MATHEMATICAL THEORY OF SCATTERING RESONANCES

Version 1.0 (August 19, 2022)

Semyon Dyatlov, UC Berkeley and MIT

Maciej Zworski, UC Berkeley

PREFACE

Mathematicians are Frenchmen of sorts: whatever one says to them they translate into their own language and then it becomes something entirely different.

Johann Wolfgang von Goethe, *Maximen und Reflexionen*, 1840

The purpose of this book is to provide a broad introduction to the theory of scattering resonances.

Scattering resonances appear in many branches of mathematics, physics and engineering. They generalize eigenvalues or bound states for systems in which energy can scatter to infinity. A typical state has then a rate of oscillation (just as a bound state does) and a rate of decay. Although the notion is intrinsically dynamical, an elegant mathematical formulation comes from considering meromorphic continuations of Green's functions or scattering matrices. The poles of these meromorphic continuations capture the physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. The resonant state, which is the corresponding wave function, then appears in the residue of the meromorphically continued operator. An example from pure mathematics is given by the zeros of the Riemann zeta function: they are, essentially, the resonances of the Laplacian on the modular surface. The Riemann hypothesis then states that the decay rates for the modular surface are all either 0 or $\frac{1}{4}$. A standard example from physics is given by shape resonances created when the interaction region is separated from free space by a potential barrier. The decay rate is then exponentially small in a way depending on the width of the barrier.

In the book we provide an introduction to mathematical techniques used in the study of scattering resonances, concentrating on simplest models but providing references to modern literature and indications of what happens in more general situations. Some chapters (such as Chapter 2 and 3) are meant to be easily accessible and others (such as Chapter 5) somewhat more demanding. The rather substantial set of appendices provides detailed accounts of most methods needed in the text. A diagram representing the dependencies of various sections is presented at the end of Chapter 1. The choice of topics is necessarily determined by the research interests of the authors and many important aspects of the subject are not covered. We also stayed away from exciting but technical developments such as precise asymptotics for shape resonances, fractal Weyl laws, resonance gaps for chaotic systems or the applications of scattering theory to hyperbolic dynamical systems – see the survey [**Zw17**] for an overview and references.

SD was introduced to scattering resonances by MZ who in turn had the good fortune to be introduced to this field by Richard Melrose. We would like to thank him for his generous guidance and insights and for his foundational results on resonance counting and trace formulas.

The view point and many discoveries of Johannes Sjöstrand changed the subject in a profound way. MZ was privileged to maintain a long collaboration with him and would like to thank him for sharing his ideas and expertise over the years.

Many other colleagues and collaborators have contributed to our understanding of the subject and special thanks are due to Ivana Alexandrova, Jean-François Bony, David Bindel, Paul Brumer, Nicolas Burq, Tanya Christiansen, Kiril Datchev, Frédéric Faure, Jeff Galkowski, Colin Guillarmou, Laurent Guillopé, Bernard Helffer, Peter Hintz, Michael Hitrik, Long Jin, Ulrich Kuhl, André Martinez, William H. Miller, Shu Nakamura, Frédéric Naud, Stéphane Nonnenmacher, Galina Perelman, Vesselin Petkov, Jim Ralston, Antônio Sá Barreto, Hart Smith, Plamen Stefanov, Siu-Hung Tang, Jared Wunsch, András Vasy and Georgi Vodev.

The project of writing this book started during lectures given at Université de Paris-Nord in the Spring of 2011 by MZ and attended by SD. We are grateful for the support of the *Chaire d'Excellence* at the Laboratoire Analyse, Géométrie et Applications there and for the generous hospitality extended by the Laboratoire to the authors in 2011. Particular thanks are due to Jean-Marc Delort, Alain Grigis, David Dos Santos Ferreira and Maher Zerzeri.

Chapter 2 developed from notes on one dimensional scattering written by Siu-Hung Tang and MZ in 2001 [**TZ01**] – we are grateful for his help on that project and for allowing us to use that material.

Simon Becker's careful reading of the final version eliminated countless mistakes: we and the readers of this book owe him a great debt. We are are also particularly grateful to Alexis Drouot, Benjamin Küster, Hari Manoharan, Alberto Parmeggiani, Euan Spence, Jian Wang, Tobias Weich, Mengxuan Yang and the three anonymous reviewers of the book for their helpful comments and corrections.

Peter Hintz helped us with the translation of Goethe's maxim for the epigraph.

During the writing of this book SD was partially supported by the Clay Research Fellowship, Sloan Research Fellowship and the National Science Foundation grant DMS-1749858. MZ was partially supported by the National Science Foundation grants DMS-1201417, DMS-1500852 and by a 2017/2018 Simons Fellowship.

Contents

Preface 3

Chapter 1. Introduction 11
- §1.1. Resonances in scattering theory 11
- §1.2. Semiclassical study of resonances 17
- §1.3. Some examples 18
- §1.4. Overview 23

Part 1. POTENTIAL SCATTERING

Chapter 2. Scattering resonances in dimension one 31
- §2.1. Outgoing and incoming solutions 32
- §2.2. Meromorphic continuation 36
- §2.3. Expansions of scattered waves 49
- §2.4. Scattering matrix in dimension one 55
- §2.5. Asymptotics for the counting function 62
- §2.6. Trace and Breit–Wigner formulas 69
- §2.7. Complex scaling in one dimension 81
- §2.8. Semiclassical study of resonances 92
- §2.9. Notes 102
- §2.10. Exercises 103

Chapter 3. Scattering resonances in odd dimensions 105
- §3.1. Free resolvent in odd dimensions 106
- §3.2. Meromorphic continuation 118

§3.3. Resolvent at zero energy 126
§3.4. Upper bounds on the number of resonances 135
§3.5. Complex valued potentials with no resonances 139
§3.6. Outgoing solutions and Rellich's theorem 141
§3.7. The scattering matrix 152
§3.8. More on distorted plane waves 165
§3.9. The Birman–Kreĭn trace formula 169
§3.10. The Melrose trace formula 187
§3.11. Scattering asymptotics 197
§3.12. Existence of resonances for real potentials 215
§3.13. Notes 217
§3.14. Exercises 220

Part 2. GEOMETRIC SCATTERING

Chapter 4. Black box scattering in $\mathbb{R}^n$ 225
§4.1. General assumptions 226
§4.2. Meromorphic continuation 231
§4.3. Upper bounds on the number of resonances 243
§4.4. Plane waves and the scattering matrix 258
§4.5. Complex scaling 275
§4.6. Singularities and resonance free regions 296
§4.7. Notes 308
§4.8. Exercises 310

Chapter 5. Scattering on hyperbolic manifolds 313
§5.1. Asymptotically hyperbolic manifolds 315
§5.2. A motivating example 322
§5.3. The modified Laplacian 325
§5.4. Phase space dynamics 331
§5.5. Propagation estimates 340
§5.6. Meromorphic continuation 349
§5.7. Applications to general relativity 359
§5.8. Notes 370
§5.9. Exercises 372

Part 3. RESONANCES IN THE SEMICLASSICAL LIMIT

Chapter 6. Resonance free regions 379
§6.1. Geometry of trapping 381
§6.2. Resonances in strips 388
§6.3. Normally hyperbolic trapping 400
§6.4. Logarithmic resonance free regions 411
§6.5. Lower bounds on resonance widths 416
§6.6. Notes 426
§6.7. Exercises 429

Chapter 7. Resonances and trapping 433
§7.1. Lower bounds on the resolvent 434
§7.2. Semiclassical growth estimates 440
§7.3. From quasimodes to resonances 445
§7.4. The Sjöstrand trace formula 454
§7.5. Resonance expansions for strong trapping 464
§7.6. Notes 475
§7.7. Exercises 476

Part 4. APPENDICES

Appendix A. Notation 483
§A.1. Basic notation 483
§A.2. Functions 484
§A.3. Spaces of functions 485
§A.4. Operators 485
§A.5. Estimates 486
§A.6. Tempered distributions 487
§A.7. Distributions on manifolds and Schwartz kernels 488

Appendix B. Spectral theory 491
§B.1. Spectral theory of self-adjoint operators 491
§B.2. Functional calculus 496
§B.3. Singular values 496
§B.4. The trace class 500
§B.5. Weyl inequalities and Fredholm determinants 505
§B.6. Lidskiĭ's theorem 512
§B.7. Notes 514

§B.8. Exercises 514

Appendix C. Fredholm theory 515

§C.1. Grushin problems 515

§C.2. Fredholm operators 517

§C.3. Meromorphic continuation of operators 521

§C.4. Gohberg–Sigal theory 524

§C.5. Notes 532

§C.6. Exercises 532

Appendix D. Complex analysis 535

§D.1. General facts 535

§D.2. Entire functions 539

Appendix E. Semiclassical analysis 543

§E.1. Pseudodifferential operators 544

§E.2. Wavefront sets and ellipticity 564

§E.3. Semiclassical defect measures 574

§E.4. Propagation estimates 577

§E.5. Hyperbolic estimates 594

§E.6. Notes 610

§E.7. Exercises 611

Bibliography 621

Index 637

Chapter 1

INTRODUCTION

1.1 Resonances in scattering theory
1.2 Semiclassical study of resonances
1.3 Some examples
1.4 Overview

1.1. RESONANCES IN SCATTERING THEORY

Scattering resonances are the replacement of discrete spectral data for problems on non-compact domains. The possibility of escape to infinity means that in addition to initial energy (the eigenvalue in the compact setting) we also have a rate of decay. It turns out that this information can be encoded as a pole of the *meromorphic continuation* of the resolvent/Green function.

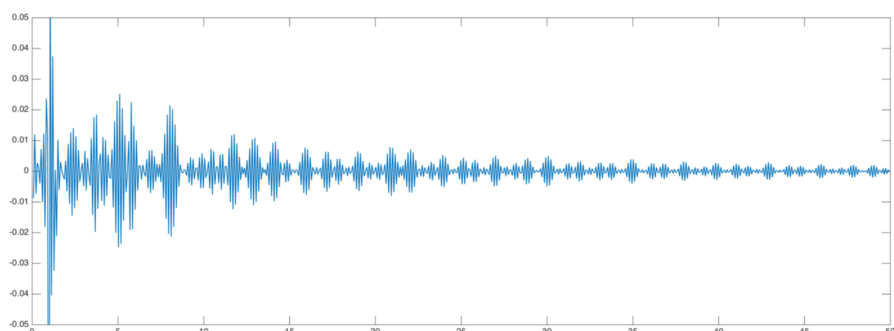


Figure 1.1. The plot of $u(t,0)$ showing oscillations and decay of the solution in the interaction region. The full $u(t,x)$ is shown in Figure 1.2.

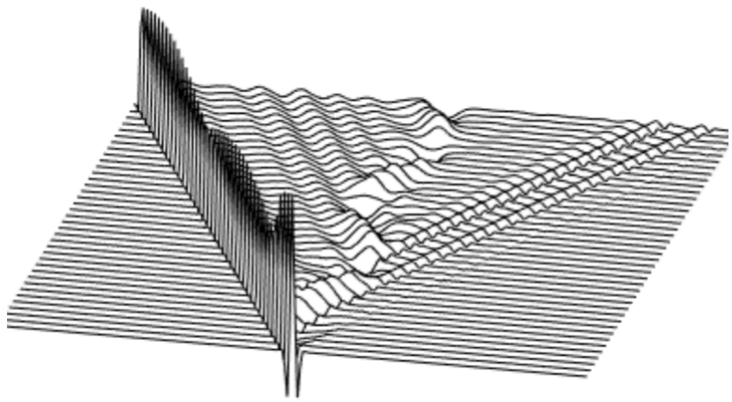


Figure 1.2. A solution of the wave equation $\partial_t^2 u - \partial_x^2 u + V(x)u = 0$ where V is shown in Figure 1.3. The initial data is localized near 0. The time axis points away from the viewer.

The simplest setting in which this can be seen is given by the operator $P = -\partial_x^2$ on the real line, $\mathbb{R}$. The resolvent $R_0(\lambda) := (P - \lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator for $\lambda \notin \mathbb{R}$. It is given explicitly as follows:

$$R_0(\lambda)f(x) = \int_{\mathbb{R}} R_0(\lambda, x, y) f(y)\, dy,$$
$$R_0(\lambda, x, y) = \frac{i}{2\lambda} e^{i\lambda|x-y|}, \quad \operatorname{Im}\lambda > 0.$$

For fixed x, y, $R_0(\lambda, x, y)$ continues to a meromorphic function of λ with one pole at $\lambda = 0$. This pole is the *scattering resonance* of P. Its dynamical significance is most easily seen in the context of the wave equation

$$(\partial_t^2 - \partial_x^2)u(t, x) = 0, \quad u(0, x) = 0, \quad \partial_t u(0, x) = f(x),$$
$$u(t, x) = \frac{1}{2} \int_{x-t}^{x+t} f(y)\, dy.$$

If $f(x) = 0$ for $|x| > R$ then

$$u(t, x) = \frac{1}{2} \int_{\mathbb{R}} f(y)\, dy \quad \text{for } t > |x| + R.$$

In terms of $R_0(\lambda, x, y)$ this can be reintepreted as follows:

$$u(t, x) = -i \int_{\mathbb{R}} (\operatorname{Res}_{\lambda=0} R_0(\lambda, x, y))\, f(y)\, dy, \quad \operatorname{Res}_{\lambda=0} R_0(\lambda, x, y) = \frac{i}{2}.$$

This means that the residue of R_0 at the pole describes *long time behaviour* (in t) of the wave in compact sets (in x).

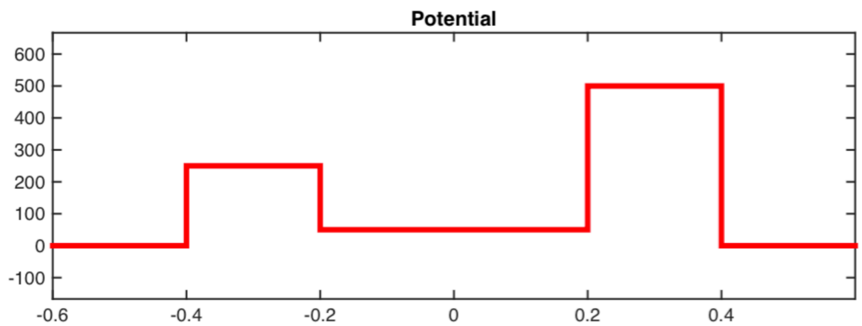


Figure 1.3. A simple one dimensional potential used to see trapping and tunneling of the wave in Figures 1.1 and 1.2.

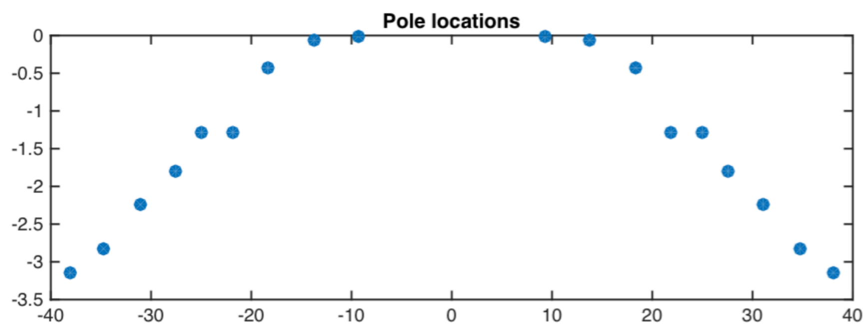


Figure 1.4. Scattering resonances of the potential shown in Figure 1.3. They are computed using the code `squarepot.m` by David Bindel [**BZ**].

A more interesting mathematical example – studied in detail in Chapter 2 – is given by scattering by a compactly supported potential, V, in dimension one, see Figure 1.3 for an example. Scattering resonances are the rates of oscillation and decay of solutions of the wave equation

$$(\partial_t^2 - \partial_x^2 + V(x))u = 0 \tag{1.1.1}$$

with localized initial data. Figures 1.1–1.2 show such a solution: we see the main wave escape and some trapped waves bounce in the well created by the potential and leak out. Instead of the eigenfunction expansion (which would hold if x took values in the circle $\mathbb{R}/\mathbb{Z}$) we have the *resonance expansion*

$$u(t,x) = \sum_{\operatorname{Im} \lambda_j > -A} e^{-i\lambda_j t} u_j(x) + \mathcal{O}_K(e^{-tA}), \quad x \in K \Subset \mathbb{R}.$$

Here *scattering resonances* λ_j are complex numbers with $\operatorname{Im} \lambda_j \leq 0$ which are independent of the initial data – see Theorem 2.9 for the precise statement. We clearly see that

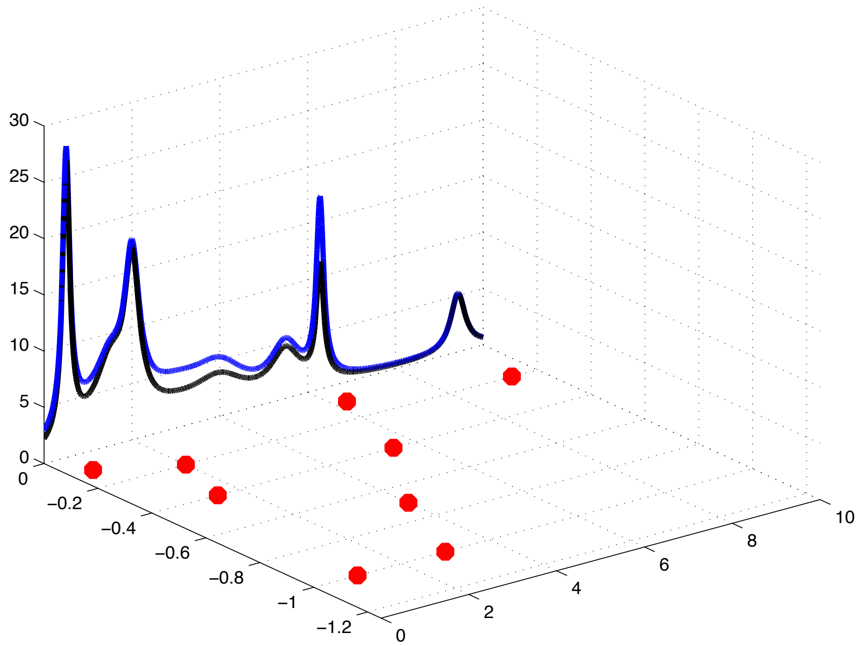


Figure 1.5. The power spectrum on the real line (Fourier transform) of correlations $\rho_{f,g}(t) = \langle U(t)f, g\rangle$: the blue and black plots show $\lambda \mapsto |\int_0^\infty \rho_{f,g}(t)e^{-i\lambda t}dt|$ for two different choices of f, g. Here $U(t)$ could be the propagator of the wave equation (1.1.1), or it could be for instance the pullback by an Anosov flow – see [**Zw17**, Chapter 4]. Resonances are the poles of the meromorphic continuation of the power spectrum to the complex plane and this figure shows a schematic correspondence between resonances and the power spectrum: $\operatorname{Re}\lambda_j$ corresponds to the location of a peak in the power spectrum and $\operatorname{Im}\lambda_j$ to its width.

$$\begin{aligned} \operatorname{Re}\lambda_j &= \text{rate of oscillation}, \\ -\operatorname{Im}\lambda_j &= \text{rate of decay}. \end{aligned}$$

The terms $u_j(x)$ are calculated using the residues of the meromorphic continuation of $R_V(\lambda) = (P - \lambda^2)^{-1}$, $P := -\partial_x^2 + V(x)$, just as we saw above in the case of $V \equiv 0$. To explain why meromorphic continuation appears we use the *power spectrum* which is the Fourier transform of u in time:

$$\widehat{u}(\lambda, x) := \int_0^\infty e^{i\lambda t} u(t, x)\, dt, \quad \operatorname{Im}\lambda > 0.$$

The resonance expansion implies that for all A

$$\widehat{u}(\lambda, x) - i \sum_{\operatorname{Im}\lambda_j > -A} \frac{u_j(x)}{\lambda - \lambda_j} \quad \text{extends holomorphically to } \{\operatorname{Im}\lambda > -A\}.$$

Thus $\widehat{u}(\lambda, x)$ extends meromorphically to $\lambda \in \mathbb{C}$ with poles at λ_j. Writing $\widehat{u}(\lambda, x)$ in terms of f and the scattering resolvent $R_V(\lambda)$ we obtain the definition of resonances used in this book:

$$\text{resonances} \;=\; \text{poles of the scattering resolvent.}$$

Harmonic inversion methods, the first being the celebrated Prony algorithm [**Pr95**], can then be used to extract scattering resonances (see for instance [**WMS88**]) from the power spectrum $\widehat{u}(\lambda, x)$ for λ real. See Figures 1.4 and 1.5.

Although this book is intended for a mathematical audience and it concentrates on rigorous presentation, physical motivation plays an essential role in the study of scattering resonances. Even when, as for instance in scattering on the modular surface, the questions have purely mathematical context, the origins lie in physics and it is easiest to relate them in the setting of quantum mechanics.

In quantum mechanics a particle is described by a wave function ψ which is normalized in L^2, $\|\psi\|_{L^2} = 1$. The probability of finding the particle in a region Ω is given by the integral of $|\psi(x)|^2$ over Ω. If $\psi = \psi_k$ is an eigenstate of a quantum Hamiltonian P (here k is a quantum number or the index of the discrete spectrum), the time evolved state is given by

$$\psi_k(t) := e^{-itP}\psi_k = e^{-itE_k}\psi_k \quad \text{where} \quad P\psi_k = E_k\psi_k. \tag{1.1.2}$$

In particular the probability density does not change when the state is propagated.

An example could be given by the Bloch electron in a quantum corral shown in Figures 1.6 and 1.9. The graphs in these figures picture the derivative of the density of states as a function of the voltage (which for us is the energy E), which measures the response of the quantum system when excited at energy E.

However the same figures also show that the measured states have nonzero "widths" – the peak is not a delta function at E_k – and hence can be more accurately modeled by resonances. The following standard argument of the physics literature explains the meaning of the real and imaginary parts: the time evolution of a pure resonant state corresponding to a resonance $\lambda_k^2 = E_k - i\Gamma_k/2$ is given by the following modification of (1.1.2)

$$\psi_k(t) = e^{-itE_k - t\Gamma_k/2}\psi_k.$$

Thus the probability of survival beyond time t is $p(t) = |\psi_k(t)|^2/|\psi_k(0)|^2 = e^{-\Gamma_k t}$. This explains why the convention for the imaginary part of a resonance is $\Gamma_k/2$. Here we neglected the issue that $\psi_k \notin L^2$ which is remedied by taking the probabilities over a bounded interaction region. In the energy

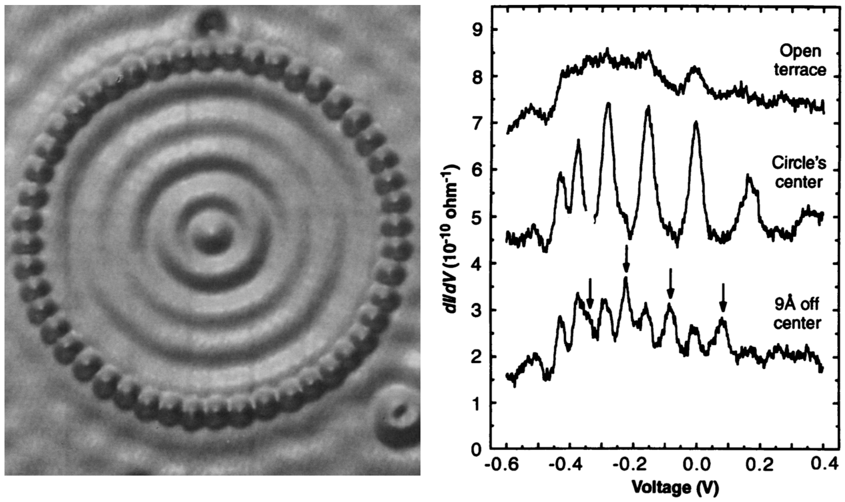


Figure 1.6. Experimental set-up and data in a scanning tunneling microscope experiment by Crommie et al [CLE93] from IBM's Almaden Research Center. The figure on the left shows a quantum corral formed by 48 iron atoms on a copper Cu(111) surface. The plot on the right shows measurements of dI/dV (I being the current) as functions of voltage V from different positions of the microscope tip: outside the corral (open terrace), the center and off center. The peaks show resonances – see Figure 1.5 for a schematic representation and Figure 2.6 for a simple mathematical example. The new states visible with an off-center measurement are shown with arrows.

representation the wave function is given by the Fourier transform in time

$$\begin{aligned}\varphi_k(E) &:= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itE} \psi_k(t)\, dt \\ &= \frac{1}{\sqrt{2\pi} i} \frac{\psi_k}{E_k - i\Gamma_k/2 - E},\end{aligned}$$

which means that the probability density of the time evolved resonant state $\psi_k(t)$ at energy E is proportional to the square of the absolute value of the right hand side. Consequently this probability density is

$$\frac{1}{2\pi} \frac{\Gamma_k}{(E - E_k)^2 + (\Gamma_k/2)^2}\, dE, \tag{1.1.3}$$

and this Lorentzian is the famous *Breit–Wigner distribution*. To see how this vague discussion works out mathematically see Theorem 2.20 and references in §§3.13, 7.6.

In practice there are many deviations from the simple formula (1.1.3), especially at high energies and in the presence of overlapping resonances. In Figures 1.6 and 1.9 we see clear Lorentzian peaks and individual resonances can be recovered. In experiment whose set up is shown in Figure 1.11 the resonances overlap and the peaks in scattering data do not have the simple interpretation using (1.1.3). Figure 2.6 illustrates a mathematical result related to the Breit–Wigner approximation.

For bound states the *Weyl law* (see [**Iv16**] for history and references) provides an asymptotic formula for the density of states. Weyl laws for counting of resonant states are more complicated and richer as they involve both energy and rates of decay. Even the leading term can be affected by dynamical properties of the system.

1.2. SEMICLASSICAL STUDY OF RESONANCES

For some very special systems resonances can be computed explicitely. One famous example is the Eckart barrier: $-\partial_x^2 + \cosh^{-2} x$. It falls into the general class of Pöschl–Teller potentials which can also be used to compute resonances of the Laplace–Beltrami operator for hyperbolic spaces or hyperbolic cylinders – see for instance [**GZ95a**] or [**Bo16**]. Another example is given by the sphere in which case scattering resonances are zeros of Hankel functions which can be described asymptotically – see [**St06**] and Figure 1.7.

In general however it is impossible to obtain an explicit description of individual resonances. Hence we need to consider their properties and their

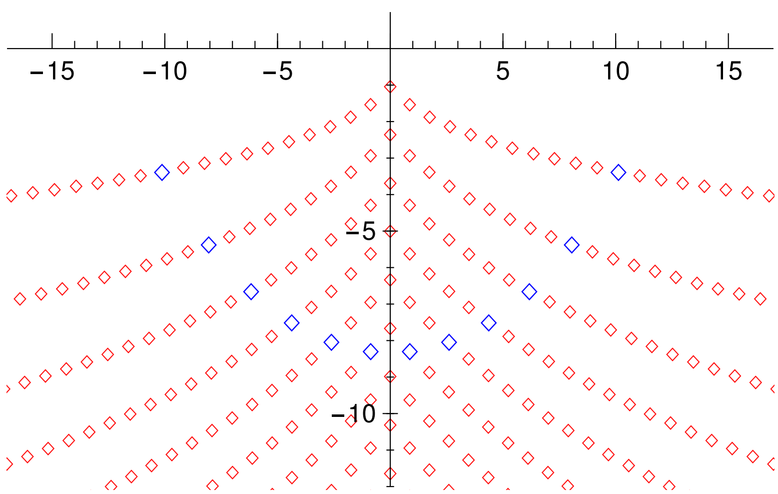


Figure 1.7. Resonances for the sphere (or radius one) in three dimensions. For each spherical momentum ℓ they are given by solutions of $H^{(2)}_{\ell+1/2}(\lambda) = 0$ where $H^{(2)}_\nu$ is the Hankel function of order ν. Each zero appears as a resonance of multiplicity $2\ell + 1$: the resonances with $\ell = 12$ are highlighted.

distribution in asymptotic régimes. For instance in the case of obstacle scattering that could mean the high energy limit. In the case of the sphere in Figure 1.7 that corresponds to letting the angular momentum $\ell \to +\infty$. For a general obstacle that means considering resonances as $|\lambda| \to +\infty$ and $|\operatorname{Im} \lambda| \ll |\lambda|$.

The high energy limit is a special case of the *semiclassical* limit. For instance we can consider resonances of the Dirichlet realization of $-h^2\Delta + V$ on $\mathbb{R}^n \setminus \mathcal{O}$ in bounded subsets of $\mathbb{C}$ as $h \to 0$. When $V \equiv 0$ that corresponds to the high energy limit for obstacle problems and when $\mathcal{O} = \emptyset$ to Schrödinger operators.

In the case of semiclassical Schrödinger operators, the properties of the classical energy surface $\xi^2 + V(x) = E$ can be used to study resonances close to $E \in \mathbb{R}$. Some aspects of that will presented in §2.8 in one dimension and in Chapters 6 and 7 in more depth. Figure 1.8 shows some of the principles in dimension one. The last set of resonances shown there and labeled as Regge resonances comes from the singularities at the boundary of the support of the potential V shown there. Roughly speaking these resonances are responsible for large energy asymptotics for the number of resonances given in Theorem 2.16.

1.3. SOME EXAMPLES

We present here a few examples of scattering resonances appearing in physical systems.

A textbook example of scattering resonances is related to tunneling through potential barriers. Resonances generated by that process are the *shape resonances* shown in Figure 1.8. Figure 7.1 illustrates a similar potential (plotted against the "reaction coordinate") motivated by an actual energy landscape of a chemical reaction. Only recently experiments caught up with this well known theory, as shown by the following quote from a survey lecture [**Cl18**]: "Quantum scattering resonances in chemical reactions have long been of interest to theoreticians but have only relatively recently been experimentally measured." We refer to [**Cl18**] for many interesting examples with varied energy landscapes of different reactions ($\mathrm{F} + \mathrm{H}_2 \to \mathrm{HF} + \mathrm{H}$[1], $\mathrm{F} + \mathrm{CHD}_3 \to \mathrm{FH} + \mathrm{CD}_3$, $\mathrm{O} + \mathrm{O}_2 \to \mathrm{O}_3$, the last a very important process in the earth's atmosphere).

[1]It is irresistible not to recall a small anecdote: one of the authors returned from a discussion with William H. Miller of Chemistry, a major contributor to the subject, to the Mathematics Department and his very pure maths French visitor mockingly asked "What reactions were you discussing?". The answer was the reaction above. Our colleague's response was "I am relieved – I was worried you were doing something useful".

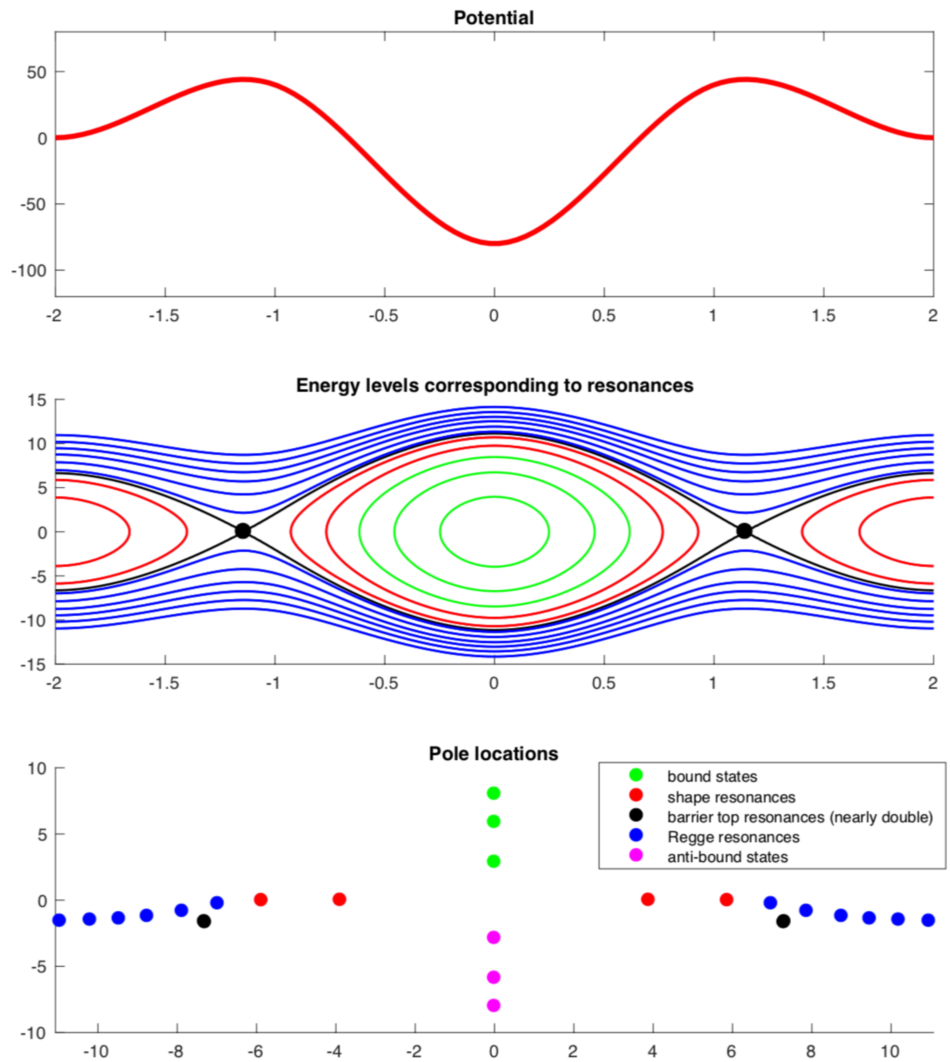


Figure 1.8. Resonances corresponding to different dynamical phenomena. The bound states are generated by negative level sets of $\xi^2+V(x)$ satisfying Bohr–Sommerfeld quantization conditions. Bounded positive level sets of $\xi^2+V(x)$ can also satisfy the quantization conditions but they cannot produce bound states – tunnelling to the unbounded components of these level sets is responsible for resonances with exponentially small ($\sim e^{-S/h}$) imaginary parts/width – see §§2.8,7.3 and references in the notes to the corresponding chapters. The unstable trapped points corresponding to maxima of the potential produce resonances which are at distance h of the real axis – see §§6.3,6.6. The Regge resonances with $\operatorname{Im}\lambda \sim -\log \operatorname{Re}\lambda$ come from the singularities at the boundary of the support of the potential – see [Re58],[Zw87]. The anti-bounds are defined here as resonances on the negative imaginary axis – see [Si00],[DG10]. The resonances are computed using the code `splinepot.m` by David Bindel [BZ].

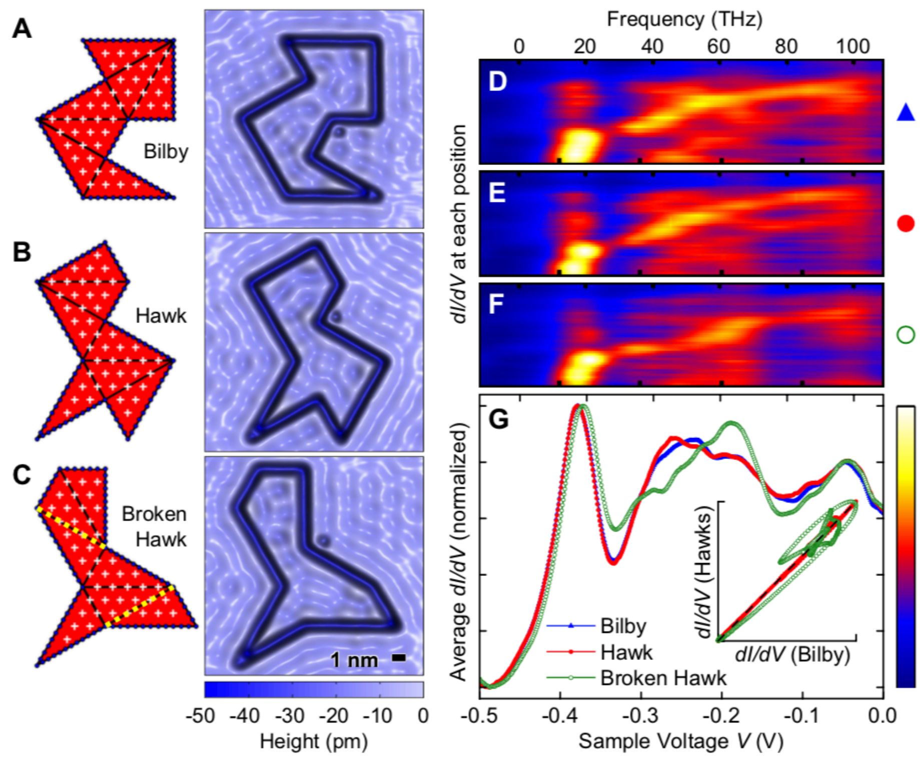


Figure 1.9. A scanning tunnelling microscope (STM) experiment from the Manoharan Lab at Stanford University [M*08]. The figure shows design and realization of quantum isospectral resonators each assembled from 90 CO molecules on the Cu(111) surface. A scanning tunneling microscope (STM) spectrum is a plot of dI/dV(I being the current) as a function of bias voltage V. Each spectrum shows the series of surface state electron resonances inside a bounded quantum corral – see Figure 1.5 for a visualization of the relation between the peaks and the complex resonances. (A to C) Schematics and STM topographies of the Bilby (A), Hawk (B), and Broken Hawk (C) domains. The seven identical $\pi/6$–$\pi/3$–$\pi/2$ triangles composing each shape are shown in red. Blue dots indicate the positions of wall molecules. White crosses mark locations where dI/dV spectroscopy was performed. STM topographs are 15 nm by 15 nm ($V = 10$ mV, $I = 1$ nA). A single CO molecule used for registration between spectra accompanies each nanostructure. (D to F) Spectral fingerprints (dI/dV spectra) acquired throughout Bilby (D), Hawk (E), and Broken Hawk (F). (G) The normalized averages of the Bilby and Hawk spectra match closely, consistent with isospectrality, whereas the average Broken Hawk spectrum clearly differs. Inset: Spectral correlation plot (dashed line denotes perfect match) quantifying Bilby-Hawk isospectrality and its departure in Broken Hawk.

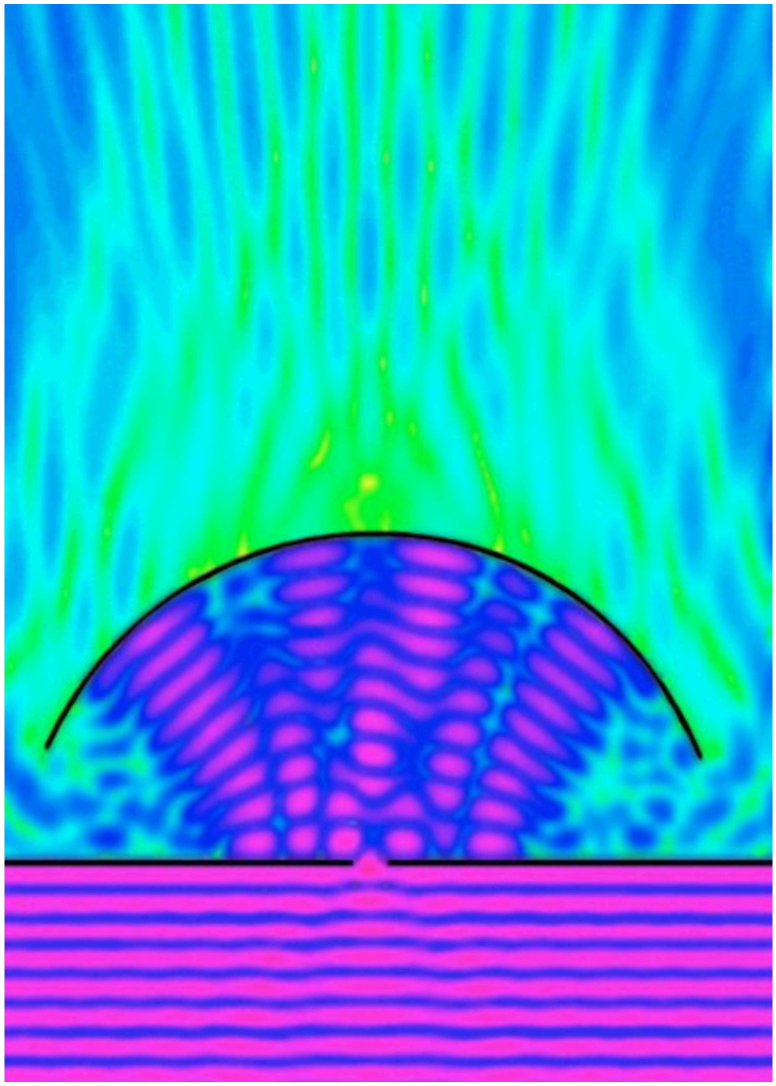


Figure 1.10. This figure shows Heller's digital visualization https://ejheller.jalbum.net of the Westervelt resonator (of size $\sim 1\mu\text{m} = 10^{-6}$meter) built at Harvard in 1995 – see [**K*97**]. A quantum wave builds up in a resonant cavity between the straight and curved walls: the waves arrive from below and most of the wave energy is reflected back. However, a surprisingly large fraction of the energy finds its way through the tiny opening if the energy of the electron corresponds to the resonant energy (real part of the scattering resonance) of the cavity: to quote https://ejheller.jalbum.net "usually it is pretty quiet in the cavity". Except for the possibility of escape through the side openings and tiny hole, classical electrons would be trapped in the cavity – the quantum waves leak out the sides. The Westervelt resonator is a quantum version of the Helmholtz resonator – see [**Zw17**, Figure 15] – where similar phenomena occur for acoustic waves.

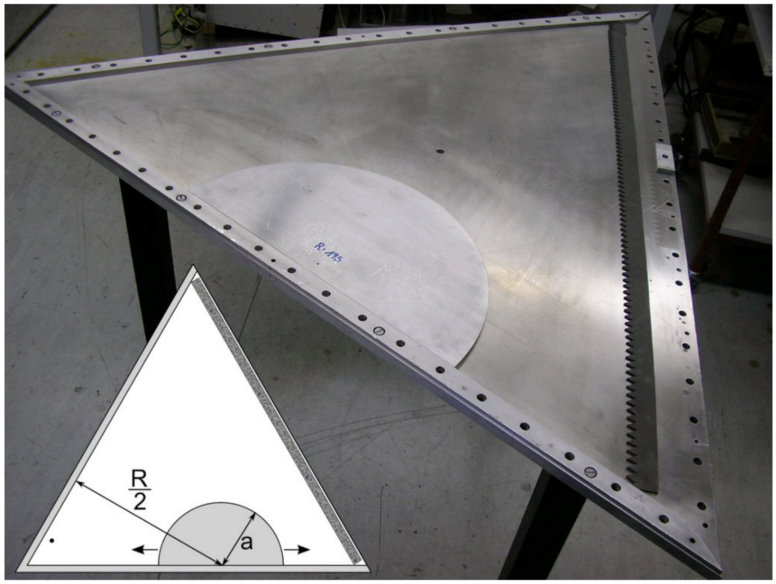


Figure 1.11. The experimental set-up used by the Stöckmann group in Marburg and the Kuhl group in Nice [**B*13**],[**P*12**]. A three disc symmetry reduced system is implemented in a microwave scattering experiment. The hard walls correspond to the Dirichlet boundary condition, that is to odd solutions (by reflection) of the full problem. The absorbing barrier, which produces negligible reflection at the considered range of frequencies, models escape to infinity.

Figure 1.6 shows resonance peaks in a scanning tunneling microscope (STM) experiment by Crommie et al [**CLE93**]: the position of the peaks corresponds to the resonant states trapped by the corral. The trapping is much more pronounced in the experiment showsn in Figure 1.9. It shows resonance peaks for a STM experiment where isospectral quantum corrals of CO molecules, instead of less densely packed iron atoms, are constructed – see [**M*08**] and references given there. The resonances are very close to eigenvalues of the Dirichlet Laplacian (rescaled by $\hbar^2/m_{\text{eff}}$ where m_{eff} is the effective mass of the Bloch electron). Mathematical results explaining existence of resonances created by a barrier (here formed by a corral of CO molecules) are presented in §7.3.

Figure 1.10 shows a visualization of the Westervelt resonator [**K*97**], a *nano*-version of the classical Helmholtz resonator (see [**DGM18**]). On the classical level, the cavity has a lot of trapping (see Figure 4.1 and §6.1) and classical electrons would get confined to the cavity for all times. The escape through the side openings and through the tiny opening at the bottom produces positive decay rates (non-zero imaginary parts of resonances).

The general mathematical mechanism of trapping producing long living resonances is described in §7.3.

Figure 1.11 shows an experimental set-up for microwave cavities used to study scattering resonances for chaotic systems. Density of resonances was investigated in this setting in [**P*12**] and that is related to semiclassical upper bounds in §§3.4,4.3,7.2. In [**B*13**] dependence of resonance free strips on dynamical quantities was confirmed experimentally and Chapter 6 contains related mathematical results and references.

Figure 1.12 shows a MEMS (the acronym for the *microelectromechanical systems*) resonator. The numerical calculations in that case are based on the complex scaling technique, presented in §4.5, adapted to the finite element methods, and known as the method of *perfectly matched layers* [**Be94**].

Figure 1.13 shows the profile of gravitational waves recently detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and originating from a binary black hole merger. Resonances for such waves are known by the name of *quasi-normal modes* in physics literature and are the characteristic frequencies of the waves emitted during the ringdown phase of the merger, when the resulting single black hole settles down to its stationary state – see for instance [**KS99**],[**Dy12**],[**DZ13**] and §§5.7,6.3.

A survey [**Zw17**] presents recent mathematical results motivated by other physical phenomena such as quantum corrals [**BZH10**],[**Ga19**], [**GS15**], Helmholtz resonators [**DGM18**] (see also Figure 1.10), or dielectric cavities (see [**CW15**] for a physics survey and [**NS08**] for some theoretical results).

There are many topics which still await mathematical treatment and one example is *long* wavelength scattering (opposite to semiclassical or *short* wavelength scattering discussed in Part 3 of the book). For instance, for two closely placed scatterers, narrow *proximity* resonances (small imaginary part) develop from broader resonances (large imaginary part) of the individual scatterers – see Heller [**He96**]. Narrower resonances due to trapping introduced by two scatterers occur in semiclassical regime (see §§6.6 and 7.3) but the mechanism described in [**He96**] is different.

1.4. OVERVIEW

To make the presentation more accessible we restrict ourselves to the simplest setting in which the theory is physically and mathematically relevant: compactly supported perturbations in odd dimensions. Many results, especially the ones based on complex scaling, are valid in all dimensions and for suitable non-compact perturbations but for the clarity of presentation we only provide pointers to the literature. The hope is that once the ideas are grasped in the technically less challenging setting the references will become

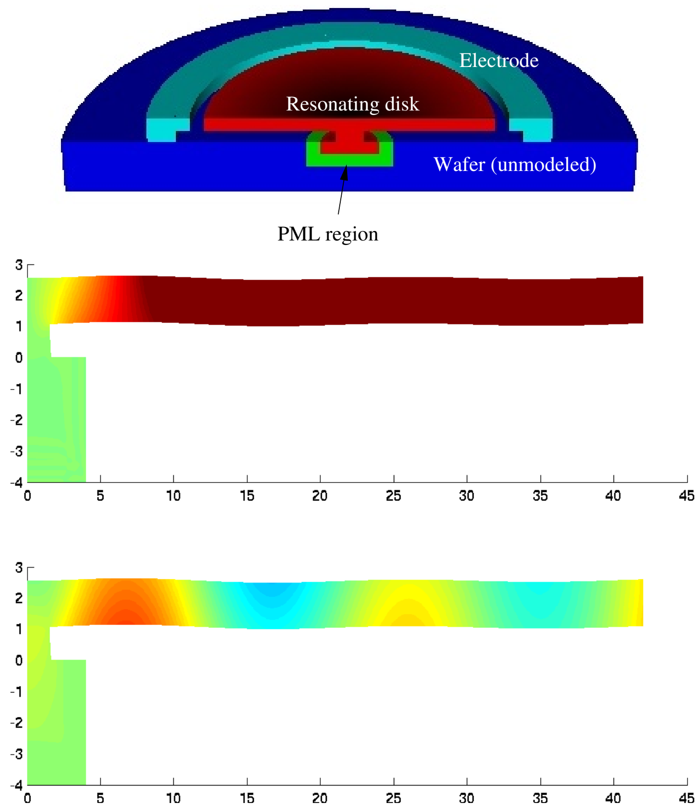


Figure 1.12. On top: a schematic representation of a MEMS (*microelectromechanical systems*) device: a disk attached by a stem to an absorbing plate. Below, a computed resonant mode for this device: the top is colored by the amount of displacement in the radial direction, and the bottom is colored by the amount of displacement in the vertical direction. The computation was done using the complex scaling/PML (*perfectly matched layer*) methods by Bindel–Govindjee [**BG05**]. As explained there, for the resonance state shown in the figure, the coupling between the radial and vertical displacements gives this state a large imaginary part – in other words a significant loss in oscillations.

accessible. In the case of scattering on asymptotically hyperbolic manifolds (Chapter 5) we present a general theory as there are few advantages in restricting our attention to the hyperbolic space alone.

We now present brief descriptions of the content of the chapters.

Chapter 2: We cover basic theory of resonances in dimension one. Many fundamental concepts such as outgoing solutions, meromorphic continuation of the resolvent, the relation of resonances to the scattering matrix,

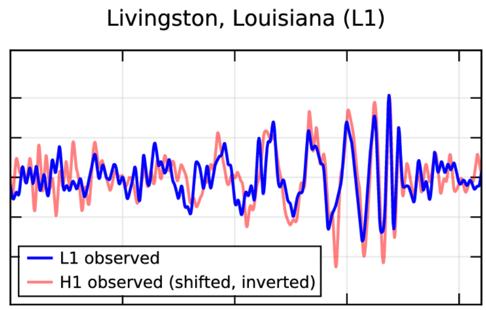


Figure 1.13. Left: an aerial view of the LIGO laboratory in Livingston, Louisiana, US. Right: the gravitational wave signal observed on September 14, 2015 simultaneously by LIGO Livingston (blue) and LIGO Hanford (red); see [**A*16**],[**V*14**]. The ringdown oscillations of the kind shown in Figure 1.1 and connected to quasi-normal modes (which is what scattering resonances are called in general relativity) have not been observed yet. What is shown here are oscillations due to a much stonger effect of black hole merger.

trace formulas, and resonance expansions of waves, appear later in more complicated settings.

Chapter 3: Here the theory of scattering by compactly supported potentials in odd dimensions is presented in detail. This chapter can be used as the introduction to the study of more general settings (for instance, in the theory of zero resonances) and to the open problems in scattering by compactly supported potentials.

Chapter 4: This chapter is devoted to *black box* scattering which allows a unified treatment of many different operators ranging from Laplacians on surfaces with constant curvature cusp ends to obstacle scattering in the Euclidean space.

Chapter 5: One of the recent advances in geometric scattering is Vasy's approach to meromorphic continuation of resolvents for (even) asymptotically hyperbolic manifolds. The method was motivated by the study of scattering for black holes and that connection is also explained.

Chapter 6: Resonance free regions have been investigated in mathematical scattering theory since the seminal work of Lax–Phillips and Vaĭnberg (see §4.6). Semiclassical scattering with its connection to classical/quantum correspondence is the natural setting for investigating resonance free regions.

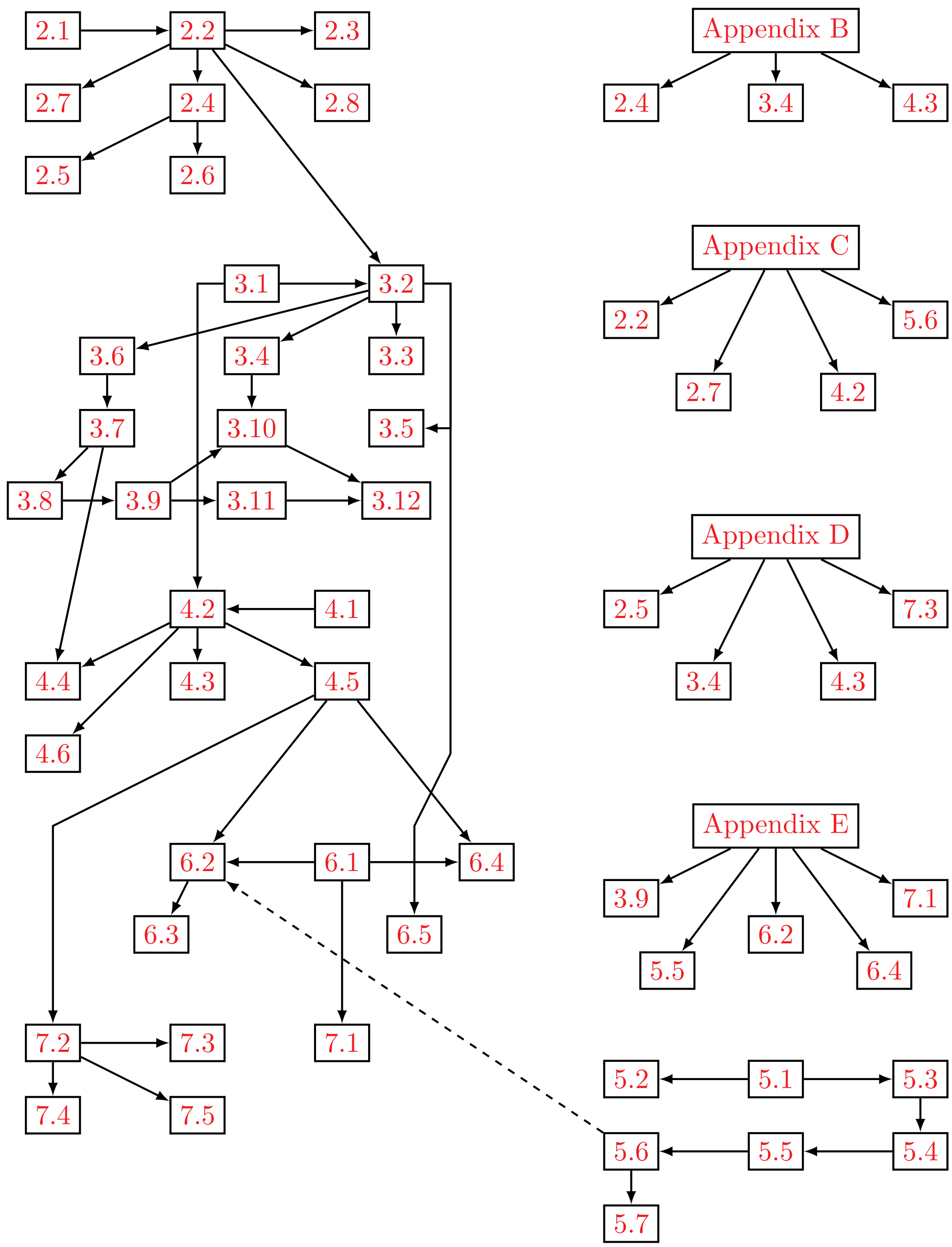
Chapter 7: This last chapter is concerned with resonances generated by strong trapping phenomena such as the presence of barriers or singularities of $E \mapsto \int_{V(x)\geq E} dx$. We conclude with expansions of waves for strong trapping.

Appendices: We present notational conventions and references to basis techniques. Proofs of various results which are crucial in the text (such as Fredholm theory or propagation of singularities in the semiclassical setting) are presented in detail.

Acknowledgments for Figures:

- Figure 1.6 comes from [**CLE93**] and is reprinted with permission from The American Association for the Advancement of Science. The authors are grateful to Mike Crommie for allowing us to use this figure.
- Figure 1.9 comes from [**M*08**] and is reprinted with permission from The American Association for the Advancement of Science. The authors are grateful to Hari Manoharan for offering us this figure as an illustration of quantum resonances in a complex STM setting.
- Figure 1.10 comes from `https://ejheller.jalbum.net` and we are grateful to Eric Heller for the permission to include it here.
- Figure 1.11 was provided to us by Ulrich Kuhl and represents an experimental set-up at his lab `http://lpmc.unice.fr/Kuhl-Ulrich.html`. We are grateful to him and to Hans-Jürgen Stöckmann for giving us access to their microwave experiments [**B*13**],[**P*12**].
- Figure 1.12 is an original figure produced by David Bindel. The authors are grateful to him for providing the figure and the caption explaining the connection to [**BG05**]. We are also grateful to Sanjay Govindjee for discussions related to [**BG05**].
- Figure 1.13 comes from [**V*14**] and made use of figures obtained from the LIGO Open Science Center `https://losc.ligo.org`, a service of LIGO Laboratory, the LIGO Scientific Collaboration and the Virgo Collaboration. LIGO is funded by the U.S. National Science Foundation. Virgo is funded by the French Centre National de Recherche Scientifique (CNRS), the Italian Istituto Nazionale della Fisica Nucleare (INFN) and the Dutch Nikhef, with contributions by Polish and Hungarian institutes.

Dependence graph of sections



Part 1

POTENTIAL SCATTERING

Chapter 2

SCATTERING RESONANCES IN DIMENSION ONE

2.1 Outgoing and incoming solutions
2.2 Meromorphic continuation
2.3 Expansions of scattered waves
2.4 Scattering matrix
2.5 Asymptotics for the counting function
2.6 Trace and Breit–Wigner formulas
2.7 Complex scaling in one dimension
2.8 Semiclassical study of resonances in dimension one
2.9 Notes
2.10 Exercises

In the simplest setting of one dimensional scattering by compactly supported potentials we can already observe many general phenomena. In particular, various notions can be explained in a very intuitive setting. Technically, there are also many advantages: we are dealing with ordinary differential equations, the methods of complex analysis apply particularly well and trace class properties hold nicely.

2.1. OUTGOING AND INCOMING SOLUTIONS

We consider the following class of operators:

$$P_V = D_x^2 + V(x)\,, \quad D_x := \frac{1}{i}\partial_x\,, \quad V \in L^\infty_{\rm comp}(\mathbb{R})\,.$$

The stationary Schrödinger equation then is

$$(P_V - z)u = f\,, \quad z \in \mathbb{C}\,, \quad f \in L^2(\mathbb{R})\,, \tag{2.1.1}$$

while the dynamical equation is given by

$$(i\partial_t - P_V)v = F\,, \quad v|_{t=0} = v_0\,, \quad v_0 \in L^2(\mathbb{R}), \quad F \in L^1_{\rm loc}(\mathbb{R}_t; L^2(\mathbb{R}_x)). \tag{2.1.2}$$

As we will see below it is sometimes important to consider initial data in different spaces than L^2.

A solution to the stationary equation (2.1.1) produces a solution to (2.1.2) corresponding to the evolution of the state u:

$$v(t,x) := e^{-izt}u(x)\,, \quad v_0(x) = u(x)\,, \quad F(x,t) = -e^{-izt}f(x)\,. \tag{2.1.3}$$

Outside the support of V and f, say for $|x| \geq R$, (assuming that f is compactly supported) the solutions of (2.1.1) are given by

$$u(x) = a_\pm e^{i\sqrt{z}|x|} + b_\pm e^{-i\sqrt{z}|x|}\,, \quad \pm x \geq R\,.$$

To consider the dependence on z we have to choose a branch of $\sqrt{z}$. We consider $\sqrt{z}$ defined on $\mathbb{C} \setminus [0,\infty)$ with $\operatorname{Im}\sqrt{z} > 0$ everywhere, so that

$$\pm \lim_{\varepsilon \to 0+} \sqrt{z \pm i\varepsilon} =: \pm\sqrt{z \pm i0} > 0\,, \quad z \in (0,\infty)\,.$$

When considering $z \in (0,\infty)$ we write $\sqrt{z} = \sqrt{z + i0}$.

Outgoing and incoming solutions. A solution to (2.1.1) with $z > 0$ is called *outgoing* if

$$u(x) = a_- e^{-i\sqrt{z}x}\,, \quad x < -R\,, \qquad u(x) = a_+ e^{i\sqrt{z}x}\,, \quad x > R\,. \tag{2.1.4}$$

This corresponds to v given by (2.1.3) moving *away* from the support of $V(x)$ – see Figure 2.1. We also note that using our convention

$$z \notin [0,\infty) \implies u(x) \in L^2(\mathbb{R})\,.$$

Similarly, the solution to (2.1.1) is called *incoming* if

$$u(x) = b_- e^{i\sqrt{z}x}\,, x < -R\,, \quad u(x) = b_+ e^{-i\sqrt{z}x}\,, x > R\,.$$

Although the physical motivation illustrated in Figure 2.1 disappears when $z \notin (0,\infty)$ we will still use the notions of outgoing and incoming solutions as defined above, paying attention to our convention for $\sqrt{z}$.

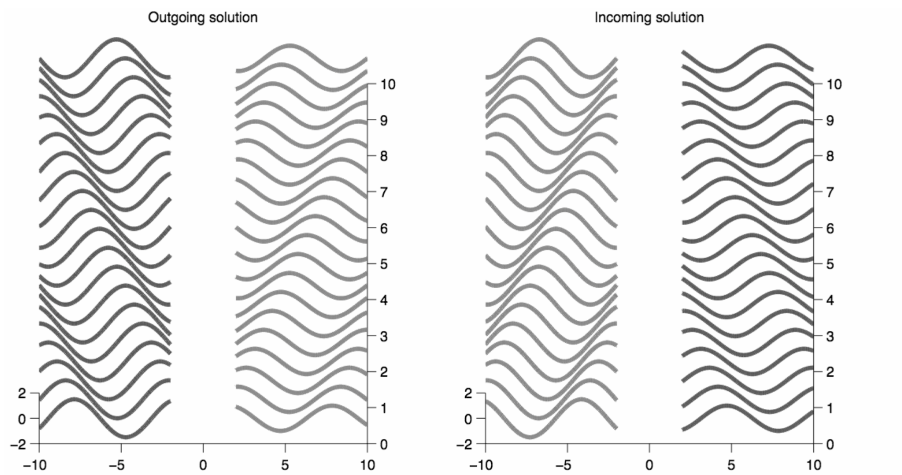


Figure 2.1. Schematic representation of the outgoing (left) and incoming (right) solutions to (2.1.3).

In Section 2.2 we will address the problem of constructing outgoing (or incoming solutions) to (2.1.1). That will lead to a natural definition of scattering resonances.

The above definition of outgoing and incoming solutions is given in terms of the Schrödinger equation. We can also consider the wave equation:

$$(-\partial_t^2 - P_V)v = F\,, \quad v|_{t=0} = v_0\,, \quad \partial_t v|_{t=0} = v_1\,. \tag{2.1.5}$$

The stationary equation, formally obtained by taking the Fourier transform in t, is given by

$$(P_V - \lambda^2)u = f\,, \quad \lambda \in \mathbb{C}\,. \tag{2.1.6}$$

In this case the convention regarding the sign of λ in the definition of outgoing and incoming solutions is somewhat arbitrary. We choose a convention consistent with the choice of $\sqrt{z}$ above:

$$\lambda^2 = z\,, \quad \lambda = \sqrt{z}\,. \tag{2.1.7}$$

In particular,

$$\lambda > 0 \implies \sqrt{(\pm\lambda + i0)^2} = \pm\lambda\,.$$

The outgoing solution to (2.1.6) with a compactly supported f is supposed to satisfy

$$u(x) = a_- e^{-i\lambda x}\,, \ x < -R\,, \quad u(x) = a_+ e^{i\lambda x}\,, \ x > R\,. \tag{2.1.8}$$

We now have

$$\operatorname{Im}\lambda > 0 \implies u(x) \in L^2(\mathbb{R})\,.$$

The solutions to (2.1.6) with $f = 0$ and $\operatorname{Im}\lambda > 0$ are the eigenfunctions of P_V corresponding to eigenvalues λ^2. Note that the equation (2.1.6) is the same when λ is replaced by $-\lambda$, but the conditions (2.1.8) change under this operation.

We will use the wave equation motivated λ convention in this chapter, except in §2.8 which is motivated by quantum mechanics[1].

To motivate the study of outgoing solutions to (2.1.6), and the importance of the poles of the meromorphic continuation of the outgoing resolvent, constructed in Section 2.2, we now briefly explain an application to the long-time asymptotics of the wave equation. The presented ideas lead to *resonance expansions* of waves, studied in detail in Section 2.3.

Consider the initial-value problem for the wave equation

$$(-\partial_t^2 - P_V)v = F, \quad v|_{t=0} = 0, \quad \partial_t v|_{t=0} = 0. \tag{2.1.9}$$

We assume that

$$V \in C^\infty_{\mathrm{c}}((-R,R);\mathbb{R}), \quad F \in C^\infty_{\mathrm{c}}((0,\infty)_t \times (-R,R)_x),$$

for some $R > 0$. We take the Fourier–Laplace transform in time

$$u(\lambda, x) := \hat{v}(\lambda, x) := \int_0^\infty e^{it\lambda} v(t,x)\, dt. \tag{2.1.10}$$

The integral (2.1.10) converges for $\operatorname{Im}\lambda > 0$, thanks to standard energy estimates for the wave equation – see for instance [**Ev98**, §7.2.4]. Taking the Fourier transform of (2.1.9) in t, we see that for $\operatorname{Im}\lambda > 0$, $u(\lambda)$ solves the equation (2.1.6):

$$(\lambda^2 - P_V)u(\lambda) = \widehat{F}(\lambda). \tag{2.1.11}$$

On the other hand the d'Alembert's formula (see [**Ev98**, §2.4]) shows

$$v(t,x) = -\frac{1}{2}\int_0^t \int_{x-s}^{x+s} (Vv + F)(t-s, y)\, dyds. \tag{2.1.12}$$

Since we assumed that V and F are supported in $\{|x| \le R\}$, we obtain

$$v(t,x) = v_\pm(x \mp t), \ \pm x \ge R, \ t \ge 0, \tag{2.1.13}$$

for some functions $v_\pm$ with

$$\operatorname{supp} v_+ \subset (-\infty, R), \quad \operatorname{supp} v_- \subset (-R, \infty).$$

see Figure 2.2 or Exercise 2.1.

[1]Different conventions for scattering resonances are due to their emergence in different fields. We will discuss those issues as they come along.

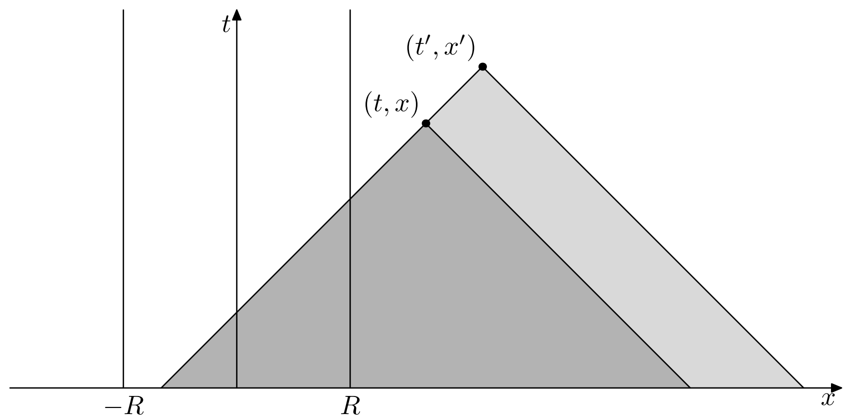


Figure 2.2. An illustration of the outgoing property for the wave equation. The values $u(t, x)$, $u(t', x')$ are obtained by integrating $-\frac{1}{2}(Vv+F)$ over the shaded triangles. For $x, x' > R$ and $t - x = t' - x'$, these triangles have the same intersection with $\{|x| \le R\} \supset \operatorname{supp}(Vv + F)$, therefore $u(t, x) = u(t', x')$.

It follows that for $\operatorname{Im}\lambda > 0$, $u(\lambda)$ is outgoing in the sense of (2.1.8): if $\pm x \ge R$ then

$$\begin{aligned} u(\lambda, x) &= \int_0^\infty v_\pm(x \mp t) e^{i\lambda t} dt = \int_{-R}^\infty v_\pm(\mp s) e^{i\lambda s} e^{\pm i\lambda x} ds \\ &= a_\pm(\lambda) e^{\pm i\lambda x}, \quad a_\pm(\lambda) := \int_{-R}^\infty v_\pm(\mp s) e^{i\lambda s}. \end{aligned}$$

Here we used the support properties and $v_\pm$ to guarantee convergence when $\operatorname{Im}\lambda > 0$.

One technique for obtaining asymptotics of $v(t, x)$ as $t \to \infty$ is to deform the contour in the Fourier inversion formula

$$v(t, x) = \frac{1}{2\pi} \int_{\operatorname{Im}\lambda = c} e^{-it\lambda} u(\lambda)(x)\, d\lambda, \quad c > 0. \tag{2.1.14}$$

For that we need to continue $u(\lambda)$ meromorphically into the lower half-plane and this is done by requiring that $u(\lambda)$ solve the equation (2.1.11) with the outgoing conditions (2.1.8), where $\widehat{F}(\lambda)$ is entire in λ since F is compactly supported. For $\operatorname{Im}\lambda > 0$, we have $u(\lambda) \in L^2(\mathbb{R}_x)$, therefore $u(\lambda)$ for general λ can be viewed as the image of $\widehat{F}(\lambda)$ under the *meromorphic continuation of the resolvent* $(\lambda^2 - P_V)^{-1} : L^2 \to L^2$, $\operatorname{Im}\lambda > 0$, through the continuous spectrum $\{\operatorname{Im}\lambda = 0\}$ of P_V to the entire complex plane. The existence of this meromorphic continuation, as an operator $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$, is established in Section 2.2. After proving additional estimates on $u(\lambda)$ in the lower half-plane, we can deform the contour in (2.1.14) to the line $\{\operatorname{Im}\lambda = -\nu\}$ with $\nu > 0$. The integral along the new contour will be $\mathcal{O}(e^{-\nu t})$, owing to the

$e^{-it\lambda}$ factor, and we accumulate residues from the poles of $u(\lambda)$. These poles, called *resonances*, will be the central objects of study in this book.

2.2. MEROMORPHIC CONTINUATION

In this section we solve (2.1.6) for $\lambda \in \mathbb{C}$, with u outgoing, that is satisfying (2.1.8). For that we first consider the case of $V = 0$. In that case $u(x)$ is given by an explicit formula:

$$u(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y) dy \,.$$

For $\operatorname{Im} \lambda > 0$ this gives the integral kernel of the *free resolvent*:

$$\begin{gathered} R_0(\lambda) := (D_x^2 - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \,, \quad \operatorname{Im} \lambda > 0 \,, \\ R_0(\lambda, x, y) = \frac{i}{2\lambda} e^{i\lambda|x-y|} \,, \end{gathered} \tag{2.2.1}$$

where we use the same notation $R_0(\lambda)$ for the operator and its integral kernel. We should stress that for $\operatorname{Im} \lambda < 0$,

$$(D_x^2 - \lambda^2)^{-1} = R_0(-\lambda), \quad (D_x^2 - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \,, \quad \operatorname{Im} \lambda < 0 \,.$$

This means that the spectrum of D_x^2 is given by $[0, \infty)$ and it is absolutely continuous as can be shown using the Fourier transform – see §B.1.

From the expression (2.2.1) we see that for fixed x and y, $R_0(\lambda, x, y)$ is a meromorphic function of $\lambda \in \mathbb{C}$ defining an operator $C_{\rm c}^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ which is not bounded on L^2 for $\operatorname{Im} \lambda \leq 0$.

Using the notion of a meromorphic family of operators (see Appendix C.3) we summarize these facts as follows.

THEOREM 2.1 (Meromorphic continuation of the free resolvent). *The operator $R_0(\lambda)$ defined by (2.2.1) for $\operatorname{Im} \lambda > 0$ extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:*

$$R_0(\lambda) := L^2_{\rm comp}(\mathbb{R}) \longrightarrow L^2_{\rm loc}(\mathbb{R}) \,.$$

We have

$$\| R_0(\lambda) \|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, [0, \infty))} \leq \frac{1}{|\lambda| \operatorname{Im} \lambda} \,, \quad \operatorname{Im} \lambda > 0 \,. \tag{2.2.2}$$

and for $\rho \in C_{\rm c}^\infty(\mathbb{R})$, $\operatorname{supp} \rho \subset [-L, L]$, $\lambda \in \mathbb{C}$,

$$\| \rho R_0(\lambda) \rho \|_{L^2(\mathbb{R}^n) \to H^j(\mathbb{R}^n)} \leq C_{L,j} e^{2L(\operatorname{Im} \lambda)_-} |\lambda|^{-1} \langle \lambda \rangle^j \,, \quad 0 \leq j \leq 2, \tag{2.2.3}$$

where $x_- := \max(0, -x)$.

REMARK. The estimate (2.2.2) and the fact that $-\Delta R_0(\lambda) = I - \lambda^2 R_0(\lambda)$ immediately imply that

$$\|R_0(\lambda)\|_{L^2 \to H^k} \leq \frac{\langle \lambda \rangle^k}{|\lambda| \operatorname{Im} \lambda}, \quad \operatorname{Im} \lambda > 0, \quad 0 \leq k \leq 2. \tag{2.2.4}$$

Proof. 1. The estimate (2.2.2) follows directly from spectral theory, since the spectrum of D_x^2 is equal to R^+:

$$d(\lambda^2, [0,\infty)) = \begin{cases} 2 \operatorname{Im} \lambda |\operatorname{Re} \lambda| & (\operatorname{Re} \lambda)^2 \geq (\operatorname{Im} \lambda)^2, \\ |\lambda|^2 & (\operatorname{Re} \lambda)^2 \leq (\operatorname{Im} \lambda)^2, \end{cases}$$

and hence $d(\lambda^2, \mathbb{R}_+) \geq |\lambda| \operatorname{Im} \lambda$.

2. The estimate (2.2.3) for $j = 0$ follows from (2.2.1) by Schur's criterion (A.5.3) since

$$\int_{-\infty}^{\infty} |\rho(x)\rho(y)R_0(\lambda)(x,y)| \, dx \leq C|\lambda|^{-1} \int_{-\infty}^{\infty} |\rho(x)\rho(y)| e^{-\operatorname{Im} \lambda |x-y|} \, dx$$

is bounded by $C_L |\lambda|^{-1} e^{2L(\operatorname{Im} \lambda)_-}$ and same is true if integration is performed in the y variable instead.

3. To obtain the estimate for $j = 2$ we use elliptic regularity estimates (see for instance [**Zw12**, Theorem 7.1]): if U and W are intervals and $U \Subset W$ then

$$\|u\|_{H^2(U)} \leq C \left(\|u\|_{L^2(W)} + \|D_x^2 u\|_{L^2(W)} \right).$$

Hence, if $\tilde{\rho} \in C_c^\infty(\mathbb{R})$ satisfies $\tilde{\rho} = 1$ near $\operatorname{supp} \rho$ then

$$\|\rho u\|_{H^2(\mathbb{R})} \leq C \left(\|\tilde{\rho} u\|_{L^2(\mathbb{R})} + \|\tilde{\rho} D_x^2 u\|_{L^2(\mathbb{R})} \right). \tag{2.2.5}$$

4. We now apply (2.2.5) to $u = R_0(\lambda)\rho f$, $f \in L^2$ so that

$$\|\rho R_0(\lambda)\rho f\|_{H^2} \leq C \|\tilde{\rho} R_0(\lambda)\rho f\|_{L^2} + C \|\tilde{\rho} D_x^2 R_0(\lambda) \rho f\|_{L^2}.$$

Since $\tilde{\rho} D_x^2 R_0(\lambda) \rho f = \rho f + \tilde{\rho} \lambda^2 R_0(\lambda) \rho f$ is bounded by $C_L \langle \lambda \rangle e^{2L(\operatorname{Im} \lambda)_-} \|f\|_{L^2}$ in L^2 the estimate (2.2.3) with $j = 2$ follows. Finally, the estimate for $j = 1$ is obtained by interpolating between the cases $j = 0$ and $j = 2$. $\square$

For $V \neq 0$ we have a result which shows that the *outgoing* resolvent of $P_V := D_x^2 + V(x)$ also has a meromorphic continuation.

THEOREM 2.2 (Meromorphic continuation of the resolvent in one dimension). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{C})$. Then the resolvent*

$$R_V(\lambda) := (D_x^2 + V - \lambda^2)^{-1} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad \operatorname{Im} \lambda > 0,$$

is a meromorphic family of operators with a finite number of poles. The family $R_V(\lambda)$ extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:

$$R_V(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}) \longrightarrow H^2_{\mathrm{loc}}(\mathbb{R}).$$

Proof. 1. We first construct $R_V(\lambda)$ for $\operatorname{Im}\lambda \gg 1$. For that we write

$$(P_V - \lambda^2)R_0(\lambda) = (D_x^2 - \lambda^2 + V)R_0(\lambda) = I + VR_0(\lambda). \tag{2.2.6}$$

For $\operatorname{Im}\lambda \gg 1$, $\|VR_0(\lambda)\|_{L^2\to L^2} \leq \|V\|_\infty(\operatorname{Im}\lambda)^{-2} \leq 1/2$, and hence $I + VR_0(\lambda)$ is invertible using the Neumann series:

$$(I + VR_0(\lambda))^{-1} = \sum_{j=0}^{\infty}(-1)^j(VR_0(\lambda))^j.$$

This shows that

$$R_V(\lambda) := (P_V - \lambda^2)^{-1} = R_0(\lambda)(I + VR_0(\lambda))^{-1}. \tag{2.2.7}$$

If $\rho \in C_c^\infty((-L,L))$, $L > 0$, then for $\operatorname{Im}\lambda > 0$, $\rho R_0(\lambda) : L^2 \to H_0^2((-L,L))$ (see (2.2.4)). Hence $\rho R_0(\lambda)$ is a compact operator on L^2 by Theorem B.4. By taking L large enough we can choose ρ which is equal to 1 on $\operatorname{supp} V$. In particular, $\rho V = V$. Hence for $\operatorname{Im}\lambda > 0$ the operator $VR_0(\lambda) = V\rho R_0(\lambda)$ is also compact and we can apply Theorem C.8 (or rather the remark after the theorem since $VR_0(\lambda)$ has a pole at 0) to see that $R_V(\lambda) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a meromorphic family of operators in $\operatorname{Im}\lambda > 0$.

2. To obtain a continuation to the entire $\mathbb{C}$, we define the following meromorphic family of operators:

$$K(\lambda) := VR_0(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{comp}}(\mathbb{R})\,. \tag{2.2.8}$$

(Strictly speaking the notion of meromorphic families operators is only defined on Banach spaces: what we mean here is that $\tilde\rho K(\lambda)\tilde\rho : L^2 \to L^2$ is a meromorphic family for any $\tilde\rho \in C_c^\infty$.)

The only pole of $K(\lambda)$ is at $\lambda = 0$. With the same $\rho \in C_c^\infty(\mathbb{R})$ as in Step 1, $(1-\rho)K(\lambda) = 0$, and hence, by inspection,

$$(I + K(\lambda)(1-\rho))^{-1} = I - K(\lambda)(1-\rho).$$

As in step 1, we see that for $\operatorname{Im}\lambda \gg 1$, $I + K(\lambda)\rho$ is invertible by a Neumann series argument. We conclude that for $\operatorname{Im}\lambda \gg 1$,

$$\begin{aligned}(I + K(\lambda))^{-1} &= \big((I + K(\lambda)(1-\rho))(I + K(\lambda)\rho)\big)^{-1} \\ &= (I + K(\lambda)\rho)^{-1}(I - K(\lambda)(1-\rho))\,.\end{aligned}$$

3. By (2.2.7), for $\operatorname{Im}\lambda \gg 1$,

$$R_V(\lambda) = R_0(\lambda)(I + K(\lambda)\rho)^{-1}(I - K(\lambda)(1-\rho))\,. \tag{2.2.9}$$

For $\lambda \in \mathbb{C} \setminus 0$, (2.2.3) shows that $\rho R_0(\lambda)\rho : L^2(\mathbb{R}) \to H_0^2((-L,L))$, and hence by Theorem B.4 this operator is compact. Since $V = V\rho$, we conclude that $K(\lambda)\rho$ is compact on $L^2(\mathbb{R})$, and hence $I + K(\lambda)\rho$ is a meromorphic family of Fredholm operators (see §C.3). Theorem C.8 gives a meromorphic continuation of

$$(I + K(\lambda)\rho)^{-1} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \tag{2.2.10}$$

to $\mathbb{C}$.

4. From (2.2.3) we also conclude that for $\operatorname{Im}\lambda \geq 0$, $\|K(\lambda)\rho\|_{L^2 \to L^2} \leq C/|\lambda|$. The Neumann series argument and (2.2.9) show that $R_V(\lambda)$ has only finitely many poles for $\operatorname{Im}\lambda > 0$. (See Theorem 2.10 for more on that.)

5. We now observe that

$$I - K(\lambda)(1-\rho) \;:\; L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{comp}}(\mathbb{R})\,,$$

and

$$(I + K(\lambda)\rho)^{-1} \;:\; L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{comp}}(\mathbb{R})\,.$$

The last property can be checked for $\operatorname{Im}\lambda \gg 1$ using the Neumann series argument: if $\chi\rho = \rho$, $\tilde{\chi}\chi = \chi$ then

$$(1-\tilde{\chi})(I + K(\lambda)\rho)^{-1}\chi = 0\,, \quad \operatorname{Im}\lambda \gg 1\,,$$

and this remains true for all λ by analytic continuation.

Combining these facts with the expression for R_V given in (2.2.9) we obtain the meromorphy of $R_V(\lambda)$ for $\lambda \in \mathbb{C}$ as a family of operators $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$. □

DEFINITION 2.3. *We call the poles of $R_V(\lambda)$* scattering resonances *or simply* resonances *. The multiplicity of a resonance at λ is defined as follows:*

$$m_R(\lambda) := \operatorname{rank} \oint_\lambda R_V(\zeta)d\zeta\,, \tag{2.2.11}$$

where the integral is over a small circle containing no other poles of R_V. We refer to the meromorphic continuation, $R_V(\lambda)$, as the scattering resolvent.

When λ is not a resonance we put $m_R(\lambda) = 0$ which is of course consistent with the above definition.

REMARKS. 1. When $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$ then the operator P_V is self-adjoint and the existence of $R_V(\lambda)$, $\operatorname{Im}\lambda > 0$, as a meromorphic operator on L^2 follows from the spectral theorem. The poles occur at $i\sqrt{-E_j}$ where E_j are the negative eigenvalues of P_V – see Figure 1.8. These statements also follow from Theorem 2.1.

2. We also have the following basic fact valid for real valued potentials

$$V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R}) \implies m_R(\lambda) = 0\,,\; \lambda \in \mathbb{R} \setminus \{0\}\,. \tag{2.2.12}$$

This implies that for $\lambda \notin \mathbb{R} \setminus \{0\}$, there exists a limit

$$\lim_{\varepsilon \to 0+} R_V(\lambda + i\varepsilon) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}.$$

This fact is known as the *limiting absorption principle*. It follows that the spectrum of P_V is given by the continuous spectrum $[0, \infty)$ and a finite number of negative eigenvalues. We will prove this *after* the proof of Theorem 2.5 below.

3. Reality of V or, equivalently, self-adjointness of P_V imply the following symmetry of resonances:

$$V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{R}) \implies m_R(\lambda) = m_R(-\bar{\lambda}), \ \lambda \in \mathbb{C} \setminus \{0\}. \tag{2.2.13}$$

In fact, we will check the following identity of Schwartz kernels:

$$R_V(-\bar{\lambda}, y, x) = \overline{R_V(\lambda, x, y)}.$$

Since both sides are meromorphic in λ we only need to check that

$$R_V(-\bar{\lambda})^* = R_V(\lambda) \ \text{ for } \operatorname{Im}\lambda > 0, \tag{2.2.14}$$

when both sides are bounded operators on L^2 (note that $\operatorname{Im}(-\bar{\lambda}) > 0$ if $\operatorname{Im}\lambda > 0$). Using the correspondence between λ and z in (2.1.7) that follows from $((P_V - z)^{-1})^* = (P_V - \bar{z})^{-1}$. □

4. It is also useful to express the operator $(I + VR_0(\lambda)\rho)^{-1}$ through $R_V(\lambda)$, as follows:

$$(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho. \tag{2.2.15}$$

This follows immediately from the identity

$$I - (I + VR_0(\lambda)\rho)^{-1} = VR_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$$

and the following formula:

$$R_V(\lambda)\rho = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}, \tag{2.2.16}$$

which in turn is true for $\operatorname{Im}\lambda \gg 1$ by considering the Neumann series in (2.2.7) and for general λ by analytic contuation.

When λ is not a pole of R_V, the operator $R_V(\lambda)$ gives solutions to the Helmholtz equation satisfying the outgoing condition:

THEOREM 2.4 ($R_V(\lambda)$ at regular points). *Assume that $\lambda \in \mathbb{C}$ is not a pole of R_V. Then for each $f \in L^2_{\mathrm{comp}}(\mathbb{R})$, $u = R_V(\lambda)f$ is the unique outgoing solution, in the sense of (2.1.8), to the equation $(P_V - \lambda^2)u = f$.*

Proof. 1. The identity $(P_V - \lambda^2)R_V(\lambda)f = f$ holds for $\operatorname{Im}\lambda > 0$ by the definition of $R_V(\lambda)$ and extends to all λ by analytic continuation. Same is true for the outgoing condition, which can be written as

$$(\partial_x \mp i\lambda)(R_V(\lambda)f)(\pm R) = 0, \tag{2.2.17}$$

where $R > 0$ is such that $\operatorname{supp} V \cup \operatorname{supp} f \subset (-R, R)$. Indeed, for $\operatorname{Im} \lambda > 0$ this condition holds as $R_V(\lambda) f \in L^2$ must be a linear combination of $e^{i\lambda|x|}$ and $e^{-i\lambda|x|}$ for $\pm x > R$, and $e^{-i\lambda|x|}$ is exponentially growing as $|x| \to \infty$.

2. It remains to show that $R_V(\lambda)f$ is the unique outgoing solution to $(P_V - \lambda^2)u = f$ and for that it suffices to prove the identity

$$u = R_V(\lambda)(P_V - \lambda^2)u \tag{2.2.18}$$

for all outgoing functions $u \in H^2_{\mathrm{loc}}$. We note that the outgoing condition guarantees that $(P_V - \lambda^2)u$ is compactly supported.

3. The equation (2.2.18) is true for $\operatorname{Im} \lambda > 0$ by the definition of $R_V(\lambda)$: $u = R_V(\lambda)f \in H^2$ in this case. To handle λ in the closed lower half-plane, we argue by analytic continuation. For that we decompose any outgoing $u \in H^2_{\mathrm{loc}}$ as

$$u(x) = u_0(x) + \chi_+(x)a_+e^{i\lambda x} + \chi_-(x)a_-e^{-i\lambda x},$$

where $u_0 \in H^2_{\mathrm{comp}}$, and where $\chi_\pm \in C^\infty(\mathbb{R})$ are equal to 1 near $\pm\infty$ and to 0 near $\mp\infty$. Then for $\operatorname{Im} \lambda > 0$ each term is in H^2 and hence

$$u_0 = R_V(\lambda)(P_V - \lambda^2)u_0, \quad \chi_\pm e^{\pm i\lambda\bullet} = R_V(\lambda)(P_V - \lambda^2)(\chi_\pm e^{\pm i\lambda\bullet}).$$

Since $(P_V - \lambda^2)(\chi_\pm e^{\pm i\lambda\bullet}) \in L^2_{\mathrm{comp}}$, the equations have to be valid for all λ at which R_V is holomorphic. Hence (2.2.18) holds for any outgoing u. $\square$

We next study in detail the singular part of $R_V(\lambda)$, starting from the following statement away from $\lambda = 0$. We use the notation in which tensor product is identified with an operator:

$$(u \otimes v)(f)(x) := u(x) \int_{\mathbb{R}} v(y)f(y)dy. \tag{2.2.19}$$

THEOREM 2.5 (Singular part of $R_V(\lambda)$ in one dimension). *Suppose $m_R(\lambda_0) > 0$, $\lambda_0 \neq 0$.*

1) There exist linearly independent $u_j \in H^2_{\mathrm{loc}}(\mathbb{R})$, $j = 1, \ldots, m_R(\lambda_0)$, such that u_1 is outgoing (see (2.1.8)) and

$$(P_V - \lambda_0^2)u_1 = 0\,, \quad (P_V - \lambda_0^2)u_j = u_{j-1}\,, \tag{2.2.20}$$

$1 < j \leq m_R(\lambda_0)$.

