EXERCISES IN SEMICLASSICAL ANALYSIS
AT SNAP 2019, §1

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Exercise 1.1. In lecture we had the following formula:

\[ u(t, x) = \frac{1}{2\pi} \int_\mathbb{R} e^{\frac{i}{\hbar}(x\xi - t\xi^2)} \hat{\chi}(\xi) \, d\xi, \quad \hat{\chi} \in C^\infty_c(\mathbb{R}). \] (1.1)

Check that this function does indeed solve Schrödinger’s equation

\[ -ih\partial_t u(t, x) = h^2 \partial_x^2 u(t, x). \]

Exercise 1.2. Go carefully through the proof of the wavefront set theorem we had in lecture: if \( u \) is given by (1.1) and supp \( \hat{\chi} \subset [-1, 2] \) then

\[ \text{WF}_h(u(t, \bullet)) \subset e^{tH_p}\{(0, \xi) \mid \xi \in [-1, 2]\}. \]

Exercise 1.3. Assume that \( u(x) \in C^\infty_c(\mathbb{R}) \) is \( h \)-independent. Show that

\[ \text{WF}_h(u) \subset \{(x, 0) \mid x \in \text{supp } u\}. \]

Exercise 1.4.* Assume that

\[ u(x; h) = e^{i\varphi(x) \hbar} a(x), \quad x \in \mathbb{R}^n \] (1.2)

where \( \varphi \in C^\infty(U; \mathbb{R}), U \subset \mathbb{R}^n \) is an open set, and \( a \in C^\infty_c(U) \). Using the method of nonstationary phase, show that

\[ \text{WF}_h(u) \subset \{(x, d\varphi(x)) \mid x \in \text{supp } a\}. \]

(Functions of the form (1.2) are a special case of Lagrangian distributions, or WKB states, which will appear again later. Note that the previous exercise was a special case of this one, with \( \varphi \equiv 0 \).)

Exercise 1.5.* Consider the Gaussian function

\[ u(x; h) := e^{-\frac{|x|^2}{2\hbar}}, \quad x \in \mathbb{R}^n. \]

(a) Show that for any (\( h \)-independent) \( \psi \in C^\infty_c(\mathbb{R}^n) \) such that \( \text{supp } \psi \cap \{0\} = \emptyset \), we have \( \psi u = O(h^\infty) \mathcal{F}(\mathbb{R}^n) \).
(b) Show that for any $\psi \in C_c^\infty(\mathbb{R})$ and any $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\mathcal{F}_h(\psi u)(\xi) = \mathcal{O}(h^\infty)$. You may use the following corollary of the convolution formula for the Fourier transform and the formula for Fourier transform of Gaussians:

$$\mathcal{F}_h(\psi u)(\xi) = (2\pi h)^{-\frac{n}{2}} \hat{\psi} u\left(\frac{\xi}{h}\right) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{\|\eta\|^2}{4h}} \hat{\psi}\left(\frac{\xi - \eta}{h}\right) d\eta.$$

(c) Combine parts (a) and (b) to show that

$$WF_h(u) \subset \{(0,0)\}.$$  

Exercise 1.6. Prove a wavefront set statement similar to the one in the lecture for more general constant coefficient operators, with the original Hamiltonian $p(x, \xi) = \xi^2$ replaced by an arbitrary real-valued polynomial in $\xi$. 