \left(\left(\tfrac{\beta}{2}\right)_k\right)^{-1}; \tag{3.29}$$

$$C_k^{(\beta)}(X+tI) = C_k^{(\beta)}(I) \sum_{s=0}^{k} t^{k-s} \binom{k}{s} \frac{C_s^{(\beta)}(X)}{C_s^{(\beta)}(I)} \tag{3.30}$$

$$= \left(\frac{m}{2}\beta\right)_k \sum_{s=0}^{k} \frac{t^{k-s}}{\left(\frac{m}{2}\beta\right)_s} \binom{k}{s} C_s^{(\beta)}(X); \tag{3.31}$$

For partitions $\kappa = (1^k)$

$$J_{(1^k)}^{(\beta)}(I_m) = \left(\tfrac{2}{\beta}\right)^k \left(\tfrac{m}{2}\beta\right)_{(1^k)}^{(\beta)} = \prod_{i=0}^{k-1}(m-i); \tag{3.32}$$

$$C_{(1^k)}^{(\beta)}(I_m) = \frac{1}{j_\kappa}\left(\tfrac{2}{\beta}\right)^k k! \prod_{i=0}^{k-1}(m-i) = \prod_{i=0}^{k-1} \frac{m-i}{i\frac{\beta}{2}+1}; \quad \text{for } \beta = 2 \text{ this is } \tbinom{m}{k}. \tag{3.33}$$

The references for the above identities are:

(3.23) This is the definition of the normalization $C_\kappa^{(\beta)}(X)$ [34, Theorem 2.3];
(3.24) [34, Theorem 5.4];
(3.25) [22, (16)], [28, (10.22), p. 381]. Then $P_\kappa^{(1)} = Q_\kappa^{(1)} = s_\kappa$, the Schur function [34, Proposition 1.2];
(3.26) [34, Propositions 5.1 and 5.5];
(3.27) [22, (33)]. This also serves as a definition for the *generalized binomial coefficient* $\binom{\kappa}{\sigma}$;
(3.28) This is a result of Macdonald [27, (4.5), (6.22)]: Both sides have $x_1^{\hat{\mu}_1} \cdots x_m^{\hat{\mu}_m}$ as the leading term and for the scalar product $\langle \cdot, \cdot \rangle_n'$ defined in [28, (10.35)]), $\langle |X|^N P_\mu(X^{-1}), |X|^N P_\nu(X^{-1}) \rangle_n' = \langle P_\mu(X), P_\nu(X) \rangle_n'$, which is 0 for $\mu \neq \nu$ [28, (10.36)].
(3.29) Directly from (3.24);
(3.31) Directly from (3.27);
(3.32) Directly from (3.24);
(3.33) Directly from (3.25).

3.4. *Multivariate Gamma function identities*

We found the following identities useful in simplifying expressions involving the multivariate Gamma function.

$$\frac{\Gamma_m^{(\beta)}(x)}{\Gamma_m^{(\beta)}\left(x-\frac{\beta}{2}\right)} = \frac{\Gamma(x)}{\Gamma\left(x-\frac{m}{2}\beta\right)}. \tag{3.34}$$

$$\lim_{x\to\infty} \frac{\Gamma_m^{(\beta)}(x+a)}{\Gamma_m^{(\beta)}(x)x^{ma}} = 1. \tag{3.35}$$

The first identity (3.34) is due to Pierre-Antoine Absil in the case $\beta = 1$. In general, from the definition (2.2)

$$\frac{\Gamma_m^{(\beta)}(x)}{\Gamma_m^{(\beta)}\left(x-\frac{\beta}{2}\right)} = \prod_{i=1}^{m} \frac{\Gamma\left(x-\frac{i-1}{2}\beta\right)}{\Gamma\left(x-\frac{i}{2}\beta\right)} = \frac{\Gamma(x)}{\Gamma\left(x-\frac{m}{2}\beta\right)}.$$

The second identity, (3.35)], follows directly from the definition (2.2). See also [13].

4. Distributions of the extreme eigenvalues

In this section we review the known explicit formulas for the densities and distributions of the extreme eigenvalues of the beta ensembles.

4.1. *Wishart*

These results are from [13]. Recall again that $r = \frac{m-1}{2}\beta + 1$. Then

$$P(\lambda_{\max}(A) < x) = \frac{\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}\left(\frac{n}{2}\beta + r\right)} \cdot \frac{\left|\frac{x}{2}\beta\Sigma^{-1}\right|^{\frac{n}{2}\beta}}{e^{\mathrm{tr}\left(\frac{x}{2}\beta\Sigma^{-1}\right)}}\ {}_1F_1^{(\beta)}\left(r; \tfrac{n}{2}\beta + r; \tfrac{x}{2}\beta\Sigma^{-1}\right); \quad (4.1)$$

and when $t \equiv \frac{n}{2}\beta - r$ is a nonnegative integer, then

$$P(\lambda_{\min}(A) < x) = 1 - e^{\mathrm{tr}(-\frac{x}{2}\beta\Sigma^{-1})} \sum_{\kappa\subseteq(m^t)} \frac{1}{|\kappa|!} C_\kappa^{(\beta)}\left(\tfrac{x}{2}\beta\Sigma^{-1}\right). \quad (4.2)$$

It is an open problem to obtain an expression that does not have the requirement for t to be a nonnegative integer—see section 7.

