Recall the standard and Weyl quantization formulas (valid as convergent integrals when $a \in \mathcal{S}(\mathbb{R}^{2n})$, $u \in \mathcal{S}(\mathbb{R}^n)$)

\[
\text{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y,\xi)} a(x,\xi) u(y) \, dy \, d\xi,
\]

(3.1)

\[
\text{Op}_h^w(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y,\xi)} a\left(\frac{x+y}{2},\xi\right) u(y) \, dy \, d\xi.
\]

(3.2)

We use the notation $\langle x \rangle := \sqrt{1 + |x|^2}$.

Exercise 3.1. Fill in the details of the proof in lecture that the formula

\[
\text{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x,\xi)} a(x,\xi) \hat{u}\left(\frac{\xi}{h}\right) \, d\xi
\]

(3.3)

implies that

(a) if $a \in \mathcal{S}(\mathbb{R}^{2n})$ then $\text{Op}_h(a) : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ (hint: use that $\xi \mapsto e^{\frac{i}{h}(x,\xi)} a(x,\xi)$ is a Schwartz function all of whose seminorms are rapidly decaying in $x$);

(b) if $a \in C^\infty(\mathbb{R}^{2n})$ and $|a(x,\xi)| \leq C \langle x \rangle^N \langle \xi \rangle^N$ for some $C, N$ then we may define $\text{Op}_h(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \langle x \rangle^N L^\infty(\mathbb{R}^n)$ (hint: the integral (3.3) converges).

Exercise 3.2. Using (3.3) and properties of the Fourier transform, verify that if $a$ is a polynomial in the $\xi$ variables

\[
a(x,\xi) = \sum_{|\alpha| \leq k} a_\alpha(x) \xi^\alpha
\]

where each $a_\alpha \in C^\infty(\mathbb{R}^n)$ is polynomially bounded, then $\text{Op}_h(a)$ is a semiclassical differential operator:

\[
\text{Op}_h(a) = \sum_{|\alpha| \leq k} a_\alpha(x) (hD_x)^\alpha, \quad D_x := -i\partial_x
\]

Here $\alpha = (\alpha_1, \ldots, \alpha_n)$ denotes multiindices and

\[
\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n, \quad (hD_x)^\alpha = h^{|\alpha|} D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}.
\]

Exercise 3.3. This exercise establishes some basic properties of the Weyl quantization.

Date: August 1, 2019.
(a) Verify that $\text{Op}_h^w(a)^* = \text{Op}_h^w(\bar{a})$ for all $a \in \mathcal{S}(\mathbb{R}^{2n})$, that is for all $u, v \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \text{Op}_h^w(a)u, v \rangle_{L^2(\mathbb{R}^n)} = \langle u, \text{Op}_h^w(\bar{a})v \rangle_{L^2(\mathbb{R}^n)}.$$ 

(b) For $a \in \mathcal{S}(\mathbb{R}^{2n})$ and $u, v \in \mathcal{S}(\mathbb{R}^n)$, show that

$$\langle \text{Op}_h^w(a)u, v \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(x, \xi) W_{u,v}(x, \xi) \, dx d\xi \tag{3.4}$$

where the function $W_{u,v}(x, \xi)$ is defined as follows:

$$W_{u,v}(x, \xi) := (\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{2i}{\hbar} \langle w, \xi \rangle} u(x - w) \overline{v(x + w)} \, dw.$$

(For $u = v$, $W_{u,v}$ is called the Wigner function of $u$.)

(c) For $u, v \in \mathcal{S}(\mathbb{R}^n)$, show that $W_{u,v} \in \mathcal{S}(\mathbb{R}^{2n})$. (Hint: write $W_{u,v}$ as the rescaled Fourier transform in $w$ of the function $B(x, w) = u(x - w)\overline{v(x + w)}$ which lies in $\mathcal{S}(\mathbb{R}^{2n})$.) Using this, show that for $a \in \mathcal{S}'(\mathbb{R}^{2n})$ we may define $\text{Op}_h^w(a) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ via (3.4).

(d) Show that $\text{Op}_h^w(1) = I$.

**Exercise 3.4.*** Finish the proof of oscillatory testing from the lecture: assuming that $e_\xi(x) = e^{i\langle x, \xi \rangle}$ (we remove $\hbar$ for simplicity, since it does not matter for this part), $B : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous, and $Be_\xi = 0$ for all $\xi$, show that $B = 0$. (Hint: by approximation it suffices to show that $Bu = 0$ for each $u \in \mathcal{S}(\mathbb{R}^n)$. Write by Fourier inversion formula

$$u = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi) e_\xi \, d\xi,$$

and use that the Riemann sums of the above integral converge to $u$ in $\langle x \rangle L^{\infty}(\mathbb{R}^n)$.)

**Exercise 3.5. (a)** Show the following product formulas for the standard quantization when $a \in \mathcal{S}(\mathbb{R}^{2n})$:

$$x_j \text{Op}_h(a) = \text{Op}_h(x_j a), \quad \text{Op}_h(a)x_j = \text{Op}_h(x_j a - ih\partial_{\xi_j} a), \quad (hD_{x_j}) \text{Op}_h(a) = \text{Op}_h(\xi_j a - ih\partial_{x_j} a),$$

$$\text{Op}_h(a)(hD_{x_j}) = \text{Op}_h(\xi_j a). \quad \tag{3.8}$$

(Hint: use the formula (3.1). For (3.6), integrate by parts in $\xi_j$. For (3.8), integrate by parts in $y_j$.)
(b) Show the following product formulas for the Weyl quantization when \(a \in \mathcal{S}(\mathbb{R}^{2n})\):

\[
x_j \text{Op}_h^w(a) = \text{Op}_h^w(x_j a + \frac{i \hbar}{2} \partial \xi_j a),
\]

\[
\text{Op}_h^w(a) x_j = \text{Op}_h^w(x_j a - \frac{i \hbar}{2} \partial \xi_j a),
\]

\[
(hD_{x_j}) \text{Op}_h^w(a) = \text{Op}_h^w(\xi_j a - \frac{i \hbar}{2} \partial x_j a),
\]

\[
\text{Op}_h^w(a)(hD_{x_j}) = \text{Op}_h^w(\xi_j a + \frac{i \hbar}{2} \partial x_j a).
\]

(Hint: use the formula (3.2). For (3.9)-(3.10), integrate by parts in \(\xi_j\). For (3.12), integrate by parts in \(y_j\).)

(c) Using (3.9)-(3.12) (which are still valid for \(a \in \mathcal{S}''(\mathbb{R}^{2n})\) via approximating it by Schwartz functions), show that \(\text{Op}_h^w(x_j \xi_j) = x_j (hD_{\xi_j}) - \frac{i \hbar}{2}\).