EXERCISES IN SEMICLASSICAL ANALYSIS
AT SNAP 2019, §6

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Exercise 6.1. Show the following versions of the Product, Commutator, and Adjoint Rules: if $a, b \in S(1)$ then

$$\text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(ab) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)},$$

$$[\text{Op}_h(a), \text{Op}_h(b)] = -ih \text{ Op}_h(\{a, b\}) + O(h^2)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)},$$

$$\text{Op}_h(a)^* = \text{Op}_h(\bar{a}) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.$$

Exercise 6.2. Assume that $a \in S(1)$, the functions $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^n)$ are $h$-independent, and $\text{supp} \chi_1 \cap \text{supp} \chi_2 = \emptyset$. Show that

$$\|\chi_1 \text{Op}_h(a) \chi_2\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = O(h^\infty).$$

This is a version of pseudolocality of pseudodifferential operators. It is a weaker property than locality of differential operators: if $a$ was a polynomial in $\xi$, then $\chi_1 \text{Op}_h(a) \chi_2 = 0$.

Exercise 6.3. Assume that $a \in S(m)$ where $m$ is an order function and $m(w) \to 0$ as $w = (x, \xi) \to 0$.

Fix $\chi \in C_c^\infty(\mathbb{R}^{2n})$ such that $\text{supp} \chi \subset B(0, 2)$ and $\chi = 1$ on $B(0, 1)$. For $R \geq 1$, define

$$a_R(w) := \chi\left(\frac{w}{R}\right)a(w), \quad w \in \mathbb{R}^{2n}.$$

(a) Show that for each multiindex $\alpha$, we have

$$\sup |\partial^\alpha (a - a_R)| \to 0 \quad \text{as} \quad R \to 0.$$

(b) Using the $L^2$ boundedness theorem (see Zworski’s book, formula (4.5.9)) show that

$$\|\text{Op}_h(a) - \text{Op}_h(a_R)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad R \to 0.$$

Exercise 6.4. For $s \in \mathbb{R}$, define the semiclassical Sobolev space $H^s_h(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n) \subset H^s_h(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, with the norm

$$\|u\|_{H^s_h(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} \langle h\xi\rangle^{2s}|\hat{u}(\xi)|^2 \, d\xi, \quad \langle h\xi\rangle := \sqrt{1 + |h\xi|^2}.$$
(a) Show that the norms \( \| \cdot \|_{H^s_h(\mathbb{R}^n)} \) are equivalent for fixed \( s \) and different values of \( h \), with equivalence constants depending on \( h \).

(b) Show that the norm \( \| u \|_{H^s_h(\mathbb{R}^n)} \) is equivalent, with equivalence constants independent of \( h \), to the norm \( \| \text{Op}_h(\langle \xi \rangle^s)u \|_{L^2(\mathbb{R}^n)} \).

(c) Assume that \( a \in S(\langle \xi \rangle^k) \). Using part (b), the Composition Theorem, and the \( L^2 \) Boundedness Theorem, show that for each \( s \) there exists a constant \( C \) such that for all \( h \)

\[
\| \text{Op}_h(a) \|_{H^s_h(\mathbb{R}^n) \to H^{s-k}_h(\mathbb{R}^n)} \leq C.
\]

Exercise 6.5.* Let \( A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) be a bounded operator. Fix a Hilbert basis \( \{ e_j \} \) of \( L^2(\mathbb{R}^n) \) and define the Hilbert–Schmidt norm of \( A \) by putting

\[
\| A \|_{\text{HS}}^2 := \sum_j \| Ae_j \|_{L^2(\mathbb{R}^n)}^2.
\]

If \( \| A \|_{\text{HS}} < \infty \) then we call \( A \) a Hilbert–Schmidt operator.

