§6. $L^2$ THEORY

§6.1. Boundedness

Recall from §5 the symbol class $S(m) = \{ a(x, \xi; h) : |\mathcal{D}_x a(x, \xi; h) | \leq C m(x, \xi) \}$

where $m$ is an order function

Theorem (Calderón–Vaiilancourt)

Assume that $a \in S(1)$. Then $\forall h, O_{ph}(a)$ defines a bounded operator on $L^2(\mathbb{R}^n)$ and $\exists C = C(a)$ such that for all $h \in (0, 1]$

$$\|O_{ph}(a)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C.$$ 

Proof: We only show the uniform norm bound for $a \in S(\mathbb{R}^{2n})$. For the harder case of $a \in S(1)$, see Zvonik's book, Theorem 4.23

1. We use the following general fact, known as Schur's inequality:

if $Au(x) = \int_{\mathbb{R}^n} K_A(x,y) u(y) \, dy$, $K_A \in S(\mathbb{R}^{2n})$

and $C_1 := \sup_{x} \int_{\mathbb{R}^n} |K_A(x,y)| \, dy$,

$C_2 := \sup_{x, y} \int_{\mathbb{R}^n} |K_A(x,y)| \, dx$,

then $\|Au\|_{L^2} \leq \sqrt{C_1 \cdot C_2}$.
To prove Schur's inequality, take \( u \in L^2(\mathbb{R}^n) \) and write
\[
\|A\|_{L^2}^2 = \int_{\mathbb{R}^{3n}} |K_A(x,y)K_A(x,z)|u(y)u(z)\,dx\,dy\,dz
\]
\[
\leq \int_{\mathbb{R}^{3n}} |K_A(x,y)| |K_A(x,z)| \left( \frac{1}{2} (\|u(y)\|^2 + \|u(z)\|^2) \right)\,dx\,dy\,dz
\]
Now we bound only instance of \( z \)
\[
\int_{\mathbb{R}^{3n}} |K_A(x,y)| |K_A(x,z)| \|u(y)\|^2\,dx\,dy\,dz
\]
\[
\leq C_1 \int_{\mathbb{R}^n} |K_A(x,y)| \|u(y)\|^2\,dx\,dy
\]
\[
\leq C_1 C_2 \int_{\mathbb{R}^n} \|u(y)\|^2 \,dy = C_1 C_2 \|u\|_{L^2}^2
\]
We handle \( \int_{\mathbb{R}^n} \|u(z)\|^2 \,dz \) similarly, giving Schur's inequality.

2. Now take \( A = \Theta_\alpha^n(a) \), \( a \in S(\mathbb{R}^{2n}) \).
Then
\[
K_A(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i\frac{x-y}{h} \cdot \xi} a(x,\xi) d\xi
\]
\[
= (2\pi h)^{-n} \hat{\Theta}(a)(x,\frac{x-y}{h}) \quad \text{where}
\]
\[
\hat{\Theta}(a)(x,z) = \int_{\mathbb{R}^n} e^{iz \cdot \xi} a(x,\xi) d\xi \quad \text{is the partial Fourier transform}
\]
Since \( a \in S(\mathbb{R}^n) \), we have \( \hat{\Theta}(a) \in S(\mathbb{R}^n) \) too.
We have
\[
\int_{\mathbb{R}^n} |K_A(x,y)|\,dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{\Theta}(a)(x,z)|\,dz \leq C
\]
And
\[
\int_{\mathbb{R}^n} |K_A(x,y)|\,dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{\Theta}(a)(y+hz,z)|\,dz \leq C.
\]
§6.2. Compactness

**Definition** A bounded operator \( A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is called **compact**, if \( V \) bounded sequence \( u_j \in L^2(\mathbb{R}^n) \), the sequence \( Au_j \) has a convergent subsequence.

**Basic property:** if \( A_k \) are compact and

\[
\lim_{k \to \infty} \| A - A_k \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = 0
\]

then \( A \) is also compact.

**Theorem** Assume \( a \in S(m) \) where \( m \) is an order function and \( m(x, \xi) \to 0 \) as \( (x, \xi) \to \infty \).

Then \( Op_h(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is compact \( \forall h \).

**Proof** We only give a sketch; see the book of Zworski, §4.6 for details.

1. Assume first that \( a \in S(\mathbb{R}^n) \). Then \( Op_h(a) : S'(\mathbb{R}^n) \to S(\mathbb{R}^n) \). So if \( u_j \) is bounded in \( L^2 \), then \( Op_h(a)u_j \) is bounded in \( S(\mathbb{R}^n) \).

In particular, \( |\hat{u_j}(x)| \leq C_x \langle x \rangle^{-N} \forall x, N \)

Then \( Op_h(a)u_j \) has a subsequence converging in \( L^2(\mathbb{R}^n) \) by the Arzela-Ascoli theorem...
2. We now consider the general case.

Take \( X \in C^\infty_c(\mathbb{R}^n) : \text{supp } X \subset B(0,2) \),
\( \chi = 1 \text{ on } B(0,1) \)

Take large \( R \) and put
\[
q_R(w) := \chi(\frac{w}{R})a(w), \quad w = (x, \xi)
\]

Then \( \chi R^2 S(\mathbb{R}^n) \implies O_{ph}(q_R) \) is compact by Step 1.

Now, each \( S(1) \) seminorm of \( a - q_R \)
goes to 0 as \( R \to \infty \) (see exercises; here we use that \( m(x, \xi) \to 0 \) as \( (x, \xi) \to \infty \)).

Thus by the \( L^2 \) boundedness statement,
\[
\| O_{ph}(a) - O_{ph}(q_R) \|_{L^2(\mathbb{R}^n)} \to \mathcal{L}^2(\mathbb{R}^n) \quad R \to \infty \to 0
\]

By the basic property above, \( O_{ph}(a) \) is compact.

§6.3. Sharp Gårding inequality

**Theorem**

Assume that \( a \in S(1) \) and
\[
a(x, \xi) > 0 \quad \text{for all } (x, \xi).
\]

Then \( \exists C = C(a) : \forall h \in (0,1], \forall \mu \in L^2(\mathbb{R}^n) \)
\[
\langle O_{ph}(a) \mu, \mu \rangle_{L^2(\mathbb{R}^n)} \geq -C h \| \mu \|_{L^2(\mathbb{R}^n)}^2
\]
Proof. We only do the simple special case when \( a = b^2 \) for some real-valued \( b \in \mathbb{R} \).

For the general case, see the book of Zworski, Theorem 4.32.

By the Adjoint Rule + \( L^2 \) boundedness

\[
O_{ph}(b)^* = O_{ph}(b) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.
\]

Then by the Product Rule + \( L^2 \) boundedness

\[
O_{ph}(b)^* O_{ph}(b) = O_{ph}(b^2) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.
\]

Thus

\[
\langle O_{ph}(a) u, u \rangle = \langle O_{ph}(b)^* O_{ph}(b) u, u \rangle
\]

\[
+ O(h) \| u \|_{L^2(\mathbb{R}^n)}
\]

\[
= \| O_{ph}(b) u \|_{L^2(\mathbb{R}^n)}^2 + O(h) \| u \|_{L^2(\mathbb{R}^n)}^2.
\]

\( \Box \)