2) The Laurent expansion of $R_V(\lambda)$ near λ_0 is given by

$$R_V(\lambda) = -\sum_{k=1}^{m_R(\lambda_0)} \frac{(P_V - \lambda_0^2)^{k-1}\Pi_{\lambda_0}}{(\lambda^2 - \lambda_0^2)^k} + A(\lambda, \lambda_0)\,, \tag{2.2.21}$$

where $\lambda \mapsto A(\lambda, \lambda_0)$ is holomorphic near λ_0,

$$\Pi_{\lambda_0} = -\frac{1}{2\pi i} \oint_{\lambda_0} R_V(\lambda) 2\lambda d\lambda\,,$$

and

$$(P_V - \lambda_0^2)^{m_R(\lambda_0)}\Pi_{\lambda_0} = 0\,, \quad \operatorname{Ran}\Pi_{\lambda_0} = \operatorname{span}\{u_1, \dots, u_{m_R(\lambda_0)}\}\,. \tag{2.2.22}$$

3) Suppose that $V \in L^\infty(\mathbb{R};\mathbb{R})$. *If* $m_R(\lambda_0) = 1$ *then*

$$\begin{gathered} \Pi_{\lambda_0} = -iu_1 \otimes u_1, \quad (P_V - \lambda_0^2)u_1 = 0, \\ u_1(x) = c_\pm e^{\pm i\lambda_0 x}, \quad \pm x \gg 1. \end{gathered} \tag{2.2.23}$$

Moreover, u_1 *is normalized as follows: for* R *large enough,*

$$-2i\lambda_0 \int_{-R}^{R} u_1(x)^2\, dx + c_+^2 e^{2i\lambda_0 R} + c_-^2 e^{-2i\lambda_0 R} = 1. \tag{2.2.24}$$

REMARK. In Section 2.7 we will find an interpretation of Π_{λ_0}, $\lambda_0 \neq 0$, as a projection. That will explain our sign convention. It will also give an alternative proof of the normalization of u_1 in (2.2.24). Generically resonances have multiplicity 1, and $m_R(0) = 0$.

DEFINITION 2.6. *In the notation of* (2.2.20), u_1 *is called a* resonant state, *and* u_j*'s,* $j > 1$, *generalized resonant states.*

Proof. 1. From the general result about meromorphic continuation in §C.3 we know that for some finite rank operators A_k, $1 \leq k \leq K$,

$$R_V(\lambda) = \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \lambda_0^2)^k} + A(\lambda, \lambda_0)\,, \quad \lambda_0 \neq 0\,,$$

where $A(\bullet, \lambda_0)$ is holomorphic near λ_0, and

$$A_1 = -\Pi_{\lambda_0} := \frac{1}{2\pi i}\oint_{\lambda_0} R_V(\lambda) 2\lambda d\lambda\,.$$

The residue theorem gives

$$\frac{1}{2\pi i}\oint_{\lambda_0} R_V(\zeta) d\zeta = \sum_{k=1}^{K} (-1)^{k-1} \frac{(2k-2)!}{(k-1)!}(2\lambda_0)^{-2k+1} A_k. \tag{2.2.25}$$

2. We now consider the equation $(P_V - \lambda^2)R_V(\lambda) = I$ near $\lambda = \lambda_0$: modulo terms holomorphic near λ_0 we have

$$\begin{aligned} (P_V - \lambda^2)R_V(\lambda) &\equiv \sum_{k=1}^{K} \left(\frac{(P_V - \lambda_0^2)A_k}{(\lambda^2 - \lambda_0^2)^k} - \frac{A_k}{(\lambda^2 - \lambda_0^2)^{k-1}} \right) \\ &\equiv \sum_{k=1}^{K} \frac{(P_V - \lambda_0^2)A_k - A_{k+1}}{(\lambda^2 - \lambda_0^2)^k}\,, \end{aligned}$$

where we use the convention that $A_k = 0$ for $k > K$.

It follows that $A_{k+1} = (P_V - \lambda_0^2)A_k$ which shows that (2.2.21) holds and $(P_V - \lambda_0^2)^K \Pi_{\lambda_0} = 0$.

3. We now need to show the existence of u_j's satisfying (2.2.20) and (2.2.22).

The operator $(P_V - \lambda_0^2)$ commutes with Π_{λ_0} and $(P_V - \lambda_0^2)^K \Pi_{\lambda_0} = 0$. Hence

$$P_V - \lambda_0^2 : \text{Ran } \Pi_{\lambda_0} \to \text{Ran } \Pi_{\lambda_0} ,$$

is nilpotent and we can put it into a Jordan normal form. That means that there exists a basis of $\operatorname{Ran}\Pi_{\lambda_0} \subset H^2_{\text{loc}}(\mathbb{R})$ of the form

$$u_{\ell,j} , \quad 1 \leq \ell \leq L , \quad 1 \leq j \leq k_\ell , \quad \sum_{\ell=1}^{L} k_\ell = K ,$$
$$(P_V - \lambda_0^2) u_{\ell,j} = u_{\ell,j-1} , \quad 1 \leq j \leq k_\ell , \quad u_{\ell,0} := 0 .$$

From (2.2.9) we see that each $u_{\ell,j}$ is a linear combination of functions in $R_0(\lambda_0)(L^2_{\text{comp}}), \ldots, \partial_\lambda^{K-1} R_0(\lambda_0)(L^2_{\text{comp}})$. Then by (2.2.1), for $\pm x \gg 1$, $u_{\ell,j}(x)$ is the product of $e^{\pm i\lambda_0 x}$ with a polynomial in x. Since $(P_V - \lambda_0^2)u_{\ell,1} = 0$, we see that $u_{\ell,1} \in R_0(\lambda)(L^2_{\text{comp}})$, that is $u_{\ell,1}$ is outgoing. But then it is unique up to a multiplicative constant. This shows that $L = 1$ and that $u_j := u_{1,j}$ satisfy (2.2.20). We also see that $K = \dim \operatorname{Ran} \Pi_{\lambda_0}$.

4. Returning to (2.2.25) we see from Step 3 that

$$N_{\lambda_0} := \sum_{k=2}^{K} (-1)^{k-1} \frac{(2k-2)!}{(k-1)!} (2\lambda_0)^{-2k+1} (P_V - \lambda_0^2)^{k-1}$$

is a nilpotent operator $N_{\lambda_0} : \operatorname{Ran} \Pi_{\lambda_0} \to \operatorname{Ran} \Pi_{\lambda_0}$. Hence

$$\begin{aligned} m_R(\lambda_0) &:= \text{rank} \oint_{\lambda_0} R_V(\zeta) d\zeta \\ &= \text{rank} \left(\sum_{k=1}^{K} (-1)^{k-1} \frac{(2k-2)!}{(k-1)!} (2\lambda_0)^{-2k+1} (P_V - \lambda_0^2)^{k-1} \right) \Pi_{\lambda_0} \\ &= \text{rank}(I + N_{\lambda_0}) \Pi_{\lambda_0} = \text{rank}\, \Pi_{\lambda_0} = K. \end{aligned}$$

This gives (2.2.22).

5. It remains to consider the case of real potentials and of resonances with multiplicity 1.

We first note that the construction in Step 1 of the proof of Theorem 2.2 shows that for V real the Schwartz kernel of $R_V(ik)$, $k \gg 1$ is real. Since $R_V(ik)$ is also self-adjoint it follows that $R_V(ik, x, y) = R_V(ik, y, x)$. By analytic continuation this is true at any value of λ. If, near λ_0,

$$R_V(\lambda) = -(\lambda - \lambda_0)^{-1} \Pi_{\lambda_0} + A(\lambda, \lambda_0), \tag{2.2.26}$$

we conclude that the Schwartz kernel of the rank one operator Π_{λ_0} is symmetric in x and y and hence, $\Pi_{\lambda_0} = iu_1 \otimes u_1$ for an outgoing solution of $(P_V - \lambda_0^2)u_1 = 0$.

To prove the normalization condition (2.2.24), fix $R > 0$ and $\chi_\pm \in C^\infty(\mathbb{R})$ such that $\chi_\pm(x) = 1$ for $\pm x \geq R$, $\chi_\pm(x) = 0$ for $\pm x \leq 0$ and such that $u_1(x) = c_\pm e^{\pm i\lambda_0 x}$ on $\operatorname{supp} \chi_\pm$. Put

$$\tilde{u}_1 = u_1 - c_+\chi_+ e^{i\lambda_0 x} - c_-\chi_- e^{-i\lambda_0 x} \in H^2_{\mathrm{comp}}(\mathbb{R}),$$

and define for $\lambda \in \mathbb{C}$ the following outgoing function

$$u_\lambda := \tilde{u}_1 + c_+\chi_+ e^{i\lambda x} + c_-\chi_- e^{-i\lambda x}, \quad u_{\lambda_0} = u_1.$$

Then, given that $(P_V - \lambda_0^2)u_1 = 0$, we find for λ near λ_0,

$$\begin{aligned}(P_V - \lambda^2)u_\lambda &= (P_V - \lambda^2)\tilde{u}_1 - c_+[\partial_x^2, \chi_+]e^{i\lambda x} - c_-[\partial_x^2, \chi_-]e^{-i\lambda x} \\ &= (\lambda - \lambda_0)\big(- 2\lambda_0 \tilde{u}_1 - ic_+[\partial_x^2, \chi_+]xe^{i\lambda_0 x} + ic_-[\partial_x^2, \chi_-]xe^{-i\lambda_0 x}\big) \\ &\qquad\qquad + \mathcal{O}((\lambda - \lambda_0)^2)_{L^2_{\mathrm{comp}}}.\end{aligned}$$

Using the identity (2.2.18) for the outgoing function $u = u_\lambda$ (λ in a punctured neighborhood of λ_0), the expansion (2.2.26) and the fact that $\chi'_\pm$ and $\tilde{u}_1$ are supported inside $(-R, R)$, we see that

$$\begin{aligned}u_\lambda &= R_V(\lambda)(P_V - \lambda^2)u_\lambda \\ &= (\lambda - \lambda_0)^{-1}u_1 \int_{-R}^{R} u_1(x)(P_V - \lambda^2)u_\lambda(x)dx + \mathcal{O}(|\lambda - \lambda_0|).\end{aligned}$$

Inserting $\lambda = \lambda_0$ gives,

$$1 = \int_{-R}^{R} u_1\big(- 2i\lambda_0\tilde{u}_1 + c_+[\partial_x^2, \chi_+]xe^{i\lambda_0 x} = c_-[\partial_x^2, \chi_-]xe^{-i\lambda_0 x}\big)\, dx.$$

From the definition of $\tilde{u}_1$ and the fact that $u_1 = c_\pm e^{\pm i\lambda_0 x}$ on $\operatorname{supp} \chi_\pm$, we then get

$$\begin{aligned}1 = -2i\lambda_0 \int_{-R}^{R} u_1^2\, dx &+ c_+^2 \int_{-R}^{R} (2i\lambda_0\chi_+ + e^{-i\lambda_0 x}[\partial_x^2, \chi_+]e^{i\lambda_0 x}x)e^{2i\lambda_0 x}\, dx \\ &+ c_-^2 \int_{-R}^{R} (2i\lambda_0\chi_- - e^{i\lambda_0 x}[\partial_x^2, \chi_-]e^{-i\lambda_0 x}x)e^{-2i\lambda_0 x}\, dx.\end{aligned}$$

Now, integration by parts, together with the fact that $\chi_\pm = 1$ near $\pm R$ and $\chi_\pm = 0$ near $\mp R$, shows that

$$\begin{aligned}\int_{-R}^{R} (2i\lambda_0\chi_+ + e^{-i\lambda_0 x}[\partial_x^2, \chi_+]e^{i\lambda_0 x}x)e^{2i\lambda_0 x}\, dx &= e^{2i\lambda_0 R}, \\ \int_{-R}^{R} (2i\lambda_0\chi_- - e^{i\lambda_0 x}[\partial_x^2, \chi_-]e^{-i\lambda_0 x}x)e^{-2i\lambda_0 x}\, dx &= e^{-2i\lambda_0 R},\end{aligned}$$

which finishes the proof of (2.2.24). □

We can now provide

Proof of (2.2.12). We need to show that there are no outgoing solutions to $(P_V - \lambda^2)u = 0$ for λ real and non-zero (at $\lambda = 0$ the example of $V = 0$ shows that a pole is possible and the outgoing solution is given by $u = 1$). Since V is real $\bar{u}$ is also a solution. Using the notation of (2.1.8) we calculate the Wronskians:

$$W(u,\bar{u}) := \begin{vmatrix} u & \bar{u} \\ u_x & \bar{u}_x \end{vmatrix} = \begin{cases} 2i\lambda|a_-|^2, & x < -R, \\ -2i\lambda|a_+|^2, & x > R. \end{cases}$$

Since the Wronskian for the equation $-\partial_x^2 + V - \lambda^2$ is constant, this is impossible for $\lambda \neq 0$ and $u \not\equiv 0$. □

For $\lambda_0 = 0$ we restrict our attention to real V's, that is to self-adjoint operators P_V. We need detailed information about the zero resonance only for resonance expansions and trace formulæ. In both cases we will assume self-adjointness of P_V so that we can use the spectral theorem.

THEOREM 2.7 (Singular part of $R_V(\lambda)$ at 0 in one dimension). *Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R})$, $V \not\equiv 0$, and that 0 is a resonance. Then $m_R(0) = 1$ and*

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda} + A(\lambda),$$

where $\lambda \mapsto A(\lambda)$ is holomorphic near 0, and

$$\Pi_0 = -iu_1 \otimes u_1, \quad P_V u_1 = 0; \quad u_1(x) = c_\pm, \quad \pm x \gg 1, \tag{2.2.27}$$

where $c_\pm \in \mathbb{R} \setminus 0$ and $c_+^2 + c_-^2 = 1$.

Proof. 1. Since P_V is self-adjoint, for $\operatorname{Im}\lambda > 0$, $|\lambda| \ll 1$ (so that we avoid possible eigenvalues which are the poles in $\operatorname{Im}\lambda > 0$), the spectral theorem gives

$$\|R_V(\lambda)\|_{L^2\to L^2} = \frac{1}{d(\lambda^2,\mathbb{R}_+)} \leq \frac{1}{|\lambda|\operatorname{Im}\lambda}.$$

This shows that

$$R_V(\lambda) = \frac{A_2}{\lambda^2} + \frac{A_1}{\lambda} + A(\lambda),$$

where A_j are finite rank operators, $L^2_{\text{comp}} \to H^2_{\text{loc}}$. By applying $P_V - \lambda^2$ to $R_V(\lambda)$ we conclude that $P_V A_j = 0$.

2. For $\psi \in C^\infty_{\text{c}}(\mathbb{R})$ and $\rho \in C^\infty_{\text{c}}(\mathbb{R};[0,1])$ and $\operatorname{Im}\lambda > 0$, $|\lambda| \ll 1$,

$$\begin{aligned}\|\rho(A_2 + \lambda A_1 + \lambda^2 A(\lambda))\psi\|_{L^2} &\leq \|(A_2 + \lambda A_1 + \lambda^2 A(\lambda))\psi\|_{L^2} \\ &= \|\lambda^2 R_V(\lambda)\psi\|_{L^2} \leq \frac{|\lambda|^2}{d(\lambda^2,\mathbb{R}_+)}\|\psi\|_{L^2}.\end{aligned}$$

Hence, letting $\lambda = it$, $t \to 0+$, we conclude that

$$\|\rho A_2 \psi\|_{L^2} \leq \|\psi\|_{L^2} .$$

Since $\rho \in C_c^\infty(\mathbb{R}; [0,1])$ is arbitrary we conclude that A_2 is bounded on $L^2(\mathbb{R})$. The range of A_2 consists of solutions to $P_V u = 0$, namely $u = a + bx$, for $x \gg 1$, so we conclude that $a = b = 0$, which implies that $A_2 = 0$.

3. We now show that $\Pi_0 := A_1$ has rank 1. Indeed, when λ is not a pole of R_V, by Theorem 2.4 $R_V(\lambda)(L^2_{\mathrm{comp}})$ consists of outgoing functions. Taking the meromorphic expansion of the outgoing condition (2.2.17) at $\lambda = 0$, we see that the range of Π_0 consists of outgoing functions. Since $P_V \Pi_0 = 0$, $\operatorname{Ran} \Pi_0$ consists of outgoing solutions to the equation $P_V u = 0$. Since the space of such solutions is at most one-dimensional, we see that Π_0 has rank one.

4. Arguing as in the proof of part 3 of Theorem 2.5, we see that

$$\Pi_0 = -i u_1 \otimes u_1,$$

where $u_1 \in H^2_{\mathrm{loc}}$ solves $P_V u_1 = 0$ and is outgoing, that is $u_1(x) = c_\pm$, $\pm x \gg 1$, for some $c_\pm \in \mathbb{C}$. We have $c_\pm \neq 0$ since otherwise u_1, a solution of an ordinary differential equation, would be identically zero. We also get the condition $c_+^2 + c_-^2 = 1$, since the proof of (2.2.24) applies for the zero resonance. Finally, by (2.2.14), we see that Π_0 is antisymmetric: for $\psi, \varphi \in L^2_{\mathrm{comp}}(\mathbb{R})$, $\langle \Pi_0 \psi, \varphi \rangle_{L^2} = -\langle \psi, \Pi_0 \varphi \rangle$. Hence $\bar{u}_1 = \pm u_1$, so either u_1 or $i u_1$ is real-valued. The second option is impossible since then $c_\pm$ are purely imaginary and cannot satisfy $c_+^2 + c_-^2 = 1$; we conclude that u_1, and thus $c_\pm$ are real. □

REMARK. We can construct a real-valued potential $V \in C_c^\infty(\mathbb{R})$ such that R_V has a resonance at $\lambda = 0$ and the constants $c_\pm$ in (2.2.27) are any given numbers in $\mathbb{R} \setminus 0$ satisfying $c_+^2 + c_-^2 = 1$. Indeed, for $c_+ c_- > 0$ consider a function $u \in C^\infty(\mathbb{R}; \mathbb{R})$ which is nonvanishing everywhere and $u(x) = c_\pm$ for $\pm x \gg 1$. Put $V = u''/u$, then $P_V u = 0$ and by Theorem 2.4, zero is a resonance of R_V. Moreover, the function u_1 from Theorem 2.7 is a multiple of u. For $c_+ c_- < 0$ we repeat the same argument, taking $u \in C^\infty(\mathbb{R}; \mathbb{R})$ which is nonvanishing except at $x = 0$, $u(x) = c_\pm$ for $\pm x \gg 1$, and $u(x) = c_+ x$ for $|x| < 1$.

EXAMPLE. We present a natural family of potentials which have resonances of multiplicity 2 for some values in the family. This is illustrated in Figure 2.3.

Consider a potential $V \in C_c^1(\mathbb{R}; \mathbb{R})$, $\operatorname{supp} V \subset [-a, a]$ with the property that $V(x) < -c < 0$ for, say, $x \in (-b, b)$, $0 < b < a$. We then consider a

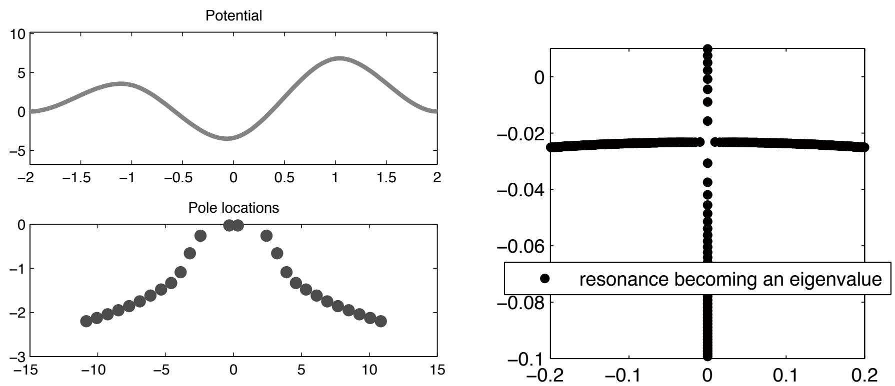


Figure 2.3. We consider resonances for τV where V is shown in the first panel on the left. The resonances for $\tau = 1$ are shown below the graph of V. On the right, we take a large discrete set of τ's, $1 < \tau < 1.12$ and see two continuous families of resonances meeting on $i\mathbb{R}_-$. Pseudospectral effects due to the non-normal nature of R_V at the point of multiplicity two (see Theorem 2.5) make the motion very rapid near at the bifurcation. Hence the double resonance is hard to pinpoint numerically. The specific potential and it resonances were obtained using

```
splinepot(3.4*[0,1,-1,2,0],[-2,-1,0,1,2])
```

see [BZ].

family of potentials τV, $\tau \geq 1$, that is we vary the coupling constant in the Schrödinger operator

$$P_\tau := D_x^2 + \tau V(x) .$$

By applying min-max methods directly (see Theorem B.12) or by using semi-classical Weyl law (with $h^2 = 1/\sqrt{\tau}$ – see for instance [Zw12, Theorem 6.8]) we see that the number of negative eigenvalues of P_τ grows (proportionally to $\sqrt{\tau}$) as τ increases.

The construction of $R_{\tau V}(\lambda)$ also shows that for any R, resonances in $D(0, R)$ are continuous as functions of τ. This means that eigenvalues, that is resonances on $i\mathbb{R}_+$, are obtained, as τ increases, from a continuous family of resonances passing through zero.

In view of the symmetry of resonances with respect to the real axis given in (2.2.13), the simplicity of the resonance at $\lambda = 0$ given in Theorem 2.7 and the absence of resonances on $\mathbb{R} \setminus \{0\}$ (see (2.2.12)) it means that two resonances meet on $i\mathbb{R}_-$ before splitting. One of them will move through 0 to become an eigenvalue. This provides a simple example of a resonance, $\lambda_0 \in i\mathbb{R}_-$ for which $m_R(\lambda_0) = 2$.

The multiplicity of a resonance can also be described using Fredholm determinants – see §B.5. For that we define

$$D(\lambda) := \det(I + VR_0(\lambda)\rho)\,. \tag{2.2.28}$$

where $\rho \in L^\infty_{\mathrm{comp}}$ and $\rho V = V$. This is allowed as $VR_0(\lambda)\rho$ is a (meromorphic) family of operators of trace class.

We note that $D(\lambda)$ is a meromorphic function of λ with a single pole at $\lambda = 0$. The multiplicity of a zero of $D(\lambda)$ is defined in the usual way and we have,

$$m_D(\lambda) := \frac{1}{2\pi i}\oint \frac{D'(\zeta)}{D(\zeta)} d\zeta\,, \tag{2.2.29}$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $D(\lambda)$.

THEOREM 2.8 (Multiplicity of a resonance in one dimension). *The multiplicities defined by (2.2.11) and (2.2.29) are related as follows*

$$m_D(\lambda) = m_R(\lambda)\,, \quad \lambda \in \mathbb{C} \setminus \{0\}\,. \tag{2.2.30}$$

For $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$,

$$m_D(0) = m_R(0) - 1. \tag{2.2.31}$$

Proof. The proof is based on the Gohberg–Sigal theory of residues for meromorphic families of operators reviewed in Section C.4.

1. We start with the case of a pole at zero, assuming that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R},\mathbb{R})$, $V \not\equiv 0$. From (2.2.15),

$$(I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho \tag{2.2.32}$$

and Theorem 2.7 we see that $(I+VR_0(\lambda)\rho)^{-1}$ has a simple pole of rank one at 0 if and only if $R_V(\lambda)$ has a pole at 0. On the other hand $I + VR_0(\lambda)\rho$ has a simple pole of rank one at 0. Hence Theorem C.10 shows that

$$I + VR_0(\lambda)\rho = U_1(\lambda)(Q_0 + \lambda^{-1}Q_{-1} + \lambda Q_1)U_2(\lambda), \tag{2.2.33}$$

where

$$\operatorname{rank} Q_{-1} = 1, \quad \operatorname{rank} Q_1 = m_R(0), \quad Q_jQ_k = \delta_{jk}Q_j,$$

and $U_j(\lambda)$ are invertible and holomorphic. The conclusion then follows from Theorem C.11.

2. Now let $V \in L^\infty_{\mathrm{comp}}(\mathbb{R},\mathbb{C})$, $V \not\equiv 0$ and assume that $m_R(\lambda_0) = 1$, $\lambda_0 \neq 0$. From (2.2.32) we see that the pole of the left hand side is simple with a rank one residue. Theorem C.10 shows that near there exist holomorphic invertible operators $U_j(\lambda)$, such that near λ_0

$$(I + VR_0(\lambda)\rho)^{-1} = U_1(\lambda)(P_0 + (\lambda - \lambda_0)^{-1}P_1)U_2(\lambda),$$

where $P_i P_j = \delta_{ij} P_j$, $\operatorname{rank} P_1 = 1$ and $\operatorname{rank}(I - P_0) < \infty$. (There are no polynomial terms as they produce poles of $I + VR_0(\lambda)\rho$ which is holomorphic near $\lambda_0 \neq 0$.) Theorem C.11 then shows that $m_D(\lambda_0) = 1$.

3. When $m_D(\lambda_0) = 1$, $\lambda_0 \neq 0$, this argument can be reversed using $R_V(\lambda)\rho = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$.

4. The case of $m_R(\lambda_0) > 1$ will be proved in §2.7 using the method of complex scaling. $\square$

2.3. EXPANSIONS OF SCATTERED WAVES

A motivation for the study of resonances is the fact that they describe oscillations and decay of waves for problems on non-compact domains. In this sense they replace eigenvalues and Fourier series expansions. Except for Theorem 2.10 we assume in this section that V is real valued. That is because we need to use methods of spectral theory of self-adjoint operators.

To explain the expansions consider first $P_V = D_x^2 + V$ on $[a, b]$ with Dirichlet (or Neumann) boundary condition. Then the problem

$$\begin{cases} (P_V - \lambda^2)u = 0 & \text{on } (a, b) \\ u(a) = u(b) = 0 \end{cases}$$

has a set of distinct solutions

$$(i\sqrt{-E_k},\, v_k)\,, \quad (\lambda_j, u_j)\,,$$
$$E_N < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty\,,$$
$$\int_a^b |u_j|^2 dx = \int_a^b |v_k|^2 dx = 1\,.$$

We then consider the wave equation

$$\begin{cases} (D_t^2 - P_V)w = 0 & \text{on } \mathbb{R} \times (a, b) \\ w(0, x) = w_0(x)\,, \quad \partial_t w(0, x) = w_1(x) & \text{on } [a, b] \\ w(t, a) = w(t, b) = 0 & \text{on } \mathbb{R}\,. \end{cases}$$

It can be solved using the eigenfunction expansion (Fourier series in the case when $V \equiv 0$):

$$\begin{aligned} w(t, x) = \sum_{k=1}^{N} \cosh(t\sqrt{-E_k}) a_k v_k(x) + \sum_{k=1}^{N} \frac{\sinh(t\sqrt{-E_k})}{\sqrt{-E_k}} b_k v_k(x) \\ + \sum_{j=0}^{\infty} \cos(t\lambda_j) c_j u_j(x) + \sum_{j=0}^{\infty} \frac{\sin(t\lambda_j)}{\lambda_j} d_j u_j(x) \end{aligned} \tag{2.3.1}$$

where

$$a_k = \int_a^b w_0(x)\overline{v_k(x)}dx \,, \qquad b_k = \int_a^b w_1(x)\overline{v_k(x)}(x)$$
$$c_j = \int_a^b w_0(x)\overline{u_j(x)}dx \,, \qquad d_j = \int_a^b w_1(x)\overline{u_j(x)}dx \,.$$

We now give the analogue of (2.3.1) when $[a,b]$ is replaced by $\mathbb{R}$:

THEOREM 2.9 (Resonance expansions of scattering waves in one dimension). *Let $V \in L^\infty(\mathbb{R};\mathbb{R})$ and suppose that $w(t,x)$ is the solution of*

$$\begin{cases} (D_t^2 - P_V)w(t,x) = 0\,, \\ w(0,x) = w_0(x) \in H^1_{\rm comp}(\mathbb{R})\,, \\ \partial_t w(0,x) = w_1(x) \in L^2_{\rm comp}(\mathbb{R})\,. \end{cases} \tag{2.3.2}$$

Then, for any $A > 0$,

$$w(t,x) = \sum_{\operatorname{Im}\lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) + E_A(t)\,, \tag{2.3.3}$$

where the sum is finite,

(2.3.4)

$$\sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x) = -\operatorname{Res}_{\mu=\lambda_j}\left((iR_V(\mu)w_1 + \mu R_V(\mu)w_0)\, e^{-i\mu t}\right)\,,$$
$$(P_V - \lambda_j^2)^{\ell+1} f_{j,\ell} = 0\,,$$

and for any $K > 0$, such that $\operatorname{supp} w_j \subset (-K,K)$*, there exist constants $C_{K,A}$ and $T_{K,A}$*

$$\|E_A(t)\|_{H^2([-K,K])} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{H^1} + \|w_1\|_{L^2}\right)\,, \quad t \geq T_{K,A}\,.$$

REMARKS. 1. For numerical illustrations of this theorem see Figures 1.2 and‘1.1.

2. It may at first seem strange that only exponential appear as contributions of negative eigenvalues (compared to sinh and cosh in (2.3.1)): the exponentially decaying terms are absorbed into the error term $E_A(t)$ when A is small and are “masked” by the resonance expansion when A is large.

3. We notice that the error term $E_A(t)$ is more regular for large times. That corresponds to propagation of singularites: when time is large all singularities leave a compact set. When $V \in C^\infty_{\rm c}(\mathbb{R})$ then an examination of the proof shows that we have the same bound with the right hand side replaced by $\|E_A(t)\|_{H^k([-K,K])}$ for any k.

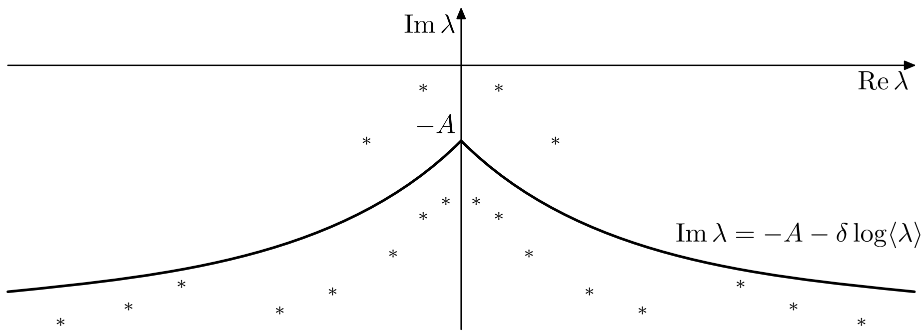


Figure 2.4. The contour used to obtain the resonance expansion.

Before proving Theorem 2.9 we need the existence of a resonance free region and an estimate for the resolvent:

THEOREM 2.10 (Resonance free regions in one dimension). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{C})$. Then for any $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R})$ and any $\delta < 1/|\mathrm{chsupp}\, V|$ (where $\mathrm{chsupp}\, V$ is the convex hull of the support of V) there exist constants A, C, T such that*

$$\|\rho R_V(\lambda)\rho\|_{L^2\to H^j} \leq C|\lambda|^{j-1}\, e^{T(\operatorname{Im}\lambda)_-}\,, \quad j = 0, 1, 2\,, \tag{2.3.5}$$

for

$$\operatorname{Im}\, \lambda \geq -A - \delta \log(1 + |\lambda|)\,, \quad |\lambda| > C_0\,.$$

In particular there are only finitely many resonances in the region

$$\{\lambda \ :\ \operatorname{Im}\lambda \geq -A - \delta \log(1+|\lambda|)\}\,.$$

for any $A > 0$.

Proof. 1. First we modify the estimate (2.2.3) for the free resolvent

$$\|\rho_1 R_0(\lambda)\rho_1\|_{L^2\to L^2} \leq C|\lambda|^{-1} e^{(b-a)|\operatorname{Im}\lambda|}, \quad 0 \leq j \leq 2, \tag{2.3.6}$$

where $\rho_1 \in L^\infty$and $\mathrm{supp}\, \rho_1 \subset [a, b]$.

We then recall (2.2.9):

$$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho_1(I + VR_0(\lambda)\rho_1)^{-1}(1 - VR_0(\lambda)(1-\rho_1))\rho \tag{2.3.7}$$

where we assumed that $\rho = 1$ on $\mathrm{supp}\, V$, and $\rho_1 \in L^\infty_{\mathrm{com}}(\mathbb{R})$ is any function satisfying $\rho_1 V = V$. In particular we can take $\rho_1 = \mathbb{1}_{\mathrm{chsupp} V}$. We see now that (2.3.5) holds for λ at which we can invert $I + VR_0(\lambda)\rho_1$ by Neumann series, and that follows from

$$\|V\rho_1 R_0(\lambda)\rho_1\|_{L^2\to L^2} \leq \frac{1}{2}\,.$$

2. To establish this estimate we put $[a, b] := \mathrm{chsupp} V$ and use (2.3.6) to see that for $\operatorname{Im}\lambda > -A - \delta \log(1 + |\lambda|)$,

$$\begin{aligned}\|V\rho_1 R_0(\lambda)\rho_1\|_{L^2\to L^2} &\leq C\|V\|_{L^\infty} e^{(b-a)|\operatorname{Im}\lambda|}/|\lambda| \\ &\leq C\|V\|_{L^\infty} e^{(A+\delta\log(1+|\lambda|))(b-a)}/|\lambda| \\ &\leq C'\|V\|_{L^\infty}|\lambda|^{-1+\delta(b-a)} \leq 1/2\,,\end{aligned}$$

provided that $\delta < 1/(b-a)$ and $|\lambda| \geq R$.

3. Returning to (2.3.7) we use the bound (2.2.3) for $\rho R_0(\lambda)\rho_1$ and $\rho R_0(\lambda)(1-\rho_1)\rho$ terms to obtain (2.3.5). $\square$

The idea for obtaining the expansion (2.3.3) is to deform the contour in the representation of the wave propagator based on the spectral theorem.

Proof of Theorem 2.9. 1. Let us first consider (2.3.2) with $w_0 \equiv 0$ and $w_1 \in H^2(\mathbb{R})$, $\operatorname{supp} w_1 \subset (-K, K)$.

By the spectral theorem, the solution of (2.3.2) can be written as

$$U(t) := \int_0^\infty \frac{\sin t\lambda}{\lambda}\, dE_\lambda + \sum_{k=1}^{K} \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k\,, \tag{2.3.8}$$

where $\mu_k^2 < 0$ ($\operatorname{Im}\mu_k > 0$) are the negative eigenvalues of P_V with v_k the corresponding real valued normalized eigenfunctions (we use the notation (2.2.19)) and dE_λ is the spectral measure on $(0, \infty)$:

$$P_V = \int_0^\infty \lambda^2 dE_\lambda + \sum_{k=1} \mu_k^2 v_k \otimes v_k, \quad I = \int_0^\infty dE_\lambda + \sum_{k=1} v_k \otimes v_k, \tag{2.3.9}$$

Since for μ near μ_k, $R_V(\mu) = (\mu_k^2 - \mu^2)^{-1}(v_k \otimes v_k) + Q_k(\mu)$, where $\mu \mapsto Q_k(\mu)$ is holomorphic near μ_k, and $\operatorname{Res}_{\mu=\mu_k}(\mu_k^2 - \mu^2)^{-1} = -(2\mu_k)^{-1}$ we have

$$\sum_{k=1}^{K} \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k = \sum_{\pm} \pm \sum_{k=1}^{K} \operatorname{Res}_{\mu=\pm\mu_k} iR_V(\pm\mu)e^{-i\mu t}. \tag{2.3.10}$$

2. Using Stone's Formula recalled in Theorem B.10 we write the spectral measure dE_λ in (2.3.9) in terms of $R_V(\lambda)$:

$$dE_\lambda = \frac{1}{\pi i}(R_V(\lambda) - R_V(-\lambda))\lambda d\lambda\,, \tag{2.3.11}$$

where we noted the change of variables (2.1.7): $z = \lambda^2$, $\pm\lambda = \sqrt{z \pm i0}$.

Hence

$$
\begin{aligned}
w(t) - \sum_{k=1}^{K} \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k &= \frac{1}{\pi i} \int_0^\infty \sin t\lambda (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&= \frac{1}{\pi i} \int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&= \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&= \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\Sigma_\varepsilon} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&\quad + \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\sigma_\varepsilon} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&= \frac{1}{2\pi} \int_{\Sigma_{\varepsilon_0}} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\
&\quad + \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\sigma_\varepsilon} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \,,
\end{aligned}
$$

where Σ_ε is the union of $\mathbb{R} \setminus (-\varepsilon, \varepsilon)$ with the semicircle $(0, \pi) \ni s \mapsto \varepsilon e^{-is}$, oriented counterclockwise, and σ_ε is the same semicircle oriented clockwise. The paremeter ε_0 is chosen so that there are no poles of R_V in $D(0, \varepsilon_0) \setminus \{0\}$.

To justify convergence of the integral over Σ_{ε_0} we use the spectral theorem (see (2.3.11)) which shows that

$$
(R_V(\lambda) - R_V(-\lambda))(D_x^2 + V) = \lambda^2 (R_V(\lambda) - R_V(-\lambda)) \,.
$$

From that we conclude that for $\rho \in C_{\rm c}^\infty$ equal to 1 on $\operatorname{supp} w_1$,

$$
\begin{aligned}
\rho(R_V(\lambda) - R_V(-\lambda))\rho w_1 &= \rho(R_V(\lambda) - R_V(-\lambda)) w_1 \\
&= \rho(R_V(\lambda) - R_V(-\lambda))(1 + \lambda^2)^{-1}(1 + D_x^2 + V) w_1 \,.
\end{aligned} \tag{2.3.12}
$$

Since $\rho(R_V(\lambda) - R_V(-\lambda))\rho = \mathcal{O}(1) : L^2 \to L^2$ this shows that the integral converges in $L^2_{\rm loc}$.

3. The integral over σ_ε converges to 0 as $\varepsilon \to 0+$ in $L^2_{\rm loc}$ unless R_V has a resonance at 0. In that case we use Theorem 2.7 to see that

$$
\begin{aligned}
\frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\sigma_\varepsilon} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda &= \frac{1}{2\pi} \lim_{\varepsilon \to 0+} \int_{\sigma_\varepsilon} e^{-it\lambda} \frac{2\Pi_0 w_1}{\lambda} d\lambda \\
&= \frac{1}{\pi} \int_0^\pi \Pi_0 w_1 (-ids) \\
&= -i\Pi_0 w_1.
\end{aligned}
$$

4. Now let $\rho \in C_{\rm c}^\infty(\mathbb{R})$ satisfy $\rho \equiv 1$ on $[-K, K]$ (recall that we assumed that $\operatorname{supp} w_1 \subset (-K, K)$). Choose R large enough so that all the resonances

with $\operatorname{Im}\lambda > -A - \delta\log(1+|\operatorname{Re}\lambda|)$ are contained in $|\lambda| \leq R$. We deform the contour of integration in the integral over Σ_{ε_0} using the following contours:

$$\Gamma := \{\lambda - i(A + \varepsilon + \delta\log(1+|\operatorname{Re}\lambda|)) \;:\; \lambda \in \mathbb{R}\}\,,$$
$$\Gamma_R := \Gamma \cap \{|\operatorname{Re}\lambda| \leq R\}\,,$$
$$\gamma_R^{\pm} = \{\pm R - it \;:\; 0 \leq t \leq A + \varepsilon + \delta\log(1+R)\}\,, \quad \gamma_R := \gamma_R^+ \cup \gamma_R^-\,,$$
$$\gamma_R^{\infty} = (-\infty, -R) \cup (R, \infty)\,.$$

Here we choose ε and so that there are no resonances on Γ. We also put

$$\Omega_A := \{\lambda \;:\; \operatorname{Im}\lambda \geq -A - \varepsilon - \delta\log(1+|\operatorname{Re}\lambda|)\} \setminus \{0\}$$

and define

$$\Pi_A(t) := -i \sum_{\lambda_j \in \Omega_A} \operatorname{Res}_{\lambda=\lambda_j}(\rho R_V(\lambda)\rho e^{-i\lambda t})\,.$$

In this notation (2.3.10) and the residue theorem show that $U(t)$ defined in (2.3.8) is given by

$$\rho U(t)\rho = -im_R(0)\Pi_0\rho + \Pi_A(t) + E_{\Gamma_R}(t) + E_{\gamma_R}(t) + E_{\gamma_R^{\infty}}(t)\,, \tag{2.3.13}$$

where (with natural orientations)

$$E_\gamma(t) := \frac{1}{2\pi}\int_\gamma e^{-it\lambda}\rho(R_V(\lambda)) - R_V(-\lambda))\rho w_1 d\lambda\,. \tag{2.3.14}$$

We note that the contributions from the poles of $R_V(-\lambda)$ at $\lambda = -\mu_k$ cancel the contributions from $\sin t\mu_k$ – see (2.3.10).

5. Since $\rho \equiv 1$ on $\operatorname{supp} w_1$

$$\|E_{\gamma_R}(t)w_1\|_{H^1}\,,\ \|E_{\gamma_R^{\infty}}(t)w_1\|_{H^1} \to 0\,, \quad R \to \infty. \tag{2.3.15}$$

In fact, using (2.3.5) and (2.3.12) we obtain

$$\|E_{\gamma_R^{\infty}}(t)w_1\|_{H^1} \leq C\int_R^\infty (1+|\lambda|^2)^{-1}\|w_1\|_{H^2}d\lambda \leq \frac{C}{R}\|w_1\|_{H^2}\,,$$

and

$$\|E_{\gamma_R}(t)w_1\|_{H^1} \leq \frac{C}{1+R^2}\|w_1\|_{H^2}\,.$$

Hence (2.3.15) holds for $w_1 \in H^2$, $\operatorname{supp} w_1 \subset (-K, K)$.

6. We now return to (2.3.13) and see that

$$\begin{gathered}\rho U(t)\rho w_1 = -i\Pi_0 w_1 + \Pi_A(t)w_1 + E_\Gamma(t)w_1,\\ \text{for} \quad w_1 \in H^2, \quad \operatorname{supp} w_1 \subset (-K,K),\end{gathered} \tag{2.3.16}$$

where E_Γ is defined using (2.3.10) and Γ defined in Step 4.

We now show that for $t \gg 1$,

$$\|E_\Gamma(t)w_1\|_{H^2} \leq Ce^{-tA}\|w_1\|_{L^2}\,. \tag{2.3.17}$$

For that we use (2.3.5) with $j = 2$ and $|\lambda| > R$, and the assumption that there are no poles of $R_V(\lambda)$ near Γ in a compact set. Thus we obtain:

$$\begin{aligned}\|E_\Gamma(t)w_1\|_{H^2} &\leq Ce^{-At}\int_{\mathbb{R}} e^{-t\delta\log(1+|\lambda|)}e^{\delta T\log(1+|\lambda|)}(1+|\lambda|)\|w_1\|_{L^2}d\lambda \\ &\leq Ce^{-At}\int_{\mathbb{R}}(1+|\lambda|)^{-\delta(t-T)+1}\|w_1\|_{L^2}d\lambda \\ &\leq C'e^{-At}\|w_1\|_{L^2}\,, \quad t > T + 3/\delta\,.\end{aligned}$$

Since $C_c^\infty((-K,K))) \subset H^2$ is dense in $L^2([-K,K])$ the decomposition (2.3.16) and the estimate (2.3.17) are valid for $w_1 \in L^2$, $\operatorname{supp} w_1 \subset [-K,K]$ proving theorem for $w_0 = 0$.

The case of arbitrary $w_0 \in H^1_{\mathrm{comp}}$ and $w_1 \equiv 0$ follows by replacing $\sin t\lambda/\lambda$ by $\cos t\lambda$ in the formula for $w(t,x)$. $\square$

2.4. SCATTERING MATRIX IN DIMENSION ONE

Outside of the support of V, a solution of

$$(P_V - \lambda^2)u = 0 \tag{2.4.1}$$

can be written as a sum of outgoing and incoming terms

$$u(x) = u_{\mathrm{in}}(x) + u_{\mathrm{out}}(x)\,, \quad |x| \geq R\,.$$

Following the conventions described in the beginning of this chapter,

$$u_{\mathrm{in}}(x) = b_{\operatorname{sgn}(x)}e^{-i\lambda|x|}\,, \quad u_{\mathrm{out}}(x) = a_{\operatorname{sgn}(x)}e^{i\lambda|x|}\,, \quad |x| \geq R\,.$$

In scattering we compare the incoming waves with the outgoing ones and mathematically that is captured by the scattering matrix which is defined as follows

$$S \;:\; \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \longmapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}\,. \tag{2.4.2}$$

To describe $S = S(\lambda)$ at frequency λ we need to find solutions to (2.4.1) of the following form:

$$u^\pm(x) = e^{\pm i\lambda x} + v^\pm(x,\lambda) \tag{2.4.3}$$

where $v^\pm(x,\lambda)$ is outgoing. The functions $v^\pm$ are easily found using the outgoing resolvent $R_V(\lambda)$:

$$v^\pm(x,\lambda) = -R_V(\lambda)\left(Ve^{\pm i\lambda x}\right)\,. \tag{2.4.4}$$

This is well defined away from the poles of $R_V(\lambda)$. In particular, in the self-adjoint case that means that $u_\pm$ exist for $\lambda \in \mathbb{R}\setminus 0$.

REMARK. The strange $\pm$ notation (which is different than the $\pm$ notation of (2.4.2)) is motivated by the higher dimensional setting in which $\pm$ is replace by $\omega \in \mathbb{S}^{n-1}$. When $n = 1$, $\mathbb{S}^0 = \{+, -\}$.

If we write

$$v^{\pm}_{\mathrm{sgn}(x)}(\lambda) := e^{-i\lambda|x|} v^{\pm}(x, \lambda) \,, \quad |x| > R \,, \tag{2.4.5}$$

then (2.4.2) shows that

$$\begin{gathered} S(\lambda) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 1 + v^{+}_{+}(\lambda) \\ v^{+}_{-}(\lambda) \end{pmatrix} , \\ S(\lambda) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} v^{-}_{+}(\lambda) \\ 1 + v^{-}_{-}(\lambda) \end{pmatrix} , \end{gathered} \tag{2.4.6}$$

which means that

$$S(\lambda) = I + A(\lambda) \,, \quad A(\lambda) = \begin{pmatrix} v^{+}_{+}(\lambda) & v^{-}_{+}(\lambda) \\ \\ v^{+}_{-}(\lambda) & v^{-}_{-}(\lambda) \end{pmatrix} . \tag{2.4.7}$$

THEOREM 2.11 (**Scattering matrix in terms of the resolvent).** *1) The coefficients of $A(\lambda)$ are meromorphic functions of λ given by the following formulæ:*

$$v^{\omega}_{\theta}(\lambda) = \frac{1}{2i\lambda} \int_{\mathbb{R}} e^{i\lambda(\omega - \theta)x} V(x)(1 - e^{-i\lambda\omega x} R_V(\lambda)(e^{i\lambda\omega \bullet} V)(x)) dx \,, \tag{2.4.8}$$

where $\theta, \omega \in \{+, -\}$.

2) If we put $E_\rho(\lambda) : L^2(\mathbb{R}) \longrightarrow \mathbb{C}^2$,

$$E_\rho(\lambda) u := \left(\int_{\mathbb{R}} e^{-i\lambda x} u(x) \rho(x) dx, \int_{\mathbb{R}} e^{i\lambda x} u(x) \rho(x) dx \right) , \tag{2.4.9}$$

where $\rho \in L^\infty_{\mathrm{comp}}$, $\rho V = V$, then

$$S(\lambda) = I + \frac{1}{2i\lambda} E_\rho(\lambda)(I + V R_0(\lambda)\rho)^{-1} V E_\rho(\bar{\lambda})^* . \tag{2.4.10}$$

Proof. 1. Since $R_V(\lambda) = R_0(\lambda)(I - V R_V(\lambda))$, we have

$$v^{\omega}_{\theta}(\lambda) = -e^{-i\lambda\theta y} R_0(\lambda)(I - V R_V(\lambda))(V e^{i\omega\lambda \bullet})(y) \,, \quad \theta y > R \,,$$

where $\operatorname{supp} V \subset [-R, R]$.

Using the explicit formula for $R_0(\lambda)$ we then notice that for f with $\operatorname{supp} f \subset [-R, R]$,

$$R_0(\lambda) f(y) = -\frac{1}{2i\lambda} e^{i\theta\lambda y} \int_{\mathbb{R}} e^{-i\theta\lambda x} f(x) dx \,, \quad \theta y > R \,.$$

Combining the two expressions we obtain (2.4.8).

2. Now we use $R_V(\lambda)V = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}V$ (see (2.2.9)) and $(I + VR_0(\lambda)\rho)^{-1}V = \rho(I + VR_0(\lambda)\rho)^{-1}V$, so that

$$v_\theta^\omega(\lambda) = -e^{-i\lambda\theta y}R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}(Ve^{i\omega\lambda\bullet}), \quad \theta y > R, \tag{2.4.11}$$

and (2.4.10) follows. $\square$

INTERPRETATION. 1. We have $v_+^+(\lambda) = v_-^-(\lambda)$. This can be seen by comparing the values of the Wronskian:

$$W(u^+, u^-) := \begin{vmatrix} u^+ & u^- \\ \partial_x u^+ & \partial_x u^- \end{vmatrix} = \begin{cases} -2i\lambda(1 + v_-^-), & x < -R, \\ -2i\lambda(1 + v_+^+), & x > R. \end{cases}$$

Since W is constant, it follows that for $\lambda \neq 0$, $v_+^+(\lambda) = v_-^-(\lambda)$.

2. The coefficients $v_\theta^\omega(\lambda)$ have important physical interpretations:

$$\begin{aligned} t(\lambda) &= 1 + v_\pm^\pm(\lambda) \ \text{ is the transmission coefficient,} \\ r_+(\lambda) &= v_+^-(\lambda) \ \text{ is the right reflection coefficient,} \\ r_-(\lambda) &= v_-^+(\lambda) \ \text{ is the left reflection coefficient.} \end{aligned} \tag{2.4.12}$$

This interpretation follows from comparing (2.4.2) and (2.4.6).

3. Changing λ to $-\lambda$ in the definition of $S(\lambda)$ shows that when $S(\lambda)$ and $S(-\lambda)$ both exist then

$$S(-\lambda) = JS(\lambda)^{-1}J, \quad J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.13}$$

4. When V is real and $\lambda \in \mathbb{R} \setminus \{0\}$ and u solves (2.4.1) we can also take Wronskians of u and $\bar{u}$:

$$W(u, \bar{u}) = \begin{cases} i\lambda(|a_-|^2 - |b_+|^2), & x < -R \\ i\lambda(|b_-|^2 - |a_+|^2), & x > R, \end{cases}$$

which means S given by (2.4.2) is unitary. Hence we obtain *unitarity of the scattering matrix*: $S(\lambda)^* = S(\lambda)^{-1}$. A meromorphic continuation of this equality gives

$$V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \implies S(\bar{\lambda})^* = S(\lambda)^{-1}, \quad \lambda \in \mathbb{C}. \tag{2.4.14}$$

This implies that $v_\theta^\omega(\lambda)$ are holomorphic for $\lambda \in \mathbb{R}$.

REMARK. As already remarked above we should think of $\pm$ as the element of the "sphere", $\mathbb{S}^0$, in one dimensional space. As we will see the same formula is valid in dimension n with $\theta, \omega \in \mathbb{S}^{n-1}$. The scattering "matrix" is then given as the sum of the idenity and an operator defined by an integral kernel (2.4.8) in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Of course the interpretation of reflected and transmited waves is then less clear.

The representation given in Theorem 2.11 gives us important estimates for the coefficients of the scattering matrix in the physical half plane, $\operatorname{Im}\lambda \geq 0$:

THEOREM 2.12 (Estimates on the scattering matrix). *For*

$$\operatorname{Im}\lambda \geq 0\,, \quad |\lambda| \geq C_0\,,$$

we have

$$\|e^{\mp i\lambda x} R_V(\lambda) V e^{\pm i\lambda \bullet}\|_{L^2\to L^\infty} \leq \frac{C_1}{|\lambda|}. \tag{2.4.15}$$

Consequently, (2.4.8) *implies that for* $\operatorname{Im}\lambda \geq 0$, $|\lambda| \geq C_0$,

$$\begin{aligned} v_+^+(\lambda) &= \frac{1}{2i\lambda}\left(\widehat{V}(0) + \mathcal{O}(1/|\lambda|)\right)\,, \\ v_-^-(\lambda) &= \frac{1}{2i\lambda}\left(\widehat{V}(0) + \mathcal{O}(1/|\lambda|)\right)\,. \end{aligned} \tag{2.4.16}$$

REMARK. The estimate (2.4.15) implies that $\|e^{\mp i\lambda x} R_V(\lambda)(V e^{\pm i\lambda\bullet})\|_{L^\infty} \leq C_1/|\lambda|$ for $\operatorname{Im}\lambda \geq 0$, $|\lambda| \geq C_0$: we simply apply the operator to $\mathbb{1}_{\operatorname{supp} V}$. That particular estimate will be used to obtain (2.4.16).

Proof. 1. Let

$$R_0^\omega(\lambda) := e^{-i\lambda\omega x} R_0(\lambda) e^{i\lambda\omega\bullet}.$$

Its Schwartz kernel given by

$$R_0^\omega(\lambda, x, y) := e^{-i\lambda\omega x} R_0(\lambda, x, y) e^{i\lambda\omega y} = \frac{i}{2\lambda} e^{i\lambda(|x-y|-\omega(x-y))}. \tag{2.4.17}$$

As $|x-y| - \omega(x-y) \geq 0$, (2.4.17) shows that for $\operatorname{Im}\lambda \geq 0$ we have

$$\|V R_0^\omega(\lambda)\rho\|_{L^2\to L^2} \leq C/|\lambda|.$$

Hence the Neumann series for $(I + V R_0^\omega(\lambda)\rho)^{-1}$ converges for $\operatorname{Im}\lambda \geq 0$, $|\lambda| > C_0$. Similarly,

$$R_0^\omega(\lambda)\rho = \mathcal{O}(1/|\lambda|) : L^2 \to L^\infty\,, \quad \operatorname{Im}\lambda \geq 0\,.$$

2. Recalling (2.2.9),

$$R_V(\lambda)V = R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1} V$$

we see

$$\begin{aligned} e^{-i\omega\lambda x} R_V(\lambda) V e^{i\omega\lambda\bullet} &= e^{-i\omega\lambda x} R_0(\lambda) e^{i\omega\lambda\bullet}\left(e^{-i\omega\lambda\bullet}(I + V R_0(\lambda)\rho)^{-1} e^{i\omega\lambda\bullet}\right) V \\ &= R_0^\omega(\lambda)(I + V R_0^\omega(\lambda)\rho)^{-1} V, \end{aligned}$$

where for $\operatorname{Im}\lambda > 0$ and $|\lambda| > C_0$ the convergence is guaranteed by estimates in Step 1. The same estimates then imply (2.4.15). The asymptotic formulas (2.4.16) then follow from the expression for v_ω^ω in (2.4.8). $\square$

REMARKS. 1. We should stress that unlike many results in this chapter the statements about the scattering matrix for λ real remain valid for real-valued potentials satisfying very weak decay conditions – see [**Me85**] for one account of that and for references.

2. The scattering matrix can also be described in the following way:

$$S(\lambda) = \begin{pmatrix} \frac{i\lambda}{\widehat{X}(\lambda)} & \frac{\widehat{Y}(\lambda)}{\widehat{X}(\lambda)} \\ \frac{\widehat{Y}(-\lambda)}{\widehat{X}(\lambda)} & \frac{i\lambda}{\widehat{X}(\lambda)} \end{pmatrix}, \tag{2.4.18}$$

where X and Y are naturally defined distributions, compactly supported in the case when V is compactly supported – see Fig. 2.5. We do not use this representation here but it can be very helpful in the study of resonances (which are then the zeros of $\widehat{X}$) and also of inverse problems – see Melin [**Me85**] and [**TZ01**],[**Zw87**],[**Zw01**].

The determinant of the scattering matrix is related to the determinant defined by (2.2.28):

THEOREM 2.13 (A determinant identity). *For $V, \rho \in L^\infty_{\rm comp}$ satisfying $\rho V = V$, let*

$$D(\lambda) := \det(I + VR_0(\lambda)\rho) .$$

Then

$$\frac{D(-\lambda)}{D(\lambda)} = \det S(\lambda) , \tag{2.4.19}$$

where $S(\lambda)$ is the scattering matrix.

Proof. 1. In the notation of (2.4.9) we write

$$\rho(R_0(\lambda) - R_0(-\lambda))\rho = \frac{i}{2\lambda} E_\rho(\bar{\lambda})^* E_\rho(\lambda) , \tag{2.4.20}$$

that is

$$E_\rho(\bar{\lambda})^* E_\rho(\lambda) = \rho(x)e^{i\lambda x} \otimes \rho(y)e^{-i\lambda y} + \rho(x)e^{-i\lambda x} \otimes \rho(y)e^{i\lambda y} . \tag{2.4.21}$$

2. We now write

$$\begin{aligned} &(I + VR_0(-\lambda)\rho) = \\ &\quad (I + VR_0(\lambda)\rho) \left(I - (I + VR_0(\lambda)\rho)^{-1} V \left(R_0(\lambda) - R_0(-\lambda)\right)\rho\right) = \\ &\quad (I + VR_0(\lambda)\rho)(I - (I + VR_0(\lambda)\rho)^{-1}(iVE_\rho(\bar{\lambda})^* E_\rho(\lambda)/2\lambda)) = \\ &\quad (I + VR_0(\lambda)\rho)\,(I + T(\lambda)) , \end{aligned}$$

where we defined

$$T(\lambda) := \frac{1}{2i\lambda}(I + VR_0(\lambda)\rho)^{-1} VE_\rho(\bar{\lambda})^* E_\rho(\lambda) . \tag{2.4.22}$$

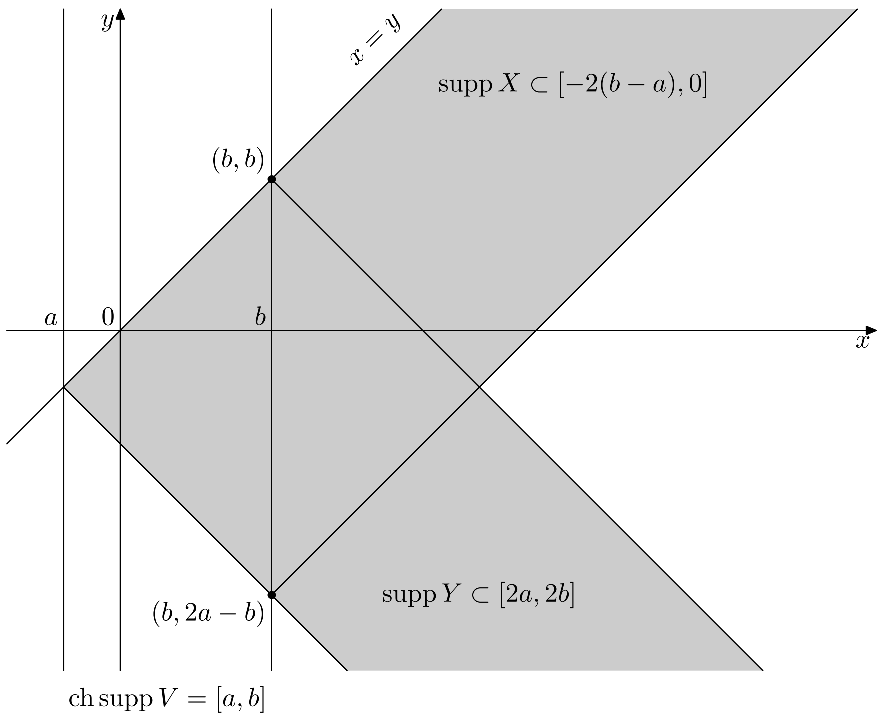


Figure 2.5. The distributions X and Y appearing in (2.4.18) are defined as follows: suppose that the convex hull of $\operatorname{supp} V$ is given by $[a,b]$ and solve $(\partial_y^2 - \partial_x^2 + V(x))A_-(x,y) = 0$ with $A_-(x,y) = \delta(x-y)$, $x < a$. The support of $\partial_y A_-(x,y)$ is shaded in the figure ($A_-(x,y)$ could be equal to a non-zero constant in the cone on the left.) For $x > b$, $\partial_y A_-(x,y)$ solves the free wave equation and hence $\partial_y A_-(x,y) = X(x-y) + Y(x+y)$ where $X, Y \in \mathcal{D}'(\mathbb{R})$ and $\operatorname{supp} X \subset [-2(b-a), 0]$, $\operatorname{supp} Y \subset [2a, 2b]$. The original proof of Theorem 2.16 in [**Zw87**] proceeded by showing that the convex hull of $\operatorname{supp} X$ is $[-2(b-a), 0]$ and then applying a theorem of Titchmarsh [**HöII**, Theorem 16.1.9] on the counting of zeros of Fourier transforms. In Melin's treatment [**Me85**] (inspired by Faddeev, Gelfand–Levitan and Deift–Trubowitz) of the inverse problem on the line, $A_-(x,y)$ is the Schwartz kernel of an operator intertwining $-\partial_x^2 + V(x)$ and $-\partial_x^2$.

We note that $T(\lambda) : L^2 \to L^2$ is a finite rank operator.

3. Hence to prove (2.4.19) we need to show that

$$\det_{\mathbb{C}^2} S(\lambda) = \det_{L^2}(I + T(\lambda)) \,, \tag{2.4.23}$$

Putting

$$A := (2i\lambda)^{-1}(I + VR_0(\lambda)\rho)^{-1})VE(\bar{\lambda})^*, \quad B = E(\lambda)$$

we have $T(\lambda) = AB$. On the hand (2.4.10) shows that $S(\lambda) = I + BA$. Hence (2.4.23) follows from (B.5.18): $\det(I + AB) = \det(I + BA)$. $\square$

The multiplicity of a pole of $S(\lambda)$ and $S(\lambda)^{-1}$ is defined using the determinant of the scattering matrix. The poles of the scattering matrix are sometimes called *scattering poles*. Theorem 2.13 combined with Theorem 2.8 gives

THEOREM 2.14 (Multiplicities of scattering poles in one dimension). *The multiplicity of a scattering pole defined by*

$$m_S(\lambda) = -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda S(\zeta)^{-1} \partial_\zeta S(\zeta) d\zeta \,, \tag{2.4.24}$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $\det S(\lambda)$*, is related to the multiplicity of a scattering resonance* (2.2.11) *as follows:*

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda) \,. \tag{2.4.25}$$

The scattering matrix is always holomorphic and unitary at zero, and thus does not 'see' the resonance at zero directly. However, we have the following

THEOREM 2.15 (Scattering matrix at zero). *For $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$ we have*

1) If 0 is not a pole of R_V, then $S(0) = -J$, where J is defined in (2.4.13).

2) If 0 is a pole of R_V and $c_\pm \in \mathbb{R}\backslash 0$, $c_+^2 + c_-^2 = 1$, are defined in Theorem 2.7, then

$$S(0) = \begin{pmatrix} 2c_- c_+ & c_+^2 - c_-^2 \\ c_-^2 - c_+^2 & 2c_- c_+ \end{pmatrix}.$$

Proof. 1. Suppose first that 0 is not a resonance. Then $v^\pm(x, 0)$, as defined by (2.4.4), is outgoing by Theorem 2.4. However, the function 1 is also outgoing at $\lambda = 0$, which means that $u^\pm(x, 0)$ are outgoing solutions to the equation $P_V u = 0$. By another application of Theorem 2.4, we see that $u^\pm = 0$. Then $v^\pm = -1$, and it remains to use (2.4.5).

2. Suppose now that 0 is a resonance. By Theorem 2.7,

$$R_V(\lambda) = A(\lambda) + \frac{i}{\lambda} u_1 \otimes u_1,$$

where $A(\lambda)$ is holomorphic near 0, $P_V u_1 = 0$, and $u_1(x) = c_\pm$ for $\pm x \gg 1$. We have

$$\int u_1(x) V(x) \, dx = \int u_1''(x) \, dx = 0$$

and thus, by (2.4.4), $v^\pm(x,\lambda)$ is holomorphic at $\lambda = 0$. Therefore, $v_+^\pm(\lambda), v_-^\pm(\lambda)$ are holomorphic at $\lambda = 0$, and for λ near 0, we have

$$u^\pm(x,\lambda) = \begin{cases} e^{\pm i\lambda x} + v_+^\pm(\lambda)e^{i\lambda x}, & x \gg 1, \\ e^{\pm i\lambda x} + v_-^\pm(\lambda)e^{-i\lambda x}, & -x \gg 1. \end{cases} \tag{2.4.26}$$

Since $P_V u^\pm(x,0) = 0$, we see that $u^\pm(x,0)$ are multiples of $u_1(x)$. Therefore,

$$c_-(1 + v_+^\pm(0)) = c_+(1 + v_-^\pm(0)). \tag{2.4.27}$$

Next, differentiating (2.4.26) in λ, we find

$$\partial_\lambda u^\pm(x,0) = \begin{cases} (\pm 1 + v_+^\pm(0))ix + \partial_\lambda v_+^\pm(0), & x \gg 1, \\ (\pm 1 - v_-^\pm(0))ix + \partial_\lambda v_-^\pm(0), & -x \gg 1. \end{cases}$$

However, since $(P_V - \lambda^2)u^\pm(x,\lambda) = 0$, we have $P_V \partial_\lambda u^\pm(x,0) = 0$. Therefore, the Wronskians $W(u_1, \partial_\lambda u^\pm)$ are constant. We compute

$$W(\partial_\lambda u^\pm, u_1) = \begin{cases} ic_+(\pm 1 + v_+^\pm(0)), & x \gg 1; \\ ic_-(\pm 1 - v_-^\pm(0)), & -x \gg 1. \end{cases}$$

Therefore,

$$\begin{aligned} c_+(1 + v_+^+(0)) &= c_-(1 - v_-^+(0)), \\ c_+(-1 + v_+^-(0)) &= c_-(-1 - v_-^-(0)), \end{aligned}$$

Combining these equations with (2.4.27), we obtain the formula for $S(0)$. $\square$

Theorem 2.15 implies the following characterization of the zero resonance in terms of the scattering matrix:

$$\det S(0) = (-1)^{m_R(0)+1}. \tag{2.4.28}$$

2.5. ASYMPTOTICS FOR THE COUNTING FUNCTION

In this section we will prove a Weyl law for the number of scattering resonances of a compactly supported, bounded, complex valued potential. In higher dimensions only weaker results are known and for the existence of resonances we need to assume that the potential is real valued: as we will see in Chapter 3 a complex valued compactly supported potential in three dimensions may have no resonances at all.

THEOREM 2.16 (Asymptotics for the number of resonances). *Suppose that $V \in L^\infty_{\rm com}(\mathbb{R};\mathbb{C})$. Then*

$$\sum \{m_R(\lambda) : |\lambda| \le r\} = \frac{2|\mathrm{chsupp} V|}{\pi} r(1 + o(1)), \tag{2.5.1}$$

as $r \longrightarrow \infty$. Here chsupp *is the convex hull of the support.*

In addition, for any $\varepsilon > 0$,

$$\sum\{m_R(\lambda) : |\lambda| \leq r \,, |\operatorname{Im}\lambda| \geq \varepsilon|\operatorname{Re}\lambda|\} = o(r)\,, \tag{2.5.2}$$

as $r \longrightarrow \infty$.

MOTIVATION. To obtain the asymptotic formula we use the theory of entire functions of finite type – see §D.2. One standard application of that theory is a proof of Titchmarsh's theorem stating that if $g \in L^1_{\text{comp}}(\mathbb{R})$ then the number of zeros of $\hat{g}$ in $D(0, r)$ is equal to $\pi^{-1}|\operatorname{chsupp} g|r(1 + o(r))$, as $r \to \infty$. This follows Theorem D.2, (D.2.9), and from the Paley-Wiener Theorem which shows that the following bound is optimal

$$|\hat{g}(\lambda)| \leq Ce^{a(\operatorname{Im}\lambda)_- + b(\operatorname{Im}\lambda)_+}, \quad \operatorname{chsupp} g = [a, b].$$

We will apply these methods to the determinant $D(\lambda) = \det(I + VR_0(\lambda)\rho)$. Using the formula (2.4.19) the growth of $D(\lambda)$ will be related to the growth of the reflection coefficients $v^{\pm}_{\mp}(\lambda)$ for $\operatorname{Im}\lambda > 0$. Formula (2.4.8) shows that the reflection coefficients can be considered as *nonlinear* Fourier transforms of V: the linearizations of $v^{\pm}_{\mp}$ at $V = 0$ are given by $\widehat{V}(\mp 2\lambda)/2i\lambda$. Hence the optimal growth of $v^{\pm}_{\mp}$, and consequently of $D(\lambda)$ can be related to the support of V.

Before proving the theorem we need some estimates for the determinant $D(\lambda) = \det(I + VR_0(\lambda)\rho)$. These estimates will also be useful in the section on trace formulas.

THEOREM 2.17 (Determinant estimates). *There exist constants* C_j, $j = 0, 1, 2, 3$ *such that the determinant* $D(\lambda)$ *defined by* (2.2.28) *satisfies*

$$\begin{gathered}\lim_{t \to +\infty} D(e^{i\theta}t) = 1\,, \quad 0 < \theta < \pi\,, \\ |D(\lambda)| \leq C_1(1 + 1/|\lambda|)\,, \quad \operatorname{Im}\lambda \geq 0\,, \\ |D(\lambda)| \geq C_2, \quad \operatorname{Im}\lambda > 0, \ |\lambda| \geq C_0, \\ |\lambda D(\lambda)| \leq C_3(1 + |\lambda|)\exp(2|\operatorname{chsupp} V|(\operatorname{Im}\lambda)_-)\,, \quad \lambda \in \mathbb{C}\,,\end{gathered} \tag{2.5.3}$$

where $\operatorname{chsupp} V$ *is the convex hull of the support of* V.

In addition, if $-\mu_K^2 < -\mu_{K-1}^2 < \cdots - \mu_1^2 < 0$, $\mu_j > 0$, *are the eigenvalues of* P_V *then the scattering matrix satisfies*

$$\left|\prod_{k=1}^{K} \frac{\lambda - i\mu_k}{\lambda + i\mu_k} \det S(\lambda)\right| \leq e^{2|\operatorname{chsupp} V|\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{2.5.4}$$

REMARK. The estimate (2.5.4) will not be needed in this section and is a by-product of the proof of the estimates on $D(\lambda)$. It will be useful in §2.6.

We start with the following lemma concerning trace class norms of the free cut-off resolvent:

LEMMA 2.18. *Suppose that* $\rho \in L^\infty(\mathbb{R})$ *and* $\operatorname{supp} \rho \subset [-L, L]$. *Then*

$$
\begin{aligned}
&\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} \leq \frac{C \exp(2L(\operatorname{Im}\lambda)_-)}{|\operatorname{Im}\lambda|}, \quad \operatorname{Im}\lambda \neq 0, \\
&\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} \leq C + \frac{C}{|\lambda|}, \quad \lambda \in \mathbb{R}.
\end{aligned} \tag{2.5.5}
$$

Proof. 1. We start with the case of $\operatorname{Im}\lambda > 0$. In that case, as operators on L^2,

$$
R_0(\lambda) = (D_x^2 - \lambda^2)^{-1} = (D_x - \lambda)^{-1}(D_x + \lambda)^{-1}.
$$

The explicit formulae for the Schwartz kernels are given by

$$
(D_x \pm \lambda)^{-1}(x, y) = \pm i e^{\pm i\lambda(x-y)} H(\pm(x-y)), \quad \operatorname{Im}\lambda > 0,
$$

where $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$.

From this we see that

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} |\rho(x)(D_x \pm \lambda)^{-1}(x, y)|^2 dx dy \leq \frac{2L\|\rho\|_\infty^2}{\operatorname{Im}\lambda}.
$$

Hence

$$
\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1}^2 \leq \|\rho(D_x + \lambda)^{-1}\|_{\mathcal{L}_2} \|(D_x - \lambda)^{-1}\rho\|_{\mathcal{L}_2} \leq \frac{C}{(\operatorname{Im}\lambda)^2}.
$$

2. To prove the estimate for $\operatorname{Im}\lambda \leq 0$ we use (2.4.20), (2.4.21) and the fact that

$$
\|u \otimes v\|_{\mathcal{L}_1} = \|u\|_{L^2} \|v\|_{L^2}. \tag{2.5.6}
$$

This gives,

$$
\begin{aligned}
\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} &\leq \|\rho R_0(-\lambda)\rho\|_{\mathcal{L}_1} + \frac{1}{|\lambda|} \|\rho e^{i\lambda \bullet}\|_{L^2} \|\rho e^{-i\lambda \bullet}\|_{L^2} \\
&\leq \frac{C}{|\operatorname{Im}\lambda|} + \frac{C e^{-2L \operatorname{Im}\lambda}}{|\lambda|} \\
&\leq \frac{2C e^{2L(\operatorname{Im}\lambda)_-}}{|\operatorname{Im}\lambda|}.
\end{aligned}
$$

3. To establish the second inequality in (2.5.5) we first use Theorem 2.1 and Proposition B.21 to see that

$$
\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} \leq \|\rho R_0(\lambda)\rho\|_{L^2 \to H^2} \leq C(|\lambda| + 1/|\lambda|). \tag{2.5.7}
$$

To see the improvement we recall (B.4.3):

$$\|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} = \max_{\{e_k\},\{f_\ell\}} \sum_{\ell,k} \langle \rho R_0(\lambda)\rho e_k, f_\ell\rangle,$$

where the maximum is taken over all pairs of orthornormal bases of $L^2(\mathbb{R})$. Hence the second inequality in (2.5.5) follows from

$$|h(\lambda)| \leq C_0 + C_0/|\lambda|, \quad h(\lambda) := \sum_{k,\ell} \langle \rho R_0(\lambda)\rho e_k, f_\ell\rangle, \tag{2.5.8}$$

as long as C_0 is independent of the choice of the bases.

4. Consider

$$h_1(\lambda) := \frac{\lambda h(\lambda)}{\lambda + 2i}.$$

Then $h_1(\lambda)$ is holomorphic in the strip $|\operatorname{Im}\lambda| < 2$. The estimate (2.5.7) shows that

$$|h_1(\lambda)| \leq C_1 + C_1|\lambda|, \quad |\operatorname{Im}\lambda| \leq 1,$$

and the first estimate in (2.5.5) (established in Steps 1 and 2 of the proof) shows that

$$|h_1(\lambda)| \leq C_2, \quad |\operatorname{Im}\lambda| = 1.$$

The inequality (2.5.8) then follows from the three line theorem (see §D.1).

5. Finally we show the lower bounds on $D(\lambda)$ for $\operatorname{Im}\lambda \geq 0$, and $|\lambda|$ large. Since $VR_0(\lambda)\rho = \mathcal{O}(1/|\lambda|)_{L^2\to L^2}$ (Theorem 2.1), $(I+VR_0(\lambda)\rho)^{-1}$ exists and is uniformly bounded on L^2 in that range of λ's. Hence

$$\begin{aligned} D(\lambda)^{-1} &= \det((I + VR_0(\lambda)\rho)^{-1}) \\ &= \det(I - VR_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}) \\ &\leq \exp\left(\|V\|_\infty \|\rho R_0(\lambda)\rho\|_{\mathcal{L}_1} \|(I + VR_0(\lambda)\rho)^{-1}\|\right) \\ &= \mathcal{O}(1), \end{aligned}$$

which gives the lower bound. $\square$

Proof of Theorem 2.17. 1. To study $D(e^{i\theta}t)$ we use (B.5.20) with $A = 0$ and $B = VR_0(e^{i\theta}t)\rho$:

$$|D(e^{i\theta}t) - 1| \leq \|V\|_\infty \|\rho R_0(e^{i\theta}t)\rho\|_{\mathcal{L}_1} e^{2+\|V\|_\infty\|\rho R_0(e^{i\theta}t)\rho\|_{\mathcal{L}_1}}$$

The first estimate in (2.5.5) shows that the right hand side goes to 0 as $t \to +\infty$ for $0 < \theta < \pi$.

2. We now write

$$VR_0(\lambda)\rho = A(\lambda) + B(\lambda), \quad A(\lambda) := \frac{i}{2\lambda} V \otimes \rho$$

where, $B(\lambda)$ is holomorphic and, by applying (2.5.6) and the estimates (2.5.5),

$$\|B(\lambda)\|_{\mathcal{L}_1} \leq C, \quad \operatorname{Im}\lambda \geq 0.$$

From (B.5.19) we see

(2.5.9)
$$\begin{aligned}|D(\lambda)| &= |\det(I + A(\lambda) + B(\lambda))| \leq \det(I + |A(\lambda)|)|\det(I + |B(\lambda)|) \\ &\leq e^{\|B(\lambda)\|_{\mathcal{L}_1}} \det(I + |A(\lambda)|) \leq C \det(I + |A(\lambda)|).\end{aligned}$$

Here (using the notation (2.2.19))

$$|A(\lambda)| := (A(\lambda)^* A(\lambda))^{\frac{1}{2}} = \frac{1}{2|\lambda|} \frac{\|V\|_2}{\|\rho\|_2} \bar{\rho} \otimes \rho$$

and hence (see Exercise B.1)

$$\det(I + |A(\lambda)|) = 1 + \frac{\|V\|_2 \|\rho\|_2}{2|\lambda|}.$$

Returning to (2.5.9) we obtain

$$D(\lambda) = \mathcal{O}(1) + \mathcal{O}(1/|\lambda|)\,, \quad \operatorname{Im}\lambda \geq 0\,. \tag{2.5.10}$$

3. For estimates in $\operatorname{Im}\lambda \leq 0$ we use Theorem 2.13:

$$D(\lambda) = \det S(-\lambda) D(-\lambda)\,. \tag{2.5.11}$$

Hence we need to estimate $\det S(-\lambda)$ for $\operatorname{Im}\lambda \leq 0$.

Using (2.4.7) and Theorem 2.12 we see that for $\operatorname{Im}\lambda \leq 0$, $|\lambda| \geq C_0$,

$$\det S(-\lambda) = 1 - v_+^-(-\lambda) v_-^+(-\lambda) + \mathcal{O}(1/|\lambda|)\,. \tag{2.5.12}$$

Now, (2.4.8) and (2.4.15) show that for $\operatorname{Im}\lambda \leq 0$, $|\lambda| \geq C_0$,

$$\begin{aligned}|v_+^-(-\lambda) v_-^+(-\lambda)| &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\operatorname{Im}\lambda(x-y)} |V(x)||V(y)| dx dy \\ &\leq C' e^{-2\operatorname{Im}\lambda |\operatorname{chsupp} V|},\end{aligned}$$

which shows that

$$|\det S(-\lambda)| \leq C e^{2|\operatorname{chsupp} V||\operatorname{Im}\lambda|}, \quad \operatorname{Im}\lambda \leq 0, \quad |\lambda| \geq C_0. \tag{2.5.13}$$

From (2.5.10) and (2.5.11) we conclude that

$$|\lambda D(\lambda)| \leq C(1 + |\lambda|) e^{2|\operatorname{chsupp} V|(\operatorname{Im}\lambda)_-}, \quad |\lambda| \geq C_0,$$

Since $\lambda D(\lambda)$ is holomorphic it follows the estimate is valid everywhere. That completes the proof of (2.5.3).

4. To establish (2.5.4) we see that (2.5.13) and unitarity of $S(\lambda)$ for $\lambda \in \mathbb{R}$ shows that

$$g(\lambda) := e^{2i\lambda|\operatorname{chsupp} V|} \prod_{k=1}^{K} \frac{\lambda - i\mu_k}{\lambda + i\mu_k} \det S(\lambda),$$

satisfies

$$|g(\lambda)| = \begin{cases} 1 & \lambda \in \mathbb{R}, \\ \mathcal{O}(1) & \operatorname{Im}\lambda \geq 0, \ |\lambda| \geq C_0. \end{cases}$$

Using Theorem 2.14 we see that product over μ_k's removed the possible singularities of $\det S(\lambda)$. This means that $g(\lambda)$ is holomorphic for $\operatorname{Im}\lambda \geq 0$. But then the Phragmén–Lindelöf theorem (see for instance [**Ti39**, §5.61] for the particular case needed here) shows that $|g(\lambda)| \leq 1$ for $\operatorname{Im}\lambda \geq 0$ which is (2.5.4). ☐

Proof of Theorem 2.16. 1. Using Theorem 2.8 we will prove the theorem by obtaining an asymptotic formula for the number of zeros of the entire function

$$f(\lambda) := \lambda D(\lambda)\,,$$

where $D(\lambda)$ is defined in (2.2.28). The factor λ removes the pole at $\lambda = 0$ – see the second estimate in (2.5.3).

2. By rescaling and translation we can assume that

$$\operatorname{chsupp} V = [-1, 1]\,. \tag{2.5.14}$$

In view of Theorem D.2 and (D.2.9) it suffices to show that

$$\int_{\mathbb{R}} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty\,, \qquad |f(\lambda)| \leq (1+|\lambda|)^N e^{4(\operatorname{Im}\lambda)_-} \tag{2.5.15}$$

and

$$\limsup_{|\lambda|\to\infty} \frac{\log|f(\lambda)|}{|\lambda|} = 4. \tag{2.5.16}$$

The conditions in (2.5.15) follow immediately from (2.5.3) and we are left to establish (2.5.16), that is to calculate the *type* of f.

3. From (2.5.3), (2.5.11) and (2.5.12) we see that for $\operatorname{Im}\lambda \leq 0$, $|\lambda| \geq C_0$,

$$|v_+^-(-\lambda)v_-^+(-\lambda)|/C - C \leq |D(\lambda)| \leq C|v_+^-(-\lambda)v_-^+(-\lambda)| + C.$$

Also, using (2.4.8), (2.4.15) and (2.5.14)

$$|v_\mp^\pm(-\lambda)| = \mathcal{O}(1) \int_{-1}^{1} e^{\mp 2\operatorname{Im}\lambda x} dx = \mathcal{O}(e^{2|\operatorname{Im}\lambda|}).$$

Hence, the type of $\lambda D(\lambda)$ will be exactly 4 if we show that we *cannot* have

$$|v_\omega^{-\omega}(-\lambda)| \leq Ce^{2(1-\delta)|\lambda|}\,, \quad \operatorname{Im}\lambda \leq 0, \quad |\lambda| \geq C\,, \tag{2.5.17}$$

with $\delta > 0$ for $\omega = +$ *or* for $\omega = -$.

4. Let us choose $\beta > 0$ such that $R_V(\lambda)$ is holomorphic for $\operatorname{Im}\lambda \geq \beta$; that is possible as there are only finitely many poles on $R_V(\lambda)$ in $\operatorname{Im}\lambda \geq 0$. Motivated by Theorem 2.12 we define

$$f_\omega(x,\lambda) := e^{-i\omega\lambda x} R_V(\lambda)(V e^{i\omega\lambda\bullet})(x),$$

which is holomorphic for $\operatorname{Im}\lambda \geq \beta$. Theorem 2.12 (see also the remark following the statement) shows that

$$|f_-(x,\lambda)| \leq C/|\lambda|, \quad \operatorname{Im}\lambda > \beta. \tag{2.5.18}$$

Since $v_+^-(-\lambda+i\beta)$ is holomorphic for $\operatorname{Im}\lambda \leq 0$, (2.5.17) with $\omega = +$ implies that

$$|v_+^-(-\lambda+i\beta)| \leq Ce^{2(1-\delta)|\lambda|}, \quad \operatorname{Im}\lambda \leq 0, \quad \delta > 0, \tag{2.5.19}$$

and we need to find a contradiction to this statement.

We start with (2.4.8): for $\operatorname{Im}\lambda \leq 0$,

$$v_+^-(-\lambda+i\beta) = -\frac{1}{2(\beta+i\lambda)}\int_{\mathbb{R}} e^{2i\lambda x}V(x)e^{2\beta x}(1-f_-(x,-\lambda+i\beta))dx.$$

We then introduce

$$\begin{aligned} V_\varepsilon^\beta(x) &:= \mathbb{1}_{[1-\varepsilon,1]}(x)V(x)e^{2\beta x}\,, \\ g_\varepsilon^\beta(x,\lambda) &= \mathbb{1}_{[1-\varepsilon,1]}(x)f_-(x,-\lambda+i\beta)\,. \end{aligned} \tag{2.5.20}$$

5. Take $\varepsilon < \delta$, and define

$$I_\varepsilon(2\lambda) := \int_{\mathbb{R}} e^{2i\lambda x}V_\varepsilon^\beta(x)(1-g_\varepsilon^\beta(x,\lambda))dx\,.$$

which holomorphic in $\operatorname{Im}\lambda \leq 0$. In the same range of λ's

$$\begin{gathered} \int_{\mathbb{R}} e^{2i\lambda x}\,\mathbb{1}_{[-1,1-\varepsilon]}(x)V(x)e^{2\beta x}(1-f_-(x,-\lambda-i\beta))dx = \\ \mathcal{O}(1)\int_{-1}^{1-\varepsilon} e^{-2\operatorname{Im}\lambda x}(1+\mathcal{O}(|\lambda|^{-1}))dx = \mathcal{O}(e^{2(1-\varepsilon)|\lambda|}). \end{gathered}$$

Hence, the fact that $\varepsilon < \delta$ and the assumption (2.5.19) show that

$$|I_\varepsilon(2\lambda)| \leq Ce^{2(1-\varepsilon)|\lambda|}\,.$$

The Paley-Wiener theorem [**HöI**, Theorem 7.3.1] then shows that

$$\widehat{I}_\varepsilon(x) = 0\,, \quad x > 1-\varepsilon\,,$$

that is

$$V_\varepsilon^\beta(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2i\lambda(y-x)}V_\varepsilon^\beta(y)g_\varepsilon^\beta(y,\lambda)d\lambda dy\,.$$

for $1-\varepsilon \leq x \leq 1$. Plancherel's theorem and the Cauchy–Schwarz inequality then imply that

$$\begin{aligned} \|V_\varepsilon^\beta\|_{L^2} &= \left\|\int_{\mathbb{R}} e^{2i\lambda y}V_\varepsilon^\beta(y)g_\varepsilon^\beta(y,\lambda)dy\right\|_{L^2(d\lambda)} \\ &\leq \left\|\|V_\varepsilon^\beta\|_{L^2}\|g_\varepsilon^\beta(\bullet,\lambda)\|_{L^2(dy)}\right\|_{L^2(d\lambda)} \\ &= \|V_\varepsilon^\beta\|_{L^2}\|g_\varepsilon^\beta\|_{L^2(dyd\lambda)}. \end{aligned} \tag{2.5.21}$$

We note that $g_\varepsilon^\beta \in L^2(d\lambda)$ because of the $\mathcal{O}(1/|\lambda|)$ decay of f_- (2.5.18).

Because of the factor $\mathbb{1}_{[1-\varepsilon,1]}$ in the definition of g_ε^β in (2.5.20), we have

$$(2\pi)^{-1}\|g_\varepsilon^\beta\|_{L^2(dyd\lambda)} \to 0\,, \quad \varepsilon \to 0+\,.$$

It follows from (2.5.21) that for ε small enough $\|V_\varepsilon^\beta\|_{L^2} = 0$. Recalling (2.5.20) this means that

$$V(x) = 0 \ \text{ for } 1-\varepsilon < x < 1,$$

contradicting the assumption that $\operatorname{chsupp} \mathrm{V} = [-1,1]$.