For the trace we have

$$\begin{aligned} \mathrm{dens}_{\,\mathrm{tr}A}(x) &= \left|\tfrac{x}{2}\beta\Sigma^{-1}\right|^{\frac{n}{2}\beta} e^{-\frac{x}{2z}\beta} \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(\frac{nm}{2}\beta + k\right)} \cdot \frac{\beta}{2z}\left(\tfrac{x}{2z}\beta\right)^{k-1} \\ &\quad \times \sum_{\kappa\vdash k} \left(\tfrac{n}{2}\beta\right)_\kappa^{(\beta)} \cdot \frac{1}{k!} \cdot C_\kappa^{(\beta)}(I - z\Sigma^{-1}), \end{aligned} \quad (4.3)$$

where z is arbitrary. Muirhead [29, p. 341] suggests the value $z = 2\sigma_1\sigma_m/(\sigma_1+\sigma_m)$, where $\sigma_1 \geq \cdots \geq \sigma_m > 0$ are the eigenvalues of Σ.

The second sum in (4.3) is the marginal sum of ${}_1F_0^{(\beta)}(\frac{n}{2}\beta; I - z\Sigma^{-1})$ thus the truncation of (4.3) for $|\kappa| \leq M$ can be computed using `mhg` as follows. If s is the vector of eigenvalues of Σ, these marginal sums (for say $k = 0$ through M) are returned in the variable c by the call:

```
[f,c]=mhg(M,α,n/α,[],1-z./s),
```

where $\alpha = 2/\beta$, making the remaining computation trivial.

Empirically, the trace can be generated off the Wishart model from [8], which upon closer observation reveals that

$$\mathrm{tr}(A) \sim \frac{1}{\beta}(\chi^2_{n\beta}\sigma_1 + \chi^2_{n\beta}\sigma_2 + \cdots + \chi^2_{n\beta}\sigma_m).$$

4.2. *Laguerre*

The eigenvalues of the Laguerre ensemble are Wishart distributed with $\Sigma = I$ and $n = a\frac{2}{\beta}$.

Interestingly, we have explicit expressions for the densities of the largest and smallest eigenvalue which are not related in an obvious way to the distributions. In these expressions the matrix argument is of size $m-1$ making them faster to compute (fewer partitions to sum over) than the straightforward derivatives of the distributions.

$$NEWSTUFF \tag{4.4}$$

$$\begin{aligned}\text{dens}_{\lambda_{\min}(L)}(x) &= \tfrac{m}{2}\beta \cdot \frac{\Gamma_m^{(\beta)}\left(\frac{m-1}{2}\beta+1\right)}{\Gamma_m^{(\beta)}(a)} \cdot \left(\tfrac{x}{2}\beta\right)^{tm} e^{-\frac{mx}{2}\beta} \\ &\quad \times {}_2F_0^{(\beta)}\left(\tfrac{m}{2}\beta+1, -t; -\tfrac{2}{\beta x} I_{m-1}\right).\end{aligned} \tag{4.5}$$

The density (4.4) is new; it generalizes the $\beta = 1$ result of Sugiyama [36]—see Theorem **??**. The result for the smallest eigenvalue (4.5) is a generalization of the same result of Krishnaiah and Chang [25] for $\beta = 1$. Dumitriu [9, Theorem 10.1.1, p. 147] established (4.5), but did not provide the scaling constant. We give full derivation in Theorem 5.5.

Forrester [15] used (3.21) to obtain the following expressions for the smallest eigenvalue

$$P(\lambda_{\min} < x) = 1 - e^{-\frac{mx\beta}{2}} {}_1F_1^{(\frac{4}{\beta})}(-m; 2t/\beta, -xI_t); \tag{4.6}$$

$$\text{dens}_{\lambda_{\min}(L)}(x) = mx^r e^{-\frac{mx\beta}{2}} \frac{\mathcal{K}_{m-1}^{(\beta)}(a+\beta)}{\mathcal{K}_m^{(\beta)}(a)} {}_1F_1^{(\frac{4}{\beta})}(-m+1; 2t/\beta+2; -xI_t), \tag{4.7}$$

with $\mathcal{K}_m^{(\beta)}(a)$ as in (2.7).

The expression (4.6) is identical (as a function of x) to (4.2)—the ${}_1F_1^{(\frac{4}{\beta})}$ function in (4.6) and the truncated ${}_0F_0^{(\beta)}$ function in (4.2) are the same polynomial in x of degree mt.

4.3. *Jacobi*

Let the $m \times m$ matrix C be Jacobi distributed with parameters a, b. Define

$$D_m^{(\beta)}(a,b) \equiv \frac{\Gamma_m^{(\beta)}(a+b)\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(a+r)\Gamma_m^{(\beta)}(b)}$$

and recall that $r = \frac{m-1}{2}\beta + 1$. Then [11]

$$P(\lambda_{\max}(C) < x) = D_m^{(\beta)}(a,b) \cdot x^{ma} \cdot {}_2F_1^{(\beta)}(a, r-b; a+r; xI); \tag{4.8}$$

$$\begin{aligned}\text{dens}_{\lambda_{\max}(C)}(x) &= D_m^{(\beta)}(a,b) \cdot ma(1-x)^{b-r} x^{ma-1} \\ &\quad \times {}_2F_1^{(\beta)}(a - \tfrac{\beta}{2}, r-b; a+r; xI_{m-1}).\end{aligned} \tag{4.9}$$

