The beta-Wishart ensemble

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We introduce a "broken-arrow" matrix model for the β-Wishart ensemble, which improves on the traditional bidiagonal model by generalizing to non-identity covariance parameters. We prove that its joint eigenvalue density involves the correct hypergeometric function of two matrix arguments, and a continuous parameter $\beta > 0$. If we choose $\beta = 1, 2, 4$, we recover the classical Wishart ensembles of general covariance over the reals, complexes, and quaternions. Jack polynomials are often defined as the eigenfunctions of the Laplace-Beltrami operator. We prove that Jack polynomials are in addition eigenfunctions of an integral operator defined as an average over a β-dependent measure on the sphere. When combined with an identity due to Stanley, we derive a definition of Jack polynomials. An efficient numerical algorithm is also presented for simulations. The algorithm makes use of secular equation software for broken arrow matrices currently unavailable in the popular technical computing languages. The simulations are matched against the cdfs for the extreme eigenvalues. The techniques here suggest that arrow and broken arrow matrices can play an important role in theoretical and computational random matrix theory including the study of corners processes. We provide a number of simulations illustrating the extreme eigenvalue distributions that are likely to be useful for applications. We also compare the $n \to \infty$ answer for all β with the free-probability prediction. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4818304]

I. INTRODUCTION

A real $m \times n$ Wishart matrix $W(D, m, n)$ is the random matrix $Z^t Z$ where Z consists of n columns of length m, each of which is a multivariate normal with mean 0 and covariance D. We can assume without loss of generality that D is a non-negative diagonal matrix. We may write Z as `randn(m,n)*sqrt(D)` using the notation of modern technical computing software. Real Wishart matrices arise in such applications as likelihood-ratio tests (summarized in Chapter 8 of Muirhead[18]), multidimensional Bayesian analysis,[2,8] and random matrix theory in general.[13]

For the special case that $D = I$, the real Wishart matrix is also known as the $\beta = 1$ Laguerre ensemble. The complex and quaternion versions correspond to $\beta = 2$ and $\beta = 4$, respectively. The method of bidiagonalization has been very successful[4] in creating matrix models that generalize the Laguerre ensemble to arbitrary β. The key features of the Dumitriu and Edelman model[4] appear in the box below:

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Beta-Laguerre model

$$
B = \begin{bmatrix}
\chi_{m\beta} & \chi_{(n-1)\beta} & & & \\
 & \chi_{(m-1)\beta} & \chi_{(n-2)\beta} & & \\
 & & \ddots & \ddots & \\
 & & & \chi_{(m-n+2)\beta} & \chi_{\beta} \\
 & & & & \chi_{(m-n+1)\beta}
\end{bmatrix}.
$$

Let n be a positive integer and m be a real greater than $n-1$. $\mathrm{eig}(B^t B)$ has density:

$$
C_\beta^L \prod_{i=1}^n \lambda_i^{\frac{m-n+1}{2}\beta - 1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \exp(-\tfrac{1}{2}\textstyle\sum_{i=1}^n \lambda_i) d\lambda
$$

</div>

For general D, it is desirable to create a general β model as well. For $\beta = 1, 2, 4$, it is obviously possible to bidiagonalize a full matrix of real, complex, or quaternionic normals and obtain a real bidiagonal matrix. However, these bidiagonal models do not appear to generalize nicely to arbitrary β. Therefore, we propose a new broken-arrow model that possesses a number of very good mathematical (and computational) possibilities. These models connect to the mathematical theory of Jack polynomials and the general-β hypergeometric functions of matrix arguments. The key features of the broken arrow models appear in the box below:

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Beta $-$ Wishart (recursive) model, $W^{(\beta)}(D,m,n)$

$$
Z = \begin{bmatrix}
\tau_1 & & & \chi_\beta D_{n,n}^{1/2} \\
 & \ddots & & \vdots \\
 & & \tau_{n-1} & \chi_\beta D_{n,n}^{1/2} \\
 & & & \chi_{(m-n+1)\beta} D_{n,n}^{1/2}
\end{bmatrix},
$$

where $\{\tau_1, \ldots, \tau_{n-1}\}$ are the singular values of $W^{(\beta)}(D_{1:n-1,1:n-1}, m, n-1)$, base case $W(D, m, 1)$ is $\tau_1 = \chi_{m\beta} D_{1,1}^{1/2}$. Let n be a positive integer and m a real greater than n - 1. Let $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\mathrm{eig}(Z^t Z)$ has density:

$$
C_\beta^W \det(D)^{-m\beta/2} \prod_{i=1}^n \lambda_i^{\frac{m-n+1}{2}\beta - 1} \Delta(\lambda)^\beta \cdot {}_0F_0^{(\beta)}\left(-\tfrac{1}{2}\Lambda, D^{-1}\right) d\lambda.
$$

</div>

Theorem 3 proves that $\mathrm{eig}(Z^t Z)$ is distributed by the formula above. This generalizes the work on the special cases $(\beta = 1)$,[13] $(\beta = 2)$,[23] and $(\beta = 4)$,[17] which found the eigenvalue distributions for full-matrix Wishart ensembles for their respective β's. This β-ensemble was discovered independently by Forrester.[11] Our proof is original, although some steps have analogs, in particular the lemma concerning Jack polynomials in n variables. This paper connects ideas through the "ghost and shadows" formalism. While the model appears straightforward, there are a number of hidden mathematical and computational complexities. The proof of Theorem 3, that the broken arrow model has the joint eigenvalue density in the box above, relies on a corollary about Jack polynomials (Corollary 1 to Theorem 2), which we have since writing learned has analogies in Forrester[10] (equation 12.210) and Okounkov and Olshanski[19] (proposition on page 8) (we thank Alexei Borodin and Peter Forrester for separate interesting discussions to help us see the connections). One advantage of our formulation is that the integral clearly takes place naturally on the unit sphere thus connecting to the geometry of the problem.

Corollary 1. Let $C_\kappa^{(\beta)}(\Lambda)$ be the Jack polynomial under the C-normalization and dq be the surface area element on the positive quadrant of the unit sphere. Then

$$C_\kappa^{(\beta)}(\Lambda) = \frac{C_\kappa^{(\beta)}(I_n)}{C_\kappa^{(\beta)}(I_{n-1})} \cdot \frac{2^{n-1}\Gamma(n\beta/2)}{\Gamma(\beta/2)^n} \cdot \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}((I - qq^t)\Lambda)dq.$$

Equivalently,

$$C_\kappa^{(\beta)}(\Lambda) \propto E_q(C_\kappa^{(\beta)}(\mathrm{Proj}_{q^\perp}\Lambda)),$$

where the expectation is taken over length n vectors of χ_β's renormalized to lie on the unit sphere.

Given a unit vector q, one can create a projected Jack polynomial of a symmetric matrix by projecting out the direction q. One can reconstruct the Jack polynomial by averaging over all directions q. This process is reminiscent of integral geometry[24] or computerized tomography. For $\beta = 1$, the measure on the sphere is the uniform measure, for other β's the measure on the sphere is proportional to $\prod_{i=1}^n q_i^{\beta-1}$. In addition, using the above formula with a fact from Stanley,[25] we derive a new definition of the Jack polynomials (see Corollary 2).

Inspiration for this broken arrow model came from the mathematical method of ghosts and shadows[7] and numerical techniques for the svd updating problem. Algorithms for updating the svd one column at a time were first considered by Bunch and Nielson.[3] Gu and Eisenstat[12] present an improvement using the fast multipole method. The random matrix context here is simpler than the numerical situation in that orthogonal invariance replaces the need for singular vector updating.

Software for efficiently computing the svd of broken arrow matrices is unavailable in the currently popular technical computing languages such as MATLAB, Mathematica, Maple, R, or Python. The sophisticated secular equation solver, LAPACK's dlasd4.f, efficiently computes the singular values of a broken arrow matrix. Using this software, we can sample the eigenvalues of $W^{(\beta)}(D, m, n)$ in $O(n^3)$ time and $O(n)$ space.

