§8. CHANGE OF VARIABLES

This section will lay the groundwork for defining semiclassical quantization on manifolds.

§8.1. Compactly supported symbols

Assume that we are given a diffeomorphism \( \varphi: U \rightarrow V \), \( U, V \subset \mathbb{R}^n \) open sets.

Let \( a \in C_c^\infty(\mathbb{R}^{2n}) \). We want to conjugate \( a \) by the pullback operator \( \varphi^* \) of \( \varphi \), i.e. study the operator

\[
\varphi^* D_p \varphi(a)(\varphi^{-1})^*.
\]

But for \( u \in \mathcal{S}(\mathbb{R}^n) \), \( (\varphi^{-1})^* u \in C_c^\infty(V) \) does not extend to a function on \( \mathbb{R}^n \) because \( \varphi \) was only defined locally. So we also fix a cutoff \( \chi \in C_c^\infty(U) \).

Then \( \chi \varphi^*: C_c^\infty(V) \rightarrow C_c^\infty(U) \),

\[
(\varphi^{-1})^* \chi: C_c^\infty(U) \rightarrow C_c^\infty(V)
\]

naturally extend to operators \( C_c^\infty(\mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n) \) (and thus \( \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n) \)).

The symbol will change by the map

\[
\hat{\varphi}: U_x \times \mathbb{R}^n_2 \rightarrow V_x \times \mathbb{R}^n_2 \quad \text{(subsets of } \mathbb{R}^{2n})
\]

\[
\hat{\varphi}(x, \xi) = (\varphi(x), d\varphi(x)^{-T} \cdot \xi)
\]

inverse of the transpose of \( d\varphi(x) \).
Theorem

Under the above assumptions, \( \varphi \in \mathcal{O}_p^h(a) \), \( (\varphi^{-1})^* \chi = \mathcal{O}_p^h(b) \)

"operator on \( V \)"  

"operator on \( U \)"

for some \( b(x, \xi, h) \in S(\mathbb{R}^{2n}) \) with an expansion

\[
b(x, \xi, h) = \sum_{j=0}^{\infty} h^j L_j \left( a \circ \tilde{\varphi} \right) \quad \text{where } L_j \text{ are differential operators of order } 2j
\]

and the leading term is

\[
b(x, \xi, h) = \chi(x)^2 a \left( \tilde{\varphi}(x, \xi) \right) + O(h) \quad \text{in } S(\mathbb{R}^{2n}).
\]

Proof

1. We use oscillatory testing:

   \[
   \text{if } e_{\xi}(x) = e^{i \langle x, \xi \rangle} \quad \text{then}
   \]

   \[
b(x, \xi, h) = e^{-i \langle x, \xi \rangle} \left( \chi \varphi \mathcal{O}_p^h(a) \mathcal{O}_p^h(b) \chi \right) = \chi(x) \left( \mathcal{O}_p^h(a) \mathcal{O}_p^h(b) \chi \right) = \chi(x) a(\varphi(x), \eta) \left( \mathcal{O}_p^h(b) \chi \right)
   \]

   \[
   = \left( 2\pi h \right)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle x, v \rangle} \int_{\mathbb{R}^{2n}} e^{i \langle \xi, v \rangle} \chi(x) a(\varphi(x), \eta) \chi(\varphi(y), \eta) \, d\xi \, dy
   \]

   \[
   = \left( 2\pi h \right)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle x, v \rangle} \int_{\mathbb{R}^{2n}} e^{i \langle \xi, v \rangle} \chi(x) a(\varphi(x), \eta) \chi(\varphi(y), \eta) \, d\xi \, dy
   \]

   \[
   = \left( 2\pi h \right)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle x, v \rangle} \int_{\mathbb{R}^{2n}} e^{i \langle \xi, v \rangle} \chi(x) a(\varphi(x), \eta) \chi(\varphi(y), \eta) \, d\xi \, dy
   \]