When $t \equiv b-r$ is a nonnegative integer, the above expressions are finite and (3.12) yields the following alternatives:

$$P(\lambda_{\max}(C) < x) = x^{ma} \sum_{k=0}^{mt} \sum_{\kappa \vdash k,\, \kappa_1 \le t} \frac{(a)_\kappa^{(\beta)} C_\kappa^{(\beta)}((1-x)I)}{k!}; \tag{4.10}$$

$$\begin{aligned} \operatorname{dens}_{\lambda_{\max}(C)}(x) &= \frac{\Gamma\big(a+b-\frac{m-1}{2}\beta\big)\Gamma(b+\frac{\beta}{2})\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(b-\frac{m-2}{2}\beta\big)\Gamma(t+1)\Gamma\big(\frac{m}{2}\beta\big)\Gamma(a)} \\ &\quad \times (1-x)^t x^{ma-1} {}_2F_1^{(\beta)}(a-\tfrac{\beta}{2}, -t; -t-\beta; xI_{m-1}). \end{aligned} \tag{4.11}$$

For the smallest eigenvalue we use that the matrix $C' = I - C$ is Jacobi distributed with parameters b, a:

$$P(\lambda_{\min}(C) < x) = 1 - P(\lambda_{\max}(C') < 1-x) \tag{4.12}$$

$$\operatorname{dens}_{\lambda_{\min}(C)}(x) = \operatorname{dens}_{\lambda_{\max}(C')}(1-x). \tag{4.13}$$

The distribution (4.8) is due to Dumitriu and Koev [11]. The density result (4.9) is new (see section 5.3). For $\beta = 1$, the above results are due to Constantine [6, (61)] (eq. (4.8)), Absil, Edelman, and Koev [1] (eq. (4.9)), and Muirhead [29, (37), p. 483] (eq. (4.10)).

5. New results

We now prove the identities (3.2), (3.7), (3.8), and (3.13) (section 5.1), and prove the formulas for the densities of the smallest eigenvalue of the Jacobi ensemble, (4.9) (section 5.3) and the Laguerre ensemble (4.5) (section 5.4).

5.1. *New identities for ${}_pF_q^{(\beta)}$*

Theorem 5.1. *With the notation from Section 3.1 the identity*

$$\begin{aligned} {}_0F_0^{(\beta)}(X,Y) &= e^{\operatorname{tr} X} {}_0F_0^{(\beta)}(X, Y-I) \\ &= e^{y_1 \operatorname{tr} X + x_1 \operatorname{tr} Y - m x_1 y_1} {}_1F_1^{(\beta)}(\tfrac{m-1}{2}\beta; \tfrac{m}{2}\beta; \bar{X}, \bar{Y}) \end{aligned} \tag{5.1}$$

holds for any $\beta > 0$.

Proof. The first part is from Baker and Forrester [2, sec. 6]. Let I_k be the identity matrix of size k. Since $C_\kappa^{(\beta)}(X - x_1 I_m) = C_\kappa^{(\beta)}(\bar{X})$ and analogously for Y,

$$\begin{aligned} {}_0F_0^{(\beta)}(X,Y) &= e^{y_1 \operatorname{tr} X} \cdot {}_0F_0^{(\beta)}(X, Y - y_1 I_m) \\ &= e^{y_1 \operatorname{tr} X + x_1 \operatorname{tr} Y - m x_1 y_1} \cdot {}_0F_0^{(\beta)}(X - x_1 I_m, Y - y_1 I_m) \\ &= e^{y_1 \operatorname{tr} X + x_1 \operatorname{tr} Y - m x_1 y_1} \cdot \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{1}{k!} \cdot \frac{C_\kappa^{(\beta)}(X - x_1 I_m) C_\kappa^{(\beta)}(Y - y_1 I_m)}{C_\kappa^{(\beta)}(I_m)} \\ &= e^{y_1 \operatorname{tr} X + x_1 \operatorname{tr} Y - m x_1 y_1} \cdot \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{1}{k!} \cdot \frac{C_\kappa^{(\beta)}(I_{m-1})}{C_\kappa^{(\beta)}(I_m)} \cdot \frac{C_\kappa^{(\beta)}(\bar{X}) C_\kappa^{(\beta)}(\bar{Y})}{C_\kappa^{(\beta)}(I_{m-1})} \\ &= e^{y_1 \operatorname{tr} X + x_1 \operatorname{tr} Y - m x_1 y_1} \cdot {}_1F_1^{(\beta)}(\tfrac{m-1}{2}\beta; \tfrac{m}{2}\beta; \bar{X}, \bar{Y}). \end{aligned}$$

□

The following two theorems use the fact that $(\frac{\beta}{2})^{(\beta)}_\kappa = 0$ for partitions κ in more than one part.

Theorem 5.2. *Let the matrices X and I be $m \times m$. Let t and n be real numbers and let $\beta > 0$. The following identities hold:*

$$ {}_1F_1^{(\beta)}\big(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X + tI\big) = {}_1F_1^{(\beta)}\big(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X\big) \cdot e^t; \tag{5.2} $$

$$ {}_2F_1^{(\beta)}(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; X) = (1-t)^n {}_2F_1^{(\beta)}(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; (1-t)X + tI), \quad t \neq 1. \tag{5.3} $$