In this paper we perform a number of numerical simulations as well to confirm the correctness and illustrate applications of the model. Among these simulations are largest and smallest eigenvalue densities. We also use free probability to histogram the eigenvalues of $W^{(\beta)}(D, m, n)$ for general β, m, n and D drawn from a prior, and show that they match the analytical predictions of free probability made by Olver and Nadakuditi.[20]

II. REAL, COMPLEX, QUATERNION, AND GHOST WISHART MATRICES

Let G_β represent a Gaussian real, complex, or quaternion for $\beta = 1, 2, 4$, with mean zero and variance one. Let χ_d be a χ-distributed real with d degrees of freedom. The following algorithm computes the singular values, where all of the random variables in a given matrix are assumed independent. We assume $D = I$ for purposes of illustration, but this algorithm generalizes. We proceed through a series of matrices related by orthogonal transformations on the left and the right:

$$\begin{bmatrix} G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \\ G_\beta & G_\beta & G_\beta \end{bmatrix} \longrightarrow \begin{bmatrix} \chi_{3\beta} & G_\beta & G_\beta \\ 0 & G_\beta & G_\beta \\ 0 & G_\beta & G_\beta \end{bmatrix} \longrightarrow \begin{bmatrix} \chi_{3\beta} & \chi_\beta & G_\beta \\ 0 & \chi_{2\beta} & G_\beta \\ 0 & 0 & G_\beta \end{bmatrix}.$$

To create the real, positive $(1, 2)$ entry, we multiply the second column by a real sign, or a complex or quaternionic phase. We then use a Householder reflector on the bottom two rows to make the $(2, 2)$ entry a $\chi_{2\beta}$. Now we take the SVD of the 2×2 upper-left block:

$$\begin{bmatrix} \tau_1 & 0 & G_\beta \\ 0 & \tau_2 & G_\beta \\ 0 & 0 & G_\beta \end{bmatrix} \longrightarrow \begin{bmatrix} \tau_1 & 0 & \chi_\beta \\ 0 & \tau_2 & \chi_\beta \\ 0 & 0 & \chi_\beta \end{bmatrix} \longrightarrow \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}.$$

We convert the third column to reals using a diagonal matrix of signs on both sides. The process can be continued for a larger matrix, and can work with one that is taller than is wide. What it proves is

that the second-to-last-matrix,

$$\begin{bmatrix} \tau_1 & 0 & \chi_\beta \\ 0 & \tau_2 & \chi_\beta \\ 0 & 0 & \chi_\beta \end{bmatrix},$$

has the same singular values as the first matrix, if $\beta = 1, 2, 4$. We call this new matrix a "broken-arrow matrix." It is reasonable to conjecture that such a procedure might work for all $\beta > 0$ in a way not yet defined. This idea is the basis of the method of ghosts and shadows.[7] We prove that for a general broken arrow matrix model, the singular value distribution is what the method of ghosts and shadows predicts for a β-dimensional algebra.

The following algorithm, which generalizes the one above for the 3×3 case, samples the singular values of the Wishart ensemble for general β and general D.

Beta-Wishart (recursive) model pseudocode

Function $\Sigma := $ Beta-Wishart(m, n, β, D)

if $n = 1$ **then**

$\quad \Sigma := \chi_{m\beta} D_{1,1}^{1/2}$

else

$\quad Z_{1:n=1,1:n-1} := $ BetaWishart$(m, n-1, \beta, D_{1:n-1,1:n-1})$,

$\quad Z_{n,1:n-1} := [0, \ldots, 0]$,

$\quad Z_{1:n-1,n} := [\chi_\beta D_{n,n}^{1/2}; \ldots; \chi_\beta D_{n,n}^{1/2}]$,

$\quad Z_{n,n} := \chi_{(m-n+1)\beta} D_{n,n}^{1/2}$,

$\quad \Sigma := \text{diag}(\text{svd}(Z))$,

end if

The diagonal of Σ contains the singular values. Since we know the distribution of the singular values of such a full matrix for $(\beta = 1, 2, 4)$,[13,17,23] we can state (using the normalization constant in Corollary 3, originally from Forrester[9]):

Theorem 1. *The distribution of the singular values* $diag(\Sigma) = (\sigma_1, \ldots, \sigma_n)$, $\sigma_1 > \sigma_2 > \cdots > \sigma_n$, *generated by the above algorithm for* $\beta = 1, 2, 4$, *is equal to*

$$\frac{2^n \det(D)^{-m\beta/2}}{\mathcal{K}_{m,n}^{(\beta)}} \prod_{i=1}^{n} \sigma_i^{(m-n+1)\beta-1} \Delta^2(\sigma)^\beta {}_0F_0^{(\beta)}\left(-\frac{1}{2}\Sigma^2, D^{-1}\right) d\sigma,$$

where

$$\mathcal{K}_{m,n}^{(\beta)} = \frac{2^{mn\beta/2}}{\pi^{n(n-1)\beta/2}} \cdot \frac{\Gamma_n^{(\beta)}(m\beta/2)\Gamma_n^{(\beta)}(n\beta/2)}{\Gamma(\beta/2)^n},$$

and the generalized gamma function $\Gamma_n^{(\beta)}$ *is defined in Definition 6.*

Theorem 3 generalizes Theorem 1 to the $\beta > 0$ case. Before we can prove Theorem 3, we need some background.

III. ARROW AND BROKEN-ARROW MATRIX JACOBIANS

Define the (symmetric) arrow matrix by

$$
A = \begin{bmatrix}
d_1 & & & c_1 \\
& \ddots & & \vdots \\
& & d_{n-1} & c_{n-1} \\
c_1 & \cdots & c_{n-1} & c_n
\end{bmatrix}.
$$

Let its eigenvalues be $\lambda_1, \ldots, \lambda_n$. Let q be the last row of its eigenvector matrix, i.e., q contains the nth element of each eigenvector. q is by convention in the positive quadrant.

Define the broken arrow matrix B by

$$
B = \begin{bmatrix}
b_1 & & & a_1 \\
& \ddots & & \vdots \\
& & b_{n-1} & a_{n-1} \\
0 & \cdots & 0 & a_n
\end{bmatrix}.
$$

Let its singular values be $\sigma_1, \ldots, \sigma_n$, and let q contain the bottom row of its right singular vector matrix, i.e., $A = B^t B$, $B^t B$ is an arrow matrix. q is by convention in the positive quadrant.

Define dq to be the surface-area element on the sphere in $\mathbb{R}^n$.

Lemma 1. For an arrow matrix A, let f be the unique map $f : (c, d) \longrightarrow (q, \lambda)$. The Jacobian of f satisfies

$$
dq\,d\lambda = \frac{\prod_{i=1}^n q_i}{\prod_{i=1}^{n-1} c_i} \cdot dc\,dd.
$$

The proof is after Lemma 3.

Lemma 2. For a broken arrow matrix B, let g be the unique map $g : (a, b) \longrightarrow (q, \sigma)$. The Jacobian of g satisfies

$$
dq\,d\sigma = \frac{\prod_{i=1}^n q_i}{\prod_{i=1}^{n-1} a_i} \cdot da\,db.
$$

The proof is after Lemma 3.

Lemma 3. If all elements of a, b, q, σ are nonnegative, and b, d, λ, σ are ordered, then f and g are bijections excepting sets of measure zero (if some $b_i = b_j$ or some $d_i = d_j$, for $i \neq j$).

Proof. We only prove it for f; the g case is similar. We show that f is a bijection using results from Dumitriu and Edelman,[4] who in turn cite Parlett.[21] Define the tridiagonal matrix by

$$
\begin{bmatrix}
\eta_1 & \epsilon_1 & 0 & 0 \\
\epsilon_1 & \eta_2 & \epsilon_2 & 0 \\
& \ddots & \ddots & \ddots \\
0 & 0 & \epsilon_{n-1} & \eta_{n-1}
\end{bmatrix}
$$

to have eigenvalues $d_1, \ldots, d_{n-1}$ and bottom entries of the eigenvector matrix $u = (c_1, \ldots, c_{n-1})/\gamma$, where $\gamma = \sqrt{c_1^2 + \cdots + c_{n-1}^2}$. Let the whole eigenvector matrix be U. $(d, u) \leftrightarrow (\epsilon, \eta)$ is a bijection[4,21] excepting sets of measure 0. Now we extend the above tridiagonal matrix further and use $\sim$ to indicate

similar matrices:

$$
\begin{bmatrix}
\eta_1 & \epsilon_1 & 0 & 0 & 0 \\
\epsilon_1 & \eta_2 & \epsilon_2 & 0 & 0 \\
 & \ddots & \ddots & \ddots & \\
0 & 0 & \epsilon_{n-1} & \eta_{n-1} & \gamma \\
0 & 0 & 0 & \gamma & c_n
\end{bmatrix}
\sim
\begin{bmatrix}
d_1 & & & & u_1\gamma \\
 & \ddots & & & \vdots \\
 & & & d_{n-1} & u_{n-1}\gamma \\
u_1\gamma & \cdots & & u_{n-1}\gamma & c_n
\end{bmatrix}
= A,
$$

