A. let \( x: I \rightarrow S \) be a curve in \( S \).\( \frac{dx}{dt} \) is the unit tangent vector of \( x \). We have \( |x'| = 1 \) at every point on \( x \).

Assume for the moment \( E \) is a smooth choice of unit normal vector. Define \( b(s) = N(\alpha(s)) \wedge \alpha'(s) \neq 0 \) so \( N \perp \alpha' \) both unit vectors.

\[
\begin{align*}
\dot{b} & = dN(\alpha') \wedge \alpha' + N \wedge \alpha'' \\
& = (\alpha') \wedge \alpha' + N \wedge (\alpha'')^T \\
& = 0
\end{align*}
\]

So \( b = \text{constant along } I \)

\[ b = b_0 \]

Thus \( \exists \ P \in b_0 \) s.t. \( \forall s \in I \)

\( x(s) + P \) is contained in the plane \( \alpha(s) + P \)

Now observe that a different choice of unit normal \( N \) leaves \( P \) unchanged.

\[
\Rightarrow \text{even if } S \text{ unbounded, the same argument shows } x \text{ locally contained in } P
\]

\[
\Rightarrow x \text{ still contained in } \alpha(s_0) + P \text{ since } I \text{ connected}
\]
The space $\alpha: I \rightarrow S$ is planar geodesic with non-zero curvature.

We have $\alpha = \text{PBAL}$.

Since $\alpha'(s) \in \text{fixed plane } P$ for

$\Rightarrow \alpha''(s) \in P$ also.

But $\alpha''(s) = \text{non-zero multiple of } N(\alpha'(s))$, for in any local chart of unit normal $P = \left( N(\alpha'(s)) \wedge \alpha'(s) \right)^\perp$.

Since $\alpha' \perp N$ at both $\alpha$, $N$ and vectors, and $(N \wedge \alpha')^\perp = \text{constant}$,

$\Rightarrow N \wedge \alpha' = \text{constant}$.

Thus $\beta \wedge (\alpha'(s)) \wedge \alpha'(s) = 0$

parallel to $H$.

$\Rightarrow \beta = 0$.

So $\beta = \lambda \alpha'$ for some $\lambda$.

So $\alpha = \lambda \alpha'$ is curve along

the open set.

The same conclusion holds even if $\{ \alpha'' \neq 0 \}$ is dense in $I$.

Since the above argument would show $\{ \frac{d}{ds}(\alpha'(s)) = 0 \}$ is dense in $I$.

$\frac{d}{ds} N(\alpha') = 0$ on all $s \in I$, by continuity.
C. If $S$ = plane, $C$ = circle

$\Rightarrow$ $C$ trivially line of curvature ($\text{smallest } d\theta = 0$)

but $C$ not a geodesic, since curvature vector lies in plane, and is non-zero

rule: The $z$-axis of the helicoid is an example of a linear geodesic which is not a line of curvature.

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Choose global coords on the plane $x: (x_1, x_2) \rightarrow (x, x')$

Then $g_{ij} = \delta_{ij} =$ constant

$-1< g_{ij} = 0 \Rightarrow \Gamma^{k}_{ij} = 0$

so $(x'(t), x''(t))$ = geodesic $\Rightarrow \ddot{x} + \dot{x}' \cdot \dot{x}' \Gamma^{k}_{ij} = 0$

$= \dddot{x}$

$\Rightarrow \dddot{x} = 0$

$\implies (x'(t), x''(t)) = (a'^{t} + b'^{t})$

is linear
endow torus with outward normal

let \( C_1 = \text{max parallel}, \ C_2 = \text{upper parallel}, \ C_3 = \text{min parallel} \)

\( \therefore, \ C_i \) generated by the points:

\[
(a,r)
\]

\[
(a-r,0) \quad \quad \quad \quad \quad \quad \quad (a+r,0)
\]

we can parameterize \( C_i \) by \( \alpha_i(\theta) \), where

\[
a_i(\theta) = (a + r)(\cos\theta, \sin\theta, 0)
\]

\[
x_i(\theta) = (a - r)(\cos\theta, \sin\theta, 0)
\]

each \( \alpha_i(\theta) \) has curvature vector:

\[
k_1 = \frac{1}{a + r} (-\cos\theta, -\sin\theta, 0)
\]

\[
k_2 = \frac{1}{a} (-\cos\theta, -\sin\theta, 0)
\]

\[
k_3 = \frac{1}{a - r} (-\cos\theta, -\sin\theta, 0)
\]

the outward unit normal \( N_i \) is:

\[
N(a_i(\theta)) = (\cos\theta, \sin\theta, 0)
\]

\[
N(x_2(\theta)) = (0, 0, 1)
\]

\[
N(x_3(\theta)) = (-\cos\theta, -\sin\theta, 0)
\]
Now $C_i = \text{geodesic} \Rightarrow \text{curvature vector} \parallel N \parallel c_i$