Proof. We transform the left hand side of (5.2) using (3.29) and (3.31):

$$ \begin{aligned} {}_1F_1^{(\beta)}\big(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X + tI\big) &= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{\left(\frac{\beta}{2}\right)_k}{\left(\frac{m}{2}\beta\right)_k} \cdot C_k^{(\beta)}(X + tI) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{C_k^{(\beta)}(X + tI)}{C_k^{(\beta)}(I)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} t^{k-s} \cdot \binom{k}{s} \cdot \frac{C_s^{(\beta)}(X)}{C_s^{(\beta)}(I)} \\ &= \sum_{i=0}^{\infty} \frac{C_i^{(\beta)}(X)}{C_i^{(\beta)}(I)} \cdot \sum_{j=i}^{\infty} \frac{t^{j-i}}{j!} \cdot \binom{j}{i} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{\left(\frac{\beta}{2}\right)_k}{\left(\frac{m}{2}\beta\right)_k} \cdot C_k^{(\beta)}(X) \cdot \sum_{j=i}^{\infty} \frac{t^{j-i}}{(j-i)!} \\ &= {}_1F_1^{(\beta)}\big(\tfrac{\beta}{2}; \tfrac{m}{2}\beta; X\big) \cdot e^t. \end{aligned} $$

We use (3.31) again to obtain (5.3):

$$ \begin{aligned} &(1-t)^n {}_2F_1^{(\beta)}\big(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; (1-t)X + tI\big) \\ &\quad = (1-t)^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{\left(\frac{\beta}{2}\right)_k (n)_k}{\left(\frac{m}{2}\beta\right)_k} \cdot C_k^{(\beta)}((1-t)X + tI) \\ &\quad = (1-t)^n \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \sum_{s=0}^{k} t^{k-s} \binom{k}{s} \frac{C_s^{(\beta)}((1-t)X)}{C_s^{(\beta)}(I)} \\ &\quad = (1-t)^n \sum_{i=0}^{\infty} \frac{C_i^{(\beta)}(X)}{C_i^{(\beta)}(I)} (1-t)^i \sum_{j=i}^{\infty} \frac{t^{j-i}}{j!} \binom{j}{i} (n)_j \\ &\quad = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{C_i^{(\beta)}(X)}{C_i^{(\beta)}(I)} (n)_i \cdot \underbrace{(1-t)^{n+i} \sum_{l=0}^{\infty} \frac{t^l}{l!} (n+i)_l}_{1} \\ &\quad = {}_2F_1^{(\beta)}(\tfrac{\beta}{2}, n; \tfrac{m}{2}\beta; X). \end{aligned} $$

□

Theorem 5.3.

$$ {}_pF_q^{(\beta)}(\tfrac{\beta}{2}, a_{2:p}; b_{1:q}; xI) = {}_pF_q(\tfrac{m}{2}\beta, a_{2:p}; b_{1:q}; x). $$

Proof. Using (3.29),

$$ \begin{aligned} {}_pF_q^{(\beta)}(\tfrac{\beta}{2}, a_{2:p}; b_{1:q}; xI) &= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{(\frac{\beta}{2})_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} C_k^{(\beta)}(xI) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{(\frac{\beta}{2})_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} x^k \frac{(\frac{m}{2}\beta)_k}{(\frac{\beta}{2})_k} \\ &= {}_pF_q(\tfrac{m}{2}\beta, a_{2:p}; b_{1:q}; x). \end{aligned} $$

□

5.2. *Density of the largest eigenvalue of the Jacobi ensemble*

We repeat the argument in [1] to extend the result for the density of the smallest eigenvalue of the Jacobi ensemble to any $\beta > 0$ (see equations (4.9) and (4.13)). We start with the joint density of the eigenvalues of the Jacobi ensemble from Table 2.2:

$$ \mathrm{dens}(\lambda_1, \lambda_2, \ldots, \lambda_m) \equiv \frac{1}{S_m^{(\beta)}(a,b)} \prod_{i=1}^{m} \lambda_i^{a-r}(1-\lambda_i)^{b-r} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m. $$

where, as before, $r \equiv \frac{m-1}{2}\beta + 1$. Let $x \equiv \lambda_1 \geq \lambda_2 \geq \cdots \lambda_1$.

$$ \begin{aligned} \mathrm{dens}(x) &= m \int_{[0,x]^{m-1}} \mathrm{dens}(x, \lambda_2, \ldots, \lambda_m) \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_m \\ &= \frac{m}{S_m^{(\beta)}(a,b)} x^{a-r}(1-x)^{b-r} \int_{[0,x]^{m-1}} \prod_{1<i<j} |\lambda_i - \lambda_j|^\beta \\ &\quad \times \prod_{i=2}^{m} |x - \lambda_i|^\beta \lambda_i^{a-r} (1-\lambda_i)^{b-r} \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_m. \end{aligned} $$

We change variables $\lambda_i = xt_i, i = 2, 3, \ldots, m$.

$$ \begin{aligned} \mathrm{dens}(x) = {}& \frac{m}{S_m^{(\beta)}(a,b)} x^{am-1}(1-x)^{b-r} \int_{[0,1]^{m-1}} |t_i - t_j|^\beta \\ & \times \prod_{i=1}^{m-1} (1-xt_i)^{b-r} t_i^{a-r} (1-t_i)^\beta \mathrm{d}t_2 \cdots \mathrm{d}t_m. \end{aligned} $$

From (3.19) we have for $T \equiv \mathrm{diag}(t_2, t_3, \ldots, t_{m-1})$, a matrix of size $m-1$,

$$ \begin{aligned} {}_2F_1^{(\beta)}\left(r-b, a+\tfrac{\beta}{2}; a+\tfrac{m+1}{2}\beta+1; xI_{m-1}\right) = {}& \frac{1}{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)} \int_{[0,1]^{m-1}} \prod_{i<j} |t_i - t_j|^\beta \\ & \times \prod_{i=2}^{m} (1-xt_i)^{b-r} t_i^{a-r} (1-t_i)^\beta \mathrm{d}t_2 \cdots \mathrm{d}t_{m-2}. \end{aligned} $$

(Note that ${}_1F_0^{(\beta)}(r-a, xI_{m-1}, T) = {}_1F_0^{(\beta)}(r-a, xT) = \prod_{i=2}^m (1-xt_i)^{a-r}$.)

