§ 2. Method of Stationary Phase

We will study asymptotics as $h \to 0$ of

$$I(h) := \int_{U} e^{\frac{i\varphi(y)}{h}} a(y) \, dy,$$

where $0 < h \leq 1$.

- $U \subset \mathbb{R}^n$ is an open set.
- $\varphi \in C^\infty(U; \mathbb{R})$, called "phase function".
- $a \in C_0^\infty(U)$, called "amplitude".

Recall this means $a \in C^\infty(U)$ & supp $a \subset U$ is compact.

**Definition.** $y \in U$ is called a critical (stationary) point of $\varphi$, if $d\varphi(y) = 0$.

§ 2.1. Method of nonstationary phase

**Theorem.** Assume supp $a$ has no critical points of $\varphi$. Then $I(h) = O(h^\infty)$, i.e.,

$$I(h) = O(h^n) \text{ \forall } n.$$ 

**Proof.** We repeatedly integrate by parts using

$$L : f(y) \mapsto -i \sum_{j=1}^{n} \frac{\partial_j \varphi(y)}{1d\varphi(y)^2} \partial_j f(y).$$

$L$ is a $1^{st}$ order differential operator on $U$. (Here we shrunk $U$ so that supp $a \subset U$, phase no critical points in $U$.)
We compute
\[ L \varphi(y) = -i \sum_{j=1}^n \frac{\partial_j \varphi(y)}{|\partial_j \varphi(y)|^2} \partial_j \varphi(y) = -i \varphi, \]
thus
\[ e^{it} = h \cdot L(e^{it}). \]
We have for each \( f \in C_c^\infty(U), \)
\[ (IBP) \int_U e^{it} f \, dy = h \int_U L(e^{it} f) \, dy = h \int_U e^{it} (L^t f) \, dy \]
where \( L^t \) is the first order differential operator given by
\[ L^t f(y) = i \sum_{j=1}^n \partial_j \left( \frac{\partial_j \varphi(y)}{|\partial_j \varphi(y)|^2} f(y) \right) \]
(the transpose of \( L \), see exercises)
Now we apply (IBP) \( N \) times:
\[ |I(h)| = |h^N \int_U e^{it} ((L^t)^N a) \, dy| = O(h^N). \]

\section{2.2. Stationary phase}

Gives a full expansion of \( I(h) \) up to \( O(h^\infty) \) remainder for under a nondegeneracy assumption.

Definition A critical point \( y_0 \) of \( \varphi \) is called nondegenerate if the Hessian \( \partial^2 \varphi(y_0) = (\partial^2_{jk} \varphi(y_0))_{j,k=1}^n \) is invertible.
If all critical points are nondegenerate, we call \( \varphi \) a Morse function.
If \( y_0 \) is a nondegenerate critical point, then \( d^2 \varphi (y_0) \) has \( k \) positive & \( n-k \) negative eigenvalues for some \( k \). We define

\[
\text{sgn } d^2 \varphi (y_0) := k - (n-k).
\]

(Signature of the Hessian)

The following result handles the contribution of a single nondegenerate critical point.

(See exercises for how to use it to get an expansion for \( I(h) \) when \( \varphi \) is any Morse function.)

Theorem (Method of stationary phase)

Assume \( \varphi, a \) are as above and \( \varphi \) has only one critical point \( y_0 \) in \( \text{supp } a \), which is moreover nondegenerate. Then as \( h \to 0 \),

\[
I(h) \sim e^{i \varphi (y_0) / h} e^{h^{1/2} \sum_{j=0}^{n/2} \frac{1}{j!} L_{\varphi, j} a(y_0)}
\]

(STPh)

where each \( L_{\varphi, j} \) is a differential operator of order \( 2j \) on \( U \) depending on \( \varphi \), but not on \( a \), and the leading term is given by

\[
L_{\varphi, 0} a(y_0) = (2\pi i)^{n/2} \sqrt{\det d^2 \varphi (y_0)} e^{i \text{sgn } d^2 \varphi (y_0) / 2} a(y_0)
\]

