OVERVIEW OF CALCULUS ON MANIFOLDS

SEMYON DYATLOV

Here is a brief overview of semiclassical pseudodifferential calculus on a manifold $M$. See §E.1 in the Dyatlov–Zworski book for details. (Note: the calculus here corresponds to symbols in $S^k_{1,0}$ in the notation of that book.)

- Distributions and general operators:
  - $\mathcal{D}'(M)$ distributions on $M$, $\mathcal{E}'(M)$ compactly supported distributions;
  - an operator $A : C^\infty_c(M) \to \mathcal{D}'(M)$ is called compactly supported, if its Schwartz kernel is compactly supported, i.e. $A = \chi A \chi$ for some $\chi \in C^\infty_c(M)$; in this case $A$ maps $C^\infty_c(M) \to \mathcal{E}'(M)$;
  - $A : C^\infty_c(M) \to \mathcal{D}'(M)$ is called properly supported, if for each $\chi \in C^\infty_c(M)$, the operators $\chi A$ and $A \chi$ are compactly supported; in this case $A$ maps $C^\infty_c(M) \to \mathcal{E}'(M)$ and $C^\infty(M) \to \mathcal{D}'(M)$;

- Pseudodifferential operators:
  - $\Psi^k(M)$, $k \in \mathbb{R}$, the class of semiclassical pseudodifferential operators of order $k$ on $M$;
  - all elements of $\Psi^k_h(M)$ map $C^\infty_c(M) \to C^\infty(M)$ and $\mathcal{E}'(M) \to \mathcal{D}'(M)$;
  - properly supported operators in $\Psi^k_h(M)$ map $C^\infty_c(M) \to C^\infty_c(M)$, $C^\infty(M) \to C^\infty(M)$, $\mathcal{E}'(M) \to \mathcal{E}'(M)$, $\mathcal{D}'(M) \to \mathcal{D}'(M)$, and thus can be multiplied with other operators;
  - $h^\infty \Psi^{-\infty} = \bigcap_k \Psi^k_h(M)$, the class of rapidly decaying smoothing operators on $M$: integral operators of the form $u \mapsto \int_M K(x, y; h) u(y) \, dy$ where $K \in C^\infty(M \times M)$ and each $C^\infty$ seminorm of $K$ is $O(h^\infty)$; such operators map $\mathcal{E}'(M) \to C^\infty(M)$;

- Symbols and quantization:
  - $S^k(T^*M)$ the space of $h$-dependent Kohn–Nirenberg symbols of order $k$ on the cotangent bundle $T^*M$ (with no uniformity in $x$ imposed when $M$ is noncompact);
  - $\sigma^k_h : \Psi^k_h(M) \to S^k(T^*M)/hS^{k-1}(T^*M)$ the principal symbol map (we usually suppress $k$ in notation, simply writing $\sigma_h$);
  - the kernel of $\sigma^k_h$ is equal to $h\Psi^{k-1}_h(M)$;

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\[ \text{– Op}_h : S^k(T^*M) \to \Psi^k_h(M) \text{ a noncanonical quantization map;} \]
\[ \text{– } \sigma^k_h(\text{Op}_h(a)) = a \mod hS^{k-1}(T^*M) \text{ for all } a \in S^k(T^*M); \]
\[ \text{– for any } a \in S^k(T^*M), \text{ Op}_h(a) \text{ is properly supported, and if } a \text{ is compactly} \]
\[ \text{– we can choose Op}_h \text{ so that Op}_h(1) = I; \]
\[ \text{– for each } A \in \Psi^k_h(M) \text{ there exists } a \in S^k(T^*M) \text{ such that } A = \text{Op}_h(a) + \mathcal{O}(h^{\infty})_{\Psi^{-\infty}}; \]

- **Algebraic properties:**
  - **Product Rule:** if \( A \in \Psi^k_h(M), B \in \Psi^\ell_h(M) \), and at least one of these operators is properly supported, then \( AB \in \Psi^{k+\ell}_h(M) \), and \( \sigma^h_k(AB) = \sigma^h_k(A)\sigma^h_k(B) \); equivalently, if \( a \in S^k(T^*M), b \in S^\ell(T^*M) \), then
    \[ \text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(ab) + \mathcal{O}(h)_{\Psi^{k+\ell-1}(M)}; \]
  - **Commutator Rule:** under the assumptions of the Product Rule we have
    \[ \sigma^h_k([A,B]) = -i\{\sigma^h_k(A), \sigma^h_k(B)\}; \]
    equivalently,
    \[ [\text{Op}_h(a), \text{Op}_h(b)] = -ih \text{Op}_h(\{a,b\}) + \mathcal{O}(h^2)_{\Psi^{k+\ell-2}(M)}; \]
  - **Adjoint Rule:** if we fix any smooth density on \( M \) (to fix an inner product on \( L^2(M) \) and thus be able to take adjoints of operators), and \( A \in \Psi^k_h(M), \) then \( A^* \in \Psi^k_h(M) \) and \( \sigma^h_k(A^*) = \overline{\sigma^h_k(A)} \); equivalently, if \( a \in S^k(T^*M) \), then
    \[ \text{Op}_h(a)^* = \text{Op}_h(\overline{a}) + \mathcal{O}(h)_{\Psi^{k-1}(M)}; \]

- **Wavefront sets:**
  - For \( A \in \Psi^k_h(M) \), its wavefront set is \( \text{WF}_h(A) \subset T^*M \), with \( T^*M \) the fiber-radial compactification of \( T^*M \);
  - \( \text{WF}_h(A) = \emptyset \iff A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}; \)
  - if \( a(x,\xi;h) \) is supported in an \( h \)-independent set \( K \), then \( \text{WF}_h(\text{Op}_h(a)) \subset K; \)
  - \( \text{WF}_h(A+B) \subset \text{WF}_h(A) \cup \text{WF}_h(B); \)
  - \( \text{WF}_h(AB) \subset \text{WF}_h(A) \cap \text{WF}_h(B); \)
  - \( \text{WF}_h(A^*) = \text{WF}_h(A); \)

- **L^2 theory, assuming for simplicity \( M \) is compact:**
  - One can define semiclassical Sobolev spaces \( H^s_h(M) \), with a noncanonical \( h \)-dependent norm, and \( H^0_h(M) = L^2(M); \)
  - if \( A \in \Psi^k_h(M) \), then \( A : H^s_h(M) \to H^{s-k}_h(M) \), with the norm bounded uniformly in \( h; \)
  - \( H^s_h(M) \) embeds compactly into \( H^t_h(M) \) for \( s > t; \)
– Sharp Gårding inequality: if \( a \in S^k(T^*M) \) and \( \text{Re} \, a \geq 0 \) everywhere, then for each \( u \in H^k_h(M) \) and \( h \)

\[
\text{Re}(\text{Op}_h(a)u, u)_{L^2} \geq -Ch\|u\|_{H^{\frac{k}{h}}_h(M)}^{k-1}
\]

where \( C \) is some constant depending on \( a \).