   \[
   = \left( 2\pi h \right)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle x, v \rangle} \int_{\mathbb{R}^{2n}} e^{i \langle \xi, v \rangle} \chi(x) a(\varphi(x), \eta) \chi(\varphi(y), \eta) \, d\xi \, dy
   \]

   \[
   = \left( 2\pi h \right)^{-n} \int_{\mathbb{R}^{2n}} e^{i \langle x, v \rangle} \int_{\mathbb{R}^{2n}} e^{i \langle \xi, v \rangle} \chi(x) a(\varphi(x), \eta) \chi(\varphi(y), \eta) \, d\xi \, dy
   \]

2. For fixed \((x, \xi)\), use stationary phase:

   \[
   \Phi = \langle z - x, \xi \rangle + \langle \varphi(z) - \varphi(x), \eta \rangle, \quad \text{integrating in } z, \eta
   \]

   \[
   \partial_{\eta} = 0 \iff x = z, \quad \partial_{z} = 0 \iff \xi = \partial_{\varphi} \varphi(z)^T \cdot \eta
   \]
Thus the only critical point is 
\( z = x, \eta = d\varphi(x)^{-T} \xi \).

This gives an asymptotic expansion of the form stated above.

Let us compute the leading term.

The Hessian of the phase is
\[
\begin{pmatrix}
  z & -d\varphi(z)^T \\
  -d\varphi(z) & 0
\end{pmatrix}
\]

so \( \det d^2 \varphi \mid z = 1 = \det d^2 \varphi(z) \).

The value of \( \varphi \) at the critical point is 0.

So the leading term is 
\[
b(x, \xi) = X(x)^2 a(x, d\varphi(x)^{-T} \xi) + O(h)
\]

3. It remains to get the expansion in \( S(\mathbb{R}^{2n}) \).

- Higher derivatives: straightforward (st. phase uniform in parameters)
- \( x \to \infty \): \( b \) is compactly supported in \( x \)
- \( \xi \to \infty \): get \( b = O(h^m <\xi^m>\infty) \) for large \( \xi \)

by integrating by parts in \( z \).

See the book of Dyatlov-Zworski, Proposition E.10 for details. □
§ 8.2. General symbols

Recall the expansion was
\[ b(x, \xi; h) = x(x) A(\psi(x), d\mu(x)^{-1} \cdot \xi) + \ldots \]

Unfortunately, this operation does not preserve the class \( S^1(1) = \{ a : \forall \xi, \partial^\infty a \text{ is bounded} \} \).

Indeed, in 1D (for simplicity)
\[ \partial_x b = x(x)^2 \partial^2_x a(\psi(x), \frac{1}{\rho(x)}, \xi, \xi) \cdot \frac{1}{\rho(x)} \cdot \xi \]

only know this is bounded

this is not bounded

To fix this, we need to require that

derivatives in \( \xi \) decay by a power of \( \xi \):

**Definition** Let \( k \in \mathbb{R} \). We say \( a(x, \xi; h) \)
is in \( S^k(\mathbb{R}^{2n}) \), if \( \forall \alpha, \beta \in \mathbb{N}_0^n \), \( \forall x, \xi, h \)

\[ |\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \leq C_{\alpha, \beta} (\xi)^{k-1} \]

\( S^k \) are called **Kohn-Nirenberg symbols**.

**Note that** \( S^k \subset S^1(\langle \xi \rangle^k) \).

**Definition** Assume that \( a \in S^k(\mathbb{R}^{2n}), a_j \in S^{k-j}(\mathbb{R}^{2n}) \)

\( j = 0, 1, \ldots \)

We write \( a \sim \sum_{j=0}^{\infty} h^j a_j \), if \( \forall N, \)

\[ a - \sum_{j=0}^{N} h^j a_j = O(h^N) S^{k-N}(\mathbb{R}^{2n}) \text{ as in } h. \]
We now revisit the calculus of \( \xi \).