$(c_1, \ldots, c_{n-1}) \leftrightarrow (u, \gamma)$ is a bijection, as is $(c_n) \leftrightarrow (c_n)$, so we have constructed a bijection from $(c_1, \ldots, c_{n-1}, c_n, d_1, \ldots, d_{n-1}) \leftrightarrow (c_n, \gamma, \eta, \epsilon)$, excepting sets of measure 0. $(c_n, \gamma, \eta, \epsilon)$ defines a tridiagonal matrix which is in bijection with (q, λ).[4,21] Hence, we have bijected $(c, d) \leftrightarrow (q, \lambda)$. The proof that f is a bijection is complete.   □

Proof of Lemma 1. By Dumitriu and Edelman,[4] Lemma 2.9,

$$
dq d\lambda = \frac{\prod_{i=1}^{n} q_i}{\gamma \prod_{i=1}^{n-1} \epsilon_i} dc_n d\gamma d\epsilon d\eta.
$$

Also by Dumitriu and Edelman,[4] Lemma 2.9,

$$
dddu = \frac{\prod_{i=1}^{n-1} u_i}{\prod_{i=1}^{n-1} \epsilon_i} d\epsilon d\eta.
$$

Together,

$$
dq d\lambda = \frac{\prod_{i=1}^{n} q_i}{\gamma \prod_{i=1}^{n-1} u_i} dc_n dd d u d\gamma.
$$

The full spherical element is, using γ as the radius,

$$
dc_1 \cdots dc_{n-1} = \gamma^{n-2} du d\gamma.
$$

Hence,

$$
dq d\lambda = \frac{\prod_{i=1}^{n} q_i}{\gamma^{n-1} \prod_{i=1}^{n-1} u_i} dc dd,
$$

which by substitution is

$$
dq d\lambda = \frac{\prod_{i=1}^{n} q_i}{\prod_{i=1}^{n-1} c_i} dc dd.
$$

Proof of Lemma 2. Let $A = B^t B$. $d\lambda = 2^n \prod_{i=1}^{n} \sigma_i d\sigma$, and since $\prod_{i=1}^{n} \sigma_i{}^2 = \det(B^t B) = \det(B)^2 = a_n^2 \prod_{i=1}^{n-1} b_i^2$, by Lemma 1,

$$
dq d\sigma = \frac{\prod_{i=1}^{n} q_i}{2^n a_n \prod_{i=1}^{n-1} (b_i c_i)} dc dd.
$$

The full-matrix Jacobian $\frac{\partial(c,d)}{\partial(a,b)}$ is

$$\frac{\partial(c,d)}{\partial(a,b)} = \begin{bmatrix} b_1 & & & 2a_1 & & & \\ & \ddots & & & \vdots & & \\ & & b_{n-1} & 2a_{n-1} & & & \\ & & & 2a_n & & & \\ a_1 & & & & 2b_1 & & \\ & \ddots & & & & \ddots & \\ & & a_{n-1} & & & & 2b_{n-1} \end{bmatrix}.$$

The determinant gives $dcdd = 2^n a_n \prod_{i=1}^{n-1} b_i^2 dadb$. So,

$$dqd\sigma = \frac{\prod_{i=1}^{n} q_i \prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n-1} c_i} dadb = \frac{\prod_{i=1}^{n} q_i}{\prod_{i=1}^{n-1} a_i} dadb.$$

IV. FURTHER ARROW AND BROKEN-ARROW MATRIX LEMMAS

Lemma 4

$$q_k = \left(1 + \sum_{j=1}^{n-1} \frac{c_j^2}{(\lambda_k - d_j)^2} \right)^{-1/2}.$$

Proof. Let v be the eigenvector of A corresponding to λ_k. Temporarily fix $v_n = 1$. Using $Av = \lambda v$, for $j < n$, $v_j = c_j/(\lambda_k - d_j)$. Renormalizing v so that $\|v\| = 1$, we get the desired value for $v_n = q_k$. $\qquad\square$

Lemma 5. For a vector x of length l, define $\Delta(x) = \prod_{i<j} |x_i - x_j|$. Then,

$$\Delta(\lambda) = \Delta(d) \prod_{k=1}^{n-1} |c_k| \prod_{k=1}^{n} q_k^{-1}.$$

Proof. Using a result in Wilkinson,[26] the characteristic polynomial of A is

$$p(\lambda) = \prod_{i=1}^{n} (\lambda_i - \lambda) = \prod_{i=1}^{n-1} (d_i - \lambda) \left(c_n - \lambda - \sum_{j=1}^{n-1} \frac{c_j^2}{d_j - \lambda} \right). \tag{1}$$

Therefore, for $k < n$,

$$p(d_k) = \prod_{i=1}^{n} (\lambda_i - d_k) = -c_k^2 \prod_{i=1, i\neq k}^{n-1} (d_i - d_k). \tag{2}$$

Taking a product on both sides,

$$\prod_{i=1}^{n} \prod_{k=1}^{n-1} (\lambda_i - d_k) = (-1)^{n-1} \Delta(d)^2 \prod_{k=1}^{n-1} c_k^2.$$

Also,

$$p'(\lambda_k) = -\prod_{i=1, i\neq k}^{n} (\lambda_i - \lambda_k) = -\prod_{i=1}^{n-1} (d_i - \lambda_k) \left(1 + \sum_{j=1}^{n-1} \frac{c_j^2}{(d_j - \lambda_k)^2} \right). \tag{3}$$

Taking a product on both sides,

$$\prod_{i=1}^{n}\prod_{k=1}^{n-1}(\lambda_i - d_k) = (-1)^{n-1}\Delta(\lambda)^2\prod_{i=1}^{n}\left(1 + \sum_{j=1}^{n-1}\frac{c_j^2}{(d_j - \lambda_i)^2}\right)^{-1}.$$

Equating expressions equal to $\prod_{i=1}^{n}\prod_{k=1}^{n-1}(\lambda_i - d_k)$, we get

$$\Delta(d)^2\prod_{k=1}^{n-1}c_k^2 = \Delta(\lambda)^2\prod_{i=1}^{n}\left(1 + \sum_{j=1}^{n-1}\frac{c_j^2}{(d_j - \lambda_i)^2}\right)^{-1}.$$

The desired result follows by the previous lemma.   □

Lemma 6. For a vector x of length l, define $\Delta^2(x) = \prod_{i<j}|x_i^2 - x_j^2|$. The singular values of B satisfy

$$\Delta^2(\sigma) = \Delta^2(b)\prod_{k=1}^{n-1}|a_k b_k|\prod_{k=1}^{n}q_k^{-1}.$$

Proof. Follows from $A = B^t B$.   □

V. JACK AND HERMITE POLYNOMIALS

As in Ref. 6, if $\kappa \vdash k$, $\kappa = (\kappa_1, \kappa_2, \dots)$ is nonnegative, ordered non-increasingly, and it sums to k. Let $\alpha = 2/\beta$. Let $\rho_\kappa^\alpha = \sum_{i=1}^{l}\kappa_i(\kappa_i - 1 - (2/\alpha)(i-1))$. We define $l(\kappa)$ to be the number of nonzero elements of κ. We say that $\mu \leq \kappa$ in "lexicographic ordering" if for the largest integer j such that $\mu_i = \kappa_i$ for all $i < j$, we have $\mu_j \leq \kappa_j$.