6. The same argument applies under the assumption that (2.5.17) holds for $\omega = -$ and it shows that

$$V(x) = 0 \ \text{ for } -1 < x < -1+\varepsilon,$$

leading again to contradiction. Hence (2.5.16) holds and Theorem D.2 gives the asymptotics of resonances. ☐

2.6. TRACE AND BREIT–WIGNER FORMULAS

We will now prove three closely related trace formulas. The first one, the Birman–Kreĭn formula, relates the scattering matrix to the trace of $f(P_V) - f(P)$. The second is a version of the Breit–Wigner approximation (1.1.3) for the effect of resonances on the spectrum. The last formula is a Poisson formula which relates the trace of the wave group to a sum of resonances $\sum e^{-i\lambda_j|t|}$ and is a special, refined, case of Melrose's trace formula described in §3.10. The Birman–Kreĭn formula is also valid in higher dimensions and for more general perturbations. However, Breit–Wigner formulas in higher dimensions are harder to formulate rigorously – see §§3.13 and 7.6 for comments on that.

From a technical point of view the trace formulas are consequences of the determinant identity presented in Theorem 2.13.

THEOREM 2.19 (Birman–Kreĭn formula in one dimension). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$. Then for $f \in \mathscr{S}(\mathbb{R})$ the operator $f(P_V) - f(P_0)$ is of trace class and*

$$\begin{aligned}\operatorname{tr}\left(f(P_V) - f(P_0)\right) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right)d\lambda \\ + \sum_{k=1}^K f(E_k) + \frac{1}{2}(m_R(0)-1)f(0)\,,\end{aligned} \tag{2.6.1}$$

where $S(\lambda)$ is the scattering matrix and $E_K < \cdots < E_1 < 0$ are the (negative) eigenvalues of P_V.

INTERPRETATION. As in the beginning of Section 2.3 we can compare this result to a result involving eigenvalues. Let us denote the Dirichlet realization of P_V on $[a,b]$ by P_V^D. The spectrum of P_V^D is discrete,

$$E_N < E_{N-1} < \cdots < E_1 \leq 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty .$$

For $f \in \mathscr{S}(\mathbb{R})$, we have

$$\operatorname{tr} f(P_V^D) = \sum_{j=0}^{\infty} f(\lambda_j^2) + \sum_{k=1}^{N} f(E_k) \tag{2.6.2}$$

which can be written as

$$\operatorname{tr} f(P_V^D) = \int_0^\infty f(\lambda^2) \frac{dN(\lambda)}{d\lambda} d\lambda + \sum_{k=1}^{N} f(E_k) \tag{2.6.3}$$

where

$$N(\lambda) = |\{\lambda_j^2 : \lambda_j^2 \leq \lambda^2\}|$$

is the counting function for the positive eigenvalues of P_V^D.

Hence we have the following correspondence between confined (discrete spectrum) and open (continuous spectrum/scattering) problems:

$$N(\lambda) \quad \longleftrightarrow \quad \sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda)$$
$$\sigma'(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \left(S(\lambda)^{-1} \partial_\lambda S(\lambda) \right) .$$

(The last identity follows from Lemma B.26.) Since $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ the right hand side is real. The natural choice of the branch of the logarithm is dictated by (2.4.28): $\sigma(0) = \frac{1}{2}(m_R(0)+1)$. (In three dimensions it is $\sigma(0) = \frac{1}{2}\widetilde{m}_R(0)$, and in odd dimension $n \geq 5$, $\sigma(0) = 0$ – see (3.7.28).)

The analogy with the counting function can however be somewhat misleading as $\sigma(\lambda)$ does not have the monotonicity properties of $N(\lambda)$ – see Figure 2.6. A more accurate spectral interpretation is given in terms of the Kreĭn spectral shift function – see Yafaev [**Ya92**, Chapter 8] and [**Ya09**, Chapter 9].

Proof of Theorem 2.19. For simplicity we assume that there are *no* negative eigenvalues as their contribution is easy to analyse.

1. Since we assume that $V \in L^\infty(\mathbb{R};\mathbb{R})$, P_V is self-adjoint and we can apply Stone's formula as we did in the proof of Theorem 2.9. That gives

$$\begin{aligned} f(P_V) &= \frac{1}{2\pi i} \int_0^\infty f(\lambda^2)(R_V(\lambda) - R_V(-\lambda)) 2\lambda d\lambda \\ &= \frac{1}{4\pi i} \int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda)) 2\lambda d\lambda , \end{aligned}$$

where we used the fact that the integrand is even in λ. The integral on the left should be understood as an operator $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$.

2. We write

$$\begin{aligned} R_V(\lambda) - R_0(\lambda) &= -R_V(\lambda) V R_0(\lambda) \\ &= -R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1} V R_0(\lambda) \,. \end{aligned} \tag{2.6.4}$$

We note that this operator has a simple pole at $\lambda = 0$. The residue is given by $(u \otimes u + 1 \otimes 1/2)/i$, where u is given in (2.2.27) and $u = 0$ if $m_R(0) = 0$.

We define

$$\begin{aligned} B(\lambda) &:= -2\lambda R_V(\lambda) V R_0(\lambda) \\ &= -2\lambda R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1} V R_0(\lambda) \;:\; L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}, \end{aligned} \tag{2.6.5}$$

which is a meromorphic family of operators, holomorphic in $\operatorname{Im}\lambda \geq 0$. (In view of (2.6.4), the simple pole at $\lambda = 0$ is cancelled by the λ factor and we assumed there were no negative eigenvalues.) With this notation we have

$$f(P_V) - f(P_0) = \frac{1}{4\pi i} \sum_{\pm} \int_{\mathbb{R}} f(\lambda^2) B(\pm\lambda) d\lambda \,. \tag{2.6.6}$$

The spectral theorem shows (again we assumed that there are no eigenvalues; easy modifications are needed otherwise) that

$$\| R_V(\lambda) \|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, \mathbb{R}_+)} \leq \frac{1}{|\lambda| \operatorname{Im}\lambda} \,, \quad \operatorname{Im}\lambda > 0 \,. \tag{2.6.7}$$

Applying this with $V = 0$ gives, for $\operatorname{Im}\lambda > 0$,

$$\begin{aligned} \| \lambda R_0(\lambda) \|_{L^2 \to H^2} &\leq C|\lambda| (\| D_x^2 R_0(\lambda) \|_{L^2 \to L^2} + \| R_0(\lambda) \|_{L^2 \to L^2}) \\ &\leq C|\lambda|(1 + |\lambda|^2) \| R_0(\lambda) \|_{L^2 \to L^2} \\ &\leq C(1 + |\lambda|^2)/\operatorname{Im}\lambda. \end{aligned}$$

This estimate and the fact that $V \in L^\infty_{\mathrm{comp}}$ show that $V R_0(\lambda)$ is of trace class for $\operatorname{Im}\lambda > 0$ (see Theorem B.4) and that

$$\| \lambda V R_0(\lambda) \|_{\mathcal{L}_1} \leq C \frac{1 + |\lambda|^2}{\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda > 0.$$

Combining this with (2.6.5) and (2.6.7)

$$\| B(\lambda) \|_{\mathcal{L}_1} \leq \frac{1 + |\lambda|^2}{|\operatorname{Im}\lambda|^2 |\lambda|} \leq \frac{1 + |\lambda|^2}{|\operatorname{Im}\lambda|^3}, \quad \operatorname{Im}\lambda > 0. \tag{2.6.8}$$

Let $g \in \mathscr{S}(\mathbb{C})$, $\operatorname{supp} g \subset \{ |\operatorname{Im}\lambda| \leq 1 \}$, be an *almost analytic extention* of $f(\lambda^2)$ (see B.2):

$$g(\lambda) = f(\lambda^2) \,, \ \lambda \in \mathbb{R} \,, \ \ \partial_{\bar{\lambda}} g(\lambda) = \mathcal{O}(|\operatorname{Im}\lambda|^\infty) \,. \tag{2.6.9}$$

The Cauchy–Green formula (D.1.1) applied to the right hand side of (2.6.6) shows that

$$\begin{aligned} f(P_V) - f(P_0) &= \frac{1}{2\pi}(t_+(f) - t_-(f)), \\ t_\pm(f) &:= \int_{\pm \operatorname{Im}\lambda > 0} \partial_{\bar\lambda} g(\lambda) B(\pm\lambda) dm(\lambda). \end{aligned} \tag{2.6.10}$$

Using (2.6.8) and (2.6.9) we conclude that for any $N > 0$, and in particular for $N \geq 4$,

$$\|t_\pm(f)\|_{\mathcal{L}_1} \leq C_N \int_{0<\pm \operatorname{Im}\lambda<1} |\operatorname{Im}\lambda|^N (1+|\lambda|)^{-N+2} |\operatorname{Im}\lambda|^{-3} dm(\lambda) < \infty.$$

This proves the claim that

$$f(P_V) - f(P_0) \in \mathcal{L}_1. \tag{2.6.11}$$

3. To calculate the trace of $f(P_V) - f(P_0)$ we use Theorem 2.13. Taking logarithmic derivatives of both sides of (2.4.19) we obtain

$$\begin{aligned} \operatorname{tr} F(-\lambda) + \operatorname{tr} F(\lambda) &= \operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1}, \\ F(\lambda) &:= -\partial_\lambda (V R_0(\lambda)\rho)(I + V R_0(\lambda)\rho)^{-1}. \end{aligned} \tag{2.6.12}$$

We note that $F(\lambda)$, $\lambda \in \mathbb{C}$, is a meromorphic family of operators in $\mathcal{L}_1(L^2)$, with no poles in $\operatorname{Im}\lambda > 0$ (we assumed that there are no negative eigenvalues). From (2.2.33) (see §C.4) we see that

$$\operatorname{tr} F(\lambda) = -\operatorname{tr}(\lambda^{-1} Q_1 - \lambda^{-1} Q_{-1}) + \varphi(\lambda) = \frac{1 - m_R(0)}{\lambda} + \varphi(\lambda), \tag{2.6.13}$$

where $\varphi(\lambda)$ is holomorphic in $\operatorname{Im}\lambda \geq 0$. To estimate $\varphi(\lambda)$ we recall (see the proof of Theorem 2.10) that

$$\|V R_0(\lambda)\rho\|_{L^2 \to H^2} \leq C|\lambda| e^{C(\operatorname{Im}\lambda)_-}, \quad |\lambda| \geq 1.$$

The Cauchy estimate (C.3.1) (or an explicit calculation) show that for $\operatorname{Im}\lambda \geq 0$, $|\lambda| > 1$,

$$\|\partial_\lambda V R_0(\lambda)\rho\|_{\mathcal{L}_1} \leq C \|\partial_\lambda V R_0(\lambda)\rho\|_{L^2 \to H^2} \leq C'|\lambda|.$$

From the definition of $F(\lambda)$ and from the invertibility of $I + V R_0(\lambda)\rho$ for $|\lambda| \gg 1$, $\operatorname{Im}\lambda \geq 0$ it now follows that

$$|\operatorname{tr} F(\lambda)| \leq C|\lambda|, \quad |\lambda| \geq C_0, \quad \operatorname{Im}\lambda \geq 0.$$

Since $\varphi(\lambda)$ is holomorphic in $\operatorname{Im}\lambda \geq 0$ we obtain.

$$|\varphi(\lambda)| \leq C(1 + |\lambda|), \quad \operatorname{Im}\lambda \geq 0.$$

4. We claim that for $\operatorname{Im}\lambda > 0$

$$\operatorname{tr} F(\lambda) = \operatorname{tr} B(\lambda), \tag{2.6.14}$$

where $B(\lambda)$ was defined by (2.6.5).

To see (2.6.14) we use the fact that $R_0(\lambda)$ is bounded on L^2 for $\operatorname{Im}\lambda > 0$ and hence

$$\partial_\lambda (V R_0(\lambda)\rho) = 2\lambda V R_0(\lambda)^2 \rho\,.$$

Using this, the cyclicity of the trace (Theorem B.4.9 applied twice) and $\rho V = V$, we obtain, always for $\operatorname{Im}\lambda > 0$,

$$\operatorname{tr} F(\lambda) = -2\lambda \operatorname{tr} R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1} V R_0(\lambda) = \operatorname{tr} B(\lambda)\,,$$

which is (2.6.14).

5. We now use (2.6.14) and (D.1.1) in (2.6.10):

$$\begin{aligned}
\operatorname{tr}\left(f(P_V) - f(P_0)\right) &= \frac{1}{2\pi}\sum_{\pm} \pm \int_{\pm \operatorname{Im}\lambda > 0} \partial_{\bar\lambda} g(\lambda) \operatorname{tr} F(\pm\lambda) dm(\lambda) \\
&= \frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)} g(\lambda) \operatorname{tr} F(\pm\lambda) d\lambda \\
&\quad + \frac{1}{2\pi}\sum_{\pm} \pm \int_{\Omega_\pm(\varepsilon)} \partial_{\bar\lambda} g(\lambda) \operatorname{tr} F(\pm\lambda) dm(\lambda),
\end{aligned} \tag{2.6.15}$$

where where

$$\begin{gathered}
\Omega_\pm(\varepsilon) = D(0,\varepsilon) \cap \mathbb{C}_\pm, \quad \mathbb{C}_\pm := \{\pm \operatorname{Im}\lambda > 0\}, \\
\gamma_+(\varepsilon) = \partial(\mathbb{C}_+ \setminus \Omega_+(\varepsilon)), \quad \gamma_-(\varepsilon) = \partial(\mathbb{C}_+ \cup \overline{\Omega_-(\varepsilon)}),
\end{gathered}$$

and the boundaries are positively oriented (as boundaries of the indicated sets).

Estimates (2.6.8) (applied using (2.6.14)) and (2.6.9) show that the last term on the right hand side of (2.6.15) is $\mathcal{O}(\varepsilon^\infty)$. Using (2.6.12) we then see that

$$\begin{aligned}
\frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)} g(\lambda) \operatorname{tr} F(\pm\lambda) d\lambda &= \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2) \operatorname{tr}\, \partial_\lambda S(\lambda) S(\lambda)^{-1} d\lambda \\
&\quad + \frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda) \operatorname{tr} F(\pm\lambda) d\lambda + \mathcal{O}(\varepsilon)\,.
\end{aligned}$$

(The error term $\mathcal{O}(\varepsilon)$ comes from passing to the limit from $\gamma_\pm(\varepsilon) \cap \mathbb{R}$ to $\mathbb{R}$.)

The structure of $F(\lambda)$ near 0 given in (2.6.13) shows that

$$\begin{aligned}
\frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda) \operatorname{tr} F(\pm\lambda) d\lambda &= (m_R(0) - 1)\frac{f(0)}{4\pi i}\int_{\partial D(0,\varepsilon)} \frac{d\lambda}{\lambda} + \mathcal{O}(\varepsilon) \\
&= \frac{1}{2}(m_R(0) - 1) f(0) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$ and noting that $\operatorname{tr} S(\lambda)^{-1}\partial S(\lambda)$ is even (see (2.6.12)) we obtain (2.6.1). $\square$

REMARK. An examination of the proof of (2.6.11) shows that $T_V : f \mapsto \operatorname{tr} f(P_V) - f(P_0)$ defines a tempered distribution, $T_V \in \mathscr{S}'(\mathbb{R})$. This will be important in higher dimensions where the properties of $\det S(\lambda)$ are less clear.

The density appearing in (2.6.1) can be expressed in terms of resonances as follows:

THEOREM 2.20 (Breit–Wigner approximation). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{R})$. Then*

$$\frac{1}{2\pi i}\operatorname{tr}\partial_\lambda S(\lambda)S(\lambda)^{-1} = -\frac{1}{\pi}|\operatorname{chsupp} V| - \frac{1}{\pi}\sum_j \frac{\operatorname{Im}\lambda_j}{|\lambda-\lambda_j|^2}, \tag{2.6.16}$$

where the sum is over all non-zero resonances of P_V and it converges for $\lambda \in \mathbb{R}$.

INTERPRETATION. 1. The operator $-i\partial_\lambda S(\lambda)S(\lambda)^{-1}$ is known as the *Eisenbud–Wigner time delay operator* – see [**TZ01**, Proposition 1.1].

2. For $\operatorname{Im}\lambda_j < 0$, the density

$$dm_{\lambda_j}(\lambda) = -\frac{1}{\pi}\frac{\operatorname{Im}\lambda_j}{|\lambda-\lambda_j|^2}d\lambda$$

defines a probability measure on $\mathbb{R}$ and is known in this context as the *Breit–Wigner Lorentzian*. Formally, if $\operatorname{Im}\lambda_j = 0$ then the Lorentzian becomes $\delta(\lambda-\lambda_j)$ which is consistent with the discussion following Theorem 2.19. The Birman–Kreĭn formula can then be written as

$$\begin{aligned}\operatorname{tr}\left(f(P_V) - f(P_0)\right) = \sum_j \int_0^\infty f(\lambda^2)dm_{\lambda_j}(\lambda) - \frac{|\operatorname{chsupp} V|}{\pi}\int_0^\infty f(\lambda^2)d\lambda \\ + \sum_{k=1}^K f(E_k) + \frac{1}{2}(m_R(0)-1)f(0).\end{aligned}$$

This means that the *spectral shift function* (see §3.13 for discussion and references) is expressed in terms of resonances. This provides a rigorous formulation of the Breit–Wigner approximation (1.1.3) – see Fig. 2.6.

3. The approximation based on (2.6.16) uses finitely many λ_j's for λ in a bounded set – see Figure 2.6. Each term in the sum on the right hand side is $\mathcal{O}(\lambda^{-2})$ as $\lambda \to \infty$ but in the proof of we will see that

$$-\frac{1}{\pi}\sum_j \frac{\operatorname{Im}\lambda_j}{|\lambda-\lambda_j|^2} = \frac{1}{\pi}|\operatorname{chsupp} V| + \mathcal{O}(\lambda^{-2}), \quad \lambda \to +\infty.$$

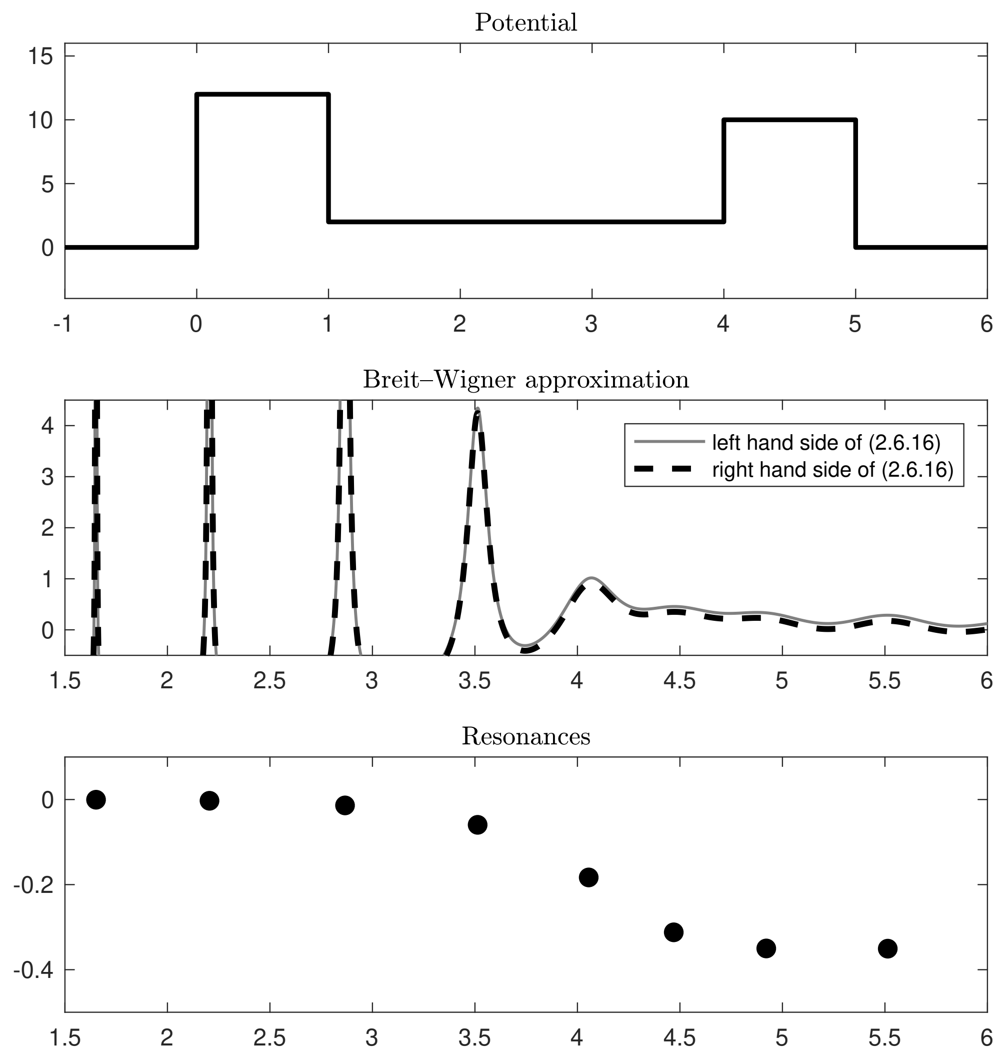


Figure 2.6. Breit–Wigner approximation in a simple example: The resonances for the potential on top are computed using a code `squarepot.m` by David Bindel [**BZ**]. The right hand side of (2.6.16) is computed using 40 of those resonances. The ones shown correspond to the peaks in $\operatorname{tr} S(\lambda)^{-1}\partial_\lambda S(\lambda)/2\pi i$. The scattering matrix was computed using the transfer matrix – see Exercise 2.6. The plot was truncated to show the fine feature agreement with (2.6.16).

When $V \in C_c^\infty(\mathbb{R};\mathbb{R})$ the asymptotics of $\sigma(\lambda) = \operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1}/2\pi i$ presented in Exercise 2.4 (see Theorem 3.67 for the general version) show that $\mathcal{O}(\lambda^{-2})$ can be replaced by a full asymptotic expansion.

Proof of Theorem 2.20. 1. Theorem 2.14 shows that the zeros of $\det S(\lambda)$ for $\operatorname{Im} \lambda \geq 0$ are given by $-\lambda_j$ where λ_j are resonances of P_V. They are then the zeros of

$$g(\lambda) := e^{2i\lambda|\operatorname{chsupp} V|} \prod_{k=1}^{K} \frac{\lambda - i\mu_k}{\lambda + i\mu_k} \det S(\lambda),$$

where we used the notation of (2.5.4). From that bound we see that $g(\lambda)$ is holomorphic for $\operatorname{Im}\lambda \geq 0$ and that $|g(\lambda)| \leq 1$ there. Carleman's estimate (D.1.12) then gives

$$\sum_j \frac{|\operatorname{Im}\lambda_j|}{|\lambda_j|^2} < \infty. \tag{2.6.17}$$

2. Hadamard's factorization theorem (D.2.7) applied to $\lambda D(\lambda)$ (where $D(\lambda)$ is given by (2.2.28) and satisfies (2.4.19)) and the symmetries of $\det S(\lambda)$ ($\det S(\lambda)^{-1} = \det S(-\lambda) = \overline{\det S(\bar\lambda)}$ – see (2.4.13) and (2.4.14)) show that

$$\det S(\lambda) = e^{ia\lambda} \frac{\overline{P(\bar\lambda)}}{P(\lambda)}, \quad P(\lambda) := \prod_j \left(1 - \frac{\lambda}{\lambda_j}\right) e^{\frac{\lambda}{\lambda_j}}, \quad a \in \mathbb{R}, \tag{2.6.18}$$

where the product is over all non-zero resonances. For $\lambda \in \mathbb{R}$, (2.6.17) and (2.6.18) give

$$\begin{aligned}
\frac{1}{2\pi i} \operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} &= \frac{a}{2\pi} - \frac{1}{2\pi i} \sum_j \left(\frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \bar\lambda_j} + \frac{1}{\lambda_j} - \frac{1}{\bar\lambda_j} \right) \\
&= \frac{a}{2\pi} - \frac{1}{\pi} \sum_j \left(\frac{\operatorname{Im}\lambda_j}{|\lambda - \lambda_j|^2} - \frac{\operatorname{Im}\lambda_j}{|\lambda_j|^2} \right) \\
&= -B - \frac{1}{\pi} \sum_j \frac{\operatorname{Im}\lambda_j}{|\lambda - \lambda_j|^2}, \quad B := -\frac{a}{2\pi} - \frac{1}{\pi} \sum_j \frac{\operatorname{Im}\lambda_j}{|\lambda_j|^2}.
\end{aligned}$$

To obtain (2.6.16) it remains to show that $B = |\operatorname{chsupp} V|/\pi$.

3. We start by estimating $\operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1}$ as $\lambda \to \pm\infty$. For that we use Theorem 2.4.23, $\det S(\lambda) = \det(I + T(\lambda))$ where $T(\lambda)$ is defined by (2.4.22). Then,

$$\begin{aligned}
|\operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1}| &= |\operatorname{tr} \partial_\lambda T(\lambda) (I + T(\lambda))^{-1}| \\
&\leq \|\partial_\lambda T(\lambda)\|_{\mathcal{L}_1} \|(I + T(\lambda))^{-1}\|_{L^2 \to L^2}.
\end{aligned}$$

The estimate $V R_0(\lambda)\rho = \mathcal{O}(|\lambda|^{-1})_{L^2 \to L^2}$ implies that

$$(I + V R_0(\lambda)\rho)^{-1} = \mathcal{O}(1)_{L^2 \to L^2},$$

$\lambda \to \pm\infty$. The definition (2.4.22) gives $T(\lambda) = \mathcal{O}(|\lambda|^{-1})_{L^2 \to L^2}$ so that

$$(I + T(\lambda))^{-1} = \mathcal{O}(1)_{L^2 \to L^2}, \quad \lambda \to \pm\infty.$$

To estimate the trace class norm of $\partial_\lambda T(\lambda)$ we write, in the notation of (2.4.22), and for $\lambda \to \pm\infty$ (real)

$$
\begin{aligned}
\|\partial_\lambda T(\lambda)\|_{\mathcal{L}_1} &\leq \|\partial_\lambda (I + VR_0(\lambda)\rho)^{-1} V(\lambda^{-1} E_\rho(\lambda)^* E_\rho(\lambda)\|_{\mathcal{L}_1} \\
&\quad + \|(I + VR_0(\lambda)\rho)^{-1} V \partial_\lambda (\lambda^{-1} E_\rho(\lambda)^* E_\rho(\lambda)\|_{\mathcal{L}_1} \\
&\leq C \|\partial_\lambda (VR_0(\lambda)\rho)\|_{L^2 \to L^2} \|E_\rho(\lambda)^* E_\rho(\lambda)\|_{\mathcal{L}_1} \qquad . \\
&\quad + C \|VR_0(\lambda)\rho\|_{L^2 \to L^2} \|\partial_\lambda (\lambda^{-1} E_\rho(\lambda)^* E_\rho(\lambda))\|_{\mathcal{L}_1} \\
&\leq C\lambda^{-2}.
\end{aligned}
$$

Here we used the bound $\|\partial_\lambda^\ell VR_0(\lambda)\rho\|_{L^2\to L^2}\| \leq C/|\lambda|$ (obtained from the explicit formula for $R_0(\lambda)$) and the fact that $E_\rho(\lambda)^* E_\rho(\lambda)$ is a finite rank operator (2.4.21) which can be differentiated with respect to λ keeping the boundedness in λ.

4. Returning to the calculation in Step 2 we see that

$$
\frac{1}{\pi} \sum_j \frac{|\operatorname{Im} \lambda_j|}{|\lambda - \lambda_j|^2} = B + \mathcal{O}(\lambda^{-2}), \quad \lambda \to \pm\infty. \tag{2.6.19}
$$

For $\delta > 0$ and $\Lambda > |\lambda|$ we split the left hand side into three terms

$$
\frac{1}{\pi} \sum_j \frac{|\operatorname{Im} \lambda_j|}{|\lambda - \lambda_j|^2} = S_{[0,(1-\delta)\Lambda]}(\lambda) + S_{(1-\delta,1+\delta)\Lambda}(\lambda) + S_{[(1+\delta)\Lambda,\infty)}(\lambda),
$$
$$
S_I := \frac{1}{\pi} \sum_{|\lambda_j| \in I} \frac{|\operatorname{Im} \lambda_j|}{|\lambda - \lambda_j|^2}, \quad I \subset [0, \infty).
$$

We then see that (2.6.17) gives (recall $|\lambda| < \Lambda$)

$$
S_{[(1+\delta)\Lambda,\infty)}(\lambda) \leq \sum_{|\lambda_j| \geq (1+\delta)\Lambda} \frac{|\operatorname{Im} \lambda_j|}{(|\lambda_j| - \Lambda)^2} \leq \delta^{-2} \sum_{|\lambda_j| \geq (1+\delta)\Lambda} \frac{|\operatorname{Im} \lambda_j|}{|\lambda_j|^2} = o_\delta(1),
$$

uniformly as $\Lambda \to \infty$. Also, the asymptotic formula for the counting function of resonances (2.5.1) gives

$$
\int_{-\Lambda}^{\Lambda} S_{[1-\delta,1+\delta]}(\lambda) d\lambda = \mathcal{O}(\delta\Lambda) + o(\Lambda).
$$

Hence integrating (2.6.19) from $-\Lambda$ to Λ we obtain

$$
\frac{1}{\pi} \sum_{|\lambda_j| \leq (1-\delta)\Lambda} \sum_{\pm} \tan^{-1} \left(\frac{\Lambda \pm \operatorname{Re} \lambda_j}{|\operatorname{Im} \lambda_j|} \right) + \mathcal{O}(\delta\Lambda) + o_\delta(\Lambda) = 2B\Lambda + \mathcal{O}(1).
$$

In view of (2.5.2) we only need to sum over $|\operatorname{Im} \lambda_j| < \delta^2 |\lambda_j|$ at an expense of adding another $o_\delta(\Lambda)$ error term.

Since for $x > 0$, $\tan^{-1}(x)/\pi = 1/2 + \mathcal{O}(\langle x\rangle^{-1})$, $|\lambda_j| \leq (1-\delta)\Lambda$ and for $|\operatorname{Im}\lambda_j| < \delta^2|\operatorname{Re}\lambda_j|$

$$\sum_{\pm} \tan^{-1}\left(\frac{\Lambda \pm \operatorname{Re}\lambda_j}{|\operatorname{Im}\lambda_j|}\right) = 1 + \mathcal{O}\left(\frac{|\operatorname{Im}\lambda_j|}{\Lambda - |\operatorname{Re}\lambda_j|}\right) = 1 + \mathcal{O}(\delta).$$

Combined with (2.5.1) this gives

$$\frac{2}{\pi}|\operatorname{chsupp} V|\Lambda + \mathcal{O}(\delta\Lambda) + o_\delta(\Lambda) = 2B\Lambda + \mathcal{O}(1).$$

Since δ is arbitrary this shows that $B = |\operatorname{chsupp} V|/\pi$ as claimed. □

REMARK. If we use the representation (2.4.18) of the scattering matrix, the transmission coefficient is given by $t(\lambda) = i\lambda/\widehat{X}(\lambda)$. Exercise (2.10.1) then shows that $\det S(\lambda) = \widehat{X}(-\lambda)/\widehat{X}(\lambda)$. Our mathematical version of the Breit–Wigner approximation (2.6.16) then follows from results of Titchmarsh [**Ti26**, Theorems IV and VI] and from [**Zw87**] where $\operatorname{chsupp} X = [-2|\operatorname{chsupp} V|, 0]$ is established. Here we used a different complex function theory to establish (2.5.1) from which we obtained (2.6.16) directly.

We now define the following distribution on $\mathbb{R}$: with the sum over all resonances and $\varphi \in C_{\mathrm{c}}^\infty((-L, L))$,

$$\text{p.v.} \sum_j e^{-i|t|\lambda_j}(\varphi) := \lim_{R\to\infty} \sum_{|\lambda_j|\leq R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j|t|} dt. \tag{2.6.20}$$

To check that (2.6.20) defines a distribution we integrate by parts twice using $e^{-i\lambda t} = \partial_t\left((i/\lambda)e^{-i\lambda t}\right)$:

$$\begin{aligned} \text{p.v.} \sum_{\lambda_j\neq 0} e^{-i|t|\lambda_j}(\varphi) &= \lim_{R\to\infty} \sum_{0<|\lambda_j|\leq R} \int_0^L (\varphi(t) + \varphi(-t)) e^{-i\lambda_j t} dt \\ &= \lim_{R\to\infty} \sum_{0<|\lambda_j|\leq R} \left(\frac{2i}{\lambda_j}\varphi(0) + \mathcal{O}\left(\frac{\sup|\varphi^{(2)}|}{|\lambda_j|^2}\right)\right) \end{aligned}$$

Thanks to (2.5.1), $\sum_{\lambda_j\neq 0}|\lambda_j|^2 < \infty$, while the symmetry of resonances $\lambda_j \to -\bar{\lambda}_j$ gives

$$\sum_{0<|\lambda_j|\leq R} \frac{2i}{\lambda_j} = \sum_{0<|\lambda_j|\leq R} \frac{2\operatorname{Im}\lambda_j}{|\lambda_j|^2}.$$

The estimate (2.6.17) shows that the series on the right converges and hence (2.6.20) is well defined.

As a consequence of Theorems 2.19 and 2.20 we have the following *trace formula for resonances*:

THEOREM 2.21 (**Trace formula for resonances**). *Suppose that* $V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R})$. *Then for* $\varphi \in C^\infty_c(\mathbb{R})$ *the operator*

$$\int_{\mathbb{R}} \varphi(t) \left(\cos t\sqrt{P_V} - \cos t\sqrt{P_0} \right) dt$$

is of trace class and, using definition (2.6.20),

$$\begin{aligned} 2\,\mathrm{tr}\left(\cos t\sqrt{P_V} - \cos t\sqrt{P_0}\right) \\ = \text{p.v.} \sum_{\lambda \in \mathbb{C}} m_R(\lambda) e^{-i\lambda|t|} - 2|\operatorname{chsupp} V|\delta_0(t) - 1, \end{aligned} \tag{2.6.21}$$

in the sense of distributions on $\mathbb{R}$.

INTERPRETATION. 1. The expansion (2.3.1) leads directly to a trace formula for, say, the Dirichlet realization of P_V on $[a,b]$. We denote that Dirichlet realization by P_V^D. Assuming for simplicity that there are no non-positive eigenvalues, we have

$$2\,\mathrm{tr} \cos t\sqrt{P_V^D} = \sum_{\lambda^2 \in \operatorname{Spec}(P_V^D)} e^{-i\lambda t}.$$

Hence the expansion (2.6.21) is an exact analogue of this well known consequence of the spectral theorem. What is remarkable is the fact that unlike the resonance wave expansions given in Theorem 2.9 the trace formula (2.6.21) is *exact*.

2. The Poisson formula (2.6.21) remains valid in higher dimensions as (3.10.2) but with a less precise statement at $t = 0$.

Proof of Theorem 2.21. 1. In the distributional sense,

$$(2\cos t\sqrt{P_V})(\varphi) = f(P_V), \quad f(z) := \widehat{\varphi}(\sqrt{z}) + \widehat{\varphi}(-\sqrt{z}),$$

where $f \in C^\infty(\mathbb{R}) \cap \mathscr{S}((0,\infty))$ (and hence Theorem 2.19 is applicable as we can replace f by $\chi f \in \mathscr{S}(\mathbb{R})$ where $\chi \in C^\infty(\mathbb{R})$ vanishes for sufficiently large negative values). Compared to the definition (2.6.20) we see that (2.6.21) is equivalent to

$$\begin{aligned} \mathrm{tr}\,(f(P_V) - f(P_0)) = \lim_{R\to\infty} \sum_{|\lambda_j| \le R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j|t|} dt \\ - 2|\operatorname{chsupp} V|\varphi(0) - \widehat{\varphi}(0). \end{aligned} \tag{2.6.22}$$

Writing $\sigma(\lambda) := \log\det S(\lambda)/2\pi i$, Theorem 3.51 shows that (since $\sigma'(\lambda)$ is even)

$$\operatorname{tr}\left(f(P_V) - f(P_0)\right) = \tfrac{1}{2}\int_{-\infty}^{\infty} f(\lambda^2)\sigma'(\lambda)d\lambda + \sum_{k=1}^{K} f(E_k) + \tfrac{1}{2}(m_R(0)-1)f(0).$$

In Step 3 of the proof of Theorem 2.20 we showed that $\sigma'(\lambda) = \mathcal{O}(\langle\lambda\rangle^{-2})$ and thus the integral converges.

Hence the proof of the theorem is reduced to showing that

$$\begin{aligned}&\int_{\mathbb{R}} \widehat{\varphi}(\lambda)\sigma'(\lambda)d\lambda + m_R(0)\widehat{\varphi}(0) + \sum_{\operatorname{Im}\lambda_j>0}(\widehat{\varphi}(\lambda_j) + \widehat{\varphi}(-\lambda_j))\\ &\qquad = \lim_{R\to\infty}\sum_{|\lambda_j|\le R}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}dt - 2|\operatorname{chsupp}V|\varphi(0).\end{aligned} \tag{2.6.23}$$

We first note that for $\operatorname{Im}\lambda_j < 0$, $t\mapsto e^{-i|t|\lambda_j}$ is a tempered distribution. The symmetry $\lambda_j \mapsto -\bar{\lambda}_j$ then shows that

$$\mathcal{F}_{t\mapsto\lambda}\left(\sum_{\substack{|\lambda_j|\le R\\ \operatorname{Im}\lambda_j<0}} e^{-i\lambda_j|t|}\right) = \sum_{\substack{|\lambda_j|\le R\\ \operatorname{Im}\lambda_j<0}} \frac{2|\operatorname{Im}\lambda_j|}{|\lambda-\lambda_j|^2}.$$

Parseval's formula [**HöI**, (7.1.8)] and (2.6.17) then show that

$$\lim_{R\to\infty}\sum_{\substack{|\lambda_j|\le R\\ \operatorname{Im}\lambda_j<0}}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}dt = \frac{1}{\pi}\int_{\mathbb{R}}\widehat{\varphi}(\lambda)\sum_{\operatorname{Im}\lambda_j<0}\frac{|\operatorname{Im}\lambda_j|}{|\lambda-\lambda_j|^2}d\lambda. \tag{2.6.24}$$

Inserting (2.6.16) (Theorem 2.20) into (2.6.23) and using $\int_{\mathbb{R}}\widehat{\varphi}(\lambda)d\lambda = 2\pi\varphi(0)$ shows that it remains to prove (recall that resonances in $\operatorname{Im}\lambda > 0$ lie on the imaginary axis and are square roots of finitely many eigenvalues of P_V)

$$\begin{aligned}&\frac{1}{\pi}\int_{\mathbb{R}}\widehat{\varphi}(\lambda)\sum_{\operatorname{Im}\lambda_j>0}\frac{\operatorname{Im}\lambda_j}{|\lambda-\lambda_j|^2}d\lambda + \sum_{\operatorname{Im}\lambda_j>0}\int_{\mathbb{R}}\varphi(t)e^{-i\lambda_j|t|}dt\\ &\qquad = \sum_{\operatorname{Im}\lambda_j>0}(\widehat{\varphi}(\lambda_j) + \widehat{\varphi}(-\lambda_j)) = 2\sum_{\operatorname{Im}\lambda_j>0}\int_{\mathbb{R}}\varphi(t)\cos(\lambda_j t)\,dt.\end{aligned} \tag{2.6.25}$$

But the same argument which led to (2.6.24) shows that the first term on the left hand side is equal to $\sum_{\operatorname{Im}\lambda_j>0}\int_{\mathbb{R}}\varphi(t)e^{i\lambda_j|t|}dt$. Since $e^{i\lambda_j|t|} + e^{-i\lambda_j|t|} = 2\cos(\lambda_j t)$ this proves (2.6.25), concluding the proof of (2.6.21). $\square$

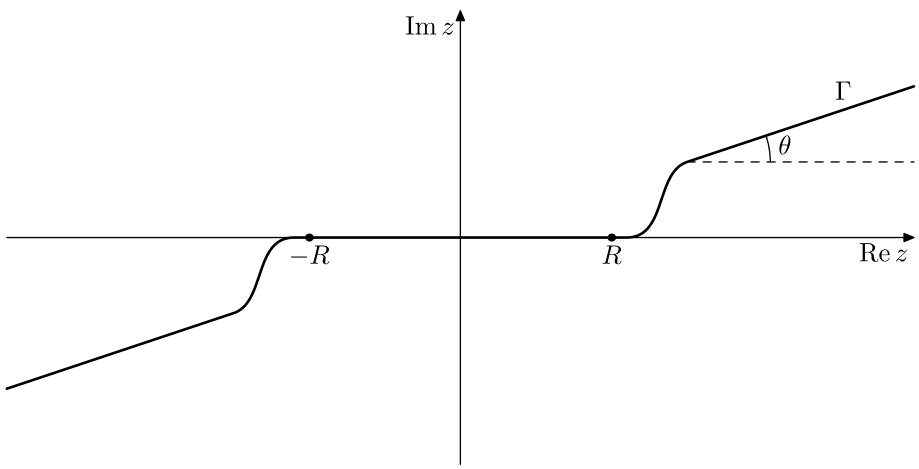


Figure 2.7. Curve Γ used in complex scaling. The curve is given by $x \mapsto x + ig(x)$ for a C^∞ function g satisfying $g(x) = 0$ for $-R \leq x \leq R$ and $g(x) = x \tan\theta$ for $|x|$ sufficiently large, where θ is a given constant.

2.7. COMPLEX SCALING IN ONE DIMENSION

In this section we present the simplest case of the method of *complex scaling* which produces a natural family of non-self-adjoint operators whose discrete spectrum consists of resonances.

The idea is to consider D_x^2 as a restriction of the complex second derivative D_z^2 to the real axis thought of as a contour in $\mathbb{C}$. This contour is then deformed away from the support of V so that P can be restricted to it. This provides ellipticity at infinity at the price of losing self-adjointness.

An account of this method in higher dimensions will be provided in §4.5. Again, in one dimension we can provide a low-tech self-contained presentation.

Let $\Gamma \subset \mathbb{C}$ be a C^1 *simple curve*. We define differentiation and integration of functions mapping Γ to $\mathbb{C}$ as follows. Let $\gamma(t)$ be a parametrization $\mathbb{R} \to \Gamma \subset \mathbb{C}$, and let $f \in C^1(\Gamma)$ in the sense that $f \circ \gamma \in C^1(\mathbb{R})$. We define

$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}\partial_t(f \circ \gamma)(t_0)\,, \quad \gamma(t_0) = z_0\,,$$

where the inverse and the multiplication are in the sense of complex numbers. We further define

$$D_z^\Gamma = \frac{1}{i}\partial_z^\Gamma\,.$$

By the chain rule, $\partial_z^\Gamma f(z_0)$ is independent of parametrization. In fact, if α is another parametrization, $\alpha(s_0) = z_0$, then

$$\gamma'(t_0) = c(s_0)\alpha'(s_0), \quad c(s_0) = \big(\partial_s(\gamma^{-1} \circ \alpha)(s_0)\big)^{-1} \in \mathbb{R}.$$

Then

$$\begin{aligned}\alpha'(s_0)^{-1}\partial_s(f\circ\alpha)(s_0) &= \alpha'(s_0)^{-1}\partial_s(f\circ\gamma\circ(\gamma^{-1}\circ\alpha))(s_0)\\ &= \alpha'(s_0)^{-1}\partial_s(\gamma^{-1}\circ\alpha)(s_0)\partial_t(f\circ\gamma)(t_0)\\ &= \alpha'(s_0)^{-1}c(s_0)^{-1}\partial_t(f\circ\gamma)(t_0)\\ &= \gamma'(t_0)^{-1}\partial_t(f\circ\gamma)(t_0).\end{aligned}$$

If f extends to a C^1 function in a neighbourhood of Γ and

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t)\,, \quad \gamma_j : \mathbb{R} \longrightarrow \mathbb{R}\,,$$

then

$$\partial_z^\Gamma f(z_0) = \gamma'(t_0)^{-1}(\partial_x f(z_0)\gamma_1'(t_0) + \partial_y f(z_0)\gamma_2'(t_0))$$

In particular, if f is holomorphic in a neighborhood of Γ, then the Cauchy-Riemann equation, $\partial_y f = i\partial_x f$, shows that

$$\partial_z^\Gamma f = \partial_x f = \partial_z f\,,$$

so in this case ∂_z^Γ coincides with the holomorphic differential operator.

To integrate along the curve we can use both the complex contour measure and the arclength measure, denoted

$$dz = \gamma'(t)dt\,, \quad |dz| = |\gamma'(t)|dt\,,$$

respectively. We assume that $\gamma \in C^2(\mathbb{R};\mathbb{C})$. The spaces $L^2(\Gamma)$ and $H^j(\Gamma)$, $j = 1, 2$, are defined using the measure $|dz|$.

Given a potential $V \in L^\infty_{\rm comp}(\mathbb{R};\mathbb{C})$ we further assume that

$$\Gamma \cap \mathbb{R} \supset [-L, L], \quad \operatorname{supp} V \subset (-L, L). \tag{2.7.1}$$

The potential V is then a well defined function on Γ, so that putting

$$P_{V,\Gamma} := (D_z^\Gamma)^2 + V(z)\,, \tag{2.7.2}$$

makes sense.

We make the following assumption on the behaviour of Γ at infinity:

$$\begin{gathered}\exists\, \theta \in (0, \pi),\ a_\pm \in \mathbb{C},\ K \Subset \mathbb{C},\\ \Gamma \setminus K = \bigcup_\pm \Big(a_\pm \pm e^{i\theta}(0,\infty)\Big) \setminus K.\end{gathered} \tag{2.7.3}$$

We orient Γ so that $\operatorname{Im}\gamma(t) \to +\infty$ as $t \to +\infty$: this can be done in view of (2.7.3). We also define

$$\Lambda_\Gamma := \{\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_- : -\theta < \arg\lambda < \pi - \theta\}. \tag{2.7.4}$$

where $\arg : \mathbb{C} \setminus \overline{\mathbb{R}}_- \to (-\pi, \pi)$. An example of Γ is shown in Fig. 2.7. We remark that one can also consider more general behaviour at infinity such as shown in Fig.2.8 where $\Gamma = \{x + ig(x) : x \in \mathbb{R}\}$ for a specific g.

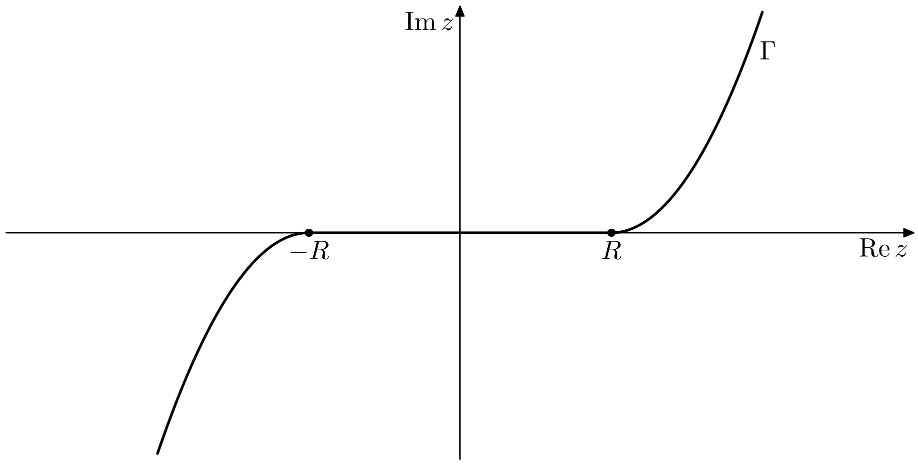


Figure 2.8. Curve Γ used in some PML (perfectly matched layer) computations. A typical curve is given by a function $\mathbb{R} \ni x \mapsto x + ig(x)$ where $g(x) = -|x+R|^\alpha$ for $x < -R$, $g(x) = 0$ for $-R \le x \le R$, and $g(x) = (x-L)^\alpha$ for $x > R$, where $\alpha > 1$.

Let us first consider the case of $V \equiv 0$.

THEOREM 2.22 (Complex scaling for the free Laplacian). *For $\lambda \in \mathbb{C} \setminus 0$ and $f \in C^1_c(\Gamma)$, define*

$$
\begin{aligned}
R_{0,\Gamma}(\lambda) f(z) &:= \frac{i}{2\lambda} \int_\Gamma e^{i\lambda\varphi(z,w)} f(w)\,dw, \\
\varphi(\gamma(t), \gamma(s)) &:= \pm(\gamma(t) - \gamma(s)), \quad \pm(t-s) \ge 0.
\end{aligned}
\tag{2.7.5}
$$

For $\lambda \in \Lambda_\Gamma$, $R_{0,\Gamma}(\lambda)$ extends to an operator $L^2(\Gamma) \to H^2(\Gamma)$ which is a two sided inverse of $(D^\Gamma_z)^2 - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma)$.

Proof. 1. For $f \in C^2_c(\Gamma)$ we check by direct calculation that

$$R_{0,\Gamma}(\lambda)((D^\Gamma_z)^2 - \lambda^2) f(z) = f(z), \quad ((D^\Gamma_z)^2 - \lambda^2) R_{0,\Gamma}(\lambda) f(z) = f(z).$$

Since $C^2_c(\Gamma)$ is dense in $L^2(\Gamma)$ and in $H^2(\Gamma)$, the result will follow once we show that $R_{0,\Gamma}(\lambda)$ is bounded on L^2 for $\lambda \in \Lambda_\Gamma$.

2. To bound $R_{0,\Gamma}(\lambda)$ on $L^2(\Gamma)$ we can, by reparametrization, assume that $\gamma(t) = b_\pm + e^{i\theta} t$ for $\pm t \ge C_0$. Then for $0 < \theta + \arg\lambda < \pi$

$$
\begin{aligned}
\operatorname{Re}(i\lambda\varphi(\gamma(t), \gamma(s))) &= -\sin(\theta + \arg\lambda)|\lambda||t-s| + \mathcal{O}(1) \\
&\le -\varepsilon|\lambda||t-s| + \mathcal{O}(1),
\end{aligned}
$$

for some $\varepsilon > 0$. This implies that

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |e^{i\lambda\varphi(\gamma(t),\gamma(s))}||\gamma'(s)|ds < \infty$$

so that boundedness of $R_{0,\Gamma}(\lambda)$ on $L^2(\Gamma)$ follows from Schur's estimate (A.5.3). $\square$

We will now use the inverse of the free operator $(D_z^\Gamma)^2 - \lambda^2$ to show that $P_{V,\Gamma} - \lambda^2$, $\lambda \in \Lambda_\Gamma$, is a Fredholm operator and to identify the values of λ for which it is not invertible with scattering resonances.

THEOREM 2.23 (Complex scaling in dimension one). *Suppose that Γ satisfies (2.7.3) and $P_{V,\Gamma}$ is defined by (2.7.2) with $V \in L^\infty_{\mathrm{comp}}(\mathbb{R};\mathbb{C})$.*

1) For $\lambda \in \Lambda_\Gamma$,

$$P_{V,\Gamma} - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma)\,, \tag{2.7.6}$$

is a Fredholm operator and the spectrum of $P_{\Gamma,V}$ in Λ_Γ is discrete.

2) We have

$$m_R(\lambda) = \mathrm{tr}_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{V,\Gamma})^{-1} 2\zeta d\zeta, \quad \lambda \in \Lambda_\Gamma, \tag{2.7.7}$$

where the integral is over any sufficiently small positively oriented curve enclosing λ (the value is constant when the curves are sufficiently small). In particular, the eigenvalues of $P_{\Gamma,V}$ in Λ_Γ are independent of Γ and coincide with scattering resonances.

3) With the same notation,

$$m_D(\lambda) = \mathrm{tr}_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{V,\Gamma})^{-1} 2\zeta d\zeta, \quad \lambda \in \Lambda_\Gamma, \tag{2.7.8}$$

where $m_D(\lambda)$ was defined in (2.2.29).

REMARKS. 1. The operator

$$\Pi_{\lambda,\Gamma} = \frac{1}{2\pi i} \oint_\lambda (\zeta^2 - P_{V,\Gamma})^{-1}\, 2\zeta\, d\zeta : L^2(\Gamma) \to L^2(\Gamma),$$

is a projection – see Theorem C.9. Since the operator $P_{V,\Gamma}$ is not normal the projection is not orthogonal. The trace in (2.7.6) then gives the rank of this projection.

As in the proof of Theorem 2.7 we then see that near an eigenvalue, $\lambda \in \Lambda_\Gamma$,

$$(P_{V,\Gamma} - \zeta^2)^{-1} = \sum_{j=1}^{m} \frac{(P_{V,\Gamma} - \lambda^2)^{j-1}\Pi}{(\lambda^2 - \zeta^2)^j} + A(\zeta, \lambda), \tag{2.7.9}$$

where, for ζ near λ, $\zeta \mapsto A(\zeta, \lambda)$ is a holomorphic family of bounded operators on $L^2(\Gamma)$. The fact that the order of the pole is equal to the rank of the projection is a consequence of the fact that we are dealing with ordinary differential equations which can have at most one L^2 solution – see step 2 of the proof below.

2. Part 3 of the theorem provides the proof of Theorem 2.8 in the case when $m_R(\lambda) > 1$, for scattering poles with $\arg\lambda > -\pi$. When V is real valued that gives the result for all λ as there are no poles on the real axis. (We can also use the symmetry (2.2.13).) For complex valued potentials we need to change the contour to $\overline{\Gamma}$ to obtain the same result for poles near $(-\infty, 0)$.

Proof. 1. For $\lambda \in \Lambda_\Gamma$, Theorem 2.22 shows

$$\rho R_{0,\Gamma}(\lambda) : L^2(\Gamma) \to H^2([-L, L]) \hookrightarrow L^2(\Gamma),$$

and hence $V R_{0,\Gamma}(\lambda) = V \rho R_{0,\Gamma}(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ is a compact operator. It follows that

$$(P_{V,\Gamma} - \lambda^2) R_{0,\Gamma}(\lambda) = (I + V R_{0,\Gamma}(\lambda))$$

which implies the Fredholm property of $P_{V,\Gamma} - \lambda^2$, $\lambda \in \Lambda_\Gamma$. Theorem C.8 shows that $(I + V R_{0,\Gamma}(\lambda))^{-1}$ is a meromorphic family of operators which means that the spectrum of $P_{V,\Gamma}$ in Λ_Γ is discrete.

2. Suppose $\lambda \in \Lambda_\Gamma$ is a resonance of multiplicity $m = m_R(\lambda)$. According to Theorem 2.5 this equivalent to the existence of $u_m : \mathbb{R} \to \mathbb{C}$ satisfying $(P_V - \lambda^2)^m u_m(x) = 0$ and

$$u_1(x) := (P - \lambda^2)^{m-1} u_m(x) = \begin{cases} A e^{i\lambda x} & x \geq L \\ B e^{-i\lambda x} & x \leq -L \end{cases}$$

This means that u_m satisfies

$$u_m(x) = \begin{cases} P(x) e^{i\lambda x}, & x \geq L, \\ Q(x) e^{-i\lambda x}, & x \leq -L, \end{cases}$$

with polynomials P and Q of degree $m - 1$. (This is easy to see on the Fourier transform side: $0 = (\xi^2 - \lambda^2) P(-D_\xi) \delta(\xi - \lambda) = (2\lambda)^m \sum_{j=0}^{\deg P} (\xi - \lambda)^m p_j \delta^{(j)}(\xi - \lambda) \Longrightarrow \deg P \leq m - 1$.)

We now define $\tilde{u} : \Gamma \to \mathbb{C}$ as follows. We write Γ as a disjoint union of connected components,

$$\Gamma = \Gamma_- \cup [-L, L] \cup \Gamma_+,$$

so that $\operatorname{Im} z \to \pm\infty$ on $\Gamma_\pm$. We then put

$$\tilde{u}_m(z) = \begin{cases} P(z) e^{i\lambda z}, & z \in \Gamma_+, \\ u_m(z), & z \in [-L, L], \\ Q(z) e^{-i\lambda z} & z \in \Gamma_-. \end{cases} \tag{2.7.10}$$

The function $\tilde{u}_m$ clearly satisfies

$$(P_{V,\Gamma} - \lambda^2)^m \tilde{u}_m = 0, \quad (P_{V,\Gamma} - \lambda^2)^{m-1} \tilde{u}_m \neq 0.$$

Since

$$\operatorname{Re}(i\lambda z|_{\Gamma_\pm}) = -\sin(\theta + \arg\lambda)|\lambda||z| + \mathcal{O}(1) < -\varepsilon|\lambda||z| + \mathcal{O}(1), \quad \varepsilon > 0.$$

It follows that

$$\|\tilde{u}_m\|^2_{L^2(\Gamma)} \leq \int_{-L}^{L} |u_m(x)|^2 dx + \sum_{\pm} C \int_{\Gamma_\pm} |z|^{m-1} e^{-\varepsilon|\lambda||z|} d|z| < \infty,$$

and the same estimate is valid for $\tilde{u}_j := (P_{V,\Gamma} - \lambda^2)^{m-j}\tilde{u}_m$, $1 \leq j \leq m$. Because $u_1 \in L^2(\Gamma)$ solving $(P_{V,\Gamma} - \lambda^2)u_1 = 0$ is unique up to a multiplicative constant (there is always a solution of the ordinary equation which is not in L^2), this shows that for every resonance of multiplicity m we obtain an eigenvalue of $P_{V,\Gamma}$ of algebraic multiplicity m and geometric multiplicity 1.

3. This argument can be reversed. Let $\lambda \in \Lambda_\Gamma$ be an eigenvalue of $P_{V,\Gamma}$ of multiplicity m (and, as explained above, geometric multiplicity 1). Since

$$e^{-\mp iz}|_{\Gamma_\pm} \notin L^2(\Gamma),$$

this forces the corresponding $\tilde{u}_m$ to be of the form (2.7.10) and by reversing the construction in step 2 we see that λ is a resonance of multiplicity m. This proves (2.7.7).

4. To prove (2.7.8) choose $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R})$ with support in $[-L, L]$ and equal to 1 on the support of V. In view of (2.7.1) ρ defines a function on Γ and the multipliation operator on $L^2(\Gamma)$: $u(z) \mapsto \rho(z)u(z)$. The definition of $R_{0,\Gamma}$ shows that

$$\rho R_{0,\Gamma}(\lambda)V = \rho R_0(\lambda)V\,,$$

where the operator on the right is well defined as an operator on $L^2(\Gamma)$. Consequently,

$$(I + \rho R_{0,\Gamma}(\lambda)V)^{-1} = (I + \rho R_0(\lambda)V)^{-1}$$

is a meromorphic family of operators on $L^2(\Gamma)$ – see (2.2.10). Arguing as in step 3 of the proof of Theorem 2.2 we obtain the following analogue of (2.2.9):

$$\begin{aligned} R_{V,\Gamma}(\lambda) &= R_{0,\Gamma}(\lambda)(I + VR_{0,\Gamma}(\lambda)\rho)^{-1}(I - VR_{0,\Gamma}(\lambda)(1-\rho)) \\ &= R_{0,\Gamma}(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_{0,\Gamma}(\lambda)(1-\rho))\,. \end{aligned} \tag{2.7.11}$$

Since

$$R_{0,\Gamma}(\lambda) = (P_{0,\Gamma} - \lambda^2)^{-1} : L^2(\Gamma) \to H^2(\Gamma)\,,$$
$$I - VR_{0,\Gamma}(\lambda)(1-\rho) = (I + VR_{0,\Gamma}(\lambda)(1-\rho))^{-1} : L^2(\Gamma) \to L^2(\Gamma)\,,$$

are holomorphic families of invertible operators, we apply Theorem C.11 to obtain (2.7.7). $\square$

INTERPRETATION. The method of complex scaling identifies scattering resonances in conic regions Λ_Γ with eigenvalues of non self-adjoint operators $P_{V,\Gamma}$. We gain the advantage of being able to use methods of spectral theory, albeit in the murkier non-normal setting. The multiplicity of a resonance is now the trace of a projection. The resonant states, that is the outgoing solutions to $(P-\lambda^2)u$, are restrictions to $\mathbb{R}$ of functions which continue holomorphically to fuctions with L^2 restrictions to Γ. In one dimension this is explicit as seen in (2.7.10). Since have dealt only with compactly supported potentials our countours Γ had to coincide with $\mathbb{R}$ near the support of V. The method generalizes to the case of potentials which are analytic and decaying to 0 in conic neighbourhoods of $\pm(L,\infty)$. As we will see later on it also generalized to higher dimensions though of course the treatment is not as explicit there.

As an application of the method we present the following result about perturbation of resonances.

THEOREM 2.24 (Continuity of resonances under perturbations). *Suppose that $V_0 \in L^\infty_{\rm comp}([-L,L])$ and that $\Omega \Subset \mathbb{C}$ is a fixed bounded open set with a C^1 boundary $\partial\Omega$ such that there are no resonances of V_0 on $\partial\Omega$.*

Denoting by $m_V(\lambda)$ the multiplicity of λ as a resonance of V there exists ε such that for $V \in L^\infty_{\rm comp}([-L,L])$ with $\|V-V_0\|_\infty < \varepsilon$, we have

$$\sum_{\lambda\in\Omega} m_{V_0}(\lambda) = \sum_{\lambda\in\Omega} m_V(\lambda). \tag{2.7.12}$$

REMARK. The dependence of ε on V_0 and Ω can be quite dramatic. The simplest example is given by taking a family $V_s = sV$, $s\in[0,1]$, where $V\neq 0$ is a fixed potential in $L^\infty_{\rm comp}$. From Theorem 2.16 we know that for $s\neq 0$ there infinitely many resonances while $s=0$ there is only one resonance at 0.

Proof of Theorem 2.24. 1. Let us first assume that $0\notin\Omega$. By writing Ω as a disjoint union of sets with piecewise C^1 boundaries and no resonances on those boundaries, we can assume that $\Omega\subset\Lambda_\Gamma$ or $\Omega\subset\Lambda_{\bar\Gamma}$ for some Γ ($\bar\Gamma := \{\bar\lambda : \lambda\in\Gamma\}$) and we will consider the first of these cases, the other being analogous. (As discussed in Remark 2 after Theorem 2.23 we need the second case for resonances near $(-\infty,0)$ in the case V is not real valued.) We take contours such that $\Gamma\cap\mathbb{R}\subset[-L-1,L+1]$.

Hence the resonances of V in Ω are the same as eigenvalues of $P_{V_0,\Gamma}$ there. From (2.7.7) we see that

$$\sum_{\lambda\in\Omega} m_{V_0}(\lambda) = \frac{1}{2\pi i}\operatorname{tr}\int_{\partial\Omega}(\zeta-P_{V_0,\Gamma})^{-1}d\zeta, \tag{2.7.13}$$

2. Since there are no resonances on $\partial\Omega$, there exists M such that

$$\|(\zeta-P_{V_0,\Gamma})^{-1}\|_{L^2(\Gamma)\to H^2(\Gamma)} \le M, \quad \zeta\in\partial\Omega.$$

For $V \in L^\infty([-L, L])$,

$$\zeta - P_{V,\Gamma} = (\zeta - P_{V_0,\Gamma}) \left(I - (\zeta - P_{V_0,\Gamma})^{-1}(V - V_0)\right)$$

Hence for $\|V - V_0\|_\infty < \varepsilon$, with epsilon sufficiently small, the last factor can be inverted by Neumann series. It follows that

$$\|(\zeta - P_{V,\Gamma})^{-1}\|_{L^2(\Gamma)\to H^2(\Gamma)} \le 2M, \quad \zeta \in \partial\Omega.$$

With $\rho \in C_c^\infty((-L-1, L+1))$, $\rho = 1$ on $[-L, L]$, we obtain a bound on the trace class norm:

$$\begin{aligned} \|\rho(\zeta - P_{V_0,\Gamma})^{-1}\|_{\mathcal{L}_1} &\le \|\rho(\zeta - P_{V_0,\Gamma})^{-1}\|_{L^2(\Gamma)\to H^2(\Gamma)} \\ &\le C(+\sup_\Omega |\zeta| M) \le C' M. \end{aligned}$$

We can now estimate

$$\begin{aligned} &\frac{1}{2\pi}\left|\operatorname{tr}\int_{\partial\Omega}\left((\zeta - P_{V_0,\Gamma})^{-1} - (\zeta - P_{V,\Gamma})^{-1}\right) d\zeta\right| \\ &\quad = \frac{1}{2\pi}\left|\operatorname{tr}\int_{\partial\Omega}\left((\zeta - P_{V_0,\Gamma})^{-1}(V - V_0)\rho(\zeta - P_{V,\Gamma})^{-1}\right) d\zeta\right| \\ &\quad \le CM^2\|V - V_0\|_{L^\infty} < 1, \end{aligned}$$

if $\|V - V_0\|_{L^\infty} < \varepsilon$ with ε small enough. Since the left hand side has to take integral values this means it has to be equal to 0. Returning to (2.7.13) that means that we can replace V_0 by V on the right hand side and that proves (2.7.12).

3. When $0 \in \Omega$ we need a different argument as the complex scaling method does not work there. However, we can assume that $\Omega = D(0, r)$ as we can apply the previous argument to $\Omega \setminus D(0, r)$. We can further assume that V_0 has a resonance at zero and that $\overline{D(0, r)}$ does not contain any other resonances. Theorem 2.8 then shows that (we use the obvious notation for multiplicities depending on the potential)

$$m_{V_0}(0) - 1 = \operatorname{tr}\int_{|\zeta|=r}(I + V_0\rho R_0(\zeta)\rho)^{-1}V_0\partial_\zeta(\rho R_0(\zeta)\rho)d\zeta,$$

where $\rho \in C_c^\infty(\mathbb{R})$ is equal to 1 on $[-L, L]$. Also,

$$\|(I + V_0\rho R_0(\zeta)\rho)^{-1}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < M, \quad |\zeta| = r.$$

Since

$$I + V\rho R_0(\lambda)\rho = (I + V_0\rho R_0(\lambda)\rho)\left(I + (I + V_0\rho R_0(\lambda)\rho)^{-1}(V - V_0)R_0(\lambda)\rho\right)$$

we see that if $\|V - V_0\|_{L^\infty} < \varepsilon$ with ε small enough then

$$\|(I + V\rho R_0(\zeta)\rho)^{-1}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < 2M, \quad |\zeta| = r.$$

We conclude that V has no resonances on $|\zeta| = r$ and that

$$\sum_{|\lambda|<r} m_V(\lambda) = 1 + \frac{1}{2\pi i} \operatorname{tr} \int_{|\zeta|=r} (I + V\rho R_0(\zeta)\rho)^{-1} V \partial_\zeta (\rho R_0(\zeta)\rho) d\zeta, \tag{2.7.14}$$

noting that right hand side, as an integral of a logarithmic derivative of a meromorphic function, $D(\lambda)$, takes only integral values. We have

$$\|\partial_\zeta(\rho R_0(\zeta)\rho)\|_{\mathcal{L}_1} \leq C, \quad |\zeta| = r,$$

see Lemma 2.18 and (C.3.1). Hence we can argue as in step 2 and see that the right hand side in (2.7.14) is equal to 1. □

REMARK. The reader have probably noticed that the proof in step 3 can be used near any resonance. But as we proved (2.2.30) using complex scaling (see Remark 2 after Theorem 2.23) it is natural to use complex scaling to give a direct proof of the continuity of resonances. One can also see that the argument in the above proof is valid for more general families of operators.

The second application is closely related and shows generic simplicity of scattering resonances. Although we have spent some time and effort discussing resonances of higher multiplicity[2] it is important to remember that higher multiplicities are vere special.

THEOREM 2.25 (Generic simplicity of resonances in dimension one). *For any L, there exists $\mathcal{V} \subset L^\infty([-L, L], \mathbb{R})$ which is an intersection of open dense sets in $L^\infty([-L, L], \mathbb{R})$ and such that*

$$\forall\, V \in \mathcal{V}\,, \ \lambda \in \mathbb{C}\,, \quad m_R(\lambda) \leq 1.$$

REMARKS. 1. An intersection of open dense set is sometimes called a *residual* set and the property that holds on a residual set is called *generic*. Complements of residual sets are called *meagre*. All this is related to the classical Baire category theorem which states that a complete metric space cannot be meagre.

2. As can be seen from the proof, we can replace $L^\infty([-L, L]; \mathbb{R})$ with $C_c^\infty((-L, L); \mathbb{R})$ or other spaces of functions.

We start the proof with the following lemma:

LEMMA 2.26 (An application of Rouché's Theorem). *Suppose that $\varepsilon \mapsto f_\varepsilon(z)$ a family of functions holomorphic in a neighbourhood of $D(0, r_0)$ and that*

$$f_\varepsilon(z) = z^m - \varepsilon + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon|z|), \quad |z| \leq r_0.$$

[2] Higher multiplicities can indeed occur as explained in the example in §2.2. For a more general construction see [Sj87, §4].

Then for ε sufficiently small $f_\varepsilon(z)$ has exactly m simple *zeros in the disc $D(0, r_0)$,*

$$z_k = \varepsilon^{\frac{1}{m}} e^{\frac{2\pi i k}{m}} \left(1 + \mathcal{O}(\varepsilon^{\frac{1}{m}})\right), \quad k = 0, \dots, m-1.$$

Proof. 1. Since

$$|z^m - f_\varepsilon(z)| = |\varepsilon + \mathcal{O}(\varepsilon^2 + \varepsilon r_0)| \le C\varepsilon < |z|^m, \quad |z| = r_0, \quad \varepsilon \ll 1$$

Rouché's Theorem shows that the number of zeros of f_ε and z^m agree (with multiplicities) in $|z| < r_0$, that is f_ε has exactly m zeros there.

2. Consider the discs $D_k := D(\varepsilon^{\frac{1}{m}} e^{\frac{2\pi i m}{k}}, \rho\varepsilon^{\frac{1}{m}})$, $0 \le k \le m-1$. We note that

$$\rho < \pi/m \implies D_k \cap D_\ell = \emptyset, \quad k \ne \ell.$$

For $z \in \partial D_k$,

$$|z^m - \varepsilon - f_\varepsilon(z)| \le C_0 \varepsilon^{1+\frac{1}{m}}.$$

On the other hand, if $\rho > 2C_0\varepsilon^{\frac{1}{m}}$ then for $z \in \partial D_k$,

$$|z^m - \varepsilon| = \varepsilon\rho(1 + \mathcal{O}(\rho^2)) > C_0\varepsilon^{1+\frac{1}{m}} \ge |z^m - \varepsilon - f_\varepsilon(z)|,$$

If ε is small enough we can choose ρ so that $C_0\varepsilon^{\frac{1}{m}} < \rho < \pi/m$, and as D_k's are then disjoint, Rouché's theorem shows that there is exactly one zero of f_ε in each D_k. This shows that all m zeros are simple. $\square$

Proof of Theorem 2.25. 1. Since for real valued potentials, there are no resonances on $\mathbb{R}\setminus\{0\}$, it is enough to show that for a residual set of potentials resonances are simple in $\mathbb{C} \setminus \mathbb{R}_-$.

We introduce the following ordering in $\mathbb{C} \setminus \mathbb{R}_-$:

$$z \preceq w \iff |z| < |w| \ \text{ or } \ |z| = |w|, \ \arg(z) \le \arg(w),$$

where $\arg : \mathbb{C} \setminus \overline{\mathbb{R}}_- \to (-\pi, \pi)$. For any V we order resonances as follows:

$$\lambda_1 \preceq \lambda_2 \preceq \dots \lambda_n \preceq \dots$$

We then define

$$\mathcal{V}_n := \{V \in L^\infty([-L, L]; \mathbb{R}) \ : \ m_R(\lambda_j) = 1, \ j = 1, \dots, n\}.$$

2. Theorem 2.24 shows that the set $\mathcal{V}_n$ is open on $L^\infty([-L, L]; \mathbb{R})$: we take Ω to be the union of n disjoint discs, each containing one resonance λ_j. Theorem 2.24 shows that for potentials in a neighbourhood of $V \in \mathcal{V}_n$ the multiplicity does not change in each disc, which means that resonances remain simple.

3. We now show that $\mathcal{V}_n$ is dense in $L^\infty([-L,L];\mathbb{R})$. That means that for $V \notin \mathcal{V}_n$, any neighbourhood of V in L^∞ contains an element of $\mathcal{V}_n$. We can choose $k \geq 0$ so that ordered resonances of V satisfy

$$\lambda_1 = \cdots = \lambda_{m_1} \prec \lambda_{m_1+1} = \cdots = \lambda_{m_1+m_2} \prec \ldots \lambda_{n+k},$$
$$|\lambda_{n+k}| < |\lambda_{n+k+1}|, \quad m_1 + m_2 + \cdots m_p = n + k.$$

Consider

$$\lambda := \lambda_{m_1 + \cdot + m_{j-1}+1}, \tag{2.7.15}$$

and assume that $m = m_j > 1$. That means that $\lambda \neq 0$ and, using symmetry (2.2.13), we can assume that $\operatorname{Re}\lambda \geq 0$. This means that λ^2 is an eigenvalue of $P_{V,\Gamma}$ for a contour Γ with $\pi/2 < \theta < \pi$ (see (2.7.3)). Hence there exists $w_m \in L^2(\Gamma)$ satisfying

$$(P_{V,\Gamma} - \lambda^2)^m w_m = 0, \quad w_1 := (P_{V,\Gamma} - \lambda^2)^{m-1} w_m \neq 0, \tag{2.7.16}$$

see (2.7.9) and (2.7.10). It is unique up to a multiplicative constant. Following §C.1 we construct the following Grushin problem for $P_{V,\Gamma}$:

$$\mathcal{P}_{V,\Gamma}(\zeta) := \begin{pmatrix} P_{V,\Gamma} - \zeta^2 & R_- \\ R_+ & 0 \end{pmatrix} : H^2(\Gamma) \oplus \mathbb{C} \longrightarrow L^2(\Gamma) \oplus \mathbb{C},$$
$$R_- : \mathbb{C} \to L^2(\Gamma), \quad R_- u_- := u_- w_m, \quad \|w_m\|_{L^2(\Gamma)} = 1,$$
$$R_+ : H^2(\Gamma) \to \mathbb{C}, \quad R_+ u := \int_\Gamma u \bar{w}_1 |dz|, \quad \|w_1\|_{L^2(\Gamma)} = 1,$$

where the normalization of w_m and w_1 was chosen for later convenience. We note that

$$\ker_{H^2(\Gamma)}(P_{V,\Gamma} - \lambda^2) = \mathbb{C} w_1 \tag{2.7.17}$$

and as the index is 0 (changing λ to ζ close to λ produces an invertible operator – see Theorem C.5), the dimension of the coker $(P_{V,\Gamma} - \lambda^2)$ is also 1. The function w_m is not in the image $P_{V,\Gamma} - \lambda^2$ as otherwise m in (2.7.16) would not be minimal, that is, the multiplicity of λ would be greater than m. Hence

$$(P_{V,\Gamma} - \lambda^2) H^2(\Gamma) + \mathbb{C} w_m = L^2(\Gamma). \tag{2.7.18}$$

Hence $\mathcal{P}_{V,\Gamma}(\lambda)$ is invertible and by Lemma C.3 so is $\mathcal{P}_{V,\Gamma}(\zeta)$ for ζ close to λ. We also check that

$$\mathcal{P}_{V,\Gamma}(\zeta)^{-1} = \begin{pmatrix} E(\zeta) & E_+(\zeta) \\ E_-(\zeta) & E_{-+}(\zeta) \end{pmatrix}, \quad E_{-+}(\zeta) = (\zeta^2 - \lambda^2)^m,$$
$$E_-(\lambda) v = \int_\Gamma v \bar{w}_m |dz|, \quad E_+(\lambda) v_+ = v_+ w_1.$$

4. Now take $W \in L^\infty([-L,L],\mathbb{R})$ and consider $\mathcal{P}_{V+\varepsilon W}(\zeta)$. If $\varepsilon > 0$ is small enough, Lemma C.3 shows that this operator is invertible for ζ close to λ and the corresponding E^ε_{-+} has an expansion given by (C.1.7):

$$\begin{aligned} E^\varepsilon_{-+}(\zeta) &= (\zeta^2-\lambda^2)^m - \varepsilon E_-(\zeta) W E_+(\zeta) + \mathcal{O}(\varepsilon^2) \\ &= (\zeta^2-\lambda^2)^m - \varepsilon \int_{-L}^{L} W(x) w_1(x) \bar{w}_m(x) dx + \mathcal{O}(\varepsilon|\lambda-\zeta| + \varepsilon^2). \end{aligned}$$

Since w_1 and w_m solve differential equations (of orders 2 and $2m$ respectively, $w_1\bar{w}_m|_{[-L,L]} \neq 0$. Hence $\int W w_1 \bar{w}_m \neq 0$ for some choice of $W \in L^\infty([-L,L];\mathbb{R})$. We can then apply Lemma 2.26 to see that for $\varepsilon > 0$ small enough all the zeros of $E^\varepsilon_{-+}(\zeta)$ near λ as simple.

5. We can apply this argument to each λ of the form (2.7.15). If $w^{m_1}_{m_j}$ and $w_1^{m_j}$ denote the corresponding w_m and w_1 the condition we need to obtain simplicity of the eigenvalues $P_{V+\varepsilon W,\Gamma}$ near λ_j^2 is

$$\int_{-L}^{L} W(x) w_1^{m_j}(x) \bar{w}^{m_j}_{m_j}(x) dx \neq 0, \quad j = 1, \dots, p.$$

But such W exists as $w_1^{m_j} \bar{w}^{m_j}_{m_1}$ do not vanish identically in $[-L,L]$. We conclude that there exist arbitrarily small $L^\infty([-L,L];\mathbb{R})$ perturbation of V such that the first n resonances of $V+\varepsilon W$ are simple. (Note that by continuity the resonances, the perturbation still satisfy $|\lambda_{n+k}| < |\lambda_{n+k+1}|$ if ε is small enough.) We now take

$$\mathcal{V} = \bigcap_{n=1}^{\infty} \mathcal{V}_n,$$

concluding the proof. $\square$

2.8. SEMICLASSICAL STUDY OF RESONANCES

In this section we will present some results concerning resonances in the semiclassical limit. This means we consider Schrödinger operators with a *small parameter* h:

$$P = P(h) := (hD_x)^2 + V, \quad V \in L^\infty_{\rm comp}(\mathbb{R};\mathbb{R}), \quad 0 < h < 1.$$

REMARK. Since this is the first appearance of h some comments are in place. Although motivated by the Planck constant

$$\hbar = 1.054571726(47) \times 10^{-34} J \cdot s,$$

our h in the Schrödinger equation plays the role of $\hbar/\sqrt{2m}$. Its effective size depends in addition on the units of length used and may vary from problem to problem. The semiclassical approximation $h \to 0$ can be applied in situations where that effective size is small. In many problems the semiclassical approximation, although mathematically valid only for h very small, produces good results for $h = 1$.

CONVENTION. Because the motivation in this section comes from quantum mechanics rather than wave propagation we use a different convention for the spectral parameter. We now consider $z \in \mathbb{C} \setminus (-\infty, 0])$ with the convention that

$$\pm \operatorname{Im} z > 0 \implies \pm \operatorname{Im} \sqrt{z} > 0.$$

From Theorem 2.2 we know that, as an operator $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$,

$$\begin{aligned} R(z,h) &:= ((hD_x)^2 + V - z)^{-1} \\ &= h^{-2}(D_x^2 + h^{-2}V - \lambda^2)^{-1}, \quad \lambda = \sqrt{z}/h, \end{aligned}$$

continues meromorphically from $\operatorname{Im} z > 0,\ \operatorname{Re} z > 0$ to $\operatorname{Im} z \leq 0,\ \operatorname{Re} z > 0$. The poles of that meromorphic continuation are denoted by

$$\operatorname{Res}(P) = \operatorname{Res}(P(h))$$

and the multiplicity is defined as in §§2.2 and 2.7:

$$m(z) := \operatorname{rank} \oint_z R(\zeta, h) d\zeta = \frac{1}{2\pi i} \operatorname{tr} \oint_z (\zeta - (hD_x^\Gamma)^2 - V)^{-1} d\zeta. \tag{2.8.1}$$

2.8.1. Truncated harmonic oscillator. As an example of resonances generated by a *well in an island* we consider the potential given in Fig. 2.10. The method applies to more general potentials having a unique positive nondegenerate minimum and step singularities at the boundary of the support. For simplicity we present the case of the truncated harmonic oscillator and of the ground state.

Thus we consider the potential $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}; \mathbb{R})$ defined by

$$V(x) = \begin{cases} x^2 + 1 & |x| \leq 1, \\ \quad 0 & |x| > 1. \end{cases}$$

and the operator

$$P(h) = (hD_x)^2 + V. \tag{2.8.2}$$

We want to describe the resonance for $P(h)$ which is close to the lowest eigenvalue of $(hD_x)^2 + x^2 + 1$,

$$\tilde{z} = 1 + h, \text{ with an eigenfunction (ground state) } \tilde{u}(x) = e^{-x^2/2h}. \tag{2.8.3}$$

As $\tilde{u}$ is even, we can study $P(h)$ on a half-line with the Neumann boundary condition at $x = 0$.

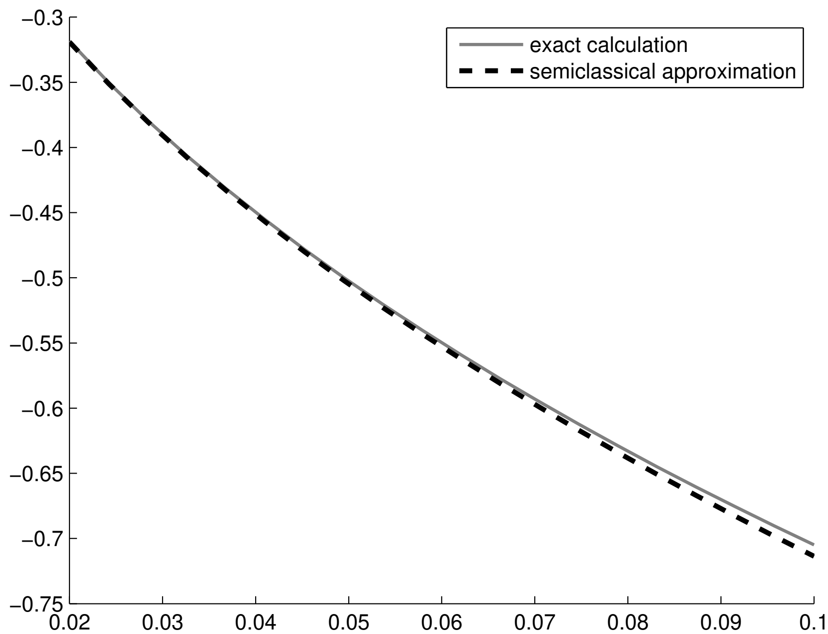


Figure 2.9. Numerically computed values $e^{1/h}\operatorname{Im} z(h)$ for $z(h)$ in Theorem 2.27 compared to the semiclassical approximation given there.

THEOREM 2.27. *For h small enough, there exists a resonance*

$$z = 1 + h - i4\pi^{-1}h^{\frac{1}{2}}e^{-1/h}(1 + \mathcal{O}(h)), \tag{2.8.4}$$

with an even resonant state.

REMARKS. 1. The correction to the real part is of order $e^{-1/2h}$ and the approximation to the imaginary part is very accurate for relatively large values of h – see Fig. 2.9

2. A finer analysis shows that for any even j bounded by a fixed h-independent J, there exists an even resonant state with

$$z_j = 1 + h(2j+1) - i\,h^{\frac{1}{2}-j}c_j e^{-1/h}(1 + \mathcal{O}(h)),$$
$$c_j := 2^{1+2j}\left(\int_0^\infty P_j(y)^2 e^{-y^2}\,dy\right)^{-1}.$$

Here P_j are Hermite polynomials defined recursively by $P_0 = 1$, $P_{j+1} = (2x - \partial_x)P_j$. A similar formula can be given for j odd, and these are the only resonances in

$$[1 - C_1 h, 1 + C_1 h] - i[0, h|\log h|/C_2],$$

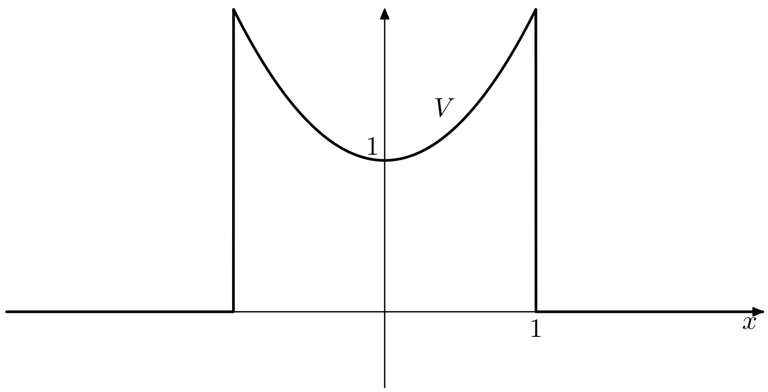


Figure 2.10. The potential V from Theorem 2.27.

where C_1 depends on J.

INTERPRETATION. We see that the eigenvalues $E_j(h) = 1 + (2j+1)h$ of the (shifted) harmonic oscillator become resonances when the potential is truncated and a finite barrier is created. The resonances are exponentially close to the eigenvalues of the (shifted) harmonic oscillator and in particular we have a perturbative formula for the resonance width (the physical name for the imaginary part):

$$-\operatorname{Im} z_1(h) = 4\pi^{-1} h^{\frac{1}{2}} e^{-1/h}(1 + \mathcal{O}(h)),$$

corresponding to tunneling through the barrier. The width is exponentially small in terms of the semiclassical parameter h. These type of resonances are called *shape* resonances as their properties are affected by the shape of the barrier.

The proof of Theorem 2.27 requires some preparation. To start we define $u_z \in C^\infty([0,1])$, $z \in \mathbb{C}$, to be the unique solution of the problem

$$(h^2 D_x^2 + x^2 + 1 - z)u_z = 0, \quad u_z(0) = 1, \; u_z'(0) = 0, \tag{2.8.5}$$

and then put

$$\mathbf{X}(z) = (\mathbf{X}_1(z), \mathbf{X}_2(z)) := (u_z(1), hu_z'(1)) \in \mathbb{C}^2.$$

Then z is a resonance if and only if

$$\mathbf{X}_2(z) - i\sqrt{z}\mathbf{X}_1(z) = 0. \tag{2.8.6}$$

We will apply a contraction mapping principle to the equation (2.8.6) to obtain a resonance exponentially close to $\tilde{z} = 1 + h$. For that, we will need to estimate the first and second derivatives of $\mathbf{X}$ in z.

To compute the first derivative $\partial_z \mathbf{X}$ at $z = \tilde{z}$ we introduce another, growing, solution to $(P(h) - \tilde{z})u = 0$:

$$((hD_x)^2 + x^2 + 1 - \tilde{z})\tilde{v} = 0, \quad \tilde{v}(0) = 0, \quad \tilde{v}'(0) = 1.$$

Since the Wronskian, $W(\tilde{u}, \tilde{v}) := \tilde{u}\tilde{v}' - \tilde{u}'\tilde{v} \equiv 1$, we obtain an expression for $\tilde{v}$ in terms of $\tilde{u}$:

$$\tilde{v}(x) = \tilde{u}(x) \int_0^x \frac{1}{\tilde{u}(y)^2} dy = e^{-x^2/2h} \int_0^x e^{y^2/h} dy.$$

It is easy to see that

$$|\tilde{v}(x)| + |h\tilde{v}'(x)| \leq Ce^{x^2/2h}, \tag{2.8.7}$$

and that[3]

$$\tilde{v}(1) = \tfrac{1}{2}he^{1/2h}(1 + \mathcal{O}(h)), \quad \tilde{v}'(1) = \tfrac{1}{2}e^{1/2h}(1 + \mathcal{O}(h)). \tag{2.8.8}$$

We define

$$\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2) := (\tilde{v}(1), h\tilde{v}'(1)), \tag{2.8.9}$$

and we see from (2.8.8) that

$$\mathbf{Y} = \tfrac{1}{2}he^{1/2h}(1 + \mathcal{O}(h))(1, 1) \tag{2.8.10}$$

With this notation we can state the following lemma.