(a) For any other Hilbert basis \( \{ f_k \} \) show the identities

\[
\sum_j \| Ae_j \|_{L^2(\mathbb{R}^n)}^2 = \sum_{j,k} |\langle Ae_j, f_k \rangle_{L^2(\mathbb{R}^n)}|^2 = \sum_k \| A^* f_k \|_{L^2(\mathbb{R}^n)}^2.
\]

Use these to show that \( \| A \|_{\text{HS}} \) does not depend on the choice of the Hilbert basis and \( \| A \|_{\text{HS}} = \| A^* \|_{\text{HS}} \).

(b) Show the inequalities

\[
\| A \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \| A \|_{\text{HS}}, \\
\| AB \|_{\text{HS}} \leq \| A \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \cdot \| B \|_{\text{HS}}, \\
\| AB \|_{\text{HS}} \leq \| A \|_{\text{HS}} \cdot \| B \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.
\]

(c) Show that the space of Hilbert–Schmidt operators is a Hilbert space with the inner product

\[
\langle A, B \rangle_{\text{HS}} := \sum_j \langle Ae_j, Be_j \rangle_{L^2(\mathbb{R}^n)}.
\]

(d) Assume that \( A \) is an integral operator:

\[
Au(x) = \int_{\mathbb{R}^n} K_A(x, y)u(y) \, dy, \quad K_A \in L^2(\mathbb{R}^{2n}).
\]

Show that \( A \) is a Hilbert–Schmidt operator and

\[
\| A \|_{\text{HS}} = \| K_A \|_{L^2(\mathbb{R}^{2n})}.
\]

You may use the fact that for any two Hilbert bases \( \{ e_j \}, \{ f_k \} \) of \( L^2(\mathbb{R}^n) \), if we define \( (e_j \otimes f_k)(x, y) = e_j(x) f_k(y) \), then \( \{ e_j \otimes f_k \}_{j,k} \) is a Hilbert basis of \( L^2(\mathbb{R}^{2n}) \).
(e) Assume that \( a \in L^2(\mathbb{R}^{2n}) \). Show that
\[
\| \text{Op}_\hbar(a) \|_{\text{HS}} = (2\pi\hbar)^{-\frac{n}{2}} \| a \|_{L^2(\mathbb{R}^{2n})}.
\]

**Exercise 6.6.** For a bounded operator \( A \) on \( L^2(\mathbb{R}^n) \), we say it is a *trace class* operator, if it can be written as \( A = BC \) where \( B, C \) are Hilbert–Schmidt operators. For a trace class operator \( A \), define its *trace* by
\[
\text{tr} A = \sum_j \langle Ae_j, e_j \rangle_{L^2(\mathbb{R}^n)}
\]
where \( \{e_j\} \) is a Hilbert basis of \( L^2(\mathbb{R}^n) \).

(a) If \( A = BC \) where \( B, C \) are Hilbert–Schmidt operators, show that
\[
\text{tr} A = \langle C, B^* \rangle_{\text{HS}}.
\]
Use this to show that \( \text{tr} A \) is independent of the choice of the Hilbert basis.

(b) For \( A = BC \) where \( B, C \) are Hilbert–Schmidt operators, show that \( \text{tr}(BC) = \text{tr}(CB) \) and \( \text{tr} A = \text{tr} A^* \).

(c) We use without proof the following fact (see Theorem C.18 in Zworski’s book): if \( A \) is an integral operator
\[
Au(x) = \int_{\mathbb{R}^n} K_A(x, y) u(y) \, dy, \quad K_A \in \mathcal{S}(\mathbb{R}^{2n}),
\]
then \( A \) is trace class and
\[
\text{tr} A = \int_{\mathbb{R}^n} K_A(x, x) \, dx. \tag{6.1}
\]
Show the formula (6.1) when \( K_A(x, y) = f(x)g(y), \ f, g \in \mathcal{S}(\mathbb{R}^n) \).

(d) Using (6.1), show that for \( a \in \mathcal{S}(\mathbb{R}^{2n}) \)
\[
\text{tr} \text{Op}_\hbar(a) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} a(x, \xi) \, dx \, d\xi.
\]