We now generalize the statements in §4 to a more general class of symbols.

**Notation:** \[ \langle x \rangle := \sqrt{1 + |x|^2} \] (Japanese bracket)

Asymptotic to \( 1 + |x| \) & smooth at \( x = 0 \)

**Definition:** \( m : \mathbb{R}^{2n} \to (0, \infty) \) is called an order function, if \( \exists C, N : \forall z, w \in \mathbb{R}^{2n} \)

\[ m(w) \leq C \langle z - w \rangle^N m(z). \]

**Definition (Symbol Classes):** Let \( m \) be an order function and \( a(x, \xi) \in C^\infty_c (\mathbb{R}^{2n}) \).

- We say \( a \in S(m) \) if any multiindex \( \alpha \in C_\xi \):

\[ \forall (x, \xi) \in \mathbb{R}^{2n}, \ |\partial_x^\alpha a(x, \xi)| \leq C_\alpha m(x, \xi) \]

- If \( a \) additionally depends on \( h \), then we require that the constants \( C_\alpha \) be \( h \)-independent (in order for \( a \) to be in \( S(m) \)).

**Caution about notation:**

- \( S(\mathbb{R}^n) \) Schwartz functions
- \( S(m) \) symbol class

**Example:** \( m(x, \xi) = \langle \xi \rangle^k \), \[ a(x, \xi) = \sum_{1 \leq \alpha \leq k} a_\alpha(x) \xi^\alpha \]

\[ 0_{Ph}(\alpha) = \sum_{1 \leq k} a_\alpha(x)(h^2)_{x}^{\alpha} \] semiclasical differential operator
§ 5.1. Mapping properties

Recall from §3 that for \( a \in S(m) \) we may define \( O_{ph}(a) : S(R^n) \to S'(R^n) \).
But this is not good enough to compose operators \( O_{ph}(a) \) with each other. We thus show

**Theorem** Let \( a \in S'(m) \) for some \( m \) and fix \( h \in (0,1] \). Then

\[
O_{ph}(a) : S(R^n) \to S'(R^n), \quad S'(R^n) \to S'(R^n)
\]

is a continuous operator.

**Proof** Will only show \( O_{ph}(a) : S(R^n) \to S'(R^n) \).

For \( S'(R^n) \to S'(R^n) \) enough to show that

\[
O_{ph}(a)^* : S(R^n) \to S(R^n)
\]

see exercises.

For notational simplicity we fix \( h := 1 \) and define

\[
O_{ph} := O_p
\]

1. The integral formula

\[
O_p(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i<x-y, \xi>} a(x, \xi) u(y) dy \, d\xi.
\]

implies that

\[
a \in S(\langle \xi \rangle^{-n+1}) \Rightarrow O_p(a) : \langle x \rangle^{-n-1} L^\infty(R^n) \to L^\infty(R^n)
\]

In particular, we have \( \forall \alpha', \beta' \)

\[
O_p(a) x^{\alpha'} D_x^{\beta'} : S(R^n) \to L^\infty(R^n) \quad (*)
\]

\[
D = \frac{1}{i\partial}
\]

when \( a \in S(\langle \xi \rangle^{-n+1}) \).
2. Fix an order function \( m \).
There exists an integer \( N \geq 0 \) such that
\[
m(x, \xi) \leq C <x>^{2N} <\xi>^{2N} <\xi>^{-n-1} \forall (x, \xi)
\]
Then each \( a \in S'(m) \) lies in
\[
<x>^{2N} <\xi>^{2N} \leq (x, \xi)^{-n-1}) \text{ and thus is a linear combination of symbols of the form}
\]
x\( \delta \xi \delta b \), \( b \in S((x, \xi)^{-n-1}) \), \( 101, 151 \leq 2N \).
Thus it suffices to show: \( \forall \alpha, \beta, \delta, \sigma, \)
\( b \in S((x, \xi)^{-n-1}) \implies x^\alpha \partial_x^\beta \partial_\tau^\sigma (x \delta \xi \delta b): S(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \).
\( (**) \)

3. It remains to show that \( (*) \implies (**) \).
For that we use the identities
\[
\begin{align*}
x_j \text{Op}(a) &= \text{Op}(a) x_j + i \text{Op}(\partial_{\xi_j} a) \\
\partial_x^j \text{Op}(a) &= \text{Op}(a) \partial_x^j - i \text{Op}(\partial_{\xi_j} a) \\
\text{Op}(x_j a) &= \text{Op}(a) x_j + i \text{Op}(\partial_{\xi_j} a) \\
\text{Op}(\xi_j a) &= \text{Op}(a) \partial_x^j.
\end{align*}
\]
For \( a \in S(\mathbb{R}^{2n}) \) these follow from the definition of \( \text{Op}_h(a) \); for general \( a \) they follow by approximation by functions in \( S(\mathbb{R}^{2n}) \).
See Exercise 3.5(a).
Iterating the above identities, we see that \( \forall \alpha, \beta, \delta, \xi, \forall b \in \mathcal{S}'(\mathbb{R}^{n-1}) \),

\[
x^{\alpha} D^{\beta}_x \mathcal{O}_p \left( x^{\delta} \xi^\delta b \right) = \text{linear combination of } \mathcal{O}_p(\partial_x^{\delta} \partial_{\xi}^{\delta} b) x^{\alpha'} D^{\beta'}_x
\]

still lies in \( \mathcal{S}'(\mathbb{R}^{n-1}) \)

which map \( \mathcal{S}(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n) \) by \((*)\).
This gives \((***)\).

\[\square\]

\section*{§5.2. The calculus}

We now give the analogs of the statements in §4. We do not provide proofs, referring to Zworski's book, Theorems 4.17-4.18

**Theorem (Composition formula)**

Assume \( a \in \mathcal{S}(m_1) \), \( b \in \mathcal{S}(m_2) \). Then

\[
\mathcal{O}_{p,h} (a) \mathcal{O}_{p,h} (b) = \mathcal{O}_{p,h} (a \# b)
\]

where

\[
a \# b(x, \xi, h) \sim \sum_{j=0}^\infty (-ih)^j \sum_{|\lambda| = j} \frac{1}{\lambda!} \partial^\lambda_x a(x, \xi) \partial_x^{\lambda} b(x, \xi)
\]

where the expansion is in \( \mathcal{S}^1(m_1 \cdot m_2) \), defined similarly to expansions in \( \mathcal{S}(\mathbb{R}^n) \) from §4.
Theorem (Adjoint formula)

Assume that $a \in S'(m)$. Then

$$\mathcal{O}_{\mathfrak{p}_n}(a)^* = \mathcal{O}_{\mathfrak{p}_n}(a^*)$$

where

$$a^*(x, \xi; h) \sim \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} \partial_x^j \partial_{\xi}^j a(x, \xi; h)$$

and the expansion is in $S'(m)$. 