LEMMA 2.28. *With* $\mathbf{Y}$ *defined by* (2.8.9) *we have*

$$\partial_z \mathbf{X}(\tilde{z}) = -h^{-\frac{3}{2}}\pi^{\frac{1}{2}}\mathbf{Y} + \mathcal{O}(e^{-1/2h}).$$

Proof. 1. For u_z defined by (2.8.5) the derivative with respect to the parameter z satisfies the following non-homogeneous equation:

$$(P(h) - z)\partial_z u_z = u_z; \quad \partial_z u_z(0) = 0, \ \partial_z u_z'(0) = 0. \tag{2.8.11}$$

2. The derivative $\partial_z \mathbf{X}(\tilde{z})$ can be written as a linear combination of $\mathbf{X}(\tilde{z}), \mathbf{Y}$ with Wronskians at $x = 1$ as coefficients:

$$\partial_z \mathbf{X}(\tilde{z}) = W(\partial_z u_{\tilde{z}}, \tilde{v})(1) \cdot \mathbf{X}(\tilde{z}) - W(\partial_z u_{\tilde{z}}, \tilde{u})(1) \cdot \mathbf{Y}. \tag{2.8.12}$$

To compute the Wronskians, we use the following identity, true for all functions w_1, w_2 and each $z \in \mathbb{C}$:

$$h^2 \partial_x W(w_1, w_2) = w_2 \cdot (P(h) - z)w_1 - w_1 \cdot (P(h) - z)w_2. \tag{2.8.13}$$

Since $(P(h) - \tilde{z})\tilde{u} = 0$, (2.8.11) and (2.8.13) show that

$$W(\partial_z u_{\tilde{z}}, \tilde{u})(1) = h^{-2} \int_0^1 \tilde{u}(x)^2 \, dx.$$

[3] The dominant contribution to the integral over $0 < y < 1$ comes from $y = 1$ and hence we can make a change of variables $y^2/h = t$ and integrate by parts over $1/2h < t < 1/h$.

Similarly

$$W(\partial_z u_{\tilde z}, \tilde v)(1) = h^{-2} \int_0^1 \tilde u(x) \tilde v(x)\, dx.$$

Since $\partial_z u_{\tilde z}(0) = \partial_z u'_{\tilde z}(0) = 0$, the bound (2.8.7) gives

$$|W(\partial_z u_{\tilde z}, \tilde v)(1)| \leq C h^{-2}.$$

Inserting the estimate $|\mathbf{X}(\tilde z)| \leq C e^{-1/2h}$ in (2.8.12) shows that

$$\partial X(\tilde z) = -\Big(h^{-2} \int_0^1 \tilde u(x)^2\, dx \Big) \mathbf{Y}(\tilde z_j) + \mathcal{O}(e^{-1/(2h)}).$$

Calculating

$$\int_0^1 \tilde u_j(x)^2\, dx = h^{\frac12} \int_0^{1/h} e^{-y^2} dy = h^{\frac12} \pi^{\frac12} (1 + \mathcal{O}(e^{-1/h})),$$

completes the proof. □

To bound the second derivative $\partial_z^2 \mathbf{X}$ we need to estimate how fast solutions to the initial value problem for the equation $(P(h) - z)u = 0$ can grow:

LEMMA 2.29. *Let C_0 be a fixed constant. Assume that z satisfies*

$$1 \leq \operatorname{Re} z \leq 1 + C_0 h, \quad |\operatorname{Im} z| \leq C_0 h^2. \tag{2.8.14}$$

Then there exists a constant C_1, depending on C_0, such that for each $u \in C^\infty([0,1])$, $f = (P(h) - z)u$, and each $x \in [0,1]$,

$$\begin{aligned} &e^{-x^2/2h}(h^{1/2}|u(x)| + |hu'(x) - xu(x)|) \\ &\quad \leq C_1 \Big(h^{1/2}|u(0)| + h|u'(0)| + h^{-1} \|e^{-y^2/2h} f(y)\|_{L^2(0,x)} \Big). \end{aligned} \tag{2.8.15}$$

Proof. 1. Put

$$v(x) := e^{-x^2/2h} u(x), \quad g(x) := e^{-x^2/2h} f(x),$$

so that

$$-h^2 v'' - 2xhv' + (1 - h - z)v = g,$$

and (2.8.15) becomes

$$h^{1/2}|v(x)| + h|v'(x)| \leq C \Big(h^{1/2}|v(0)| + h|v'(0)| + h^{-1}\|g\|_{L^2(0,x)} \Big). \tag{2.8.16}$$

2. We now put $h + \operatorname{Re} z - 1 = \nu h$, where $\nu \geq 1$ is bounded uniformly in h and independent of x. We have the following estimate valid for $x \geq 0$:

$$\begin{aligned}\frac{1}{2}\partial_x(h^2|v'|^2 + h\nu|v|^2) &= \operatorname{Re}(\overline{v'}(h^2v'' + \nu hv)) \\ &= -2xh|v'|^2 + \operatorname{Im} z \operatorname{Im}(v\overline{v'}) - \operatorname{Re}(g\overline{v'}) \\ &\leq C(h^2|v'|^2 + h^2|v|^2 + h^{-2}|g|^2) \\ &\leq C(h^2|v'|^2 + h\nu|v|^2 + h^{-2}|g|^2).\end{aligned}$$

Since $\partial_x F \leq CF + G$ implies $F(x) \leq e^{Cx}F(0) + \int_0^x e^{C(x-y)}G(y)dy$ (Gronwall's inequality) we arrive to (2.8.16) for $x \in [0,1]$. $\square$

With Lemma 2.29 the bound on the second derivative is easy:

LEMMA 2.30. *Assume that z satisfies (2.8.14). Then for some constant C we have*

$$|\partial_z^2 \mathbf{X}(z)| \leq Ch^{-2}e^{1/2h}.$$

Proof. This follows directly by applying Lemma 2.29 to (2.8.5), (2.8.11), and the equation

$$(P(h) - z)\partial_z^2 u_z = 2\partial_z u_z, \quad \partial_z^2 u_z(0) = 0,\ \partial_z^2 u_z'(0) = 0$$

and putting $x = 1$. $\square$

We can now give the proof of Theorem 2.27. The basic idea is to solve equation (2.8.6) by Newton's method. From Lemma 2.28 we know that the first derivative at $\tilde{z}$ is exponentially large and Lemma 2.30 provides an estimate for the second derivative in a neighbourhood of $\tilde{z}$. Hence we expect that there is a zero exponentially close to $\tilde{z}$ and the proof below shows that this is in fact the case.

Proof of Theorem 2.27. 1. Put

$$\Theta(z) := \mathbf{X}_2(z) - i\sqrt{z}\mathbf{X}_1(z),$$

so that by (2.8.6), z is a resonance if and only if $\Theta(z) = 0$. Using the explicit formula for the ground state (2.8.3) we see that

$$\Theta(\tilde{z}) = -(1+i)e^{-1/2h}(1 + \mathcal{O}(h)).$$

This, Lemma 2.28 and (2.8.10) give

$$\begin{aligned}\frac{\partial_z \Theta(\tilde{z})}{\Theta(\tilde{z})} &= (1+i)^{-1}e^{1/2h}(1 + \mathcal{O}(h))h^{-\frac{3}{2}}\pi^{\frac{1}{2}}(\mathbf{Y}_2(\tilde{z}) - i\mathbf{Y}_1(\tilde{z}) + \mathcal{O}(h)|\mathbf{Y}(z)|) \\ &= 2^{-1}i\pi^{\frac{1}{2}}h^{-\frac{1}{2}}e^{1/h}(1 + \mathcal{O}(h).\end{aligned}$$

2. The equation $\Theta(z) = 0$ is equivalent to $z = \Psi(z)$, where

$$\Psi(z) := z - \frac{\Theta(z)}{d_z\Theta(\tilde{z})}.$$

Take large N and define the disc

$$\Omega := \{|z - \tilde{z}| \leq h^N\}.$$

Then by Lemma 2.30,

$$|\partial_z\Theta(z) - \partial_z\Theta(\tilde{z})| \ll |\partial_z\Theta(\tilde{z})|, \quad z \in \Omega,$$

and thus

$$|\partial_z\Psi(z)| \leq 1/2, \quad z \in \Omega.$$

Then $\Psi : \Omega \to \Omega$ and $|\Psi(z) - \Psi(z')| \leq |z - z'|/2$ for $z, z' \in \Omega$. By contraction mapping principle, the equation $z = \Psi(z)$ has a unique solution z in Ω; this z is then the unique resonance in Ω.

3. To see the asymptotic expansion for z, we use that it is the limit of the sequence $z^{(k)}$ defined by

$$z^{(0)} = \tilde{z}, \quad z^{(k+1)} = \Psi(z^{(k)}).$$

It is then enough to prove that $z = z^{(1)} + \mathcal{O}(h^{\frac{3}{2}}e^{-1/h})$. Since $|z^{(k+1)} - z^{(k)}| \leq |z^{(k)} - z^{(k-1)}|/2$,

$$|z - z^{(1)}| = \left|\sum_{k=1}^{\infty} z^{(k+1)} - z^{(k)}\right| \leq 2|z^{(2)} - z^{(1)}|,$$

and it suffices to show that

$$z^{(2)} = z^{(1)} + \mathcal{O}(h^{\frac{3}{2}}e^{-1/h}).$$

But that is the same as showing that $\Theta(z^{(1)})/\partial_z\Theta(z^{(1)}) = \mathcal{O}(h^{\frac{3}{2}}e^{-1/h})$. The definition of $z^{(1)}$, Lemma 2.30 and the expressions for $\Theta(\tilde{z})$ and $\partial_z\Theta(\tilde{z})$ is step 1, show that

$$\begin{aligned}
\frac{\Theta(z^{(1)})}{\partial_z\Theta(z^{(1)})} &= \frac{\Theta(\tilde{z}) + \partial_z\Theta(\tilde{z})(z^{(1)} - \tilde{z}) + \mathcal{O}(h^{-2}e^{1/2h})(z^{(1)} - \tilde{z})^2}{\partial_z\Theta(\tilde{z}) + \mathcal{O}(h^{-2}e^{1/2h})(z^{(1)} - \tilde{z})} \\
&= \frac{\mathcal{O}(h^{-2}e^{1/2h}(h^{\frac{1}{2}}e^{-1/h})^2)}{2^{-1}(1-i)h^{-\frac{1}{2}}\pi^{\frac{1}{2}}e^{1/2h}(1 + \mathcal{O}(h))} \\
&= \mathcal{O}(h^{-\frac{1}{2}}e^{-2/h}),
\end{aligned}$$

which is an even stronger estimate. This completes the proof of Theorem 2.27. $\square$

2.8.2. A general bound on resonance width. The next result is an easy one dimensional version of a theorem due to Burq (see §6.5 below). It states that for any compactly supported potential the modulus of the imaginary part of the resonance is bounded from below by $\exp(-C/h)$. Theorem 2.27 shows that this bound is optimal. Except for the simple characterization of outgoing solutions, we avoid the use of one dimensional methods to prepare the reader for the proof of the higher dimensional case in §6.5.

To estimate the imaginary we use the following basic fact. Suppose that $V \in L^\infty(\mathbb{R};\mathbb{R})$ and that $u \in H^2([-R,R])$ solves

$$((hD)_x^2 + V(x) - z)u(x) = 0, \quad x \in [-R,R], \quad z \in \mathbb{C}.$$

Then

$$\operatorname{Im} z \int_{-R}^{R} |u(x)|^2 dx = -h^2 \operatorname{Im} u_x \,\overline{u}\big|_{-R}^{R}. \tag{2.8.17}$$

Proof of (2.8.17). Since $u \in C^1([-R,R])$, the following integration by parts argument is justified:

$$\begin{aligned} 0 &= \int_{-R}^{R} \Big(((hD_x)^2 + V - z)u\bar{u} - u\overline{((hD_x)^2 + V - z)u} \Big) dx \\ &= \int_{-R}^{R} ((hD_x)^2 u\bar{u} - u(hD_x)^2\bar{u})dx - (z - \bar{z}) \int_{-R}^{R} |u|^2 dx \\ &= -h^2 u_x \,\bar{u}\big|_{-R}^{R} + h^2 u\, \bar{u}_x\big|_{-R}^{R} - (z-\bar{z}) \int_{-R}^{R} |u|^2 dx. \end{aligned}$$

The formula (2.8.17) follows from dividing this by $2i$. □

Before we use (2.8.17) to estimate resonance width we need the following simple lemma:

LEMMA 2.31. *Suppose that $M > 0$. Then for $u \in H^2_{\rm comp}(\mathbb{R})$ we have*

$$\| e^{-Mx/h}(hD_x)^2 e^{Mx/h} u\|_{L^2} \geq M^2 \|u\|_{L^2}. \tag{2.8.18}$$

Proof. We define the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) := \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} u(x) e^{-ix\xi/h} dx,$$

and recall its basic properties:

$$\mathcal{F}_h(hD_x u)(\xi) = \xi \mathcal{F}_h u(\xi), \quad \|u\|_{L^2(\mathbb{R})} = \|\mathcal{F}_h u\|_{L^2(\mathbb{R})},$$

see for instance [**Zw12**, §3.3]. We then have

$$\begin{aligned}\|e^{-Mx/h}(hD_x)^2 e^{Mx/h} u\|_{L^2} &= \|(hD_x - iM)^2 u\|_{L^2} \\ &= \|(\xi - iM)^2 \mathcal{F}_h u\|_{L^2} \\ &\geq M^2 \|\mathcal{F}_h u\|_{L^2} = M^2 \|u\|_{L^2},\end{aligned}$$

where we used the fact that

$$|(\xi - iM)^2| = |\xi - iM|^2 = |\xi|^2 + M^2 \geq M^2. \qquad \square$$

We are now ready to prove

THEOREM 2.32 (Lower bounds on resonance width in dimension one). *Suppose that* $P(h) := -h^2\Delta + V$, $V \in L^\infty_{\rm comp}(\mathbb{R};\mathbb{R})$ *and that* $E > 0$. *Then there exists* $c = c(V,E)$ *such that for* $0 < h < h_0$,

$$\operatorname{Re} z \in [E/2, E], \ z \in \operatorname{Res}(P(h)) \implies |\operatorname{Im} z| > e^{-c/h}. \tag{2.8.19}$$

Proof. 1. Suppose z is a resonance with $E/2 \leq \operatorname{Re} z \leq E$. In view of Theorem 2.2 this means that there exists u, a resonant state, satisfying

$$((hD_x)^2 + V - z)u = 0, \quad u(x) = A_\pm e^{\pm i\sqrt{z}x/h}, \quad \pm x \gg 1, \quad A_\pm \neq 0.$$

In view of (2.8.17) the lower bound (2.8.19) will follow from showing that for some R,

$$\int_{-R}^{R} |u(x)|^2 dx \leq C e^{c/h} \big| \operatorname{Im} u_x \bar{u}\big|_{-R}^{R}\big|. \tag{2.8.20}$$

.

2. We can assume that $\operatorname{Im} z > -h$ as otherwise there is nothing to prove. Note that as $\operatorname{Re} z \geq E/2$ this implies that $\operatorname{Im}\sqrt{z} > -h/C$. Hence for R sufficiently large,

(2.8.21)

$$\begin{aligned}\big|\operatorname{Im} u_x \bar{u}\big|_{-R}^{R}\big| &= \operatorname{Re} z(|A_+|^2 + |A_-|^2)e^{-2R\operatorname{Im}\sqrt{z}/h} \\ &\geq \frac{1}{C}\int_R^{R+1} |A_+ e^{i\sqrt{z}x/h}|^2 dx + \frac{1}{C}\int_{-R-1}^{-R} |A_- e^{i\sqrt{z}x/h}|^2 dx \\ &= \frac{1}{C}\int_{R\leq|x|\leq R+1} |u(x)|^2 dx,\end{aligned}$$

where the constant C depends on R and E.

3. From (2.8.21) we see that (2.8.20) follows from the estimate

$$\int_{-R}^{R} |u(x)|^2 dx \leq C e^{c/h} \int_{R\leq|x|\leq R+1} |u(x)|^2 dx. \tag{2.8.22}$$

(This is easy by using ODE methods but we proceed in a more complicated way indicating some aspects of higher dimensional methods such as (6.5.30).)

To establish (2.8.22) we use Lemma 2.31: for $M^2 > \|V\|_{L^\infty} + 1$, (2.8.18) shows that for $v \in H^2_{\mathrm{comp}}$,

$$\begin{aligned}\|e^{-Mx/h}((hD_x)^2 + V)e^{Mx/h}v\|_{L^2} &\geq \|e^{-Mx/h}(hD_x)^2 e^{Mx/h}v\|_{L^2} - \|Vv\|_{L^2} \\ &\geq (M^2 - \|V\|_{L^\infty})\|v\|_{L^2} \geq \|v\|_{L^2}.\end{aligned}$$

We apply this to $v = e^{Mx/h}\chi u$ where $\chi \in C^\infty_{\mathrm{c}}((-R-1, R+1); [0,1])$ is equal to 1 on $[-R, R]$:

$$\begin{aligned}\int_{-R}^{R} |u(x)|^2 dx &\leq e^{2MR/h}\|e^{-Mx/h}\chi u\|^2_{L^2(\mathbb{R})} \\ &\leq e^{2MR/h}\|e^{-Mx/h}((hD_x)^2 + V)\chi u\|^2_{L^2} \\ &\leq 2e^{4MR/h}h\|\chi' hD_x u\|^2_{L^2} \\ &\leq Ce^{4MR/h}\int_{R\leq|x|\leq R+1} |u(x)|^2 dx,\end{aligned} \tag{2.8.23}$$

where the last inequality follows from the fact that $\operatorname{supp}\chi' \subset \{R \leq |x| \leq R+1\}$ and $u(x) = A_\pm e^{i\sqrt{z}|x|/h}$ there. Combining this estimate with (2.8.23) gives (2.8.22) completing the proof. $\square$

2.9. NOTES

For more information about the structure of the resolvent at $\lambda = 0$ see Jensen–Nenciu [**JN01**] and references given there.

Theorem 2.16 was proved in some special cases in Regge [**Re58**] and in general (for $V \in L^1_{\mathrm{comp}}(\mathbb{R}; \mathbb{R})$) in [**Zw87**] (see Figure 2.5 for indication how it was done there). Different proofs were given by Froese [**Fr97**] and Simon [**Si00**]. Here we followed [**Fr97**], where complex valued potentials were allowed. That paper also treats certain non-compactly supported potentials.

The importance of the Carleman estimate (2.6.17),(D.1.12) in one dimensional scattering seems to go back to Selberg [**Se53**] in scattering on finite volume hyperbolic surfaces. The reason why scattering on finite volume surfaces is effectively one dimensional will be explained in Example 3 in §4.1 (see also Example 3 in §4.2).

For recent advances in the study of resonances for potentials in dimension one see Korotyaev [**Ko04**],[**Ko05**],[**Ko14**], Bledsoe–Weikard [**BW15**] and references given there.

The presentation of complex scaling in Section 2.7 owes a lot to unpublished notes of Kiril Datchev.

For more general one dimensional "well-in-an-island" potentials the shape resonances were described by Helffer–Sjöstrand [**HS86**, §11] – see also Servat [**Se04**] and Dalla Venezia–Martinez [**DM17**] for more recent accounts and references.

2.10. EXERCISES

Section 2.1

1. Show that in the notation of (2.1.13) we have

$$\begin{aligned} v_+(z) &= -\tfrac{1}{2}\int_0^\infty \int_{z+\tau}^R (Vv+F)(\tau,y)dyd\tau, \\ v_-(z) &= -\tfrac{1}{2}\int_0^\infty \int_{-R}^{z-\tau} (Vv+F)(\tau,y)dyd\tau. \end{aligned}$$

Section 2.2

2. Find an approximation for resonances for a step potential,

$$V(x) = \begin{cases} 0 & |x| > L, \\ V & |x| \leq L, \end{cases}$$

Hint: Use the characterization of a resonant state

$$(D_x^2 + V(x) - \lambda^2)u = 0, \quad u(x) = a_\pm e^{i\lambda|x|}, \quad |x| \geq L.$$

Section 2.4

3. Use (2.4.13) to show that in the notation of (2.4.12),

$$\det S(\lambda) = \frac{t(\lambda)}{t(-\lambda)}. \tag{2.10.1}$$

4. Suppose that $V \in C_c^\infty(\mathbb{R};\mathbb{R})$. Use Theorem 2.13 to show that

$$\sigma(\lambda) := \frac{1}{2\pi i}\log\det S(\lambda) = \sum_{j=1}^J a_j \lambda^{-j} + \mathcal{O}(\lambda^{-J-1}), \quad \lambda \to \infty,$$

where

$$a_1 = -\frac{1}{2\pi}\int V(x)dx, \quad a_2 = \frac{1}{8\pi}\int V(x)^2 dx.$$

Hint: For operators with $\|A\| < 1$ and of trace class (see §B.4 for a review of the trace class) we have $\log\det(I-A) = \operatorname{tr}\log(I-A) = -\sum_{k=1}^\infty k^{-1}\operatorname{tr} A^k$ and this can be applied with $A := -VR_0(\lambda)\rho$ for $\lambda \gg 1$. To evaluate the traces split integrals involving $|x-y|$ to integrals over $x > y$ and $y < x$ and integrate by parts. This result is a special case of Theorem 3.67.

Section 2.6

5. Check carefully that the proof of Theorem 2.19 applies when there *are* negative eigenvalues.

6. Suppose that $x_0 < x_1 < \cdots < x_N$ and $V_j \in \mathbb{R}$, $j = 1, \cdots J$. Define

$$V(x) := \begin{cases} 0 & x \leq 0 \\ V_j & x_{j-1} < x \leq x_j,\ 0 < j \leq N \\ 0 & x > x_N. \end{cases} \tag{2.10.2}$$

Find an expression for $S(\lambda)$ using transfer matrices (which should also be computed):

$$M_{\text{step}}(k_-, k_+) : \begin{pmatrix} a_- \\ b_- \end{pmatrix} \to \begin{pmatrix} a_+ \\ b_+ \end{pmatrix},$$
$$u(x) = a_\pm e^{ik_\pm x} + b_\pm e^{-ik_\pm x}, \quad \pm x \geq 0, \quad u \in C^1(\mathbb{R}),$$
$$M_{\text{free}}(K) = \begin{pmatrix} e^{iK} & 0 \\ 0 & e^{-iK} \end{pmatrix}.$$

This method was used to compute the scattering phase for the data in Fig. 2.6.

Hint: Compute the transfer matrix for the scattering problem with the potential V:

$$M_V(\lambda) : \begin{pmatrix} a_- \\ b_- \end{pmatrix} \to \begin{pmatrix} a_+ \\ b_+ \end{pmatrix},$$
$$u(x) = u(x) = a_\pm e^{ik_\pm x} + b_\pm e^{-ik_\pm x}, \quad \pm x \gg 1, \quad (D_x^2 + V(x) - \lambda^2)u = 0$$

as a product of M_{step} and M_{free} for appropriate choices of the parameters. The scattering matrix sends incoming data to outgoing data:

$$S(\lambda) : \begin{pmatrix} a_- \\ b_+ \end{pmatrix} \to \begin{pmatrix} a_+ \\ b_- \end{pmatrix}$$

(see (2.4.2) where the notation is different!) and M_V and S can be related.

Chapter 3

SCATTERING RESONANCES IN ODD DIMENSIONS

3.1 Free resolvent in odd dimensions
3.2 Meromorphic continuation
3.3 Resolvent at zero energy
3.4 Upper bounds on the number of resonances
3.5 Complex valued potentials with no resonances
3.6 Outgoing solutions and Rellich's Theorem
3.7 The scattering matrix
3.8 More on distorted plane waves
3.9 The Birman–Kreĭn trace formula
3.10 The Melrose trace formula
3.11 Scattering asymptotics
3.12 Existence of resonances for real potentials
3.13 Notes
3.14 Exercises

In this section we will consider the simplest higher dimensional situation: scattering by compactly supported potentials in odd dimensions. Some of the results presented in Chapter 2 are valid in this case with proofs requiring only small modifications. Other results, such as asymptotics, or even sharp lower bounds, for the number of scattering poles, are not known.

The main advantage of odd dimensions greater than one is the strong Huyghens principle for the wave equation: if $\Box u = 0$ and the support of initial data lies in $|x| < R$ then support of $u(t, \bullet)$ lies in $|t|-R < |x| < |t|+R$. The weak Huyghens principle valid in all dimensions says only that the support of $u(t, \bullet)$ lies in $|x| < |t| + R$.

One consequence of the strong Huyghens principle is the analytic continuation of $(-\Delta - \lambda^2)^{-1}$ from $\operatorname{Im}\lambda > 0$ to $\mathbb{C}$.

3.1. FREE RESOLVENT IN ODD DIMENSIONS

The outgoing resolvent of the free Laplacian is defined just as in the case of dimension one:

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} \ : \ L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)\,, \quad \operatorname{Im}\lambda > 0\,. \tag{3.1.1}$$

Its existence follows from using the Fourier transform which provides an explicit diagonalization of $-\Delta$:

$$\begin{gathered} R_0(\lambda)\varphi(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\langle \xi, x\rangle}}{|\xi|^2 - \lambda^2}\widehat{\varphi}(\xi)d\xi\,, \quad \operatorname{Im}\lambda > 0\,, \\ \widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x)e^{-i\langle x, \xi\rangle}dx. \end{gathered} \tag{3.1.2}$$

This formula is of course valid in all dimension but the operator $R_0(\lambda)$ has much nicer properties when n is odd.

3.1.1. Relation to the wave equation. We will starts our presentation with the properties of the wave equation. Thus we consider its unique forward fundamental solution:

$$\Box E_+ := (\partial_t^2 - \Delta)E_+ = \delta_0(x)\delta_0(t)\,, \quad \operatorname{supp} E_+ \subset \{t \geq 0\}\,. \tag{3.1.3}$$

For n odd we have a particularly nice expression for the distribution E_+. Its action on $\varphi \in C_c^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$ is given by

$$\begin{gathered} \langle E_+, \varphi\rangle = \int_0^\infty \langle E_+(t), \varphi(t, \bullet)\rangle dt\,, \\ \langle E_+(t), \psi\rangle := \frac{1}{4\pi^k}\left(\frac{d}{ds}\right)^{k-1} \widetilde{\psi}(\sqrt{s})|_{s=t^2}, \quad n = 2k+1, \quad t > 0, \\ \widetilde{\psi}(r) := r^{n-2}\int_{|\omega|=1} \psi(r\omega)d\omega\,, \end{gathered} \tag{3.1.4}$$

and we have a distributional convergence $E_+(t) \to 0$, $E_+'(t) \to \delta_0$ as $t \to 0+$ – see [**Ev98**, §2.4.1] or [**HöI**, Section 6.2].

The crucial fact seen from this expression is the support property of E_+: for odd $n \geq 3$

$$\operatorname{supp} E_+ = \{(x,t) \; : \; |x| = |t|, \; t \geq 0\} \,. \tag{3.1.5}$$

This implies the *strong Huyghens principle*:

$$\begin{gathered} \Box u = f, \; \operatorname{supp} f \subset [-R,R]_t \times B_{\mathbb{R}^n_x}(0,R) \,, \; u|_{t<-R} = 0 \implies \\ u(t,x) = 0 \,, \quad \text{for } |x| < t - 2R. \end{gathered}$$

The *weak Huyghens principle* valid in all dimensions says that $u(t,x) = 0$ for $|x| > t + 2R$.

The distribution $E_+(t)$ appearing in (3.1.4) is used to solve the initial value problem:

$$\begin{gathered} \Box u = 0 \,, \; u(0,x) = \varphi_0(x) \,, \quad \partial_t u(0,x) = \varphi_1(x) \,, \\ u(t,x) = E_+(t) * \varphi_1(x) + \partial_t E_+(t) * \varphi_0(x) \,, \quad \varphi_j \in C^\infty(\mathbb{R}^n) \,, \quad t \geq 0. \end{gathered} \tag{3.1.6}$$

Here $u * v$ denotes the convolution of a compactly supported distribution u with a smooth function v. Putting

$$E(t) := \begin{cases} E_+(t), & t > 0, \\ -E_+(-t), & t < 0, \end{cases}$$

$E(0) := 0$, gives

$$E(t) \in C^\infty(\mathbb{R}_t; \mathscr{D}'(\mathbb{R}^n)).$$

The solution of (3.1.6) can also be given using the spectral decomposition of $-\Delta$ and the functional calculus – this corresponds to the Fourier transform decomposition:

$$\begin{gathered} u(t,x) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \varphi_1(x) + \cos(t\sqrt{-\Delta})\varphi_0(x) \,, \\ f(\sqrt{-\Delta})\varphi(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(|\xi|) e^{i\langle x,\xi\rangle} \widehat{\varphi}(\xi) d\xi \,, \end{gathered} \tag{3.1.7}$$

where $f(\rho) = \sin t\rho/\rho$ or $f(\rho) = \cos t\rho$.

If we write

$$U(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \,,$$

then a comparison with (3.1.6) gives the Schwartz kernel of $U(t)$:

$$U(t,x,y) = E(t, x-y), \tag{3.1.8}$$

see [**HöI**, Section 6.1] for the details on the pull back (by $(x,y) \mapsto x - y$ here) of distributions.

The strong Huyghens principle (3.1.5) implies that

$$\left(U(t)v\right)(x) = 0, \quad t > \sup\{|x-y| \colon y \in \operatorname{supp} v\}. \tag{3.1.9}$$

For future reference we note that the spectral representation immediately gives

$$\partial_t^k U(t) \;:\; H^s(\mathbb{R}^n) \longrightarrow H^{s-k+1}(\mathbb{R}^n)\,, \quad k \in \mathbb{N}\,, \quad s \in \mathbb{R}\,. \tag{3.1.10}$$

The free resolvent $R_0(\lambda)$ given by (3.1.2) can be written using $U(t)$:

$$R_0(\lambda) = \int_0^\infty e^{i\lambda t} U(t) dt\,. \tag{3.1.11}$$

In fact, for $\operatorname{Im}\lambda > 0$,

$$\frac{1}{|\xi|^2 - \lambda^2} = \int_0^\infty \frac{\sin t|\xi|}{|\xi|} e^{it\lambda} dt\,, \quad \operatorname{Im}\lambda > 0\,,$$

where the integral converges since $\sup_{\lambda\in\mathbb{R}} |\sin t\lambda/\lambda| = |t|$. The formula (3.1.11) then follows from (3.1.2) and (3.1.7).

This representation gives us the following important result:

THEOREM 3.1 (**Free resolvent in odd dimensions**). *Suppose that $n \geq 3$ is odd. Then the resolvent defined by*

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1} \;:\; L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,,$$

for $\operatorname{Im}\lambda > 0$, continues analytically to an entire family of operators

$$R_0(\lambda) : L^2_{\mathrm{comp}}(\mathbb{R}^n) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

For any $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ we have the following estimates:

$$\rho R_0(\lambda)\rho = \mathcal{O}((1+|\lambda|)^{j-1} e^{L(\operatorname{Im}\lambda)_-}) \;:\; L^2(\mathbb{R}^n) \longrightarrow H^j(\mathbb{R}^n)\,, \tag{3.1.12}$$

$j = 0, 1, 2$, where $L > \operatorname{diam}(\operatorname{supp}\rho) := \sup\{|x-y| : x, y \in \operatorname{supp}\rho\}$.

Proof. 1. For the statement about holomorphy it suffices show that for any $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$,

$$\rho R_0(\lambda)\rho : L^2 \longrightarrow L^2$$

continues from $\operatorname{Im}\lambda > 0$ to an entire family of bounded operators.

2. If $L > \operatorname{diam}\operatorname{supp}\rho$ then (3.1.9) gives $\rho U(t)\rho = 0$ for $t \geq L$. Then (3.1.11) shows that, for $\operatorname{Im}\lambda > 0$ at first,

$$\rho R_0(\lambda)\rho = \int_0^L e^{i\lambda t} \rho U(t)\rho\, dt\,. \tag{3.1.13}$$

The right hand side is now defined and, as an operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, holomorphic for $\lambda \in \mathbb{C}$.

3. Since $U(t) = \sin t\sqrt{-\Delta}/\sqrt{-\Delta}$, and $\sup_{\lambda\in\mathbb{R}} |\sin t\lambda/\lambda| = |t|$, we have

$$\begin{aligned}\|U(t)\|_{L^2\to H^1} &\leq C\|U(t)\|_{L^2\to L^2} + C\|\sqrt{-\Delta}U(t)\|_{L^2\to L^2} \\ &= \mathcal{O}(|t|+1)\,.\end{aligned}$$

This and (3.1.13) give the bound (3.1.12) for $j = 1$. For $j = 0$ we write

$$\lambda\rho R_0(\lambda)\rho = \int_0^L D_t(e^{i\lambda t})\rho U(t)\rho dt = -\int_0^L e^{i\lambda t}\rho D_t U(t)\rho dt\,.$$

We have

$$D_t U(t) = -i\cos t\sqrt{-\Delta} = \mathcal{O}_{L^2\to L^2}(1)\,,$$

and the bound (3.1.12) for $j = 0$ follows.

4. Finally, we consider (3.1.12) for $j = 2$. Suppose that $\rho_1 \in C_c^\infty(\mathbb{R}^n)$ satisfies

$$\rho_1 = 1 \quad \text{near } \operatorname{supp}\rho, \quad \operatorname{diam}(\operatorname{supp}\rho_1) < L.$$

Since $(-\Delta - \lambda^2)R_0(\lambda) = I$, we have

$$\begin{aligned}
\|\rho R_0(\lambda)\rho\|_{L^2\to H^2} &\leq C\|\Delta\rho R_0(\lambda)\rho\|_{L^2\to L^2} + C\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\\
&\leq C\|\rho\Delta R_0(\lambda)\rho\|_{L^2\to L^2} + C\|[\Delta,\rho](\rho_1 R_0(\lambda)\rho_1)\rho\|_{L^2\to L^2}\\
&\qquad + C\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\\
&\leq C|\lambda|^2\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} + C\|\rho_1 R_0(\lambda)\rho_1\|_{L^2\to H^1}\\
&\qquad + C\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} + C\,,
\end{aligned}$$

for some constants C (which may change from line to line). Hence (3.1.12) for $j = 2$ follows from the estimates for $j = 0, 1$. $\square$

The wave equation representation and the formulæ for $E(t)$ (and hence $U(t)$ in view of (3.1.8)) given in (3.1.4) can be used to derive an explicit formula for the Schwartz kernel of $R_0(\lambda)$, $R_0(\lambda, x, y)$. Instead we take a direct approach based on the Fourier transform representation (3.1.2).

3.1.2. An explicit formula for $R_0(\lambda)$ in odd dimensions. We start with the following

LEMMA 3.2 (Oscillatory integrals over $\mathbb{S}^{n-1}$). *Suppose that $n \geq 3$ is odd and $d\omega$ denotes the standard measure on $\mathbb{S}^{n-1}$ (induced from the Lebesgue measure on $\mathbb{R}^n$, $\mathbb{S}^{n-1} := \{x : |x| = 1, x \in \mathbb{R}^n\}$).*

Then for $\zeta \in \mathbb{R}$, and $x \in \mathbb{R}^n$,

$$\int_{\mathbb{S}^{n-1}} e^{i\zeta\langle\omega,x\rangle}d\omega = 2\pi^{\frac{n-1}{2}}\left(e^{i\zeta|x|}F_n(\zeta|x|) + e^{-i\zeta|x|}F_n(-\zeta|x|)\right), \tag{3.1.14}$$

where $F_n(k)$ is given by

$$F_n(k) := e^{-2ik}(-\partial_k)^{\frac{n-3}{2}}\left(e^{2ik}/ik^{\frac{n-1}{2}}\right), \tag{3.1.15}$$

REMARK. The integral in (3.1.14) can be expressed using the Bessel function $J_{\frac{n-2}{2}}$ but we take a direct approach.

Proof. 1. It is clear that the right hand side is a function of ζ and $|x|$. Hence we can assume that $x = |x|e_1$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Using the parametrization of $\mathbb{S}^{n-1}$ by $B_{\mathbb{R}^{n-1}}(0, 1)$:

$$\omega = (\pm\sqrt{1-r^2}, r\theta), \quad \theta \in \mathbb{S}^{n-2}, \quad 0 \leq r \leq 1,$$

and putting $c_n := \operatorname{vol}(\mathbb{S}^{n-2})$ and $k := \zeta|x|$, we have

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} e^{ik\langle\omega,e_1\rangle} d\omega &= c_n \int_0^1 \left(e^{ik\sqrt{1-r^2}} + e^{-ik\sqrt{1-r^2}}\right) \frac{r^{n-2}}{\sqrt{1-r^2}} dr \\
&= 2c_n \int_0^1 \cos(ky)(1-y^2)^{\frac{n-3}{2}} dy \\
&= 2c_n (1+\partial_k^2)^{\frac{n-3}{2}} \int_0^1 \cos(ky) dy \\
&= c_n (1+\partial_k^2)^{\frac{n-3}{2}} \left(\frac{e^{ik} - e^{-ik}}{ik}\right).
\end{aligned}$$

2. Putting $D_k := (1/i)\partial_k$, we factorize

$$\begin{aligned}
(1+\partial_k^2)^{\frac{n-3}{2}} &= (1+D_k)^{\frac{n-3}{2}}(1-D_k)^{\frac{n-3}{2}} \\
&= (e^{-ik} D_k e^{ik})^{\frac{n-3}{2}} (-e^{ik} D_k e^{-ik})^{\frac{n-3}{2}} \\
&= e^{-ik} D_k^{\frac{n-3}{2}} e^{2ik} (-D_k)^{\frac{n-3}{2}} e^{-ik} \\
&= e^{-ik} (-\partial_k)^{\frac{n-3}{2}} e^{2ik} (-\partial_k)^{\frac{n-3}{2}} e^{-ik}.
\end{aligned}$$

Hence

$$\begin{aligned}
(1+\partial_k^2)^{\frac{n-3}{2}} \left(\frac{e^{ik}}{ik}\right) &= e^{-ik}(-\partial_k)^{\frac{n-3}{2}} e^{2ik} (-\partial_k)^{\frac{n-3}{2}} (1/ik) \\
&= \left(\tfrac{n-3}{2}\right)! \, e^{-ik} (-\partial_k)^{\frac{n-3}{2}} \left(e^{2ik}/ik^{\frac{n-1}{2}}\right),
\end{aligned}$$

with the action on $-e^{-ik}/ik$ obtained by taking complex conjugates. We now recall that $c_n = 2\pi^{(n-1)/2}/\left(\frac{n-3}{2}\right)!$ which gives (3.1.14). $\square$

From Lemma 3.2 we obtain a formula for the Schwartz kernel of $R_0(\lambda)$:

THEOREM 3.3 (Schwartz kernel of the resolvent in odd dimensions). *Suppose that* $n \geq 3$ *is odd. Then the Schwartz kernel of the resolvent* $R_0(\lambda)$ *defined in Theorem 3.1 is given by*

$$\begin{aligned}
R_0(\lambda, x, y) &= \frac{e^{i\lambda|x-y|}}{|x-y|^{n-2}} P_n(\lambda|x-y|), \\
P_n(k) &:= i2^{-n+1}\pi^{-\frac{n-1}{2}} k^{n-2} F_n(k),
\end{aligned} \tag{3.1.16}$$

where F_n is given in (3.1.14); P_n is a polynomial of degree $(n-3)/2$ with

$$\begin{aligned} P_n(0) &= \frac{(n-3)!}{\pi^{\frac{n-1}{2}} 2^{n-1} \left(\frac{n-3}{2}\right)!}, \\ \left(z^{\frac{n-3}{2}} P_n(1/z)\right)|_{z=0} &= \frac{1}{4\pi} \frac{1}{(2\pi i)^{\frac{n-3}{2}}}. \end{aligned} \tag{3.1.17}$$

REMARK. Formulæ (3.1.17) give the extremal coefficients of the polynomial $P_n(z)$: $P_n(0)$ determines the leading asymptotic of $R_0(\lambda, x, y)$ as $|x-y| \to 0$ and, $(z^{\frac{n-3}{2}} P_n(1/z))_{z=0}$, the leading asymptotic as $|x-y| \to \infty$. In dimension $n = 3$, (3.1.16) takes the simple form

$$R_0(\lambda, x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}.$$

Proof. 1. We prove the formula for $\operatorname{Im}\lambda > 0$ and continue both sides analytically in λ. We start by rewriting (3.1.2) using polar coordinates $\xi = \zeta\omega$, $\omega \in \mathbb{S}^{n-1}$, $\zeta \in \mathbb{R}$ and Lemma 3.2:

$$\begin{aligned} R_0(\lambda, x, y) &= \frac{1}{(2\pi)^n} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{e^{i\zeta\langle\omega, x-y\rangle}}{\zeta^2 - \lambda^2} d\omega \zeta^{n-1} d\zeta \\ &= \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{e^{i\zeta\langle\omega, x-y\rangle}}{\zeta^2 - \lambda^2} d\omega \zeta^{n-1} d\zeta \\ &= \frac{1}{2^n \pi^{\frac{n+1}{2}}} \int_{\mathbb{R}} e^{i\zeta|x-y|} F_n(\zeta|x-y|) \frac{\zeta^{n-1}}{\zeta^2 - \lambda^2} d\zeta \\ &\qquad + \frac{1}{2^n \pi^{\frac{n+1}{2}}} \int_{\mathbb{R}} e^{-i\zeta|x-y|} F_n(-\zeta|x-y|) \frac{\zeta^{n-1}}{\zeta^2 - \lambda^2} d\zeta. \end{aligned}$$

We note that switching to the integral over $(0, \infty)$ to the integral over $\mathbb{R}$ was justified as for n odd $\zeta^{n-1} = (-\zeta)^{n-1}$.

2. For $|x-y| \neq 0$, the functions $F_n(\pm\zeta|x-y|)\zeta^{n-1}$ are holomorphic in ζ and hence we can deform the contours in the two integrals on the right hand side: to $\operatorname{Im}\zeta = N \to \infty$ and $\operatorname{Im}\zeta = -N \to \infty$, respectively. For the first integral we obtain a contribution from the pole at $\zeta = \lambda$ and for the second from the pole at $\zeta = -\lambda$. The residue theorem then gives

$$R_0(\lambda, x, y) = 2^{-n+1} \pi^{-\frac{n-1}{2}} i e^{i\lambda|x-y|} \lambda^{n-2} F_n(\lambda|x-y|), \tag{3.1.18}$$

and this gives (3.1.16).

3. To compute the highest and lowest coefficients of P_n we use (3.1.15) which gives

$$\begin{aligned} P_n(0) &= \pi^{-\frac{n-1}{2}} 2^{-n+1} \left(\tfrac{n-1}{2}\right) \left(\tfrac{n-1}{2}+1\right) \cdots \left(\tfrac{n-1}{2}+\tfrac{n-5}{2}\right) \\ &= \pi^{-\frac{n-1}{2}} 2^{-n+1} \frac{(n-3)!}{\left(\frac{n-3}{2}\right)!}. \end{aligned}$$

Similarly we obtain the formula for the highest coefficient of $P_n(z)$. $\square$

We finish with an explicit version of Stone's formula (B.1.12) for the free Laplacian in odd dimensions:

THEOREM 3.4 (**Stone's formula for the free Laplacian**). *Suppose $n \geq 3$ is odd and $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, $\operatorname{Im} \lambda > 0$. Then the analytic continuation of the Schwartz kernel $R_0(\lambda, x, y)$ satisfies*

$$R_0(\lambda, x, y) - R_0(-\lambda, x, y) = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} e^{i\lambda\langle \omega, x-y\rangle} d\omega, \quad \lambda \in \mathbb{C}, \tag{3.1.19}$$

where $d\omega$ denotes the standard measure on $\mathbb{S}^{n-1}$.

REMARK. We refer to (3.1.19) as Stone's formula as it is special case of a formula valid for all self-adjoint operators – see Theorem B.10. The right hand side is related to the spectral measure of $-\Delta$ obtained using the Fourier transform and the left hand side is the difference of boundary values of resolvents at the real axis: for $\lambda > 0$,

$$R_0(\lambda) = \lim_{\varepsilon \to 0+} (-\Delta - \lambda^2 - i\varepsilon)^{-1}, \quad R_0(-\lambda) = \lim_{\varepsilon \to 0+} (-\Delta - \lambda^2 + i\varepsilon)^{-1},$$

where the limits are taken in the sense of operators $C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ (with finer formulations possible).

Proof. We could use contour deformation starting with (3.1.2) as in §3.1.4 below but we can use (3.1.18) and Lemma 3.2.

Indeed, from (3.1.18) we see that

$$\begin{aligned} &R_0(\lambda, x, y) - R_0(-\lambda, x, y) = \\ &\qquad i 2^{-n+1} \pi^{-\frac{n-1}{2}} \lambda^{n-2} (e^{ik} F_n(k) + e^{-ik} F_n(-k))|_{k=\lambda|x-y|}, \end{aligned}$$

which combined with (3.1.14) gives (3.1.19). $\square$

3.1.3. Asymptotic behaviour of $R_0(\lambda)f$.

The next theorem gives asymptotics of $R_0(\lambda)f$, $f \in \mathcal{E}'(\mathbb{R}^n)$ as $|x| \to \infty$ for $\lambda \neq 0$.

This result does not depend on the parity of the dimension.

THEOREM 3.5 (**Outgoing asymptotics**). *Suppose that $n \geq 3$ is odd and that $f \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution (or $f \in \mathscr{S}(\mathbb{R}^n)$).*

Then for $\lambda \in \mathbb{R} \setminus 0$,

$$R_0(\lambda) f(|x|\theta) = e^{i\lambda|x|}|x|^{-\frac{n-1}{2}} h(|x|, \theta)\,,$$

$$h(x,\theta) \sim \sum_{j=0}^{\infty} |x|^{-j} h_j(\theta)\,, \quad h_0(\theta) = \frac{1}{4\pi}\left(\frac{\lambda}{2\pi i}\right)^{\frac{1}{2}(n-3)} \hat{f}(\lambda\theta)\,, \tag{3.1.20}$$

as $|x| \to \infty$.

Proof. 1. The proof is based on the formula (3.1.16) for the Schwartz kernel of $R_0(\lambda)$ and the following expansions valid as $|x| \to \infty$:

(3.1.21)

$$\begin{aligned} |x-y| &= |x|(1 - 2\langle x/|x|, y\rangle/|x| + |y|^2/|x|^2)^{\frac{1}{2}} \\ &= |x| - \langle x/|x|, y\rangle + (|y|^2/2 - \langle x/|x|, y\rangle^2/2)/|x| + \cdots \\ &= |x| - \langle x/|x|, y\rangle + \sum_{k=1}^{K-1} a_k(y, x/|x|)|x|^{-k} + \mathcal{O}(|y|^{K+1}|x|^{-K}). \end{aligned}$$

The last bound is valid for $y \in \mathbb{R}^n$, $|x| > 1$ and $|a_k(y,\omega)| \leq C_k|y|^{k+1}$. Similarly,

$$\begin{aligned} |x-y|^{-p} &= |x|^{-p}(1 - 2\langle x/|x|, y\rangle/|x| + |y|^2/|x|^2)^{-p/2} \\ &= |x|^{-p}\,(1 - p\langle x/|x|, y\rangle/|x| + \cdots) \\ &= |x|^{-p}\left(1 + \sum_{k=1}^{K-1} b_k(y, x/|x|)|x|^{-k} + \mathcal{O}(|y|^K|x|^{-K})\right), \end{aligned} \tag{3.1.22}$$

where $|b_k(y,\omega)| \leq C_k|y|^k$.

2. We now use (3.1.16) and (3.1.17) (see the remark following the theorem) to write

$$R_0(\lambda, x, y) = \frac{1}{4\pi}\frac{\lambda^{\frac{n-3}{2}}}{(2\pi i)^{\frac{n-3}{2}}}\frac{e^{i\lambda|x-y|}}{|x-y|^{\frac{n-1}{2}}}\left(1 + \cdots + a_n|x-y|^{-\frac{n-3}{2}}\right).$$

Pairing this with $f(y) \in \mathcal{E}'(\mathbb{R}^n)$ (or integrating against $f(y) \in \mathscr{S}(\mathbb{R}^n)$) and using expansions (3.1.21) and (3.1.22) gives (3.1.20). □

3.1.4. Continuation of the resolvent using contour deformation. We will now consider another way of continuing the resolvent kernel $R_0(\lambda, x, y)$. To streamline the notation we will write

$$R_0(\lambda, x, y) = R_0(\lambda, x - y),$$

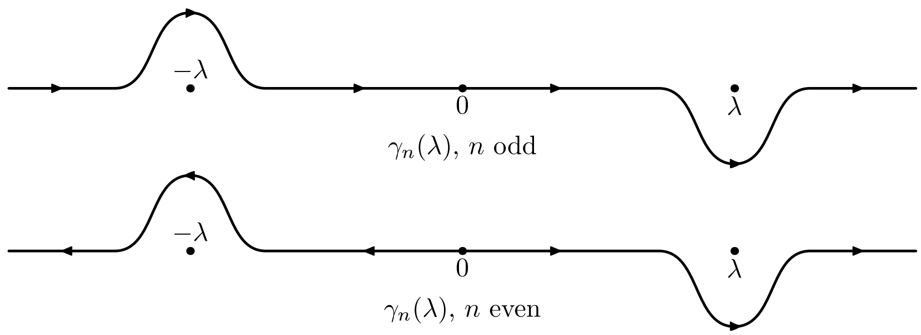


Figure 3.1. Contour deformation used to define $R_0(\lambda)$ for $\operatorname{Im}\lambda \geq 0$.

and we think of $R_0(\lambda, x)$ as a distribution in the x variables. Then for $\varphi \in C_c^\infty(\mathbb{R}^n)$ (for any n, odd or even)

$$R_0(\lambda)(\varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\widehat{\varphi}(\xi)}{|\xi|^2 - \lambda^2} d\xi, \quad \operatorname{Im}\lambda > 0, \tag{3.1.23}$$

where the left hand side is understood as the distributional pairing of $R_0(\lambda)(x)$ and $\varphi(x)$.

REMARK. The spectral representation (3.1.23) immediately implies bounds on the free resolvent for $\operatorname{Im}\lambda > 0$:

$$\|R_0(\lambda)\|_{L^2 \to H^k} \simeq \sup_{t>0} \frac{(1+t)^{k/2}}{|t - \lambda^2|} \leq C' \frac{\langle\lambda\rangle^k}{|\lambda| \operatorname{Im}\lambda}, \quad k = 0, 1, 2. \tag{3.1.24}$$

This estimate is independent of the dimension.

We can re-write the integral in (3.1.23) using polar coordinates in $\xi = \rho\theta$, $\rho \in (0, \infty)$, $\theta \in \mathbb{S}^{n-1}$ so that

$$R_0(\lambda)(\varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{S}^n} \int_0^\infty \frac{\widehat{\varphi}(\rho\theta)}{\rho^2 - \lambda^2} \rho^{n-1} d\rho d\theta, \quad \operatorname{Im}\lambda > 0,$$

where $d\theta$ is the element of integration on $\mathbb{S}^{n-1}$.

We can rewrite the integral in ρ as a contour integral over the following contours:

$$\gamma_n = \begin{cases} \mathbb{R}, \ \text{ oriented from } -\infty \text{ to } +\infty, & \text{for } n \text{ odd} \\ \mathbb{R}_- + \mathbb{R}_+, \ \ \mathbb{R}_\pm \text{ oriented from } 0 \text{ to } \pm\infty, & \text{for } n \text{ even,} \end{cases}$$

$$R_0(\lambda)(\varphi) = \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{S}^n} \int_{\gamma_n} \frac{\widehat{\varphi}(\rho\theta)}{\rho^2 - \lambda^2} \rho^{n-1} d\rho d\theta, \quad \operatorname{Im}\lambda > 0. \tag{3.1.25}$$

We can now deform the contours in a λ-dependent way, as shown in Fig. 3.1 for $\lambda \in \mathbb{R} \setminus \{0\}$. For $\operatorname{Im}\lambda < 0$ the further deformation leads to

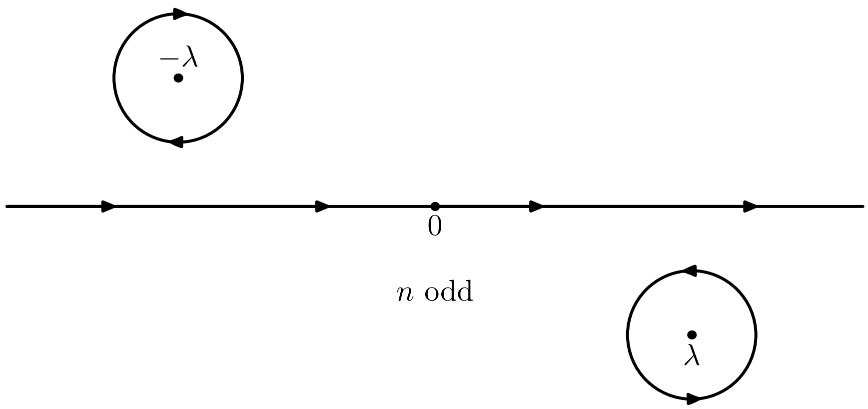


Figure 3.2. Contour deformations used to define $R_0(\lambda)$ for $\lambda > 0$ for n odd. Because of the orientation $R_0(e^{-\pi i}\lambda) = R_0(\lambda)$ for $\lambda > 0$ and the operator is defined in $\mathbb{C}$.

contours shown in Fig. 3.2 (n odd) and Fig. 3.3 (n even):

$$\gamma_n(\lambda) = \gamma_n + \delta_n(\lambda) + \delta_n(-\lambda),$$

where, for some $r < |\operatorname{Im}\zeta|$,

$$\delta_n(\zeta) = \begin{cases} \partial D(\zeta, r) & \text{for } \operatorname{Im}\zeta < 0 \text{ and } n \text{ odd, or for } n \text{ even,} \\ -\partial D(\zeta, r) & \operatorname{Im}\zeta > 0 \text{ and } n \text{ odd,} \end{cases}$$

where the boundary of a disc is positively oriented.

For n odd the contour integrals over $\delta_n(\pm\lambda)$ can be absorbed into γ_n as λ crosses the real axis again and this shows that we can continue $R_0(\lambda)\varphi$, $\varphi \in C_c^\infty$ to a holomorphic function in $\mathbb{C} \setminus \{0\}$. This gives the holomorphic continuation of the distributional kernel, $R_0(\lambda, x, y)$.

For n even the contour integrals over $\delta_n(\pm\lambda)$ cannot be absorbed into γ_n as λ crosses the real axis again due to the wrong orientation: that means that $R_0(\lambda, x, y)$ continues to the logarithmic cover of $\mathbb{C} \setminus \{0\}$ when n is even. The integrals over $\delta_n(\pm\lambda)$ can be evaluated by the residue theorem and that shows that for n even

$$R_0(\lambda e^{i\ell\pi})(x) = R_0(\lambda)(x) + \frac{\ell}{2i}(-1)^{\frac{n-2}{2}(\ell+1)} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} e^{i\lambda\langle x, \omega\rangle} d\omega.$$

3.1.5. Additional estimates. We conclude this section with two low energy estimates which will be useful in §3.9. In particular, they can be omitted till that section is reached.

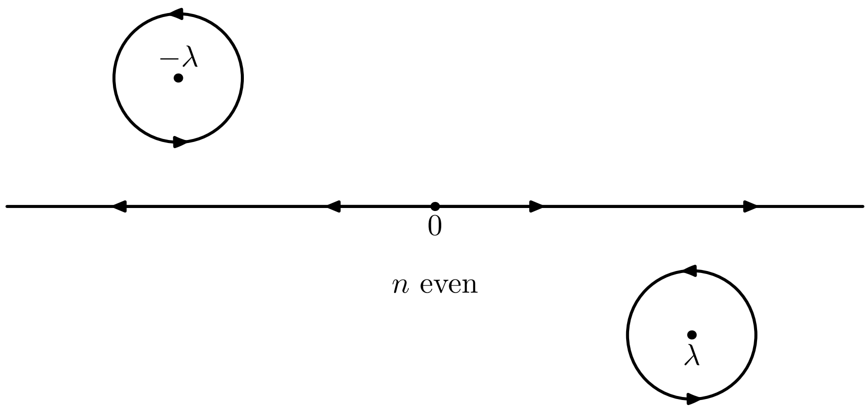


Figure 3.3. Contour deformations used to define $R_0(\lambda)$ for $\lambda > 0$ for n even. Now $R_0(e^{-\pi i}\lambda) = R_0(\lambda)$ can be expressed using an integral over the circular contour which doubles rather than gets absorbed. The resolvent is defined on the logarithmic plane.

LEMMA 3.6. *Suppose that $\rho \in C_c^\infty(\mathbb{R}^n)$, $n \geq 3$, odd. Then for any $C_0 > 0$ and $k \in \mathbb{N}$ there exist C_1 such that*

$$\begin{gathered} \|\lambda \rho R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda)\rho\|_{L^2 \to H^{2k}} \leq C_1, \\ \text{for } 0 \leq |\lambda| \leq C_0 \leq \tfrac{1}{2}\operatorname{Im}\lambda_0 \leq 2C_0, \quad \operatorname{Im}\lambda \geq 0. \end{gathered} \tag{3.1.26}$$

Proof. 1. It is convenient to prove a stronger estimate for the same range of λ and λ_0:

$$\|e^{-\langle x \rangle} \lambda R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) e^{-\langle x \rangle}\|_{L^2 \to H^{2k}} \leq C. \tag{3.1.27}$$

2. We first prove

$$\|e^{-\langle x \rangle} \lambda R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) e^{-\langle x \rangle}\|_{L^2 \to L^2} \leq C, \tag{3.1.28}$$

for any $k \geq 0$. In fact, using $R_0(\lambda_0)R_0(\lambda) = (\lambda^2 - \lambda_0^2)^{-1}(R_0(\lambda) - R_0(\lambda_0))$

$$\begin{aligned} R_0(\lambda)R_0(\lambda_0)^k R_0(\lambda) = a_k(\lambda^2 - \lambda_0^2)^{-k} R_0(\lambda)^2 + b_k(\lambda^2 - \lambda_0^2)^{-k-1} R_0(\lambda) \\ + \sum_{\ell=0}^{k-1} c_{k\ell}(\lambda^2 - \lambda_0^2)^{-2-\ell} R_0(\lambda_0)^{k-\ell}. \end{aligned}$$

Since $|\lambda^2 - \lambda_0|^2 > (\operatorname{Im}\lambda_0)^2 - |\lambda|^2 > 3C_0^2 > 0$ and since $\|R_0(\lambda_0)\|_{L^2 \to L^2} \leq C|\lambda_0|^{-1}(\operatorname{Im}\lambda_0)^{-1} \leq C' C_0^{-2}$ (see (3.1.24)) we only need to show that

$$\|e^{-\langle x \rangle} R_0(\lambda) e^{-\langle x \rangle}\|_{L^2 \to L^2} \leq C \quad \text{and} \quad \|e^{-\langle x \rangle} \lambda R_0(\lambda)^2 e^{-\langle x \rangle}\|_{L^2 \to L^2} \leq C.$$

The first estimate follows the representation of the Schwartz kernel of $R_0(\lambda)$ given in Theorem 3.3 and the Schur criterion (A.5.3). For the second estimate we note that

$$\lambda R_0(\lambda)^2 = \tfrac{1}{2}\partial_\lambda R_0(\lambda)$$

which gives an expression for the Schwartz kernel of $\lambda R_0(\lambda)^2$. Then the Schur criterion gives the second estimate.

This gives (3.1.26) for $k=0$.

3. To move to $k \in \mathbb{N}$, we start with the following observation: suppose that $e^{-\langle x\rangle}u \in H^2(\mathbb{R}^n)$. Then with $\| \bullet \| := \| \bullet \|_{L^2}$

$$\|\Delta(e^{-\langle x\rangle}u)\| \leq C\|e^{-\langle x\rangle}\Delta u\| + C\|e^{-\langle x\rangle}u\|. \tag{3.1.29}$$

In fact,

$$\begin{aligned}\Delta(e^{-\langle x\rangle}u) &= e^{-\langle x\rangle}\left(\Delta u - 2\langle x\rangle^{-1}x\cdot\nabla u + (n+(n-1)|x|^2)\langle x\rangle^{-3}u\right)\\ &= e^{-\langle x\rangle}\Delta u - 2\langle x\rangle^{-1}x\cdot\nabla(e^{-\langle x\rangle}u) + \mathcal{O}(1)e^{-\langle x\rangle}u.\end{aligned}$$

Thus,

$$\begin{aligned}\|\Delta(e^{-\langle x\rangle})u\|^2 &\leq 2\|e^{-\langle x\rangle}\Delta u\|^2 + 8\|\nabla(e^{-\langle x\rangle}u)\|^2 + C\|e^{-\langle x\rangle}u\|^2\\ &= 2\|e^{-\langle x\rangle}\Delta u\|^2 - 8\langle\Delta(e^{-\langle x\rangle}u), e^{-\langle x\rangle}u\rangle + C\|e^{-\langle x\rangle}u\|^2\\ &\leq 2\|e^{-\langle x\rangle}\Delta u\|^2 + \tfrac{1}{2}\|\Delta(e^{-\langle x\rangle}u)\|^2 + (C+32)\|e^{-\langle x\rangle}u\|^2.\end{aligned}$$

Since we assumed $e^{-\langle x\rangle}u \in H^2$, the integration by parts was justified. Hence, we can move the $\Delta(e^{-\langle x\rangle}u)$ term to the left hand side, proving (3.1.29).

4. The estimate (3.1.29) shows that

$$\begin{aligned}&\|e^{-\langle x\rangle}\lambda R_0(\lambda)R_0(\lambda)^k R_0(\lambda)e^{-\langle x\rangle}\|_{L^2\to H^{2k}} \leq\\ &\qquad C\|e^{-\langle x\rangle}\lambda R_0(\lambda)\Delta^k\, R_0(\lambda_0)^k R_0(\lambda)e^{-\langle x\rangle}\|_{L^2\to L^2}\\ &\qquad\quad + C\|e^{-\langle x\rangle}\lambda R_0(\lambda)R_0(\lambda)^k R_0(\lambda)e^{-\langle x\rangle}\|_{L^2\to L^2}.\end{aligned}$$

But then (3.1.26) follows from (3.1.28) by iterating the identity $-\Delta R_0(\lambda_0) = I + \lambda_0^2 R_0(\lambda_0)$ k times. □

The second lemma provides a basic weighted estimate on the resolvent up to the real axis near 0 energy. It can be refined in many ways – see [**JK79**],[**Je80a**],[**Mu82**] for classical estimates and [**Va18**] for recent developments.

LEMMA 3.7. *Suppose that $n \geq 3$ is odd and $s \geq 0$, $s \notin \mathbb{N}$. Then, for $C_0, C_1 > 0$ there exists C_3 such that for*

$$|\lambda| \leq C_0 \operatorname{Im}\lambda \leq C_1,$$

$$\|\langle x\rangle^{-s}R_0(\lambda)\langle x\rangle^{-s}\|_{L^2\to L^2} \leq C_2 + C_2|\lambda|^{s-2}. \tag{3.1.30}$$

Proof. From Theorem 3.3 we know that for $\operatorname{Im}\lambda \geq 0$,

$$|R_0(\lambda,x,y)| \leq \sum_{k=0}^{\frac{n-3}{2}} c_k|\lambda|^k|x-y|^{2+k-n}e^{-\operatorname{Im}\lambda\,|x-y|}.$$

Using $2\langle x\rangle\langle y\rangle \geq \langle x-y\rangle$ we have for $s \geq 0$,

$$\langle x\rangle^{-s}|x-y|^{2+k-n}\langle y\rangle^{-s} \leq C|x-y|^{2+k-n}\langle x-y\rangle^{-s}.$$

Hence, for $s \geq 0$, $s \notin \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}} \langle x\rangle^{-s}|R_0(\lambda,x,y)|\langle y\rangle^{-s}dx &\leq \sum_{k=0}^{\frac{n-3}{2}} c_k|\lambda|^k \int_0^\infty (1+r)^{-s}r^{1+k}e^{-\operatorname{Im}\lambda\, r}dr \\ &\leq \sum_{k=0}^{\frac{n-3}{2}} c_k|\lambda|^k \left(C + \int_1^\infty r^{1+k-s}e^{-\operatorname{Im}\lambda\, r}dr\right) \\ &\leq \sum_{k=0}^{\frac{n-3}{2}} c_k|\lambda|^k (C + C(\operatorname{Im}\lambda)^{s-k-2}) \\ &\leq C + C|\lambda|^{s-2}.\end{aligned}$$

The Schur criterion (A.5.3) then gives (3.1.30). □

3.2. MEROMORPHIC CONTINUATION

In this chapter we define scattering resonances for compactly supported potentials in odd dimension and present some of their basic properties.

3.2.1. Continuation of the resolvent. Once we have established the properties of the free resolvent in odd dimensions the properties of

$$R_V(\lambda) := (P_V - \lambda^2)^{-1}, \quad P_V = -\Delta + V, \quad \operatorname{Im}\lambda \gg 0,$$

$$V \in L^\infty(\mathbb{R}^n, \mathbb{C}), \quad n \geq 3, \text{ odd}, \quad \Delta = \sum_{j=1}^n \partial_{x_j}^2,$$

follow exactly as in one dimension. The situation is even simpler as we do not have a resonance at zero for $R_0(\lambda)$.

In particular the proof of the following theorem is exactly the same as in the one dimensional case:

THEOREM 3.8 (Meromorphic continuation of the resolvent). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{C})$ and that $n \geq 3$ is odd. Then the*

$$R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1} : L^2 \longrightarrow L^2, \quad \operatorname{Im}\lambda > 0,$$

is a meromorphic family of operators with finitely many poles. It extends to a meromorphic family of operators for:

$$R_V(\lambda) := L^2_{\mathrm{comp}} \longrightarrow L^2_{\mathrm{loc}}, \quad \lambda \in \mathbb{C}.$$

Just as in the proof of Theorem 2.2 we have a formula for the meromorphically continued resolvent. Let $\rho \in L^\infty_{\rm comp}(\mathbb{R}^n)$ satisfy $\rho \equiv 1$ on $\operatorname{supp} V$. Then

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho)) \,. \tag{3.2.1}$$

We also have

$$\begin{aligned} R_V(\lambda)\rho &= R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}, \\ \rho R_V(\lambda) &= (I + \rho R_0(\lambda)V)^{-1}\rho R_0(\lambda). \end{aligned} \tag{3.2.2}$$

Since the operator $\rho R_0(\lambda)V$ is compact, $I + \rho R_0(\lambda)V$ is invertible unless it has a non-trivial kernel – see (C.2.6). Hence, if $R_V(\lambda)$ is singular at λ_0 then there exists U such that $U = -\rho R_0(\lambda_0)VU$. Since ρ with $\rho V = V$ is arbitrary this means that

$$R_V(\lambda) \text{ has a pole at } \lambda_0 \implies \exists\, u \in H^2_{\rm loc}, \ u = R_0(\lambda_0)Vu. \tag{3.2.3}$$

The converse also follows from (3.2.2) and a more precise statement is given in Theorem 3.15.

REMARK. For future reference we make the following observation about the Schwartz kernel of $R_V(\lambda)$: if $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})$ then

$$R_V(\lambda, x, y) = R_V(\lambda, y, x). \tag{3.2.4}$$

In fact, since $R_0(\lambda, x, y) = R_0(\lambda, y, x)$ this follows from (3.2.2) as the two expressions for $\rho R_V(\lambda)\rho$ are transposes of each other and ρ can be chosen to be equal to 1 on arbitrarily large sets.

Scattering resonances are the poles of $R_V(\lambda)$ and their multiplicities, $m_R(\lambda)$ are defined by

$$m_V(\lambda) = m_R(\lambda) := \dim \operatorname{span}\{A_1(L^2_{\rm comp}), \cdots A_J(L^2_{\rm comp})\}, \tag{3.2.5}$$

where

$$R_V(\zeta) = \sum_{j=1}^J \frac{A_j}{(\zeta - \lambda)^j} + A(\zeta, \lambda),$$

with $\zeta \mapsto A(\zeta, \lambda)$ holomorphic near λ. We refer to the continuation, $R_V(\lambda)$, as the *scattering resolvent*. .

As we will see in Theorem 3.9 this definition coincides with the definition (2.2.11) away from 0:

$$\begin{aligned} m_V(\lambda) = m_R(\lambda) &:= \dim \operatorname{span} A_1(L^2_{\rm comp}) \\ &= \operatorname{rank} \oint_\lambda R_V(\zeta) 2\zeta d\zeta, \quad \lambda \neq 0. \end{aligned} \tag{3.2.6}$$

We use notation m_V to emphasize the dependence on the potential V. When there is no ambiguity m_R is used to distinguish this multiplicity from the multiplicity $m_S(\lambda)$ defined in §3.7 using the scattering matrix.

The structure of the singular part of the resolvent at a pole is described by following the proof of 1) in Theorem 2.5:

THEOREM 3.9 (Singular part of $\boldsymbol{R_V(\lambda)}$). *Suppose $m_R(\mu) > 0$, $\mu \neq 0$. Then for some integer $K(\mu) \leq m_R(\mu)$,*

$$R_V(\lambda) = -\sum_{k=1}^{K(\mu)} \frac{(P_V - \mu^2)^{k-1}}{(\lambda^2 - \mu^2)^k} \Pi_\mu + A(\lambda, \mu) \,, \tag{3.2.7}$$

where $\lambda \mapsto A(\lambda, \mu)$ is holomorphic near μ, and

$$\Pi_\mu = -\frac{1}{2\pi i} \oint_\mu R_V(\lambda) 2\lambda d\lambda, \quad (P_V - \mu^2)^{K(\mu)} \Pi_\mu = 0. \tag{3.2.8}$$

The elements of the range of $(P_V - \mu^2)^{K(\mu)-1} \Pi_\mu$ satisfy $(P_V - \mu^2)u = 0$ and are called *resonant states*. See Definition 4.8 and Theorem 4.9 for a treatment in a more general setting.

REMARKS. 1. The expansion (3.2.7) takes a particularly simple form when $m_R(\mu) = 1$, $\mu \neq 0$:

$$R_V(\lambda) = \frac{u \otimes u}{\lambda - \mu} + A(\lambda, \mu), \quad u \in H^2_{\rm loc}(\mathbb{R}^n), \tag{3.2.9}$$

where for $f \in L^2_{\rm comp}(\mathbb{R}^n)$, $(u \otimes u) f(x) := u(x) \int_{\mathbb{R}^n} u(y) f(y) dy$. In fact, since $K(\mu) = 1$ and Π_μ is of rank one the Schwartz kernel of the residue is given by $u(x)v(y)$ for some $u, v \in H^2_{\rm loc}(\mathbb{R}^n)$. But the Schwartz kernel of the resolvent satisfies (3.2.4) (see also Exercise 3.4) which means that we can choose $v(x) = u(x)$.

2. A generalization of Theorem 3.23 to a large class of compactly supported perturbations of $-\Delta$ is given in §4.2. A discussion of outgoing solutions is also provided there.

3.2.2. Resonance expansions of scattered waves. The proofs of Theorem 2.10 on resonance free regions and of Theorem 2.9 apply without any modifications to the case of higher odd dimensions. Thus we obtain

THEOREM 3.10 (Resonance free regions). *Suppose that*

$$V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{C}) \,, \quad n \geq 3 \,, \quad \textit{odd.}$$

Then for any $\rho \in C^\infty_{\rm c}(\mathbb{R}^3)$ there exist constants A, C, T depending on ρ such that

$$\|\rho R_V(\lambda) \rho\|_{L^2 \to H^j} \leq C |\lambda|^{j-1} \, e^{T(\operatorname{Im} \lambda)_-} \,, \quad j = 0, 1, 2 \,, \tag{3.2.10}$$

for

$$\operatorname{Im} \lambda \geq -A - \delta \log(1 + |\lambda|)\,, \quad |\lambda| > C_0\,, \quad \delta < 1/\operatorname{diam}(\operatorname{supp} V)\,.$$

In particular there are only finitely many resonances in the region

$$\{\lambda \ : \ \operatorname{Im} \lambda \geq -A - \delta \log(1 + |\lambda|)\}\,.$$

for any $A > 0$.

THEOREM 3.11 (Resonance expansions of scattering waves). *Let* $V \in L^\infty(\mathbb{R}^n; \mathbb{R})$ *for* $n \geq 1$ *odd, and suppose that* $w(t,x)$ *is the solution of*

$$\left\{ \begin{array}{l} (D_t^2 - P_V) w(t,x) = 0\,, \\ w(0,x) = w_0(x) \in H^1_{\mathrm{comp}}(\mathbb{R}^n)\,, \\ \partial_t w(0,x) = w_1(x) \in L^2_{\mathrm{comp}}(\mathbb{R}^n)\,. \end{array} \right. \tag{3.2.11}$$

Let $\{\lambda_j\}$ *be the set of resonances of* P_V *(including* $\{i\sqrt{-E_k}\}_{k=1}^N$*, where* $E_N < \cdots \leq E_2 \leq E_1 \leq 0$ *are the eigenvalues of* P_V*).*

Then, for any $A > 0$,

$$w(t,x) = \sum_{\operatorname{Im} \lambda_j > -A} \ \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} w_{j,\ell}(x) + E_A(t)\,, \tag{3.2.12}$$

where the sum is finite and

(3.2.13)
$$\sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) = -\operatorname{Res}_{\lambda=\lambda_j} \Big((iR_V(\lambda) w_1 + \lambda R_V(\lambda) w_0)\, e^{-i\lambda t} \Big)\,,$$

$$(P_V - \lambda_j)^{\ell+1} w_{j,\ell} = 0\,,$$

and for any $K > 0$*, such that* $\operatorname{supp} w_j \subset [-K, K]$*, there exist constants* $C_{K,A}$ *and* $T_{K,A}$

$$\|E_A(t)\|_{H^2([-K,K])} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{H^1} + \|w_1\|_{L^2} \right)\,, \quad t \geq T_{K,A}\,.$$

3.2.3. Finite rank perturbations and multiplicities. To handle agreement of multiplicities it is convenient to work with simple poles. This section as well as Theorem 3.45 below deal with these thorny issues and can be safely skipped at first reading.

The theory presented above applies without changes to V of the form

$$V = V_0 + V_1, \quad V_0 \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n, \mathbb{C}),$$

$$V_1 = \sum_{j=1}^J f_j \otimes g_j, \quad f_j, g_j \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C}),$$

that is to potentials replaced by a sum of a potential and a finite rank perturbation. The advantage of this approach is the ease with which the multiplicities can be split. That will allow a simple treatment of the relation between multiplicities of the scattering matrix and the resolvent.

We have already seen that the structure of the singular part of the resolvent can be quite complicated and both statements and proofs are easier when the resonances are simple, that is when multiplicity is equal to 1.

We start with a simple lemma which is a more precise version of Theorem C.4 in a special case. From (C.2.2) we see that for a compact operator on a Hilbert space $\dim\ker(I+K) = \dim \operatorname{coker}(I+K) = \dim\ker(I+K^*)$.

LEMMA 3.12 (Grushin problem for Fredholm operators). *Suppose that $K : H \to H$ is a compact operator on a Hilbert space H. Let $v_1, \cdots, v_m$ be an orthonormal basis of $\ker(I+K)$ and $w_1, \cdots, w_m$ be an orthormal basis of $\ker(I+K^*)$. Define*

$$R_- : \mathbb{C}^m \to H, \quad R_- u_- := \sum_{j=1}^m u_{-,j} w_j,$$
$$R_+ : H \to \mathbb{C}^m, \quad R_+ u := (\langle u, v_j\rangle_H)_{j=1}^m.$$

Then

$$\begin{bmatrix} I+K & R_- \\ R_+ & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E & E_+ \\ E_- & 0 \end{bmatrix} : H \oplus \mathbb{C}^m \to H \oplus \mathbb{C}^m, \tag{3.2.14}$$

where

$$E_+ : \mathbb{C}^m \to H, \quad E_+ z = \sum_{j=1}^m z_j v_j, \quad z \in \mathbb{C}^m,$$
$$E_- : H \to \mathbb{C}^m, \quad E_- u = (\langle u, w_j\rangle_H)_{j=1}^m.$$

The next lemma deals with simplicity of zeros of determinants:

LEMMA 3.13 (Simplicity of zeros). *Suppose that $M(\lambda)$ is a holomorphic family of $m \times m$ matrices and for some holomorphic function g,*

$$\det M(\lambda) = \lambda^p g(\lambda), \quad g(0) \neq 0, \quad M(0) = 0.$$

There exists a matrix A such that for any holomorphic family of $m \times m$ matrices, $\lambda \mapsto M(\lambda, \varepsilon)$ satisfying

$$M(\lambda, \varepsilon) = M(\lambda) + \varepsilon A + f(\lambda, \varepsilon), \quad \|f(\lambda,\varepsilon)\| = \mathcal{O}(\varepsilon^2 + |\lambda|\varepsilon),$$

there exists ε_0 and r_0 such that for $0 < \varepsilon < \varepsilon_0$, $\det M(\lambda, \varepsilon)$ has exactly p simple *zeros in $D(0, r_0)$.*

Proof. 1. We note that for $0 < |\lambda| \le r_0$ with r_0 small enough, $\det M(\lambda) \neq 0$ and hence $M(\lambda)$ is invertible. It follows that for $|\lambda| = r_0$

$$\|M(\lambda)^{-1}(M(\lambda,\varepsilon) - M(\lambda))\| \le C_{r_0}\varepsilon < 1,$$

if $\varepsilon < \varepsilon_0$ for some ε_0 chosen small enough depending on r_0 and A. The matrix valued version of Rouché's Theorem C.12 now shows that the number of zeros of $\det M(\lambda,\varepsilon)$ in $|\lambda| < r_0$ is equal to p, the multiplicity of the only zero of $\det M(\lambda)$ there.

2. We can apply Lemma C.13 to $M(\lambda)$, so that

$$M(\lambda) = E(\lambda)M_0(\lambda)F(\lambda),$$
$$M_0(\lambda) = \lambda^{k_1}P_1 + \lambda^{k_2}P_2 + \cdots \lambda^{k_r}P_r, \quad k_j > 0,$$
$$P_kP_j = P_jP_k = \delta_{jk}P_k, \quad \sum_{j=1}^{r} P_j = I_{\mathbb{C}^m}, \quad \sum_{j=1}^{m} k_j \operatorname{rank} P_j = p,$$

and $E(\lambda), F(\lambda)$ are holomorphic and invertible near $\lambda = 0$. We can make identifications $\operatorname{Im} P_j \simeq \mathbb{C}^{m_j}$, $m_j = \operatorname{rank} P_j$.

3. Let $0 < \theta \ll 1$ and put $D_m = \operatorname{diag}(1, e^{i\theta}, \cdots, e^{i(m-1)\theta})$. We then put

$$A_0 = D_{m_1}P_1 + D_{m_2}P_2 + \cdots + D_{m_r}P_r,$$

where we identified $\operatorname{Im} P_j$ with $\mathbb{C}^{m_j}$ and $\mathbb{C}^m$ with $\mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_r}$. If θ is small enough, the zeros of $\det(M_0(\lambda) - \varepsilon A_0)$ are simple and are given by

$$\lambda_{k_j,q,\ell} = \varepsilon^{1/k_j} e^{2\pi i\ell/k_j + i\theta q/k_j},$$
$$\ell = 0, \cdots, k_j, \quad q = 0, \cdots, m_j - 1, \quad j = 1, \cdots, r.$$

One can check that there exists $c_0 > 0$ such that for $(k,q,\ell) \neq (k',q',\ell')$,

$$|\lambda_{k,q,\ell} - \lambda_{k',q',\ell'}| > c_0 \max(\varepsilon^{\frac{1}{k}}, \varepsilon^{\frac{1}{k'}}), \tag{3.2.15}$$

provided ε and θ are smaller than some fixed constant (depending on k_j's and m_j's).

4. Putting $A = E(0)^{-1}A_0F(0)^{-1}$ we have

$$\begin{gathered} E(\lambda)^{-1}M(\lambda,\varepsilon)F(\lambda)^{-1} = M_0(\lambda) - \varepsilon A_0 + e(\lambda,\varepsilon), \\ \|e(\lambda,\varepsilon)\| = \mathcal{O}(\varepsilon^2 + |\lambda|\varepsilon) \end{gathered} \tag{3.2.16}$$

and we are looking for zeros of the determinant of the right hand side. Consider $U = D(\lambda_{k_j,q,\ell}, \varepsilon^{1/k_j}\rho)$. In view of (3.2.15) different U's are disjoint from the set where $\rho < c_0$. Then for $\lambda \in \partial U$, and ρ small enough (independently of ε),

$$\|(M_0(\lambda) - \varepsilon A_0)^{-1}\| = \left(\min_{j=1,\cdots r,\ 0\le q\le m_j-1} \min |\lambda^{k_j} - e^{i\theta q}\varepsilon| \right)^{-1} \le C\varepsilon^{-1}\rho^{-1}.$$

Hence, for $\lambda \in \partial U$ and using (3.2.16),

$$\|(M_0(\lambda) - \varepsilon A_0)^{-1} e(\lambda, \varepsilon)\| \leq C(|\lambda| + \varepsilon)\rho^{-1} \leq C\varepsilon^{\frac{1}{k_j}}\rho^{-1} < 1,$$

if $C\varepsilon^{\frac{1}{k_j}} < \rho$. An application of Theorem C.12 shows that the determinant of $M(\lambda, \varepsilon)$ has exactly one zero in U. In view of (3.2.15) there exists p disjoint discs with such properties for the corresponding $\lambda_{k_j,q,\ell}$'s. It follows that all the zeros of this determinant in $D(0, r_0)$ are simple. $\square$

We can now prove

THEOREM 3.14 (Multiplicity splitting). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$, odd, and that for some $\lambda_0 \in \mathbb{C}$, $m_V(\lambda_0) > 1$. Then there exists a finite rank perturbation*

$$W = \sum_{i,j=1}^{m} f_i \otimes g_j, \quad f_i, g_j \in C^\infty_{\mathrm{c}}(\mathbb{R}^n; \mathbb{C}),$$

and constants ε_0 and r_0 such that for $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \sum_{|\lambda - \lambda_0| < r_0} m_{V+\varepsilon W}(\lambda) &= m_V(\lambda_0), \\ m_{V+\varepsilon W}(\lambda) \leq 1, \quad &|\lambda - \lambda_0| < r_0. \end{aligned} \tag{3.2.17}$$

REMARK. In Theorem 4.39 we will prove a higher dimensional version of Theorem 2.25 and show that resonances are generically simple. In this book that is only done for resonances in a conic neighbourhood of the real axis (see (4.5.45)) but as remarked there large angle complex scaling of [**SZ91**] gives the result for all resonances.

Proof. 1. Since

$$\begin{aligned} R_V(\lambda) &= R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho))\,, \\ (I + VR_0(\lambda)\rho)^{-1} &= I - VR_V(\lambda)\rho\,, \end{aligned} \tag{3.2.18}$$

we see that simplicity of a pole of R_V is equivalent to the simplicity of a pole of $(I + VR_0(\lambda)\rho)^{-1}$ and we will consider that operator.

2. We apply Lemma 3.12 with $K = VR_0(\lambda_0)\rho$. From (3.1.12) we have

$$\begin{aligned} &\|(V + \varepsilon W)R_0(\lambda)\rho - VR_0(\lambda_0)\rho\|_{L^2 \to L^2} \\ &\quad \leq \|V\|_{L^\infty}\|\rho(R_0(\lambda) - R_0(\lambda_0))\rho\|_{L^2\to L^2} + \varepsilon\|WR_0(\lambda)\rho\|_{L^2\to L^2} \\ &\quad \leq C_1(|\lambda - \lambda_0| + \varepsilon)e^{C_0|\lambda|}, \end{aligned}$$

where C_1 depends on V and W. Hence, for λ sufficiently close to λ_0 and ε small enough we can use the same $R_\pm$ as for $VR_0(\lambda_0)\rho$ to obtain a well posed Grushin problem:

$$\begin{bmatrix} I + (V+\varepsilon W)R_0(\lambda)\rho & R_- \\ R_+ & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E^\varepsilon(\lambda) & E_+^\varepsilon(\lambda) \\ E_-^\varepsilon(\lambda) & E_{-+}^\varepsilon(\lambda) \end{bmatrix},$$

3. The expansion (C.1.7) gives

$$E_{-+}^\varepsilon(\lambda) = E_{-+}(\lambda) - \varepsilon A + \mathcal{O}(\varepsilon|\lambda| + \varepsilon^2),$$

where

$$A = E_- W E_+ = B^T C, \quad B_{ij} := \int_{\mathbb{R}^n} f_i(x)\overline{w_j(x)}, \quad C_{ij} := \int_{\mathbb{R}^n} g_i(x) v_j(x) dx.$$

Since the sets $\{w_j\}_{j=1}^m$ and $\{v_j\}_{j=1}^m$ are linearly independent we can find f_j's and g_j's so that B and C are arbitrary matrices. This means that we can choose A as in Lemma 3.13. The poles of $I + (V+\varepsilon W)R_0(\lambda)\rho$ are the zeros of $\det E_{-+}^\varepsilon(\lambda)$ and hence the conclusion follows. ☐

As the first application we record the following fact:

THEOREM 3.15 (Multiplicity as a trace). *Suppose that $n \geq 3$ is odd. Let $m_R(\lambda)$ be defined by (3.2.5) and suppose $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfies $\rho V = V$. Then*

$$m_R(\lambda) = \frac{1}{2\pi i} \operatorname{tr} \oint_\lambda (I + VR_0(\zeta)\rho)^{-1} \partial_\zeta (VR_0(\zeta)\rho) d\zeta, \tag{3.2.19}$$

where the integral is over a positively oriented circle containing λ and no other possible pole of R_V. The same result holds with $VR_0(\lambda)\rho$ replaced by $\rho R_0(\lambda) V$.

Proof. 1. As explained in the beginning of this section we can replace V by more general operators of the form $V = V_0 + V_1$ where V_0 is a potential V_1 is a finite rank perturbation,

$$V_0 \in L^\infty(\mathbb{R}^n; \mathbb{C}), \quad V_1 = \sum_{j=1}^J \varphi_j \otimes \psi_j, \quad \varphi_j, \psi_j \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C}).$$

Let us denote by $n_V(\lambda)$ the right hand side of (3.2.19).

2. Suppose that that $W = \sum_{j=1}^J f_j \otimes g_j$, for some $f_j, g_j \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C})$. Then for $\rho = 1$ on a sufficiently large compact set and for $1/C_0 \leq |\mu| < C_0$,

$$\|VR_0(\mu)\rho - (V+\varepsilon)R_0(\mu)\rho\|_{\mathcal{L}_1(\mathbb{R}^n)} \leq \varepsilon \|\rho R_0(\mu)\rho\| \|W\|_{\mathcal{L}_1(\mathbb{R}^n)} \leq C_1 \varepsilon.$$

Hence for $\varepsilon > 0$ small enough we can apply Theorem C.12 (with $A(\lambda) = I + VR_0(\lambda)\rho$ and $B(\lambda) = I + (V+\varepsilon W)R_0(\lambda)\rho$) to see that exist $\varepsilon_0, r_0 > 0$ such that

$$\sum_{\mu \in D(\lambda, r_0)} n_{V+\varepsilon W}(\mu) = n_V(\lambda), \quad \varepsilon < \varepsilon_0. \tag{3.2.20}$$

3. Theorem 3.14 shows that for any V there exists W such that the poles of $R_{V+\varepsilon W}(\lambda)$, $0<\varepsilon<\varepsilon_0$ near λ_0 are all simple.