Definition 1. As in Dumitriu, Edelman and Shuman,[6] we define the Jack polynomial of a matrix argument, $C_\kappa^{(\beta)}(X)$, as follows: Let $x_1, \dots, x_n$ be the eigenvalues of X. $C_\kappa^{(\beta)}(X)$ is the only homogeneous polynomial eigenfunction of the Laplace-Beltrami-type operator:

$$D_n^* = \sum_{i=1}^{n}x_i^2\frac{\partial^2}{\partial x_i^2} + \beta\cdot\sum_{1\leq i\neq j\leq n}\frac{x_i^2}{x_i - x_j}\cdot\frac{\partial}{\partial x_i},$$

with eigenvalue $\rho_k^\alpha + k(n-1)$, having highest order monomial basis function in lexicographic ordering (see Ref. 6, Section 2.4) corresponding to κ. In addition,

$$\sum_{\kappa\vdash k, l(\kappa)\leq n}C_\kappa^{(\beta)}(X) = \text{trace}(X)^k.$$

Lemma 7. If we write $C_\kappa^{(\beta)}(X)$ in terms of the eigenvalues $x_1, \dots, x_n$, as $C_\kappa^{(\beta)}(x_1, \dots, x_n)$, then $C_\kappa^{(\beta)}(x_1, \dots, x_{n-1}) = C_\kappa^{(\beta)}(x_1, \dots, x_{n-1}, 0)$ if $l(\kappa) < n$. If $l(\kappa) = n$, $C_\kappa^{(\beta)}(x_1, \dots, x_{n-1}, 0) = 0$.

Proof. The $l(\kappa) = n$ case follows from a formula in Stanley,[25] Propositions 5.1 and 5.5, that only applies if $\kappa_n > 0$,

$$C_\kappa^{(\beta)}(X) \propto \det(X)C_{\kappa_1-1,\dots,\kappa_n-1}^{(\beta)}(X).$$

If $\kappa_n = 0$, from (3.8) in Koev,[14] $C_\kappa^{(\beta)}(x_1, \dots, x_{n-1}) = C_\kappa^{(\beta)}(x_1, \dots, x_{n-1}, 0)$.   □

Definition 2. The Hermite polynomials (of a matrix argument) are a basis for the space of symmetric multivariate polynomials over eigenvalues $x_1, \dots, x_n$ of X which are related to the Jack polynomials by (Dumitriu, Edelman, and Shuman,[6] page 17)

$$H_\kappa^{(\beta)}(X) = \sum_{\sigma\subseteq\kappa}c_{\kappa,\sigma}^{(\beta)}\cdot\frac{C_\sigma^{(\beta)}(X)}{C_\sigma^{(\beta)}(I_n)},$$

where $\sigma \subseteq \kappa$ means for each i, $\sigma_i \leq \kappa_i$, and the coefficients $c_{\kappa,\sigma}^{(\beta)}$ are given by (Dumitriu, Edelman, and Shuman,[6] page 17). Since Jack polynomials are homogeneous, that means

$$H_\kappa^{(\beta)}(X) \propto C_\kappa^{(\beta)}(X) + L.O.T.$$

Furthermore, by (Dumitriu, Edelman, and Shuman,[6] page 16), the Hermite polynomials are orthogonal with respect to the measure

$$\exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\prod_{i\neq j}|x_i - x_j|^\beta.$$

Lemma 8. Let

$$A(\mu, c) = \begin{bmatrix} \mu_1 & & & c_1 \\ & \ddots & & \vdots \\ & & \mu_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix} = \begin{bmatrix} & & & c_1 \\ & M & & \vdots \\ & & & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix},$$

and let for $l(\kappa) < n$,

$$Q(\mu, c_n) = \int \prod_{i=1}^{n-1} c_i^{\beta-1} H_\kappa^{(\beta)}(A(\mu, c)) \exp(-c_1^2 - \cdots - c_{n-1}^2) dc_1 \cdots dc_{n-1}.$$

Q is a symmetric polynomial in μ with leading term proportional to $H_\kappa^{(\beta)}(M)$ plus terms of order strictly less than $|\kappa|$.

Proof. If we exchange two c_i's, $i < n$, and the corresponding μ_i's, $A(\mu, c)$ has the same eigenvalues, so $H_\kappa^{(\beta)}(A(\mu, c))$ is unchanged. So, we can prove $Q(\mu, c_n)$ is symmetric in μ by swapping two μ_i's, and seeing that the integral is invariant over swapping the corresponding c_i's.

Now since $H_\kappa^{(\beta)}(A(\mu, c))$ is a symmetric polynomial in the eigenvalues of $A(\mu, c)$, we can write it in the power-sum basis, i.e., it is in the ring generated by $t_p = \lambda_1^p + \cdots + \lambda_n^p$, for $p = 0, 1, 2, 3,$ $\ldots$, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A(\mu, c)$. But $t_p = \text{trace}(A(\mu, c)^p)$, so it is a polynomial in μ and c,

$$H_\kappa^{(\beta)}(A(\mu, c)) = \sum_{i \geq 0} \sum_{\epsilon_1, \ldots, \epsilon_{n-1} \geq 0} p_{i,\epsilon}(\mu) c_n^i c_1^{\epsilon_1} \cdots c_{n-1}^{\epsilon_{n-1}}.$$

Its order in μ and c must be $|\kappa|$, the same as its order in λ. Integrating, it follows that

$$Q(\mu, c_n) = \sum_{i \geq 0} \sum_{\epsilon_1, \ldots, \epsilon_{n-1} \geq 0} p_{i,\epsilon}(\mu) c_n^i M_\epsilon,$$

for constants M_ϵ. Since $\deg(H_\kappa^{(\beta)}(A(\mu, c))) = |\kappa|$, $\deg(p_{i,\epsilon}(\mu)) \leq |\kappa| - |\epsilon| - i$. Writing

$$Q(\mu, c_n) = M_{\vec{0}} p_{0,\vec{0}}(\mu) + \sum_{(i,\epsilon) \neq (0,\vec{0})} p_{i,\epsilon}(\mu) c_n^i M_\epsilon,$$

we see that the summation has degree at most $|\kappa| - 1$ in μ only, treating c_n as a constant. Now

$$p_{0,\vec{0}}(\mu) = H_\kappa^{(\beta)}\left(\begin{bmatrix} M & \vec{0} \\ \vec{0} & 0 \end{bmatrix}\right) = H_\kappa^{(\beta)}(\mu) + r(\mu),$$

where $r(\mu)$ has degree at most $|\kappa| - 1$. This follows from the expansion of $H_\kappa^{(\beta)}$ in Jack polynomials in Definition 2 and the fact about Jack polynomials in Lemma 7. The new lemma follows. □

Lemma 9. Let the arrow matrix below have eigenvalues in $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$ and have q be the last row of its eigenvector matrix, i.e., q contains the nth element of each eigenvector,

$$A(\Lambda, q) = \begin{bmatrix} \mu_1 & & & c_1 \\ & \ddots & & \vdots \\ & & \mu_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix} = \begin{bmatrix} & & & c_1 \\ & M & & \vdots \\ & & & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix},$$

by Lemma 3 this is a well-defined map except on a set of measure zero. Then, for $U(X)$ a symmetric homogeneous polynomial of degree k in the eigenvalues of X,

$$V(\Lambda) = \int \prod_{i=1}^{n} q_i^{\beta-1} U(M) dq,$$

is a symmetric homogeneous polynomial of degree k in $\lambda_1, \ldots, \lambda_n$.

Proof. Let e_n be the column vector that is 0 everywhere except in the last entry, which is 1. $(I - e_n e_n^t) A(\Lambda, q)(I - e_n e_n^t)$ has eigenvalues $\{\mu_1, \ldots, \mu_{n-1}, 0\}$. If the eigenvector matrix of $A(\Lambda, q)$ is Q, so must

$$Q^t(I - e_n e_n^t) Q \Lambda Q^t (I - e_n e_n^t) Q$$

have those eigenvalues. But this is

$$(I - qq^t)\Lambda(I - qq^t).$$

So

$$U(M) = U(\text{eig}((I - qq^t)\Lambda(I - qq^t))\backslash\{0\}). \tag{4}$$

It is well known that we can write $U(M)$ in the power-sum ring, $U(M)$ is made of sums and products of functions of the form $\mu_1^p + \cdots + \mu_{n-1}^p$, where p is a positive integer. Therefore, the RHS is made of functions of the form

$$\mu_1^p + \cdots + \mu_{n-1}^p + 0^p = \text{trace}(((I - qq^t)\Lambda(I - qq^t))^p),$$

which if $U(M)$ is order k in the μ_i's, must be order k in the λ_i's. So $V(\Lambda)$ is a polynomial of order k in the $\lambda_i's$. Switching λ_1 and λ_2 and also q_1 and q_2 leaves

$$\int \prod_{i=1}^{n} q_i^{\beta-1} U(\text{eig}((I - qq^t)\Lambda(I - qq^t))\backslash\{0\}) dq$$

invariant, so $V(\Lambda)$ is symmetric. $\square$

Theorem 2 is a new theorem about Jack polynomials.