Therefore

$$\begin{aligned}\mathrm{dens}(x) &= m\frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{(a-r)m}(1-x)^{mb-1} \\ &\quad \times\ {}_2F_1^{(\beta)}\left(-a+r, \tfrac{\beta}{2}+r; b+r; \tfrac{x-1}{x} I_{m-1}\right),\end{aligned}$$

which using (3.11) gives

$$\begin{aligned}\mathrm{dens}(x) &= m\frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{(a-r)m}(1-x)^{mb-1} \\ &\quad \times\ {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right) x^{-(a-r)(m-1)} \\ &= m\frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{a-r}(1-x)^{mb-1} \\ &\quad \times\ {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right), \\ &= mb\cdot D_m^{(\beta)}(b,a)\cdot x^{a-r}(1-x)^{mb-1} \\ &\quad \times\ {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right),\end{aligned}$$

which yields (4.9) and (4.13).

5.3. *Density of the smallest eigenvalue of the Jacobi ensemble*

We repeat the argument in [1] to extend the result for the density of the smallest eigenvalue of the Jacobi ensemble to any $\beta > 0$ (see equations (4.9) and (4.13)). We start with the joint density of the eigenvalues of the Jacobi ensemble from Table 2.2:

$$\mathrm{dens}(\lambda_1, \lambda_2, \ldots, \lambda_m) \equiv \frac{1}{S_m^{(\beta)}(a,b)} \prod_{i=1}^m \lambda_i^{a-r}(1-\lambda_i)^{b-r} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_m.$$

where, as before, $r \equiv \frac{m-1}{2}\beta + 1$. We integrate all but the smallest eigenvalue (call it x) out of the above density:

$$\begin{aligned}\mathrm{dens}(x) &= m\int_{[x,1]^{m-1}} \mathrm{dens}(\lambda_1, \ldots, \lambda_{m-1}, x)\mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{m-1} \\ &= \frac{m}{S_m^{(\beta)}(a,b)} x^{a-r}(1-x)^{b-r} \int_{[x,1]^{m-1}} \prod_{i<j\le m-1} |\lambda_i - \lambda_j|^\beta \\ &\quad \times \prod_{i=1}^{m-1} |\lambda_i - x|^\beta \lambda_i^{a-r}(1-\lambda_i)^{b-r} \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{m-1}.\end{aligned}$$

We change variables $\lambda_i = (1-x)t_i + x, i = 1, 2, \ldots, m-1$.

$$\begin{aligned}\mathrm{dens}(x) = &\frac{m}{S_m^{(\beta)}(a,b)} x^{(a-r)m}(1-x)^{mb-1} \int_{[0,1]^{m-1}} |t_i - t_j|^\beta \\ &\times \prod_{i=1}^{m-1} t_i^\beta \left(1 - \frac{x-1}{x} t_i\right)^{a-r} (1-t_i)^{b-r} \mathrm{d}t_1 \cdots \mathrm{d}t_{m-1}.\end{aligned}$$

From (3.19) we have for $T \equiv \mathrm{diag}(t_1, t_2, \ldots, t_{m-1})$, a matrix of size $m-1$,

$$\begin{aligned}{}_2F_1^{(\beta)}\left(r-a, \tfrac{\beta}{2}+r; b+r; zI_{m-1}\right) = &\frac{1}{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)} \int_{[0,1]^{m-1}} \prod_{i<j} |t_i - t_j|^\beta \\ &\times \prod_{i=1}^{m-1} (1-zt_i)^{a-r} t_i^\beta (1-t_i)^{b-r} \mathrm{d}t_1 \cdots \mathrm{d}t_{m-1}.\end{aligned}$$

(Note that ${}_1F_0^{(\beta)}(-a+r, zI_{m-1}, T) = {}_1F_0^{(\beta)}(-a+r, zT) = \prod_{i=1}^{m-1}(1-zt_i)^{a-r}$.)

Now, for $z = \frac{x-1}{x}$

$$\begin{aligned}\mathrm{dens}(x) = &m \frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{(a-r)m}(1-x)^{mb-1} \\ &\times {}_2F_1^{(\beta)}\left(-a+r, \tfrac{\beta}{2}+r; b+r; \tfrac{x-1}{x} I_{m-1}\right),\end{aligned}$$

which using (3.11) gives

$$\begin{aligned}\mathrm{dens}(x) &= m \frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{(a-r)m}(1-x)^{mb-1} \\ &\quad\times {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right) x^{-(a-r)(m-1)} \\ &= m \frac{S_{m-1}^{(\beta)}\left(\frac{\beta}{2}+r, b-\frac{\beta}{2}\right)}{S_m^{(\beta)}(a,b)} x^{a-r}(1-x)^{mb-1} \\ &\quad\times {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right), \\ &= mb \cdot D_m^{(\beta)}(b,a) \cdot x^{a-r}(1-x)^{mb-1} \\ &\quad\times {}_2F_1^{(\beta)}\left(-a+r, b-\tfrac{\beta}{2}; b+r; (1-x)I_{m-1}\right),\end{aligned}$$

which yields (4.9) and (4.13).