This holds for $C_1$, $C_3$, but not $C_2$

$\Rightarrow C_1$, $C_3 = \text{geodesics}$, $C_2$ not geodesic

We compute: $\frac{d}{dt} N(a(t), \theta \approx (\theta_a(a(t)), 0, 0)) = A N(a(t), \theta \approx (\theta_a(a(t)), 0, 0))$

\[
\begin{align*}
\frac{d}{dt} N(a(t), \theta \approx (\theta_a(a(t)), 0, 0)) &= \frac{1}{a(t)} \left( -\sin \theta, \cos \theta, 0 \right) \\
&= \left( 0, 0, 0 \right) \\
&= 0 \cdot a(t) \left( 0, 0, 0 \right) \\
&= -\frac{1}{a(t)} \left( \sin \theta, -\cos \theta, 0 \right)
\end{align*}
\]

$\Rightarrow$ all 3 curves are lines of curvature

Since corresponding principle curvature is 0 for $C_2$

$\Rightarrow C_2$ also an asymptotic curve (but $C_1$, $C_3$ are not)
we compute geodesic curvature of \( C_2 \) w.r.t. outer normal 
\[
\mathbf{t}_2(\theta) = x_2 \left( \frac{\theta}{a} \right) = \text{parameterized by arclength}
\]
\[
\mathbf{t}_2(\theta) = \left. \left| dl_2 \right| = N(x_2) \wedge x_2 \right| = (0,0,1) \wedge \left( \frac{\cos \theta}{a}, \frac{\sin \theta}{a}, 0 \right)
\]
\[
= -\left( \frac{\cos \theta}{a}, \frac{\sin \theta}{a}, 0 \right)
\]
\[
\Rightarrow k_g = \left< x_2, \mathbf{t}_2' \right> = \frac{1}{a}
\]

we show \( S = \text{umbilic} \) — this will suffice to prove required conclusion, by a theorem from class

take \( p \in S \rightarrow \mathbf{V} \in T_pS, \exists \) geodesic \( \gamma_p \) s.t. \( \gamma_p(0) = p \) \( \gamma_p'(0) = \mathbf{V} \)

with \( |\mathbf{V}| = 1 \)

by assumption, \( \mathbf{V} \in \text{plane} \)

\( \Rightarrow \) either curvature \( \kappa_{p,v}''/|\mathbf{V}| = 0 \)

\( \Rightarrow \) DH(\( \gamma_p', 1, 0 \)) = \( \kappa_{p,v}(0) \)

by 0.1

\( \Rightarrow \) either \( \mathbf{v} \) = asymptotic direction \( \Rightarrow \mathbf{v} \) is eigenvalue of \( DH \)

if \( p \) not umbilic \( \Rightarrow \) \( \exists \) precisely 2 eigen directions and 2 asymptotic directions

\( \Rightarrow \) contradiction! \( \Rightarrow \) so \( p = \text{umbilic} \) \( \forall p \in S \)
Define \( x(s, v) : I \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3 \)

\[
= a(s) + v b(s)
\]

we have \( \frac{\partial s}{\partial s} \bigg|_{(s, 0)} = a'(s) \)

\( \frac{\partial v}{\partial s} \bigg|_{(s, 0)} = b(s) \)

since \( u(0) = \varepsilon, b \neq 0 \) \( \Rightarrow \) \( a', b \) linearly independent

\( \Rightarrow D(s) \) has rank 2 \( \forall s \in I \)

(5, 0)

by inverse function theorem, \( \forall s \in I \) \( \exists \) open intervals \( I_s \supset \varepsilon \)

\((I_m) \times (-\varepsilon, \varepsilon) \)

such that \( x \mid (I_m) \times (-\varepsilon, \varepsilon) \) is diffeomorphism onto its image

since \( I \) compact, \( \exists \) finite subcover \( \{ I_0, I_1, \ldots, I_n \} \) of \( I \)

- let \( \varepsilon = \min_{i=1}^{n} \varepsilon_i \)
- \( \exists \) \( \varphi \mid (I_m \times (-\varepsilon, \varepsilon)) \) is local diffeomorphism onto its image

(immerses surface)

just need to show \( x \) can be made injective, by shrinking \( \varepsilon \) further.
spz, towards a contradiction, \( \exists (s; t, i), (s'; t', i') \in I \times \mathbb{R}_+ \)

\[ s; (s; t, i) \neq (s'; t', i') \text{ but } x(s; t, i) \neq x(s'; t', i') \]

\[ t \to 0, t' \to 0 \]

since I is a connected interval and its image

\[ \Rightarrow \exists \text{ and } U \ni (s, 0) \text{ st } x|_{U} \text{ is injective} \]

\[ \Rightarrow (s, t, i) \notin U \forall i \text{ large} \]

\[ \Rightarrow s \neq s' \]

but then, we must have \( \partial(s) = \partial(i') \), contradicting injectivity of \( x \)

so \( \exists l > 0 \text{ s.t. } x|_{I \times (0, l)} = \text{ parameterization of } \gamma \text{ on surface} \)

now \( \alpha'' = \text{ covariant vector } \parallel \alpha \), \( b \)

\( \Rightarrow \alpha''(s) \perp T_{x(s, 0)} S \)

\( \Rightarrow \alpha(s) = x(s, 1) = \text{ geodesic in } S \)