Theorem 2. *Let the arrow matrix below have eigenvalues in $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$ and have q be the last row of its eigenvector matrix, i.e., q contains the nth element of each eigenvector,*

$$A(\Lambda, q) = \begin{bmatrix} \mu_1 & & & c_1 \\ & \ddots & & \vdots \\ & & \mu_{n-1} & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix} = \begin{bmatrix} & & & c_1 \\ & M & & \vdots \\ & & & c_{n-1} \\ c_1 & \cdots & c_{n-1} & c_n \end{bmatrix},$$

by Lemma 3 this is a well-defined map except on a set of measure zero. Then, if for a partition κ, $l(\kappa) < n$, and q on the first quadrant of the unit sphere,

$$C_\kappa^{(\beta)}(\Lambda) \propto \int \prod_{i=1}^{n} q_i^{\beta-1} C_\kappa^{(\beta)}(M) dq.$$

Proof. Define

$$\eta_\kappa^{(\beta)}(\Lambda) = \int \prod_{i=1}^{n} q_i^{\beta-1} H_\kappa^{(\beta)}(M) dq.$$

This is a symmetric polynomial in n variables (Lemma 9). Thus, it can be expanded in Hermite polynomials with max order $|\kappa|$ (Lemma 9):

$$\eta_\kappa^{(\beta)}(\Lambda) = \sum_{|\kappa^{(0)}| \le |\kappa|} c(\kappa^{(0)}, \kappa) H_{\kappa^{(0)}}^{(\beta)}(\Lambda),$$

where $|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_{l(\kappa)}$. Using orthogonality, from the previous definition of Hermite polynomials,

$$c(\kappa^{(0)}, \kappa) \propto \int_{\Lambda \in \mathbb{R}^n} \int_q \prod_{i=1}^{n} q_i^{\beta-1} H_\kappa^{(\beta)}(M) H_{\kappa^{(0)}}^{(\beta)}(\Lambda),$$

$$\times \exp(-\frac{1}{2}\text{trace}(\Lambda^2)) \prod_{i \ne j} |\lambda_i - \lambda_j|^\beta dq d\lambda.$$

Using Lemmas 1 and 3,

$$c(\kappa^{(0)}, \kappa) \propto \int \prod_{i=1}^{n} q_i^{\beta-1} H_\kappa^{(\beta)}(M) H_{\kappa^{(0)}}^{(\beta)}(\Lambda)$$

$$\times \exp(-\frac{1}{2}\text{trace}(\Lambda^2)) \prod_{i \ne j} |\lambda_i - \lambda_j|^\beta \frac{\prod_{i=1}^{n} q_i}{\prod_{i=1}^{n-1} c_i} d\mu dc.$$

Using Lemma 6,

$$c(\kappa^{(0)}, \kappa) \propto \int \prod_{i=1}^{n-1} c_i^{\beta-1} H_\kappa^{(\beta)}(M) H_{\kappa^{(0)}}^{(\beta)}(\Lambda)$$

$$\times \exp(-\frac{1}{2}\text{trace}(\Lambda^2)) \prod_{i \ne j} |\mu_i - \mu_j|^\beta d\mu dc,$$

and by substitution

$$c(\kappa^{(0)}, \kappa) \propto \int \prod_{i=1}^{n-1} c_i^{\beta-1} H_\kappa^{(\beta)}(M) H_{\kappa^{(0)}}^{(\beta)}(A(\Lambda, q))$$

$$\times \exp(-\frac{1}{2}\text{trace}(A(\Lambda, q)^2)) \prod_{i \ne j} |\mu_i - \mu_j|^\beta d\mu dc.$$

Define

$$Q(\mu, c_n) = \int \prod_{i=1}^{n-1} c_i^{\beta-1} H_{\kappa^{(0)}}^{\beta}(A(\Lambda, q)) \exp(-c_1^2 - \cdots - c_{n-1}^2) dc_1 \cdots dc_{n-1}.$$

$Q(\mu, c_n)$ is a symmetric polynomial in μ (Lemma 8). Furthermore, by Lemma 8,

$$Q(\mu, c_n) \propto H_{\kappa^{(0)}}^{(\beta)}(M) + L.O.T.,$$

where the Lower Order Terms are of lower order than $|\kappa^{(0)}|$ and are symmetric polynomials. Hence, they can be written in a basis of lower order Hermite polynomials, and as

$$c(\kappa^{(0)}, \kappa) \propto \int H_\kappa^{(\beta)}(M) Q(\mu, c_n)$$

$$\times \prod_{i \neq j} |\mu_i - \mu_j|^\beta \exp\left(-\frac{1}{2}(c_n^2 + \mu_1^2 + \cdots + \mu_{n-1}^2)\right) d\mu \, dc_n,$$

we have by orthogonality

$$c(\kappa^{(0)}, \kappa) \propto \delta(\kappa^{(0)}, \kappa),$$

where δ is the Dirac delta. So

$$\eta_\kappa^{(\beta)}(\Lambda) = \int \prod_{i=1}^n q_i^{\beta-1} H_\kappa^{(\beta)}(M) dq \propto H_\kappa^{(\beta)}(\Lambda).$$

By Lemma 9, coupled with Definition 2,

$$C_\kappa^{(\beta)}(\Lambda) \propto \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}(M) dq.$$

$\square$

Corollary 1. Finding the proportionality constant: For $l(\kappa) < n$,

$$C_\kappa^{(\beta)}(\Lambda) = \frac{C_\kappa^{(\beta)}(I_n)}{C_\kappa^{(\beta)}(I_{n-1})} \cdot \frac{2^{n-1}\Gamma(n\beta/2)}{\Gamma(\beta/2)^n} \cdot \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}((I - qq^t)\Lambda) dq.$$

Proof. By Theorem 2 with Eq. (4) (in the proof of Lemma 9),

$$C_\kappa^{(\beta)}(\Lambda) \propto \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}(\text{eig}((I - qq^t)\Lambda(I - qq^t)) - \{0\}) dq,$$

which by Lemma 7 and properties of matrices is

$$C_\kappa^{(\beta)}(\Lambda) \propto \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}((I - qq^t)\Lambda) dq.$$

Now to find the proportionality constant. Let $\Lambda = I_n$, and let c_p be the constant of proportionality:

$$C_\kappa^{(\beta)}(I_n) = c_p \cdot \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}(I - qq^t) dq.$$

Since $I - qq^t$ is a projection, we can replace the term in the integral by $C_\kappa^{(\beta)}(I_{n-1})$, which can be moved out. So

$$c_p = \frac{C_\kappa^{(\beta)}(I_n)}{C_\kappa^{(\beta)}(I_{n-1})} \left(\int \prod_{i=1}^n q_i^{\beta-1} dq\right)^{-1}.$$

Now

$$\int \prod_{i=1}^{n} q_i^{\beta-1} dq = \frac{2}{\Gamma(n\beta/2)} \int_0^\infty r^{n\beta-1} e^{-r^2} dr \int \prod_{i=1}^{n} q_i^{\beta-1} dq$$

$$= \frac{2}{\Gamma(n\beta/2)} \iint_0^\infty \prod_{i=1}^{n} (rq_i)^{\beta-1} e^{-r^2} (r^{n-1} dr\, dq)$$

$$= \frac{2}{\Gamma(n\beta/2)} \int \prod_{i=1}^{n} x_i^{\beta-1} e^{-x_1^2-\cdots-x_n^2} dx$$

$$= \frac{2}{\Gamma(n\beta/2)} \left(\int_0^\infty x_1^{\beta-1} e^{-x_1^2} dx_1 \right)^n$$

$$= \frac{2}{\Gamma(n\beta/2)} \left(\frac{\Gamma(\beta/2)}{2} \right)^n$$

$$= \frac{\Gamma(\beta/2)^n}{2^{n-1}\Gamma(n\beta/2)},$$

and the corollary follows.   □

Corollary 2. The Jack polynomials can be defined recursively using Corollary 1 and two results in the compilation.[15]

Proof. By Stanley,[25] Proposition 4.2, the Jack polynomial of *one* variable under the *J* normalization is

$$J_{\kappa_1}^{(\beta)}(\lambda_1) = \lambda_1^{\kappa_1}(1 + (2/\beta))\cdots(1 + (\kappa_1 - 1)(2/\beta)).$$