5.4. *The density of the smallest eigenvalue of the Laguerre ensemble*

We will obtain this result as a limiting argument from the density of the smallest eigenvalue of the Jacobi ensemble. The connection is that if λ_i are the eigenvalues of a Jacobi matrix, then $\bar{\lambda}_i \equiv \lim_{b\to\infty} \frac{2b\lambda_i}{\beta(1-\lambda_i)}$ are the eigenvalues of a Laguerre matrix.

Proposition 5.1. *Let $L_m^{(\beta)}(a;\Lambda)$ be the joint eigenvalue density of the Laguerre ensemble and $J_m^{(\beta)}(a,b;\Lambda)$ be the joint eigenvalue density of the Jacobi ensemble as defined in Table 2.2. Then*

$$\lim_{b\to\infty} \frac{1}{(\frac{2}{\beta}b)^m} \cdot J_m^{(\beta)}\big(a,b;\Lambda(\tfrac{2}{\beta}bI+\Lambda)^{-1}\big) = L_m^{(\beta)}(a;\Lambda).$$

Proof.

$$\begin{aligned}
\frac{J_m^{(\beta)}\big(a,b;\Lambda(\frac{2}{\beta}bI+\Lambda)^{-1}\big)}{(\frac{2}{\beta}b)^m} &= \frac{1}{(\frac{2}{\beta}b)^{ma}}\cdot\frac{1}{S_m^{(\beta)}(a,b)} \\
&\quad \times \prod_{i=1}^m \lambda_i^{a-\frac{m-1}{2}\beta-1}\big(1+\tfrac{\beta}{2b}\lambda_i\big)^{2-a-b}\prod_{j<k}|\lambda_j-\lambda_k|^\beta \\
&= \frac{\pi^{\frac{m(m-1)}{2}\beta}\Big(\Gamma\big(\frac{\beta}{2}\big)\Big)^m}{(\frac{2}{\beta})^{ma} m!\,\Gamma_m^{(\beta)}\big(\frac{m}{2}\beta\big)\Gamma_m^{(\beta)}(a)}\cdot\frac{\Gamma_m^{(\beta)}(a+b)}{\Gamma_m^{(\beta)}(b)b^{ma}} \\
&\quad \times \prod_{i=1}^m \lambda_i^{a-\frac{m-1}{2}\beta-1}\big(1+\tfrac{\lambda_i}{2b}\beta\big)^{2-a-b}\prod_{j<k}|\lambda_j-\lambda_k|^\beta.
\end{aligned}$$

We then take the limit as $b\to\infty$ and use (3.35) to obtain the desired result. □

Next, we derive the density of the largest eigenvalue of the Laguerre ensemble.

Theorem 5.4. *The density of the largest eigenvalue of the Laguerre ensemble is*

$$\frac{nm\beta^2\Gamma_m^{(\beta)}(r)}{4\Gamma_m^{(\beta)}\big(\frac{n\beta}{2}+r\big)}e^{-\frac{m\beta x}{2}}\left(\frac{\beta x}{2}\right)^{\frac{nm\beta}{2}-1}{}_1F_1^{(\beta)}\big(\tfrac{m\beta}{2}+1;\tfrac{n\beta}{2}+r;\tfrac{x\beta}{2}I_{m-1}\big). \tag{5.4}$$

Proof. The density of the maximum eigenvalue of the Jacobi ensemble with parameters a and b is given by

$$\begin{aligned}
f(x) &= \frac{\Gamma_m^{(\beta)}(a+b)\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(a+r)\Gamma_m^{(\beta)}(b)}ma(1-x)^{b-r}x^{ma-1}{}_2F_1^{(\beta)}\left(a-\frac{\beta}{2},r-b;a+r;xI_{m-1}\right) \\
&= \frac{\Gamma_m^{(\beta)}(a+b)\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(a+r)\Gamma_m^{(\beta)}(b)}ma(1-x)^{b-1-a+\frac{\beta}{2}}x^{ma-1}{}_2F_1^{(\beta)}\left(a-\frac{\beta}{2},a+b;a+r;-\frac{x}{1-x}I_{m-1}\right)
\end{aligned} \tag{5.5}$$

where $r=(m-1)\beta/2+1$. Let $y=(2b/\beta)x/(1-x)$, we have

$$\frac{1}{1-x} = 1+\frac{\beta y}{2b}, \tag{5.6}$$

$$x = \frac{\frac{\beta y}{2b}}{1+\frac{\beta y}{2b}}, \tag{5.7}$$

$$\mathrm{d}x = \frac{\beta(1-x)^2}{2b}\mathrm{d}y, \tag{5.8}$$

and the density function of y is given by

$$\begin{aligned} f(y) &= \frac{\Gamma_m^{(\beta)}(a+b)\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(a+r)\Gamma_m^{(\beta)}(b)}\frac{ma\beta}{2b}(1-x)^{b+1-a-\frac{\beta}{2}}x^{ma-1}{}_2F_1^{(\beta)}\left(a-\frac{\beta}{2},a+b;a+r;-\frac{\beta y}{2b}I_{m-1}\right) \\ &= \frac{b^{-ma}\Gamma_m^{(\beta)}(a+b)\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(b)\Gamma_m^{(\beta)}(a+r)}\frac{ma\beta}{2}\left(\frac{\beta y}{2}\right)^{ma-1}\left(1+\frac{\beta y}{2b}\right)^{-(m-1)a-b-\frac{\beta}{2}} \\ &\quad\times {}_2F_1^{(\beta)}\left(a-\frac{\beta}{2},a+b;a+r;-\frac{\beta y}{2b}I_{m-1}\right). \end{aligned} \tag{5.9}$$