In view of (3.2.18) the same holds for the poles of $(I+VR_0(\lambda)\rho)^{-1}$. Then (3.2.20) and (3.2.17) show that it is sufficient to prove (3.2.19) in the case of simple poles of $R_V(\lambda)$ and $(I+VR_0(\lambda)\rho)^{-1}$.

3. Suppose that λ is such a pole. Then (C.4.6) (the first part of Theorem C.11) gives

$$\frac{1}{2\pi i}\operatorname{tr}\oint_\lambda (I+VR_0(\zeta)\rho)^{-1}\partial_\zeta(VR_0(\zeta)\rho)d\zeta = 1 = m_R(\lambda),$$

proving the claim. □

3.3. RESOLVENT AT ZERO ENERGY

Consider the operator P_V, $V\in L^\infty_{\text{comp}}(\mathbb{R}^n;\mathbb{R})$, where $n>1$ is odd. Denote by H_0 the eigenspace of P_V at 0:

$$H_0 := \{v\in H^2(\mathbb{R}^n) : P_V v = 0\},$$

and let

$$\Pi_0 : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$$

be the orthogonal projector onto H_0.

We describe the structure of the resolvent $R_V(\lambda)$ at $\lambda=0$, starting with the following

LEMMA 3.16. *We have*

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda), \tag{3.3.1}$$

as operators $L^2_{\text{comp}}\to L^2_{\text{loc}}$, where $\lambda\mapsto A(\lambda)$ is holomorphic near 0 and $A_1 : L^2_{\text{comp}}\to L^2_{\text{loc}}$ is a symmetric operator such that $P_VA_1=0$.

REMARK. In the Lemma, the symmetry of A_1 means that $\langle A_1\varphi,\psi\rangle = \langle\varphi, A_1\psi\rangle$, for $\varphi,\psi\in L^2_{\text{comp}}(\mathbb{R}^n)$.

Proof. 1. The upper bound on the resolvent in the upper half-plane,

$$\|R_V(it)\|_{L^2\to L^2}\le\frac{1}{t^2},\quad t\in(0,\varepsilon), \tag{3.3.2}$$

and the fact that $\lambda\mapsto R_V(\lambda)$ is a meromorphic family of operators imply that we have the following decomposition:

$$R_V(\lambda) = -\frac{A_2}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda),$$

where $A_j : L^2_{\rm comp} \to H^2_{\rm loc}$ are finite rank operators and $A(\lambda) : L^2_{\rm comp} \to H^2_{\rm loc}$ is holomorphic near $\lambda = 0$. Since $R_V(it)$ is self-adjoint for $t > 0$, we see that for $\varphi, \psi \in L^2_{\rm comp}(\mathbb{R}^n)$,

$$\langle A_2\varphi, \psi\rangle = \lim_{t\to 0+} \langle t^2 R_V(it)\varphi, \psi\rangle = \lim_{t\to 0+} \langle \varphi, t^2 R_V(it)\psi\rangle = \langle \varphi, A_2\psi\rangle, \tag{3.3.3}$$

and

$$\begin{aligned}\langle A_1\varphi, \psi\rangle &= \lim_{t\to 0+} \langle (tR_V(it) + t^{-1}A_2)\varphi, \psi\rangle \\ &= \lim_{t\to 0+} \langle \varphi, (tR_V(it) + t^{-1}A_2)\psi\rangle = \langle \varphi, A_1\psi\rangle.\end{aligned}$$

Hence A_j are symmetric on $L^2_{\rm comp}$. Since for $\psi \in L^2_{\rm comp}(\mathbb{R}^n)$,

$$\begin{aligned}\psi &= (P_V - \lambda^2)R_V(\lambda)\psi \\ &= -\lambda^{-2}P_V A_2\psi + \lambda^{-1}P_V A_1\psi + (A_2 - \lambda A_1 + (P_V - \lambda^2)A(\lambda))\psi,\end{aligned}$$

we obtain $P_V A_j = 0$.

2. We now observe that (3.3.2) shows that A_2 is bounded $L^2 \to L^2$ and that (3.3.3) is valid for all $\varphi, \psi \in L^2$, that is, A_2 is selfadjoint. Since $P_V A_2 = 0$, the range of A_2 is contained in H_0. To show that $A_2 = \Pi_0$, it remains to verify that for each $v \in H_0$, we have $A_2 v = v$, and this follows by substituting $\varphi = v$ in (3.3.3):

$$\langle v, \psi\rangle = \langle t^2 R_V(it)v, \psi\rangle \to \langle A_2 v, \psi\rangle \quad \text{as } t \to 0+. \qquad \square$$

In dimensions 5 or greater, we have $A_1 = 0$:

THEOREM 3.17. *Assume that $n \geq 5$ is odd. Then*

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + A(\lambda),$$

where $A(\lambda) : L^2_{\rm comp} \to L^2_{\rm loc}$ is holomorphic at $\lambda = 0$.

Proof. 1. Since $R_0(\lambda)$ is injective on $L^2_{\rm comp}$ ($(-\Delta - \lambda^2)$ is its left inverse), it follows from (3.2.1) that, for λ near 0,

$$R_V(\lambda) = R_0(\lambda)(-\widetilde{A}_2/\lambda^2 + \widetilde{A}_1/\lambda + \widetilde{A}(\lambda)), \tag{3.3.4}$$

where $\widetilde{A}_j, \widetilde{A}(\lambda) : L^2_{\rm comp} \to L^2_{\rm comp}$ and $\widetilde{A}(\lambda)$ is holomorphic near 0. For $n \geq 5$, in the notation of Theorem 3.3, and for $\varphi \in L^2_{\rm comp}(\mathbb{R}^n)$,

$$\begin{aligned}R_0(0)\varphi(x) &= P_n(0) \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}} dy \in L^2(\mathbb{R}^n), \\ \partial_\lambda R_0(0)\varphi(x) &= (iP_n(0) + P_n'(0)) \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-3}} dy \in L^2(\mathbb{R}^n),\end{aligned}$$

and the asymptotics are given by $c/|x|^{n-2}$ and $c'/|x|^{n-3}$, respectively. For $n \geq 7$ this means that $R_0(0)\varphi, \partial_\lambda R_0(0)\varphi \in L^2(\mathbb{R}^n)$. This is also true for $n = 5$ as using (3.1.16) we check that $iP_5(0) + P_5'(0) = 0$. Hence

$$R_0(0)\varphi, \ \ \partial_\lambda R_0(0)\varphi \in L^2, \quad \text{for } \varphi \in L^2_{\text{comp}}, \, n \geq 5.$$

From this we conclude that

$$\begin{aligned} A_1(L^2_{\text{comp}}) &= \left(R_0(0)\widetilde{A}_1 - \partial_\lambda R_0(0)\widetilde{A}_2 \right)(L^2_{\text{comp}}) \\ &\subset R_0(0)(L^2_{\text{comp}}) + \partial_\lambda R_0(0)(L^2_{\text{comp}}) \subset L^2. \end{aligned}$$

Since $P_V A_1 = 0$, it follows that $A_1 : L^2_{\text{comp}} \to H_0$.

2. We now take $\psi \in L^2_{\text{comp}}$, $v \in H_0$, and consider, for $t > 0$,

$$\begin{aligned} 0 &= t\langle R_V(it)v, \psi\rangle - t^{-1}\langle v, \psi\rangle = \langle v, tR_V(it)\psi - t^{-1}\langle v, \psi\rangle \\ &= i\langle v, A_1\psi\rangle + t\langle v, A(it)\psi\rangle + t^{-1}\langle v, \Pi_0\psi\rangle - t^{-1}\langle v, \psi\rangle \\ &= i\langle v, A_1\psi\rangle + t\langle v, A(it)\psi\rangle + t^{-1}\langle \Pi_0 v, \psi\rangle - t^{-1}\langle v, \psi\rangle \\ &\to i\langle v, A_1\psi\rangle, \quad t \to 0+, \end{aligned}$$

as $\Pi_0 v = v$. Since $A_1\psi \in H_0$ we can take $v = A_1\psi$ to conclude that $A_1 \equiv 0$. □

We now concentrate on the interesting case[1] of $n = 3$. We start by analysing the asymptotic behaviour of the elements of H_0:

LEMMA 3.18. *Assume that $n = 3$ and $v \in H_0$. Then:*

1. *$v = R_0(0)f$, where $f = -Vv = -\Delta v \in L^2_{\text{comp}}(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} f = 0$.*
2. *Uniformly in $\theta \in \mathbb{S}^2$ and locally uniformly in $y \in \mathbb{R}^3$,*

$$\begin{aligned} v(y + r\theta) &= \frac{1}{4\pi r^2}\sum_{j=1}^{3} b_j\theta_j + \frac{3}{8\pi r^3}\sum_{j,k=1}^{3}(B_{jk} - 2b_j y_k)\,\theta_j\theta_k \\ &\quad - \frac{1}{8\pi r^3}\sum_{j=1}^{3}(B_{jj} - 2b_j y_j) + \mathcal{O}\Big(\frac{1}{r^4}\Big), \quad r \to +\infty, \end{aligned} \tag{3.3.5}$$

where

$$b_j(v) = \int_{\mathbb{R}^3} x_j f(x)\,dx, \quad B_{jk}(v) = \int_{\mathbb{R}^3} x_j x_k f(x)\,dx. \tag{3.3.6}$$

3. *For $y \in \mathbb{R}^3$ and $r > 0$,*

$$I_v(r, y) := \int_{\mathbb{S}^2} v(y + r\theta)\,d\theta = \mathcal{O}(r^{-4}), \quad r \to +\infty \tag{3.3.7}$$

[1] This detailed analysis will not be needed in the rest of the book but it contains ideas behind the important study of more general potentials – see references in §3.13.

locally uniformly in y.

Proof. 1. We have for all $\lambda \in \mathbb{C}$, $(-\Delta - \lambda^2)R_V(\lambda) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}$ and, usings (3.2.1),

$$R_V(\lambda) = R_0(\lambda)(-\Delta - \lambda^2)R_V(\lambda). \tag{3.3.8}$$

From (3.3.1), we see that $\Pi_0 = R_0(0)(-\Delta)\Pi_0$, and, since $P_V\Pi_0 = 0$, that

$$(-\Delta)\Pi_0 = -V\Pi_0 : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}.$$

By Lemma 3.16, $v \in H_0$ is in the image of Π_0. Therefore, $v = R_0(0)f$, where $f = (-\Delta)v = -Vv \in L^2_{\mathrm{comp}}$.

2. Since $R_0(0)(x,y) = \frac{1}{4\pi|x-y|}$, we write

$$\begin{aligned} v(y + r\theta) &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(x)}{|x - y - r\theta|}\,dx = \frac{1}{4\pi r}\int_{\mathbb{R}^3} \frac{f(x)}{|\theta - r^{-1}(x-y)|}\,dx \\ &= \frac{1}{4\pi r}\int_{\mathbb{R}^3} f(x)(1 - 2r^{-1}\langle\theta, x - y\rangle + r^{-2}|x-y|^2)^{-1/2}\,dx. \end{aligned}$$

We now use the Taylor expansion $(1+s)^{-1/2} = 1 - \frac{1}{2}s + \frac{3}{8}s^2 + \mathcal{O}(s^3)$. In particular, we get

$$v(r\theta) = \frac{1}{4\pi r}\int_{\mathbb{R}^3} f + \mathcal{O}\Big(\frac{1}{r^2}\Big); \tag{3.3.9}$$

since $v \in L^2$, this implies that $\int_{\mathbb{R}^3} f = 0$. Expanding $(1 - 2r^{-1}\langle\theta, x-y\rangle + r^{-2}|x-y|^2)^{-1/2}$ up to an $\mathcal{O}(r^{-3})$ remainder, we get (3.3.5).

3. Using the formulas

$$\int_{\mathbb{S}^2} \theta_j\,d\theta = 0, \quad \int_{\mathbb{S}^2} \theta_j\theta_k\,d\theta = \frac{4\pi}{3}\delta_{jk},$$

we see that the spherical integrals of the terms on the right-hand side of (3.3.5) are zero, except for the $\mathcal{O}(r^{-4})$ remainder. □

We now want to understand the asymptotics, as $t \to 0+$, of the function $R_V(it)v \in L^2$ for $v \in H_0$. We first consider $R_0(it)v$:

LEMMA 3.19. *Suppose that* $n = 3$ *and assume that* $v \in H_0$. *Then we have the following asymptotic expansion in* L^2_{loc} *as* $t \to 0+$:

$$R_0(it)v = K_v + tJ_v + \mathcal{O}(t^{3/2}), \tag{3.3.10}$$

where, using the notation of (3.3.7),

$$K_v(y) = \frac{1}{4\pi}\int_0^\infty rI_v(r,y)\,dr, \quad J_v(y) = -\frac{1}{4\pi}\int_0^\infty r^2 I_v(r,y)\,dr.$$

Moreover, $J_v(y)$ is a polynomial of order 1 *in y and*

$$\partial_{y_j} J_v(y) = -\frac{b_j(v)}{12\pi}, \quad 1 \le j \le 3, \tag{3.3.11}$$

where $b_j(v)$ are defined in (3.3.6).

Proof. We write

$$R_0(it)v(y) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-t|x-y|}}{|x-y|} v(x)\,dx = \frac{1}{4\pi}\int_0^\infty re^{-tr} I_v(r,y)\,dr.$$

We now use the expansion $e^{-s} = 1-s+\mathcal{O}(s^{3/2})$, valid uniformly in $s \in [0,\infty)$, with $s = tr$. By part 3 of Lemma 3.18, we have locally uniformly in y, $I_v(r,y) = \mathcal{O}(r^{-4})$ as $r \to +\infty$. We moreover have $I_v(r,y) = \mathcal{O}(1)$ uniformly in r, y, since $v \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. Then the integral

$$\int_0^\infty r^{1+\alpha} I_v(r,y)\,dr$$

converges absolutely for $\alpha = 0, 1, 3/2$, which gives (3.3.10).

To show (3.3.11), we use the divergence theorem for the vector field $v(x)e_j$ to obtain

$$\begin{aligned}\partial_{y_j}\int_0^R r^2 I_v(r,y)\,dr = \partial_{y_j}\int_{B(y,R)} v(x)\,dx &= \int_{B(y,R)} \partial_j v(x)\,dx \\ &= R^2\int_{\mathbb{S}^2} \theta_j v(y+R\theta)\,d\theta.\end{aligned} \tag{3.3.12}$$

By (3.3.5), this converges as $R \to +\infty$ to $b_j/3$, locally uniformly in y, proving (3.3.11). □

The operator $v \mapsto K_v$ can be characterized in terms of the principal value integrals:

LEMMA 3.20. *Assume that $n = 3$ and let $v \in H_0$ and $\varphi \in L^2_{\mathrm{comp}}$ and put $u := R_0(0)\varphi \in L^2_{\mathrm{loc}}$. Then the limit*

$$\langle v, u\rangle_0 := \lim_{R\to+\infty}\int_{B(y,R)} v(x)\overline{u(x)}\,dx \tag{3.3.13}$$

exists, is independent of y, and

$$\langle v, u\rangle_0 = \langle K_v, \varphi\rangle. \tag{3.3.14}$$

Proof. For $y, y' \in \mathbb{R}^3$ and large R, we proceed as in (3.3.12) to obtain

$$\partial_{y'_j}\int_{B(y',R)} \frac{v(x)}{|x-y|}\,dx = R^2\int_{\mathbb{S}^2} \theta_j \frac{v(y'+R\theta)}{|y'-y+R\theta|}\,d\theta. \tag{3.3.15}$$

Since $v(x) = \mathcal{O}(|x|^{-2})$ by (3.3.5), we see that (3.3.15) is $\mathcal{O}(R^{-1})$, locally uniformly in y, y', implying that

$$\int_{B(y,R)} \frac{v(x)}{|x-y|}\, dx - \int_{B(y',R)} \frac{v(x)}{|x-y|}\, dx = \mathcal{O}(R^{-1}).$$

Then for each fixed y',

$$\begin{aligned}\langle K_v, \varphi\rangle &= \frac{1}{4\pi} \lim_{R\to+\infty} \int_{\mathbb{R}^3} \overline{\varphi(y)} \int_{B(y,R)} \frac{v(x)}{|x-y|}\, dxdy \\ &= \frac{1}{4\pi} \lim_{R\to+\infty} \int_{\mathbb{R}^3} \overline{\varphi(y)} \int_{B(y',R)} \frac{v(x)}{|x-y|}\, dxdy \\ &= \lim_{R\to+\infty} \int_{B(y',R)} \overline{u(x)} v(x)\, dx,\end{aligned}$$

yielding (3.3.13). $\square$

To characterize A_1, we consider the following space of resonant states at zero:

$$\widetilde{H}_0 := \{v \in H^2_{\mathrm{loc}}(\mathbb{R}^3) : P_V v = 0,\ v = R_0(0)(-\Delta v)\}.$$

By part 1 of Lemma 3.18, we see that $H_0 \subset \widetilde{H}_0$. Moreover, by (3.3.9) (which applies to any function of the form $R_0(0)f$ with $f \in L^2_{\mathrm{comp}}$) and since $|x|^{-2}$ lies in L^2 near infinity, we see that

$$H_0 = \left\{ v \in \widetilde{H}_0 \;\Big|\; \int_{\mathbb{R}^3} \Delta v(x)\, dx = 0 \right\}. \tag{3.3.16}$$

Since $\Delta v = Vv$ it follows immediately that H_0 has codimension at most one in $\widetilde{H}_0$:

$$\widetilde{m}_R(0) := \dim(\widetilde{H}_0/H_0) \le 1. \tag{3.3.17}$$

LEMMA 3.21. *1. The image of A_1 lies inside $\widetilde{H}_0$.*

2. If $v \in \widetilde{H}_0$, then

$$v - \frac{A_1(V)}{4\pi} \int_{\mathbb{R}^3} \Delta v(x) dx \in H_0. \tag{3.3.18}$$

Proof. 1. It follows from (3.3.8) that

$$iA_1 = R_0(0)(-\Delta) iA_1 + \partial_\lambda R_0(0)\Delta\Pi_0.$$

Since

$$\partial_\lambda R_0(0)(x,y) = \frac{i}{4\pi} \quad \text{and} \quad \int_{\mathbb{R}^3} \Delta v = 0, \quad v \in H_0,$$

we have

$$\partial_\lambda R_0(0)\Delta\Pi_0 = 0.$$

Therefore, $A_1 = R_0(0)(-\Delta)A_1$. Together with $P_V A_1 = 0$ this shows that the image of A_1 lies in $\widetilde{H}_0$.

2. From by the resolvent identity, $R_V(\lambda) = R_0(\lambda) - R_V(\lambda)VR_0(\lambda)$, we see that for $\rho \in C_c^\infty(\mathbb{R}^3)$ equal to one near $\operatorname{supp} V$

$$R_0(\lambda)V = R_V(\lambda)\rho(P_V - \lambda^2)R_0(\lambda)V : L^2 \to L^2_{\mathrm{loc}}. \tag{3.3.19}$$

We now apply (3.3.19) to $-v$, recalling from Lemma 3.18 that $v = -R_0(0)Vv$:

$$\begin{aligned} -R_0(\lambda)Vv &= -\left(-\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda)\right)\rho(P_V - \lambda^2)R_0(\lambda)Vv \\ &= \Pi_0\rho v + \tfrac{1}{2}\Pi_0\rho P_V\partial_\lambda^2 R_0(0)Vv - iA_1\rho P_V\partial_\lambda R_0(0)Vv + \lambda g(\lambda), \end{aligned}$$

where $g(\lambda) \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ is holomorphic. (The singular terms have to cancel out as the left hand side is holomorphic in λ). Putting $\lambda = 0$ we obtain

$$v = \Pi_0\left(\rho v + \tfrac{1}{2}\rho P_V\partial_\lambda^2 R_0(0)Vv\right) - iA_1\rho P_V\partial_\lambda R_0(0)Vv.$$

3. It follows that for $g = -i\rho P_V\partial_\lambda R_0(0)Vv$, we have $v - A_1 g \in H_0$. We now calculate

$$g(x) = \frac{V(x)}{4\pi}\int_{\mathbb{R}^3} V(y)v(y)dy = \frac{V(x)}{4\pi}\int_{\mathbb{R}^3} \Delta v(y)dy$$

completing the proof. $\square$

We finally split A_1 into two parts, one produced by functions from H_0 and one orthogonal to it in a certain sense:

LEMMA 3.22. *Define the operator*

$$T : L^2_{\mathrm{comp}}(\mathbb{R}^3) \to L^2_{\mathrm{loc}}(\mathbb{R}^3), \quad Tf(x) = \frac{1}{12\pi}\int_{\mathbb{R}^3}\langle x, y\rangle f(y)\,dy. \tag{3.3.20}$$

In the notation of (3.3.17) we have:

1. *If $\widetilde{m}_R(0) = 0$, then $A_1 = \Pi_0 VTV\Pi_0$.*
2. *If $\widetilde{m}_R(0) = 1$, then*

$$A_1 = \Pi_0 VTV\Pi_0 + 4\pi(u_0 \otimes \bar{u}_0), \tag{3.3.21}$$

where u_0 is the unique element of $\widetilde{H}_0$ satisfying

$$u_0(x) = \frac{1}{4\pi|x|} + \mathcal{O}\Big(\frac{1}{|x|^2}\Big), \quad |x| \to \infty, \quad \langle u_0, v\rangle_0 = 0 \text{ for } v \in H_0, \tag{3.3.22}$$

where $\langle \bullet, \bullet\rangle_0$ is defined by (3.3.13).

REMARK. Since Lemma 3.18 (part 1) shows that $\Pi_0(V) = 0$ we could replace the operator T in (3.3.21) by the operator

$$G_3 f(x) := -\frac{1}{24\pi}\int_{\mathbb{R}^3}|x - y|^2 f(y)\,dy. \tag{3.3.23}$$

Indeed, $|x-y|^2 = |x|^2 - 2\langle x, y\rangle + |y|^2$ and if $T_1 f(x) := \int_{\mathbb{R}} f(y)|y|^2 dy/12\pi$ (a constant function) then

$$\Pi_0 VTV\Pi_0 = \Pi_0 VG_3 V\Pi_0 - \Pi_0 VT_1 V\Pi_0 - \Pi_0 VT_1^* V\Pi_0.$$

But $\Pi_0 VT_1 f = (T_1 f)\Pi_0(V) = 0$.

Proof. 1. Suppose that $v \in H_0$ and $\psi \in L^2_{\mathrm{comp}}$ and take $\rho \in C_c^\infty(\mathbb{R}^3)$ such that $\rho V = V$ and $\rho\psi = \psi$. Since $(P + t^2)v = t^2 v$, we have $v = t^2 R_0(it)v$ for $t > 0$. Hence,

$$\begin{aligned} \langle v, \psi\rangle &= t^2\langle R_V(it)v, \psi\rangle = t^2\langle v, R_V(it)\psi\rangle \\ &= t^2\langle v, R_0(it)(-\Delta + t^2)R_V(it)\psi\rangle \\ &= t^2\langle R_0(it)v, \rho(-\Delta + t^2)R_V(it)\psi\rangle, \end{aligned} \tag{3.3.24}$$

where in the last equality we used that

$$\begin{aligned} (-\Delta + t^2)R_V(it)\psi &= (I - VR_V(it))\psi = \rho(I - VR_V(it))\psi \\ &= \rho(-\Delta + t^2)R_V(it)\psi. \end{aligned}$$

2. We next write the following expansion in L^2_{comp} as $t \to +0$ (note that $(-\Delta)\Pi_0 = -V\Pi_0$ and $(-\Delta)A_1 = -VA_1$):

$$\rho(-\Delta + t^2)R_V(it)\psi = \frac{(-\Delta)\Pi_0\psi}{t^2} + \frac{(-\Delta)A_1\psi}{t} + \mathcal{O}(1). \tag{3.3.25}$$

3. Combining (3.3.10) (Lemma 3.19) with (3.3.24) and (3.3.25) gives

$$\langle v, \psi\rangle = \langle K_v, (-\Delta)\Pi_0\psi\rangle + t\langle J_v, (-\Delta)\Pi_0\psi\rangle + t\langle K_v, (-\Delta)A_1\psi\rangle + \mathcal{O}(t^{3/2}).$$

The terms next to the first power of t give in the limit $t \to 0+$,

$$\langle J_v, (-\Delta)\Pi_0\psi\rangle + \langle K_v, (-\Delta)A_1\psi\rangle = 0. \tag{3.3.26}$$

4. We evaluate the first term in (3.3.26). By part 1 of Lemma 3.18, we have $\int_{\mathbb{R}^3}(-\Delta)\Pi_0\psi = 0$. By part 3 of the same lemma, we then have

$$\begin{aligned} \langle J_v, (-\Delta)\Pi_0\psi\rangle &= -\frac{1}{12\pi}\sum_{j=1}^3 b_j(v)\int_{\mathbb{R}^3} x_j\overline{(-\Delta\Pi_0\psi)(x)}\,dx \\ &= -\frac{1}{12\pi}\sum_{j=1}^3 b_j(v)b_j(\overline{\Pi_0\psi}). \end{aligned}$$

Since $b_j(v)$ is equal to the integral of $-x_j Vv$, we have

$$\begin{aligned} \langle J_v, (-\Delta)\Pi_0\psi\rangle &= -\frac{1}{12\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle x, y\rangle V(x)V(y)v(x)\overline{\Pi_0\psi(y)}\,dxdy \\ &= -\langle v, \Pi_0 VTV\Pi_0\psi\rangle. \end{aligned}$$

5. Returning to (3.3.26), by Lemma 3.20

$$\langle K_v, (-\Delta)A_1\psi\rangle = \langle v, A_1\psi\rangle_0,$$

which inserted into (3.3.26) gives

$$\langle v, (A_1 - \Pi_0 VTV\Pi_0)\psi\rangle_0 = 0.$$

Thus the image of $\widetilde{A}_1 := A_1 - \Pi_0 VTV\Pi_0$ lies inside $\widetilde{H}_0$ and is orthogonal to H_0 with respect to the generalization $\langle\cdot,\cdot\rangle_0$ of the L^2 inner product on H_0. In the case $\widetilde{m}_R(0) = 0$, we then have $\widetilde{A}_1 = 0$.

6. We now consider the case $\widetilde{m}_R(0) = 1$. Let $u_0 \in \widetilde{H}_0$ be defined by (3.3.22): from (3.3.16) we see that such u_0 exists and is unique. The discussion in Step 5 shows that the image of $\widetilde{A}_1$ is contained in the span of u_0. Since $\widetilde{A}_1$ is symmetric, we have for some $c \in \mathbb{C}$,

$$\widetilde{A}_1 = c(u_0 \otimes \bar{u}_0).$$

To find c, we apply (3.3.18) to u_0. Since V is L^2 orthogonal to the space H_0 by part 1 of Lemma 3.3.5, we have $\Pi_0(V) = 0$. Moreover, using the expansion in (3.3.22) and the fact that $u_0 = R_0(0)(-\Delta u_0)$ we obtain (see (3.3.9))

$$\int_{\mathbb{R}^3} (-\Delta u_0)(x)dx = \int_{\mathbb{R}^3} (-V(x)u_0(x))dx = 1. \tag{3.3.27}$$

From (3.3.18) we obtain

$$u_0 + \frac{\widetilde{A}_1(V)}{4\pi} \in H_0.$$

However, $\widetilde{A}_1(V) = -cu_0$ and thus $c = 4\pi$, finishing the proof. $\square$

We summarize the findings of this section in

THEOREM 3.23 ($\boldsymbol{R_V(\lambda)}$ near 0 for $n \geq 3$ odd). *1. Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R};\mathbb{R})$ and that $m_R(0) > 0$. Then*

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda)\,, \tag{3.3.28}$$

where $\lambda \mapsto A(\lambda)$ is holomorphic near 0, Π_0 is the orthogonal projection onto the space of L^2 solutions to $P_V u = 0$ and A_1 is described by Lemma 3.22.

2. For $n \geq 5$, (3.3.28) holds with $A_1 = 0$.

We can now reinterpret $\widetilde{m}_R(0)$ defined in (3.3.17) as the multiplicity of the "genuine" resonance at zero, corresponding to a resonant state which is not an eigenstate:

$$\widetilde{m}_R(0) = m_R(0) - \operatorname{tr}\Pi_0. \tag{3.3.29}$$

We note that $0 \leq \widetilde{m}_R(0) \leq 1$ and that $\widetilde{m}_R(0) = 0$ for $n \geq 5$

3.4. UPPER BOUNDS ON THE NUMBER OF RESONANCES

As in the case of dimension one we will estimate the number of resonances using a suitable determinant. We start with

LEMMA 3.24 (**Trace class properties**). *For $V, \rho \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$, odd,*

$$(VR_0(\lambda)\rho)^p \ , \quad p \geq \frac{n+1}{2} \, ,$$

is an entire family of trace class operators.

Proof. 1. We first estimate the characteristic values of $\rho_1 R_0(\lambda) \rho_1$ where $\rho_1 \in C^\infty_{\rm c}(\mathbb{R}^n)$. If $\operatorname{supp} \rho_1 \subset B(0, R)$ we can consider

$$\rho_1 R_0(\lambda) \rho_1 : L^2(\mathbb{T}^n_R) \longrightarrow L^2(\mathbb{T}^n_R) \, , \quad \mathbb{T}_R := \mathbb{R}^n / R\mathbb{Z}^n \, . \tag{3.4.1}$$

2. Then, using (B.3.7) and then (B.3.9), we have

$$\begin{aligned} s_j(\rho_1 R_0(\lambda)\rho_1) &\leq s_j((-\Delta_{\mathbb{T}^n_R} + 1)^{-\ell}) \|(-\Delta_{\mathbb{T}^n_R} + 1)^{\ell} \rho_1 R_0(\lambda) \rho_1\| \\ &\leq C j^{-2\ell/n} \|\rho_1 R_0(\lambda) \rho_1\|_{L^2 \to H^{2\ell}} \, . \end{aligned} \tag{3.4.2}$$

Theorem 3.1 gives

$$s_j(\rho_1 R_0(\lambda)\rho_1) \leq C \min(|\lambda|^{-1}, j^{-1/n}, |\lambda| j^{-2/n}) \exp(C(\operatorname{Im} \lambda)_-) \, . \tag{3.4.3}$$

3. By taking $\rho_1 = 1$ on $\operatorname{supp} \rho \cup \operatorname{supp} V$ we can use (B.3.7) again to see that (3.4.3) holds for $s_j(VR_0(\lambda)\rho)$. Using (B.3.6) we see that

$$s_j\left((VR_0(\lambda)\rho)^p\right) \leq C_1 |\lambda|^p j^{-2p/n} \exp(C_1 (\operatorname{Im} \lambda)_-) \, .$$

when $p \geq (n+1)/2$

$$\sum_j s_j\left((VR_0(\lambda)\rho)^p\right) < \infty$$

which means the operator is of trace class. $\square$

Since for $n \geq 2$, $VR_0(\lambda)$ is no longer of trace class we cannot use the determinant defined by (2.2.28).

DEFINITION 3.25. *Suppose that $n \geq 3$ is odd. Using Lemma 3.24 the following definition is justified: for $\rho \in C^\infty_{\rm c}$ equal to 1 near the support of V,*

$$H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}) \, . \tag{3.4.4}$$

We sometimes write $H = H_V$ to emphasize the dependence on the potential.

THEOREM 3.26 (Multiplicity of a resonance). *Let the functions H be given by (3.4.4) and let $m_H(\lambda)$ be the multiplicity of λ as a zero of $H(\lambda)$.*

Then, in the notation of (3.2.5),

$$m_R(\lambda) \leq m_H(\lambda), \quad \lambda \in \mathbb{C}. \tag{3.4.5}$$

Proof. 1. Arguing as in the proof of Theorem 3.15 it is enough to prove (3.4.5) when $m_R(\lambda) \leq 1$.

2. As n is odd,

$$I - (VR_0(\lambda)\rho)^{n+1} = \sum_{j=0}^{n} (-VR_0(\lambda)\rho)^j (I + VR_0(\lambda)\rho) \, . \tag{3.4.6}$$

Hence if λ is a simple pole of $(I + VR_0(\lambda)\rho)^{-1}$ then the operator $I - (VR_0(\lambda)\rho)^{n+1}$ has a non-empty kernel. That implies that $H(\lambda) = 0$, that is, $m_H(\lambda) \geq 1 = m_R(\lambda)$ completing the proof of (3.4.5). ☐

DISCUSSION. To obtain a determinant for which the zeros would agree with resonances *with multiplicities* we could use regularized determinants – see [**Si79b**] – and put

$$D(\lambda) := \det_p (I + VR_0(\lambda)\rho), \quad p \geq \frac{n+1}{2}.$$

However one can show that, except when $n = 3$, $D(\lambda)$ grows too fast as $\operatorname{Im}\lambda \to -\infty$. This makes estimates on the number of zeros unwieldy.

The determinant $H(\lambda)$ is introduced to remedy the growth problem but we pay by introducing additional zeros. For bounds on the growth of the number of resonances, which is all we are able to do precisely, that of course does not matter. The choice of $n+1$ as the power of $VR_0(\lambda)\rho$ was arbitrary as in view of Lemma 3.24 we could have taken any $p \geq (n+1)/2$. It turns out convenient in the proof of Theorem 3.28.

The main result of this section is the following upper bound

THEOREM 3.27 (Upper bounds on the number of resonances). *Suppose that $n \geq 3$ is odd and that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C})$. Let $m_R(\lambda)$ be the multiplicity of a resonance at λ as defined in (2.2.11).*

Then

$$\sum_{|\lambda| \leq r} m_R(\lambda) \leq C_V r^n \, . \tag{3.4.7}$$

INTERPRETATION. In the case of $-\Delta + V$ on a bounded domain, for instance on $\mathbb{T}^n$, the spectrum is discrete and for $V \in L^\infty(\mathbb{T}^n; \mathbb{R})$ we have the asymptotic Weyl law for the number of eigenvalues:

$$
\begin{gathered}
|\{\lambda : \lambda^2 \in \operatorname{Spec}(-\Delta_{\mathbb{T}^n} + V)\,,\ |\lambda| \leq r\}| = c_n \operatorname{vol}(\mathbb{T}^n) r^n (1 + \mathcal{O}(1/r))\,, \\
c_n = 2\operatorname{vol}(B_{\mathbb{R}^n}(0,1))/(2\pi)^n\,,
\end{gathered}
$$

where the eigenvalues are included according to their multiplicities.

In the case of $-\Delta + V$ on $\mathbb{R}^n$ the discrete spectrum is replaced by the discrete set of resonances. Hence the bound (3.4.7) is an analogue of the Weyl law. Except in dimension one (see Theorem 2.16) the issue of asymptotics or even optimal lower bounds remains unclear at the time of writing (see Section 3.13 for references).

Jensen's formula, see (D.1.9) in §D.2, and (3.4.5) show that Theorem 3.27 is an immediate consequence of an estimate on $H(\lambda)$:

THEOREM 3.28 (Determinant bounds I). *If $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{C})$ and $\rho \in C^\infty_{\rm c}(\mathbb{R}^n)$ is equal to one on $\operatorname{supp} V$, then for some constant A,*

$$
H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}),
$$

satisfies

$$
|H(\lambda)| \leq A \exp(A|\lambda|^n)\,. \tag{3.4.8}
$$

In particular, we have

$$
\sum_{|\lambda| \leq r} m_H(\lambda) \leq C_V r^n.
$$

Proof. 1. We use the Weyl inequality (B.5.8) to see that

$$
|H(\lambda)| \leq \prod_{k=1}^\infty \left(1 + s_k((VR_0(\lambda)\rho)^{n+1})\right)\,. \tag{3.4.9}
$$

We then use (B.3.6) to see that

$$
s_k((VR_0(\lambda)\rho)^{n+1}) \leq \|V\|_\infty^{n+1} \left(s_{[k/(n+1)]}(\rho R_0(\lambda)\rho)\right)^{n+1}\,. \tag{3.4.10}
$$

Hence we need to estimate $s_j(\rho R_0(\lambda)\rho)$ for $\rho \in C^\infty_{\rm c}(\mathbb{R}^n)$.

2. We start with easier estimates in the physical half-plane $\operatorname{Im}\lambda \geq 0$. We apply (3.4.3) to obtain

$$
s_j(\rho R_0(\lambda)\rho) \leq C j^{-1/n},
$$

which inserted in (3.4.10) gives

$$
s_k((VR_0(\lambda)\rho)^{n+1}) \leq C_1 k^{-(n+1)/n}.
$$

Using this in (3.4.9) we then get

$$\begin{aligned} H(\lambda) &\leq \exp\left(\sum_{k=1}^{\infty} s_k((VR_0(\lambda)\rho)^{n+1})\right) \\ &\leq \exp\left(C_1 \sum_{k=1}^{\infty} k^{-(n+1)/n}\right) \\ &\leq C_2 \,, \end{aligned}$$

that is, $H(\lambda)$ is uniformly bounded for $\operatorname{Im}\lambda \geq 0$.

3. To obtain estimates for $\operatorname{Im}\lambda < 0$ we use (3.1.19) to write

$$\begin{gathered} \rho(R_0(\lambda) - R_0(-\lambda))\rho = a_n \lambda^{n-2} E_\rho(\bar{\lambda})^* E_\rho(\lambda) \,, \\ E_\rho(\lambda)u(\omega) := \int_{\mathbb{R}^n} e^{i\lambda\langle\omega,x\rangle} \rho(x)u(x)dx \,, \\ E_\rho(\lambda) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{S}^{n-1}) \,. \end{gathered} \tag{3.4.11}$$

Hence for $\operatorname{Im}\lambda < 0$ (B.3.5) gives

$$\begin{aligned} s_j(\rho R_0(\lambda)\rho) &\leq a_n|\lambda|^{n-2}\|E_\rho(\lambda)\| s_{[j/2]}(E_\rho(\lambda)) + s_{[j/2]}(\rho R_0(-\lambda)\rho) \\ &\leq C\exp(C|\lambda|) s_{[j/2]}(E_\rho(\lambda)) + Cj^{-1/n} \,. \end{aligned} \tag{3.4.12}$$

4. To estimate $s_j(E_\rho(\lambda))$ we use the Laplacian on the sphere, $-\Delta_{\mathbb{S}^{n-1}}$, and (B.3.6):

$$\begin{aligned} s_j(E_\rho(\lambda)) &\leq s_j((-\Delta_{\mathbb{S}^{n-1}}+1)^{-\ell})\|(-\Delta_{\mathbb{S}^{n-1}}+1)^\ell E_\rho(\lambda)\| \\ &\leq C^\ell j^{-2\ell/(n-1)} \|(-\Delta_{\mathbb{S}^{n-1}}+1)^\ell E_\rho(\lambda)\| \\ &\leq C_1^\ell j^{-2\ell/(n-1)} \exp(C_1|\lambda|)(2\ell)! \,. \end{aligned} \tag{3.4.13}$$

Here we used the fact that for ρ with support in $B(0,R)$,

$$\|(-\Delta_{\mathbb{S}^{n-1}}+1)^\ell E_\rho(\lambda)\| \leq C_\rho \sup_{\omega\in\mathbb{S}^{n-1},|x|\leq R} \left|(-\Delta_\omega+1)^\ell e^{i\lambda\langle x,\omega\rangle}\right| \,,$$

and we estimated sup using, essentially, the Cauchy estimates.

We now optimize the estimate (3.4.13) in ℓ: since $(2\ell)! \leq (2\ell)^{(2\ell)}$,

$$\begin{aligned} C_1^\ell j^{-2\ell/(n-1)}(2\ell)! &\leq (j/(C_3\ell^{n-1}))^{-2\ell/(n-1)} \\ &= \exp(-j^{\frac{1}{n-1}}/C_4), \quad \text{if } \ell = (j/C_3 e)^{\frac{1}{n-1}}. \end{aligned}$$

This gives

$$s_j(E_\rho(\lambda)) \leq C_2 \exp\left(C_2|\lambda| - j^{\frac{1}{n-1}}/C_2\right) \,. \tag{3.4.14}$$

5. Going back to (3.4.10) and (3.4.12) we obtain

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq C_3 \exp\left(C_3|\lambda| - k^{\frac{1}{n-1}}/C_3\right) + C_3 k^{-\frac{n+1}{n}}.$$

In particular,

$$s_k((VR_0(\lambda)\rho)^{n+1}) \leq \begin{cases} C_4 \exp(C_4|\lambda|), & k \leq C_4|\lambda|^{n-1} \\ C_4 k^{-\frac{n+1}{n}}, & k \geq C_4|\lambda|^{n-1}. \end{cases} \tag{3.4.15}$$

Returning to (3.4.9) we use (3.4.15) as follows

$$\begin{aligned} |H(\lambda)| &\leq \prod_{k \leq C_k|\lambda|^{n-1}} \exp(C_4|\lambda|) \left(\exp \sum_{k \geq C_4|\lambda|^{n-1}} C_4 k^{-(n+1)/n}\right) \\ &\leq \exp(C_5|\lambda|^n), \end{aligned}$$

which completes the proof. $\square$

REMARK. The exponent n in (3.4.7) is optimal as shown by the case of radial potentials. Let $V(x) = v(|x|)(R-|x|)^0_+$, where v is a C^2 even function, and $v(R) > 0$. Then, see [**Zw89a**],

$$\sum\{m_R(\lambda) \ : \ |\lambda| \leq r\} = C_R r^n(1 + o(1)). \tag{3.4.16}$$

The constant C_R and its appearance in (3.4.7) is explained and discussed in [**St06**].

3.5. COMPLEX VALUED POTENTIALS WITH NO RESONANCES

As we have seen in Theorem 2.16 one dimensional complex valued compactly supported non-zero potentials always have infinitely many resonances with a counting functions satisfying a nice asymptotic formula. The situation is dramatically different in higher dimensions where complex valued potentials may have *no* resonances at all.

THEOREM 3.29 (Complex valued potentials with no resonances). *Let (r, θ, x') be cylindrical coordinates in $\mathbb{R}^{k+2}$, where $k \geq 1$ is odd:*

$$x = (x_1, x_2, x'), \quad x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x' \in \mathbb{R}^k.$$

Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R}^{k+2}; \mathbb{C})$ is of the following form:

$$V(x) = e^{i\theta m} W(r, x'), \quad W \in L^\infty_{\rm comp}([0,\infty) \times \mathbb{R}^k).$$

Then, if $m \neq 0$, the resolvent $R_V(\lambda)$ is entire in $\mathbb{C}$, that is the operator $-\Delta + V$ has no resonances.

REMARK. We can easily place conditions on W so that

$$V \in C_c^\infty(\mathbb{R}^{2+k}; \mathbb{C}) .$$

Before starting the proof we need two simple lemmas

LEMMA 3.30 (Fourier decomposition of the resolvent). *Let Π_ℓ be the projection onto the ℓ'th Fourier mode:*

$$\Pi_\ell u(r, \theta, x') := e^{i\ell\theta} \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi, x') e^{-i\ell\varphi} d\varphi . \tag{3.5.1}$$

Then for $\rho \in C_c^\infty(\mathbb{R}^{2+k})$, $\rho = \rho(r, x')$, we have

$$\|\Pi_\ell \rho R_0(\lambda) \rho \Pi_\ell\|_{L^2 \to L^2} \leq \frac{C e^{C(\operatorname{Im}\lambda)_-}}{\langle \ell \rangle} , \quad \ell \in \mathbb{Z} . \tag{3.5.2}$$

Proof. 1. Because we chose ρ to be independent of θ, Π_ℓ commutes with $\rho R_0(\lambda) \rho$. Put

$$u := \rho R_0(\lambda) \rho \Pi_\ell f , \quad f \in L^2 .$$

Then (3.1.12) gives

$$\|u\|_{H^1} \leq C e^{C(\operatorname{Im}\lambda)_-} \|f\|_{L^2} . \tag{3.5.3}$$

2. On the other hand

$$\begin{aligned}
\|u\|_{H^1}^2 &\geq \langle -\Delta u, u \rangle \\
&= \int_{\mathbb{R}^{n-2}} \int_0^\infty \int_0^{2\pi} (D_r^2 - (i/r) D_r - \Delta_{x'} + \ell^2/r^2) u) \bar{u} d\theta r dr dx' \\
&= \int_{\mathbb{R}^k} \int_0^\infty \int_0^{2\pi} (|D_r u|^2 + |D_{x'} u|^2 + \ell^2/r^2) u) \bar{u} d\theta r dr dx' \\
&\geq \langle (\ell^2/r^2) u, u \rangle_{L^2} \geq \ell^2 \|u\|_{L^2}^2 / C ,
\end{aligned}$$

where the last inequality followed from the fact that r is bounded on the support of u by $\max_{x \in \operatorname{supp}\rho} |x|$ so that $\ell^2/r^2 \geq \ell^2/C$. Combining this with (3.5.3) proves (3.5.2). □

The next lemma is an elementary statement about sequences:

LEMMA 3.31 (Two sided sequences). *Let $\{a_j\}_{j=-\infty}^\infty$ be a sequence satisfying $a_j \to 0$, $j \to \pm\infty$. Suppose that for some $m \in \mathbb{Z} \setminus \{0\}$ and $J \in \mathbb{N}$ we have the following property: for each j there exists $C_j \geq 0$ such that*

$$|a_{j+m}| \leq C_j |a_j| , \quad \text{and} \quad C_j \leq 1 \text{ for } |j| \geq J , \tag{3.5.4}$$

for some J.

Then $a_j \equiv 0$, $j \in \mathbb{Z}$.

Proof. Fix $j \in \mathbb{Z}$ and use (3.5.4) to obtain

$$\begin{aligned}|a_j| &\le C_{j-m}|a_{j-m}| \le \cdots \le \prod_{k=1}^{p} C_{j-km}|a_{j-mp}| \\ &\le K|a_{j-mp}| \to 0\,, \quad p \to \infty\,, \quad K := \prod_{|\ell|<J} C_\ell \ge \prod_{|j-mk|<J} C_{j-km}\,.\end{aligned}$$

This shows that $a_j = 0$ as claimed. □

Proof of Theorem 3.29. 1. In view of (3.2.18) if $m_R(\lambda) > 0$ for some λ then $(I + VR_0(\lambda)\rho)^{-1}$ has a pole for any $\rho \in C_c^\infty(\mathbb{R}^{2+k})$ such that $\rho = 1$ on $\operatorname{supp} V$. In particular we can take $\rho = \rho(r, x')$.

Hence there exists $u \in L^2$ such that

$$u = -VR_0(\lambda)\rho u = -V\rho R_0(\lambda)\rho u\,.$$

2. We now use the structure of V, $V(r, \theta, x') = e^{im\theta}W(r, x')$, to calculate

$$\begin{aligned}\Pi_{j+m}u &= \Pi_{j+m}\left(e^{im\theta}W\rho R_0(\lambda)\rho u\right) \\ &= e^{im\theta}\Pi_j W\rho R_0(\lambda)\rho\Pi_j u\,.\end{aligned}$$

Lemma 3.30 now shows that

$$\|\Pi_{j+m}u\|_{L^2} \le \frac{C\langle\lambda\rangle e^{C|\lambda|}}{\langle j\rangle}\|\Pi_j u\|_{L^2}\,.$$

If we put

$$a_j := \|\Pi_j u\|_{L^2}\,, \quad C_j := \frac{C\langle\lambda\rangle e^{C|\lambda|}}{\langle j\rangle}\,,$$

then the assumptions of Lemma 3.31 are satisfied. Thus $\Pi_j u = 0$ for all j which means that $u = 0$ and there is no resonance at λ. □

3.6. OUTGOING SOLUTIONS AND RELLICH'S THEOREM

In Section 2.4 the scattering matrix mapped incoming to outgoing components of solutions to

$$(P_V - \lambda^2)w = 0\,. \tag{3.6.1}$$

The intuition behind the notion of incoming and outgoing components of a solution to (3.6.1) was presented in §2.1.

A conceptually similar procedure is used in the case of scattering in higher dimensions with asymptotic formulae such as (3.1.20) replacing explicit representations involving $\exp(i\lambda|x|)$. The starting point is the same as in (2.4.3): we consider solutions to (3.6.1) of the form

$$w(x,\lambda,\omega) = e^{-i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)\,, \tag{3.6.2}$$

where u is outgoing in the sense defined below. It is obtained using the resolvent $R_V(\lambda)$, except at the possible poles:

$$u(x,\lambda,\omega) := -R_V(\lambda)(Ve^{-i\lambda\langle\bullet,\omega\rangle})\,. \tag{3.6.3}$$

First we need to define outgoing solutions and show that $R_V(\lambda)$ is well defined for $\lambda \in \mathbb{R}\setminus\{0\}$.

DEFINITION 3.32. *A solution u to $(P_V - \lambda^2)u = f$, $\lambda \in \mathbb{R}\setminus\{0\}$, $f \in L^2_{\text{comp}}(\mathbb{R}^n)$ is called* outgoing *if there exists $g \in L^2_{\text{comp}}(\mathbb{R}^n)$ such that*

$$u = R_0(\lambda)g\,, \tag{3.6.4}$$

where $R_0(\lambda)$ is the resolvent described in Theorem 3.1.

A solution u is called incoming *if $u = R_0(-\lambda)g$, $\lambda \in \mathbb{R}\setminus\{0\}$, for some $g \in L^2_{\text{comp}}(\mathbb{R}^n)$*

INTERPRETATION. The asymptotic expansion in Theorem (3.5) shows that the outgoing $(+)$ and incoming $(-)$ solutions satisfy

$$u(x) = \frac{e^{\pm i\lambda|x|}}{|x|^{\frac{n-1}{2}}}a\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|^{\frac{n+1}{2}}}\right), \quad |x|\to\infty.$$

Hence u can be interpreted as a *spherical wave* with $a(x/|x|)$ giving the intensity at different directions $x/|x|$. The different signs λ show that $R_0(\pm)g$ is holomorphic in $\pm\operatorname{Im}\lambda \in \mathbb{R}\setminus\{0\}$. The wave equation interpretation discussed in §2.1 shows that the corresponding time dependent solutions are supported in $\pm t > C$, that is are outgoing/incoming. See also the self-contained discussion after Theorem 4.9 and Exercise 4.3.

In particular we see that u given by (3.6.3) is outgoing provided that λ is not a pole of $R_V(\lambda)$:

$$u(x,\lambda,\omega) = -R_V(\lambda)(Ve^{-i\lambda\langle\bullet,\omega\rangle}) = R_0(\lambda)f\,,$$
$$f = -(I + VR_0(\lambda)\rho)^{-1}(Ve^{-i\lambda\langle\bullet,\omega\rangle}) \in L^2_{\text{comp}}(\mathbb{R}^n)\,.$$

When V is real valued and $\lambda \in \mathbb{R}\setminus\{0\}$ then Rellich's important result (Theorem 3.33) states that there are no outgoing solutions to (3.6.1). In other words, $R_V(\lambda)$ has no non-zero real poles:

THEOREM 3.33 (Rellich's uniqueness theorem I). *Suppose that* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$ *is real valued. Then for* $\lambda \in \mathbb{R} \setminus \{0\}$ *there are* no *outgoing solutions to*

$$(P_V - \lambda^2)u = 0 .$$

Equivalently, $R_V(\lambda)$ *has no poles for* $\lambda \in \mathbb{R} \setminus \{0\}$*.*

Proof. 1. We first show that having an outgoing solution for $\lambda > 0$ is equivalent to R_V having a pole at λ.

One implication follows directly from (3.2.1): if R_V has a pole at λ then $I + VR_0(\lambda)\rho$ is not invertible which by the Fredholm property (see §C.2) means that it has a non-empty kernel. If $g = -VR_0(\lambda)\rho g$, then $g = \rho g \in L^2_{\mathrm{comp}}$ and $u := R_0(\lambda)g$ solves

$$(P_V - \lambda^2)u = (-\Delta - \lambda^2)R_0(\lambda)g + VR_0(\lambda)g = g + VR_0(\lambda)g = 0.$$

Hence (3.6.4) holds.

Conversely, suppose that $u = R_0(\lambda)g$, $g \in L^2_{\mathrm{comp}}$ solves $(P_V - \lambda^2)u = 0$. By the same argument we see $(I + VR_0(\lambda)\rho)g = 0$ and by Theorem 3.26, R_V has a pole at λ.

2. The proof now proceeds by contradiction. So suppose that R_V has a pole at $\lambda > 0$. From (3.2.1) we see (as in Step 1) that there exists $g \in L^2$ such that $g = -VR_0(\lambda)g$. Defining $w = -R_0(\lambda)g$ we obtain $Vw = g$, and hence,

$$(P_V - \lambda^2)w = 0 , \quad w = -R_0(\lambda)Vw .$$

Theorem 3.5 shows that

$$w = R_0(\lambda)(Vw)(x) = \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} \left(h\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right) , \tag{3.6.5}$$

where

$$h(\theta) = c_n \lambda^{\frac{n-3}{2}} \widehat{Vw}(\lambda\theta) .$$

In particular,

$$(\partial_r - i\lambda)w = \mathcal{O}(r^{-\frac{n+1}{2}}), \quad r := |x|. \tag{3.6.6}$$

3. Since λ is real we have

$$\begin{aligned} 0 &= \int_{B(0,R)} (w(P_V - \lambda^2)\bar{w} - (P_V - \lambda^2)w\bar{w})dx \\ &= \int_{B(0,R)} (\bar{w}\Delta w - w\Delta\bar{w})dx = \int_{\partial B(0,R)} (\partial_r w\bar{w} - w\partial_r\bar{w})dS \end{aligned} \tag{3.6.7}$$

Using (3.6.5) and (3.6.6) we obtain

$$0 = 2i\lambda \int_{\partial B(0,R)} |w|^2 dS + \mathcal{O}(R^{-n}) \int_{\partial B(0,R)} dS$$

which gives

$$\int_{\partial B(0,R)} |w|^2 dS = \mathcal{O}(R^{-1}).$$

Taking $R \to \infty$ this implies, in the notation of (3.6.5), that

$$0 = \int_{\mathbb{S}^{n-1}} |h(\theta)|^2 d\theta = |c_n|^2 |\lambda|^{n-3} \int_{\mathbb{S}^{n-1}} |\widehat{Vw}(\lambda\theta)|^2 d\theta \,.$$

4. We conclude that

$$\widehat{Vw}(\xi) = 0 \,, \quad \langle \xi, \xi \rangle = \lambda^2 \,, \quad \xi \in \mathbb{R}^n \,.$$

If we put

$$\Sigma := \{ \xi \in \mathbb{C}^n \; : \; \langle \xi, \xi \rangle = \lambda^2 \} \,,$$

then Σ is a connected complex hypersurface in $\mathbb{C}^n$ and the entire function $\widehat{Vw}(\xi)$ vanishes on $\Sigma \cap \mathbb{R}^n$. It follows that $\widehat{Vw}(\xi) = 0$ on Σ. From that we see that

$$\frac{\widehat{Vw}(\xi)}{\langle \xi, \xi \rangle - \lambda^2} \quad \text{is an entire function of } \xi \in \mathbb{C}^n.$$

Since

$$(\langle \xi, \xi \rangle - \lambda^2) \widehat{w}(\xi) = \widehat{Vw}(\xi) \,,$$

Paley-Wiener theorem as applied in [**HöI**, Theorem 7.3.2] shows that $w \in \mathcal{E}'$.

To complete the proof we need the following lemma which is a simple version of a Carleman estimate (see [**Zw12**, §7.2] and references given there):

LEMMA 3.34. *For every $R > 0$ there exists $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that for $h > 0$ and $u \in H^2(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B(0, R)$ we have*

$$\|h^2 e^{\varphi/h} \Delta e^{-\varphi/h} u\|_{L^2} \geq c h^{\frac{1}{2}} \|u\|_{L^2}. \tag{3.6.8}$$

Proof. 1. Let us first assume that $u \in C^\infty_{\rm c}(B(0, R))$. Put

$$P_\varphi := -h^2 e^{\varphi/h} \Delta e^{-\varphi/h}.$$

Then

$$\begin{aligned} \|P_\varphi u\|^2_{L^2} &= \langle P_\varphi u, P_\varphi u \rangle = \langle P^*_\varphi P_\varphi u, u \rangle \\ &= \langle P_\varphi P^*_\varphi u, u \rangle + \langle [P^*_\varphi, P_\varphi] u, u \rangle \\ &= \|P^*_\varphi u\|^2_{L^2} + \langle [P^*_\varphi, P_\varphi] u, u \rangle \\ &\geq \langle [P^*_\varphi, P_\varphi] u, u \rangle. \end{aligned} \tag{3.6.9}$$

2. From (3.6.9) we see that it suffices to construct φ such that for all $u \in C^\infty_c(B(0, R))$

$$\langle [P^*_\varphi, P_\varphi] u, u \rangle \geq c^2 h \|u\|^2_{L^2}.$$

A calculation shows that

$$P_\varphi u = -h^2\Delta u + 2h\langle \partial\varphi, \partial u\rangle - |\partial\varphi|^2 u + h(\Delta\varphi)u,$$
$$P_\varphi^* u = -h^2\Delta u - 2h\langle \partial\varphi, \partial u\rangle - |\partial\varphi|^2 u - h(\Delta\varphi)u.$$

Using the identity

$$[P_\varphi^*, P_\varphi] = \frac{1}{2}[P_\varphi + P_\varphi^*, P_\varphi - P_\varphi^*]$$

we compute

$$\begin{aligned}[P_\varphi^*, P_\varphi]u = &- 8h^3 \sum_{j,k=1}^n \partial^2_{x_j x_k}\varphi \cdot \partial^2_{x_j x_k} u + 4h\langle \partial\varphi, \partial|\partial\varphi|^2\rangle u \\ &- 8h^3\langle \partial(\Delta\varphi), \partial u\rangle - 2h^3(\Delta^2\varphi)u.\end{aligned} \tag{3.6.10}$$

We choose $\varphi(x) := |x|^2/2 + Mx_1$ for some large constant M. Then the contribution from the second line of (3.6.10) is zero and we compute

$$[P_\varphi^*, P_\varphi]u = 8h(-h^2\Delta u + |x + Me_1|^2 u).$$

If $M \geq R + 1$ then for $u \in C_c^\infty(B(0, R))$,

$$\langle [P_\varphi^*, P_\varphi]u, u\rangle \geq 8h\big(\|hD_x u\|^2_{L^2} + \|u\|^2_{L^2}\big) \geq 8h\|u\|^2_{L^2}.$$

3. The last inequality proved (3.6.8) for $u \in C_c^\infty(B(0, R))$. Since both sides are finite for $u \in H^2$, $\operatorname{supp} u \subset B(0, R)$ an approximation argument completes the proof. $\square$

4. To complete the proof of Theorem 3.33 we now apply Lemma 3.34 to $u = e^{\varphi/h}w$ where w comes from Step 4 of the proof: $w \in H^2$, $\operatorname{supp} w \subset B(0, R)$. We have

$$\begin{aligned} 0 = h^2\|e^{\varphi/h}(P_V - \lambda^2)w\|_{L^2} &= \|e^{\varphi/h}(-h^2\Delta + h^2 V - h^2\lambda^2)e^{-\varphi/h}u\|_{L^2} \\ &\geq \|e^{\varphi/h}(-h^2\Delta)e^{-\varphi/h}u\|_{L^2} - Ch^2\|u\|_{L^2} \\ &\geq ch^{\frac12}\|u\|_{L^2} - Ch^2\|u\|_{L^2} \geq (c/2)h^{\frac12}\|u\|_{L^2},\end{aligned}$$

if h is small enough. But this means that $u \equiv 0$ which implies that $w \equiv 0$. Hence we have no outgoing solutions to the homogeous equation when $\lambda \in \mathbb{R} \setminus \{0\}$. $\square$

Rellich's uniqueness theorem holds in a stronger form which will be useful later in this section and also when we consider more general perturbatons:

THEOREM 3.35 (Rellich's uniqueness theorem II). *Suppose P is a self-adjoint operator with domain $H^2(\mathbb{R}^n)$ such that for $\chi \in C_c^\infty(B(0, 2R))$,*

$\chi = 1$ *in* $B(0, R)$ *we have* $P(1 - \chi) = -\Delta(1 - \chi)$. *Suppose that* $\lambda > 0$ *and* $u \in H^2_{\mathrm{loc}}$ *satisfies*

$$(P - \lambda^2)u = 0, \qquad \lim_{R\to\infty} \int_{\partial B(0,R)} |(\partial_r - i\lambda)u|^2 dS = 0. \tag{3.6.11}$$

Then

$$u(x) = 0 \ \ \textit{for } |x| > R. \tag{3.6.12}$$

REMARKS. 1. The second condition in (3.6.11) is implied by a stronger condition that

$$(\partial_r - i\lambda)u = o(r^{-\frac{n-1}{2}}),$$

which is called the *Sommerfeld radiation condition.* As we saw in (3.6.6) (Step 2 of the proof of Theorem 3.33) it typically arises in an even stronger form

$$(\partial_r - i\lambda)u = \mathcal{O}(r^{-\frac{n+1}{2}}).$$

2. The specific structure of P is unimportant and this is our first encounter with more general operators than $-\Delta + V$. What matters is the fact that P coincides with $-\Delta$ outside a compact set and that it is self-adjoint. The assumptions about the domain of P can also be relaxed as we will see in the chapter on *black box* scattering.

Proof. 1. With χ as in the statement of theorem we have

$$(-\Delta - \lambda^2)(1 - \chi)u = [\Delta, \chi]u =: f \in C^\infty_{\mathrm{c}}(\mathbb{R}^n).$$

We claim that $(1 - \chi)u = R_0(\lambda)f$. To see that put

$$w := (1 - \chi)u - R_0(\lambda)f, \quad (-\Delta - \lambda^2)w = 0.$$

Then from (3.6.11) and the asymptotics of $R_0(\lambda)f$ in (3.1.20) we see that

$$\int_{\partial B(0,R)} |G|^2 dS = o(1), \quad G := \tfrac{1}{2i\lambda}(\partial_r - i\lambda)w. \tag{3.6.13}$$

We now use Green's formula (see Step 3 of the proof of Theorem 3.33 for a similar argument):

$$\begin{aligned} 0 &= \tfrac{1}{2i\lambda} \int_{B(0,R)} (w(-\Delta - \lambda^2)\bar{w} - \bar{w}(\Delta - \lambda^2)w)dx \\ &= \tfrac{1}{2i\lambda} \int_{B(0,R)} (\bar{w}\Delta w - w\Delta\bar{w})dx = \tfrac{1}{2i\lambda} \int_{\partial B(0,R)} (\partial_r w\bar{w} - w\partial_r\bar{w})dS \\ &= \int_{\partial B(0,R)} \left(|w|^2 + 2\,\mathrm{Re}\, G\bar{w}\right) dS \geq \tfrac{1}{2} \int_{\partial B(0,R)} |w|^2 dS - 2\int_{\partial B(0,R)} |G|^2 dS. \end{aligned}$$

This and (3.6.13) (which was derived from the assumption (3.6.11)) imply that

$$\varphi(R) := \int_{\partial B(0,R)} |w|^2 dS \to 0, \quad R \to \infty.$$

It follows that $\frac{1}{R}\int_{B(0,R)} |w(x)|^2 dx = \frac{1}{R}\int_0^R \varphi(r)dr \to 0, \quad R \to \infty$. (Note that this implies that $w \in \mathscr{S}'(\mathbb{R}^n)$ and hence $\widehat{w}$ makes sense as a distribution.) From $(-\Delta - \lambda^2)w = 0$ we have $\operatorname{supp}\widehat{w} \subset \{|\xi|^2 = \lambda^2\}$. To see that $w = 0$ we apply the following result to $u = \widehat{w}$. It is a special case of [**HöI**, Theorem 7.1.27]:

LEMMA 3.36. *Suppose that* $u \in \mathcal{E}'(\mathbb{R}^n)$ *satisfies, for* $\lambda > 0$,

$$\operatorname{supp} u \subset \partial B(0, \lambda^2) \quad \textit{and} \quad \lim_{R\to\infty} \frac{1}{R}\int_{B(0,R)} |\widehat{u}(\xi)|^2 d\xi = 0.$$

Then $u \equiv 0$.

Proof. Without loss of generality we can assume that $\lambda = 1$.

1. Since u is compactly supported we see that $\widehat{u} \in C^\infty(\mathbb{R}^n)$. Suppose that $\chi \in C^\infty_{\rm c}(B(0,1))$ and $\int \chi(x)dx = 1$. Put $\chi_\varepsilon(x) := \varepsilon^{-n}\chi(x/\varepsilon)$ and define

$$C^\infty_{\rm c}(\mathbb{R}^n) \ni u_\varepsilon := u * \chi_\varepsilon \to u \in \mathcal{D}'(\mathbb{R}^n).$$

Since $\widehat{u}_\varepsilon(\xi) = \widehat{u}(\xi)\widehat{\chi}_\varepsilon(\xi)$, Plancherel's formula gives

$$\begin{aligned}
(2\pi)^n\|u_\varepsilon\|^2 &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2|\widehat{\chi}(\varepsilon\xi)|^2 d\xi \\
&\leq \int_{\varepsilon|\xi|\leq 1} |\widehat{u}(\xi)|^2|\widehat{\chi}(\varepsilon\xi)|^2 d\xi + \sum_{j=1}^\infty \int_{2^{j-1}\leq\varepsilon|\xi|\leq 2^j} |\widehat{u}(\xi)|^2|\widehat{\chi}(\varepsilon\xi)|^2 d\xi \\
&\leq \frac{1}{\varepsilon}\sup_{|\eta|\leq 1}|\widehat{\chi}(\eta)|^2 \ \varepsilon\int_{|\xi|\leq 1/\varepsilon} |\widehat{u}(\xi)|^2 d\xi \\
&\qquad + \frac{1}{\varepsilon}\sum_{j=1}^\infty \sup_{2^{j-1}\leq|\eta|\leq 2^j} 2^j|\widehat{\chi}(\eta)|^2 \ 2^{-j}\varepsilon\int_{|\xi|\leq 2^j/\varepsilon} |\widehat{u}(\xi)|^2 d\xi \\
&\leq \frac{1}{\varepsilon}C_\chi \sup_{R>1/\varepsilon}\frac{1}{R}\int_{B(0,R)} |\widehat{u}(\xi)|^2 d\xi,
\end{aligned}$$

where

$$C_\chi = \sup_{|\eta|\leq 1}|\widehat{\chi}(\eta)|^2 + \sum_{j=1}^\infty \sup_{2^{j-1}\leq|\eta|\leq 2^j} 2^j|\widehat{\chi}(\eta)|^2.$$

The sum converges as $\widehat{\chi} \in \mathscr{S}$ and thus $\widehat{\chi}(\eta) = \mathcal{O}(\langle\eta\rangle^{-\infty})$.

Consequently, the hypothesis of the lemma gives

$$\|u_\varepsilon\|^2 \leq \frac{C}{\varepsilon}\sup_{R>1/\varepsilon}\frac{1}{R}\int_{B(0,R)} |\widehat{u}(\xi)|^2 d\xi =: K(\varepsilon)/\varepsilon, \quad \lim_{\varepsilon\to 0} K(\varepsilon) = 0. \tag{3.6.14}$$

2. Now suppose that $\psi \in C_c^\infty(\mathbb{R}^n)$. Because of the support condition on u we know that $\operatorname{supp} u_\varepsilon \subset \operatorname{supp} u + \operatorname{supp} \chi_\varepsilon \subset \partial B(0,1) + B(0,\varepsilon) =: A_\varepsilon$. Using (3.6.14) we see that

$$\begin{aligned}
|u(\psi)|^2 &= \lim_{\varepsilon\to 0} |u_\varepsilon(\psi)|^2 \le \lim_{\varepsilon\to 0} \|u_\varepsilon\|^2 \int_{A_\varepsilon} |\psi(x)|^2 dx \\
&\le \lim_{\varepsilon\to 0} K(\varepsilon) \times \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{A_\varepsilon} |\psi(x)|^2 dx = \lim_{\varepsilon\to 0} K(\varepsilon) \int_{\partial B(0,1)} |\psi(x)|^2 dS \\
&= 0.
\end{aligned}$$

Since ψ was an arbitrary smooth function it follows $u \equiv 0$. $\square$

2. In Step 1 of the proof of Lemma 3.36 we showed (using Lemma 3.36) that

$$(1-\chi)u = R_0(\lambda)f, \quad f = [\Delta, \chi]u \in C_c^\infty(\mathbb{R}^n).$$

For $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on the support of χ, the expansion (3.1.20) and Green's formula give (see (3.6.7))

$$\frac{1}{i}\langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f\rangle = |c_n|^2 \lambda^{n-2} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\theta)|^2 d\theta. \tag{3.6.15}$$

3. On the other hand,

$$\begin{aligned}
\langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f\rangle &= \langle [-\Delta, \chi_1] u, u\rangle \\
&= \langle [-\Delta - \lambda^2, \chi_1] u, u\rangle \\
&= \langle [P - \lambda^2, \chi_1] u, u\rangle \\
&= \langle \chi_1 u, (P-\lambda^2) u\rangle - \langle (P-\lambda^2) u, \chi_1 u\rangle \\
&= 0.
\end{aligned} \tag{3.6.16}$$

Returning to (3.6.15) we see that $\widehat{f}(\lambda\theta) \equiv 0$, $\theta \in \mathbb{S}^{n-1}$.

4. We now argue as in Step 4 of the proof of Theorem 3.33: $\widehat{f}(\xi)/(\langle\xi,\xi\rangle - \lambda^2)$ is entire and hence $(1-\chi)u$ is compactly supported. $\square$

INTERPRETATION. In the formula (3.6.15) the left hand side depends on χ_1 while the right hand side does not. Another way to of stating this formula is

$$\frac{1}{i}\langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f\rangle = \operatorname{Im}\langle R_0(\lambda) f, f\rangle, \tag{3.6.17}$$

which follows from (3.6.16) applied with $-\Delta$ in place of P and with $\chi_1 = 1$ on $\operatorname{supp} f$. The Stone's formula (3.1.19) the right hand side can be expressed using the spectral pojection of $-\Delta$. From this (3.6.15) follows directly without using the expansion (3.1.20).

The expression on the left of (3.6.17) is called the *quantum flux* of $R_0(\lambda)f$ (or of more general solutions). The fact that $R_0(\lambda)f$ is outgoing is reflected in the fact that the flux is positive; it is negative for an incoming solution.

Using Rellich's uniqueness theorem the condition of being outgoing can be formulated in the following equivalent ways. For the proof see the hints in Exercise 3.5.

THEOREM 3.37 (Outgoing solutions). *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$, $f \in \mathcal{E}'(\mathbb{R}^n)$ *is a compactly supported distribution and that* u *solves*

$$(P_V - \lambda^2)u = f\,, \quad \lambda \in \mathbb{R}\setminus\{0\}\,. \tag{3.6.18}$$

Then the following conditions are equivalent:

(i) $u(x) = e^{i\lambda|x|}a(x/|x|)|x|^{-(n-1)/2} + \mathcal{O}(|x|^{-(n+1)/2})$, *as* $|x| \to \infty$, *where the expansion can be differentiated,*

(ii) $(\partial/\partial r - i\lambda)u = o(r^{-(n-1)/2})$, *as* $r \to \infty$, $r = |x|$,

(iii) $u = R_V(\lambda)f$,

(iv) $u = R_0(\lambda)g$, *for some* $g \in \mathcal{E}'(\mathbb{R}^n)$.

When $V \in C^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R})$ *and* $f \in C^\infty_{\rm c}(\mathbb{R}^n)$ *then* $g \in C^\infty_{\rm c}(\mathbb{R}^n)$.

As in dimension one we want to decompose the solution (3.6.2) into incoming and outgoing terms. The scattering matrix will then relate these two terms.

THEOREM 3.38 (Decomposition of free plane waves). *For* $\lambda \in \mathbb{R}\setminus\{0\}$, *we have, in the sense of distributions in* $x/|x| \in \mathbb{S}^{n-1}$

$$e^{-i\lambda\langle x,\omega\rangle} \sim \frac{1}{(\lambda|x|)^{\frac{n-1}{2}}}\left(c_n^+ e^{-i\lambda|x|}\delta_\omega(x/|x|) + c_n^- e^{i\lambda|x|}\delta_{-\omega}(x/|x|)\right)\,, \tag{3.6.19}$$

as $|x| \to \infty$, *where*

$$c_n^\pm = (2\pi)^{\frac{n-1}{2}} e^{\pm\frac{\pi}{4}(n-1)i}\,.$$

More precisely for $\varphi \in C^\infty(\mathbb{S}^{n-1})$,

$$(\lambda r)^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} e^{-i\lambda r\langle\omega,\theta\rangle}\varphi(\theta)d\theta = c_n^+ e^{-i\lambda r}\varphi(\omega) + c_n^- e^{i\lambda r}\varphi(-\omega) + \mathcal{O}(1/r)\,,$$

as $r \to \infty$, *with a full expansion in powers of* r.

INTERPRETATION. We consider

$$\lambda^{-\frac{n-1}{2}} c_n^\pm \delta_{\pm\omega}(\theta)$$

as leading coefficients of the incoming $(+)$ and outgoing $(-)$ components of $\exp(-i\lambda\langle x,\omega\rangle)$, even though that is valid only in the sense of distributions.

This is an analogue of the decomposition of $\exp(\pm i\lambda x)$, $x \in \mathbb{R}$, into the incoming and outgoing components:

$$e^{\pm i\lambda x} = e^{-i\lambda|x|}(\pm x)_-^0 + e^{i\lambda|x|}(\pm x)_+^0 \,, \quad x \neq 0 \,.$$

Proof. To prove the results we use the method of stationary phase – see for instance [**Zw12**, §3.5] or [**HöI**, §7.7].

1. We can assume that $\omega = (1, 0, \cdots, 0)$. Then the function $\langle \theta, \omega \rangle = \theta_1$ has two critical points on $\mathbb{S}^{n-1}$, correspoding to $\theta_1 = \pm 1$. Hence we can assume that φ is supported near the two poles $\theta_1 = \pm$ – the other contributions are $O((\lambda r)^{-\infty})$ as the phase is non-stationary.

2. Near the two poles we can use coordinates $t \in \mathbb{R}^{n-1}$, $\theta = (\pm\sqrt{1 - |t|^2}, t) \in \mathbb{S}^{n-1}$. Then, for φ supported near $\theta_1 = \pm 1$ ($t = 0$) we have

$$\int_{\mathbb{S}^{n-1}} e^{-i\lambda r \langle \omega, \theta \rangle} \varphi(\theta) d\theta = \int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\mp i\lambda r \sqrt{1-|t|^2}} \varphi(\pm\sqrt{1-|t|^2}, t) J(t) dt \,,$$

where $J(t) = 1 + \mathcal{O}(t^2)$.

3. The Hessian of the phase at $t = 0$ is given by $\pm I_{\mathbb{R}^{n-1}}$ and hence the method of stationary phase gives

$$\begin{aligned} &\int_{B_{\mathbb{R}^{n-1}}(0,1)} e^{\mp i\lambda r \sqrt{1-|t|^2}} \varphi(\pm\sqrt{1-|t|^2}, t) J(t) dt \\ &\sim \left(\frac{2\pi}{r\lambda} \right)^{\frac{n-1}{2}} e^{\pm i \frac{\pi}{4}(n-1)} \left(\varphi(\pm 1, 0) + \mathcal{O}\left(\frac{1}{r\lambda} \right) \right) \,, \end{aligned}$$

with a full assymptotic expansion in powers of $(r\lambda)^{-1}$.

4. A general φ can be written as a sum of functions which are supported near $\theta_1 = \pm 1$, and in the non-stationary region. That gives the result. □

REMARK. The proof gives a more precise result which we formulate as follows: for $\varphi \in C^\infty(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} e^{-i\lambda r \langle \omega, \theta \rangle} \varphi(\theta) d\theta = e^{-i\lambda r} a^+(\lambda r, \omega)(\varphi) + e^{i\lambda r} a^-(\lambda r, \omega)(\varphi), \tag{3.6.20}$$

where

$$a^\pm(\rho, \omega, \theta) \in S_{\mathrm{phg}}^{-\frac{n-1}{2}}((0,\infty)_\rho; C^\infty(\mathbb{S}^{n-1}_\omega, \mathscr{D}'(\mathbb{S}^{n-1}_\theta))$$

which means that for every $k \geq 1$,

$$
\begin{gathered}
|a^{\pm}(\rho, \bullet)(\varphi)| \leq C\rho^{-\frac{n-1}{2}} \|\varphi\|_{C^{n+1}}, \quad \rho > 0, \\
a^{\pm}(\rho, \omega)(\varphi) = \rho^{-\frac{n-1}{2}} \sum_{j=0}^{k-1} \rho^{-j} a_j^{\pm}(\omega)(\varphi) + \rho^{-\frac{n-1}{2}-k} A_k^{\pm}(\rho, \omega)(\varphi), \\
a_0^{\pm}(\omega) = c_n^{\pm} \delta_{\pm\omega}, \quad |a_j^{\pm}(\omega)(\varphi)| \leq C_j \|\varphi\|_{C^{2j}}, \\
|A_k(\rho, \omega)(\varphi)| \leq C_{pk\ell} \|\varphi\|_{C^{2k+n}}, \quad \text{uniformly in } \rho > 0.
\end{gathered} \tag{3.6.21}
$$

This means in particular that $a_j^{\pm}(\omega)$ is a family of distribution of order $2j$. The expansion (3.6.21) is a distributional formulation of the stationary phase estimate [**HöI**, (7.7.13)]. It will be useful in the proof of Theorem 3.51.

The next result shows that the incoming and outgoing scattering patterns are naturally paired. Later it will allow us to establish the unitarity of the scattering matrix.

THEOREM 3.39 (**Boundary pairing**). *Let P be a self-adjoint operator with domain $H^2(\mathbb{R}^n)$, and such that for $\chi \in C_c^{\infty}(B(0, 2R); \mathbb{R})$, $\chi = 1$ in $B(0, R)$ we have $P(1 - \chi) = -\Delta(1 - \chi)$.*

Suppose that $u_\ell \in H^2_{\text{loc}}(\mathbb{R}^n)$, $\ell = 1, 2$ satisfy

$$
\begin{gathered}
(P - \lambda^2) u_\ell = F_\ell \in \mathscr{S}(\mathbb{R}^n), \quad \lambda \in \mathbb{R} \setminus \{0\}, \\
u_\ell(r\theta) = r^{-\frac{n-1}{2}} \left(e^{i\lambda r} f_\ell(\theta) + e^{-i\lambda r} g_\ell(\theta) \right) + \mathcal{O}(r^{-\frac{n+1}{2}}), \quad \theta \in \mathbb{S}^{n-1},
\end{gathered}
$$

with $f_\ell, g_\ell \in C^{\infty}(\mathbb{S}^{n-1})$, and the expansion is also valid for derivatives with respect to ∂_r.

Then

$$
2i\lambda \int_{\mathbb{S}^{n-1}} \left(g_1 \bar{g}_2 - f_1 \bar{f}_2 \right) d\omega = \int_{\mathbb{R}^n} \left(F_1 \bar{u}_2 - u_1 \bar{F}_2 \right) dx. \tag{3.6.22}
$$

Proof. 1. We note that the integral on the right hand side is well defined as $F_\ell \in \mathscr{S}$ and $u_\ell \in L^\infty$ (in view of the expansions). If $\chi \in C_c^{\infty}(\mathbb{R}^n; \mathbb{R})$ is as in the statement of the theorem and $\tilde{\chi} \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ is equal to 1 on $\operatorname{supp}\chi$, then

$$\begin{aligned}\int_{\mathbb{R}^n} \big(F_1 \bar{u}_2 - u_1 \bar{F}_2\big)\, dx = \langle (P-\lambda^2)u_1, \chi u_2\rangle - \langle u_1, (P-\lambda^2)\chi u_2\rangle \\ + \lim_{r\to\infty} \int_{B(0,r)} \big((-\Delta-\lambda^2)u_1(1-\chi)\bar{u}_2 - u_1(-\Delta-\lambda^2)((1-\chi)\bar{u}_2)\big)\, dx \\ = \langle P\tilde{\chi}u_1, \chi u_2\rangle - \langle \tilde{\chi}u_1, P\chi u_2\rangle \\ + \lim_{r\to\infty} \int_{B(0,r)} (-\Delta u_1 (1-\chi)\bar{u}_2 + u_1\Delta((1-\chi)\bar{u}_2))\, dx \\ = \lim_{r\to\infty} \int_{B(0,r)} (-\Delta u_1 (1-\chi)\bar{u}_2 + u_1\Delta((1-\chi)\bar{u}_2))\, dx.\end{aligned}$$

Here we used the self-adjointness of P and the facts $\tilde{\chi}u_1, \chi u_2 \in H^2(\mathbb{R}^n)$, $[P, \tilde{\chi}]\chi = 0$.

2. Hence we need to show that

$$\begin{aligned}\lim_{r\to\infty} \int_{B(0,r)} (-\Delta u_1 (1-\chi)\bar{u}_2 + u_1\Delta((1-\chi)\bar{u}_2))\, dx \\ = 2i\lambda \int_{\mathbb{S}^{n-1}} \big(g_1 \bar{g}_2 - f_1 \bar{f}_2\big)\, d\omega.\end{aligned} \tag{3.6.23}$$

For that we apply Green's formula which shows that the integral on the left hand side is equal to

$$\begin{aligned}&\int_{\partial B(0,r)} u_1(r\theta)\partial_r \bar{u}_2(r\theta) - \partial_r u_1(r\theta)\bar{u}_2(r\theta) d\theta \\ &= i\lambda \int_{\mathbb{S}^{n-1}} (e^{i\lambda r} f_1(\theta) + e^{-i\lambda r} g_1(\theta))(-e^{-i\lambda r}\bar{f}_2(\theta) + e^{i\lambda r}\bar{g}_2(\theta)) d\theta \\ &\quad - i\lambda \int_{\mathbb{S}^{n-1}} (e^{i\lambda r} f_1(\theta) - e^{-i\lambda r} g_1(\theta))(e^{-i\lambda r}\bar{f}_2(\theta) + e^{i\lambda r}\bar{g}_2(\theta)) d\theta + \mathcal{O}(r^{-1}) \\ &= 2i\lambda \int_{\mathbb{S}^{n-1}} \big(g_1(\theta)\bar{g}_2(\theta) - f_1(\theta)\bar{f}_2(\theta)\big)\, d\theta + \mathcal{O}(r^{-1}).\end{aligned}$$

This proves (3.6.23) and hence (3.6.22). □

3.7. THE SCATTERING MATRIX

In this section we will define and describe the scattering matrix for $V \in L^{\infty}_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$, $n \geq 3$, odd. Except for the behaviour near $\lambda = 0$ and the fact that we use the properties of the resolvent, the parity of the dimension is not very important here.

To define the scattering matrix we go back to to (3.6.2)

$$e(\lambda,\omega,x) = e^{-i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)\,, \quad (P_V-\lambda^2)w = 0\,,$$
$$u(x,\lambda,\omega) := -R_V(\lambda)(Ve^{-i\lambda\langle\bullet,\omega\rangle})\,. \tag{3.7.1}$$

Theorem 3.33 shows that w is defined for $\lambda \in \mathbb{R}\setminus\{0\}$. We see from *ii)* in Theorem 3.37 that u is an outgoing spherical wave The scattering matrix will be defined as the operator relating the leading incoming and outgoing terms, normalized so that it is the identity when $V = 0$.

Using Theorems 3.37 and 3.38 we write the leading terms in w of (3.7.1) as follows:

(3.7.2)
$$e(\lambda,\omega,r\theta) \sim c_n^+(\lambda r)^{-\frac{n-1}{2}}\left(e^{-i\lambda r}\delta_\omega(\theta) + e^{i\lambda r}i^{1-n}\left(\delta_{-\omega}(\theta)+b(\lambda,\theta,\omega)\right)\right).$$

Here $b(\lambda,\theta,\omega)$ gives the leading part of the asymptotics of $u(r\theta,\lambda,\omega)$ as $r\to\infty$:

$$u(r\theta,\lambda,\omega) = c_n^-(\lambda r)^{-\frac{n-1}{2}}e^{i\lambda r}b(\lambda,\theta,\omega) + \mathcal{O}(r^{-\frac{n+1}{2}}), \tag{3.7.3}$$

and the constants are

$$c_n^\pm = e^{\pm\frac{\pi}{4}(n-1)i}(2\pi)^{\frac{n-1}{2}}. \tag{3.7.4}$$

DEFINITION 3.40. *The* absolute scattering matrix *maps the incoming terms to the outgoing terms in* (3.7.2)*:*

$$S_{\text{abs}}(\lambda) : \delta_\omega(\theta) \longmapsto i^{1-n}\left(\delta_{-\omega}(\theta)+b(\lambda,\theta,\omega)\right)\,. \tag{3.7.5}$$

The relative scattering matrix is defined as

$$S(\lambda) : \delta_\omega(\theta) \longmapsto \delta_\omega(\theta) + b(\lambda,\theta,-\omega), \tag{3.7.6}$$

where b is given in (3.7.2)*.*

The action on delta functions determines the integral kernel defining $S(\lambda)$ as an operator on $C^\infty(\mathbb{S}^{n-1})$:

$$f(\theta) = \int_{\mathbb{S}^{n-1}}\delta_\omega(\theta)f(\omega)d\omega \longmapsto \int_{\mathbb{S}^{n-1}}\left(\delta_{-\omega}(\theta)+b(\lambda,\theta,-\omega)\right)f(\omega)d\omega,$$

that

$$S(\lambda)f(\theta) = f(\theta) + \int_{\mathbb{S}^{n-1}} b(\lambda,\theta,-\omega)f(\omega)d\omega.$$

See Theorem 3.42 for an equivalent definition of scattering matrices.

We observe that for $V=0$ we have

$$S_{\text{abs},0}(\lambda)f(\theta) = i^{1-n}f(-\theta)\,.$$

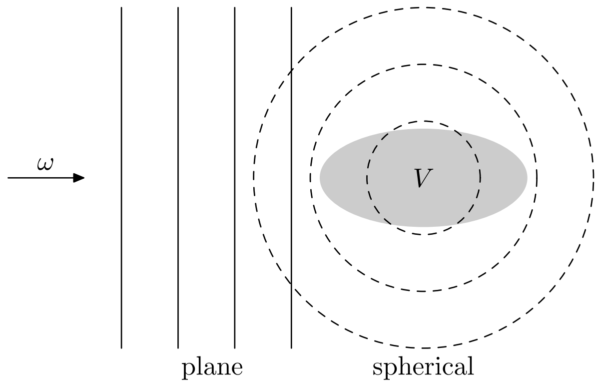


Figure 3.4. Schematic presentation of plane and spherical waves: a plane wave hits the perturbation and produces an additional spherical wave.

The scattering matrix was obtained by normalizing S_{abs} by this free absolute scattering matrix:

$$S(\lambda) = S_{\mathrm{abs}}(\lambda) S_{\mathrm{abs},0}(\lambda)^{-1} = i^{n-1} S_{\mathrm{abs}}(\lambda) J, \quad Jf(\theta) := f(-\theta).$$

INTERPRETATION. The function $b(\lambda, \theta, \omega)$ is called the *scattering amplitude* (up to a normalizing factor). It measures the intensity of the spherical scattered wave in the direction θ, following an interaction of the incident plane wave in the direction of $-\omega$ (coming from the point ω at infinity) – see Fig.3.4. We will see below that

$$b(\lambda, \theta, \omega) = b(\lambda, \omega, \theta) \text{ for } V \in L^\infty(\mathbb{R}^n; \mathbb{R}). \tag{3.7.7}$$

This is a higher dimensional version of the symmetry of the left and right transmission coefficients – see (2.4.12) and (2.4.13).