There exists another recursion for Jack polynomials under the *J* normalization:

$$J_\kappa^{(\beta)}(\Lambda) = \det(\Lambda) J_{(\kappa_1-1,\ldots,\kappa_n-1)} \prod_{i=1}^{n} (n - i + 1 + (2/\beta)(\kappa_i - 1)),$$

if $\kappa_n > 0$. Note that if $\kappa_n > 0$ we can use the above formula to reduce the size of κ in a recursive expression for a Jack polynomial, and if $\kappa_n = 0$ we can use Corollary 1 to reduce the number of variables in a recursive expression for a Jack polynomial. Using those facts together and the conversion between *C* and *J* normalizations in Ref. 6, we can define all Jack polynomials.   □

VI. HYPERGEOMETRIC FUNCTIONS

Definition 3. We define the hypergeometric function of two matrix arguments and parameter β, $_0F_0{}^\beta(X, Y)$, for $n \times n$ matrices X and Y, by

$$_0F_0{}^\beta(X, Y) = \sum_{k=0}^\infty \sum_{\kappa \vdash k, l(\kappa) \leq n} \frac{C_\kappa^{(\beta)}(X) C_\kappa^{(\beta)}(Y)}{k! C_\kappa^{(\beta)}(I)},$$

as in Koev and Edelman.[14] *It is efficiently calculated using the software described in Koev and Edelman,*[14] mhg, *which is available online.*[16] *The C's are Jack polynomials under the C normalization, $\kappa \vdash k$ means that κ is a partition of the integer k, so $\kappa_1 \geq \kappa_2 \geq \cdots \geq 0$ have $|\kappa| = k = \kappa_1 + \kappa_2 + \cdots = k$.*

Lemma 10

$$_0F_0{}^{(\beta)}(X, Y) = \exp(s \cdot \mathrm{trace}(X)) \, _0F_0{}^{(\beta)}(X, Y - sI).$$

Proof. The claim holds for $s = 1$ by Baker and Forrester.[1] Now, using that fact with the homogeneity of Jack polynomials,

$$_0F_0{}^{(\beta)}(X, Y - sI) = {}_0F_0{}^{(\beta)}(X, s((1/s)Y - I)) = {}_0F_0{}^{(\beta)}(sX, (1/s)Y - I)$$

$$= \exp(s \cdot \text{trace}(X)) \, _0F_0{}^{(\beta)}(sX, (1/s)Y) = \exp(s \cdot \text{trace}(X)) \, _0F_0{}^{(\beta)}(X, Y).$$

$\square$

Definition 4. We define the generalized Pochhammer symbol to be, for a partition $\kappa = (\kappa_1, \ldots, \kappa_l)$,

$$(a)_\kappa^{(\beta)} = \prod_{i=1}^{l} \prod_{j=1}^{\kappa_i} \left(a - \frac{i-1}{2}\beta + j - 1 \right).$$

Definition 5. As in Koev and Edelman,[14] we define the hypergeometric function $_1F_1$ to be

$$_1F_1{}^{(\beta)}(a; b; X, Y) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k, l(\kappa) \leq n} \frac{(a)_\kappa^\beta}{(b)_\kappa^{(\beta)}} \cdot \frac{C_\kappa^{(\beta)}(X) C_\kappa^{(\beta)}(Y)}{k! C_\kappa^{(\beta)}(I)}.$$

The best software available to compute this function numerically is described in Koev and Edelman,[14] mhg.

Definition 6. We define the generalized Gamma function to be

$$\Gamma_n^{(\beta)}(c) = \pi^{n(n-1)\beta/4} \prod_{i=1}^{n} \Gamma(c - (i-1)\beta/2)$$

for $\Re(c) > (n-1)\beta/2$.

VII. THE β-WISHART ENSEMBLE AND ITS SPECTRAL DISTRIBUTION

The β-Wishart ensemble for $m \times n$ matrices is defined iteratively; we derive the $m \times n$ case from the $m \times (n-1)$ case.

Definition 7. We assume n is a positive integer and m is a real greater than $n - 1$. Let D be a positive-definite diagonal $n \times n$ matrix. For $n = 1$, the β-Wishart ensemble is

$$Z = \begin{bmatrix} \chi_{m\beta} D_{1,1}^{1/2} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

with $n - 1$ zeros, where $\chi_{m\beta}$ represents a random positive real that is χ-distributed with $m\beta$ degrees of freedom. For $n > 1$, the β-Wishart ensemble with positive-definite diagonal $n \times n$ covariance matrix D is defined as follows: Let $\tau_1, \ldots, \tau_{n-1}$ be one draw of the singular values of the $m \times (n-1)$ β-Wishart ensemble with covariance $D_{1:(n-1), 1:(n-1)}$. Define the matrix Z by

$$Z = \begin{bmatrix} \tau_1 & & & \chi_\beta D_{n,n}^{1/2} \\ & \ddots & & \vdots \\ & & \tau_{n-1} & \chi_\beta D_{n,n}^{1/2} \\ & & & \chi_{(m-n+1)\beta} D_{n,n}^{1/2} \end{bmatrix}.$$

All the χ-distributed random variables are independent. Let $\sigma_1, \ldots, \sigma_n$ be the singular values of Z. They are one draw of the singular values of the $m \times n$ β-Wishart ensemble, completing the recursion. $\lambda_i = \sigma_i^2$ are the eigenvalues of the β-Wishart ensemble.

Theorem 3. *Let $\Sigma = diag(\sigma_1, \ldots, \sigma_n)$, $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. The singular values of the β-Wishart ensemble with covariance D are distributed by a pdf proportional to*

$$\det(D)^{-m\beta/2} \prod_{i=1}^{n} \sigma_i^{(m-n+1)\beta-1} \Delta^2(\sigma)^\beta {}_0F_0^{(\beta)} \left(-\frac{1}{2}\Sigma^2, D^{-1} \right) d\sigma.$$

It follows from a simple change of variables that the ordered λ_i's are distributed as

$$C_\beta^W \det(D)^{-m\beta/2} \prod_{i=1}^{n} \lambda_i^{\frac{m-n+1}{2}\beta-1} \Delta(\lambda)^\beta {}_0F_0^{(\beta)} \left(-\frac{1}{2}\Lambda, D^{-1} \right) d\lambda.$$

Proof. First we need to check the $n = 1$ case: the one singular value σ_1 is distributed as $\sigma_1 = \chi_{m\beta} D_{1,1}^{1/2}$, which has pdf proportional to

$$D_{1,1}^{-m\beta/2} \sigma_1^{m\beta-1} \exp\left(-\frac{\sigma_1^2}{2D_{1,1}} \right) d\sigma_1.$$

We use the fact that

$${}_0F_0^{(\beta)} \left(-\frac{1}{2}\sigma_1^2, D_{1,1}^{-1} \right) = {}_0F_0^{(\beta)} \left(-\frac{1}{2D_{1,1}}\sigma_1^2, 1 \right) = \exp\left(-\frac{\sigma_1^2}{2D_{1,1}} \right).$$

The first equality comes from the expansion of ${}_0F_0$ in terms of Jack polynomials and the fact that Jack polynomials are homogeneous, see the definition of Jack polynomials and ${}_0F_0$ in this paper, the second comes from (2.1) in Koev,[15] or in Forrester.[10] We use that ${}_0F_0^{(\beta)}(X, I) = {}_0F_0^{(\beta)}(X)$, by definition.[14]

Now we assume $n > 1$. Let

$$Z = \begin{bmatrix} \tau_1 & & & \chi_\beta D_{n,n}^{1/2} \\ & \ddots & & \vdots \\ & & \tau_{n-1} & \chi_\beta D_{n,n}^{1/2} \\ & & & \chi_{(m-n+1)\beta} D_{n,n}^{1/2} \end{bmatrix} = \begin{bmatrix} \tau_1 & & & a_1 \\ & \ddots & & \vdots \\ & & \tau_{n-1} & a_{n-1} \\ & & & a_n \end{bmatrix},$$

so the a_i's are χ-distributed with different parameters. By hypothesis, the τ_i's are a β-Wishart draw. Therefore, the a_i's and the τ_i's are assumed to have joint distribution proportional to

$$\det(D)^{-m\beta/2} \prod_{i=1}^{n-1} \tau_i^{(m-n+2)\beta-1} \Delta^2(\tau)^\beta {}_0F_0^{(\beta)} \left(-\frac{1}{2}T^2, D_{1:n-1,1:n-1}^{-1} \right)$$

$$\times \left(\prod_{i=1}^{n-1} a_i \right)^{\beta-1} a_n^{(m-n+1)\beta-1} \exp\left(-\frac{1}{2D_{n,n}} \sum_{i=1}^{n} a_i^2 \right) da\, d\tau,$$

where $T = diag(\tau_1, \ldots, \tau_{n-1})$. Using Lemmas 2 and 3, we can change variables to

$$\det(D)^{-m\beta/2} \prod_{i=1}^{n-1} \tau_i^{(m-n+2)\beta-1} \Delta^2(\tau)^\beta {}_0F_0^{(\beta)} \left(-\frac{1}{2}T^2, D_{1:n-1,1:n-1}^{-1} \right)$$

$$\times \left(\prod_{i=1}^{n-1} a_i \right)^{\beta} a_n^{(m-n+1)\beta-1} \exp\left(-\frac{1}{2D_{n,n}} \sum_{i=1}^{n} a_i^2 \right) \prod_{i=1}^{n} q_i^{-1} \cdot d\sigma\, dq.$$

