Recall the Kohn–Nirenberg symbols

\[ S^k(\mathbb{R}^{2n}) = \{ a: \forall \alpha, \beta \partial_\xi^\alpha \partial_x^\beta a(x, \xi) = O(\langle \xi \rangle^{k-|\beta|}) \}. \]

**Exercise 8.1.** (a) Assume that \( a \in C^\infty(\mathbb{R}^{2n}) \) is compactly supported in \( x \) and satisfies the following homogeneity condition:

\[ a(x, \tau \xi) = \tau^k a(x, \xi) \quad \text{when} \quad \tau \geq 1, \ |\xi| \geq 1. \]

Show that \( a \in S^k(\mathbb{R}^{2n}) \).

(b) Show that \( \langle \xi \rangle^k \in S^k(\mathbb{R}^{2n}) \).

**Exercise 8.2.** (a) Assume that \( a \in S^k(\mathbb{R}^{2n}), b \in S^\ell(\mathbb{R}^{2n}) \). Show that \( ab \in S^{k+\ell}(\mathbb{R}^{2n}) \).

(b) Assume additionally that there exists a constant \( c > 0 \) such that \( |b(x, \xi)| \geq c \langle \xi \rangle^\ell \) for all \((x, \xi) \in \text{supp} \ a \). Show that \( a/b \in S^{k-\ell}(\mathbb{R}^{2n}) \).

**Exercise 8.3.** Assume that \( U, V \subset \mathbb{R}^n \) are open sets, \( \varphi: U \to V \) is a diffeomorphism, and \( \chi \in C^\infty_c(U) \). Let \( a \in S^k(\mathbb{R}^{2n}) \). Show that \( b(x, \xi) := \chi(x) a(\varphi(x), d\varphi(x)^{-T} \xi) \) lies in \( S^k(\mathbb{R}^{2n}) \) as well.

**Exercise 8.4.** Give the following extension of the elliptic parametrix construction of Exercise 7.1 to the Kohn–Nirenberg classes: assume that \( a \in S^k, p \in S^\ell \), and there exists a constant \( c > 0 \) such that \( |p(x, \xi)| \geq c \langle \xi \rangle^\ell \) on \( \text{supp} \ a \). Construct \( q, q' \in S^{k-\ell} \) such that

\[ a = q \# p + O(h^\infty)_{S^{-\infty}}, \quad a = p \# q' + O(h^\infty)_{S^{-\infty}} \]

where \( S^{-\infty} := \cap_N S(\langle \xi \rangle^{-N}) \). (You may assume that Borel’s Theorem is still valid, see §E.1.2 in the Dyatlov–Zworski book.)

**Exercise 8.5.** Assume that \( U \subset \mathbb{R}^n \) is an open set and

\[ P = \sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha, \quad a_\alpha \in C^\infty(U), \]

is a (nonsemiclassical) differential operator of order \( k \) on \( U \) whose principal symbol

\[ p_0(x, \xi) := \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha \]

\( Date: \) August 1, 2019.
is (nonsemiclassically) elliptic, namely $p_0(x,\xi) \neq 0$ for all $x \in U$, $\xi \in \mathbb{R}^n \setminus \{0\}$. (Examples include the Laplacian and, in dimension 2, the Cauchy–Riemann operator $\partial_{x_1} + i\partial_{x_2}$.)

We will show the following elliptic regularity theorem: if $u \in \mathcal{D}'(U)$ (i.e. $u$ is a distribution on $U$, that is a continuous linear functional on $C_\infty(U)$; any element of $\mathcal{S}'(\mathbb{R}^n)$ would define such a distribution), then

$$Pu \in C_\infty(U) \implies u \in C_\infty(U).$$

(a) Fix an arbitrary cutoff function $\chi \in C_\infty_c(U)$. Take $\chi' \in C_\infty_c(U)$ such that $\operatorname{supp}(1 - \chi') = \emptyset$ and define the rescaled cut off operator

$$P_h := h^k \chi' P.$$

Show that $P_h = \operatorname{Op}_h(p)$ for some $p \in S^k(\mathbb{R}^{2n})$ such that $p(x,\xi) = \chi'(x)p_0(x,\xi) + \mathcal{O}(h)^{s_k-1}(\mathbb{R}^{2n})$.

(b) Fix $\psi \in C_\infty_c(\mathbb{R}^n)$ with $\psi = 1$ near 0, and put

$$a(x,\xi) := \chi(x)(1 - \psi(\xi)) \in S^0(\mathbb{R}^{2n}).$$

Using Exercise 8.4, construct $q \in S^{-k}(\mathbb{R}^{2n})$ such that

$$\operatorname{Op}_h(a) = \operatorname{Op}_h(q)P_h + \operatorname{Op}_h(r), \quad r = \mathcal{O}(h^\infty)_{s_{-\infty}}. \tag{8.1}$$

(c) Put $v := \chi'u \in \mathcal{S}'(\mathbb{R}^n)$. Applying (8.1) to $v$, obtain

$$\chi \operatorname{Op}_h(a)v = \chi \operatorname{Op}_h(q)\chi'P_hu + \chi \operatorname{Op}_h(q)[P_h,\chi']u + \chi \operatorname{Op}_h(r)v.$$

Show that all three terms on the right-hand side are in $C_\infty_c(\mathbb{R}^n)$:

- for the first term, use the assumption $Pu \in C_\infty(U)$;
- for the second term, use the pseudolocality statement from the lecture and the fact that the coefficients of $[P_h,\chi']$ are supported away from $\operatorname{supp} \chi$;
- for the last term, use the properties of $r$ to see that $\chi \operatorname{Op}_h(r) : \mathcal{S}'(\mathbb{R}^n) \to C_\infty_c(\mathbb{R}^n)$.

(d) Now write

$$\chi^2u = \chi \operatorname{Op}_h(\chi(x))v = \chi \operatorname{Op}_h(a)v + \chi \operatorname{Op}_h(\chi(x)\psi(\xi))v.$$

Using that $\operatorname{Op}_h(\chi(x)\psi(\xi)) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ show that $\chi^2u \in C_\infty_c(\mathbb{R}^n)$. Since $\chi$ was arbitrary, this gives $u \in C_\infty(U)$.

(Note: in the above arguments $h$ was completely irrelevant, in fact we could have fixed $h := 1$.)