We have the following description of $S(\lambda)$:

THEOREM 3.41 (Description of the scattering matrix I). *The scattering matrix given by (3.7.6) defines an operator*

$$S(\lambda) = I + A(\lambda) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1}),$$

where $A(\lambda) : \mathcal{D}'(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})$, is given by

$$A(\lambda) = a_n \lambda^{n-2} E_\rho(\lambda)(I + V R_0(\lambda)\rho)^{-1} V E_\rho(\bar{\lambda})^*, \tag{3.7.8}$$

where $E_\rho(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$ are defined by Schwartz kernels:

$$E_\rho(\lambda)(\omega, x) := \rho(x) e^{-i\lambda\langle x, \omega\rangle},$$

with $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to one on $\operatorname{supp} V$ *and $a_n = (2\pi)^{-n+1}/2i$.*

In particular the Schwartz kernel $(\theta, \omega) \mapsto A(\lambda, \theta, \omega)$ is real analytic and can be written as

$$a_n \lambda^{n-2} \int_{\mathbb{R}^n} e^{i\lambda\langle \omega - \theta, x\rangle} V(x)(1 - e^{-i\lambda\langle \omega, x\rangle} R_V(\lambda)(e^{i\lambda\langle \omega, \bullet\rangle} V(\bullet))(x))dx. \tag{3.7.9}$$

Proof. 1. The definition of u in (3.7.1) gives

$$u = R_0(\lambda) f, \quad f(x) := -(I + V R_0(\lambda)\rho)^{-1}(V e^{-i\lambda\langle \bullet, \omega\rangle})(x).$$

From Theorem 3.5 and (3.7.3) we obtain

$$\begin{aligned} b(\lambda, \theta, \omega) &= -\frac{1}{4\pi} \frac{\lambda^{n-2}}{(2\pi i)^{\frac{n-3}{2}}} \frac{1}{c_n^-} \widehat{f}(\lambda\theta) \qquad \left(c_n^- = (2\pi)^{-n+1} i^{-\frac{n-1}{2}}\right) \\ &= \frac{1}{2i} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{R}^n} e^{-i\lambda\langle x, \theta\rangle} (I + V R_0(\lambda)\rho)^{-1}(V e^{-i\lambda\langle \bullet, \omega\rangle})(x) dx, \end{aligned}$$

which gives (3.7.8).

2. To see (3.7.9) we recall that

$$\begin{aligned} (I + V R_0(\lambda)\rho)^{-1} V &= (I - V R_0(\lambda)\rho(I + V R_0(\lambda)\rho)^{-1})V \\ &= (I - V R_V(\lambda))V, \end{aligned}$$

from which the formula follows immediately. $\square$

The next result gives an intrinsic definition of the scattering matrix as the operator mapping incoming to outgoing data at infinity:

THEOREM 3.42 (Solution with a prescribed incoming part). *Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, $\lambda \in \mathbb{R} \setminus \{0\}$. Then for any $g \in C^\infty(\mathbb{S}^{n-1})$ there exist* unique *$f \in C^\infty(\mathbb{S}^{n-1})$ and $v \in H^2_{\text{loc}}(\mathbb{R}^n)$ such that*

$$\begin{aligned} &(P_V - \lambda^2) v = 0, \\ &v(r\theta) = r^{-\frac{n-1}{2}} \left(e^{i\lambda r} f(\theta) + e^{-i\lambda r} g(\theta)\right) + \mathcal{O}(r^{-\frac{n+1}{2}}). \end{aligned} \tag{3.7.10}$$

Moreover, the scattering matrices defined by (3.7.5) and (3.7.6) relate g and f as follows:

$$\begin{aligned} S_{\text{abs}}(\lambda) &: g(\theta) \longmapsto f(\theta), \\ S(\lambda) J &: g(-\theta) \longmapsto i^{n-1} f(\theta). \end{aligned} \tag{3.7.11}$$

Proof. 1. For $g \in C^\infty(\mathbb{S}^{n-1})$ consider

$$\begin{aligned} u_0(x) &:= b_n \lambda^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} g(\omega) e^{-i\lambda\langle x, \omega\rangle} d\omega, \\ b_n &= 1/c_n^+ = (2\pi)^{-\frac{n-1}{2}} e^{-\frac{\pi}{4}(n-1)i}. \end{aligned}$$

Then $(-\Delta - \lambda^2) u_0 = 0$ and Theorem 3.38 gives asymptotics of u_0.

2. We then put

$$v(x) = u_0(x) - R_V(\lambda)(Vu_0)(x) = b_n \lambda^{\frac{n-1}{2}} \int_{\mathbb{S}^{n-1}} g(\omega) w(x, \lambda, \omega) d\omega,$$

where w is given by (3.7.1). This gives the desired v and (3.7.6) (that is, the fact that $S(\lambda) = i^{n-1} S_{\mathrm{abs}}(\lambda) J$, $Jg(-\theta) = g(\theta)$) shows that (3.7.11) holds.

3. The uniqueness of v follows from Theorem 3.35. $\square$

REMARK. For any given $g \in C^\infty(\mathbb{S}^{n-1})$ we can find $v_0 \in C^\infty(\mathbb{R}^n)$ such that, for $\lambda \in \mathbb{R} \setminus \{0\}$,

$$(-\Delta - \lambda^2) v_0 \in \mathscr{S}(\mathbb{R}^n),$$

$$v_0(r\theta) = r^{-\frac{n-1}{2}} e^{-i\lambda r} F(r, \theta), \quad F(r, \theta) \sim \sum_{j=0}^{\infty} F_j(\theta) r^{-j}, \quad F_0 = g, \tag{3.7.12}$$

$$F_{j+1} = \frac{1}{2i(j+1)\lambda} \left(-\Delta_{\mathbb{S}^{n-1}} + \tfrac{(n-1)(n-3)}{4} - j(j+1) \right) F_j.$$

In particular, the terms in the asymptotic expansions of F are determined by the leading term, g.

To obtain (3.7.12) we write

$$r^{\frac{n-1}{2}} \partial_r \, r^{-\frac{n-1}{2}} = \partial_r - \frac{n-1}{2r},$$

so that, in polar coordinates,

$$-r^{\frac{n-1}{2}} \Delta r^{-\frac{n-1}{2}} = -\partial_r^2 + \tfrac{(n-1)(n-3)}{4r^2} - \tfrac{1}{r^2} \Delta_{\mathbb{S}^{n-1}},$$

and hence

$$\begin{aligned} & - r^{-\frac{n-1}{2}} e^{i\lambda r} \Delta (r^{-\frac{n-1}{2}} e^{-i\lambda r} F_j(\theta) r^{-j}) = \\ & \qquad - r^{-j-1} 2ij\lambda F_j(\theta) + r^{-j-2} \left(-\Delta_\theta + \tfrac{(n-1)(n-3)}{4} - j(j+1) \right) F_j(\theta). \end{aligned}$$

Hence, we see that $v_0^J(r, \theta) := r^{-\frac{n-1}{2}} e^{i\lambda r} \sum_{j=0}^J F_j(\theta) r^{-j}$, satisfies $(-\Delta - \lambda^2) v_0^J = \mathcal{O}(r^{-J-1})$, $r \to \infty$. Constructing v_0 using Borel's argument (see for instance [**Zw12**, Theorem 4.15]) gives

$$(-\Delta - \lambda^2) v_0 \in \mathscr{S}(\mathbb{R}^n). \tag{3.7.13}$$

Conversely, if v_0 satisfies (3.7.13), and has an expansion in (3.7.12), then the terms are determined by the leading term, F_0, as in (3.7.12). The solution v in (3.7.10) is obtained by putting

$$v = v_0 - R_V(\lambda)(P_V - \lambda^2) v_0.$$

We can now state basic properties of the scattering matrix:

THEOREM 3.43 **(Properties of the scattering matrix).** *For $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})$, $n \geq 3$, odd, the scattering matrix, $S(\lambda)$, is meromorphic in $\mathbb{C}$ with poles of finite rank, and it satisfies*

$$S(\lambda)^{-1} = JS(-\lambda)J\,, \quad Jf(\theta) := f(-\theta), \quad \lambda \in \mathbb{C}\,. \tag{3.7.14}$$

There are only finitely many poles in the closed upper half plane and for $\operatorname{Im}\lambda > 0$, $\lambda^2 \in \operatorname{Spec}(P_V)$.

When $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$ then

$$S(\lambda)^{-1} = S(\bar\lambda)^*\,, \quad \lambda \in \mathbb{C}\,. \tag{3.7.15}$$

In particular, $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ and holomorphic on $\mathbb{R}$.

REMARK. Since $S(\lambda) = I + A(\lambda)$ where $A(\lambda)$ is of trace class (see Theorem 3.41) the determinant, $\det S(\lambda)$, is well defined – see §B.5. From (3.7.15), (B.5.17) and Proposition B.30 we see that

$$(\det S(\lambda))^{-1} = \overline{\det S(\bar\lambda)} \ \text{ for } V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R}). \tag{3.7.16}$$

We also have,

$$(\det S(\lambda))^{-1} = \det S(-\lambda), \ \text{ for } V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{C})\ . \tag{3.7.17}$$

In fact, from (3.7.14), (B.5.17) and Proposition B.30 (or the fact that $\operatorname{tr}(JA(-\lambda)J)^k = \operatorname{tr} A(-\lambda)^k$)

$$\det S(\lambda)^{-1} = \det(I + JA(-\lambda)J) = \det(I + A(-\lambda)) = \det S(-\lambda).$$

Proof of Theorem 3.43. 1. The meromorphy of $\lambda \mapsto S(\lambda)$ follows from the meromorphy of $\lambda \mapsto R_V(\lambda)$ and the representation (3.7.8).

2. Theorem 3.42, and (3.7.11) in particular, show that $S_{\rm abs}(\lambda)^{-1} = S_{\rm abs}(-\lambda)$ for $\lambda \in \mathbb{R}\setminus\{0\}$ and hence for all λ, in the sense of meromorphic families of operators. Since $S(\lambda) = i^{n-1}S_{\rm abs}(\lambda)J$, we have

$$\begin{aligned} S(-\lambda) &= i^{n-1}S_{\rm abs}(-\lambda)J = i^{n-1}S_{\rm abs}(\lambda)^{-1}J \\ &= i^{n-1}(i^{1-n}S(\lambda)J)^{-1}J = i^{2(n-1)}JS(\lambda)^{-1}J \\ &= JS(\lambda)^{-1}J, \end{aligned}$$

since n is odd.

3. The unitarity of $S_{\rm abs}(\lambda)$ for real valued V follows from Theorems 3.39 and 3.42. In the notation of (3.7.10), we apply (3.6.22) with $u_1 = u_2 = u$, $g_1 = g_2 = g$ and $f_1 = f_2 = S_{\rm abs}(\lambda)g$ to obtain

$$\|S(\lambda)g\|^2_{L^2(\mathbb{S}^{n-1})} = \|g\|^2_{L^2(\mathbb{S}^{n-1})}.$$

It follows that $S(\lambda)$ is unitary for $\lambda \in \mathbb{R} \setminus \{0\}$. Since Theorem 3.41 shows that $\lambda \mapsto S(\lambda)$ is a meromorphic family of operators, 0 must be a removable singularity and $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$.

4. The equation (3.7.15) holds for $\lambda \in \mathbb{R}$ and, as both sides are meromorphic it extends to all of $\mathbb{C}$. $\square$

Proof of (3.7.7). We can now prove the symmetry of the scattering amplitudes. For $\lambda = ik$, $k \gg 1$, $R_V(ik) = (P_V + k^2)^{-1}$ is self-adjoint (we assume here V is real valued). This and (3.2.4) show that $R_V(ik, x, y)$ is real valued. The expression for the scattering matrix kernel(3.7.8) shows that $i^{1-n} b(ik, \theta, \omega)$ is real – the i factors cancel.

On the other hand,

$$S_{\text{abs}}(ik)^* = S_{\text{abs}}(-ik)^{-1} = S_{\text{abs}}(ik), \quad k \in \mathbb{R},$$

and hence $S_{\text{abs}}(ik)$ is self-adjont. From (3.7.5) we see that the Schwartz kernel is real and hence b is symmetric, $b(ik, \theta, \omega) = b(ik, \omega, \theta)$. By analytic continuation this remains valid for all λ. $\square$

We now present a different formula for the scattering matrix. The potential V does not appear explicitely and hence it can be used for more general (compactly supported) perturbations. In fact, it holds for any *black box* perturbation which we will consider later in this book.

THEOREM 3.44 **(Description of the scattering matrix).** *Let P_V, ρ, and $E_\rho(\lambda)$ be as in Theorem 3.41. Choose $\chi_i \in C_c^\infty(\mathbb{R}^n)$, $i = 1, 2, 3$,*

$$\chi_i|_{\text{supp } V} = 1, \quad \chi_{i+1}|_{\text{supp } \chi_i} = 1, \quad \chi_3 = \rho.$$

Then the scattering matrix is given by

$$S(\lambda) = I + a_n \lambda^{n-2} E_\rho(\lambda)[\Delta, \chi_1] R_V(\lambda)[\Delta, \chi_2] E_\rho(\bar{\lambda})^* , \tag{3.7.18}$$

where $a_n := (2\pi)^{-n+1}/2i$.

Proof. 1. For $h_1, h_2 \in C^\infty(\mathbb{S}^{n-1})$ let us put

$$\begin{aligned} u_1 &= \left((1 - \chi_2)E(\bar{\lambda})^* - R_V(\lambda)[\Delta, \chi_2]E_\rho(\bar{\lambda})^*\right) h_1 , \\ u_2 &= (1 - \chi_1)E(\bar{\lambda})^* h_2 , \end{aligned}$$

where

$$E(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1}), \quad E(\lambda)u(\omega) := \int_{\mathbb{R}^n} u(x)e^{-i\lambda\langle x,\omega\rangle}dx,$$

and $E(\bar{\lambda})^* : L^2(\mathbb{S}^{n-1}) \to L^2_{\text{loc}}(\mathbb{R}^n)$.

We check that, thanks to the support properties of χ_j's and ρ,

$$F_1 := (P_V - \lambda^2)u_1 = 0$$

and

$$F_2 := (P_V - \lambda^2)u_2 = [\Delta, \chi_1]E_\rho(\bar\lambda)^* h_2 \,.$$

2. We first assume that $\lambda \in \mathbb{R} \setminus \{0\}$. We see that u_1 satisfies the assumptions of Theorem 3.39 with

$$\begin{aligned} g_1(\theta) &= c_n^- \lambda^{-\frac{n-1}{2}} h_1(-\theta)\,, \quad c_n^- = e^{-\frac{i}{4}\pi(n-1)}(2\pi)^{\frac12(n-1)}, \\ f_1(\theta) &= S_{\rm abs}(\lambda) g_1(\theta) = c_n^- i^{1-n} \lambda^{-\frac{n-1}{2}} S(\lambda) h_1(\theta)\,. \end{aligned}$$

In fact, since $R_V(\lambda)$ is the outgoing resolvent, the only incoming contribution comes from the free term $(1-\chi_1)E_\rho(\bar\lambda)^* h_1$ and this follows from Theorems 3.38 (formula for g_1) and 3.42 (formula for f_1). Note the change of sign in the exponent in $E(\bar\lambda)^*$.

Theorem 3.38 then shows that for u_2 we have asymptotic expansion with

$$g_2(\theta) = c_n^- \lambda^{-\frac{n-1}{2}} h_2(-\theta), \quad f_2(\theta) = c_n^- i^{1-n} \lambda^{-\frac{n-1}{2}} h_2(\theta).$$

3. We define

$$G(\lambda) := E_\rho(\lambda)[\Delta, \chi_1] R_V(\lambda) [\Delta, \chi_2] E_\rho(\bar\lambda)^*, \tag{3.7.19}$$

which is an operator $L^2(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})$.

Using the fact that $(1-\chi_2)[\Delta, \chi_1] = 0$, we have

$$u_1 \bar F_2 = -R_V(\lambda)[\Delta, \chi_2] E_\rho(\bar\lambda)^* h_1 \times [\Delta, \chi_1] E_\rho(\lambda)^* \bar h_2,$$

and since $F_1 = 0$ and $[\Delta, \chi_1]^* = -[\Delta, \chi_1]$,

(3.7.20)
$$\begin{aligned} \int_{\mathbb{R}^n} (F_1 \bar u_2 - u_1 \bar F_2) dx &= \langle R_V(\lambda)[\Delta, \chi_2] E_\rho(\bar\lambda)^* h_1, [\Delta, \chi_1] E_\rho(\lambda)^* h_2 \rangle_{L^2(\mathbb{R}^3)} \\ &= -\langle E_\rho(\lambda)[\Delta, \chi_1] R_V(\lambda)[\Delta, \chi_2] E_\rho(\bar\lambda)^* h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})} \\ &= -\langle G(\lambda) h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

4. On the other hand the pairing formula (3.6.22) and the expressions for f_ℓ and g_ℓ from Step 2, give

$$\begin{aligned} \int_{\mathbb{R}^n} (F_1 \bar u_2 - u_1 \bar F_2) dx &= 2i\lambda \left(\langle g_1, g_2 \rangle_{L^2(\mathbb{S}^{n-1})} - \langle f_1, f_2 \rangle_{L^2(\mathbb{S}^{n-1})} \right) \\ &= 2i \lambda^{-n+2} (2\pi)^{n-1} \langle (I - S(\lambda)) h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})}. \end{aligned} \tag{3.7.21}$$

Comparing (3.7.21) with (3.7.20) we see that

$$A(\lambda) h_2 = (S(\lambda) - I) h_2 = a_n \lambda^{n-2} G(\lambda),$$

which, recalling the definition of $G(\lambda)$ (3.7.19), proves (3.7.18) for $\lambda \in \mathbb{R} \setminus \{0\}$. Analytic continuation shows that the formula is valid, as an equality of two meromorphic families of operators, for $\lambda \in \mathbb{C}$. $\square$

REMARK. It is interesting to note that the representation (3.7.18) does not depend on the cut-off functions, and that we can reverse the condition $\chi_2 \equiv 1$ on the support of χ_1 to $\chi_1 \equiv 1$ on the support of χ_2. Both facts follow directly from the properties of the scattering matrix but here we propose a direct argument based on considering *quantum flux*.

Suppose that χ_2 is equal to one on the supports of functions χ_1, $\tilde{\chi}_1$, which are equal to 1 near $\operatorname{supp} V$. We claim that

$$E_\rho(\lambda)[\Delta, \chi_2] R_V(\lambda)[\Delta, \chi_1 - \tilde{\chi}_1] E_\rho(\bar{\lambda})^* \equiv 0 \,.$$

This will follow from showing that

$$\begin{gathered} (\Delta - \lambda^2) v_j = 0 \,, \ j = 1, 2 \implies \\ \langle R_V(\lambda)[\Delta, \chi_1 - \tilde{\chi}_1] v_1, [\Delta, \chi_2] v_2 \rangle_{L^2(\mathbb{R}^n)} = 0 \,. \end{gathered}$$

This however is clear as the left hand side is equal to

$$\begin{aligned} &\langle R_V(\lambda)(-P_V(\chi_1 - \tilde{\chi}_1) - (\chi_1 - \tilde{\chi}_1)\Delta) v_1, [\Delta, \chi_2] v_2 \rangle_{L^2(\mathbb{R}^n)} \\ &\qquad = -\langle (\chi_1 - \tilde{\chi}_1) v_1, [\Delta, \chi_2] v_2 \rangle_{L^2(\mathbb{R}^n)} = 0 \,, \end{aligned}$$

since $(\chi_1 - \tilde{\chi}_1)[\Delta, \chi_2] = 0$. Similarly, if $\chi_1 \equiv 1$ on the support of $\tilde{\chi}_1$, and $\tilde{\chi}_1 \equiv 1$ near $\operatorname{supp} V$, then

$$E_\rho(\lambda)[\Delta, \chi_2 - \tilde{\chi}_1] R_V(\lambda)[\Delta, \chi_1] E_\rho(\bar{\lambda})^* \equiv 0 \,,$$

which shows that we can switch the conditions on χ_1 and χ_2. Yet another argument of the same type shows that for the free resolvent we have

$$E_\rho(\lambda)[\Delta, \chi_2] R_0(\lambda)[\Delta, \chi_1] E_\rho(\bar{\lambda})^* \equiv 0 \,.$$

In the study of resonances the following theorem provides a crucial connection between singularities of the scattering matrix and the resolvent.

THEOREM 3.45 (Multiplicities of scattering poles). *Suppose that $S(\lambda)$ is the scattering matrix for $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$, odd.*

If we define

$$m_S(\lambda) = -\frac{1}{2\pi i} \operatorname{tr} \oint S(\zeta)^{-1} \partial_\zeta S(\zeta) d\zeta \,, \tag{3.7.22}$$

where the integral is over a positively oriented circle which includes λ and no other pole or zero of $\det S(\lambda)$, *then*

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda) \,. \tag{3.7.23}$$

Proof. 1. The results in this section apply equally well to V replaced by $V = V_0 + V_1$ where

$$V_0 \in L^\infty, \quad V_1 = \sum_{j=1}^{m} f_j \otimes g_j, \tag{3.7.24}$$
$$f_j, g_j \in L^\infty, \quad \operatorname{supp} V_0, \operatorname{supp} f_j, \operatorname{supp} g_j \subset B(0,R),$$

for some fixed R (we then choose $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 on $B(0,R)$.

2. We first see that if $(I + VR_0(\zeta)\rho)^{-1}$ is holomorphic on $|\zeta \pm \lambda| = r$ then for V' close to V, in operator norm (and with a fixed bound on the support as in (3.7.24))

$$\sum_{|\lambda-\zeta|<r} m_{S_V}(\zeta) = \sum_{|\zeta-\lambda|<r} m_{S_{V'}}(\zeta). \tag{3.7.25}$$

In fact, we can use (3.7.8) and (3.7.14) to see that $S_V(\zeta)^{-1} = JS_V(-\zeta)J$ exists and is bounded on $|\zeta - \lambda| = r$. Since

$$(I + VR_0(\lambda)\rho)^{-1} - (I + V'R_0(\lambda)\rho)^{-1} =$$
$$(I + VR_0(\lambda)\rho)^{-1}(V' - V)\rho R_0(\lambda)\rho(I + V'R_0(\lambda)\rho)^{-1}$$

we see that if $\|V' - V\|_{L^2\to L^2}$ is sufficiently small invertibility of $I + VR_0(\lambda)\rho$ implies invertibility of $I + V'R_0(\lambda)\rho$ and we can estimate the norm of the difference of the inverses. The the formula for $A(\zeta)$ in (3.7.8) then shows that

$$\|S_V(\zeta)^{-1}(S_V(\zeta) - S_{V'}(\zeta))\| < 1, \quad |\zeta - \lambda| = r.$$

This and Theorem C.12 give (3.7.25).

We can always find arbitrary small r's for which $(I + VR_0(\zeta)\rho)^{-1}$ is holomorphic on $|\zeta \pm \lambda| = r$. The formula (3.7.25) implies that the poles and zeros of $\det S_V(\lambda)$ depend continuously on V in compact sets.

3. The continuity statement above and Theorem 3.14 show that we only need to prove (3.7.23) if $m_R(\pm\lambda) \le 1$ (the proof of that theorem applies to splitting multiplicities of any finite number of resonances). Hence suppose that $R_V(\zeta)$ has a pole of multiplicity 1 at λ.

Let us consider the case of $\lambda \neq 0$ first. Then by (3.2.9),

$$R_V(\zeta) = \frac{u \otimes u}{\lambda - \zeta} + H(\zeta, \lambda), \tag{3.7.26}$$

where $\zeta \mapsto H(\zeta,\lambda)$ is holomorphic near λ, $(P_V - \lambda^2)u = 0$, $u = R_0(\lambda)f$, $f \in L^\infty_{\mathrm{comp}}$, $f \neq 0$. Theorem 3.44 shows that

$$S(\zeta) = S_0(\zeta,\lambda) - a_n\lambda^{n-2}\frac{U_1 \otimes U_2}{\zeta - \lambda}, \tag{3.7.27}$$

where $\zeta \mapsto S_0(\zeta, \lambda)$ is holomorphic near λ and

$$U_j := E_\rho(\lambda)[\Delta, \chi_j]u.$$

(Note that the change of sign between (3.7.26) and (3.7.27) comes from $[\Delta, \chi_2]^* = -[\Delta, \chi_2]$.)

We now claim that

$$U_1(\theta) = U_2(\theta) = U(\theta) := \widehat{Vu}(\lambda\theta).$$

In fact, using the equation $P_V u = \lambda^2 u$,

$$[\Delta, \chi_j]u = \Delta\chi_j u - \chi_j \Delta u = (\Delta + \lambda^2)\chi_j u + Vu.$$

The contribution of the first term vanishes since

$$\mathcal{F}\left((\Delta + \lambda^2)\chi_j u\right)(\lambda\theta) = (-|\xi|^2 + \lambda^2)\widehat{\chi_j u}(\lambda\theta) = 0.$$

Step 4 of the proof of Theorem 3.33 shows that $U \equiv 0$ would imply that u is compactly supported which is impossible, as shown in Step 5 of that proof.

4. This means that a simple pole of $R_V(\zeta)$ at λ, implies (3.7.27) with $U_1 = U_2 \neq 0$. Theorem C.10 shows that near λ

$$S(\zeta) = G(\zeta)(Q_{-1}(\zeta - \lambda)^{-1} + Q_k(\zeta - \lambda)^k + \cdots + Q_0)F(\zeta),$$

where Q_{-1} has rank 1 and G and F are holomorphic and invertible near λ. Since we assumed that $R_V(\zeta)$ has at most a simple pole at $-\lambda$ we also have, for ζ near λ

$$S(-\zeta) = \widetilde{E}(\zeta)(\widetilde{Q}_{-1}(\zeta - \lambda)^{-1} + \widetilde{Q}_k(\zeta - \lambda)^k + \cdots + \widetilde{Q}_0)\widetilde{F}(\zeta),$$

where $\operatorname{rank} \widetilde{Q}_{-1} = m_R(-\lambda)$.

By (3.7.14)

$$\begin{aligned} S(-\zeta) &= JS(\zeta)^{-1}J \\ &= JF(\zeta)^{-1}(Q_{-1}(\zeta - \lambda) + Q_k(\zeta - \lambda)^{-k} + \cdots + Q_0)G(\zeta)^{-1}J \end{aligned}$$

which means that $k \leq 1$ and $\widetilde{Q}_1 = Q_{-1}$, $\widetilde{Q}_{-1} = Q_1$. We conclude that

$$m_S(\lambda) = \operatorname{rank} Q_{-1} - \operatorname{rank} Q_1 = m_R(\lambda) - m_R(-\lambda),$$

which, by our reduction to the case of simple poles, proves (3.7.23) for all $\lambda \neq 0$.

5. It remains to discuss the case of $\lambda = 0$. We can again assume that $m_R(\lambda) \leq 1$. Then (3.7.18) shows that $S(\lambda)$ is holomorphic near 0 and using $S(\lambda)^{-1} = JS(\lambda)J$ so is its inverse. (When V is real valued the scattering matrix is unitary on the real axis and hence holomorphic and invertible near 0.) That means that $m_S(0) = 0$. $\square$

REMARK. When V is real valued we can use the results of §3.3 to describe the scattering matrix at $\lambda = 0$. When $n \geq 5$, Theorem 3.17 and (3.7.18) show that $S(0) = I$. When $n = 3$

$$S(0) = I - \frac{\widetilde{m}_R(0)}{2\pi} 1 \otimes 1, \tag{3.7.28}$$

where $\widetilde{m}_R(0)$ is the multiplicity of the zero resonance – see (3.3.29).

Proof of (3.7.28). From Theorem 3.23 and (3.7.18) we see that near $\lambda = 0$,

$$\begin{aligned} S(\lambda) = I &- a_3 \lambda^{-1} E_\rho(0)[\Delta, \chi_1]\Pi_0[\Delta, \chi_2]E_\rho(0)^* \\ &- a_3 \partial_\lambda E_\rho(\lambda)|_{\lambda=0}[\Delta, \chi_1]\Pi_0[\Delta, \chi_2]E_\rho(0) \\ &- a_3 E_\rho(0)[\Delta, \chi_1]\Pi_0[\Delta, \chi_2]\partial_\lambda E_\rho(\bar{\lambda})^*|_{\lambda=0} \\ &+ i a_3 E_\rho(0)[\Delta, \chi_1]A_1[\Delta, \chi_2]E_\rho(0)^* + \mathcal{O}(\lambda). \end{aligned} \tag{3.7.29}$$

We have $\Pi_0 = \sum_{j=1}^J u_j \otimes \bar{u}_j$ where $u_j \in H^2(\mathbb{R}^3)$ and $P_V u_j = 0$. Then, using $\chi_\ell \Delta u_j = \chi_\ell V u_j = V u_j = \Delta u_j$ and part 1 of Lemma 3.18 we have

$$E_\rho(0)[\Delta, \chi_\ell]u_j = \int_{\mathbb{R}^3} (\Delta(\chi_\ell u_j)(x) - \Delta u_j(x))\, dx = 0, \quad \ell = 1, 2.$$

Hence $E_\rho(0)[\Delta, \chi_\ell]\Pi_0 = 0$, and most terms in (3.7.29) disappear. In view of Lemma 3.22 and (3.3.27),

$$S(0) = I + (2\pi)^{-1} E_\rho(0)[\Delta, \chi_1](u_0 \otimes \bar{u}_0)[\Delta, \chi_2]E_\rho(0)^*,$$
$$P_V u_0 = 0, \quad -\int_{\mathbb{R}^3} V(x) u_0(x) dx = 1.$$

Now,

$$E_\rho(0)[\Delta, \chi_\ell]u_0 = \int_{\mathbb{R}^3} (\Delta(\chi_\ell u_0)(x) - V(x)u_0(x))\, dx = 1,$$

and this gives (3.7.28). The change of sign comes for $[\Delta, \chi_2]^* = -[\Delta, \chi_2]$. (We also note that once we see that $S(0) - I$ is of the form $c 1 \otimes 1$ then c is determined by $S(0) = S(0)^{-1}$.) $\square$

We conclude this section with a useful results relating the determinant of the scattering matrix to the determinant of an operator acting on $L^2(\mathbb{R}^n)$.

THEOREM 3.46 (Trace identities). *Suppose that*

$$V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{C}), \quad n \geq 3, \text{ odd},$$

and that $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is equal to 1 on $\operatorname{supp} V$. *Let*

$$T(\lambda) := (I + V R_0(\lambda)\rho)^{-1}(V(R_0(\lambda) - R_0(-\lambda))\rho) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Then $T(\lambda)$ is of trace class and

$$\det S(\lambda) = \det(I - T(\lambda)). \tag{3.7.30}$$

Proof. 1. The operator $T(\lambda)$ is of trace class since

$$\rho(R_0(\lambda) - R_0(-\lambda))\rho : L^2(\mathbb{R}^n) \longrightarrow H^k(B(0,R))\,,$$

for any k, provided that $\rho \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\rho \subset B(0,R)$.

2. We will prove the formula for $\lambda \in \mathbb{R}$. For that we first write $S(\lambda) = I - Z(\lambda)$ where, using (3.7.9),

$$\begin{aligned} Z(\lambda) &= b_n \lambda^{n-2} E_\rho(\lambda)(I - VR_V(\lambda)\rho)VE_\rho(\lambda)^* \\ &= b_n \lambda^{n-2} E_\rho(\lambda)(I + VR_0(\lambda)\rho)^{-1} VE_\rho(\lambda)^*\,, \end{aligned} \tag{3.7.31}$$

where $b_n = -a_n = i(2\pi)^{1-n}/2$.

3. To prove (3.7.30) all we need to show is that for all $k \in \mathbb{N}$

$$\operatorname{tr} T(\lambda)^k = \operatorname{tr} Z(\lambda)^k\,. \tag{3.7.32}$$

In fact, (3.7.32) shows for $t \in \mathbb{C}$, $|t| \ll 1$ (so that the log can be defined),

$$\begin{aligned} \log\det(I - tT(\lambda)) &= \operatorname{tr}\log(I - tT(\lambda)) \\ &= \operatorname{tr}\log(I - tZ(\lambda)) \\ &= \log\det(I - tZ(\lambda))\,. \end{aligned}$$

It follows that $\det(I - tZ(\lambda)) = \det(I - tT(\lambda))$ for $|t|$ small enough, and by analytic continuation in t, for $t = 1$.

4. To establish (3.7.32) we use (3.1.19) for $\lambda \in \mathbb{R}$:

$$\rho(R_0(\lambda) - R_0(-\lambda))\rho = b_n \lambda^{n-2} E_\rho(\lambda)^* E_\rho(\lambda)$$

in the definition of $T(\lambda)$:

$$T(\lambda) = b_n \lambda^{n-2}(I + VR_0(\lambda)\rho)^{-1} VE_\rho(\lambda)^* E_\rho(\lambda)\,.$$

Let $A = b_n \lambda^{n-2} E_\rho(\lambda)$, $B = (I + VR_0(\lambda)\rho)^{-1}V$, $C = E_\rho(\lambda)^*$ so that

$$T = BCA \quad \text{and } Z = ABC,$$

see (3.7.31). The operators A and C are of trace class as operators between different spaces:

$$A : H_1 \to H_2, \quad B : H_1 \to H_1\,, \quad C : H_2 \to H_1\,,$$

and B is a bounded operator. Cyclicity of trace shows that

$$\begin{aligned} \operatorname{tr}_{H_1}(ABC)^k &= \operatorname{tr}_{H_1} A(BCA)^{k-1}BC \\ &= \operatorname{tr}_{H_2} BCA(BCA)^{k-1} \\ &= \operatorname{tr}_{H_2}(BCA)^k\,. \end{aligned}$$

This gives (3.7.32). □

3.8. MORE ON DISTORTED PLANE WAVES

Distorted plane waves were already defined in (3.7.1) were used to define the scattering matrix. Here we will study them further.

We start with an explicit form of Stone's formula. Its abstract form for P_V given in (B.1.12) follows from general spectral theory but the results of §3.6 give us an analogue of Theorem 3.4 which was stated for $P_0 = -\Delta$.

THEOREM 3.47 (Stone's formula for P_V). *Let $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$. For $\lambda \in \mathbb{R}\setminus\{0\}$ and $\omega \in \mathbb{S}^{n-1}$ define $e(\lambda,\omega,x)$ by (3.6.2):*

$$e(\lambda,\omega,x) := e^{-i\lambda\langle x,\omega\rangle} - R_V(\lambda)(Ve^{-i\lambda\langle \bullet,\omega\rangle})(x). \tag{3.8.1}$$

Then

$$\overline{e(\lambda,x,\omega)} = e(-\lambda,x,\omega), \tag{3.8.2}$$

and

$$R_V(\lambda,x,y) - R_V(-\lambda,x,y) = \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}\int_{\mathbb{S}^{n-1}} e(\lambda,\omega,x)\overline{e(\lambda,\omega,y)}d\omega. \tag{3.8.3}$$

The Schwartz kernel of spectral measure of P_V corresponding to the continuous spectrum is given by

$$\begin{gathered} dE_\lambda(\lambda,x,y) = \int_{\mathbb{S}^{n-1}} e(\lambda,\omega,x)\overline{e(\lambda,\omega,y)}d\omega\,\frac{\lambda^{n-1}d\lambda}{(2\pi)^n}, \\ P_V = \sum_{k=1}^K E_k u_j\otimes \bar u_j + \int_0^\infty \lambda^2 dE_\lambda, \quad I = \sum_{k=1}^K u_k\otimes\bar u_k + \int_0^\infty dE_\lambda, \end{gathered} \tag{3.8.4}$$

where u_k's are normalized eigenfunctions of P_V corresponding to eigenvalues E_k, $E_K < E_{k-1} \le \cdots \le E_1 \le 0$.

INTERPRETATION. The functions defined in (3.8.1) (and earlier in (3.6.2)) are called *distorted plane waves*. That is because $\exp(-i\lambda\langle\omega,x\rangle)$ is a *free plane wave* in the sense that its inverse Fourier transform in λ is equal to $2\pi\delta(t-\langle\omega,x\rangle)$ which is a plane wave in the direction of ω: a wave in the sense of solving the wave equation, $(\partial_t^2-\Delta)\widehat{e}=0$.

Formula (3.8.3) combined with (B.1.12) provides a description of the continuous spectral measure of P_V in terms of distorted plane waves. As we will see in Theorem 3.49 the scattering matrix intertwines $e(-\lambda,-\bullet,x)$ and $e(\lambda,\bullet,x)$.

REMARK. The definition (3.8.1) shows that $\lambda\mapsto\lambda^{\frac{n-1}{2}}e(\lambda,\omega,x)$ extends to a meromorphic family for $\lambda\in\mathbb{C}$:

$$\lambda^{\frac{n-1}{2}}e(\lambda,\omega,x) \in C^\infty(\mathbb{R}_\lambda\times\mathbb{S}^{n-1}_\omega; H^2_{\rm loc}(\mathbb{R}^n_x)), \tag{3.8.5}$$

This follows from Theorem 3.23.

When $n = 3$, then the term Π_0 does does not contribute a pole of order two, λ^{-2}, in (3.8.1) as $\Pi_0(V) = 0$ – see Step 1 of the proof of Lemma 3.18. But, due to a possible resonance at 0, we may have a singularity λ^{-1} canceled by $\lambda^{\frac{n-1}{3}} = \lambda$.

For $n \geq 5$ there is $\Pi_0(V)$ may not be zero and hence we may have a pole λ^{-2}. Since $\frac{n-1}{2} \geq 2$ it is cancelled by $\lambda^{\frac{n-1}{2}}$ and $\lambda \mapsto \lambda^{\frac{n-1}{2}} e(\lambda, \omega, x)$ is smooth.

Proof of Theorem 3.47. 1. To see (3.8.2) we note that self-adjointness of P_V shows that for $\operatorname{Im} \lambda > 0$, $R_V(\lambda)^* = R_V(-\bar{\lambda})$. Combining this with (3.2.4) we obtain

$$\overline{R_V(\lambda)\bar{u}} = R_V(\lambda)^* u = R_V(-\bar{\lambda})u,$$

and this remains valid for $\lambda \in \mathbb{R}$ and $u \in L^2_{\mathrm{comp}}$. Hence, for $\lambda \in \mathbb{R}$,

$$\begin{aligned}\overline{e(\lambda, \omega, x)} &= e^{i\lambda\langle x, \omega\rangle} - \overline{R_V(\lambda)\left(\overline{V e^{i\lambda\langle x,\omega\rangle}}\right)} \\ &= e^{i\lambda\langle x, \omega\rangle} - R_V(-\lambda)\left(V e^{i\lambda\langle x,\omega\rangle}\right) \\ &= e(-\lambda, \omega, x).\end{aligned}$$

2. We now note that (3.8.3) is equivalent to

(3.8.6)
$$\langle (R_V(\lambda) - R_V(-\lambda))\varphi, \varphi\rangle = \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} e(\lambda, \omega, x)\varphi(x)dx\right|^2 d\omega,$$

for any $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

For $\operatorname{Im} \lambda < 0$, $R_V(-\lambda)^* = R_V(\bar{\lambda})$ and hence by analytic continuation,

$$\langle R_V(-\lambda)\varphi, \varphi\rangle = \langle \varphi, R_V(\bar{\lambda})\varphi\rangle, \quad \lambda \in \mathbb{C}. \tag{3.8.7}$$

If $\operatorname{supp}\varphi \subset B_R := B(0, R)$ then, using (3.8.7) and $(P_V - \lambda^2)R_V(\pm\lambda)\varphi = \varphi$, we obtain

$$\begin{aligned}\langle (R_V(\lambda) - R_V(-\lambda))\varphi, \varphi\rangle &= \langle R_V(\lambda)\varphi, \varphi\rangle - \langle \varphi, R_V(\lambda)\varphi\rangle \\ &= \langle R_V(\lambda)\varphi, \varphi\rangle_{L^2(B_R)} - \langle \varphi, R_V(\lambda)\varphi\rangle_{L^2(B_R)} \\ &= \langle R_V(\lambda)\varphi, (P_V - \lambda^2)R_V(\lambda)\varphi\rangle_{L^2(B_R)} \\ &\qquad - \langle (P_V - \lambda^2)R_V(\lambda)\varphi, R_V(\lambda)\varphi\rangle_{L^2(B_R)} \\ &= \langle \Delta R_V(\lambda)\varphi, R_V(\lambda)\varphi\rangle_{L^2(B_R)} \\ &\qquad - \langle R_V(\lambda)\varphi, \Delta R_V(\lambda)\varphi\rangle_{L^2(B_R)}.\end{aligned}$$

3. Since $R_V(\lambda)\varphi(x) \in C^\infty(\mathbb{R}^n \setminus \operatorname{supp} V)$ we can apply Green's formula which shows that the left hand side of (3.8.6) is equal to

$$2i \operatorname{Im} \int_{\partial B(0,R)} \partial_r \left[R_V(\lambda)\varphi \right](y) \overline{R_V(\lambda)\varphi(y)} dS(y), \tag{3.8.8}$$

where $dS(y)$ is the standard measure on the sphere $\partial B(0,R)$.

Hence we need to find asymptotics of the resolvent kernels. The answer is given in the following analogue of (3.1.20):

LEMMA 3.48. *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{R})$ and $\lambda \in \mathbb{R} \setminus \{0\}$. For $y \in \mathbb{R}^n$ we have*

$$\begin{gathered} R_V(\lambda, r\omega, y) = \frac{e^{i\lambda r}}{r^{\frac{n-1}{2}}} \lambda^{\frac{n-3}{2}} c_n e(\lambda, y, \omega) + \mathcal{O}(r^{-\frac{n+1}{2}}), \quad r \to \infty, \\ c_n = \frac{1}{4\pi} \left(\frac{1}{2\pi i} \right)^{\frac{1}{2}(n-3)}, \end{gathered} \tag{3.8.9}$$

with a full (differentiable) expansion in powers of $1/r$ valid uniformly for y in compact subsets of $\mathbb{R}^n$.

Proof. 1. We recall that

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda) V R_V(\lambda).$$

Since

$$R_V(\lambda, r\omega, y) = R_V(\lambda)(\delta(\bullet - y))(r\omega), \quad \delta(\bullet - y) \in \mathcal{E}'(\mathbb{R}^n)$$

we can apply Theorem 3.5 to see that

$$\begin{aligned} \lambda^{-\frac{n-3}{2}} c_n^{-1} e^{-i\lambda r} r^{\frac{n-1}{2}} R_V(\lambda, r\omega, y) &= e^{-i\lambda\langle y,\omega\rangle} \\ &- \int_{\mathbb{R}^n} e^{-i\lambda\langle y',\omega\rangle} V(y') R_V(\lambda, y', y) dy' + \mathcal{O}(r^{-1}), \end{aligned}$$

with a full asymptotic expansion in powers of r.

2. The symmetry of $R_V(\lambda, y, y')$ (see (3.2.4)) shows that the integral on the right hand side is equal to $R_V(\lambda)(Ve^{-i\lambda\langle\bullet,\omega\rangle})(y)$ which means that the right hand side is equal to $e(\lambda, y, \omega) + \mathcal{O}(1/r)$. ∎

4. We complete the proof of (3.8.6), and hence of (3.8.3) by inserting (3.8.9) into (3.8.8) and letting $R \to \infty$. More precisely,

$$\begin{aligned} & r\partial_r \left[R_V(\lambda)\varphi \right](R\omega) \overline{R_V(\lambda)\varphi(R\omega)} = \\ & \qquad i|c_n|^2 R^{-n+2} \lambda^{n-2} \left| \int_{\mathbb{R}^n} e(\lambda, \omega, y)\varphi(y) dy \right|^2 + \mathcal{O}(R^{-n+1}). \end{aligned}$$

Hence the integral in (3.8.8) is equal to

$$2i|c_n|^2\lambda^{n-2}\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{R}^n} e(\lambda,\omega,y)\varphi(y)dy\right|^2 d\omega + \mathcal{O}(R^{-1}).$$

Inserting the value of c_n from (3.8.9) and letting $R\to\infty$ gives (3.8.6) and that is equivalent to (3.8.3).

5. To obtain (3.8.4) we apply Stone's formula given in Theorem B.10. □

The scattering matrix intertwines distorted plane waves:

THEOREM 3.49. *In the notation of Theorem 3.47 define*

$$E_V(\lambda)f(\omega) := \int_{\mathbb{R}^n} e(\lambda,\omega,x)f(x)dx, \quad f\in L^2_{\mathrm{comp}}(\mathbb{R}^n), \tag{3.8.10}$$

$E_V(\lambda): L^2_{\mathrm{comp}}(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1})$. *Then,*

$$E_V(\lambda) = S(\lambda)JE_V(-\lambda) \tag{3.8.11}$$

where $S(\lambda)$ is the scattering matrix and $Jf(\theta)=f(-\theta)$. In other words,

$$S(\lambda)e(-\lambda,-\bullet,x) = e(\lambda,\bullet,x). \tag{3.8.12}$$

INTERPRETATION. With $E_V(\lambda): L^2_{\mathrm{comp}}(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1})$. Theorem 3.47 can be stated as follows:

$$\begin{aligned} R_V(\lambda)-R_V(-\lambda) &= \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}E_V(\bar\lambda)^*E_V(\lambda), \quad \lambda\in\mathbb{C},\\ dE_\lambda &= \frac{\lambda^{n-1}}{(2\pi)^n}E_V(\lambda)^*E_V(\lambda), \quad \lambda>0. \end{aligned} \tag{3.8.13}$$

In fact, using (3.8.2) and (3.8.3) we see that

$$R_V(\lambda)-R_V(-\lambda) = \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}E_V(-\lambda)^*E_V(-\lambda), \quad \lambda\in\mathbb{R}\setminus\{0\}, \tag{3.8.14}$$

and changing λ to $-\lambda$ gives (3.8.13) for λ real once we remember that n is odd. We then continue both sides meromorphically noting that $\lambda\mapsto E_V(\bar\lambda)^*$ is meromorphic.

We can use (3.8.11) to obtain a formula in which the singularities are "pushed into" the scattering matrix:

$$R_V(\lambda)-R_V(-\lambda) = \frac{i}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}E_V(\bar\lambda)^*S(\lambda)JE_V(-\lambda), \quad \lambda\in\mathbb{C}.$$

This means that for $\operatorname{Im}\lambda<0$, except for $S(\lambda)$, all the factors on the right hand side are holomorphic (except for finitely many poles coming from negative eigenvalues).

Proof of Theorem 3.49. 1. We start by writing the definition of E_V in terms of operators:

$$E_V(\lambda) = E_0(\lambda)(I - VR_V(\lambda)), \tag{3.8.15}$$

where we used definitions (3.8.1), (3.8.10) and (3.2.4). Recalling from (3.7.9) that

$$S(\lambda) = I + A(\lambda), \quad A(\lambda) = a_n \lambda^{n-2} E_0(\lambda) V (I - R_V(\lambda)V) E_0(\lambda)^*,$$

$a_n = (2\pi)^{1-n}/2i$, $\lambda \in \mathbb{R}$, we need to show that

$$E_V(\lambda) - JE_V(-\lambda) = A(\lambda) JE_V(-\lambda), \tag{3.8.16}$$

and it is enough to consider $\lambda \in \mathbb{R}$.

2. For $\lambda \in \mathbb{R}$, we apply Theorem 3.47 (as rephrased in (3.8.14)) to obtain

$$\begin{aligned} E_V(\lambda) - JE_V(-\lambda) &= -E_0(\lambda)V(R_V(\lambda) - R_V(-\lambda)) \\ &= a_n \lambda^{n-2} E_0(\lambda) V E_V(-\lambda)^* E_V(-\lambda) \\ &= a_n \lambda^{n-2} E_0(\lambda) V (I - R_V(\lambda)V) E_0(-\lambda)^* E_V(-\lambda) \\ &= A(\lambda) JE_V(-\lambda) \end{aligned}$$

where we also used $R_V(-\lambda)^* = R_V(\lambda)$ and $E_0(-\lambda)^* = E_0(\lambda)^* J$. This gives (3.8.16) completing the proof. □

3.9. THE BIRMAN–KREĬN TRACE FORMULA

The Birman–Kreĭn formula gives an expression for $\mathrm{tr}(f(P_V) - f(P_0))$ in terms of the determinant of the scattering matrix. It was given in one dimension in Theorem 2.19 and we now proceed to the case of potential scattering in all odd dimensions. We consider the case of *real* potentials so that the Schrödinger operator P_V is self-adjoint.

Theorems 2.19 below is valid without much change in all dimensions and for much less restrictive classes of potentials. That is not the case with the trace formulæ of Theorem 2.21 and 3.53 which cannot hold in even dimensions and are delicate for more general perturbations.

We start with[2]

THEOREM 3.50 (Trace class property of $f(P_V) - f(P_0)$). *Suppose that $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$. For $f \in \mathscr{S}(\mathbb{R})$*

$$f(P_V) - f(P_0) \in \mathcal{L}_1(L^2(\mathbb{R}^n)), \tag{3.9.1}$$

[2] If the reader is interested in the case of dimension 3 only then Theorem 3.50 can be skipped as the proof of Theorem 3.51 for $n = 3$ provides a direct argument for the trace class property.

that is, the operator on the left hand side is of trace class and

$$T_V : f \mapsto \operatorname{tr}\left(f(P_V) - f(P_0)\right), \quad f \in \mathscr{S}(\mathbb{R})$$

defines an element of $\mathscr{S}'(\mathbb{R})$*.*

In addition if $\mathbb{1}_{B(0,R)}$ *is the indicator function of* $B(0,R)$*, then*

$$\mathbb{1}_{B(0,R)}\, f(P_V) \in \mathcal{L}_1(L^2(\mathbb{R}^n)), \tag{3.9.2}$$

and

$$\operatorname{tr}(f(P_V) - f(P_0)) = \lim_{R\to\infty} \operatorname{tr} \mathbb{1}_{B(0,R)}(f(P_V) - f(P_0)). \tag{3.9.3}$$

Proof. 1. Since $f \in \mathscr{S}$, we can write f as

$$f(z) = (z+i)^{-N} g(z), \quad g \in \mathscr{S}(\mathbb{R}).$$

We then apply the Helffer-Sjöstrand formula for functions of self-adjoint operators (see §B.2):

(3.9.4)

$$f(P_V) - f(P_0) =$$
$$\frac{1}{\pi}\int_{\mathbb{C}} \left((P_V - z)^{-1}(P_V+i)^{-N} - (P_0 - z)^{-1}(P_0+i)^{-N}\right) \bar\partial_z \widetilde{g}(z) dm(z),$$

where $dm(z)$ is the Lebesgue measure on $\mathbb{C}$ and

$$\widetilde{g} \in \mathscr{S}(\mathbb{C}), \quad \operatorname{supp} g \subset \{|\operatorname{Im} z| < 1\},$$

is an almost analytic extension of g.

2. We write

$$(P_V - z)^{-1}(P_V+i)^{-N} - (P_0 - z)^{-1}(P_0+i)^{-N} = A + B,$$

where

$$\begin{aligned} A &:= ((P_V - z)^{-1} - (P_0 - z)^{-1})(P_0+i)^{-N} \\ &= -(P_V - z)^{-1} V (P_0+i)^{-N}(P_0 - z)^{-1}, \end{aligned}$$

and

$$\begin{aligned} B &:= (P_V - z)^{-1}\left((P_V+i)^{-N} - (P_0+i)^{-N}\right) \\ &= -(P_V - z)^{-1}\sum_{k=1}^{N}(P_V+i)^{-N+k-1} V (P_0+i)^{-k}, \end{aligned} \tag{3.9.5}$$

where the formula for B is easily proved by induction on N.

3. Arguing as in (B.3.10) we see that for $\rho \in C^\infty_{\rm c}(\mathbb{R}^n)$, singular values satisfy

$$s_j(\rho(P_0+i)^{-k}) = s_j((P_0+i)^{-k}\rho) \le C j^{-2k/n}. \tag{3.9.6}$$

We claim that the same estimate is valid with P_0 replaced by P_V:

$$s_j(\rho(P_V+i)^{-k}) = s_j((P_V+i)^{-k}\rho) \le C j^{-2k/n}. \tag{3.9.7}$$

In fact, we prove this by induction. The case of $k = 0$ is immediate and assume that (3.9.7) holds for k. Without loss of generality we can take ρ satisfying $\rho V = V$, using the same decomposition as in (3.9.5), applying (B.3.5), (3.9.6) and the induction hypothesis, ($N := k + 1$)

$$\begin{aligned} s_j((P_V + i)^{-k-1}\rho) &\leq s_{[j/N]}((P_0 + i)^{-k-1}\rho) \\ &\qquad + C \sum_{\ell=1}^{k+1} s_{[j/N]}((P_V + i)^{-k+\ell-1} V (P_0 + i)^{-\ell}\rho) \\ &\leq Cj^{-2k/n} \\ &\qquad + C \sum_{\ell=1}^{k+1} s_{[j/2N]}((P_V + i)^{-k+\ell-1}\rho) s_{[j/2N]}((P_0 + i)^{-\ell}\rho) \\ &\leq Cj^{-2(k+1)/n} + C' \sum_{\ell=1}^{k+1} j^{-2(k+1-\ell)/n} j^{-2\ell/n} \\ &\leq C'' j^{-2(k+1)/n}. \end{aligned}$$

4. Returning to step 2, we use (3.9.7) to obtain

$$\begin{aligned} & s_j((P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)^{-1}(P_0 + i)^{-N}) \\ & \quad \leq s_{[j/2]}(A) + s_{[j/2]}(B) \\ & \quad \leq \|(P_V - z)^{-1}\| \|(P_0 - z)^{-1}\| s_{[j/2]}(V(P_0 + i)^{-N}) \\ & \quad\quad + \|(P_V - z)^{-1}\| \sum_{k=1}^{N} s_{[j/2N]}((P_V + i)^{-N+k-1} V (P_0 + i)^{-k}) \\ & \quad \leq C|\operatorname{Im} z|^{-2} j^{-2N/n}. \end{aligned}$$

Expressing the trace class norm (B.4.2) using singular values we see that

$$(P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)^{-1}(P_0 + i)^{-N} = \mathcal{O}(|\operatorname{Im} z|^{-2})_{\mathcal{L}_1}, \tag{3.9.8}$$

if $N > n/2$. Combined with (3.9.4) we obtain (3.9.1).

5. The estimates (3.9.7) also show that for $\rho \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 on $B(0, R)$ and $N > n/2$,

$$\begin{aligned} \mathbb{1}_{B(0,R)}(P_V + i)^{-N}(P_V - z)^{-1} &= \mathbb{1}_{B(0,R)}\, \rho (P_V + i)^{-N}(P_V - z)^{-1} \\ &= \mathcal{O}(|\operatorname{Im} z|^{-1})_{\mathcal{L}_1}, \end{aligned}$$

which gives (3.9.2).

6. To see (3.9.3) we claim that

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R)}(P_0 - z)^{-1} V = \mathcal{O}(R^{-M} |\operatorname{Im} z|^{-M-1})_{\mathcal{L}^1}, \tag{3.9.9}$$

for sufficiently large M. From this (3.9.3) follows with a quantitative estimate: let $\widetilde{f}$ be an almost analytic extension of f (with the same properties as $\widetilde{g}$ for $N=0$). Then using (3.9.4) with $N=0$ and the resolvent identity,

$$\begin{aligned}
&\operatorname{tr} \mathbb{1}_{\mathbb{R}^n\setminus B(0,R)} \left(f(P_V)-f(P_0)\right)\\
&\quad = -\frac{1}{\pi}\operatorname{tr}\int_{\mathbb{C}} \left(\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)}(P_0-z)^{-1}V(P_V-z)^{-1}\right)\bar\partial_z \widetilde{f}(z)dm(z)\\
&\quad = \operatorname{tr}\int_{\mathbb{C}} \mathcal{O}(R^{-M}|\operatorname{Im} z|^{-M-1})_{\mathcal{L}_1}\mathcal{O}(|\operatorname{Im} z|^{-1})_{\mathcal{L}}\mathcal{O}\left(|\operatorname{Im} z|^{\infty}\langle z\rangle^{-\infty}\right)dm(z)\\
&\quad = \mathcal{O}(R^{-M}).
\end{aligned}$$

This gives a quantitative version of (3.9.3).

7. To see (3.9.9) we choose $\rho\in C^\infty_{\rm c}$, $\rho\equiv 1$ on $\operatorname{supp} V$ (independent of R) and $\psi_{j,R}\in C^\infty_{\rm c}$, $1\le j\le J$, such that $\operatorname{supp}\psi_{j+1,R}\equiv 1$ on $\operatorname{supp}\psi_{j,R}$, $j<J$, $\operatorname{supp}\psi_{1,R}\equiv 1$ on $\operatorname{supp}\rho$ and

$$\partial^\alpha\psi_{j,R}=\mathcal{O}_{\alpha,J}(R^{-|\alpha|}),\quad \operatorname{supp}\psi_{j,R}\Subset B(0,R).$$

In particular $\operatorname{supp}\psi_{j,R}\cap\operatorname{supp}\partial\psi_{j+1,R}=\emptyset$. This fact will be crucial in the next calculation.

Since the estimate (3.9.9) is equivalent to the estimate for the adjoint, we can estimate the trace class norm of

$$\begin{aligned}
\rho(P_0-z)^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)} &= \rho\psi_{1,R}\cdots\psi_{J,R}(P_0-z)^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)}\\
&= \rho\psi_{1,R}\cdots\psi_{J-1,R}(P_0-z)^{-1}[P_0,\psi_{J,R}](P_0-z)^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)}\\
&= \rho(P_0-z)^{-1}[P_0,\psi_{1,R}](P_0-z)^{-1}\cdots[P_0,\psi_{J,R}](P_0-z)^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)}\,.
\end{aligned}$$

From estimates on derivatives of $\psi_{j,R}$ we see that

$$[P_0,\psi_{j,R}]=\mathcal{O}(R^{-1})_{H^{s+2}(\mathbb{R}^n)\to H^{s+1}(\mathbb{R}^n)}.$$

Also $(P-z)^{-1}=\mathcal{O}(|\operatorname{Im} z|^{-1})_{H^s\to H^{s+2}}$ and hence

$$[P_0,\psi_{j,R}](P_0-z)^{-1}=\mathcal{O}(R^{-1}|\operatorname{Im} z|^{-1})_{H^s(\mathbb{R}^n)\to H^{s+1}(\mathbb{R}^n)}.$$

We conclude that

$$\rho(P_0-z)^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R)}=\mathcal{O}(|\operatorname{Im} z|^{-J-1}R^{-J})_{L^2(\mathbb{R}^n)\to H^{J+2}(B(0,R_0))},$$

for R_0 such that $\operatorname{supp}\rho\subset B(0,R_0)$. For J large enough $\|\,\mathbb{1}_{B(0,R_0)}\,A\|_{\mathcal{L}_1}\le C\|A\|_{L^2\to H^{J+2}}$, and that concludes the proof of (3.9.9).

8. The proof that $f\mapsto\operatorname{tr}(f(P_V)-f(P_0))$ defines a tempered distribution follows from estimates on $\bar\partial g$ in terms of the a finite number of semi-norms $\sup|\langle\lambda\rangle^m\partial^k_\lambda f|$. These follow from the construction of g in §B.2. □

We are now ready for the main result of this section:

THEOREM 3.51 (The Birman–Kreĭn formula). *Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n;\mathbb{R})$, where $n \geq 3$ is odd.*

Then for $f \in \mathscr{S}(\mathbb{R})$ the operator $f(P_V) - f(P_0)$ is of trace class and

$$\begin{aligned} \operatorname{tr}\left(f(P_V) - f(P_0)\right) &= \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}\left(S(\lambda)^{-1}\partial_\lambda S(\lambda)\right)d\lambda \\ &\quad + \sum_{k=1}^K f(E_k) + \tfrac{1}{2}\widetilde{m}_R(0)f(0)\,, \end{aligned} \tag{3.9.10}$$

where $S(\lambda)$ is the scattering matrix and $E_K < E_{K-1} \leq \cdots \leq E_1 \leq 0$ are the eigenvalues of P_V and $\widetilde{m}_R(0)$ is defined by (3.3.29) (and can be non-zero only when $n = 3$).

REMARKS. 1. For a heuristic interpretation of

$$\sigma'(\lambda) := \frac{1}{2\pi i}\operatorname{tr} S(\lambda)^{-1}\partial_\lambda S(\lambda)$$

see the discussion after Theorem 2.19. We also mention that the operator $-i\partial_\lambda S(\lambda)S(\lambda)^{-1}$ is known as the *Eisenbud–Wigner time delay operator* and has an interesting physical interpretation – see Jensen [**Je81**]. From (B.5.23) we also see that $\operatorname{tr} S(\lambda)^{-1}\partial_\lambda S(\lambda)$ is the logarithmic derivative of $\det S(\lambda)$.

2. At this stage the integral on the right hand side of (3.9.10) is meant as *distributional pairing* of $\sigma' \in \mathscr{S}'$ with $f(\lambda^2)$ – see Theorem 3.50. In §3.11 we will see that σ' is polynomially bounded so that the integral converges in the usual sense.

3. In Lemma 3.52 §3.11 we will use $f(s) = e^{-ts}$: that is allowed as $f(P_V) = (\chi f)(P_V)$ for $\chi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\chi \subset (\min\operatorname{Spec}(P_V) - 1, \infty)$, $\chi \equiv 1$ on $[\min\operatorname{Spec}(P_V), \infty)$. We then have $\chi f \in \mathscr{S}(\mathbb{R})$.

In dimension three a complication arises from the possibility of the resonance at zero – see Theorem 3.23. On the other hand the trace class properties are easier and hence the proof we presented in dimension one (see §2.6) applies and in fact is somewhat easier as $R_0(\lambda)$ is now holomorphic at zero.

Proof of Theorem 3.51 for $n = 3$. 1. As in (3.8.4), the spectral theorem and Stone's formula show that

$$f(P_V) = \sum_{k=1}^K f(E_k)u_k \otimes \bar{u}_k + \frac{1}{4\pi i}\int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda d\lambda,$$

where we used the fact that the integrand is even to change the integration from $(0,\infty)$ to $\mathbb{R}$. The operator $(R_V(\lambda) - R_V(-\lambda))2\lambda$ is smooth in $\lambda \in \mathbb{R}$ as an operator $L^2_{\text{comp}} \to L^2_{\text{loc}}$. For simplicity we will now assume that there are no negative eigenvalues: their contribution in the formula is clear.

2. We write

$$\begin{aligned} R_V(\lambda) - R_0(\lambda) &= -R_V(\lambda) V R_0(\lambda) \\ &= -R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} V R_0(\lambda) \,. \end{aligned}$$

This operator can have a pole at $\lambda = 0$ as described in Lemma 3.16. In the notation of that lemma we define

$$\begin{aligned} B(\lambda) &:= 2\lambda(R_V(\lambda) - R_0(\lambda)) + \frac{2\Pi_0}{\lambda} \\ &= -2\lambda R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} V R_0(\lambda) + \frac{2\Pi_0}{\lambda}, \\ B(\lambda) &: L^2_{\rm comp}(\mathbb{R}^3) \to L^2_{\rm loc}(\mathbb{R}^3)\,, \end{aligned} \tag{3.9.11}$$

which is a meromorphic family of operators, holomorphic for $\operatorname{Im}\lambda \geq 0$. The possible $1/\lambda$ term in $R_V(\lambda)$ at $\lambda = 0$ is cancelled by the λ factor and the spectral pole $-\Pi_0/\lambda^2$ by the last term on the right hand side.

With this notation we have (we assumed for simplicity of exposition that there are no negative eigenvalues)

$$f(P_V) - f(P_0) = f(0)\Pi_0 + \frac{1}{4\pi i}\sum_{\pm}\int_{\mathbb{R}} f(\lambda^2) B(\pm\lambda) d\lambda. \tag{3.9.12}$$

We recall from (3.1.24) that for $\operatorname{Im}\lambda \geq 0$ $\|R_0(\lambda)\|_{L^2\to H^2} \leq C\langle\lambda\rangle^2/|\lambda|\operatorname{Im}\lambda$. Arguing as in (3.4.2) (with $n = 3$) this gives

$$\begin{aligned} s_j(R_0(\lambda)\rho) &= s_j(\rho R_0(-\bar\lambda)) \\ &= s_j((-\Delta_{\mathbb{T}^3_R} + 1)^{-1}(-\Delta_{\mathbb{T}^3_R} + 1)\rho R_0(-\bar\lambda)) \\ &\leq s_j((-\Delta_{\mathbb{T}^3_R} + 1)^{-1})\|R_0(-\bar\lambda)\|_{L^2\to H^2} \\ &= \mathcal{O}(\langle\lambda\rangle^2/|\lambda|\operatorname{Im}\lambda) j^{-2/3}. \end{aligned}$$

Since there are no poles for $\lambda \neq 0$, $\operatorname{Im}\lambda \geq 0$,

$$\|(I + VR_0(\lambda)\rho)^{-1}\|_{L^2\to L^2} = \mathcal{O}\left(\langle\lambda\rangle^2/|\lambda|^2\right), \quad \operatorname{Im}\lambda > 0. \tag{3.9.13}$$

In fact, for $|\lambda| \gg 1$ this follows from (3.1.12) and for λ near 0 from (3.3.4). Hence,

$$\begin{aligned} \|B(\lambda)\|_{\mathcal{L}_1} &\leq \frac{2}{|\lambda|} + 2|\lambda|\sum_{j=1}^\infty s_j(R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} V\rho R_0(\lambda)) \\ &\leq \frac{2}{|\lambda|} + \frac{C\langle\lambda\rangle^2}{|\lambda|}\sum_{j=1}^\infty s_{[j/2]}(R_0(\lambda)\rho)^2 \\ &\leq \frac{C\langle\lambda\rangle^6}{|\lambda|^3(\operatorname{Im}\lambda)^2}\sum_{j=1}^\infty j^{-4/3} \leq \frac{C\langle\lambda\rangle^6}{|\lambda|^3(\operatorname{Im}\lambda)^2} \leq \frac{C\langle\lambda\rangle^6}{(\operatorname{Im}\lambda)^5}. \end{aligned} \tag{3.9.14}$$

3. Let $g \in \mathscr{S}(\mathbb{C})$, $\operatorname{supp} g \subset \{|\operatorname{Im}\lambda| \leq 1\}$, be an *almost analytic extension* of $f(\lambda^2)$, see §B.2. The Cauchy–Green formula (D.1.1) applied to the right hand side of (3.9.12) shows that (with $dm(\lambda)$ the Lebesgue measure on $\mathbb{C}$)

$$\begin{aligned} f(P_V) - f(P_0) &= \frac{1}{2\pi}(t_+(f) - t_-(f)), \\ t_\pm(f) &:= \int_{\pm \operatorname{Im}\lambda > 0} \partial_{\bar\lambda} g(\lambda) B(\pm\lambda) dm(\lambda). \end{aligned} \tag{3.9.15}$$

Since $\partial_{\bar\lambda} g(\lambda) = \mathcal{O}(|\operatorname{Im}\lambda|^\infty \langle\lambda\rangle^{-\infty})$, the estimate (3.9.14) shows that $t_\pm(f) \in \mathcal{L}_1$. In particular, this implies directly that the $f(P_V) - f(P_0)$ is of trace class and that $f \mapsto \operatorname{tr}(f(P_V) - f(P_0))$ defines a tempered distribution – of course this also follows from Theorem 3.50.

3. To relate the trace of (3.9.15) to the scattering matrix we reformulate Theorem 3.46 as follows:

$$\det S(\lambda) = \det((I + VR_0(\lambda)\rho)^{-1}(I + VR_0(-\lambda)\rho)). \tag{3.9.16}$$

This is valid since, in the notation of Theorem 3.46,

$$\begin{aligned} I - T(\lambda) &= I - (I + VR_0(\lambda)\rho)^{-1}(V(R_0(\lambda) - R_0(-\lambda))\rho) \\ &= (I + VR_0(\lambda)\rho)^{-1}(I + VR_0(-\lambda)\rho). \end{aligned}$$

Since $(-\Delta - \lambda^2)\partial_\lambda R_0(\lambda) = 2\lambda R_0(\lambda)$, elliptic estimates (see for instance [**Zw12**, Theorem 7.1]) show that

$$\rho\partial_\lambda R_0(\lambda)\rho = \mathcal{O}\left(\langle\lambda\rangle^2 e^{C(\operatorname{Im}\lambda)_-}\right) : L^2(\mathbb{R}^3) \to H^4(\mathbb{R}^3)$$

and that implies that

$$\|\partial_\lambda(VR_0(\lambda)\rho)\|_{\mathcal{L}_1(L^2(\mathbb{R}^3))} \leq C\langle\lambda\rangle^2 e^{C(\operatorname{Im}\lambda)_-}, \quad \lambda \in \mathbb{C}. \tag{3.9.17}$$

Taking logarithmic derivatives of both sides of (3.9.16) gives

$$\begin{aligned} \operatorname{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} &= \operatorname{tr} F(-\lambda) + \operatorname{tr} F(\lambda), \\ F(\lambda) &:= -\partial_\lambda(VR_0(\lambda)\rho)(I + VR_0(\lambda)\rho)^{-1}. \end{aligned} \tag{3.9.18}$$

We note that $F(\lambda)$, $\lambda \in \mathbb{C}$, is a meromorphic family of operators in $\mathcal{L}_1(L^2)$, with no poles in $\operatorname{Im}\lambda > 0$.

Theorem 3.15 and the Gohberg–Sigal theory, see §C.4, show that, near $\lambda = 0$,

$$I + VR_0(\lambda)\rho = U_1(\lambda)(Q_2\lambda^2 + Q_1\lambda + Q_0)U_2(\lambda), \tag{3.9.19}$$

where the operators $U_j(\lambda)$ are invertible and holomorphic as function of λ,

$$\begin{gathered} Q_i Q_j = \delta_{ij} Q_{ij}, \quad \operatorname{rank}(I - Q_0) < \infty, \\ \operatorname{rank} Q_2 = m_R(0) - \widetilde{m}_R(0) = \operatorname{tr}\Pi_0, \quad \operatorname{rank} Q_1 = \widetilde{m}_R(0). \end{gathered}$$

(We have $(I + VR_0(\lambda))^{-1} = U_2(\lambda)^{-1}(Q_2\lambda^{-2} + Q_1\lambda^{-1} + Q_0)U_1(\lambda)^{-1}$; the total multiplicity of the pole is m_R while the rank of the λ^{-2} has to be $\operatorname{tr}\Pi_0$.)

Taking a logarithmic derivative of (3.9.19) gives

$$\begin{aligned} \operatorname{tr} F(\lambda) &= -\operatorname{tr}(2\lambda^{-1}Q_2 + \lambda^{-1}Q_1) + \varphi(\lambda) \\ &= -\frac{1}{\lambda}(2\operatorname{tr}\Pi_0 + \widetilde{m}_R(0)) + \varphi(\lambda), \end{aligned} \tag{3.9.20}$$

where $\varphi(\lambda)$ is holomorphic in $\operatorname{Im}\lambda \geq 0$. In view of (3.9.13) and (3.9.17), we have

$$|\varphi(\lambda)| \leq C\langle\lambda\rangle^2, \quad \operatorname{Im}\lambda \geq 0. \tag{3.9.21}$$

4. We claim that for $\operatorname{Im}\lambda > 0$

$$\operatorname{tr} F(\lambda) = \operatorname{tr}\left(B(\lambda) - \frac{2\Pi_0}{\lambda}\right), \tag{3.9.22}$$

where $B(\lambda)$ was defined by (3.9.11). To see this we use the fact that $R_0(\lambda)$ is bounded on L^2 for $\operatorname{Im}\lambda > 0$ and hence $\partial_\lambda(VR_0(\lambda)\rho) = 2\lambda VR_0(\lambda)^2\rho$. Using this, the cyclicity of the trace, and $\rho V = V$, we obtain, always for $\operatorname{Im}\lambda > 0$,

$$\operatorname{tr} F(\lambda) = -2\lambda \operatorname{tr} R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda) = \operatorname{tr}\left(B(\lambda) - \frac{2\Pi_0}{\lambda}\right),$$

which is (3.9.22).

5. Let

$$h(\lambda) := \operatorname{tr} F(\lambda) + \frac{2\operatorname{tr}\Pi_0}{\lambda} = -\frac{\widetilde{m}_R(0)}{\lambda} + \varphi(\lambda). \tag{3.9.23}$$

In this notation, (3.9.22) and (D.1.1) used in (3.9.15) give

$$\begin{aligned} \operatorname{tr}(f(P_V) - f(P_0)) &= \frac{1}{2\pi}\sum_{\pm} \pm \int_{\pm\operatorname{Im}\lambda>0} \partial_{\bar\lambda} g(\lambda) h(\pm\lambda)\, dm(\lambda) \\ &= \frac{1}{4\pi i}\sum_{\pm} \int_{\gamma_\pm(\varepsilon)} g(\lambda)h(\pm\lambda)\, d\lambda \\ &\quad + \frac{1}{2\pi}\sum_{\pm} \pm \int_{\Omega_\pm(\varepsilon)} \partial_{\bar\lambda} g(\lambda)h(\pm\lambda)\, dm(\lambda), \end{aligned} \tag{3.9.24}$$

where

$$\Omega_\pm(\varepsilon) = D(0,\varepsilon) \cap \mathbb{C}_\pm, \quad \mathbb{C}_\pm := \{\pm\operatorname{Im}\lambda > 0\},$$
$$\gamma_+(\varepsilon) = \partial(\mathbb{C}_+ \setminus \Omega_+(\varepsilon)), \quad \gamma_-(\varepsilon) = \partial(\mathbb{C}_+ \cup \overline{\Omega_-(\varepsilon)}),$$

and the boundaries are positively oriented (as boundaries of the indicated sets).

Estimates (3.9.21) and $\partial_{\bar\lambda} g(\lambda) = \mathcal{O}(|\operatorname{Im}\lambda|^\infty \langle\lambda\rangle^{-\infty})$ show that the last term on the right hand side of (3.9.24) is $\mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. Also, (3.9.18) and (3.9.23) imply that

$$\begin{aligned}
&\frac{1}{4\pi i}\sum_{\pm}\int_{\gamma_\pm(\varepsilon)\cap\mathbb{R}} g(\lambda)h(\pm\lambda)d\lambda \\
&\quad = \frac{1}{4\pi i}\sum_{\pm}\int_{\gamma_\pm(\varepsilon)\cap\mathbb{R}} f(\lambda^2)\left(\operatorname{tr} F(\pm\lambda) \pm 2\operatorname{tr}\Pi_0\lambda^{-1}\right)d\lambda \\
&\quad = \frac{1}{2\pi i}\int_\varepsilon^\infty f(\lambda^2)\operatorname{tr}\,\partial_\lambda S(\lambda)S(\lambda)^{-1}d\lambda \\
&\quad = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}\,\partial_\lambda S(\lambda)S(\lambda)^{-1}d\lambda + \mathcal{O}(\varepsilon),
\end{aligned}$$

as the λ^{-1} terms cancel. Hence,

$$\begin{aligned}
\frac{1}{4\pi i}\sum_{\pm}\int_{\gamma_\pm(\varepsilon)} g(\lambda)h(\pm\lambda)d\lambda &= \frac{1}{2\pi i}\int_0^\infty f(\lambda^2)\operatorname{tr}\,\partial_\lambda S(\lambda)S(\lambda)^{-1}d\lambda \\
&\quad + \frac{1}{4\pi i}\sum_{\pm}\int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda)h(\pm\lambda)d\lambda + \mathcal{O}(\varepsilon)\,.
\end{aligned}$$

The structure of $h(\lambda)$ near 0 given in (3.9.23) shows that

$$\begin{aligned}
\frac{1}{4\pi i}\sum_{\pm}\int_{\gamma_\pm(\varepsilon)\setminus\mathbb{R}} g(\lambda)h(\pm\lambda)d\lambda &= \frac{\widetilde{m}_R(0)f(0)}{4\pi i}\int_{\partial D(0,\varepsilon)}\frac{d\lambda}{\lambda} + \mathcal{O}(\varepsilon) \\
&= \tfrac{1}{2}\widetilde{m}_R(0)f(0) + \mathcal{O}(\varepsilon).
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain (3.9.10). $\square$

For odd dimensions $n \geq 5$ we do not have a possibility of a zero resonance but the arguments are complicated by weaker trace class properties. The following somewhat laborious lemma will be used to establish (3.9.10) for functions supported near 0:

LEMMA 3.52. *Suppose that $n \geq 5$ and $E_K < E_{K-1} \leq \cdots \leq E_1 \leq 0$ are the eigenvalues of P_V included according to their multiplicities. Then for sufficiently large M and N,*

$$\begin{aligned}
&\operatorname{tr}((P_V + M)^{-N}e^{-tP_V} - (P_0+M)^{-N}e^{-tP_0}) = \\
&\qquad \sum_{k=1}^K (E_k + M)^{-N}e^{-tE_k} + o(1), \quad t \to +\infty.
\end{aligned} \tag{3.9.25}$$

REMARK. A more precise asymptotics for $t \to +\infty$ are valid once we establish (3.9.10) and restrictions on N or M are not needed either – see Exercise 3.11.

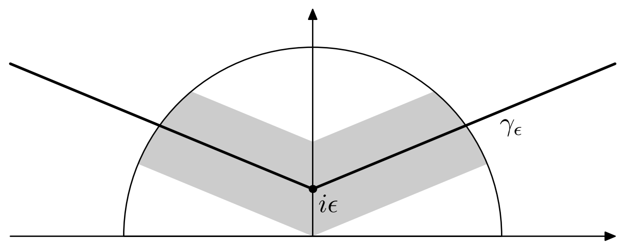


Figure 3.5. The contour γ_ε in the proof of Lemma 3.52 and the domain Ω (shaded).

Proof. 1. For ε small enough define the contour (see Figure 3.5)

$$\gamma_\varepsilon = (i\varepsilon - e^{-\frac{\pi}{8}i}[0,\infty)) \cup (i\varepsilon + e^{\frac{\pi}{8}i}[0,\infty))$$

oriented from left to right[3]. We also define

$$\Omega := \bigcup_{0\le\varepsilon\le\varepsilon_0} \gamma_\varepsilon \cap D(0, 2\varepsilon_0), \ \ \varepsilon_0 \ll 1, \tag{3.9.26}$$

We choose ε_0 small enough so that for no poles of $R_V(\lambda)$ other than 0 belong to Ω. With this notation, for $\operatorname{Im}\lambda_0 \gg 1$,

$$e^{-tP_V}(P_V - \lambda_0^2)^{-N} - e^{-tP_0}(P_0 - \lambda_0^2)^{-N} = \sum_{E_k<0} (E_k - \lambda_0^2)^{-N} e^{-tE_k} u_k \otimes \bar{u}_k$$
$$+ \frac{1}{\pi i}\int_{\gamma_\varepsilon} (\lambda R_V(\lambda) R_V(\lambda_0)^N - \lambda R_0(\lambda) R_0(\lambda_0)^N) e^{-t\lambda^2} d\lambda.$$

(We first write the left hand side as a contour integral over γ_M with M large – that gives a contour enclosing the spectrum – and then deform it to γ_ϵ picking up contribution from negative eigenvalues.)

In (3.9.8) (in the proof of Theorem 3.50) we showed that

$$\|R_V(\lambda)R_V(\lambda_0)^N - R_0(\lambda)R_0(\lambda_0)^N\|_{\mathcal{L}_1} \le C|\operatorname{Im}\lambda|^{-4}, \quad \lambda \in \gamma_\varepsilon, \tag{3.9.27}$$

which shows that

$$\begin{aligned} &\operatorname{tr}((P_V + M)^{-N} e^{-tP_V} - (P_0 + M)^{-N} e^{-tP_0}) = \\ &\sum_{E_k<0} (E_k + M)^{-N} e^{-tE_k} + \frac{1}{\pi i}\int_{\gamma_\varepsilon} f(\lambda) e^{-t\lambda^2} d\lambda, \end{aligned} \tag{3.9.28}$$

where

$$f(\lambda) := \operatorname{tr}\left(\lambda R_V(\lambda)R_V(\lambda_0)^N - \lambda R_0(\lambda)R_0(\lambda_0)^N\right), \quad \lambda_0 := i\sqrt{M}. \tag{3.9.29}$$

Away from a neighbourhood of 0 the estimate (3.9.27) is all that we need and we now concentrate on $\lambda \in \Omega$, with Ω given in (3.9.26).

[3] This choice of orientation comes from the fact that we are integrating $(P_V - z)^{-1}dz$, $z = \lambda^2$ which is then consistent with the residue theorem.

2. Since $n \geq 5$, Theorem 3.17 shows that

$$\begin{aligned}\lambda R_V(\lambda)\rho &= \lambda R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} = \lambda R_0(\lambda)\rho(I - VR_V(\lambda)\rho)\\ &= \lambda^{-1}R_0(\lambda)V\Pi_0\rho + \lambda R_0(\lambda)\rho B(\lambda)\rho_1,\end{aligned}$$

where $\rho_1 \in C_c^\infty(\mathbb{R}^n)$, $\rho_1\rho = \rho$, and

$$\begin{aligned}B(\lambda) &:= I - VA(\lambda)\rho : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\\ A(\lambda) &:= R_V(\lambda) + \lambda^{-2}\Pi_0,\end{aligned} \tag{3.9.30}$$

where $A(\lambda)$ and $B(\lambda)$ are holomorphic family of operators for $\lambda \in \Omega$ (see (3.9.26)). Using

$$R_0(\lambda)V\Pi_0 = -R_0(\lambda)((-\Delta - \lambda^2) + \lambda^2)\Pi_0 = -(I + \lambda^2 R_0(\lambda))\Pi_0, \tag{3.9.31}$$

and the adjoint of this identity for $\bar{\lambda}$, we obtain

$$\begin{aligned}\lambda R_V(\lambda) - \lambda R_0(\lambda) &= -\lambda R_V(\lambda)VR_0(\lambda)\\ &= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}R_0(\lambda)V\Pi_0 VR_0(\lambda)\\ &= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}(I + \lambda^2 R_0(\lambda))\Pi_0(I + \lambda^2 R_0(\lambda))\\ &= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}\Pi_0\\ &\quad - \lambda R_0(\lambda)\Pi_0 - \lambda\Pi_0 R_0(\lambda) - \lambda^3 R_0(\lambda)\Pi_0 R_0(\lambda).\end{aligned} \tag{3.9.32}$$

The goal of the above identity is to obtain as few factors of $R_0(\lambda)$ and as many powers of λ as possible.

3. We rewrite f in (3.9.29) as follows:

$$f(\lambda) = \operatorname{tr}\big((\lambda R_V(\lambda) - \lambda R_0(\lambda))R_0(\lambda_0)^N + \lambda R_V(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N)\big).$$

This gives the decomposition

$$\begin{gathered}f(\lambda) = a(\lambda) + b(\lambda) + c(\lambda),\\ a(\lambda) := -\lambda^{-1}\operatorname{tr}\Pi_0 R_V(\lambda_0)^N, \quad b(\lambda) = b_1(\lambda) + b_2(\lambda)\\ b_1(\lambda) := -\lambda\operatorname{tr} R_0(\lambda)\rho B(\lambda)VR_0(\lambda)R_0(\lambda_0)^N,\\ b_2(\lambda) := -\lambda\operatorname{tr}\left(2R_0(\lambda)\Pi_0 + \lambda^2 R_0(\lambda)\Pi_0 R_0(\lambda)\right)R_0(\lambda_0)^N,\\ c(\lambda) := \operatorname{tr}\lambda A(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N).\end{gathered} \tag{3.9.33}$$

The contribution of $a(\lambda)$ is straightforward:

$$\begin{aligned}\frac{1}{\pi i}\int_{\gamma_\varepsilon} a(\lambda)e^{-t\lambda^2}d\lambda &= -m_R(0)(-\lambda_0^2)^{-N}\frac{1}{\pi i}\int_{\gamma_\varepsilon}\lambda^{-1}e^{-t\lambda^2}d\lambda\\ &= -m_R(0)(-\lambda_0^2)^{-N}\frac{1}{\pi i}\int_{\gamma_\varepsilon^1}\lambda^{-1}e^{-t\lambda^2}d\lambda\\ &= m_R(0)(-\lambda_0^2)^{-N},\end{aligned} \tag{3.9.34}$$

where we deformed γ_ε to $\gamma_\varepsilon^1 := (\mathbb{R} \setminus (-\varepsilon, \varepsilon)) \cup (\partial D(0,\varepsilon) \cap \operatorname{Im} \lambda > 0)$ oriented from left to right: the contributions over the real axis cancel and integration over the clockwise oriented half circle produces $-\pi i$.

4. We now move to the analysis of $b_1(\lambda)$. Since $\rho R_0(\lambda) R_0(\lambda_0)^N$, $\lambda \in \gamma_\varepsilon$, $\varepsilon > 0$, is of trace class, cyclicity of the trace shows that

$$b_1(\lambda) = \lambda \operatorname{tr} B(\lambda) V \rho R_0(\lambda) R_0(\lambda_0)^N R_0(\lambda) \rho.$$

If we take N sufficiently large, then by Lemma 3.6 we have

$$\begin{aligned} |b_1(\lambda)| &\leq C \|B(\lambda)\|_{L^2 \to L^2} \|\lambda \rho R_0(\lambda) R_0(\lambda_0)^N R_0(\lambda) \rho\|_{\mathcal{L}_1} \\ &\leq C \|B(\lambda)\|_{L^2 \to L^2} \|\lambda \rho R_0(\lambda) R_0(\lambda_0)^N R_0(\lambda) \rho\|_{L^2 \to H^{n+1}} \leq C, \end{aligned}$$

for $\lambda \in \Omega$. This shows that we can deform γ_ε to γ_0 and hence

$$\begin{aligned} \int_{\gamma_\varepsilon} b_1(\lambda) e^{-t\lambda^2} d\lambda &= \int_{\gamma_0} b_1(\lambda) e^{-t\lambda^2} d\lambda = \mathcal{O}(1) \int_{\gamma_0} e^{-t \operatorname{Re} \lambda^2} d|\lambda| \\ &= \mathcal{O}(t^{-\frac{1}{2}}) = o(1). \end{aligned} \tag{3.9.35}$$

5. To analyse b_2 let $\Pi_0 = \sum_{j=1}^J u_j \otimes \bar{u}_j$,

$$u_j = -R_0(0) V u_j = \mathcal{O}(\langle x \rangle^{2-n}) \in L^p(\mathbb{R}^n), \quad p > \frac{n}{n-2},$$

in particular for some $p < 2$ when $n \geq 5$. We then use Exercise 3.2 and Young's inequality (A.5.2) to see that for $\lambda \in \Omega$,

$$\|R_0(\lambda) u_j\|_{L^2} \leq C |\lambda|^{2 - n(q-1)/q} \|u_j\|_{L^p}, \quad \frac{1}{q} + \frac{1}{p} = \frac{3}{2}.$$

But since we can use $p < 2$ this means that we can take $q > 1$ and hence,

$$\|R_0(\lambda) \Pi_0\|_{L^2 \to L^2} \leq C |\lambda|^{-2-\delta}, \quad \delta > 0. \tag{3.9.36}$$

It follows that

$$\lambda R_0(\lambda) \Pi_0 R_0(\lambda_0)^N = (|\lambda|^{-1+\delta}) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \lambda \in \Omega,$$

and since the operator of finite rank the same estimate is valid for the trace class norm. The same estimate holds for the term involving $\lambda^3 R_0(\lambda) \Pi_0 R_0(\lambda)$ (or an even better estimate if we use Exercise 3.3). We can now deform the contour to γ_0 and

$$\begin{aligned} \int_{\gamma_\varepsilon} b_2(\lambda) e^{-t\lambda^2} d\lambda &= \int_{\gamma_0} b_2(\lambda) e^{-t\lambda^2} d\lambda = \int_{\gamma_0} \mathcal{O}(|\lambda|^{-1+\delta}) e^{-t \operatorname{Re} \lambda^2} d|\lambda| \\ &= \mathcal{O}(t^{-\delta/2}) = o(1). \end{aligned} \tag{3.9.37}$$

6. We continue with the analysis of the term $c(\lambda)$. For that we use (3.9.30) and (3.9.32) to write

$$\begin{gathered}\lambda A(\lambda) = T_1(\lambda) + T_2(\lambda),\\ T_1(\lambda) := \lambda R_0(\lambda) + \lambda R_0(\lambda) V A(\lambda) V R_0(\lambda) - \lambda R_0(\lambda) V R_0(\lambda),\\ T_2(\lambda) := -\lambda R_0(\lambda)\Pi_0 - \lambda \Pi_0 R_0(\lambda) - \lambda^3 R_0(\lambda)\Pi_0 R_0(\lambda).\end{gathered}$$

We then write

$$c(\lambda) = c_1(\lambda) + c_2(\lambda), \quad c_j(\lambda) := \operatorname{tr} T_j(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N).$$

7. To analyse $c_1(\lambda)$ we use (3.9.5):

$$c_1(\lambda) = -\sum_{k=1}^{N} \operatorname{tr} T_1(\lambda) R_V(\lambda_0)^{N+1-k} V R_0(\lambda_0)^k. \tag{3.9.38}$$

We claim the following: for $s \geq \frac{3}{2}$,

$$\|\langle x\rangle^{-s} T_1(\lambda) \langle x\rangle^{-s}\|_{L^2\to L^2} = \mathcal{O}(1), \quad \lambda \in \Omega, \tag{3.9.39}$$

and for any $r \in \mathbb{R}$,

$$s_j(\langle x\rangle^r R_V(\lambda_0)^k \rho), \quad s_j(\rho R_0(\lambda_0)^k \langle x\rangle^r) \leq C j^{-k/n}. \tag{3.9.40}$$

Assuming (3.9.39) and (3.9.40) for now we can estimate $c_1(\lambda)$ as follows. We first note that (3.9.40) give

$$s_j(\langle x\rangle^{-s} R_V(\lambda_0)^{N+1-k} V R_0(\lambda_0)^k \langle x\rangle^s) \leq C j^{-(N+1)/k}.$$

Combining this with (3.9.39) and (3.9.38) give

$$\|T_1(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N)\|_{\mathcal{L}_1(\langle x\rangle^s L^2)} \leq C, \quad \lambda \in \gamma_0, \quad s \geq s_0.$$

Using Lemma B.33 we see that

$$c_1(\lambda) = \operatorname{tr}_{\langle x\rangle^s L^2} T_1(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N) = \mathcal{O}(1), \qquad \lambda \in \Omega.$$

Hence we can deform γ_ε to γ_0 and that gives, as in (3.9.35)

$$\int_{\gamma_\varepsilon} c_1(\lambda) e^{-t\lambda^2} d\lambda = \mathcal{O}(t^{-\frac{1}{2}}). \tag{3.9.41}$$

7. The term c_2 is treated the same as b_2 in Step 5. We now put (3.9.41), (3.9.37), (3.9.35), (3.9.34) and (3.9.33) in (3.9.28) to obtain (3.9.25).