Using Lemma 6 this becomes

$$\det(D)^{-m\beta/2} \prod_{i=1}^{n-1} \tau_i^{(m-n+1)\beta-1} \Delta^2(\sigma)^\beta {}_0F_0^{(\beta)}(-\frac{1}{2}T^2, D_{1:n-1,1:n-1}^{-1})$$

$$\times \exp\left(-\frac{1}{2D_{n,n}} \sum_{i=1}^n a_i^2\right) \prod_{i=1}^n q_i^{\beta-1} d\sigma dq.$$

Using properties of determinants this becomes

$$\det(D)^{-m\beta/2} \prod_{i=1}^{n} \sigma_i^{(m-n+1)\beta-1} \Delta^2(\sigma)^\beta {}_0F_0^{(\beta)}\left(-\frac{1}{2}T^2, D_{1:n-1,1:n-1}^{-1}\right)$$

$$\times \exp\left(-\frac{1}{2D_{n,n}} \sum_{i=1}^n a_i^2\right) \prod_{i=1}^n q_i^{\beta-1} \cdot d\sigma dq.$$

To complete the induction, we need to prove

$${}_0F_0^{(\beta)}\left(-\frac{1}{2}\Sigma^2, D^{-1}\right) \propto \int \prod_{i=1}^n q_i^{\beta-1} e^{-\|a\|^2/(2D_{n,n})} {}_0F_0^{(\beta)}\left(-\frac{1}{2}T^2, D_{1:(n-1),1:(n-1)}^{-1}\right) dq.$$

We can reduce this expression using $\|a\|^2 + \sum_{i=1}^{n-1} \tau_i^2 = \sum_{i=1}^{n} \sigma_i^2$ that it suffices to show

$$\exp\left(\mathrm{trace}(\Sigma^2)/(2D_{n,n})\right) {}_0F_0^{(\beta)}\left(-\frac{1}{2}\Sigma^2, D^{-1}\right)$$

$$\propto \int \prod_{i=1}^n q_i^{\beta-1} \exp\left(\mathrm{trace}(T^2)/(2D_{n,n})\right) {}_0F_0^{(\beta)}\left(-\frac{1}{2}T^2, D_{1:(n-1),1:(n-1)}^{-1}\right) dq,$$

or moving some constants and signs around,

$$\exp\left((-1/D_{n,n})\mathrm{trace}(-\Sigma^2/2)\right) {}_0F_0^{(\beta)}\left(-\frac{1}{2}\Sigma^2, D^{-1}\right)$$

$$\propto \int \prod_{i=1}^n q_i^{\beta-1} \exp\left((-1/(D_{n,n}))\mathrm{trace}(-T^2/2)\right) {}_0F_0^{(\beta)}\left(-\frac{1}{2}T^2, D_{1:(n-1),1:(n-1)}^{-1}\right) dq,$$

or using Lemma 10,

$${}_0F_0^{(\beta)}\left(-\frac{1}{2}\Sigma^2, D^{-1} - \frac{1}{D_{n,n}}I_n\right)$$

$$\propto \int \prod_{i=1}^n q_i^{\beta-1} {}_0F_0^{(\beta)}\left(-\frac{1}{2}T^2, D_{1:(n-1),1:(n-1)}^{-1} - \frac{1}{D_{n,n}}I_{n-1}\right) dq.$$

We will prove this expression termwise using the expansion of ${}_0F_0$ into infinitely many Jack polynomials. The (k, κ) term on the right-hand side is

$$\int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}\left(-\frac{1}{2}T^2\right) C_\kappa^{(\beta)}\left(D_{1:(n-1),1:(n-1)}^{-1} - \frac{1}{D_{n,n}}I_{n-1}\right) dq,$$

where $\kappa \vdash k$ and $l(\kappa) < n$. The (k, κ) term on the left-hand side is

$$C_\kappa^{(\beta)}\left(-\frac{1}{2}\Sigma^2\right) C_\kappa^{(\beta)}\left(D^{-1} - \frac{1}{D_{n,n}}I_{n-1}\right),$$

where $\kappa \vdash k$ and $l(\kappa) \leq n$. If $l(\kappa) = n$, the term is 0 by Lemma 7, so either it has a corresponding term on the right-hand side or it is zero. Hence, using Lemma 7 again it suffices to show that for $l(\kappa) < n$,

$$C_\kappa^{(\beta)}(\Sigma^2) \propto \int \prod_{i=1}^n q_i^{\beta-1} C_\kappa^{(\beta)}(T^2) dq.$$

This follows by Theorem 2, and the proof of Theorem 3 is complete.   □

Corollary 3. The normalization constant, for $\lambda_1 > \lambda_2 > \cdots > \lambda_n$:

$$C_\beta^W = \frac{1}{\mathcal{K}_{m,n}^{(\beta)}},$$

where

$$\mathcal{K}_{m,n}^{(\beta)} = \frac{2^{mn\beta/2}}{\pi^{n(n-1)\beta/2}} \cdot \frac{\Gamma_n^{(\beta)}(m\beta/2)\Gamma_n^{(\beta)}(n\beta/2)}{\Gamma(\beta/2)^n},$$

Proof. We have used the convention that elements of D do not move through $\propto$, so we may assume D is the identity. Using ${}_0F_0^{(\beta)}(-\Lambda/2, I) = \exp(-\text{trace}(\Lambda)/2)$ ((2.1) in Koev[15]), the model becomes the β-Laguerre model studied in Forrester.[9]   □

Corollary 4. Using Definition 6 of the generalized gamma, the distribution of λ_{max} *for the* β-*Wishart ensemble with general covariance in diagonal D,* $P(\lambda_{max} < x)$, *is*

$$\frac{\Gamma_n^{(\beta)}(1+(n-1)\beta/2)}{\Gamma_n^{(\beta)}(1+(m+n-1)\beta/2)} \det\left(\frac{x}{2}D^{-1}\right)^{m\beta/2} {}_1F_1^{(\beta)}\left(\frac{m}{2}\beta; \frac{m+n-1}{2}\beta+1; -\frac{x}{2}D^{-1}\right).$$