**Theorem (Composition Formula)**

Let \( a \in S^k(\mathbb{R}^{2m}) \), \( b \in S^\ell(\mathbb{R}^{2m}) \). Then

\[
O_{ph}(a)O_{ph}(b) = O_{ph}(a \# b), \quad a \# b \in S^{k+\ell}(\mathbb{R}^{2m}),
\]

\[
a \# b \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{\|\alpha\| = j} \frac{1}{\alpha!} \partial_\alpha a \cdot \partial_\alpha b
\]

Here the expansion is in \( S^{k+\ell} \), so we get

**Product Rule:**

\[
a \# b = ab + O(h) S^{k+\ell-1}
\]

**Commutator Rule:**

\[
a \# b - b \# a = -ih \{a, b\} + O(h^2) S^{k+\ell-2}
\]

Why do we get an expansion with improved remainders?

An informal explanation is that the terms in the expansions decay faster in \( \xi \), owing to the \( \xi \)-derivatives:

\[
\partial_\xi a \cdot \partial_\xi b \in S^{k+\ell-1-1}
\]

For the actual proof see Zworski’s book,

**Theorem (Adjoint Formula)**

Let \( a \in S^k(\mathbb{R}^{2m}) \). Then

\[
O_{ph}(a)^* = O_{ph}(a^*), \quad a^* \in S^k(\mathbb{R}^{2m}),
\]

\[
a \sim \sum_{j=0}^{\infty} (-ih)^j \sum_{\|\alpha\| = j} \frac{1}{\alpha!} \partial_\alpha a \cdot \partial_\alpha \overline{a} \leftarrow \text{expansion in } S^{k}
\]

**Adjoint Rule:**

\[
a^* = \overline{a} + O(h) S^{k-1}
\]
Theorem (Change of variables)

Assume that \( U, V \subseteq \mathbb{R}^n \) are open sets, \( \varphi : U \to V \) is a diffeomorphism, \( X \in C^\infty_c(U) \), and \( \tilde{\varphi}(x, \xi) := (\varphi(x), d\varphi(x)^{-1} \cdot \xi) \).

Let \( a \in S^k(\mathbb{R}^{2n}) \). Then

\[
X \varphi^* Op_h(a)(\varphi^{-1})^* X = Op_h(b), \quad b \in S^k(\mathbb{R}^{2n}),
\]

\[ b = \sum_{j=0}^{\infty} h^j L_j(a \circ \tilde{\varphi}) \]

expansion in \( S^k \) diff. operator of order \( 2j \) mapping \( S^k \to S^{k-j} \).

In particular

\[ b(x, \xi) = X(x)^\top a(\tilde{\varphi}(x, \xi)) + O(h^\infty), \quad S^k \to S^{k-1}. \]

Theorem (Pseudodolocality) Assume \( a \in S^k(\mathbb{R}^{2n}) \) and \( X_1, X_2 \in C^\infty_c(\mathbb{R}^{2n}) \), \( \text{supp } X_1 \cap \text{supp } X_2 = \emptyset \).

Then \( X_1 Op_h(a) X_2 = O(h^{\infty}) S^k \to S \), namely it has the form \( u \mapsto \int_{\mathbb{R}^{2n}} K(x, y; h) u(y) \, dy \)

where \( \forall N \geq C_n \| (x, y) \|_{\mathbb{R}^{2n}} \leq C_N \).

Proof follows from the composition formula:

\[ X_1 Op_h(a) X_2 = Op_h(b) \]

where \( b = O(h^{\infty}) S(\langle x \rangle^{-n}, \langle y \rangle^{-n}) \)

for all \( N \) (the \( \langle x \rangle^{-n} \) is because \( X_1, X_2 \in C^\infty_c \)).

Interpretation: \( \text{supp } u \subseteq \bar{U} \Rightarrow Op_h(a) u = O(h^\infty) \to \text{outside of } U. \)