It remains to provide the two missing proofs.

8. *Proof of* (3.9.39). We use Lemma 3.7 so that (writing $\|Q\| := \|Q\|_{L^2\to L^2}$), so that on γ_0,

$$\begin{aligned}\|\langle x\rangle^{-s} T_1(\lambda)\langle x\rangle^{-s}\| &\leq |\lambda|(1 + \|\langle x\rangle^{-s} R_0(\lambda)\langle x\rangle^{-s}\|)^2 (1 + \|\langle x\rangle^s V \langle x\rangle^s\|)\\ &\leq C|\lambda|(1 + |\lambda|^{-2+s})^2 \leq C, \quad s \geq \tfrac{3}{2}.\end{aligned}$$

9. *Proof of* (3.9.40). For $r = 0$ the estimates were proved in Step 3 of the proof of Theorem 3.50. Since $s_j(A) = s_j(A^*A)^{\frac{1}{2}}$,

$$
\begin{aligned}
s_j(\langle x\rangle^r R_V(\lambda_0)^k \rho)^2 &= s_j(\rho R_V(\lambda_0)^k \langle x\rangle^{2r} R_V(-\overline{\lambda}_0)^k \rho) \\
&\leq s_j(\rho R_V(\lambda_0)^k) \|\langle x\rangle^{2r} R_V(-\bar{\lambda}_0)\rho\|_{L^2\to L^2} \\
&\leq C j^{-2k/n} \|\langle x\rangle^{2r} R_V(-\bar{\lambda}_0)\langle x\rangle^{-2r}\|_{L^2\to L^2},
\end{aligned} \tag{3.9.42}
$$

where we used the estimate with $r = 0$. To see that the norm on the right hand side is finite we first note that if M ($\lambda_0 = i\sqrt{M}$) is large enough[4] then $\|\langle x\rangle^{2r} V R_0(\lambda_0)\rho\langle x\rangle^{-2r}\|_{L^2\to L^2} \leq C_r M^{-1} < \frac{1}{2}$. A Neumann series argument then shows that

$$\|\langle x\rangle^{2r}(I + V R_0(\lambda_0)\rho)^{-1}\langle x\rangle^{-2r}\|_{L^2\to L^2} \leq 2,$$

and using (3.14.1) and (3.2.1) we obtain

$$\|\langle x\rangle^{2r} R_V(\lambda_0)\langle x\rangle^{-2r}\|_{L^2\to L^2} \leq C.$$

Returning to (3.9.42) we obtain the first inequality in (3.9.40). The second inequality follows by taking $V = 0$. $\square$

We are now ready for the somewhat involved and computational proof of (3.9.10) in higher dimensions.

Proof of Theorem 3.51 for $n \geq 5$. In the proof we first assume that $f \in C_c^\infty(\mathbb{R} \setminus \{0\})$ and prove (3.9.10) in that case. Let $T_V \in \mathscr{S}'(\mathbb{R})$ denote $T_V(f) = \operatorname{tr}(f(P_V) - f(P_0))$ as defined in Theorem 3.50. This means that

$$T_V|_{\mathbb{R}\setminus\{0\}}(E) = \sum_{E_k<0} \delta_{E_k}(E) + E_+^0\, \partial_E \left[\sigma(\sqrt{E})\right],$$

$$\sigma(\lambda) := \frac{1}{2\pi i} \log\det(I + S(\lambda)), \quad E_+^0 = \begin{cases} 1 & E > 0 \\ 0 & E \leq 0. \end{cases}$$

with $\sigma(\lambda)$ defined up to a constant – see Theorem 3.67 for more on that. Hence

$$T_V - \sum_{E_k<0} \delta_{E_k}(E) + E_+^0\, \partial_E \left[\sigma(\sqrt{E})\right] = \sum_{j=0}^{J} c_j \delta_0^{(j)}(E).$$

We test this against $(E + M)^{-N} e^{-tE}$ (see Remark 3 after Theorem 3.51) and apply Lemma 3.52 to see that $c_0 = m_R(0)$ and $c_j \equiv 0$ for $j > 0$.

1. We first observe that on the right hand side (3.9.10) we have

$$\operatorname{tr} S(\lambda)^{-1}\partial_\lambda S(\lambda) = \operatorname{tr} S(\lambda)^*\partial_\lambda S(\lambda) = \operatorname{tr} S_{\mathrm{abs}}(\lambda)^*\partial_\lambda S_{\mathrm{abs}}(\lambda), \tag{3.9.43}$$

and hence we can work with the absolute scattering matrix given by (3.7.5).

[4] This is the only place that the requirement that M is large is used; we are not interested in optimality of estimates here since the conclusions of the lemma will be strengthened once Theorem 3.51 is proved.

2. We define

$$\tilde{e}(\lambda,\omega,r\theta) = (2\pi)^{-\frac{n-1}{2}} e^{\frac{\pi}{4}(n-1)i} \lambda^{\frac{n-1}{2}} e(\lambda,\omega,r\theta), \tag{3.9.44}$$

with a similar definition for e_0. By (3.8.5) $\lambda \mapsto e(\lambda)$ is holomorphic on $\mathbb{R}$.

We then rewrite (3.6.20) as follows:

$$\tilde{e}_0(\lambda,\omega,r\theta) := e^{i\lambda r} a(r,\theta,\omega) + e^{-i\lambda r}\tilde{a}(r,\omega,\theta),$$

where we suppressed the dependence on λ in a and $\tilde{a}$. The coefficients are distribution valued symbols

$$r^{\frac{n-1}{2}} a, r^{\frac{n-1}{2}}\tilde{a} \in S^0_{\mathrm{phg}}((0,\infty)_r, C^\infty(\mathbb{S}^{n-1}_\omega, \mathcal{D}'(\mathbb{S}^{n-1}_\theta))),$$

where the notation means that we have a full asymptotic expansions in r,

$$a(r,\omega,\theta) \sim r^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} r^{-j} a_j(\theta,\omega), \quad a_j \in C^\infty(\mathbb{S}^{n-1}_\omega, \mathcal{D}'(\mathbb{S}^{n-1}_\theta)),$$

with error bounds described in (3.6.21) and

$$a_0(\omega,\theta) = \delta_{-\omega}(\theta).$$

The difference,

$$\tilde{e}(\lambda,\omega,r\theta) - \tilde{e}_0(\lambda,\omega,r\theta) = e^{i\lambda r} B(r,\omega,\theta),$$

is given by

$$B \in S^{-(n-1)/2}_{\mathrm{phg}}((R_0,\infty)_r, C^\infty(\mathbb{S}^{n-1}_\omega \times \mathbb{S}^{n-1}_\theta)),$$

$$B(r,\omega,\theta) \sim r^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} r^{-j} B_j(\theta,\omega), \quad B_j \in C^\infty(\mathbb{S}^{n-1}_\omega \times \mathbb{S}^{n-1}_\theta)),$$

and

$$B_0(\omega,\theta) = b(\lambda,\omega,\theta), \quad B_0(\theta,\omega) = B_0(\omega,\theta), \tag{3.9.45}$$

where b apears in (3.7.5), the definition of $S_{\mathrm{abs}}(\lambda)$ and the symmetry (3.7.7) was proved after Theorem 3.43. All of the expansions above are *uniform for* $\lambda > \varepsilon > 0$ which is sufficient as we assume that $f \in C^\infty_{\mathrm{c}}((0,\infty))$.

The condition that $r > R_0$ comes from the fact that $x \mapsto \tilde{e}(\lambda,x,\omega)$ may not be smooth for $x \in \operatorname{supp} V$, if V is not smooth. We only use the above expressions asymptotically so this restriction is not important.

3. In the above notation the Schwartz kernel of the absolute scattering matrix (3.7.5) is given by

$$S_{\mathrm{abs}}(\lambda,\theta,\omega) = i^{1-n}(a_0(\theta,\omega) + B_0(\theta,\omega)). \tag{3.9.46}$$

Recalling (3.9.43) and using the symmetry (3.9.45), we have

$$\begin{aligned}&\operatorname{tr} S(\lambda)^{-1}\partial_\lambda S(\lambda)\\ &\quad = \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \big(\bar a_0(\theta,\omega)+\bar B_0(\theta,\omega)\big)\,\partial_\lambda B_0(\theta,\omega)d\theta d\omega,\end{aligned} \tag{3.9.47}$$

where the integral is meant in the sense of distributional pairing in θ.
4. Using (3.8.13), we rewrite the spectral measure in (3.8.4) as

$$dE_\lambda = \widetilde{E}_V(\lambda)^*\widetilde{E}_V(\lambda)\frac{d\lambda}{2\pi}, \quad \lambda>0, \tag{3.9.48}$$

where

$$\widetilde{E}_V(\lambda)f(\theta) := \int_{\mathbb{R}^n}\tilde e(\lambda,\theta,x)f(x)dx, \quad \widetilde{E}_V(\lambda): L^2_{\rm comp}(\mathbb{R}^n)\to L^2(\mathbb{S}^{n-1}).$$

Putting $B_r := B(0,r)$ We note that (3.8.5) gives

$$\mathbb{1}_{B_r}\,\widetilde{E}_V(\lambda)^*\widetilde{E}_V(\lambda)\,\mathbb{1}_{B_r}\in\mathcal{L}_1(L^2(\mathbb{R}^n)). \tag{3.9.49}$$

Applying (3.9.48) we get

$$f(P_V)-\sum_{k=1}^K f(E_k)u_k\otimes\bar u_k = \frac{1}{2\pi}\int_0^\infty f(\lambda^2)\widetilde{E}_V(\lambda)^*\widetilde{E}_V(\lambda)d\lambda,$$

which combined with (3.9.3) gives

$$\begin{aligned}&\operatorname{tr}(f(P_V)-f(P_0))-\sum_{k=1}^K f(E_k)\\ &\quad=\lim_{r\to\infty}\operatorname{tr}\mathbb{1}_{B_r}(f(P_V)-f(P_0))\,\mathbb{1}_{B_r}-\sum_{k=1}^K f(E_k)\\ &\quad=\lim_{r\to\infty}\frac{1}{2\pi}\int_0^\infty f(\lambda^2)\operatorname{tr}\mathbb{1}_{B_r}(\widetilde{E}_V(\lambda)^*\widetilde{E}_V(\lambda)-\widetilde{E}_0(\lambda)^*\widetilde{E}_0(\lambda))\,\mathbb{1}_{B_r}\,d\lambda.\end{aligned}$$

Using (3.9.49) and (B.4.12) (the example at the end of §B.4)

$$\operatorname{tr}\mathbb{1}_{B_r}\,\widetilde{E}_V(\lambda)^*\widetilde{E}_V(\lambda)\,\mathbb{1}_{B_r} = \int_{B_r}\int_{\mathbb{S}^{n-1}}|\tilde e(\lambda,x,\omega)|^2 d\omega dx.$$

We conclude that

$$\begin{aligned}&\operatorname{tr}(f(P_V)-f(P_0))-\sum_{k=1}^K f(E_k)\\ &\quad=\lim_{r\to\infty}\frac12\frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda^2)d\lambda\int_{\mathbb{S}^{n-1}}d\omega\int_{B_r}dx(|\tilde e|^2-|\tilde e_0|^2).\end{aligned} \tag{3.9.50}$$

(Of course $|\tilde e_0|^2=(\lambda/2\pi)^{n-1}$ but it is useful to keep it as is.)

5. We now use the *Maaß–Selberg method* for converting the integral in $x \in B_r$ to an integral over ∂B_r. It is based on the following simple identity:

$$(P_V - \lambda^2)\partial_\lambda \tilde{e} = 2\lambda \tilde{e},$$

with an analogue valid for P_0 and $\tilde{e}_0$.

Hence, using the fact that $\lambda \neq 0$ is real, we use Green's formula and put $D = (1/i)\partial$:

$$\begin{aligned}\int_{B_r} |\tilde{e}|^2 dx &= \frac{1}{2\lambda}\int_{B_r} (P_V - \lambda^2)\partial_\lambda \tilde{e}\bar{\tilde{e}} dx \\ &= \frac{1}{2\lambda}\int_{B_r} \left((P_V - \lambda^2)\partial_\lambda \tilde{e}\bar{\tilde{e}} - \partial_\lambda \tilde{e}(P_V - \lambda^2)\bar{\tilde{e}}\right) dx \\ &= \frac{1}{2\lambda}\int_{B_r} \left(-\Delta \partial_\lambda \tilde{e}\,\bar{\tilde{e}} + \partial_\lambda \tilde{e}\Delta \bar{\tilde{e}}\right) dx \\ &= \frac{1}{2\lambda}\int_{\mathbb{S}^{n-1}} \left((D_r D_\lambda e)\bar{\tilde{e}} + D_\lambda \tilde{e}\overline{D_r \tilde{e}}\right) r^{n-1} d\theta, \quad x = r\theta.\end{aligned} \tag{3.9.51}$$

6. Before inserting (3.9.51) into (3.9.50) we make three observations:

A) if we write $\widetilde{e} = \widetilde{e}_0 + \widetilde{w}$ then, applying (3.9.51) in (3.9.50) shows that no terms that are quadratic in $\widetilde{e}_0$ remain;

B) since $\widetilde{w}(r\theta, \omega)$ is smooth in θ, products of terms derived from $\widetilde{w}$ and $\widetilde{e}_0$ can be expressed using the expansions in Step 2, with the integrals of products understood as distributional pairings;

C) all terms with factors of $e^{\pm 2i\lambda r}$ vanish in the $r \to \infty$ limit due to the integration against $f(\lambda^2) \in C^\infty_{\rm c}(\mathbb{R}_\lambda)$; this means the only distributional pairings come from terms involving a and B.

D) Noting that

$$D_r e^{i\lambda r} = e^{i\lambda r}(D_r + \lambda), \quad D_\lambda e^{i\lambda r} = e^{i\lambda r}(D_\lambda + r),$$

the observations made in Step 6 and the expansions in Step 4 show that

$$\begin{aligned}&\operatorname{tr}(f(P_V) - f(P_0)) - \sum_{k=1}^{K} f(E_k) \\ &\qquad = \lim_{r\to\infty} \int_{\mathbb{R}} f(\lambda^2)\frac{d\lambda}{4\pi\lambda}\int_{\mathbb{S}^{n-1}} d\omega \int_{\mathbb{S}^{n-1}} d\theta\, r^{n-1}\, C(r),\end{aligned} \tag{3.9.52}$$

where

$$C(r) = C(r, \theta, \omega) \in S^{-n+2}_{\rm phg}((0,\infty)_r; \mathcal{D}'(\mathbb{S}^{n-1}_\theta \times \mathbb{S}^{n-1}_\omega)),$$

is given by

$$\begin{aligned}C(r) := {} & (D_r+\lambda)(D_\lambda+r)a\bar{B} + (D_r+\lambda)(D_\lambda+r)B\bar{a} \\ & + (D_r+\lambda)(D_\lambda+r)B\bar{B} + (D_\lambda+r)a\overline{(D_r+\lambda)B} \\ & + (D_\lambda+r)B\overline{(D_r+\lambda)a} + (D_\lambda+r)B\overline{(D_r+\lambda)B}.\end{aligned}$$

We note that differentiation in r decreases the order in r and differentiation in λ preserves it. We now group terms according to their order in r: all terms of size $\mathcal{O}(r^{-n})$ will disappear in the limit (remember the r^{n-1} factor in (3.9.52)). Hence,

$$\begin{aligned}C(r) &= 2\lambda ra\bar{B} + 2\lambda rB\bar{a} + 2\lambda r|B|^2 + D_r(ra)\bar{B} + 2\lambda D_\lambda a\bar{B} \\ &\quad + D_r(rB)\bar{a} + 2\lambda D_\lambda B\bar{a} + D_r(rB)\bar{B} + \lambda D_\lambda B\bar{B} \\ &\quad + ar\overline{D_rB} + Br\overline{D_ra} + \lambda D_\lambda B\bar{B} + Br\overline{D_rB} + \mathcal{O}(r^{-n}) \\ &= 2\operatorname{Re}\left(rD_ra\bar{B} + rD_rB\bar{B} + rD_rB\bar{a} + \lambda D_\lambda ar\overline{D_rB}\right) \\ &\quad + 2r\lambda\operatorname{Re}\left(2a\bar{B} + |B|^2\right) - i\operatorname{Re}\left(2a\bar{B} + |B|^2\right) + 2\lambda D_\lambda a\bar{B} + 2\lambda D_\lambda B\bar{a} \\ &\quad + 2\lambda D_\lambda B\bar{B} + \mathcal{O}(r^{-n}),\end{aligned}$$

where $\mathcal{O}(r^{-n})$ is meant in the sense of distributional expansion in Step 4.

8. The coefficient of r^{-n+2} in the expansion of $C(r)$ is given by

$$2\lambda\operatorname{Re}(2a_0(\theta,\omega)\bar{B}_0(\theta,\omega) + |B_0(\theta,\omega)|^2).$$

We claim that the integral of this term with respect to ω (or θ) is equal to 0. In fact, unitarity of the scattering matrix $S_{\mathrm{abs}}(\lambda)$ and symmetry of b imply that

$$\int_{\mathbb{S}^{n-1}} (\delta_{-\omega}(\theta) + b(\theta,\omega))(\delta_{-\gamma}(\omega) + \overline{b(\omega,\gamma)})d\omega = \delta_\gamma(\theta),$$

which means that

$$b(\theta,-\gamma) + \bar{b}(\gamma,-\theta) + \int_{\mathbb{S}^{n-1}} b(\theta,\omega)\overline{b(\gamma,\omega)}d\omega \equiv 0. \tag{3.9.53}$$

On the other hand, putting $\gamma=\theta$ in (3.9.53),

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} \left(2\operatorname{Re}(a_0(\theta,\omega)\bar{B}_0(\theta,\omega)) + |B_0(\theta,\omega)|^2\right) d\omega \\ &\qquad = 2\operatorname{Re} b(\theta,-\theta) + \int_{\mathbb{S}^{n-1}} |b(\theta,\omega)|^2 d\omega \equiv 0.\end{aligned}$$

9. To compute the coefficient of r^{-n+1} we note that $D_\lambda a_0 = 0$ and that one of the terms vanishes due to the unitarity of S_{abs}. Another satisfies

$$\begin{aligned}&2\operatorname{Re}\left(rD_ra\bar{B} + rD_rB\bar{B} + rD_rB\bar{a} + \lambda D_\lambda ar\overline{D_rB}\right) \\ &\quad = 2\operatorname{Re}\left(\tfrac{n-1}{2}i(a_0\bar{B}_0 + \bar{a}_0B_0 + |B_0|^2)\right)r^{-n+1} + \mathcal{O}(r^{-n}) = \mathcal{O}(r^{-n}).\end{aligned}$$

Hence we are left with

$$4\lambda\,\mathrm{Re}(a_0\bar{B}_1 + a_1\bar{B}_0 + B_1\bar{B}_0) + \tfrac{2\lambda}{i}\partial_\lambda B_0(\bar{a}_0 + \bar{B}_0).$$

In view of (3.9.47) and (3.9.52) the second term is exactly what appears in (3.9.10). Hence it remains to show that the first vanishes. For that we use (3.7.12) to see that

$$a_1 = \frac{1}{2i\lambda}\left(-\Delta_{\mathbb{S}^{n-1}} + b_n\right) a_0, \quad B_1 = \frac{1}{2i\lambda}\left(-\Delta_{\mathbb{S}^{n-1}} + b_n\right) B_0.$$

where $b_n := \frac{(n-1)(n-3)}{4}$ is real. This means that (in the sense of distibutional pairing),

$$\begin{aligned} &4\lambda\,\mathrm{Re}\int_{\mathbb{S}^{n-1}}(a_0\bar{B}_1 + a_1\bar{B}_0 + B_0\bar{B}_1)d\theta \\ &\quad= 2\,\mathrm{Re}\,i\int_{\mathbb{S}^{n-1}}\left(-a_0\Delta_{\mathbb{S}^{n-1}}\bar{B}_0 + \Delta_{\mathbb{S}^{n-1}}a_0\bar{B}_0 + \Delta_{\mathbb{S}^{n-1}}B_0\bar{B}_0 - b_n|B_0|^2\right)d\theta \\ &\quad= 2\,\mathrm{Re}\,i\int_{\mathbb{S}^{n-1}}\left(-\Delta_{\mathbb{S}^{n-1}}a_0\bar{B}_0 + \Delta_{\mathbb{S}^{n-1}}a_0\bar{B}_0 - |\nabla B_0|^2\right)d\theta \\ &\quad= -2\,\mathrm{Re}\,i\int_{\mathbb{S}^{n-1}}|\nabla B_0|^2 d\theta = 0. \end{aligned}$$

And this completes the proof. $\square$

3.10. THE MELROSE TRACE FORMULA

The next theorem is the odd dimensional generalization of Theorem 2.21 and it connects resonances with the trace of the wave group. We first observe that for we can define the distribution

$$\varphi \longmapsto \sum_{\lambda\in\mathbb{C}} m_R(\lambda)\int_{\mathbb{R}} t^n\varphi(t)e^{-i\lambda|t|}dt\,, \quad \varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}).$$

To see this, suppose that $\operatorname{supp}\varphi \subset [-R,R]$. With $(n+1)$ integrations by parts based on $(i/\lambda)\partial_t e^{-i\lambda t} = e^{-i\lambda t}$ we see

$$\left|\int_0^\infty t^n\varphi(\pm t)e^{-i\lambda t}dt\right| \le CR(1+|\lambda|)^{-(n+1)}e^{R(\operatorname{Im}\lambda)_+}\sup_{0\le k\le n+1}|\varphi^{(k)}|.$$

Let

$$N(r) := \sum\{m_R(\lambda) : 0 < |\lambda| \le r\}, \tag{3.10.1}$$

so that by Theorem 3.27 we have $N(r) \le C_V r^n$. Since there are only finitely many λ's with $\operatorname{Im}\lambda > 0$,

$$\begin{aligned}\Big| \sum_{\lambda\in\mathbb{C}} & m_R(\lambda) \int_0^\infty t^n (\varphi(t)+\varphi(-t)) e^{-i\lambda t} dt \Big| \\ &\le C \sup_{0\le k\le n+1} |\varphi^{(k)}| \sum_{\lambda\neq 0} m_R(\lambda)\langle\lambda\rangle^{-n-1} \\ &\le C \sup_{0\le k\le n+1} |\varphi^{(k)}| \left(C + \int_1^\infty r^{-n-1} dN(r) \right) \\ &\le C \sup_{0\le k\le n+1} |\varphi^{(k)}| \left(C + C\int_1^\infty r^{-2} dr \right) \le C \sup_{0\le k\le N} |\varphi^{(k)}| \,,\end{aligned}$$

where the constants depend on R and V.

We now present a trace formula in which resonances appear in an almost the same way as eigenvalues:

THEOREM 3.53 (**Trace formula for resonances**). *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$*, where* $n \ge 3$ *is odd. Then,*

$$2\, t^n \operatorname{tr}\left(\cos t\sqrt{P}_V - \cos t\sqrt{P}_0\right) = t^n \sum_{\lambda\in\mathbb{C}} m_T(\lambda) e^{-i\lambda|t|}\,, \tag{3.10.2}$$

in the sense of distributions on $\mathbb{R}$ *and where (in the notation of* (3.2.5) *and* (3.3.29)*)*

$$m_T(\lambda) := \begin{cases} m_R(\lambda) & \lambda\neq 0 \\ 2m_R(0) - \widetilde{m}_R(0) & \lambda = 0. \end{cases} \tag{3.10.3}$$

REMARKS. 1. As explained after Theorem 2.21 in the case of $-\Delta+V$ on a bounded domain in $\mathbb{R}^n$ (with, say, Dirichlet boundary conditions) this result is an immediate consequence of the spectral theorem for self-adjoint operators with discrete spectra. It is quite remarkable that the same theorem holds (in odd dimensions, and for compactly supported perturbations) in an exactly the same form for resonances. The formula remains valid for any "black box" (see §4.1) compactly supported perturbations in odd dimensions [**SZ94**],[**Zw97**].

2. A power of t in (3.10.2) is needed as there are many possible extensions of the distribution $\sum_{\lambda\in\mathbb{C}} m_T(\lambda)\exp(-i|t|\lambda)$ from $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}$.

3. The trace formula (3.10.2) is a consequence of the Birman–Kreĭn formula and of the Hadamard factorization the scattering determinant, $\det S(\lambda)$, as a meromorphic function – see Theorem 3.54 below. That was not the original proof – see §3.13. In applications it is sometimes easier to use the factorization of $\det S(\lambda)$ directly, see §3.12.

THEOREM 3.54 (Factorization of the scattering matrix). *Suppose that $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$ where $n \geq 1$ is odd. Then*

$$\det S(\lambda) = (-1)^m e^{g(\lambda)} \frac{P(-\lambda)}{P(\lambda)} \,,$$
$$P(\lambda) := \prod_{\mu \neq 0} E_n(\lambda/\mu)^{m_R(\mu)} \,, \quad E_n(z) := (1-z)e^{z+z^2/2+\cdots+z^n/n} \,, \tag{3.10.4}$$
$$g(\lambda) = a_n \lambda^n + a_{n-2}\lambda^{n-2} + \cdots + a_1 \lambda$$

When $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{R})$ then

$$P(-\lambda) = \overline{P(\bar{\lambda})}, \quad a_n \in i\mathbb{R}, \quad m = \widetilde{m}_R(0), \tag{3.10.5}$$

where $\widetilde{m}_R(0)$ is defined by (3.3.29).

Proof. 1. Theorem 3.45 already shows that (3.10.4) holds with g being an entire function. From (3.7.17) we see that g has to be odd. Hence all we need to show is that g is a polynomial of degree at most n.

2. We first establish two preliminary bounds

$$|\det S(\lambda)| \leq \begin{cases} C \exp C|\lambda|^n \,, & \operatorname{Im} \lambda \geq 0 \,, |\lambda| > C \,, \\ C \exp C|\lambda|^{n^2} \,, & \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle \mu \rangle^{-n-\varepsilon}) \,, \end{cases} \tag{3.10.6}$$

where $m_H(R)$ was defined before (3.4.5). In view of (3.7.30) this will follow from estimates on the characteristic values of the operator $T(\lambda)$ appearing there, in the spirit of the proof of Theorem 3.27.

3. To apply estimates on characteristic values we use (3.4.11) to write

$$T(\lambda) = c_n (I + VR_0(\lambda)\rho)^{-1} V E_\rho(\bar{\lambda})^* E_\rho(\lambda) \,.$$

From (3.1.12) we see that

$$\|(I + VR_0(\lambda)\rho)^{-1}\| \leq C \,, \quad \operatorname{Im} \lambda \geq 0 \,, \quad |\lambda| \geq C \,.$$

Hence in the same range of λ's we have

$$s_j(T(\lambda)) \leq C|\lambda|^{n-2} \|E_\rho(\bar{\lambda})^*\| s_j(E_\rho(\lambda)) \,.$$

Applying (3.4.14) we obtain (with a different constant C)

$$s_j(T(\lambda)) \leq C \exp\left(C|\lambda| - j^{\frac{1}{n-1}}/C \right) \,, \quad \operatorname{Im} \lambda \geq 0 \,, \quad |\lambda| \geq C \,.$$

The Weyl inequality can now be applied as in part 5 of the proof of Theorem 3.27 and that gives

$$
\begin{aligned}
|\det(I - T(\lambda))| &\leq \prod_{j=0}^{\infty}(1 + s_j(T(\lambda))) \\
&\leq \prod_{j \leq C|\lambda|^{n-1}} (1 + e^{C|\lambda|}) \exp \sum_{j \geq 1} e^{-j^{1/n-1}/C} \\
&\leq C \exp C|\lambda|^n , \quad \operatorname{Im}\lambda \geq 0 \quad |\lambda| \geq C.
\end{aligned}
$$

In view of (3.7.30) this proves the first part of (3.10.6).

4. We now consider the case of λ outside of a union of discs containing resonances. First we note that for any $\varepsilon > 0$, there exists a sequence $r_k \to \infty$, such that

$$
\forall\, k , \quad \partial D(0, r_k) \cap \bigcup_{m_H(\mu)>0} D(\mu, \langle \mu \rangle^{-n-\varepsilon}) = \emptyset , \tag{3.10.7}
$$

which follows from Theorem 3.28 as it implies that

$$
\sum_{\mu \in \mathbb{C}} m_H(\mu) \langle \mu \rangle^{-n-\varepsilon} < \infty.
$$

(This estimate shows that the sum of radii of the discs in (3.10.7) is finite; if $\partial D(0, r)$ intersected at least one of the discs for all $r > r_0$, the sum of radii would have to be infinite.)

To estimate $\|(I + VR_0(\lambda)\rho)^{-1}\|$ away from resonances we use (B.5.21) and (3.4.6) to obtain

$$
\|(I + VR_0(\lambda)\rho)^{-1}\| \leq \frac{G_0(\lambda) G_1(\lambda)}{|H(\lambda)|} ,
$$

where

$$
G_0(\lambda) := \sum_{k=0}^{n} \|VR_0(\lambda)\rho\|_{L^2 \to L^2}^k , \quad G_1(\lambda) := \prod_{j=0}^{\infty} (1 + s_j(VR_0(\lambda))^{n+1}) ,
$$
$$
H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}) .
$$

Theorem 3.27 shows that $H(\lambda)$ is an entire function of order n, and its proof shows that

$$
G_1(\lambda) \leq C \exp(C|\lambda|^n) ,
$$

while $G_0(\lambda) \leq Ce^{C|\lambda|}$.

The minimum modulus theorem for entire functions of order n (see (D.2.6)) shows that

$$|H(\lambda)| \geq \exp(-C_\varepsilon |\lambda|^{n+\varepsilon}) , \quad \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle \mu \rangle^{-n-\varepsilon}) .$$

Hence for λ's in the same set we obtain

$$\|(I + VR_0(\lambda)\rho)^{-1}\| \leq C \exp(C|\lambda|^{n+\varepsilon}) .$$

Returning to singular values of $T(\lambda)$ this gives

$$s_j(T(\lambda)) \leq C \exp\left(C|\lambda|^{n+\varepsilon} - j^{\frac{1}{n-1}}/C \right) , \quad \lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle \mu \rangle^{-n-\varepsilon}) .$$

The same argument as before proves the second part of (3.10.6).

5. We now recall the estimates on Weierstrass products (see (D.2.5) and (D.2.6) in §D.2):

$$e^{-C_\varepsilon |\lambda|^{n+\varepsilon}} \leq |P(\pm\lambda)| \leq e^{C_\varepsilon |\lambda|^{n+\varepsilon}} , \quad \pm\lambda \notin \bigcup_{m_H(\mu)>0} D(\mu, \langle \mu \rangle^{-n-\varepsilon}) .$$

Hence in the same set of λ's we have

$$\begin{aligned} |\exp(g(\lambda))| &= |\det S(\lambda)| \frac{|P(\lambda)|}{|P(-\lambda)|} \\ &\leq C \exp(C|\lambda|^{n^2} + C|\lambda|^{n+\varepsilon}) \\ &\leq C \exp C|\lambda|^{n^2} . \end{aligned} \tag{3.10.8}$$

We use this estimate on circles of radius r_k satisfying (3.10.7). The maximum principle then shows that the above estimate holds everywhere. Hence,

$$\operatorname{Re} g(\lambda) \leq C|\lambda|^{n^2},$$

and an application of the Borel-Carathéodory inequality (D.1.6) gives

$$|g(\lambda)| \leq C|\lambda|^{n^2} ,$$

which implies that g is a polynomial.

6. To see that $g(\lambda)$ is a polynomial of degree n we apply the same strategy as in the proof of (3.10.8) but for $\operatorname{Im} \lambda \geq 0$, $|\lambda| \geq C$. This way we can use the first estimate in (3.10.6). That gives

$$\operatorname{Re} g(\lambda) \leq C_\varepsilon |\lambda|^{n+\varepsilon} , \quad \operatorname{Im} \lambda \geq 0 , \quad |\lambda| \geq C .$$

For $n \geq 1$ any polynomial satisfying this bound has to have degree at most n.

7. The statement (3.10.5) about $P(\lambda)$ and the polynomial g when V is real valued comes from the unitarity of the scattering matrix. For the factor $(-1)^m$ see (3.7.28) and Exercise 3.9. $\square$

Before proving Theorem 3.53 we need a bound on $\log\det S(\lambda)$ for $\lambda \in \mathbb{R}$. A much more precise result will be presented in Theorem 3.67 but the point is that the lemma depends *only* on the upper bound on the counting function in Theorem 3.27 and the factorization of the scattering matrix in Theorem 3.54. Hence, it can be used in more general situations.

LEMMA 3.55. *Suppose that* $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$ *and* $n \geq 1$ *is odd. Then for any* $\varepsilon > 0$ *there exists* C_ε *such that*

$$|\log\det S(\lambda)| \leq C_\varepsilon \langle\lambda\rangle^{n+\varepsilon}, \quad \lambda \in \mathbb{R}. \tag{3.10.9}$$

Proof. 1. Let

$$s(\lambda) := -i\log\det S(\lambda), \quad \lambda \in \mathbb{R}, \quad s(0) = 0.$$

(We do not divide by 2π in order to simplify the subsequent formulas.) In view of (3.10.5),

$$\det S(\lambda) = (-1)^m e^{g(\lambda)}\frac{\overline{P(\bar\lambda)}}{P(\lambda)}.$$

Since for $\lambda \in \mathbb{R}$ and $\mu \neq 0$,

$$\partial_\lambda\left(\log\left(1-\frac{\lambda}{\bar\mu}\right) - \log\left(1-\frac{\lambda}{\mu}\right)\right) = -\frac{2i\operatorname{Im}\mu}{|\lambda-\mu|^2},$$

we have

$$\partial_\lambda^{n+1}s(\lambda) = -2\sum_{\mu\neq 0} m_R(\mu)\,\partial_\lambda^n\frac{\operatorname{Im}\mu}{|\lambda-\mu|^2}, \tag{3.10.10}$$

where the sum converges uniformly for λ in compact sets. That is guaranteed by Theorem 3.27 as, with $N(r)$ defined by (3.10.1),

$$\sum_{\mu\neq 0} m_R(\mu)|\mu|^{-n-1} = C + \int_0^\infty r^{-n-1}dN(r) < \infty. \tag{3.10.11}$$

2. Choose $\zeta \in C^\infty_{\rm c}((-3,3);[0,1])$ equal to 1 on $[-2,2]$ and define $s_1(\lambda)$ by

$$s_1'(\lambda) := -2\sum_{\mu\neq 0} m_R(\mu)\zeta\left(\frac{|\mu|}{\lambda}\right)\frac{\operatorname{Im}\mu}{|\lambda-\mu|^2}, \quad s_1(0) = 0. \tag{3.10.12}$$

The upper bound on the number of resonances (3.4.7) and the fact that $\int_{\mathbb{R}}|\operatorname{Im}\mu|/|\lambda-\mu|^2 d\lambda = \pi$, show

$$|s_1(\lambda)| \leq C\langle\lambda\rangle^n. \tag{3.10.13}$$

3. We also define

$$s_2(\lambda) := s(\lambda) - s_1(\lambda), \tag{3.10.14}$$

and use (3.10.10) and (3.10.12) to obtain

$$\begin{aligned}\partial_\lambda^{n+1} s_2(\lambda) &= -2 \sum_{\mu \neq 0} m_R(\mu) \partial_\lambda^n \left[\left(1 - \zeta \left(\frac{|\mu|}{\lambda} \right) \right) \frac{\operatorname{Im} \mu}{|\lambda - \mu|^2} \right] \\ &= A + B\end{aligned}$$

where

$$A := -2 \sum_{\mu \neq 0} m_R(\mu) \left(1 - \zeta \left(\frac{|\mu|}{\lambda} \right) \right) \partial_\lambda^n \frac{\operatorname{Im} \mu}{|\lambda - \mu|^2}$$

and

$$B := 2 \sum_{\mu \neq 0} m_R(\mu) \sum_{k=1}^{n} \binom{n}{k} \partial_\lambda^k \left[\zeta \left(\frac{|\mu|}{\lambda} \right) \right] \partial_\lambda^{n-k} \left(\frac{\operatorname{Im} \mu}{|\lambda - \mu|^2} \right).$$

The sum in the definition of B is finite as the support of ζ' restricts μ's to $2|\lambda| \leq |\mu| \leq 3|\lambda|$. Moreover,

$$\partial_\lambda^k \left[\zeta \left(|\mu|/\lambda \right) \right] = \mathcal{O}_k \left(\langle \lambda \rangle^{-k} \sup_{1 \leq \ell \leq k} |r^\ell \zeta^{(\ell)}(r)| \right) = \mathcal{O}(|\lambda|^{-k}).$$

We also have

$$\begin{aligned}\partial_\lambda^{n-k} \left(\frac{2i \operatorname{Im} \mu}{|\lambda - \mu|^2} \right) &= \partial_\lambda^{n-k} \left((\lambda - \bar{\mu})^{-1} - (\lambda - \mu)^{-1} \right) \\ &= \mathcal{O}(|\lambda - \mu|^{-n+k-1}) \\ &= \mathcal{O}(|\mu|^{-n+k-1}), \quad |\mu| > 2|\lambda|.\end{aligned}$$

Hence, using (3.4.7),

$$|B| \leq C \sum_{2|\lambda| \leq |\mu| \leq 3|\lambda|} m_R(\mu) |\lambda|^{-k} |\mu|^{-n+k-1} = \mathcal{O}(|\lambda|^{-1}).$$

To estimate A we also use (3.4.7):

$$|A| \leq C \sum_{|\mu| \geq 2|\lambda|} m_R(\mu) |\mu|^{-n-1} = C \int_{2|\lambda|}^{\infty} r^{-n-1} dN(r) = \mathcal{O}(|\lambda|^{-1}).$$

We conclude that

$$|\partial_\lambda^{n+1} s_2(\lambda)| = \mathcal{O}(\langle \lambda \rangle^{-1}) \implies s_2(\lambda) = \mathcal{O}_\varepsilon(\langle \lambda \rangle^{n+\varepsilon}).$$

Combined with (3.10.13) this concludes the proof. $\square$

Proof of Theorem 3.53. 1. Let $\varphi := t^n \psi$, $\psi \in C_c^\infty(\mathbb{R})$. In the distributional sense,

$$(2 \bullet^n \cos \bullet \sqrt{P_V})(\psi) = f(P_V), \quad f(z) := \widehat{\varphi}(\sqrt{z}) + \widehat{\varphi}(-\sqrt{z}),$$

where $f \in C^\infty(\mathbb{R}) \cap \mathscr{S}((0,\infty))$. This means that (3.10.2) is equivalent to

$$\operatorname{tr}\left(f(P_V) - f(P_0)\right) = \sum_{\lambda \in \mathbb{C}} m_T(\lambda) \int_0^\infty (\varphi(t) + \varphi(-t)) e^{-i\lambda t} dt, \tag{3.10.15}$$

where m_T is defined in (3.10.3).

We define $\sigma(\lambda) := \log\det S(\lambda)/2\pi i$, noting that Lemma 3.55 shows that σ, and hence σ', are elements of $\mathscr{S}'(\mathbb{R})$. Since $\sigma'(\lambda)$ is even (see (3.7.17)), Theorem 3.51 shows that

$$\operatorname{tr}\left(f(P_V) - f(P_0)\right) = \tfrac{1}{2}\int_{-\infty}^\infty f(\lambda^2)\sigma'(\lambda) d\lambda + \sum_{k=1}^K f(E_k) + \tfrac{1}{2}\widetilde{m}_R(0) f(0),$$

where the integral is understood as a pairing of $f(\lambda^2) \in \mathscr{S}(\mathbb{R}_\lambda)$ and $\sigma' \in \mathscr{S}'(\mathbb{R})$. We claim that

$$\int_{-\infty}^\infty f(\lambda^2)\sigma'(\lambda) d\lambda = \lim_{r\to\infty} \int_{-r}^r f(\lambda^2)\sigma'(\lambda) d\lambda.$$

In fact, using (3.10.9) and the distributional definition of $\sigma'(\lambda)$, we have

$$\begin{aligned}\int_{\mathbb{R}} f(\lambda^2)\sigma'(\lambda) d\lambda &= -\int_{\mathbb{R}} \sigma(\lambda) f'(\lambda^2) 2\lambda d\lambda = -\lim_{r\to\infty}\int_{-r}^r \sigma(\lambda) f'(\lambda^2) 2\lambda d\lambda \\ &= \lim_{r\to\infty}\left(\int_{-r}^r f(\lambda^2)\sigma'(\lambda) d\lambda + 2f(r^2)\sigma(r)\right) \\ &= \lim_{r\to\infty}\int_{-r}^r f(\lambda^2)\sigma'(\lambda) d\lambda.\end{aligned}$$

Hence the proof of the theorem is reduced to showing that for some $r_j \to \infty$,

$$\begin{aligned}\frac{1}{2\pi i}\lim_{j\to\infty}\int_{-r_j}^{r_j} \widehat{\varphi}(\lambda)\partial_\lambda(\log\det S(\lambda)) d\lambda + (2m_R(0) - \widetilde{m}_R(0))\widehat{\varphi}(0) \\ + \sum_{\operatorname{Im}\mu > 0} m_R(\mu)\left(\widehat{\varphi}(\mu) + \widehat{\varphi}(-\mu)\right) \\ = \sum_{\mu\in\mathbb{C}} m_T(\mu)\int_0^\infty (\varphi(t) + \varphi(-t)) e^{-i\mu t} dt.\end{aligned} \tag{3.10.16}$$

2. We define $\varphi_\pm(t) := t_\pm^n \psi(t) \in C_{\mathrm{c}}^{n-1}(\mathbb{R})$, so that $\varphi = \varphi_+ - \varphi_-$. Integration by parts shows that

$$|\partial_\lambda^\ell \widehat{\varphi}_\pm(\lambda)| \le C(1 + |\lambda|)^{-n-1-\ell} e^{C(\operatorname{Im}\lambda)_\pm}. \tag{3.10.17}$$

It is enough to prove (3.10.16) for φ replaced by $\varphi_\pm$ and we will present the $+$ case, the $-$ case being analogous.

3. In the notation of (3.10.4),

$$\partial_z \log E_n(z) = -(1-z)^{-1} + 1 + \cdots + z^{n-1} = -z^n(1-z)^{-1}.$$

Hence the factorization in (3.10.4) gives

$$\begin{aligned}\partial_\lambda(\log\det S(\lambda)) &= g'(\lambda) + \sum_{\mu\neq 0} m_R(\mu)\partial_\lambda\left(\log E_n(-\lambda/\mu) - \log E_n(\lambda/\mu)\right)\\ &= g'(\lambda) + \sum_{\mu\neq 0} m_R(\mu)\left(\frac{\lambda}{\mu}\right)^n ((\lambda+\mu)^{-1} - (\lambda-\mu)^{-1}).\end{aligned}$$

Since $m_R(\mu) = 0$ for $\mu \in \mathbb{R}\setminus\{0\}$ the sum converges uniformly in $\lambda \in [-r, r]$. Also, if $\psi_+(t) := i^n t_+^0 \psi$ then $\widehat{\varphi}_+(\lambda) = \partial_\lambda^n \widehat{\psi}_+$ and

$$\lim_{r\to\infty}\int_{-r}^{r} \widehat{\varphi}_+(\lambda) g'(\lambda) d\lambda = \lim_{r\to\infty}\int_{-r}^{r} \partial_\lambda^n \widehat{\psi}_+(\lambda) g'(\lambda) d\lambda = 0.$$

(The polynomial g' has degree at most $n-1$ and $\partial_\lambda^\ell \widehat{\psi}_+(\lambda) = \mathcal{O}(\langle\lambda\rangle^{-1-\ell})$ for $0 \le \ell \le n$.) Hence,

$$\lim_{r\to\infty}\int_{-r}^{r} \widehat{\varphi}_+(\lambda)\partial_\lambda(\log\det S(\lambda)) d\lambda = \lim_{r\to\infty}\sum_{\mu\neq 0} m_R(\mu)\Phi(\mu, r), \tag{3.10.18}$$

where

$$\Phi(\mu, r) := \int_{-r}^{r} \widehat{\varphi}_+(\lambda)\,(\lambda/\mu)^n\,((\lambda+\mu)^{-1} - (\lambda-\mu)^{-1}) d\lambda.$$

4. We now claim that there exists a sequence $r_j \to \infty$ such that for all $\mu \neq 0$ satisfying $m_R(\mu) > 0$ and $\mp \operatorname{Im}\mu > 0$

$$\frac{1}{2\pi i}\Phi(\mu, r_j) = \begin{cases} \pm\widehat{\varphi}_+(\pm\mu) + \mathcal{O}(|\mu|^{-n-\frac12} r_j^{-1/2}\log r_j), & |\mu| < r_j, \\ \mathcal{O}(|\mu|^{-n-\frac12} r_j^{-1/2}\log r_j), & |\mu| > r_j \end{cases}. \tag{3.10.19}$$

Let

$$\Gamma_r = \partial D(0, r) \cap \{\operatorname{Im}\lambda \le 0\},$$

be oriented counterclockwise. If $r \neq |\mu|$ then the residue theorem shows that for $\pm \operatorname{Im}\mu < 0$,

$$\begin{aligned}\Phi(\mu, r) &= \pm 2\pi i \widehat{\varphi}_+(\pm\mu)(r - |\mu|)_+^0 \\ &\quad + \int_{\Gamma_r} \widehat{\varphi}_+(\lambda)\,(\lambda/\mu)^n\,((\lambda+\mu)^{-1} - (\lambda-\mu)^{-1}) d\lambda,\end{aligned}$$

and we need to estimate the last term.

We first note that if $|\mu| < r/2$ or $|\mu| > 2r$ then

$$\int_{\Gamma_r} \widehat{\varphi}_+(\lambda)\,(\lambda/\mu)^n\,(\lambda \pm \mu)^{-1} d\lambda = \mathcal{O}(|\mu|^{-n-\frac12} r^{-\frac12}). \tag{3.10.20}$$

In fact, using (3.10.17) we see that for $\lambda \in \Gamma_r$

$$\begin{aligned}\widehat{\varphi}_+(\lambda)\,(\lambda/\mu)^n\,(\lambda \pm \mu)^{-1} &= \mathcal{O}(|r/\mu|^n r^{-n-1})\min(|\mu|^{-1}, r^{-1}) \\ &= \mathcal{O}(|\mu|^{-n-\frac12} r^{-\frac32}).\end{aligned}$$

Since the length of Γ_r is πr, (3.10.20) follows.

5. For $r/2 < \mu < 2r$ we will use the following

LEMMA 3.56. *Suppose that $r > 1$, h is holomorphic in a neighbourhood of Γ_r and $\mu \notin \Gamma_r$. Then*

$$\int_{\Gamma_r} h(\lambda)(\lambda-\mu)^{-1}d\lambda = \mathcal{O}\left(\langle \log d(\mu,\Gamma_r)\rangle + \log r\right) \max_{\Gamma_r}(|h| + r|h'|), \tag{3.10.21}$$

where $d(\mu,\Gamma_r) = \min_{\lambda\in\Gamma_r}|\lambda-\mu|$.

Proof. We define $\log(\lambda-\mu)$ for $\lambda \in \mathbb{C}\setminus(\mu+i[0,\infty))$ if $|\mu| < r$ or $\operatorname{Im}\mu > 0$ and for $\lambda \in \mathbb{C}\setminus(\mu - i[0,\infty))$ otherwise. In particular, $\log(\lambda-\mu)$ is well-defined and holomorphic on a neighbourhood of Γ_r. Then,

$$\begin{aligned}
\int_{\Gamma_r} h(\lambda)(\lambda-\mu)^{-1}d\lambda &= \int_{\Gamma_r} h(\lambda)\partial_\lambda \log(\lambda-\mu)d\lambda \\
&= h(r)\log(r-\mu) - h(-r)\log(-r-\mu) - \int_{\Gamma_r} h'(\lambda)\log(\lambda-\mu)d\lambda \\
&= \mathcal{O}(|\log|r-\mu|| + |\log|r+\mu|| + 4\pi)\max_{\Gamma_r}|h| \\
&\qquad + \mathcal{O}(\max_{\lambda\in\Gamma_r}|\log(\lambda-\mu)|) r \max_{r\in\Gamma_r}|h'|.
\end{aligned}$$

Since

$$\max_{\lambda\in\Gamma_r}|\log(\lambda-\mu)| \le 2\pi + |\log d(\mu,\Gamma_r)| + \log(2r)$$

and

$$|\log|\mu \pm r|| \le |\log d(\mu,\Gamma_r)| + \log(4r),$$

(3.10.21) follows. $\square$

We now choose a sequence $r_j \to \infty$ so that

$$\forall\, j, \quad \Gamma_{r_j} \cap \bigcup_{m_R(\pm\mu)>0} D(\mu, \langle\mu\rangle^{-n-1}) = \emptyset.$$

As in the case of (3.10.7) this follows from (3.4.7). We then apply Lemma 3.56 with

$$h(\lambda) := \widehat{\varphi}_+(\lambda)(\lambda/\mu)^n, \quad \max_{\Gamma_r}(|h| + r|h'|) = \mathcal{O}(|\mu|^{-n}r^{-1}). \tag{3.10.22}$$

where to get the estimate we used (3.10.17). We have

$$m_R(\pm\mu) > 0 \implies d(\mu,\Gamma_{r_j}) > \langle\mu\rangle^{-n-1},$$

and (3.10.21) gives, for $r_j/2 \le |\mu| \le 2r_j$,

$$\begin{aligned}
\int_{\Gamma_{r_j}} \widehat{\varphi}_+(\lambda)\,(\lambda/\mu)^n\,(\lambda\pm\mu)^{-1}d\lambda &= \mathcal{O}(r_j^{-n-1}\log r_j) \\
&= \mathcal{O}(|\mu|^{-n-\frac12} r_j^{-\frac12}\log r_j).
\end{aligned}$$

Combining this with (3.10.20) gives (3.10.19).

6. Returning to (3.10.18) we see that (3.10.19) and (3.10.11) give

$$\frac{1}{2\pi i}\int_{-r_j}^{r_j} \widehat{\varphi}_+(\lambda)\partial_\lambda(\log\det S(\lambda))d\lambda =$$
$$\sum_{\substack{\operatorname{Im}\mu<0\\ |\mu|<r_j}} m_R(\mu)\widehat{\varphi}_+(\mu) - \sum_{\substack{\operatorname{Im}\mu>0\\ |\mu|<r_j}} m_R(\mu)\widehat{\varphi}_+(-\mu) + o(1)_{r_j\to\infty}.$$

This proves (3.10.16) which, as explained in Step 1 gives (3.10.2). □

REMARK. The proof of Theorem 3.53 relies only on the upper bound on the counting function for resonances and the factorization of $\det S(\lambda)$ in Theorem 3.54. We *did not* use any specific results about the distribution of resonances.

3.11. SCATTERING ASYMPTOTICS

Our next result about the scattering matrix concern asymptotics of the scattering phase, $\log\det S(\lambda)/2\pi i$. As explained after Theorem 2.19 the scattering phase, also known as the scattering winding number, is the analogue of the counting function for eigenvalues of a Schrödinger operator on a bounded domain. It is of intrinsic interest but it will also play an important role in establishing existence of infinitely many resonances – see §3.12.

The proof consists of a number of steps, each of independent interest and each useful in other situations. We first describe the structure of the scattering amplitude as $\lambda \to \infty$. That is done by viewing the resolvent dynamically and relating it to the Schrödinger propagator. We then prove that the scattering phase has an expansion as $\lambda \to \infty$. Using heat trace asymptotics we then compute leading coefficients of that expansions.

In this section we for the first time in the book use *microlocal/semiclassical* methods – we will refer to Appendix E for what is needed but will assume familiarity with basic notation for classes of pseudodifferential operators and symbols (see §E.1).

3.11.1. Semiclassical structure of the scattering matrix. The semiclassical parameter h, in the notation of Appendix E is $h := 1/\lambda$. The semiclassical Hamiltonian is

$$P := h^2 P_V = -h^2\Delta + h^2 V, \quad V \in C_c^\infty(\mathbb{R}^n;\mathbb{R}).$$

The potential term, h^2V, is a very weak perturbation of the free Hamiltonian $-h^2\Delta$. That is in contrast to the examples considered in §2.8 and to the theory presented in Part 3 of this book.

THEOREM 3.57 (Scattering amplitude as a pseudodifferential operator). *Suppose that $V \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$. Let $A(\lambda)$ be as in Theorem 3.41. Then*

$$A_h(E) := A(\sqrt{E}/h) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \quad E > 1.$$

is a family of semiclassical pseudodifferential operators depending smoothly on E,

$$\partial_E^k A_h(E) \in h\Psi_h^{\text{comp}}(\mathbb{S}^{n-1}), \quad k = 0, 1 \cdots .$$

Moreover,

$$\sigma_h(h^{-1}A_h(E))(\theta, \xi) = \frac{1}{2i} E^{-1/2} \int_{\mathbb{R}} V(-\xi/\sqrt{E} + s\theta)\, ds, \tag{3.11.1}$$

where $\xi \in \{\eta \in \mathbb{R}^n : \langle \eta, \theta \rangle = 0\} = T_\theta \mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $T^*\mathbb{S}^{n-1}$ is identified with $T_\theta \mathbb{S}^{n-1}$ using the standard metric on the sphere.

The first step of the proof is yet another formula for the scattering matrix, similar to that in Theorem 3.44:

THEOREM 3.58 (Description of the scattering matrix). *Let P_V, ρ, and $E_\rho(\lambda)$ be as in Theorem 3.41. Take $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi = 1$ near* $\operatorname{supp} V$ *and $\rho = 1$ near* $\operatorname{supp} \chi$. *Then*

$$A(\lambda) = c_n \lambda^{n-2} E_\rho(\lambda)[\Delta, \chi] R_V(\lambda) V E_\rho(\bar{\lambda})^*, \quad c_n = (2\pi)^{1-n}/2i. \tag{3.11.2}$$

Proof. Recall that the operator $A(\lambda)$ is given by

$$A(\lambda) f(\theta) = \int_{\mathbb{S}^{n-1}} b(\lambda, \theta, -\omega) f(\omega)\, d\omega,$$

with $b(\lambda, \theta, \omega)$ defined in (3.7.3). The function u in that definition is given by

$$u(x, \lambda, -\omega) = -R_V(\lambda)(V e^{i\lambda\langle \bullet, \omega\rangle}) = -R_V(\lambda) V E_\rho(\bar{\lambda})^* \delta_\omega.$$

We have $(-\Delta - \lambda^2)u = (P_V - \lambda^2)u = 0$ on $\operatorname{supp}(1 - \chi)$ and thus

$$(-\Delta - \lambda^2)(1 - \chi(x)) u(x, \lambda, -\omega) = [\Delta, \chi] u(x, \lambda, -\omega).$$

The function $(1 - \chi)u$ is outgoing and Theorem 3.37 shows that

$$(1 - \chi(x)) u(x, \lambda, -\omega) = -R_0(\lambda)[\Delta, \chi] R_V(\lambda) V E_\rho(\bar{\lambda})^* \delta_\omega(x).$$

The formula (3.11.2) follows from Theorem 3.5. $\square$

We now analyse the expression (3.11.2) for $\lambda = \sqrt{E}/h$, where $E > 0$ varies in a fixed compact set, say $E \in [1, 2]$, and $h \to 0$. The operators $[\Delta, \chi]$ and multiplication by V (denoted by V) are differential and $E_\rho(\lambda), E_\rho(\lambda)^*$ have an explicit oscillatory integral form.

Therefore the only component of (3.11.2) which needs to be understood further is the resolvent $R_V(\lambda)$. We rewrite in the semiclassical form:

$$R_h(E) := h^{-2} R_V(\sqrt{E}/h), \quad E \in [1,2].$$

We start with the following microlocalization statement about the free resolvent away from the diagonal:

LEMMA 3.59. *For fixed $E > 0$, the free semiclassical resolvent*

$$R_{0,h}(E) := h^{-2} R_0(\sqrt{E}/h),$$

satisfies

$$\begin{aligned} &\mathrm{WF}'_h(R_{0,h}(E)) \cap (T^*\mathbb{R}^n)^2 \cap \{x \neq y\} \\ &\subset \{(x,\xi,x+t\xi,\xi) : |\xi|^2 = E,\ t \geq 0\}. \end{aligned} \tag{3.11.3}$$

Proof. 1. Let $\chi_\delta \in C^\infty(\mathbb{R}^{2n})$ satisfy

$$\chi_\delta(x,y) = \begin{cases} 0 & |x-y| < \delta, \\ 1 & |x-y| > 2\delta. \end{cases}$$

If $R_{0,h}(E,x,y)$ is the Schwartz kernel of $R_{0,h}(E)$ it is enough to show that

$$\mathrm{WF}_h(\chi_\delta R_{0,h}) \subset \{(x, x+t\xi, \xi, -\xi) : |\xi|^2 = E, t \geq 0, x \in \mathbb{R}^n\}. \tag{3.11.4}$$

(See §E.2 for a review of wave front sets.)

2. Theorem 3.3 shows that the smooth function $(\chi_\delta R_{0,h}(E))(x,y)$ can be written as

$$h^{-\frac{n+1}{2}} e^{\frac{i}{h}\varphi(x,y)} a_\delta(x,y,h),$$

where $\varphi(x,y) = \sqrt{E}|x-y|$ and $\partial^\alpha_{x,y} a_\delta = \mathcal{O}_{\alpha,\delta}(1)$. Its wave front set is then given by the standard formula (see for instance [**Zw12**, Example (iii), §8.4]):

$$\{(x,y,\partial_x\varphi(x,y), \partial_y\varphi(x,y), (x,y) \in \operatorname{supp} a_\delta\}.$$

But that gives (3.11.4) and hence (3.11.3). $\square$

We now write $R_h(E)$ using the semiclassical Schrödinger propagator,

$$\exp(-itP/h), \quad P = h^2 P_V.$$

This is motivated by the following formula valid for $\operatorname{Im} E > 0$

$$R_h(E) = \frac{i}{h} \int_0^\infty e^{itE/h} e^{-itP/h}\, dt. \tag{3.11.5}$$

The integral (3.11.5) converges as an operator $L^2 \to L^2$ since $e^{-itP/h}$ is unitary and $e^{itE/h}$ is exponentially decaying for $\operatorname{Im} E > 0$ as $t \to +\infty$. Hence (3.11.5) follows from the spectral theorem.

This is no longer true when $E \in \mathbb{R}$, however we have a microlocal approximation statement given in the next theorem. It relies strongly on a translation of the estimate (2.3.5) to the semiclassical setting: for $k = 0, 1, 2$,

$$\|\rho R_h(E)\rho\|_{L^2 \to H_h^k} \leq C_k/h, \quad E \in [1, 2]. \tag{3.11.6}$$

We call this a *non-trapping* resolvent estimate – more general versions for more complicated operators will be studied in Chapter 6, see Theorems 6.10, 6.16 and 6.22.

LEMMA 3.60 (Parametrix for $R_h(E)$). *Assume that $B \in \Psi^{\mathrm{comp}}(\mathbb{R}^n)$, $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, and V are all supported in $\{|x| < R\}$ and that*

$$\mathrm{WF}_h(B) \subset \{1/2 \leq |\xi| \leq 2\}.$$

Then for $T \geq 8R$,

$$\chi R_h(E) B = \frac{i}{h} \int_0^T \chi e^{itE/h} e^{-itP/h} B \, dt + \mathcal{O}(h^\infty)_{\mathcal{D}' \to C_{\mathrm{c}}^\infty}. \tag{3.11.7}$$

Proof. 1. Consider an h-tempered family $f = f(h) \in \mathcal{D}'(\mathbb{R}^n)$ (see §E.2.3) and define a family $v = v(h)$ by

$$v := \Big(R_h(E) B - \frac{i}{h} \int_0^T \chi_1 e^{itE/h} e^{-itP/h} B \, dt \Big) f \in \mathcal{D}'(\mathbb{R}^n).$$

Here $\chi_1 \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is a cutoff such that $\chi_1 = 1$ on $B(0, R + 10T)$. Note that by the nontrapping estimate (3.11.6), v is also h-tempered. We calculate

$$\begin{aligned} g &:= (P - E) v \\ &= Bf + \chi_1 \int_0^T \partial_t (e^{itE/h} e^{-itP/h}) Bf \, dt - \frac{i}{h} \int_0^T e^{itE/h} [P, \chi_1] e^{-itP/h} Bf \, dt \\ &= \chi_1 e^{iTE/h} e^{-iTP/h} Bf - \frac{i}{h} \int_0^T e^{itE/h} [P, \chi_1] e^{-itP/h} Bf \, dt \in C_{\mathrm{c}}^\infty(\mathbb{R}^n). \end{aligned}$$

From [**Zw12**, Theorem 12.5] we have

$$\mathrm{WF}_h(e^{-itP/h} Bf) \subset \{(x + 2t\xi, \xi) : |x| \leq R, \ |\xi| \in [1/2, 2]\}. \tag{3.11.8}$$

Then $\mathrm{WF}_h(e^{-itP/h} Bf) \cap \operatorname{supp}(1 - \chi_1) = \emptyset$ for $t \in [0, T]$. Therefore,

$$[P, \chi_1] e^{-itP/h} Bf = \mathcal{O}(h^\infty)_{C_{\mathrm{c}}^\infty}.$$

Putting $t = T \geq 8R$ in (3.11.8) we see that $(x, \xi) \in \mathrm{WF}_h(g)$ implies that $x = y + 2R\xi$, $|y| \leq R$ and hence

$$\langle x, \xi \rangle = T|\xi|^2 + \langle y, \xi \rangle \geq T/4 - 2R \geq 0.$$

In other words,

$$\mathrm{WF}_h(g) \subset \{(x, \xi) : |x| \geq R, \ \langle x, \xi \rangle \geq 0, \ |\xi| \in [1/2, 2]\}. \tag{3.11.9}$$

2. We next claim that

$$v = R_{0,h}(E)g + \mathcal{O}(h^\infty)_{C^\infty}. \tag{3.11.10}$$

Indeed, both v and $R_{0,h}(E)g$ are outgoing, therefore by Theorem 3.37

$$\begin{aligned} v - R_{0,h}(E)g &= R_h(E)(P-E)(v - R_{0,h}(E)g) \\ &= -R_h(E)(h^2 V R_{0,h}(E)g). \end{aligned}$$

From (3.11.3) and (3.11.9), we obtain that

$$\mathrm{WF}_h(R_{0,h}(E)g) \cap \{|x| < R\} = \emptyset. \tag{3.11.11}$$

Thus $VR_{0,h}(E)g = \mathcal{O}(h^\infty)_{C_c^\infty}$ and (3.11.10) follows from (3.11.6).

3. Finally, by (3.11.11), we have $\chi R_{0,h}(E)g = \mathcal{O}(h^\infty)_{C_c^\infty}$, therefore $\chi v = \mathcal{O}(h^\infty)_{C_c^\infty}$. Since this is true for any h-tempered family $f(h)$, (3.11.7) follows. □

To write an oscillatory integral expression for $R_h(E)$ and thus for $A_h(E)$, we need to write such an expression for $e^{-itP/h}$:

THEOREM 3.61 (Parametrix for the Schrödinger propagator). *Suppose that $V \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$ and $B \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$. Then for $|t| \le T$,*

$$\exp(-itP/h)B = \exp(-ithP_V)B = U_B(t) + \mathcal{O}_T(h^\infty)_{\mathscr{S}' \to C^\infty}, \tag{3.11.12}$$

where for $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} U_B(t)f(x) &:= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle - \frac{i}{h}t|\xi|^2} b(t,x,\xi,h) f(y)\,dy\,d\xi, \\ b(t,x,\xi,h) &\sim \sum_{j=0}^\infty h^j b_j(t,x,\xi), \quad b_0(t,x,\xi) = \sigma_h(B)(x - 2t\xi, \xi). \end{aligned} \tag{3.11.13}$$

Proof. The proof follows the WKB construction – see [**Zw12**, §10.2] – much simplified due to the fact that the flow is explicit and linear.

1. We are looking for $b(t,x,\xi,h)$ solving the following equation asymptotically in h:

$$\begin{aligned} &(hD_t - h^2\Delta + h^2 V)\left(e^{\frac{i}{h}\langle x,\xi\rangle - \frac{i}{h}t|\xi|^2} b(t,x,\xi,h)\right) = 0, \\ &b(0,x,\xi,h) = b(x,\xi), \quad B = b(x,hD). \end{aligned} \tag{3.11.14}$$

We will solve for b as an asymptotic expansion,

$$b(t,x,\xi,h) \sim \sum_{j=0} h^j b_j(t,x,\xi), \quad b_0(0,x,\xi) = b(x,\xi), \quad b_j(0,x,\xi) = 0.$$

Since

$$e^{-\frac{i}{h}\langle x,\xi\rangle + \frac{i}{h}t|\xi|^2}(hD_t - h^2\Delta) e^{\frac{i}{h}\langle x,\xi\rangle - \frac{i}{h}t|\xi|^2} = \frac{h}{i}(\partial_t + 2\langle \xi, \partial_x\rangle) - h^2\Delta,$$

we have the following set of equations for the terms in the expansion ($b_{-1} \equiv 0$):

$$(\partial_t + 2\langle \xi, \partial_x \rangle) b_j(t, x, \xi) = -i(-\Delta_x + V(x)) b_{j-1}(t, x, \xi).$$

These are solved by

$$\begin{aligned} & b_0(t, x, \xi) = b(x - 2t\xi, \xi) \\ & b_j(t, x, \xi) = \frac{1}{i} \int_0^t (-\Delta_x + V(x - 2s\xi)) b_{j-1}(s, x - 2s\xi, \xi) ds, \quad j \geq 1. \end{aligned}$$

Since b is compactly supported in (x, ξ) so are b_j's (with the size of the support depending on t) which shows that $U_B(t) : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ – see [**Zw12**, Theorem 4.1].

2. It remains to justify the estimate on the remainder in (3.11.13) and for that we will use Duhamel's formula and the mapping properties of $\exp(-itP/h)$ – see [**Zw12**, §10.1]. We have shown so far that

$$(ih\partial_t - P) U_B(t) = r(t) \in \mathcal{O}_T(h^\infty)_{\mathscr{S}' \to \mathscr{S}}, \quad U_B(0) = B.$$

Hence, by Duhamel's formula,

$$e^{-itP/h} B - U_B(t) = -\frac{i}{h} \int_0^t e^{-i(t-s)P/h} r(s) ds.$$

Using [**Zw12**, (10.1.10)] with $m(x, \xi) = 1 + |\xi|^2$ we see that $e^{-i(t-s)P/h} : H_h^k \to H_h^k$ for all k which implies that $e^{-i(t-s)P/h} r(s) = \mathcal{O}(h^\infty)_{\mathscr{S}' \to H^N}$ for all N. Inserting this into (4.6.15) shows that $e^{-itP/h} B - U_B(t) = \mathcal{O}_T(h^\infty)_{\mathscr{S}' \to C^\infty}$ as claimed.

□

Proof of Theorem 3.57. 1. To simplify notation we assume that $E = 1$ and drop E from all the formulas. Since $A_h(E) = A_{h/\sqrt{E}}(1)$ that is justified if $E \in K \Subset (0, \infty)$.

By (3.11.2), we write

$$A_h := A_h(1) = \tfrac{1}{2i} (2\pi)^{1-n} h^{4-n} \mathscr{E}_h [\Delta, \chi] R_h V \mathscr{E}_h^*, \quad R_h := R_h(1).$$

where (with $d\omega$ denoting the measure on $\mathbb{S}^{n-1}$ induced from $\mathbb{R}^n$)

$$\mathscr{E}_h^* f(x) := \int_{\mathbb{S}^{n-1}} e^{\frac{i}{h} \langle x, \omega \rangle} f(\omega) \, d\omega.$$

As in the proof of (3.11.3) we see that

$$\mathrm{WF}_h'(\mathscr{E}_h^*) \subset \{(x, \omega; \omega, x - \langle x, \omega \rangle \omega)\} \subset T^*\mathbb{R}^n \times T^*\mathbb{S}^{n-1},$$

where we identified $T_\theta^* \mathbb{S}^{n-1}$ with $T_\theta \mathbb{S}^{n-1} \subset T_\theta \mathbb{R}^n$, $\theta \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$.

Take $B \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ such that

$$
\begin{aligned}
B = 1 \quad &\text{microlocally on } \{x \in \operatorname{supp} V,\ |\xi| \in [2/3, 3/2]\}; \\
&\mathrm{WF}_h(B) \subset \{|x| < R,\ |\xi| \in [1/2, 2]\}.
\end{aligned} \tag{3.11.15}
$$

This and the estimate of the wave front set of $\mathscr{E}_h^*$ imply that

$$V\mathscr{E}_h^* = BV\mathscr{E}_h^* + \mathcal{O}(h^\infty)_{\mathcal{D}'(\mathbb{S}^{n-1}) \to C_c^\infty(\mathbb{R}^n)}.$$

Therefore, by Lemma 3.60, for fixed large $T > 0$,

$$
\begin{aligned}
A_h(E) = \frac{\pi h^{3-n}}{(2\pi)^n} \mathscr{E}_h[\Delta, \chi] \int_0^T e^{it/h} e^{-itP/h} BV\mathscr{E}_h^* \, dt \\
+ \mathcal{O}(h^\infty)_{\mathcal{D}'(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})}.
\end{aligned} \tag{3.11.16}
$$

2. We may replace $e^{-itP/h}B$ in (3.11.16) by $U_B(t)$ from Theorem 3.61. Also, we may replace the integral from 0 to T in (3.11.16) by the integral against a function $\psi(t) \in C_c^\infty(0, T)$ such that $\psi = 1$ on $[\delta, T-\delta]$ for δ small enough. (The supports of $[\Delta, \chi]$ and V are disjoint and we can use propagation result [**Zw12**, Theorem 12.5].) Applying the differential operator $[\Delta, \chi]$ to the formula (3.11.13) we find

$$
\begin{aligned}
&[\Delta, \chi]U_B(t)f(x) = \frac{2i}{h}(2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle x-y, \xi\rangle - t|\xi|^2)} \tilde{b}(t, x, \xi, h) f(y) \, dy d\xi, \\
&\tilde{b}(t, x, \xi, h) \sim \sum_{j=0}^\infty h^j \tilde{b}_j(t, x, \xi), \quad \tilde{b}_0(t, x, \xi) = \langle \xi, d\chi(x)\rangle b_0(t, x, \xi).
\end{aligned}
$$

We thus obtain the following integral formula, valid modulo an $\mathcal{O}(h^\infty)_{\mathcal{D}' \to C^\infty}$ remainder:

$$
\begin{aligned}
A_h g(\theta) &\equiv \frac{ih}{(2\pi h)^{2n-1}} \int_{\mathbb{R}^{3n+1} \times \mathbb{S}^{n-1}} e^{\frac{i}{h}\Phi} \tilde{a} g(\omega) \, dt \, dx \, dy \, d\xi \, d\omega, \\
\Phi(t, x, y, \xi, \omega, \theta) &= \langle x - y, \xi\rangle + t(1 - |\xi|^2) + (\langle y, \omega\rangle - \langle x, \theta\rangle), \\
\tilde{a} &\sim \sum_{j=0}^\infty h^j \tilde{a}_j, \\
\tilde{a}_0(t, x, y, \xi) &= \psi(t) V(y) \langle \xi, d\chi(x)\rangle \sigma_h(B)(x - 2t\xi, \xi).
\end{aligned} \tag{3.11.17}
$$

The support of the symbol $\tilde{a}(t, x, y, \xi, h)$ is contained in

$$\{t \in (0, T),\ y \in \operatorname{supp} V,\ x \in \operatorname{supp} \chi,\ |\xi| \in [1/2, 2]\}.$$

Moreover, we note for future reference that for $|\xi|^2 = 1 \in [1, 2]$, it follows from (3.11.15) that

$$\tilde{a}_0(t, x, x - 2t\xi, \xi) = \mathbb{1}_{\mathbb{R}_+}(t) V(x - 2t\xi) \langle \xi, d\chi(x)\rangle. \tag{3.11.18}$$

3. We now use the integral formula for $A_h(E)$ to show that it is an h-pseudodifferential operator. We refer to §E.1.7 for the definition of a pseudodifferential operator on a manifold ($\mathbb{S}^{n-1}$ in our case).

First we note that if $\theta \neq \omega$ then $|\xi-\theta|^2 + |\xi-\omega|^2 \geq \frac{1}{2}|\theta-\omega|^2 > 0$ and

$$-h^2(|\xi-\theta|^2 + |\xi-\omega|^2)^{-1}(\Delta_x + \Delta_y)e^{\frac{i}{h}\Phi} = e^{\frac{i}{h}\Phi}.$$

Hence, repeated integration by parts in x, y in the integral in (3.11.17) show that for $\chi', \chi'' \in C^\infty(\mathbb{S}^{n-1})$ and $\operatorname{supp}\chi' \cap \operatorname{supp}\chi'' = \emptyset$, we have $\chi' A \chi'' = \mathcal{O}(h^\infty)_{\mathcal{D}' \to C^\infty}$.

It then remains to show that for χ' supported in a small coordinate neighbourhood, $\chi' A_h \chi'$ is a classical pseudodifferential operator. Without loss of generality, we consider the coordinate neighbourhood $\{\theta_n > 0\}$ with the coordinate θ', where $\theta = (\theta', \sqrt{1-|\theta'|^2})$. For χ' supported in this neighbourhood, the symbol of $\chi' A_h \chi'$ is given by oscillatory testing:

$$a_{\chi'}(\theta', \eta) := e^{-\frac{i}{h}\langle \theta', \eta\rangle}\chi'(\theta')(A_h\chi')(e^{\frac{i}{h}\langle \bullet, \eta\rangle})(\theta'), \quad \theta', \eta \in \mathbb{R}^{n-1}.$$

We need to calculate $a(\theta', \eta)$ and show that it is indeed a classical symbol. Then $\chi' A_h \chi' = a_{\chi'}(\theta', hD'_\theta, h)$ (note that we are using here the standard quantization (E.1.18) and not the Weyl quantization).

4. Using the oscillatory integral expression for A_h, we write

$$a_{\chi'}(\theta', \eta) = (2\pi)^{1-2n} i h^{2-2n} \int_{\mathbb{R}^{4n}} e^{\frac{i}{h}(\Phi(t,x,y,\xi,\omega,\theta) + \langle \omega' - \theta', \eta\rangle)}$$
$$\frac{\chi'(\omega')\chi'(\theta')\tilde{a}(t,x,y,\xi)}{\sqrt{1-|\omega'|^2}}\, dt\, dx\, dy\, d\xi\, d\omega'.$$

The phase in the integral is

$$\langle x-y, \xi\rangle + \langle \omega'-\theta', \eta\rangle + t(1-|\xi|^2) + (\langle y, \omega\rangle - \langle x, \theta\rangle).$$

The critical points in (y, ξ) are given by $\xi = \omega$, $y = x - 2t\xi$ and the critical Hessian has determinant 1 and signature 0. Hence, the stationary phase method (see Proposition E.7) in the (y, ξ) variables gives (with $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$)

$$a_{\chi'}(\theta', \eta) = (2\pi)^{1-n} i h^{2-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\Phi^\sharp} a^\sharp \, dt dx d\omega',$$

$$\Phi^\sharp(t, x, \omega', \theta') = \langle \omega' - \theta', \eta\rangle + \langle x', \omega' - \theta'\rangle + x_n\left(\sqrt{1-|\omega'|^2} - \sqrt{1-|\theta'|^2}\right),$$

$$a^\sharp(t, x, \omega', \theta') \sim \sum_{j=0}^\infty h^j a_j^\sharp(t, x, \omega', \theta'),$$

$$a_0^\sharp(t, x, \omega', \theta') = \frac{\chi'(\omega')\chi'(\theta')\tilde{a}_0(t, x, x-2t\omega, \omega)}{\sqrt{1-|\omega'|^2}}.$$

We next apply the stationary phase method in the (x', ω') variables: the critical points are given by $\omega' = \theta'$ and $x' = -\eta$ and the Hessian has determinant 1 and signature 0. Hence,

$$a_{\chi'}(\theta', \eta) = ih \int_{\mathbb{R}^2} a^\flat \, dt dx_n, \quad a^\flat \sim \sum_{j=0}^{\infty} h^j a_j^\flat,$$

where $a_0^\flat(t, x_n, \theta', \eta) =$

$$\theta_n^{-1} \chi'(\theta')^2 \tilde{a}_0 \left(t, -\eta + x_n\theta'/\theta_n, x_n, -\eta + x_n\theta'/\theta_n - 2t\theta', x_n - 2t\theta_n, \theta\right),$$

with $\tilde{a}_0$ given in (3.11.17). Note that $a^\flat$ is compactly supported, as follows from the support condition on $\tilde{a}$. Integrating in t, x_n, we see that $h^{-1}a_{\chi'}$ is a compactly supported classical symbol and thus $A_h \in h\Psi^{\text{comp}}(\mathbb{S}^n)$.

5. It remains to calculate the principal symbol of $h^{-1}A_h$. For that recall that χ has properties listed in Theorem 3.58 and that note that

$$\begin{aligned}\int_0^\infty \langle \theta, d\chi\left(-(\eta, 0) + (s + 2t)\theta\right)\rangle \, dt &= \tfrac{1}{2} \int_0^\infty \partial_t \left[\chi\left(-(\eta, 0) + (s + 2t)\theta\right)\right] dt \\ &= -\tfrac{1}{2}\chi((-(\eta, 0) + s\theta)\,,\end{aligned}$$

which is equal to $-\frac{1}{2}$ for $-(\eta, 0) + s\theta \in \operatorname{supp} V$. Hence, a change of variables $x_n = \theta_n(s + 2t)$ gives

$$\begin{aligned}&\sigma_h(h^{-1}A_h)(\theta', \eta) \\ &\quad= i \int_0^\infty dt \int_{\mathbb{R}} ds V\left(-(\eta, 0) + s\theta\right) \langle \theta, d\chi\left(-(\eta, 0) + (s + 2t)\theta\right)\rangle \\ &\quad= \frac{1}{2i} \int_{\mathbb{R}} V\left(-(\eta, 0) + s\theta\right) \, ds,\end{aligned}$$

which proves (3.11.1) with $E = 1$.

6. We now consider the derivatives of $A_h(E)$ with respect to E. The relation $A_h(E) = A_{h/\sqrt{E}}(1)$ (or the construction of the expansions) shows that the terms in the expansion depend smoothly on E. On the other hand, in the notation of Step 3, $\chi'\partial_E^k A_h(E)\chi'' = \mathcal{O}_{\mathcal{D}' \to C^\infty}(h^\infty)$, and $\chi'\partial_E^k A_h(E)\chi' = a_{\chi',k}(\theta', hD_{\theta'}, h)$, where

$$\partial_{\theta'}^\alpha \partial_\eta^\beta a_{\chi',k} = \mathcal{O}(h^{-N(k,|\alpha|+|\beta|)})$$

(that follows from (3.11.2) and (3.11.6)). Also $a_{\chi'}(\theta', \eta) = \tilde{a}_{\chi'}(\theta', \eta) + \mathcal{O}(h^\infty)$, and $\tilde{a}_{\chi'}$ has differentiable expansions in h.

The interpolation inequality (A.5.4) (applied with $m = 0$, $\ell = M + k$ and $p = M + k + 1$) now gives

$$\sup_{|\alpha|+|\beta|=M} \|\partial^\alpha\partial^\beta(a_{\chi',k} - \partial_E^k \tilde{a}_{\chi'})\|_{L^\infty} \leq$$

$$C_{M,k}\|a_{\chi'}-\tilde a_{\chi'}\|_{L^\infty}^{\frac{1}{M+k+1}}\left(\sup_{|\alpha|+|\beta|=M+1}(\|\partial^\alpha\partial^\beta a_{\chi',k+1}\|_{L^\infty}+C'_{M,k})\right)^{\frac{M+k}{M+k+1}}$$
$$\leq \mathcal{O}(h^\infty)\mathcal{O}(h^{-N_{M+1,k+1}})=\mathcal{O}(h^\infty).$$

It follows that $a_{\chi',k}$ is a symbol with a full asymptotic expansion. □

An important consequence of Theorem 3.57 is the existence of the expansion for the scattering phase:

THEOREM 3.62 (Existence of an asymptotic expansion). *Define the derivative of the scattering phase, $\sigma'(\lambda)$, by*

$$\sigma'(\lambda):=\frac{1}{2\pi i}\operatorname{tr}S(-\lambda)\partial_\lambda S(\lambda)\,. \tag{3.11.19}$$

Then there exists a sequence $b_1, b_2, \dots$ such that

$$\sigma'(\lambda)\sim\sum_{j=1}^\infty b_j\lambda^{n-2-j},\quad \lambda\to+\infty\,, \tag{3.11.20}$$

and

$$b_1=-\frac{(n-2)\operatorname{Vol}(\mathbb{S}^{n-1})}{2(2\pi)^n}\int_{\mathbb{R}^n}V(x)\,dx. \tag{3.11.21}$$

REMARK. In Theorem 3.67 we will show that only even terms appear in the expansion and will provide a method for computing the coefficients. That will show that the integrated expansion is also valid – see (3.11.40).

Proof. 1. Since $S(\lambda)=I+A(\lambda)$ and $A_h(E)=A(\sqrt{E}/h)$,

$$\partial_\lambda S(\lambda)=2h\partial_E A_h(1),\quad \lambda=1/h,$$

and

$$\sigma'(\lambda)=\frac{1}{2\pi i}\operatorname{tr}(S(\lambda)^*\partial_\lambda S(\lambda))=\frac{h}{\pi i}\operatorname{tr}((I+A_h(1)^*)\partial_E A_h(1)) \tag{3.11.22}$$

Theorem 3.57 shows that $A_h(1),\partial_E A_h(1)\in h\Psi_h^{\mathrm{comp}}(\mathbb{S}^{n-1})$. Its proof, and the composition formula for pseudodifferential operators (see Proposition E.8), also gives asymptotic expansions of full symbols of operators localized to coordinate patches:

$$\begin{gathered} b_{\chi'}(y,hD_y,h)=\chi'(I+A_h(1))\partial_E A_h(1)\chi',\\ b_{\chi'}(y,\eta,h)\sim\sum_{k=1}^\infty h^k b_{\chi',k}(y,\eta),\\ b_{\chi',k}\in C_{\mathrm{c}}^\infty(B(0,1)_y\times(B(0,2)\setminus B(0,\tfrac12))_\eta), \end{gathered} \tag{3.11.23}$$

$y = \theta' \in \mathbb{R}^{n-1}$, $(\theta', (1-|\theta'|^2)^{\frac{1}{2}}) \in \mathbb{S}^{n-1}$ – see Step 3 of that proof for the notation. Choosing a partition of unity $\sum_{j=1}^J \chi_j^2 = 1$, $\chi_j \in C^\infty(\mathbb{S}^{n-1})$, (3.11.23) gives

$$
\begin{aligned}
\operatorname{tr}_{L^2(\mathbb{S}^{n-1})}(I + A_h(1))\partial_E A_h(1) &= \sum_{j=1}^J \operatorname{tr}_{L^2(\mathbb{R}^{n-1})} b_{\chi_j}(y, hD_y, h) \\
&= \frac{1}{(2\pi h)^{n-1}} \sum_{j=1}^J \int_{T^*\mathbb{R}^{n-1}} b_{\chi_j}(y, \eta, h) dy d\eta \\
&\sim \sum_{k=1}^\infty a_k h^{-n-1+k},
\end{aligned}
$$

$$
a_k := \sum_{j=1}^J \int_{T^*\mathbb{R}^{n-1}} b_{\chi_j, k}(y, \eta) dy d\eta.
$$

Returning to (3.11.22) we obtain (3.11.20).

2. The first coefficient is given by the integral of the principal symbol of $h^{-1}\partial_E A_h(1)$ which we compute using (3.11.1) (note that $\xi \in \{\eta : \langle \eta, \theta \rangle = 0\} = T_\theta \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $d\xi d\theta = \omega^n/n!$ is the volume form obtained from the symplectic form ω):

$$
\begin{aligned}
b_1 &= \frac{2}{(2\pi)^n i} \int_{T^*\mathbb{S}^{n-1}} \partial_E|_{E=1} \sigma_h(h^{-1}A_h(E))(\theta, \xi) d\xi d\theta \\
&= -\frac{1}{(2\pi)^n} \partial_E|_{E=1} \left(E^{-1/2} \int_{\mathbb{R}} ds \int_{T^*\mathbb{S}^{n-1}} d\xi d\theta \, V(-\xi/\sqrt{E} + s\theta) \right) \\
&= -\frac{1}{2(2\pi)^n} \int_{\mathbb{R}} ds \int_{T^*\mathbb{S}^{n-1}} d\xi d\theta \, (\xi \cdot \nabla V(-\xi + s\theta) - V(-\xi + s\theta)) \, .
\end{aligned}
$$

We can integrate parts in ξ by assuming, without loss of generality that $\theta = (0, 1) = e_n$ and $\xi = (\xi', 0)$, $\xi' \in \mathbb{R}^{n-1}$:

$$
\int_{\mathbb{R}^{n-1}} (\xi', 1) \cdot \nabla V(-(\xi', 0) + s e_n) d\xi' = \int_{\mathbb{R}^{n-1}} (n-1) V(\xi + s\theta) d\xi'.
$$

Hence,

$$
\begin{aligned}
b_1 &= -\frac{n-2}{2(2\pi)^n} \int_{\mathbb{R}} ds \int_{T^*\mathbb{S}^{n-1}} d\theta d\xi \, V(\xi + s\theta) \\
&= -\frac{n-2}{2(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{S}^{n-1}} V(x) \, d\theta dx = -\frac{(n-2)\operatorname{Vol}(\mathbb{S}^{n-1})}{2(2\pi)^n} \int_{\mathbb{R}^n} V(x) dx,
\end{aligned}
$$

concluding the proof of (3.11.21). □

It would be tempting to compute the coefficients of the expansions in Theorem 3.57 directly using symbolic calculus and the structure of the parametrix for the propagator and we did this for the first term in the expansion (3.11.21) However, it is more convenient to compute the terms appearing in the expansion by using the Birman–Kreĭn in formula (3.9.10) and *heat trace asymptotics* of Theorem 3.64 below.

