Recall the Composition and Adjoint Theorems: for \( a, b \in \mathcal{S}(\mathbb{R}^{2n}), \)
\[
\text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(a\# b), \quad \text{Op}_h(a) = \text{Op}_h(a^*)
\]
where we have the asymptotic expansions in \( \mathcal{S}(\mathbb{R}^{2n}) \), as \( h \to 0 \)
\[
a\# b(x, \xi; h) \sim \infty \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial^{\alpha}_\xi a(x, \xi) \partial^{\alpha}_x b(x, \xi),
\]
(4.1)
\[
a^*(x, \xi; h) \sim \infty \sum_{j=0}^{\infty} (-ih)^j \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial^{\alpha}_x \partial^{\alpha}_\xi a(x, \xi).
\]
(4.2)

Exercise 4.1. (a) Check by hand that an expansion similar to (4.1) holds for \( a = \xi_j \), \( b = x_j \). (Of course the expansion will no longer be in \( \mathcal{S}(\mathbb{R}^{2n}); \) the next section will address this.) Check the Product Rule and the Commutator Rule in this case.

(b) Check by hand that an expansion similar to (4.2) holds for \( a = x_j \xi_j \).

(c)* By direct computation (using the Leibniz rule) show that expansions of the form (4.1)–(4.2) hold in the case when \( a, b \) are polynomials in \( \xi \), and thus \( \text{Op}_h(a), \text{Op}_h(b) \) are semiclassical differential operators, see Exercise 3.2.

Exercise 4.2. Verify that the \( j = 0, 1 \) terms of (4.1) give the Product Rule and the Commutator Rule, and the \( j = 0 \) term of (4.2) gives the Adjoint Rule.

Exercise 4.3. Using the multinomial theorem, show the following identities used in the proof of the Composition Theorem and the Adjoint Theorem:
\[
\frac{1}{j!} \langle \partial_y, \partial_\eta \rangle^j (a(y, \eta)b(y, \xi)) |_{y=x, \eta=\xi} = \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial^\alpha_\xi a(x, \xi) \partial^\alpha_x b(x, \xi),
\]
\[
\frac{1}{j!} \langle \partial_x, \partial_\xi \rangle^j a(x, \xi) = \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial^\alpha_x \partial^\alpha_\xi a(x, \xi).
\]

Exercise 4.4. In lecture, we only established the expansion (4.1) for any fixed \( (x, \xi) \). Show that this expansion is valid in \( \mathcal{S}(\mathbb{R}^{2n}) \), in particular the remainder is controlled

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uniformly in \((x, \xi)\) and the expansion can be differentiated. (Here \(a \# b\) is compactly supported and thus there is no need to get asymptotics as \((x, \xi) \to \infty\).)

**Exercise 4.5.** (a) Assume that \(Q\) is a \(2n \times 2n\) invertible symmetric real-valued matrix, \(a \in C^\infty_c(\mathbb{R}^{2n})\) is supported in the ball \(B_{\mathbb{R}^{2n}}(0, R)\) for some \(R \geq 1\), and
\[
\tilde{a}(\rho; h) := \int_{\mathbb{R}^{2n}} e^{\frac{i}{2h}(Q w, w)} a(\rho + w) \, dw.
\]
Show that for each multiindices \(\alpha, \beta\) and each \(N\) there exists a constant \(C_{\alpha \beta N}\) such that for all \(h \in (0, 1]\)
\[
|\rho^\alpha \partial_\rho^\beta \tilde{a}(\rho; h)| \leq C_{\alpha \beta N} h^N \quad \text{for all } \rho \in \mathbb{R}^{2n}, \ |ho| \geq 2R.
\]
(Hint: integrate by parts using the identity \(e^{\frac{i}{2h}(Q w, w)} = h L e^{\frac{i}{2h}(Q w, w)}\) where \(L := -\frac{i}{|w|^2}(Q^{-1} w, \partial_w)\).)

(b) Explain how part (a) gives the last part of the proof of the Adjoint Theorem in the lecture.

**Exercise 4.6.** Following the proof of the Adjoint Theorem, show the following change of quantization formula: if \(a \in C^\infty_c(\mathbb{R}^{2n})\), then
\[
\text{Op}_h^w(a) = \text{Op}_h (a_w)
\]
where \(a_w(x, \xi; h)\) has the asymptotic expansion in \(\mathcal{S}(\mathbb{R}^{2n})\)
\[
a_w(x, \xi; h) \sim \sum_{j=0}^\infty \left(-\frac{ih}{2}\right)^j \sum_{|\alpha| = j} \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha a(x, \xi).
\]
In particular, \(a_w = a + \mathcal{O}(h)\) in \(\mathcal{S}(\mathbb{R}^{2n})\). For a more general change of quantization statement, see Theorem 4.13 in Zworski’s book.