Using the fact that

$$\lim_{b\to\infty}\frac{b^{-ma}\Gamma_m^{(\beta)}(a+b)}{\Gamma_m^{(\beta)}(b)} = 1, \tag{5.10}$$

$$\lim_{b\to\infty}\left(1+\frac{\beta y}{2b}\right)^{-(m-1)a-b-\frac{\beta}{2}} = e^{-\frac{\beta y}{2}}, \tag{5.11}$$

$$\begin{aligned} \lim_{b\to\infty}{}_2F_1^{(\beta)}\left(a-\frac{\beta}{2},a+b;a+r;-\frac{\beta y}{2b}I_{m-1}\right) &= {}_1F_1^{(\beta)}\left(a-\frac{\beta}{2};a+r;-\frac{\beta y}{2}I_{m-1}\right) \\ &= e^{-\frac{\beta y(m-1)}{2}}{}_1F_1^{(\beta)}\left(r+\frac{\beta}{2};a+r;\frac{\beta y}{2}I_{m-1}\right) \end{aligned} \tag{5.12}$$

we have

$$\lim_{b\to\infty}f(y) = \frac{\Gamma_m^{(\beta)}(r)}{\Gamma_m^{(\beta)}(a+r)}\left(\frac{ma\beta}{2}\right)\left(\frac{\beta y}{2}\right)^{ma-1}e^{-\frac{m\beta y}{2}}{}_1F_1^{(\beta)}\left(\frac{m\beta}{2}+1;a+r;\frac{\beta y}{2}I_{m-1}\right), \tag{5.13}$$

and this is the density of the maximum eigenvalue of Laguerre ensemble with parameter a. Putting $a = n\beta/2$, we prove (5.4). □

We now derive the scaling constant in the formula for the density of the Laguerre ensemble, (4.5). The proportionality result is due to Dumitriu [9, Theorem 10.1.1, p. 146].

Theorem 5.5. *When $t \equiv a - \frac{m-1}{2}\beta - 1$ is a nonnegative integer, the density of the smallest eigenvalue of a Laguerre matrix is*

$$\begin{aligned} \mathrm{dens}_{\lambda_{\min}(A)}(y) = \tfrac{m}{2}\beta\cdot\frac{\Gamma_m^{(\beta)}(\frac{m-1}{2}\beta+1)}{\Gamma_m^{(\beta)}(a)}\cdot e^{-\frac{my}{2}\beta}(\tfrac{y}{2}\beta)^{mt} \\ \times {}_2F_0^{(\beta)}(\tfrac{m}{2}\beta+1,-t;-\tfrac{2}{\beta y}I_{m-1}). \end{aligned} \tag{5.14}$$

Proof. Let C be Jacobi distributed. From (4.9), (4.13), and the identity (3.11) we have

$$\begin{aligned} \mathrm{dens}_{\lambda_{\min}(C)}(x) &= mb\cdot D_m^{(\beta)}(b,a)\cdot(1-x)^{mb-1}x^t \\ &\quad\times {}_2F_1^{(\beta)}(b-\tfrac{\beta}{2},-t;b+\tfrac{m-1}{2}\beta+1;(1-x)I_{m-1}) \\ &= mb\cdot D_m^{(\beta)}(b,a)\cdot(1-x)^{mb-1}x^{mt} \\ &\quad\times {}_2F_1^{(\beta)}(\tfrac{m}{2}\beta+1,-t;b+\tfrac{m-1}{2}\beta+1;-\tfrac{1-x}{x}I_{m-1}). \end{aligned}$$

Substituting $y = \frac{2b}{\beta} \cdot \frac{x}{1-x}$ (with the Jacobian being $\frac{\beta}{2b}(1-x)^2$) we get:

$$\begin{aligned}\mathrm{dens}_{\lambda_{\min}(\frac{2b}{\beta}C(I-C)^{-1})}(y) = {} & \tfrac{m}{2}\beta \cdot \frac{\Gamma_m^{(\beta)}\left(\frac{m-1}{2}\beta+1\right)}{\Gamma_m^{(\beta)}(a)} \cdot \frac{\Gamma_m^{(\beta)}(a+b)\cdot b^{-mt}}{\Gamma_m^{(\beta)}\left(b+\frac{m-1}{2}\beta+1\right)} \\ & \times \left(1+\tfrac{\beta y}{2b}\right)^{-mb-1}\left(\tfrac{y}{2}\beta\right)^{mt}\left(1+\tfrac{\beta y}{2b}\right)^{-mt} \\ & \times {}_2F_1^{(\beta)}(\tfrac{m}{2}\beta+1,-t;b+\tfrac{m-1}{2}\beta+1;b(-\tfrac{2}{\beta y})I_{m-1}).\end{aligned}$$

We use Proposition 5.1, (3.17), and (3.35) to take the limit as $b \to \infty$ and obtain (5.14). □

6. Numerical experiments

All formulas for the distributions and densities in this paper can be approximated (within the limits of the computational power of modern computers) using the software `mhg` [24].