Note: one has a formula for all \( L_{\varphi, j} \), see e.g. Hörmander, Vol I, Theorem 7.7.5 or Zworski, (3.4.11) (in dimension 1)
Remark: the expansion \((STPh)\) is an asymptotic one: namely
\[\forall N, \quad \| I(h) - \sum_{j=0}^{N-1} (...) \|_h = O(h^{\frac{1}{2} + N}).\]
Typically the series \(\sum_{j=0}^{\infty} (...)\) does not converge for any fixed \(h > 0\). Such style asymptotic expansions will appear a lot in the course.
In fact, we can say more about the remainder:
\[\forall N, \forall h \in (0,1], \quad \| I(h) - \sum_{j=0}^{N-1} (...) \|_h \leq C_N, \forall y_0, \| y \|_{C_{2N+1}} \leq h^{\frac{1}{2} + N}\]
where \(C_N, y_0\) is some constant depending only on \(N, y, y_0, h\), and \(\| y \|_{C_{2N+1}} := \max_{|l| \leq 2N+1} \| \partial^l y \|_{L^2}\).

See: Zworski, Theorem 3.16
Hörmander, Vol I, Theorem 7.7.5

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\(\S 2.3.\) Quadratic stationary phase.

Here we consider a special case of the method of stationary phase, useful for the proof of the general case:
\[I(h) = \int_{\mathbb{R}^n} e^{i\frac{1}{h} \langle Qy, y \rangle} a(y) dy\]
where
- \(Q\) is an invertible symmetric real \(n \times n\) matrix
- \(a \in C^\infty_c (\mathbb{R}^n)\).
Note: \( \varphi(y) = \frac{1}{2} \langle Qy, y \rangle \) is a Morse function. The only critical point is \( y = 0 \) and \( \lambda^2 \varphi(0) = 0 \).

**Theorem (Quadratic stationary phase)**

We have
\[
I(h) \sim (2\pi h)^{-\frac{n}{2}} e^{\frac{im}{h}} \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \frac{h^j}{j!} \left( \frac{\langle a^j \partial_y, a^j \partial_y \rangle}{i} \right)_{a(0)}
\]

Here for an \( n \times n \) matrix \( A \) we write
\[
\langle A \partial_y, \partial_y \rangle = \sum_{k, \ell = 1}^{n} A_{k \ell} \partial_y^k \partial_y^\ell, \quad \partial_y^k := -i \partial_y^k
\]

The expansion has the same remainder estimate as the general stationary phase: \( \forall N \)
\[
|I(h) - \sum_{j=0}^{N-1} | \leq C_{N, Q} \frac{h^{\frac{n}{2} + N}}{\| a \|_{2N+1}}
\]

depends on \( N, Q \), and a compact set containing \( \text{supp} a \).

**Proof** 1. We first express \( I(h) \) in terms of the Fourier transform \( \hat{a} \).

We only give here an outline, details might appear in the Fourier Transform course.

Start from the 1D formula: \( \forall a \in \mathcal{S}(\mathbb{R}) \),
\[
\int_{\mathbb{R}} e^{-\frac{x^2}{2}} a(y) dy = (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{y^2}{2}} \hat{a}(y) dy
\]

(because the Fourier transform of \( e^{-\frac{x^2}{2}} \) is \( (2\pi)^{\frac{1}{2}} e^{-\frac{y^2}{2}} \))
Arguing by analytic continuation in \( \lambda \) (putting \( \lambda = -\frac{i\mu}{\hbar} \)), get for \( \mu > 0 \),

\[
\int_{\mathbb{R}} e^{-\frac{iy^2}{2\hbar}} a(y) \, dy = \frac{\hbar^{\frac{1}{2}}}{\sqrt{2\pi}} \cdot e^{\frac{i\pi}{4}} \int_{\mathbb{R}} e^{\frac{-i\hbar y^2}{2\mu}} \hat{a}(y) \, dy
\]

where \( e^{\frac{i\pi}{4}} \) come from taking \( \left( \frac{\pm i\mu}{\hbar} \right)^{-\frac{1}{2}} \).