Proof. See page 14 of Koev,[15] Theorem 6.1. A factor of β is lost due to differences in nomenclature. The best software to calculate this is described in Koev and Edelman,[14] mhg. Convergence is improved using formula (2.6) in Koev.[15]   □
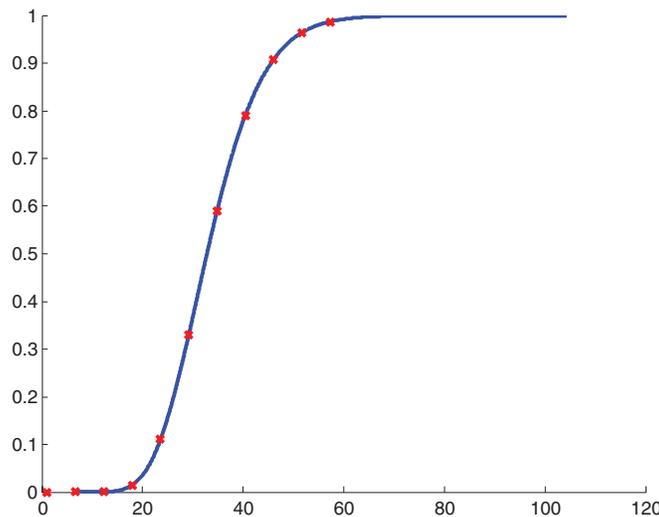
FIG. 1. The line is the empirical cdf created from many draws of the maximum eigenvalue of the β-Wishart ensemble, with $m = 4$, $n = 4$, $\beta = 2.5$, and $D = \text{diag}(1.1, 1.2, 1.4, 1.8)$. The x's are the analytically derived values of the cdf using Corollary 4 and mhg.
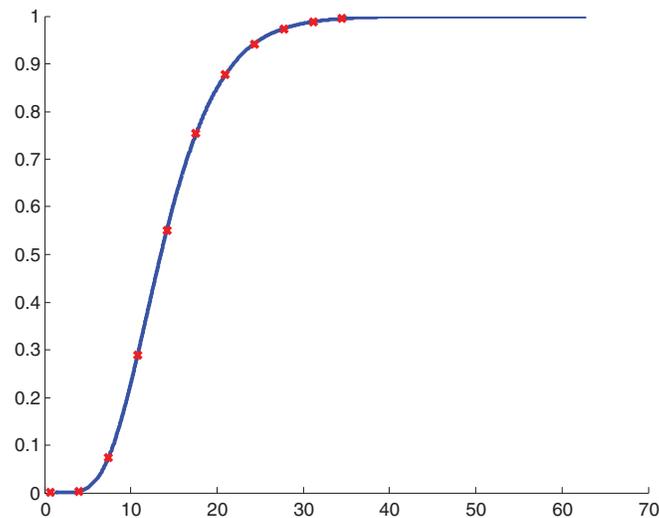
FIG. 2. The line is the empirical cdf created from many draws of the maximum eigenvalue of the β-Wishart ensemble, with $m = 6$, $n = 4$, $\beta = 0.75$, and $D = \text{diag}(1.1, 1.2, 1.4, 1.8)$. The x's are the analytically derived values of the cdf using Corollary 4 and mhg.

Corollary 5. The distribution of λ_{min} for the β-Wishart ensemble with general covariance in diagonal D, $P(\lambda_{min} < x)$, is

$$1 - \exp\left(\text{trace}(-xD^{-1}/2)\right) \sum_{k=0}^{nt} \sum_{\kappa \vdash k, \kappa_1 \leq t} \frac{C_\kappa^{(\beta)}(xD^{-1}/2)}{k!}.$$

It is only valid when $t = (m - n + 1)\beta/2 - 1$ is a nonnegative integer.

Proof. See pages 14 and 15 of Koev,[15] Theorem 6.1. A factor of β is lost due to differences in nomenclature. The best software to calculate this is described in Koev and Edelman,[14] mhg.   □
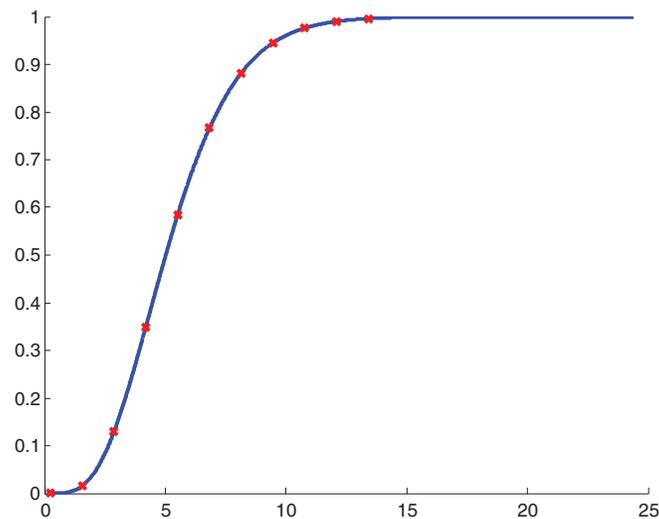
FIG. 3. The line is the empirical cdf created from many draws of the minimum eigenvalue of the β-Wishart ensemble, with $m = 4$, $n = 3$, $\beta = 5$, and $D = \text{diag}(1.1, 1.2, 1.4)$. The x's are the analytically derived values of the cdf using Corollary 5 and mhg.
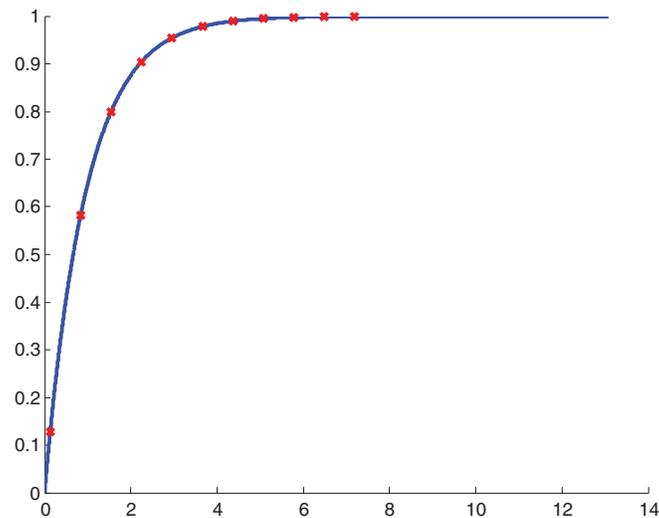
FIG. 4. The line is the empirical cdf created from many draws of the minimum eigenvalue of the β-Wishart ensemble, with $m = 7$, $n = 4$, $\beta = 0.5$, and $D = \text{diag}(1, 2, 3, 4)$. The x's are the analytically derived values of the cdf using Corollary 5 and `mhg`.

Theorem 6.2 in Ref. 15 gives a formula for the distribution of the trace of the β-Wishart ensemble.

Figures 1–4 demonstrate the correctness of Corollaries 4 and 5, which are derived from Theorem 3.

VIII. THE β-WISHART ENSEMBLE AND FREE PROBABILITY

Given the eigenvalue distributions of two large random matrices, free probability allows one to analytically compute the eigenvalue distributions of the sum and product of those matrices (a good summary is Nadakuditi and Edelman[22]). In particular, we would like to compute the eigenvalue histogram for $X^t X D/(m\beta)$, where X is a tall matrix of standard normal reals, complexes, quaternions, or ghosts, and D is a positive definite diagonal matrix drawn from a prior. Dumitriu[5] proves that for
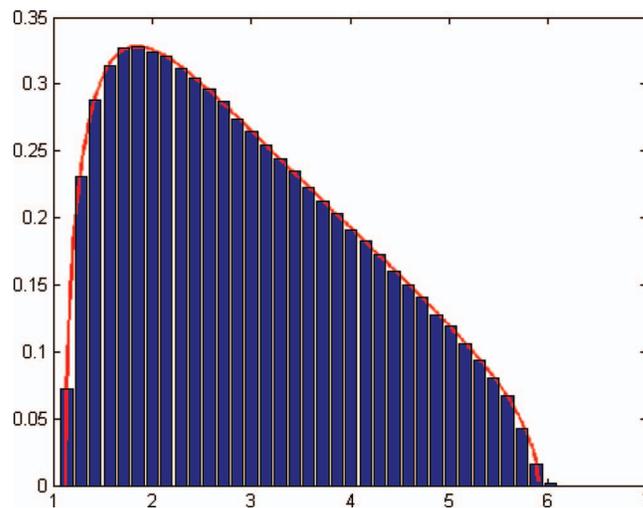
FIG. 5. The analytical product of the semicircle and Marcenko-Pastur laws is the red line, the histogram is 1000 draws of the β-Wishart ($\beta = 3$) with covariance drawn from the shifted semicircle distribution. They match perfectly.

the $D = I$ and $\beta = 1$, 2, 4 case, the answer is the Marcenko-Pastur law, invariant over β. So it is reasonable to assume that the value of β does not figure into hist(eig($X^t XD$)), where D is random.

We use the methods of Olver and Nadakuditi[20] to analytically compute the product of the Marcenko-Pastur distribution for $m/n \longrightarrow 10$ and variance 1 with the semicircle distribution of width $2\sqrt{2}$ centered at 3. Figure 5 demonstrates that the histogram of 1000 draws of $X^t XD/(m\beta)$ for $m = 1000$, $n = 100$, and $\beta = 3$, represented as a bar graph, is equal to the analytically computed red line. The β-Wishart distribution allows us to draw the eigenvalues of $X^t XD/(m\beta)$, even if we cannot sample the entries of the matrix for $\beta = 3$.

ACKNOWLEDGMENTS

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