It returns the truncation

$$\sum_{k=0}^{M}\sum_{\kappa\vdash k}\frac{1}{k!}\cdot\frac{(a_1)_\kappa^{(\beta)}\cdots(a_p)_\kappa^{(\beta)}}{(b_1)_\kappa^{(\beta)}\cdots(b_q)_\kappa^{(\beta)}}\cdot\frac{C_\kappa^{(\beta)}(X)C_\kappa^{(\beta)}(Y)}{C_\kappa^{(\beta)}(I)}.$$

of (2.5) as well as the partial sums

$$\sum_{\kappa\vdash k}\frac{1}{k!}\cdot\frac{(a_1)_\kappa^{(\beta)}\cdots(a_p)_\kappa^{(\beta)}}{(b_1)_\kappa^{(\beta)}\cdots(b_q)_\kappa^{(\beta)}}\cdot\frac{C_\kappa^{(\beta)}(X)C_\kappa^{(\beta)}(Y)}{C_\kappa^{(\beta)}(I)}$$

for $k = 0, 1, \ldots, M$ with the additional option to restrict the summation for $\kappa_1 \le t$ (for, e.g., (4.2)). All expressions in this paper are trivially approximated then using these partial sums.

We performed extensive numerical tests to verify the correctness of the formulas in this paper and present four examples in Figure 1.

7. Open problems and future work

We finish with two open problems.

Several formulas in this paper, e.g., (4.2), (4.10), are only valid when certain parameters are integers. It is an open problem to derive formulas that do not have that restriction. The central problem there is to find an explicit expression (perhaps in terms of ${}_pF_q^{(\beta)}$) for the integral

$$\int_{\mathbb{R}_m^+} {}_0F_0^{(\beta)}(-X,Y)|X|^a|I+X|^b \mathrm{d}\mu(X).$$

This is the multivariate version of the function Ψ, the confluent hypergeometric function of the second kind.

The second open problem we pose is to find explicit formulas for the extreme eigenvalues of the Hermite ensemble, which are analogous to the ones for the other

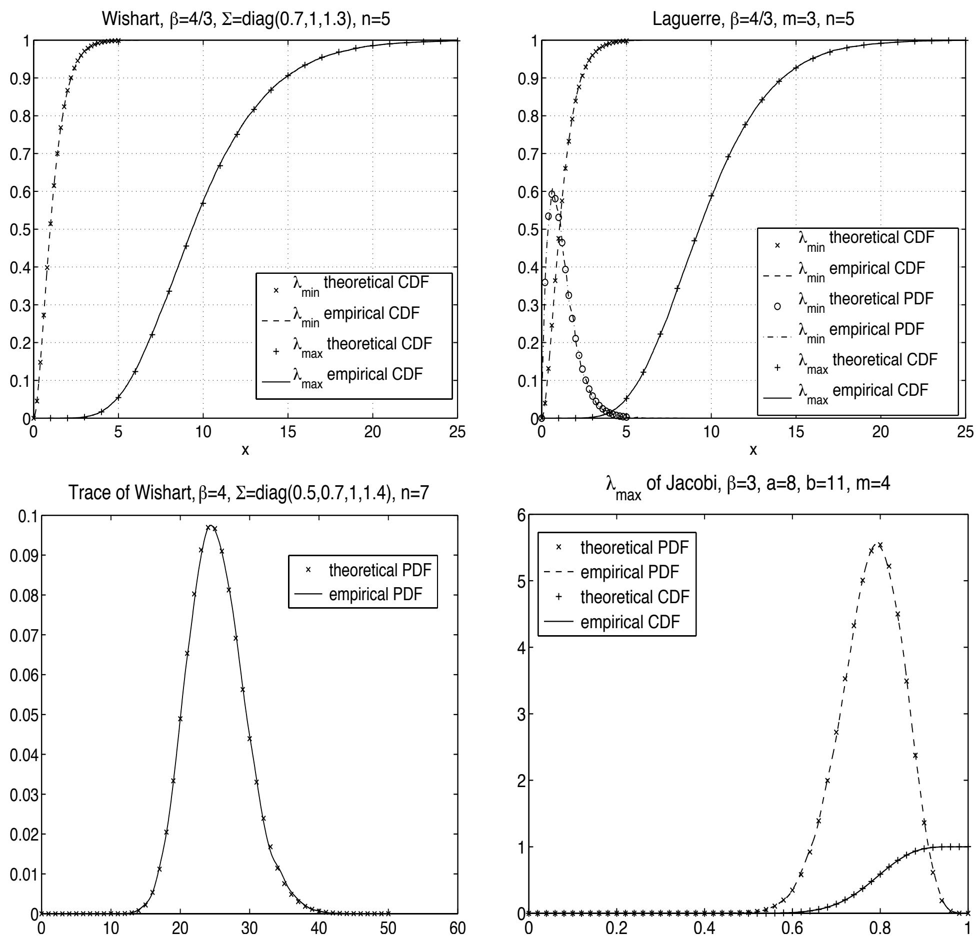

Fig. 1. Numerical experiments comparing the theoretical predictions of the eigenvalue distributions for various ensembles vs. the empirical results.

ensembles. This problem boils down to evaluating the integral

$$\int_{[x,\infty)^m} e^{-\frac{\mathrm{tr}(X^2)}{2}} \mathrm{d}\mu(X).$$

Bornemann [4] presents algorithms for numerical evaluation of distributions in Random Matrix Theory that are expressible as Painlevé transcendents or Fredholm determinants in the $\beta = 1, 2, 4$ cases. It is an open problem to devise good numerical routines for the general β versions of some of these quantities, especially the bulk, hard edge, and soft edge statistics.

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