Replacing the number \( \mu \) by a matrix \( Q \) (e.g., taking tensor products + diagonalizing \( Q \)) we get

\[
\forall a \in S(\mathbb{R}^n), \quad \left( \mathcal{A} \right) \int_{\mathbb{R}^n} e^{\frac{i\langle Qy, y \rangle}{2\hbar}} a(y) \, dy = \left( \frac{\hbar}{2\pi} \right)^{\frac{n}{2}} \cdot e^{\frac{i\pi}{4} \text{sgn} Q} \cdot \frac{1}{|\det Q|^{1/2}} \cdot \mathcal{J}(\hbar)
\]

where \( \mathcal{J}(\hbar) := \int_{\mathbb{R}^n} e^{\frac{\hbar}{2i} \langle a^{-1}, y \rangle} \hat{a}(y) \, dy \)

2. We now use the Taylor expansion of \( e^{\frac{h}{2i} \langle a^{-1}, y \rangle} \) at \( h \to 0 \).

Here we do it formally, see exercises for how to get the precise remainder

\[
e^{\frac{h}{2i} \langle a^{-1}, y \rangle} \approx 1 + \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{h}{2i} \langle a^{-1}, y \rangle \right)^j
\]

\[\Rightarrow \mathcal{J}(\hbar) \approx \int_{\mathbb{R}^n} \sum_{j=0}^{\infty} \frac{h^j}{j!} \int_{\mathbb{R}^n} \left( \frac{\langle a^{-1}, y \rangle}{2i} \right)^j \hat{a}(y) \, dy
\]

\[
(2\pi)^n \left( \frac{\langle a^{-1}, Dy, Dy \rangle}{2i} \right)^{\frac{1}{2}} a(0) \quad \text{by property of Fourier transform.}
\]
This gives the quadratic stationary phase expansion. □

§2.4. Proof of general stationary phase

Reduces to quadratic stationary phase using Theorem (Morse Lemma) Assume that \( \varphi \in C^\infty (U; \mathbb{R}) \) has a nondegenerate stationary point at \( y_0 \in U \), and \( d^2\varphi(y_0) \) has signature \( k = (n-k) \). Then there exist:

- a neighborhood \( U' \) of \( y_0 \), and
- a diffeomorphism \( F: V \to U' \), such that

\[
\forall x \in V, \quad \varphi(F(x)) = \varphi(y_0) + \frac{1}{2} (x_1^2 + \cdots + x_{k-1}^2 - x_{k+1}^2 - \cdots - x_n^2)
\]

Moreover, \( \det dF(0) = |\det d^2\varphi(y_0)|^{-\frac{1}{2}} \).

Proof See Zworski, Theorem 3.15. □

Returning to general stationary phase, write

\[
I(h) = \int_U e^{\frac{i}{h} \varphi(y)} a(y) dy = \int_{U'} e^{\frac{i}{h} \varphi(y)} a_1(y) dy + \int_{U''} e^{\frac{i}{h} \varphi(y)} a_2(y) dy
\]

where \( a = a_1 + a_2 \), \( \text{supp } a_1 \subset U' \), \( \text{supp } a_2 \subset \text{supp } \varphi \).

By nonstationary phase, \( I_2(h) = O(h^\infty) \).

Changing variables in \( I_1 \), get

\[
I_1(h) = \int_V e^{\frac{i}{h} \varphi(F(x))} a_1(F(x)) |\det dF(x)| dx
\]

\( \sqrt{\cdot} \) can be handled by quadratic stationary phase!