EXERCISES IN SEMICLASSICAL ANALYSIS
AT SNAP 2019, §7
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Exercise 7.1. This exercise finishes the details of the elliptic parametrix construction from the lecture. We assume that $m_1, m_2$ are order functions, $a(x,\xi;h) \in S(m_1)$, $p(x,\xi;h) \in S(m_2)$, and we have the following ellipticity condition: there exists $c > 0$ such that for all $h$

$$|p(x,\xi;h)| \geq cm_2(x,\xi) \quad \text{for all } (x,\xi) \in \text{supp } a(\bullet;h).$$

(a) Show that $a/p \in S(m_1/m_2)$. (Hint: prove first that every derivative $\partial^{\alpha}(p^{-1})$ is a linear combination of expressions of the form $p^{-\ell-1}\partial^{\alpha_1}p \cdots \partial^{\alpha_\ell}p$ where the multiindices $\alpha_1, \ldots, \alpha_\ell$ add up to $\alpha$.)

(b) Recall that $q_0 := a/p \in S(m_1/m_2)$ and $\text{supp } q_0 \subset \text{supp } a$. Recall from the Composition Theorem that

$$q_0 \# p = q_0 p - hr_1 + O(h^2)_{S(m_1)}, \quad r_1 := i \sum_{k=1}^{n} (\partial_{\xi_k} q_0)(\partial_{x_k} p) \in S(m_1).$$

Put $q_1 := r_1/p$. Show that $q_1 \in S(m_1/m_2)$, $\text{supp } q_1 \subset \text{supp } a$, and

$$(q_0 +hq_1) \# p = a + O(h^2)_{S(m_1)}.$$

(c) Iterating the argument in part (b), construct symbols $q_2, q_3, \ldots \in S(m_1/m_2)$, $\text{supp } q_j \subset \text{supp } a$, such that for each $k$

$$(q_0 +hq_1 + \cdots + h^{k-1} q_{k-1}) \# p = a + O(h^k)_{S(m_1)}.$$

(d) Using Borel’s Theorem, choose

$$q \in S(m_1/m_2), \quad q \sim \sum_{j=0}^{\infty} h^j q_j.$$

Show that $q \# p = a + O(h^\infty)_{S(m_1)}$.

Exercise 7.2. Assume that we are in the setting of Exercise 7.1 and $m_1 = 1$, $a = 1$. The elliptic parametrix construction gives two symbols $q, q' \in S(1/m_2)$ such that

$$1 = q \# p + O(h^\infty)_{S(1)}, \quad 1 = p \# q' + O(h^\infty)_{S(1)}.$$
Show that $q = q' + \mathcal{O}(h^\infty)_{S(1/m_2)}$ and thus $q = p\#q + \mathcal{O}(h^\infty)_{S(1)}$. (Hint: compute the product $q\#p\#q'$.)

**Exercise 7.3.** Consider the Schrödinger operator on $\mathbb{R}^n$

$$P = -h^2 \Delta + V(x)$$

where $V \in C^\infty(\mathbb{R}^n)$ satisfies the following assumptions for some $\ell > 0$:

- Bounded derivatives: $\partial^\alpha V(x) = \mathcal{O}(\langle x \rangle^{\ell})$ for all $\alpha$;
- Ellipticity at infinity: $V(x) \geq C^{-1} \langle x \rangle^{\ell} - C$ for some $C > 0$.

Assume that we are given a family of eigenfunctions:

$$(P - E_h)u_h = 0, \quad \|u_h\|_{L^2(\mathbb{R}^n)} = 1, \quad E_h \xrightarrow{h \to 0} E \in \mathbb{R}.$$ 

Define the classically allowed region

$$\Omega_E := \{x \in \mathbb{R}^n \mid V(x) \leq E\}$$

and fix an open set $U \supset \Omega_E$. Using the elliptic estimate, show that

$$\|u_h\|_{L^2(\mathbb{R}^n \setminus U)} = \mathcal{O}(h^\infty) \quad \text{as} \quad h \to 0.$$ 

(Hint: take $a(x, \xi) = \chi(x)$ where $\chi \in C^\infty(\mathbb{R}^n)$, supp $\chi \cap \Omega_E = \emptyset$, and $\chi = 1$ on $\mathbb{R}^n \setminus U$.)

**Exercise 7.4.** This advanced exercise provides estimates which may be used to establish functional calculus for pseudodifferential operators in the course on eigenfunctions. Assume that

$$P = \text{Op}_h(p), \quad p \in S(m)$$

where $m$ is an order function such that $m(x, \xi) \to \infty$ as $(x, \xi) \to \infty$, and $p$ is real-valued and satisfies the following ellipticity at infinity assumption: there exists a constant $C$ such that for all $(x, \xi)$

$$p(x, \xi; h) \geq \frac{m(x, \xi)}{C} - C.$$ 

Assume that $z \in \mathbb{C}$ varies in a compact set.

(a) Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Show that the symbol $p - z$ is elliptic everywhere. Define the symbols $q_0, q_1, \ldots \in S(1/m)$ using Exercise 7.1(c) such that for all $k$

$$(q_0 + hq_1 + \cdots + h^{k-1}q_{k-1})\#(p - z) = 1 + \mathcal{O}(h^k)_{S(1)}. \quad (7.1)$$

(b) We now allow $z$ to approach the real line. Show the derivative bounds for each $\alpha$, $j$, and $(x, \xi)$

$$|\partial^\alpha q_j(x, \xi, z; h)| \leq \frac{C_{\alpha, j}}{|\text{Im} z|^{2j + 1 + |\alpha|}} \cdot m(x, \xi).$$
(Hint: first of all, for any symbol \( q \) write the composition formula in the form

\[
q \# (p - z) \sim q(p - z) - \sum_{j=1}^{\infty} \hbar^j L_j q
\]

where each \( L_j \) is a differential operator of order \( j \) with \( z \)-independent coefficients which are in \( S(m) \). Now, to get (7.1) we put

\[
q_0 := \frac{1}{p - z}; \quad q_k := \frac{1}{p - z} \sum_{j=1}^{k} L_j q_{k-j}, \quad k \geq 1.
\]

From here obtain the formula

\[
q_k = \sum_{r=1}^{2k+1} \tilde{q}_{kr} \frac{1}{(p - z)^r}
\]

where \( \tilde{q}_{kr} \in S(m^{r-1}) \) are \( z \)-independent, and deduce the needed estimate.)

(c) Using \( L^2 \) boundedness (see formula (4.5.10) in Zworski’s book) and analyzing the remainder in (7.1) similarly to part (b) of this exercise, show that there exists some \( M_k \) depending only on \( n, k \) such that

\[
Q(z)(P - z) = I + \mathcal{O}\left(\frac{h^k}{|\text{Im } z|^{3M_k}}\right)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}
\]

where \( Q(z) := \text{Op}_h(q_0 + hq_1 + \cdots + h^{k-1} q_k) \).

(d) Show that the above statements are still true when \( p \) is not real-valued but \( \text{Im } p = \mathcal{O}(\hbar)_{S(m)} \), by dividing by \( \text{Re } p - z \) instead of \( p - z \) and putting the imaginary part of \( p \) into the next step of the iteration.