3.11.2. Heat trace asymptotics. We start with a representation of the resolvent useful for $\operatorname{Im}\lambda > 0$:

LEMMA 3.63 (**Another expansion for the resolvent**). *Suppose $V \in C^\infty_{\rm c}(\mathbb{R}^n;\mathbb{C})$. Then for any $M \in \mathbb{N}$,*

$$R_V(\lambda) = \sum_{m=0}^{M} X_m R_0(\lambda)^{m+1} + R_V(\lambda) X_{M+1} R_0(\lambda)^{M+1}, \tag{3.11.24}$$

where the operators X_m are defined by induction as follows:

$$X_0 := I, \quad X_{m+1} = -V X_m + [X_m, P_0], \tag{3.11.25}$$

where V (as elsewhere) means the multiplication operator $f \mapsto Vf$. For $m > 0$, X_m have order $\leq m-1$ and compactly supported coefficients.

Proof. 1. For $M = 0$ (3.11.24) states that

$$R_V(\lambda) - R_0(\lambda) = R_V(\lambda) X_1 R_0(\lambda) = -R_V(\lambda) V R_0(\lambda),$$

which is the resolvent identity.

2. Assuming that (3.11.24) holds for M replaced with $M-1$, the inductive step means proving that

$$R_V(\lambda) X_M R_0(\lambda)^M - X_M R_0(\lambda)^{M+1} = R_V(\lambda) X_{M+1} R_0(\lambda)^{M+1}. \tag{3.11.26}$$

To see this we use the definition of X_{M+1} to write (we suppress the λ dependence)

$$\begin{aligned} R_V X_M - R_V X_{M+1} R_0 &= R_V(X_M + V X_M R_0 + P_0 X_M R_0 - X_M P_0 R_0) \\ &= R_V(X_M + (P_V - \lambda^2) X_M R_0 - X_M (P_0 - \lambda^2) R_0) \\ &= R_V X_M + X_M R_0 - R_V X_M \\ &= X_M R_0. \end{aligned}$$

Multiplying this on the right by R_0^M gives (3.11.26). ☐

We can now study heat trace asymptotics.

THEOREM 3.64. *Suppose that $V \in C^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R})$. Then,*

$$e^{-tP_V} - e^{-tP_0} \in \mathcal{L}_1(L^2(\mathbb{R}^n)), \quad t > 0,$$

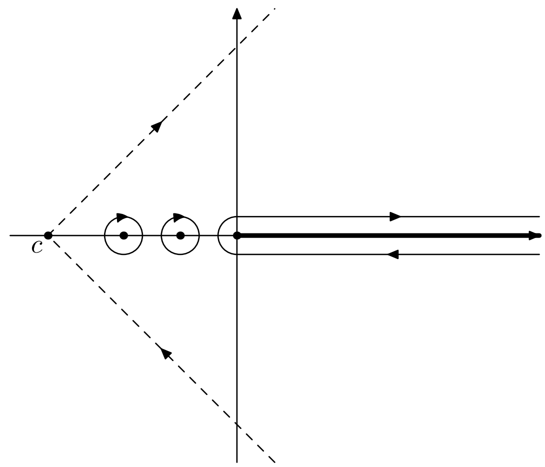


Figure 3.6. Contours in the z-plane ($z = \lambda^2$) used to express $e^{-tP_V} - e^{-tP_0}$ in terms of the resolvent. The solid contour provides the usual expression of e^{-tP_V} and e^{-tP_0}. The dashed contour $s \mapsto c + |s|e^{i\,\mathrm{sgn}(s)\frac{\pi}{4}}$, $s \in \mathbb{R}$, provides an expression for $e^{-tP_V} - e^{-tP_0}$.

and for any $K \in \mathbb{N}$,

$$\operatorname{tr}\left(e^{-tP_V} - e^{-tP_0}\right) = \frac{1}{(4\pi t)^{n/2}} \sum_{k=1}^{K} a_k(V) t^k + \mathcal{O}(t^{K+1-n/2}), \tag{3.11.27}$$

where

$$a_1(V) = -\int V(x)dx, \quad a_2(V) = \frac{1}{2}\int V(x)^2 dx. \tag{3.11.28}$$

Proof. 1. Functional calculus for self-adjoint operators shows that

$$\begin{gathered} e^{-tP_V} - e^{-tP_0} = \frac{1}{2\pi i}\int_{\Gamma_c} e^{-tz}((P_V - z)^{-1} - (P_0 - z)^{-1})dz, \\ \Gamma_c : s \mapsto z(s) := c + i|s|e^{i\,\mathrm{sgn}(s)\pi/4}, \quad s \in \mathbb{R}, \quad c < E_K. \end{gathered} \tag{3.11.29}$$

That follows from a deformation of a contour involving $(P_V - z)^{-1}$ and $(P_0 - z)^{-1}$ separately (where $t > 0$ is needed) – see Fig. 3.6. The integral on the right hand side converges in operator norm on L^2 as $\operatorname{Re} z(s) \sim |s|$, $s \to \pm\infty$, and

$$\begin{aligned} \|(P_V - z)^{-1} - (P_0 - z)^{-1})\| &= \|(P_V - z)^{-1} V (P_0 - z)^{-1}\| \\ &\leq C|z|^{-2}, \quad z \in \Gamma_c. \end{aligned}$$

Theorem 3.50 or a direct argument based on (3.11.29) also show that $e^{-tP_V} - e^{-tP_0}$ is of trace class. The trace can then be computed by integrating the Schwartz kernel over the diagonal.

2. The formula

$$\frac{1}{2\pi i}\int_{\Gamma_c} e^{-tz}(P_0 - z)^{-m-1}dz = \frac{t^m}{m!}e^{-tP_0}$$

and Lemma 3.63 show

$$\operatorname{tr}(e^{-tP_V} - e^{-tP_0}) = \sum_{m=1}^{M} \frac{t^m}{m!} \operatorname{tr}\left(X_m e^{-tP_0}\right) + \operatorname{tr} e_M(t), \tag{3.11.30}$$

where

$$e_M(t) := \frac{1}{2\pi i}\int_{\Gamma_c} e^{-tz}(P_V - z)^{-1} X_{M+1}(P_0 - z)^{-M-1}dz. \tag{3.11.31}$$

3. We first analyze the terms in the sum on the right hand side of (3.11.30). The Schwartz kernel of e^{-tP_0} is given by

$$K(t,x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-|x-y|^2/4t},$$

see for instance [**Ev98**, Chapter 1] or [**HöI**, Theorem 3.3.3].

Since for $m \geq 1$, X_m is a differential operator of order $m-1$, with compactly supported coefficients,

$$\begin{aligned}\frac{t^m}{m!}\operatorname{tr}\left(X_m e^{-tP_0}\right) &= \frac{t^m}{m!(4\pi t)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(X_m e^{-|x-y|^2/4t}\right)|_{x=y}dx \\ &= \frac{1}{(4\pi t)^{\frac{n}{2}}} t^{m-\left[\frac{m-1}{2}\right]} \sum_{k=0}^{\left[\frac{m-1}{2}\right]} a_{m,k}t^k = \mathcal{O}(t^{\left[\frac{m}{2}\right]+1-\frac{n}{2}}),\end{aligned} \tag{3.11.32}$$

which means that the expansion makes formal sense. Grouping the coefficients according to the powers of t gives the coefficients in the expansion and a calculation based on (3.11.25) gives (3.11.28). In fact,

$$X_1 = V, \quad X_2 = V^2 - 2\nabla V\cdot\nabla - \Delta V,$$
$$X_3 = -2\sum_{j,k}\partial_{x_j}\partial_{x_k}V\partial_{x_j}\partial_{x_k} + \widetilde{X}_3,$$

where $\widetilde{X}_3$ is an operator of order 1. Hence,

$$a_1(V) = (4\pi t)^{\frac{n}{2}}\operatorname{tr}(X_1 e^{-tP_0}) = -\int_{\mathbb{R}^n}(V(x)e^{-|x-y|^2/4t})|_{x=y}dx = -\int_{\mathbb{R}^3}V(x)dx.$$

Also,

$$(4\pi t)^{\frac{n}{2}}\operatorname{tr}(X_2 e^{-tP_0}) = \int_{\mathbb{R}^n}\left(X_2 e^{-|x-y|^2/4t}\right)|_{x=y}dx = \int_{\mathbb{R}^n}V(x)^2dx,$$

and

$$
\begin{aligned}
(4\pi t)^{\frac{n}{2}} \operatorname{tr}(X_3 e^{-tP_0}) &= -2 \int_{\mathbb{R}^n} \left(\sum_{j,k} V_{x_j x_k} \partial_{x_j} \partial_{x_k} e^{-|x-y|^2/t} \right) |_{x=y} dx + \mathcal{O}(t^{-1}) \\
&= 4 \int_{\mathbb{R}^n} \Delta V dx + \mathcal{O}(t^{-1}) = \mathcal{O}(t^{-1}).
\end{aligned}
$$

Using this in (3.11.30) gives $a_2(V) = \frac{1}{2} \int_{\mathbb{R}^n} V(x)^2 dx$.

4. To estimate the trace class norm of the remainder $e_M(t)$ in (3.11.31) we start with estimates on the integrand. Uniformly for $\operatorname{Re} z \leq -1$ we have $\|(-\Delta - z)^{-1}\|_{L^2 \to L^2} \leq 1/|z|$ and

$$
\begin{aligned}
\|(-\Delta - z)^{-1}\|_{L^2 \to H^2} &\simeq \|\Delta(-\Delta - z)^{-1}\|_{L^2 \to L^2} + \|(-\Delta - z)^{-1}\|_{L^2 \to L^2} \\
&\leq C.
\end{aligned}
$$

For $0 \leq s \leq 2$, we can use Hölder's inequality with $p = 2/s$ and $q = 2/2 - s$ to obtain

$$
\begin{aligned}
\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 d\xi &= \int_{\mathbb{R}^n} |\widehat{u}(\xi)|^{2-s} \left(\langle \xi \rangle^2 |\widehat{u}(\xi)| \right)^s d\xi \\
&\leq \|\widehat{u}\|_{L^2}^{\frac{2-s}{2}} \|\langle \xi \rangle^2 \widehat{u}\|_{L^2}^{\frac{s}{2}} = \|u\|_{L^2}^{\frac{2-s}{2}} \|u\|_{H^2}^{\frac{s}{2}}.
\end{aligned}
$$

Commuting Δ through the resolvent, it follows that

$$
\|(-\Delta - z)^{-1}\|_{H^r \to H^{s+r}} \leq C|z|^{-1+\frac{s}{2}}, \quad 0 \leq s \leq 2,
$$

on the contour, uniformly with respect to $\operatorname{Re} z < -1$. Consequently as long

$$
M \geq n \implies s := \frac{n+1}{M+1} \leq 2,
$$

we can iterate the estimate to obtain

$$
\|(-\Delta - z)^{-M-1}\|_{L^2 \to H^{n+1}} \leq C_M |z|^{-(1-\frac{n+1}{2(M+1)})(M+1)} = C_M |z|^{-M+\frac{n-1}{2}}.
$$

Since X_{M+1} is a differential operator with coefficients supported in $|x| \leq R$ we obtain

$$
\begin{aligned}
\|X_{M+1}(-\Delta - z)^{-M-1}\|_{\mathcal{L}_1} &\leq C \|X_{M+1}(-\Delta - z)^{-M-1}\|_{L^2(\mathbb{R}^n) \to H^{n+1}(B(0,R))} \\
&\leq C'' |z|^{-M+\frac{n-1}{2}}.
\end{aligned}
$$

5. Returning to (3.11.31) we deform the contour of integration to $s \mapsto -1/t + is$, $s \in \mathbb{R}$. Then, using the uniformity of the above estimates for

$\operatorname{Re} z < E_K - 1$, and the bound $\|(P_V - z)^{-1}\|_{L^2 \to L^2} \leq C/|z|$, we obtain

$$\begin{aligned}
\|e_M(t)\|_{\mathcal{L}_1} &\leq C \int_{-1/t-i\infty}^{-1/t+i\infty} e^{t \operatorname{Re} z} \|X_{M+1}(-\Delta - z)^{-M-1}\|_{\mathcal{L}_1} \frac{|dz|}{|z|} \\
&\leq C' \int_{-1/t-i\infty}^{-1/t+i\infty} |z|^{-M+\frac{n-1}{2}} \frac{|dz|}{|z|} \\
&\leq C' t^{M+\frac{1-n}{2}} \int_{-1-i\infty}^{-1+i\infty} |w|^{-M+\frac{n-1}{2}} \frac{|dw|}{|w|} \\
&= \mathcal{O}(t^{M-\frac{n-1}{2}}),
\end{aligned}$$

where the integral converges if $M \geq n$. Combined with (3.11.32) that gives an estimate of the remainder in (3.11.27). $\square$

3.11.3. Asymptotic expansion. To relate the asymptotic expansion of σ' in Theorem 3.62 with the heat trace asymptotics in Theorem 3.64 we will use the following elementary fact:

LEMMA 3.65. *For $m \in \mathbb{Z}$ and $t > 0$ define*

$$u_m(t) := \int_1^\infty \lambda^m e^{-t\lambda^2} d\lambda \in C^\infty((0,\infty)_t).$$

Then

$$\begin{aligned}
&u_m(t) - \tfrac{1}{2}\Gamma\left(\tfrac{m+1}{2}\right) t^{-\frac{m+1}{2}} \in C^\infty([0,\infty)_t), \quad m \geq 0 \quad \text{or} \quad m \in -2\mathbb{N}, \\
&u_m(t) - \tfrac{(-1)^{k+1}}{2\,k!} t^k \log t \in C^\infty([0,\infty)_t), \quad m = -2k-1, \quad k \in \mathbb{N},
\end{aligned} \tag{3.11.33}$$

where $C^\infty([0,\infty)_t)$ denotes functions which are smooth up to 0 *in t, $\mathbb{N} = \{0, 1, \cdots\}$.*

Proof. 1. We first consider the case of $m \geq 0$. Then

$$u_m(t) - \int_0^\infty \lambda^m e^{-t\lambda^2} d\lambda \in C^\infty([0,\infty)_t),$$

and

$$\int_0^\infty \lambda^m e^{-t\lambda^2} d\lambda = t^{-\frac{m+1}{2}} \int_0^\infty x^m e^{-x^2} dx = \tfrac{1}{2}\Gamma\left(\tfrac{m+1}{2}\right) t^{-\frac{m+1}{2}},$$

proving the claim.

2. When $m = -2k$, $k \in \mathbb{N}$ then,

$$(-\partial_t)^k u_m(t) = u_0(t) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) t^{-\frac{1}{2}} + w_0(t), \quad w_0 \in C^\infty([0,\infty)_t).$$

Integrating k times from 1 to t ($t \to 0+$) gives the desired expression up to a smooth additive term.

3. When $m = -2k-1$, $k \in \mathbb{N}$ then,

$$(-\partial_t)^{k+1} u_m(t) = u_1(t) = \tfrac{1}{2} t^{-1} + w_1(t), \quad w_1 \in C^\infty([0,\infty)_t).$$

We again integrate $k+1$ times from 1 to t ($t \to 0+$) which gives the second case of (3.11.33). ☐

THEOREM 3.66 (**Asymptotics of $\boldsymbol{\sigma'(\lambda)}$**). *Let $\sigma'(\lambda)$ be given by (3.11.19). Then, only even powers appear in the expansion (3.11.20):*

$$\sigma'(\lambda) \sim \sum_{k=1}^{\infty} c_k'(V) \lambda^{n-2k-1}, \quad c_k'(V) := \frac{2a_k(V)}{\Gamma(\frac{n}{2}-k)(4\pi)^{\frac{n}{2}}}, \tag{3.11.34}$$

where $a_k(V)$ are given by (3.11.27). Moreover,

$$\int_0^\infty \left(\sigma'(\lambda) - \sum_{k=1}^{\frac{n-1}{2}} c_k'(V) \lambda^{n-2k-1} \right) d\lambda = -K - \tfrac{1}{2} \widetilde{m}_R(0), \tag{3.11.35}$$

where K is the number of eigenvalues of P_V and $\widetilde{m}_R(0)$ is the multiplicity of the zero resonance (3.3.29).

Proof. 1. To obtain (3.11.34) we apply Theorem 3.51 to the function $f(\lambda) = e^{-t\lambda^2}$ (see Remark after the statement of Theorem 3.51):

$$\int_0^\infty e^{-t\lambda^2} \sigma'(\lambda) d\lambda = \operatorname{tr}\left(e^{-tP_V} - e^{-tP_0}\right) - \sum_{k=1}^{K} e^{-tE_k} - \frac{1}{2} \widetilde{m}_R(0). \tag{3.11.36}$$

The right hand side has an expansion

$$\frac{1}{(4\pi t)^{n/2}} \sum_{k=1}^{\infty} a_k t^k + \sum_{k=1}^{\infty} b_k t^k - (K + \tfrac{1}{2} \widetilde{m}_R(0)), \quad t \to 0+ \tag{3.11.37}$$

where the second (smooth) sum comes from the Taylor expansion of the eigenvalue contributions.

2. The asymptotic expansion (3.11.20) means that for $\lambda > 1$ and any J,

$$\sigma'(\lambda) = \sum_{j=1}^{J} b_j \lambda^{n-j-2} + \mathcal{O}(\lambda^{n-J-3}).$$

In the notation of Lemma 3.65 we then see that for $J \geq n-2$,

$$\int_0^\infty \sigma'(\lambda) e^{-t\lambda^2} d\lambda - \sum_{j=1}^{J} b_j u_{n-j-2}(t) \in C^{[\frac{J+3-n}{2}]}([0,\infty)_t). \tag{3.11.38}$$

Comparison of (3.11.33) with (3.11.36) shows that $b_j = 0$ for $j = 2k$, and $b_{2k-1} = c_k'(V)$ where $c_k'(V)$ are given in (3.11.34).

3. It remains to show that (3.11.35) holds. Applying (3.11.38) with $J = n-2$ (which corresponds to $k = (n-1)/2$) shows that

$$G(t) := \int_0^\infty \left(\sigma'(\lambda) - \sum_{k=1}^{\frac{n-1}{2}} c_k'(V)\lambda^{n-2k-1} \right) e^{-t\lambda^2} d\lambda$$

is a continuous function on $[0,\infty)$. The integrand in the definition of $G(t)$ is uniformly bounded by $C\langle\lambda\rangle^{-2}$ as $t \geq 0$ and hence $G(0+)$ is equal to the left hand side of (3.11.35). Comparison with (3.11.37) gives $G(0+) = K + \frac{1}{2}\widetilde{m}_R(0)$, completing the proof. $\square$

For completeness we include a result about the asymptotic behaviour of the actual scattering phase:

THEOREM 3.67 (Asymptotics of the scattering phase). *Suppose that $V \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$ where $n \geq 1$ is odd. Define the* scattering phase

$$\sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda), \tag{3.11.39}$$

by demanding that $\sigma'(\lambda)$ is given by (3.11.19) and that $\sigma(0) = \widetilde{m}_R(0)/2$ where $\widetilde{m}_R(0)$ is given in (3.3.29).

Then there exists a sequence $c_k(V)$ such that, as $\lambda \to +\infty$,

$$\sigma(\lambda) \sim \sum_{k=1}^{\frac{n-1}{2}} c_k(V)\lambda^{n-2k} - K + \sum_{k=\frac{n+1}{2}}^{\infty} c_k(V)\lambda^{n-2k}, \tag{3.11.40}$$

where K is the number of eigenvalues of P_V and

$$c_k(V) = \frac{a_k(V)}{\Gamma(\frac{n}{2}-k+1)(4\pi)^{\frac{n}{2}}},$$

with $a_k(V)$ are given in (3.11.36). In particular,

$$\begin{aligned} c_1(V) &= -\frac{1}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} V(x)dx\,, \\ c_2(V) &= \frac{1}{2\Gamma(\frac{n}{2}-1)(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} V(x)^2 dx\,. \end{aligned} \tag{3.11.41}$$

Proof. 1. We know from (3.7.28) that $\det S(0) = (-1)^{\widetilde{m}_R(0)}$ (see also (3.10.4)). Hence our choice of the value of $\sigma(0)$ determines a branch of the logarithm.

2. We know that $\sigma'(\lambda)$ is an even function and hence $\sigma(\lambda) - \frac{1}{2}\widetilde{m}_R(0)$ is odd. In view of (3.11.35), we have for $\lambda > 0$,

$$\sigma(\lambda) = \sum_{k=1}^{\frac{n-1}{2}} \lambda^{n-2k} - K - \int_\lambda^\infty \left(\sigma'(\tau) - \sum_{k=1}^{\frac{n-1}{2}} c_k'(V)\tau^{n-2k-1} \right) d\tau.$$

Integrating (3.11.34) and noting that

$$\frac{2}{(n-2k)\Gamma(\frac{n}{2}-k)} = \frac{1}{\Gamma(\frac{n}{2}-k+1)}$$

gives (3.11.40). □

3.12. EXISTENCE OF RESONANCES FOR REAL POTENTIALS

Theorem 2.16 implies that any complex valued compactly supported potential in one dimension has infinitely many resonances. In Section 3.5 earlier in this chapter we have shown that there exist complex valued compactly supported potentials in higher dimensions with *no* resonances. We will now prove existence of infinitely many resonances for arbitrary potentials $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$. It is based on the Birman–Kreĭn formula (2.6.1) applied with $f(s) = e^{-ts}$ and *heat trace asymptotics* as $t \to 0$.

THEOREM 3.68 (Existence of resonances). *Suppose that*

$$V \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), \quad \textit{with } n \geq 3, \textit{ odd, } V \neq 0.$$

Then

$$\sum_{\lambda \in \mathbb{C}} m_R(\lambda) = \infty\,,$$

that is, V has infinitely many scattering resonances.

REMARK. To show that there have to be *some* resonances we only use the Birman–Krein formula and the factorization of the scattering matrix from §§3.9 and 3.10 respectively. The asymptotic analysis of §3.11 is needed to show that there are infinitely many resonances.

Proof. 1. We first assume that of $R_V(\lambda)$ has no poles at all. Then Theorem 3.54 implies that

$$\begin{aligned} \sigma(\lambda) &:= \frac{1}{2\pi i}\log\det S(\lambda) \\ &= b_n\lambda^n + b_{n-2}\lambda^{n-2} + \cdots + b_1\lambda\,, \quad b_j \in \mathbb{R}\,. \end{aligned} \tag{3.12.1}$$

Using (2.6.1) with $f(s) := e^{-st}$ we obtain

$$\begin{aligned} \operatorname{tr}(e^{-tP_V} - e^{-tP_0}) &= \int_0^\infty f(\lambda^2)\sigma'(\lambda)d\lambda + m_0 \\ &= \alpha_n b_n t^{-\frac{n}{2}} + \alpha_{n-2}b_{n-2}t^{-\frac{n}{2}-1} + \cdots \\ &\quad + \alpha_1 b_1 t^{-\frac{1}{2}} + m_0, \end{aligned} \tag{3.12.2}$$

where $m_0 = m_R(0) - \frac{1}{2}\widetilde{m}_R(0)$ and $\alpha_k = k\int_0^\infty x^{k-1}e^{-x^2}dx = \Gamma\left(\frac{k}{2}+1\right) \neq 0$. Comparison with (3.11.40) and (3.11.41) shows that $b_n = 0$ and

$$b_{n-2} = -\beta_{n-2}\int_{\mathbb{R}^n} V(x)dx\,, \quad b_{n-4} = \beta_{n-4}\int_{\mathbb{R}^n} V(x)^2dx \neq 0.$$

This gives an immediate contradiction when $n = 3$ as (3.12.2) contradicts the formula (3.11.27).

2. To obtain a contradiction for $n > 3$. We consider the behaviour of $\sigma(\lambda)$ as $\lambda \to 0$. Suppose that $R_V(\lambda)$ is entire. Theorem 3.17 shows

$$R_V(\lambda)(Ve^{i\lambda\langle\bullet,\omega\rangle})(x) = B(\lambda, x, \omega),$$

where B is holomorphic in λ near 0 and smooth in $x \in \mathbb{R}^n$, $\omega \in \mathbb{S}^{n-1}$. The formula (3.7.8) then shows that

$$\|A(\lambda)\|_{\mathcal{L}_1} = \mathcal{O}(\lambda^{n-2})\,, \quad \lambda \to 0\,.$$

Using this we see that near 0,

$$\begin{aligned} 2\pi i\sigma(\lambda) &= \log\det(I + A(\lambda)) = \operatorname{tr}\log(I + A(\lambda)) \\ &= \mathcal{O}(\|A(\lambda)\|_{\mathcal{L}_1}) = \mathcal{O}(\lambda^{n-2})\,. \end{aligned}$$

Comparing this with (3.12.1) we see that $\sigma(\lambda) = b_{n-2}\lambda^{n-2}$. But this contradicts the fact that $b_{n-4} \neq 0$.

3. It remains to show that the number of resonances is infinite. Again we proceed by contradiction using Theorems 3.54 and 3.67. Suppose that there exists a finite number of non-zero resonances. Hence suppose that $-\mu_1^2 < -\mu_2^2 \leq \cdots \leq -\mu_{K'}^2 < 0$, $\mu_k > 0$, are the negative eigenvalues of P_V and that $i\rho_j$, $\rho_j < 0$, $j = 1, \cdots, J_1$, $\lambda_j \neq -\bar{\lambda}_j$, $j = 1\cdots, J_2$ are a finite set of resonances with $\operatorname{Re}\lambda_j > 0$. Factorization (3.10.4) shows that

$$\det S(\lambda) = (-1)^m e^{g(\lambda)}\prod_{k=1}^{K'}\frac{\lambda + i\mu_k}{\lambda - i\mu_k}\prod_{j=1}^{J_1}\frac{\lambda + i\rho_j}{\lambda - i\rho_j}\prod_{j=1}^{J_2}\frac{\lambda - \bar{\lambda}_j}{\lambda + \lambda_j}\frac{\lambda - \lambda_j}{\lambda + \bar{\lambda}_j}.$$

Hence for $\lambda \in \mathbb{R}$ and with $b(\lambda) := g'(\lambda)/2\pi i$,

$$\begin{aligned} \sigma'(\lambda) - b(\lambda) &= -\frac{1}{\pi}\sum_{k=1}^{K'}\frac{\mu_j}{\lambda^2 + \mu_j^2} - \frac{1}{\pi}\sum_{j=1}^{J_1}\frac{\rho_j}{\lambda^2 + \rho_j^2} \\ &\quad - \frac{1}{\pi}\sum_{j=1}^{J_2}\left(\frac{\operatorname{Im}\lambda_j}{|\lambda - \lambda_j|^2} + \frac{\operatorname{Im}\lambda_j}{|\lambda + \lambda_j|^2}\right), \end{aligned} \tag{3.12.3}$$

That means that

$$\int_0^\infty \left(\sigma'(\lambda) - b(\lambda)\right)d\lambda = -\tfrac{1}{2}K' + \tfrac{1}{2}J_1 + J_2.$$

On the other hand, (3.11.35) shows that the left hand side is equal to $-K' - m_0$, $m_0 := m_R(0) + \frac{1}{2}\widetilde{m}_R(0) \geq \frac{1}{2}m_R(0) \geq 0$, and hence,

$$0 \geq -\tfrac{1}{2}K' - m_0 = \tfrac{1}{2}J_1 + J_2 \geq 0,$$

which means that both sides vanish. Since we showed that some resonances exist this gives a contradiction. □

3.13. NOTES

This chapter presented odd dimensional potential scattering from the perspective of the study of resonances. For a direct treatment of obstacle scattering in dimension three see Taylor [**TaII**, Chapter 9].

Results in §3.1 are classical. The proof of Theorem 3.1 comes from Vodev [**Vo92**]. The contour deformation argument in §3.1.4 is a baby version of homological conditions for the existence of lacunas for hyperbolic equations due to Petrovsky – see Atiyah–Bott–Gårding [**ABG70**] for a detailed presentation. We learned this many years ago from Johannes Sjöstrand – see [**Sj02**, §2.1]. Exercise 3.3 was suggested by Gilles Carron.

The discussion of the resonance of zero in §3.3 can be used as an introduction, in the spirit of PDE, to the now classical work of Jensen–Kato [**JK79**] (where (3.3.21) is described using the operator given in (3.3.23)). For further discussion of the threshold behaviour for non-compactly supported potentials see Jensen–Nenciu [**JN01**], Rodnianski–Tao [**RT15**] and references given there. The survey by Schlag [**Sc07**] can be consulted for the role of threshold resonances for non-linear equations, and [**HZ09**, Figure 6] for an example of a linearly counterintuitive phenomenon involving a resonance at zero.

The proof of Theorem 3.27 is based on ideas of Melrose who proved the bound

$$\sum \{ m_R(\lambda) \; : \; |\lambda| \leq r \} \leq C_V r^{n+1} .$$

The optimal bound (3.4.7) was proved in [**Zw89b**]. Our presentation uses a substantial simplification of the argument due to Vodev [**Vo92**] – see Chapter 4 for further applications of these methods.

For the early history of Rellich's uniqueness theorem, Sommerfeld radiation patterns and of outgoing solution see Wilcox [**Wi56**]. Our definition that $u = R_0(\lambda)g$ for $g \in \mathcal{E}'(\mathbb{R}^n)$ is equivalent to the more classical definition which in dimension three states that for R_0 sufficiently large and $|x| > R_0$,

$$u(x) = \frac{1}{4\pi} \int_{\partial B(0,R_0)} \left(u(y) \partial_r \left(\frac{e^{i\lambda|x-y|}}{|x-y|} \right) - \frac{e^{i\lambda|x-y|}}{|x-y|} \partial_r u(y) \right) dS(y), \tag{3.13.1}$$

see Exercise 3.6.

The class of examples in Theorem 3.29 was constructed by Christiansen [**Ch06**].

The interpretation of the (absolute) scattering matrix as mapping the the incoming "boundary data" to incoming "boundary data" was emphasized by Melrose in many geometric settings [**Me95**]. Here the boundary refers to the boundary at infinity. The more traditional interpretation in which the scattering matrix provides a mapping between distorted plane waves (see Theorem 3.47) is given in (3.8.12). In time dependent scattering theory the *scattering operator*, S, is defined using wave operators:

$$W_{\pm}u := \lim_{t\to\pm\infty} e^{itP_V}e^{-itP_0}u, \quad u \in L^2(\mathbb{R}^n), \quad S := W_+^* W_-.$$

The scattering operator S commutes with $P_0 = -\Delta$ (as formally follows from the definitions of $W_{\pm}$ which intertwine P_V and P_0) and hence we can decompose it using the spectral decomposition of $-\Delta$:

$$S = \int_0^\infty S(\lambda) dE_\lambda^0, \quad f(-\Delta) = \int_0^\infty f(\lambda^2) dE_\lambda^0,$$

and our scattering matrix $S(\lambda)$ is produced. Reed–Simon [**RS79**], Hörmander [**HöII**, §14.4], Newton [**N02**], Yafaev [**Ya92**], [**Ya09**], Melrose–Uhlmann [**MU**] and Taylor [**TaII**, Chapter 8] can be consulted for different mathematical perspectives on the subject. Analytic properties of the scattering matrix were discussed early on by Jensen [**Je80b**].

A different point of view, rooted in the wave equation and three dimensional obstacle scattering rather than quantum mechanics, was proposed by Lax–Phillips [**LP68**]. The key object in their theory is the Lax–Phillips semigroup, $Z(t)$, obtained by truncating the wave group $U(t)$ by the orthogonal projection, π, onto the *interaction space* which is the orthogonal complement of spaces of incoming and outgoing data: $Z(t) = \pi U(t)\pi$. This provides a beautiful dynamical definition of resonances as eigenvalues of B, the generator of $Z(t) = e^{-itB}$. This point of view played a crucial role in the understanding of the relation between the distribution of resonances and trapping (see §4.6), the developments of trace formulas (see later in this section) and automorphic scattering (see Examples 2 and 3 in §4.4.3). For a brief, self-contained presentation of generalized Lax-Phillips theory see Sjöstrand–Zworski [**SZ94**, §2]. Various subtle issues such as multiplicities and domains of operators were also clarified there.

Since this book is devoted to the study of scattering resonance we focus on the stationary, energy dependent scattering matrix. The general formula (3.7.18) giving $S(\lambda)$ in terms of the resolvent is valid for general compactly supported perturbations (see Theorem 4.26) and comes from Petkov–Zworski [**PZ01**].

The trace identity in Theorem 3.46 was proved by Buslaev in [**Bu62**]. Theorem 3.67 and further references can be found in [**Gu84**]. The Birman–Kreĭn formula goes back to the classical paper [**BK62**] and is related to the more general study of spectral shift functions. For that connection and for references see Yafaev [**Ya92**, Chapter 8] and [**Ya09**, Chapter 9].

Trace formulas relating the wave group to the resonances were established by Lax–Phillips [**LP78**] and Bardos–Guillot–Ralston [**BGR82**] (in the closely related obstacle case) but for $t > 2R$, where $\operatorname{supp} V \subset B(0,R)$. The case of $t > 0$ was proved by Melrose [**Me82**] and generalized by Sjöstrand–Zworski [**SZ94**]. A different proof with a more precise statement at $t = 0$ was given in [**Zw97**] (see also [**GZ97**]) and our exposition provides a more detailed account of the argument there. The idea for obtaining the correct power of t in (3.10.2) based on Lemma 3.56 was suggested by Jeff Galkowski. For a trace formula in even dimensions see Christiansen [**Ch17b**] and Zworski [**Zw98**].

Lemma 3.55 reverse engineers [**Me88**, §4]. The terms in the definition (3.10.12) there can be recognized as the Breit–Wigner Lorentzians (1.1.3), see also Theorem 2.20 and Figure 2.6. Further analysis leads to a justification of the Breit–Wigner formulas in the presence of many resonances – see [**PZ99**],[**PZ01**]. For the Breit-Wigner approximation in the semiclassical limit and for isolated resonances see Gérard–Martinez–Robert [**GMR89**].

Theorem 3.67 is due to Colin de Verdière [**CdV81a**] when $n = 3$ and Guillopé [**Gu84**] for other dimensions, see also Buslaev [**Bu75**] and Christiansen [**Ch98**] and references given there. Analysis of heat trace asymptotics follows Hitrik–Polterovich [**HP03**] where more general potentials were considered – see that paper for references and also [**SZ16**] for a more direct approach to heat trace expansions.

That a smooth compactly supported potential (or any superexponentially decaying potential) in any odd dimension has infinitely many resonances was proved by Sá Barreto–Zworski [**SZ96**] but the method there was less direct.

There have been many improvement since. Christiansen and Hislop [**CH05**] proved that for a generic $L^\infty_{\rm com}(\mathbb{R}^n, \mathbb{R})$ (or $C^\infty_{\rm c}(\mathbb{R}^n, \mathbb{R})$) potential the exponent n in the polynomial bound (3.4.7) is optimal. That relied on the existence of a lower bound given by (3.4.16) and on results from the theory of several complex variables. It is also true for generic *complex valued* potentials. The paper [**CH05**] can be consulted for intermediate results on lower bounds. Here we mention Christiansen [**Ch99**] and Sá Barreto [**Sá01**] who proved that

$$\limsup_{r\to\infty} \frac{N(r)}{r} > 0,$$

and their methods inspired our presentation. Dinh–Vu [**DV13**] also used several complex variables techniques and showed that a large class of potentials supported in a ball enjoys the same counting asymptotics as radial potentials; see also Dinh–Nguyen [**DN17**].

More recently Smith–Zworski [**SZ16**] showed that any

$$V \in H^{\frac{n-3}{2}}(\mathbb{R}^n) \cap L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{R}),$$

has infinitely many resonances and any $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{R})$ has some resonances (all for n odd).

When magnetic field is added interested new phenomena occur which are not studied here or in the semiclassical setting of Chapter 5.16. Alexandrova–Tamura [**AT14**], Bony–Bruneau–Raikov [**BBR14**] and Tamura [**Ta15**] can be consulted for some recent results and for pointers to the literature.

3.14. EXERCISES

Section 3.1

We assume here that $n \geq 3$ is odd and write $R_0(\lambda, x, y) = R_0(\lambda, x-y)$ for the Schwartz kernel of the free resolvent.

1. Show that for any $r \in \mathbb{R}$ and c_0 there exists a constant C_1 such that for $\operatorname{Im}\lambda \geq c_0$ we have

$$\|\langle x\rangle^r R_0(\lambda)\langle x\rangle^{-r}\|_{L^2\to L^2} \leq C_1, \tag{3.14.1}$$

that is,

$$R_0(\lambda) : \langle x\rangle^{-r} L^2(\mathbb{R}^2) \to \langle x\rangle^{-r} L^2(\mathbb{R}^n).$$

Hint. Use Theorem 3.3 and Schur's criterion (see the proof of the Lemma 3.7) noting that $\langle x\rangle^{-r}\langle y\rangle^r \leq \langle x-y\rangle^{|r|}$.

2. Show that for any $C_0 > 0$ there exists C_1 such that for such that for $|\lambda|/C_0 \leq \operatorname{Im}\lambda \leq C_0$,

$$\|R_0(\lambda, \bullet)\|_{L^q(\mathbb{R})} \leq C_1|\lambda|^{-2+n(q-1)/q}, \quad 1 \leq q < \frac{n}{n-2}. \tag{3.14.2}$$

3. Use the definition of P_n in Theorem 3.3 to show that $\lambda \mapsto R_0(\lambda)^2 = \partial_\lambda R_0(\lambda)/2\lambda$ is analytic near 0 as an operator from $L^2_{\rm comp} \to L^2_{\rm loc}$. Conclude that for $n \geq 5$, Lemma 3.6 can be improved to

$$\begin{gathered}\|\rho R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda)\rho\|_{L^2 \to H^{2k}} \leq C_1, \quad \rho \in C^\infty_{\rm c}(\mathbb{R}^n), \\ \text{for } 0 \leq |\lambda| \leq C_0 \leq \tfrac{1}{2} \operatorname{Im} \lambda_0 \leq 2C_0, \ \ \operatorname{Im}\lambda \geq 0.\end{gathered} \tag{3.14.3}$$

Section 3.2

4. Suppose that $V \in L^\infty_{\rm comp}(\mathbb{R}^n; \mathbb{C})$. Show directly that the Schwartz kernel of $R_V(\lambda)$ satisfies

$$R_V(\lambda, x, y) = R_V(\lambda, y, x), \tag{3.14.4}$$

in the sense of distributions – that is $R_V(\lambda) = j^* R_V(\lambda) \in \mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^n_y)$, $j(x,y) = (y,x)$.

Hint: it is enough to prove (3.14.4) for $\operatorname{Im}\lambda \gg 1$ and that follows from having

$$\int_{\mathbb{R}^n} R_V(\lambda) f_1(x) f_2(x) dx = \int_{\mathbb{R}^n} f_1(x) R_V(\lambda) f_2(x) dx, \ \ f_j \in L^2(\mathbb{R}^n), \ \operatorname{Im}\lambda \gg 1.$$

But for that we can put $F_j(x) := R_V(\lambda) f_j \in H^2$, and use integration by parts: $\int F_1 (P_V - \lambda^2) F_2 = \int (P_V - \lambda^2) F_1 F_2$.)

Section 3.6

5. Prove Theorem 3.37.

Hint: $(i) \Rightarrow (ii)$ is obvious and $(iv) \Rightarrow (i)$ follows from the expansion (3.1.20). Since for $\lambda \in \mathbb{R} \setminus \{0\}$, $R_V(\lambda)\rho = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$, and hence $(I + VR_0(\lambda)\rho)^{-1}$ ($R_0(\lambda)$ is injective on $L^\infty_{\rm comp}(\mathbb{R}^n)$) have not no poles (Theorem 3.33), we see that $(iii) \Rightarrow (iv)$. We finally get $(ii) \Rightarrow (iii)$ by applying Theorem 3.35 to $u - R_V(\lambda) f$. For the last statement use elliptic regularity: for V and f smooth, u is smooth.

6. Let $R_0(\lambda, x, y)$ be the Schwartz kernel of $R_0(\lambda)$ and suppose that $u = R_0(\lambda) f$ where $f \in \mathcal{E}'(\mathbb{R}^n)$ and $\operatorname{Im}\lambda > 0$. Suppose that $\mathcal{O}$ is a bounded open set with a smooth boundary, $\operatorname{supp} f \subset \mathcal{O}$. Show that for $u \notin \overline{\mathcal{O}}$,

$$u(x) = \int_{\partial\mathcal{O}} \left(u(y) \partial_\nu R_0(\lambda, x, y) - R_0(\lambda, x, y) \partial_\nu u(y)\right) dS(y), \tag{3.14.5}$$

where $\partial_\nu = \langle \nu(y), \partial \rangle$, $\nu(y)$ is the outward unit normal vector to $\partial\mathcal{O}$ and $dS(y)$ is the surface measure on $\partial\mathcal{O}$. Deduce (3.13.1).

7. Show that if $(P_V - \lambda^2) u = g$, $g \in \mathcal{E}'(\mathbb{R}^n)$ and (3.14.5) holds then there exists $f \in \mathcal{E}'(\mathbb{R}^n)$ such that $u = R_0(\lambda) f$.

Section 3.7

8. Suppose that $\lambda > 0$. For a given $f \in C^\infty(\mathbb{S}^{n-1})$ construct v_0 such that

$$(-\Delta - \lambda^2)v_0 \in \mathscr{S}(\mathbb{R}^n), \quad v_0(r\theta) \sim \frac{e^{i\lambda r}}{r^{\frac{n-1}{2}}} f(\theta).$$

Show that $v_0 = R_V(\lambda)(P_V - \lambda^2)v_0$.

9. Show that for $n = 3$, $\det S(0) = (-1)^m$ where m is the multiplicity of resonance at zero and that for $n \geq 5$, $\det S(0) = 1$.

10. Suppose that $V \in L^\infty(B(0,R);\mathbb{R})$ and that that $i\mu_j$, $j = 1, \ldots, J$, are the poles of the scattering matrix with $\mu_j > 0$. Show that

$$\|S(\lambda)\| \leq e^{2R \operatorname{Im} \lambda} \prod_{j=1}^{J} \frac{|\lambda + i\mu_j|}{|\lambda - i\mu_j|}, \quad \operatorname{Im} \lambda > 0. \tag{3.14.6}$$

Hint: Consider

$$S_1(\lambda) := e^{2Ri\lambda} \prod_{j=1}^{J} \frac{\lambda - i\mu_j}{\lambda + i\mu_j} S(\lambda),$$

which is holomorphic in $\operatorname{Im} \lambda \geq 0$ and $\|S_1(\lambda)\| = 1$ for $\operatorname{Im} \lambda = 0$. The bound (3.14.6) is equivalent to $\|S_1(\lambda)\| \leq 1$. Representation of the scattering matrix (3.7.8) shows that $\|S_1(\lambda)\| \leq C(1 + |\lambda|)^N$ for $\operatorname{Im} \lambda > 0$. Applying the Phragmén–Lindelöf principle to $S_1(\lambda)$ in the upper half plane (see for instance [**Ti86**, §5.61]) gives $\|S_1(\lambda)\| \leq 1$, $\operatorname{Im} \lambda \geq 0$, and that proves (3.14.6).

Section 3.11

11. Prove the following refinement of Lemma 3.52:

$$\begin{aligned} \operatorname{tr}(e^{-tP_V} - e^{-tP_0}) &= \sum_{k=1}^{K} e^{-tE_k} + \tfrac{1}{2}\widetilde{m}_R(0) \\ &\qquad + t^{-\frac{1}{2} - \frac{(n-5)_+}{2}} \sum_{k=0}^{K} t^{-k} h_k + \mathcal{O}(t^{-K-\frac{n}{2}}), \end{aligned} \tag{3.14.7}$$

as $t \to +\infty$. What are the improvements if there is no zero eigenvalue or resonance?

Hint: Use Theorem 3.51 and the facts that $\sigma'(\lambda)$ is polynomially bounded (Theorem 3.62), $\sigma'(\lambda) = \mathcal{O}(\lambda^{n-5})$ for $n \geq 5$ and $\mathcal{O}(1)$ for $n = 3$ near 0 (use Theorem 3.58 and the structure of the resolvent at 0 from §3.3).

Part 2

GEOMETRIC SCATTERING

Chapter 4

BLACK BOX SCATTERING IN $\mathbb{R}^n$

4.1 General assumptions
4.2 Meromorphic continuation
4.3 Upper bounds on the number of resonances
4.4 Plane waves and the scattering matrix
4.5 Complex scaling
4.6 Singularities and resonance free regions
4.7 Notes
4.8 Exercises

In Chapters 2 and 3 we studied general properties of resonances in scattering by compactly supported potential. More general compactly supported perturbations include metric perturbations and obstacle scattering. They offer many new interesting and relevant physical features such as presence of trapping – see Figure 4.1.

For general results of the type seen in the case of potential scattering it is convenient to replace a specific perturbation by an abstractly defined *black box* perturbation. The table below shows the basic differences and analogies in the case when n is odd.

Here P denotes a self-adjoint operator equal to $-\Delta$ outside $B(0, R_0)$ – see Section 4.1 for precise assumptions. The operator P is assumed to act on a Hilbert space $\mathcal{H}$ with an orthogonal decomposition $\mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$. The orthogonal projection onto the first component – the *black box* – is denoted $\mathbb{1}_{B(0,R_0)}$.

An example is given by a compactly supported metric perturbation. More precisely, let g, $g_{jk} - \delta_{jk} \in C^\infty_{\rm c}(\mathbb{R}^n)$, be a metric on $\mathbb{R}^n$ and let P be the corresponding *Laplace–Beltrami operator*:

$$P = -\Delta_g := -\frac{1}{\sqrt{|g|}} \sum_{j,k} \partial_{x_j} \sqrt{|g|} g^{jk} \partial_{x_k},$$
$$(g^{jk}) := (g_{jk})^{-1}, \quad |g| := \det(g_{jk}).$$

$-\Delta + V$	Black Box
Meromorphy of the resolvent $R_V(\lambda) : L^2_{\rm comp} \to L^2_{\rm loc}$; Theorem 3.8.	If $\mathbb{1}_{B(0,R_0)}(P-i)^{-1}$ is compact then $R(\lambda) : \mathcal{H}_{\rm comp} \to \mathcal{H}_{\rm loc}$ is meromorphic; Theorem 4.4.
Upper bound on the number of resonances, $N(r) \leq Cr^n$; Theorem 3.27.	Upper bounds using bounds for eigenvalues of a reference operator: $N(r) \leq Cr^{n^\#}$; Theorem 4.13.
Trace formula for resonances; Theorem 3.53.	Trace formulae hold if for some k $\mathbb{1}_{B(0,R_0)}(P-i)^{-k} \in \mathcal{L}_1(\mathcal{H}, \mathcal{H})$; [**SZ94**],[**Zw97**].
Pole free regions; Theorem 3.10	Geometric assumptions about the classical flow are needed; Theorems 4.43, 6.10, 6.16, [**Ma02b**],[**SZ07a**, §3].
Resonance expansions of waves; Theorem 3.11	Delicate when there are no large pole free regions; Theorem 7.20, [**TZ00**].

4.1. GENERAL ASSUMPTIONS

Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)), \tag{4.1.1}$$

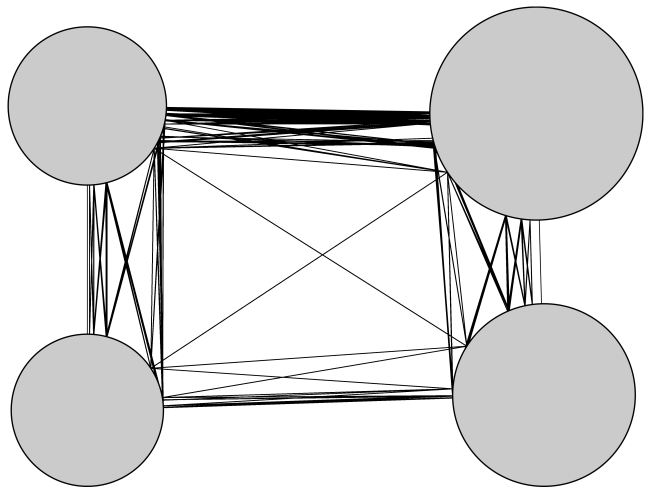


Figure 4.1. An example of trapped trajectories in obstacle scattering.

Here $R_0 > 0$ is fixed and

$$B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}.$$

The orthogonal projections on the first and the second summands in (4.1.1) can be thought of as restrictions of elements of $\mathcal{H}$ to $B(0, R_0)$ and $\mathbb{R}^n \setminus B(0, R_0)$:

$$u \longmapsto \mathbb{1}_{B(0,R_0)}\, u =: u|_{B(0,R_0)} \in \mathcal{H}_{R_0},$$
$$u \longmapsto \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}\, u =: u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2(\mathbb{R}^n \setminus B(0, R_0)).$$

If $\chi \in L^\infty(\mathbb{R}^n)$ and $\chi \equiv c_0 \in \mathbb{C}$ (is equal to a constant) on $B(0, R_0)$ then we define

$$\chi u := c_0 \left(u|_{B(0,R_0)}\right) + \left(\chi|_{\mathbb{R}^n \setminus B(0,R_0)}\right) \left(u_{\mathbb{R}^n \setminus B(0,R_0)}\right),$$

where the restriction of χ is the restriction of a function in $L^\infty(\mathbb{R}^n)$ to a subset of $\mathbb{R}^n$.

We define a smaller space of compactly supported elements of $\mathcal{H}$ as

$$\mathcal{H}_{\text{comp}} := \left\{u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2_{\text{comp}}(\mathbb{R}^n \setminus B(0, R_0))\right\}, \tag{4.1.2}$$

and a larger spaces of vectors locally in $\mathcal{H}$:

$$\mathcal{H}_{\text{loc}} := \mathcal{H}_{R_0} \oplus L^2_{\text{loc}}(\mathbb{R}^n \setminus B(0, R_0)). \tag{4.1.3}$$

We now assume that $P(h)$, $0 < h \leq 1$, is a family of unbounded self-adjoint operators, $P(h) : \mathcal{H} \longrightarrow \mathcal{H}$, with the domain $\mathcal{D} \subset \mathcal{H}$, independent of h. We assume that

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}\, \mathcal{D} \subset H^2(\mathbb{R}^n \setminus B(0, R_0)). \tag{4.1.4}$$

Outside of the "black box" $B(0, R_0)$ the operator is equal to the semiclassical Laplacian in the following sense:

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}(P(h)u) = -h^2 \Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \quad u \in \mathcal{D}, \tag{4.1.5}$$

where the right hand side defines an element of $L^2(\mathbb{R}^n \setminus B(0, R_0))$ thanks to (4.1.4).

Condition (4.1.4) is complemented by the condition

$$v \in H^2(\mathbb{R}^n), \quad v|_{B(0,R_0+\varepsilon)} \equiv 0 \ \text{ for some } \varepsilon > 0 \quad \Longrightarrow \quad v \in \mathcal{D}. \tag{4.1.6}$$

(Since $v = \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}\, v$, v defines an element of $\mathcal{H}$ and hence this statement makes sense.)

For vectors v satisfying the condition in (4.1.6) we have

$$P(h)v = -h^2 \Delta v. \tag{4.1.7}$$

In fact, if $u \in \mathcal{D}$ then $u|_{\mathbb{R}^n \setminus B(0,R_0)} \in H^2(\mathbb{R}^n \setminus B(0, R_0))$ and (recalling that $v \in H^2(\mathbb{R}^n)$ and that it vanishes in $B(0, R_0 + \varepsilon)$)

$$\begin{aligned}
\langle P(h)v, u\rangle_{\mathcal{H}} = \langle v, P(h)u\rangle_{\mathcal{H}} &= \langle v, (P(h)u)|_{\mathbb{R}^n \setminus B(0,R_0)}\rangle_{L^2(\mathbb{R}^n \setminus B(0,R_0))} \\
&= \langle v, -h^2\Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)})\rangle_{L^2(\mathbb{R}^n \setminus B(0,R_0))} \\
&= \langle -h^2\Delta v, u|_{\mathbb{R}^n \setminus B(0,R_0)}\rangle_{L^2(\mathbb{R}^n \setminus B(0,R_0))} \\
&= \langle -h^2\Delta v, u\rangle_{\mathcal{H}}.
\end{aligned}$$

We equip $\mathcal{D}$ with (h-dependent) Hilbert space norms given by

$$\|u\|^2_{\mathcal{D}_h} := \|u\|^2_{\mathcal{H}} + \|P(h)u\|^2_{\mathcal{H}}, \quad u \in \mathcal{D}, \tag{4.1.8}$$

using $\|u\|_{\mathcal{D}}$ when $h = 1$. Using the functional calculus of $P(h)$ (see §B.2) we can define more general spaces $\mathcal{D}^\alpha$ with norms

$$\|u\|_{\mathcal{D}^\alpha_h} := \|(P(h) + i)^\alpha u\|_{\mathcal{H}}, \quad u \in \mathcal{D}, \tag{4.1.9}$$

From (4.1.5) it follows that for $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n \setminus B(0, R_0))$,

$$u \in \mathcal{D}^\alpha_h \implies \varphi u \in H^{2\alpha}_h(\mathbb{R}^n) \tag{4.1.10}$$

The spaces $\mathcal{D}_{\mathrm{comp}}$ and $\mathcal{D}_{\mathrm{loc}}$ are defined using (4.1.2) and (4.1.3)

$$\begin{gathered}
\mathcal{D}_{\mathrm{comp}} := \mathcal{D} \cap \mathcal{H}_{\mathrm{comp}}, \\
\mathcal{D}_{\mathrm{loc}} := \{u \in \mathcal{H}_{\mathrm{loc}} : \chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \chi|_{B(0,R_0)} \equiv 1 \Rightarrow \chi u \in \mathcal{D}\}.
\end{gathered} \tag{4.1.11}$$

Since $\mathcal{D}$ is dense in $\mathcal{H}$ (P is a self-adjoint operator – see §B.1.2) we also see that $\mathcal{D}_{\mathrm{comp}}$ is dense in $\mathcal{H}_{\mathrm{comp}}$ in the sense that for each $u \in \mathcal{H}_{\mathrm{comp}}$ there exists $u_j \in \mathcal{D}_{\mathrm{comp}}$ such that $u_j \to u$ in $\mathcal{H}$. The same conclusion holds for $\mathcal{D}^k_{\mathrm{comp}}$ where $\mathcal{D}^k$ is the domain of P^k.

Finally we assume

$$\mathbb{1}_{B(0,R_0)}(P(h) + i)^{-1} \text{ is compact.} \tag{4.1.12}$$

DEFINITION 4.1 (Black box Hamiltonians). *A family of unbounded selfadjoint operators, $P(h)$, $0 < h < 1$, on a complex Hilbert $\mathcal{H}$ satisfying* (4.1.1) *is called a* semiclassical black box Hamiltonian *if* (4.1.4),(4.1.5),(4.1.6) *and* (4.1.12) *hold. If P satisfies* (4.1.4),(4.1.5),(4.1.6) *and* (4.1.12) *for $h = 1$ we call it a* black box Hamiltonian.

REMARK. A black box formalism is also possible for other operators than $-h^2\Delta$ – see Sjöstrand [**Sj96a**] for the development of that theory and §4.7 for more references.

EXAMPLES. 1. **Potential scattering.** Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^n;\mathbb{R})$ with $\operatorname{supp} V \subset B(0,R_0)$. If $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{D} = H^2(\mathbb{R}^n)$ and $P(h) := -h^2\Delta + V(x)$ then all the black box assumptions are satisfied. This is the case of scattering by compactly supported potentials presented in Chapters 2 and 3. Assumptions (4.1.1), (4.1.4), (4.1.5) and (4.1.6) are also satisfied for more singular compactly supported potentials for which $P(h) = -h^2\Delta + V(x)$ has self-adjoint extensions. For instance we can take $V \geq 0$, $V \in L^2(B(0,R_0))$. However, (4.1.12) may not hold.

2. **Obstacle scattering.** Suppose that $\mathcal{O} \subset \overline{B(0,R_0)}$ is an open set such that $\partial\mathcal{O}$ is a smooth hypersurface in $\mathbb{R}^n$. Let $\mathcal{H} = L^2(\mathbb{R}^n \setminus \mathcal{O})$, and

$$\mathcal{D} = H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O}) = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial\mathcal{O}} = 0\}$$

and $P = -\Delta$ (the self-adjoint Dirichlet Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$). This is the case of Dirichlet obstacle scattering.

We can also take the Neumann Laplacian in which case $\mathcal{D} = \{u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : \partial_\nu u|_{\partial\mathcal{O}} = 0\}$, where ∂_ν is the normal derivative with respect to $\partial\mathcal{O}$.

3. **Scattering on finite volume surfaces.** Let (X,g) be a complete Riemannian surface with the following decomposition

$$X = X_1 \cup X_0, \quad \partial X_0 = \partial X_1, \quad \text{smooth.}$$

and

$$(X_1, g|_{X_1}) = (\mathbb{S}^1_\theta \times [a,\infty)_r,\, dr^2 + e^{-2r}d\theta^2)), \quad a > 0, \quad \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

We define

$$\mathcal{H} = \mathcal{H}_a \oplus L^2([a,\infty), dr), \quad \mathcal{H}_a = L^2(X_0) \oplus \mathcal{H}^0_a, \tag{4.1.13}$$

where (with $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$)

$$\mathcal{H}^0_a = \left\{ \{a_n(r)\}_{n\in\mathbb{Z}^*} \ : \ a_n \in L^2([a,\infty)), \ \sum_{n\in\mathbb{Z}^*} \int_a^\infty |a_n(r)|^2 dr < \infty \right\}.$$

For

$$u = (u|_{X_0}, \{a_n(r)\}_{n\in\mathbb{Z}^*}, a_0(r)) \in L^2(X_0) \oplus \mathcal{H}_a^0 \oplus L^2([a,\infty)) = \mathcal{H},$$

the projections are defined by

$$\mathbb{1}_{[0,a)}\, u = u|_{[0,a)} u := (u|_{X_0}, \{a_n(r)\}_{n\in\mathbb{Z}^*}) \in \mathcal{H}_a, \quad \mathbb{1}_{[a,\infty)} = u_{[a,\infty)} = a_0(r).$$

The norm on $\mathcal{H}$ is given by

$$\|u\|_{\mathcal{H}}^2 := \int_{X_0} |u|_{X_0}|^2 d\operatorname{Vol}_g + \sum_{n\in\mathbb{Z}} \int_a^\infty |a_n(r)|^2 dr. \tag{4.1.14}$$

The space $H^1(X)$ is defined by requiring that $|du|_g^2$ is integrable with respect to $d\operatorname{Vol}_g$, where $|\bullet|_g^2$ is defined using the dual metric $\sum_{i,j} g^{ij}(x)\xi_i\xi_j$, $(g^{ij}) = (g_{ij})^{-1}$. We can indentify $H^1(X)$ with the subset of $\mathcal{H}$ since for $u \in H^1(X)$, the restriction of u to any circle $\{r_0\} \times \mathbb{S}^1_\theta$, $r_0 > a$, is in $L^2(\mathbb{S}^1_\theta)$ and can be expanded into Fourier series:

$$\begin{gathered} H^1(X) \ni u \stackrel{\iota}{\longmapsto} \left(u|_{X_0}, \{e^{-r/2}u_n(r)\}_{n\in\mathbb{Z}^*}, e^{-r/2}u_0(r)\right), \\ u_n(r) := \frac{1}{2\pi}\int_{\mathbb{S}^1} u(r,\theta)e^{-in\theta}d\theta, \quad r > a. \end{gathered} \tag{4.1.15}$$

The factor $e^{-r/2}$ comes from the change of the Riemannian volume $d\operatorname{Vol}_g|_{X_1} = e^{-r}drd\theta$ to the volume $drd\theta$ used in (4.1.14).

Fubini's theorem shows for $u \in L^2(X)$, $u_n(r)$ in (4.1.15) exist for almost every $r > a$ and define a function in L^2. Hence the map in (4.1.15) extends to an isomorphism

$$\iota : L^2(X) \simeq \mathcal{H}.$$

For $u \in C_c^\infty(X)$,

$$\begin{aligned} \Delta_g u|_{X_1} &= \left(e^{-r}\partial_r e^r \partial_r + e^{-2r}\partial_\theta^2\right) u(r,\theta) \\ &= \sum_{n\in\mathbb{Z}} (\partial_r^2 - \tfrac14 - e^{2r}n^2)(e^{-r/2}u_n(r))e^{in\theta + r/2}. \end{aligned} \tag{4.1.16}$$

We now define P as the Friedrichs extension of $-\Delta_g - \frac14$ on $C_c^\infty(X) \subset H^1(X)$ identified using (4.1.15) with a subset of $\mathcal{H}$, obtained from the quadratic form

$$\begin{aligned} Q_g(u,u) &= \sum_{n\in\mathbb{Z}^*} \int_a^\infty \left(|\partial_r a_n(r)|^2 + n^2 e^{2r}|a_n(r)|^2\right) dr \\ &\quad + \int_a^\infty |\partial_r a_0(r)|^2 dr + \int_{X_0} \left(|du|_g^2 - \tfrac14 |u|^2\right) d\operatorname{Vol}_g, . \end{aligned} \tag{4.1.17}$$

where $a_n(r) := e^{-r/2}\frac{1}{2\pi}\int_{\mathbb{S}^1} u(r,\theta)$. This gives a self-adjoint operator with the domain $\mathcal{D}$ which is the image of $H^2(X)$ under the map ι in (4.1.15).

The spaces $\mathcal{H}$ and the operator P satisfy the black box assumptions (4.1.1),(4.1.4),(4.1.5) and (4.1.6) with $\mathbb{R}^n \setminus B(0, R_0)$ replaced by a half-line $[a, \infty)$. To check the condition (4.1.12), that is the fact that

$$\mathbb{1}_{[0,a)}(P+i)^{-1} \text{ is a compact operator on } \mathcal{H}, \tag{4.1.18}$$

we first note that on the first component of $\mathcal{H}_a$, the operator is

$$\mathbb{1}_{X_0}(P+i)^{-1} = \mathbb{1}_{X_0}(-\Delta_g - \tfrac{1}{4} + i)^{-1}\iota^{-1} : \mathcal{H} \to H^2(X_0), \tag{4.1.19}$$

and hence it is compact on $\mathcal{H} \simeq L^2(X)$. Denoting by $\mathbb{1}_{\mathcal{H}_a^0}$ the orthogonal projection onto $\mathcal{H}_a^0$ in (4.1.13) and using (4.1.16) we see that

$$\begin{gathered} \mathbb{1}_{\mathcal{H}_a^0}(P+i)^{-1}u = \{b_n(r)\}_{n\in\mathbb{Z}^*}, \quad \mathbb{1}_{\mathcal{H}_a^0}\, u = \{a_n(r)\}_{n\in\mathbb{Z}^*}, \\ (-\partial_r^2 + e^{2r}n^2 + i)b_n(r) = a_n(r), \quad n \neq 0, \quad r > a. \end{gathered} \tag{4.1.20}$$

Since $\iota^{-1}(P+i)^{-1}u \in H^2(X) \subset H^1(X)$, we see from (4.1.17) that

$$\sum_{n\in\mathbb{Z}^*} \int_a^\infty (|\partial_r b_n(r)|^2 + n^2 e^{2r}|b_n(r)|^2)dr \leq C\|u\|_{\mathcal{H}}^2$$

But this and an adaptation of the Rellich–Kondrachev criterion (Theorem B.4, Exercise 4.2) shows that the map

$$u \longmapsto \{b_n(r)\}_{n\in\mathbb{Z}^*} \in \mathcal{H}_a^0 \tag{4.1.21}$$

is compact. That completes the proof of (4.1.18) and shows that P satisfies the assumptions of Definition 4.1.

The same procedure work for surfaces of the form

$$\begin{gathered} X = X_0 \cup X_1 \cup \cdots X_N, \qquad \partial X_0 = \bigsqcup_{j=1}^{N} \partial X_j, \\ (X_j, g|_{X_j}) \simeq ([a_j, \infty)_r \times (\mathbb{R}/b_j\mathbb{Z})_\theta, dr^2 + e^{-2r}d\theta^2), \end{gathered} \tag{4.1.22}$$

$a_j \in \mathbb{R}$, $b_j > 0$.

Example 3 shows that $\mathcal{H}_{R_0}$ can be a genuinely abstract Hilbert space unlike the geometrically simple spaces $\mathcal{H}_{R_0} = L^2(B(0, R_0))$ and $\mathcal{H}_{R_0}(B(0, R_0)\setminus \mathcal{O})$ in Examples 1 and 2 respectively. We will use Example 3 to illustrate other interesting facts such as the *Fermi Golden Rule*.

4.2. MEROMORPHIC CONTINUATION

In this section we will prove that for a black box Hamiltonians, $P(h)$, defined in §4.1, the resolvent

$$(P(h) - \lambda^2)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}, \quad \operatorname{Im}\lambda > 0, \quad \lambda^2 \notin \operatorname{Spec}(P(h)),$$

continues meromorphically as an operator from $\mathcal{H}_{\text{comp}}$ to $\mathcal{D}_{\text{loc}}$. When n is odd the continuation is to $\lambda \in \mathbb{C}$ and when n is even to the logarithmic cover of $\mathbb{C}$:

$$\Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\}). \tag{4.2.1}$$

Since it does not require much additional work, we temporarily break our simplifying assumption that n is odd.

To obtain meromorphic continuation the condition (4.1.12) is essential. Our taking λ^2 as the spectral parameter is discussed in §2.1. We recall that $\operatorname{Im} \lambda > 0$ corresponds to the spectral parameter λ^2 in $\mathbb{C} \setminus [0, \infty)$.

Since the statement about meromorphic continuation is purely functional analytic in nature we consider $P = P(1)$ only. We start with two lemmas.

LEMMA 4.2 (Compactness with a larger cut-off). *Suppose that P is a black box Hamiltonian. For $R > R_0$ let $\mathbb{1}_{B(0,R)}$ be the orthogonal projection onto $\mathcal{H}_{R_0} \oplus L^2(B(0,R) \setminus B(0,R_0))$. Then for $\lambda^2 \notin \operatorname{Spec}(P)$, $\operatorname{Im} \lambda > 0$, the operators*

$$\mathbb{1}_{B(0,R)}(P - \lambda^2)^{-1}, \quad (P - \lambda^2)^{-1}\, \mathbb{1}_{B(0,R)}, \quad R \geq R_0, \tag{4.2.2}$$

are compact $\mathcal{H} \to \mathcal{H}$.

Proof. 1. For $R = R_0$ and the first operator in (4.2.2) compactness follows from (4.1.12) and the resolvent identity:

$$\mathbb{1}_{B(0,R_0)}(P - \lambda^2)^{-1} = \mathbb{1}_{B(0,R_0)}(P + i)^{-1} + \mathbb{1}_{B(0,R_0)}(P + i)^{-1}(i + \lambda^2)(P - \lambda^2)^{-1}.$$

(A compact operator composed with a bounded operator gives a compact operator.)

2. To handle the case of $R > R_0$ we note that the inclusion

$$H^2(B(0,R) \setminus B(0,R_0)) \hookrightarrow L^2(B(0,R) \setminus B(0,R_0)),$$

is compact. Since $(P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D}$, (4.1.4) shows that

$$(\mathbb{1}_{B(0,R)} - \mathbb{1}_{B(0,R_0)})(P - \lambda^2)^{-1} \ \text{ is compact.}$$

Hence the first operator in (4.2.2) is compact.

3. For $\operatorname{Im} \lambda > 0$ we have $\operatorname{Im}(-\bar{\lambda}) > 0$. Hence $\mathbb{1}_{B(0,R)}(P - (-\bar{\lambda})^2)^{-1}$ is compact. By taking the adjoint we see that the second operator in (4.2.2) is compact. $\square$

The next lemma is a version of the free resolvent estimate (3.1.24). For future reference we formulate it in the semiclassical version:

LEMMA 4.3 (**Estimates on the resolvent in the physical half plane**). *Suppose that $P(h)$ is a semiclassical black box Hamiltonian. Then for $k = 0, 1, 2$ and $\tau > 0$, we have*

$$\| \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} (P(h) - i\tau)^{-1} \|_{\mathcal{H} \to H_h^k(\mathbb{R}^n \setminus B(0,R_0))} \leq C \langle \tau \rangle^{k/2} \tau^{-1}, \tag{4.2.3}$$

where $\|u\|^2_{H_h^k(\Omega)} := \sum_{|\alpha| \leq k} \|(hD)^\alpha u\|^2_{L^2(\Omega)}$.

Proof. 1. Self-adjointness of $P(h)$ implies that

$$\|(P(h) - i\tau)^{-1}\|_{\mathcal{H} \to \mathcal{H}} = \frac{1}{\tau}.$$

This and (4.1.1) give (4.2.3) for $k = 0$. Since

$$P(h)(P(h) - i\tau)^{-1} = I + i\tau(P(h) - i\tau)^{-1}$$

we see that

$$\|(P(h) - i\tau)^{-1}\|_{\mathcal{H} \to \mathcal{D}_h} \leq C \frac{\langle \tau \rangle}{\tau},$$

where the norm on $\mathcal{D}_h = \mathcal{D}$ is given by (4.1.8). This and (4.1.4) give (4.2.3) for $k = 2$.

2. From the interpolation estimate

$$\|u\|_{H_h^1(\mathbb{R}^n \setminus B(0,R_0))} \leq C \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n \setminus B(0,R_0)} \|u\|^{\frac{1}{2}}_{H_h^2(\mathbb{R}^n \setminus B(0,R_0)},$$

(see for instance [**Ev98**, §5.4]) we obtain (4.2.3) for $k = 1$ from the estimates for $k = 0, 2$. ☐

We are now ready for the main result of this section. We state it for $h = 1$:

THEOREM 4.4 (**Meromorphic continuation for black box Hamiltonians**). *Suppose that P is a black box Hamiltonian in the sense of Definition 4.1. Then*

$$R(\lambda) := (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D} \quad \text{is meromorphic for } \operatorname{Im} \lambda > 0. \tag{4.2.4}$$

In particular, the spectrum of P in $(-\infty, 0)$ is discrete.

Moreover, when n is odd, the resolvent in (4.2.4) extends to a meromorphic family

$$R(\lambda) : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}, \quad \lambda \in \mathbb{C}. \tag{4.2.5}$$

When n is even (4.2.5) holds with $\mathbb{C}$ replaced by the logarithmic plane Λ defined in (4.2.1).

Proof. 1. We first consider the case of $\operatorname{Im}\lambda > 0$. We choose $\chi_0 \in C_c^\infty(\mathbb{R}^n; [0,1])$ with the property that

$$\chi_0(x) \equiv 1 \quad \text{for } x \in B(0, R_0 + \varepsilon), \ \ \varepsilon > 0.$$

We then choose $\chi_j \in C_c^\infty(\mathbb{R}^n; [0,1])$, $j = 1, 2, 3$ so that

$$\chi_j(x) \equiv 1 \quad \text{for } x \in \operatorname{supp}\chi_{j-1}, \quad \operatorname{supp}\chi_j \subset B(0,R), \tag{4.2.6}$$

for some fixed $R > R_0$.

We also choose λ_0 with $\operatorname{Im}\lambda_0 > 0$, $\lambda_0^2 \notin \operatorname{Spec}(P)$ and define

$$\begin{gathered} Q(\lambda, \lambda_0) := Q_0(\lambda) + Q_1(\lambda_0), \\ Q_0(\lambda) := (1-\chi_0) R_0(\lambda)(1-\chi_1), \quad Q_1(\lambda_0) := \chi_2 (P - \lambda_0^2)^{-1}\chi_1. \end{gathered} \tag{4.2.7}$$

Here $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ is the free resolvent – see §3.1.

From (4.1.7) we deduce that $P(1-\chi_0) = -\Delta(1-\chi_0)$, and hence

$$(P - \lambda^2) Q_0(\lambda) = 1 - \chi_1 + K_0(\lambda), \quad K_0(\lambda) := -[\Delta, \chi_0] R_0(\lambda)(1-\chi_1). \tag{4.2.8}$$

Since $\operatorname{Im}\lambda > 0$ and $\operatorname{supp}\chi_0 \subset B(0,R)$, we have

$$[\Delta, \chi_0] R_0(\lambda)(1-\chi_1) : L^2 \to H^2(B(0,R) \setminus B(0,R_0)).$$

We conclude that $K_0(\lambda)$ is a compact operator on $\mathcal{H}$.

The corresponding contribution of $Q_1(\lambda_0)$ is

$$\begin{gathered} (P - \lambda^2) Q_1(\lambda_0) = \chi_1 + K_1(\lambda, \lambda_0), \\ K_1(\lambda, \lambda_0) := (\lambda_0^2 - \lambda^2)\chi_2 (P - \lambda_0^2)^{-1}\chi_1 + [P, \chi_2](P - \lambda_0^2)^{-1}\chi_1. \end{gathered} \tag{4.2.9}$$

Since $\operatorname{supp}\chi_2 \subset B(0,R)$, compactness of operators (4.2.2) shows that

$$K_1(\lambda, \lambda_0) : \mathcal{H} \to \mathcal{H} \text{ is a compact operator.}$$

We remark that this conclusion is valid for any $\lambda \in \mathbb{C}$ for n odd and $\lambda \in \Lambda$ for n even.

2. Putting (4.2.7),(4.2.8) and (4.2.9) together gives

$$(P - \lambda^2) Q(\lambda, \lambda_0) = I + K(\lambda, \lambda_0), \quad K(\lambda, \lambda_0) := K_0(\lambda) + K_1(\lambda, \lambda_0), \tag{4.2.10}$$

where $\operatorname{Im}\lambda > 0$, $K(\lambda, \lambda_0)$ is a compact operator. By Theorem C.8, $(I + K(\lambda, \lambda_0))^{-1} : \mathcal{H} \to \mathcal{H}$ will form a meromorphic family of operators in $\operatorname{Im}\lambda > 0$ if we show that that inverse exists for a suitably chosen λ. That will be achieved after making a suitable choice of λ_0.

We choose

$$\lambda_0 = e^{i\pi/4}\mu, \quad \mu \gg 1, \tag{4.2.11}$$

so that Lemma 4.3 (applied with $h = 1$) gives

$$\begin{aligned} &\|[P, \chi_2](P - \lambda_0^2)^{-1} \chi_1\|_{\mathcal{H} \to \mathcal{H}} \\ &\qquad \leq C\| \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} (P - i\mu^2)^{-1} \|_{\mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0,R_0))} \leq \frac{C}{\mu} \ll 1, \end{aligned}$$

and

$$\begin{aligned} &\|[\Delta, \chi_0] R_0(\lambda_0)(1 - \chi_1)\|_{\mathcal{H} \to \mathcal{H}} \\ &\qquad \leq C\| \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} (-\Delta - i\mu^2)^{-1} \|_{\mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0,R_0))} \leq \frac{C}{\mu} \ll 1. \end{aligned}$$

Definitions of $K_0(\lambda)$, $K_1(\lambda, \lambda_0)$ and $K(\lambda, \lambda_0)$ in (4.2.8),(4.2.9) and (4.2.10) respectively then give

$$\|K(\lambda_0, \lambda_0)\|_{\mathcal{H} \to \mathcal{H}} \leq \|K_0(\lambda_0)\|_{\mathcal{H} \to \mathcal{H}} + \|K(\lambda_0, \lambda_0)\|_{\mathcal{H} \to \mathcal{H}} \ll 1.$$

This implies that for our choice of λ_0 and for $\lambda = \lambda_0$, $I + K(\lambda, \lambda_0)$ is invertible on $\mathcal{H}$. Hence, Theorem C.8 implies that

$$(I + K(\lambda, \lambda_0))^{-1} : \mathcal{H} \to \mathcal{H}, \quad \operatorname{Im} \lambda > 0,$$

is a meromorphic family of operators. Hence (4.2.10) shows that for $\operatorname{Im} \lambda > 0$ and $\lambda^2 \notin \operatorname{Spec}(P)$,

$$(P - \lambda^2)^{-1} = Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0))^{-1}. \tag{4.2.12}$$

That provides the meromorphy of the left hand side as an operator $\mathcal{H} \to \mathcal{H}$ in $\operatorname{Im} \lambda > 0$. Thus we have proved (4.2.4).

3. We now consider meromorphic extension to the lower half plane and start with the case of n odd. We will use (4.2.10) with λ_0 given by (4.2.11).

For χ_3 with the property (4.2.6) and $K(\lambda, \lambda_0)$ given in (4.2.10) we have

$$(1 - \chi_3) K(\lambda, \lambda_0) = 0.$$

Hence

$$\begin{gathered} I + K(\lambda, \lambda_0) = (I + K(\lambda, \lambda_0)(1 - \chi_3))(I + K(\lambda, \lambda_0)\chi_3), \\ (I + K(\lambda, \lambda_0)(1 - \chi_3))^{-1} = I - K(\lambda, \lambda_0)(1 - \chi_3). \end{gathered} \tag{4.2.13}$$

From (4.2.12) we see that for $\operatorname{Im} \lambda > 0$ (as meromorphic families of operators)

$$(P - \lambda^2)^{-1} = Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0)\chi_3)^{-1}(I - K(\lambda, \lambda_0)(1 - \chi_3)). \tag{4.2.14}$$

4. To conclude the proof of (4.2.5) we observe the following facts valid for n odd:

$$\mathbb{C} \ni \lambda \longmapsto K_0(\lambda)\chi_3, \quad \mathbb{C} \ni \lambda \longmapsto K_1(\lambda, \lambda_0)$$

are meromorphic families of compact operators on $\mathcal{H}$.

We note that if $n \geq 3$ then the families are in fact holomorphic – see §3.1. When $n = 1$ the only pole is at $\lambda = 0$, see §2.2.

In Step 2 of the proof we showed that $I + K(\lambda_0, \lambda_0)$ is invertible on $\mathcal{H}$. The first equation in (4.2.13) gives

$$(I + K(\lambda_0, \lambda_0)\chi_3)^{-1} = (I + K(\lambda_0, \lambda_0))^{-1}(I + K(\lambda_0, \lambda_0)(1 - \chi_3)).$$

Consequently Theorem C.8 shows that

$$\lambda \mapsto (I + K(\lambda, \lambda_0)\chi_3)^{-1} : \mathcal{H} \to \mathcal{H} \tag{4.2.15}$$

is a meromorphic family of operators on $\mathbb{C}$. We then note that

$\mathbb{C} \ni \lambda \mapsto Q(\lambda, \lambda_0)$ is a meromorphic family of operators $\mathcal{H} \to \mathcal{D}_{\rm loc}$,

and

$$\lambda \mapsto I - K(\lambda, \lambda_0)(1 - \chi_3) : \mathcal{H}_{\rm comp} \to \mathcal{H}_{\rm comp} \subset \mathcal{H}.$$

In the last mapping property, the only term which is not bounded on $\mathcal{H}$ comes from $K_0(\lambda)$. But we do have $[-\Delta, \chi_0]R_0(\lambda)(1-\chi_1) : \mathcal{H}_{\rm comp} \to \mathcal{H}_{\rm comp}$.

Using (4.2.15) we conclude that

$$\lambda \mapsto R(\lambda) := Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0)\chi_3)^{-1}(I - K(\lambda, \lambda_0)(1 - \chi_3)),$$

is a meromorphic family of operators $\mathcal{H}_{\rm comp} \to \mathcal{D}_{\rm loc}$ for $\lambda \in \mathbb{C}$. In view of (4.2.14) this gives the meromorphic extension of $(P - \lambda^2)^{-1}$, $\operatorname{Im} \lambda > 0$, acting between these spaces.

5. For n even we use the fact that for the logarithmic plane Λ,

$$\Lambda \ni \lambda \to R_0(\lambda) : L^2_{\rm comp}(\mathbb{R}^n) \to H^2_{\rm loc}(\mathbb{R}^n),$$

is a holomorphic family of operators. Hence all the statement in steps 3 and 4 of the proof hold if $\mathbb{C}$ is replaced by Λ and lead to the construction of the meromorphic extension $R(\lambda) : \mathcal{H}_{\rm comp} \to \mathcal{D}_{\rm loc}$, $\lambda \in \Lambda$. $\square$

As an immediate consequence of Theorem 4.4 we have a general result about spectra of black box Hamiltonians:

THEOREM 4.5 (Spectrum of black box Hamiltonians). *Let P satisfy the assumptions of Theorem 4.4 and let n be odd. Then*

$$\begin{gathered} \operatorname{Spec}(P) = \operatorname{Spec}_{\rm cont}(P) \cup \operatorname{Spec}_{\rm pp}(P), \quad \operatorname{Spec}_{\rm cont}(P) = [0, \infty), \\ \operatorname{Spec}_{\rm pp}(P) = \{z_j\}_{j=N_-}^{N_+}, \quad z_j \leq z_{j+1}, \end{gathered} \tag{4.2.16}$$

where $N_\pm$ can take values $\pm\infty$, the multiplicities are finite, and the set $\{z_j\}_{j=N_-}^{N_+}$ is discrete.

REMARK. The theorem is also valid for n even but a more detailed analysis is needed near 0 – see Vodev [Vo94a],[Vo94b].

EXAMPLES. 1. **Potential scattering.** For $P(h) = -h^2\Delta + V$, $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ we obtain meromorphic extensions of $(P(h)-\lambda^2)^{-1} : L^2_{\mathrm{comp}}(\mathbb{R}^n) \to H^2_{\mathrm{loc}}(\mathbb{R}^n)$ to $\lambda \in \mathbb{C}$ for n odd and to $\lambda \in \Lambda$ for n even. The proof of Theorem 4.4 is a generalization of the proofs of Theorems 2.2 and 3.8.

2. **Obstacle scattering.** Denote by $-\Delta_{\mathcal{O}}$ the Dirichlet realization of $-\Delta$ on $\mathbb{R}^n \setminus \mathcal{O}$ – see Example 2 in §4.2. Then Theorem 4.4 shows that

$$(-\Delta_{\mathcal{O}} - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \mathcal{O}) \to H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O}),$$

continues meromorphically to $\mathbb{C}$ when n is odd, and to Λ when n is even.

One can formulate this result in terms of *Green's function* of $-\Delta_{\mathcal{O}}$. For $\operatorname{Im}\lambda > 0$ we consider the Schwartz kernel $G(\lambda, x, y)$ defined by

$$(-\Delta_{\mathcal{O}} - \lambda^2)^{-1} f(x) = \int_\Omega G(\lambda, x, y) f(y) dy, \quad f \in C^\infty_{\mathrm{c}}(\Omega), \quad \Omega := \mathbb{R}^n \setminus \overline{\mathcal{O}},$$

where by local elliptic theory (see for instance [TaI, §5.11])

$$\begin{gathered} G(\lambda, x, y) \in C(\Omega_x, L^1_{\mathrm{loc}}(\Omega)), \quad G(\lambda, x, y) = G(\lambda, y, x), \\ G(\lambda, x, y) \in C^\infty(\Omega \times \Omega \setminus \Delta(\Omega)), \quad \Delta(\Omega) := \{(x, x) : x \in \Omega\}. \end{gathered} \tag{4.2.17}$$

Theorem 4.4 shows that for fixed $x \neq y$ the function $\lambda \mapsto G(\lambda, x, y)$ extends to a meromorphic function on $\mathbb{C}$ or Λ depending on n being odd or even. There are no poles for $\operatorname{Im}\lambda > 0$. Since for fixed y, $u(x) := G(\lambda, x, y) - G(-\lambda, x, y)$ solves the equation $(-\Delta_{\mathcal{O}} - \lambda^2)u = 0$, it follows that $u \in C^\infty(\Omega)$ and that the properties (4.2.17) hold for all λ.

In the case of obstactle scattering a direct approach to meromorphic continuation can be given using boundary layer potentials –see [TaII, §9.7].

3. **Scattering on finite volume surfaces.** Let (X, g) be a Riemannian surface with finitely many cusps – see Example 3 in §4.1 and (4.1.22). Theorem 4.4 shows that the resolvent of the Laplacian

$$(-\Delta_g - \tfrac{1}{4} - \lambda^2)^{-1} : L^2(X) \to H^2(X), \quad \operatorname{Im}\lambda > 0,$$

continues meromorphically to $\mathbb{C}$ as an operator $L^2_{\mathrm{comp}}(X) \to H^2_{\mathrm{loc}}(X)$. (Strictly speaking we should use for $R_0(\lambda)$ the resolvent for, say, Dirichlet realization of $-\partial_s^2$ on $[a-1, \infty)$ – the straightforward modifications are left to the reader.) Since $-\Delta_g \geq 0$, Theorem 4.5 show that there are only finitely many eigenvalues of $-\Delta_g$ in $[0, \frac{1}{4}]$ and hence

$$\operatorname{Spec}_{\mathrm{cont}}(-\Delta_g) = [\tfrac{1}{4}, \infty), \quad \operatorname{Spec}_{\mathrm{pp}}(-\Delta_g) = \{E_j\}_{j=0}^N \cup \{z_j\}_{j=1}^M,$$

$$0 = E_0 < E_1 \leq \cdots \leq E_N \leq \frac{1}{4}, \quad \frac{1}{4} < z_1 \leq z_2 \leq \cdots ,$$

and M can take the value $+\infty$ (infinitely many eigenvalues embedded in the continuous spectrum) or 0 (no embedded eigenvalues). A famous example for which $M = +\infty$ is given by the modular surface $X = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ – see [**LP76**]. As we will see for a generic metric g, $M = 0$.

The significance of the poles of the continuation of $(-\Delta_g - \frac{1}{4} - \lambda^2)^{-1} : L^2_{\text{comp}}(X) \to H^2_{\text{loc}}(X)$ will be discussed in §4.4 when we define distorted plane waves and the scattering matrix.

DEFINITION 4.6 (Resonances for black box Hamiltonians). *Let P be a black box Hamiltonian ($P = P(h)$ or is independent of h). Then $\lambda \in \mathbb{C}$ for n odd and $\lambda \in \Lambda$ (the logarithmic plane (4.2.1)) for n even is a* resonance *of P if λ is a poles of the meromophic extension of $(P - \lambda^2)^{-1}$ given in Theorem 4.4. The multiplicity of a pole at λ is defined as*

$$m_R(\lambda) := \operatorname{rank} \oint_\lambda R(\zeta) d\zeta, \tag{4.2.18}$$

where the integral is over a circle containing no other pole of $R(\zeta)$ than λ.

The set of resonances will be denoted by $\operatorname{Res}(P)$.

REMARKS. 1. The convention of meromorphically extending $(P-\lambda^2)^{-1}$, that is taking λ^2 as a spectral parameter, is useful when global extensions are considered and is motivated by the wave equation – see 2.1. Sometimes, especially when considering problems motivated by quantum mechanics and the Schrödinger equation, it is more convenient to use $z = \lambda^2$ as spectral parameter. Away from 0 the two conventions are clearly equivalent. When confusion is unlikely we will sometimes write $\operatorname{Res}(P)$ for the image of the set of resonances under the map $\lambda \mapsto z = \lambda^2$. As we will see in Theorem 4.7

$$\lambda \neq 0 \implies m_R(\lambda) = \operatorname{rank} \oint_\lambda R(\zeta) 2\zeta d\zeta,$$

so the multiplicities agree when consider R as a function of λ or λ^2.

We remark that the different conventions are unavoidable in a subject touching different disciplines – see §1.1.

2. Self-adjointness of P_V shows that for $\operatorname{Im} \lambda = -\operatorname{Im} \bar{\lambda} > 0$, $(P_V - \lambda^2)^{-1} = (P_V - (-\bar{\lambda})^2)^{-1}$. Hence by meromorphic continuation,

$$R(\lambda) = R(-\bar{\lambda})^*, \quad m_R(\lambda) = m_R(-\bar{\lambda}). \tag{4.2.19}$$

The next result describes the structure of the singular part of the resolvent.

THEOREM 4.7 (Singular part of $R_V(\lambda)$ for black box Hamiltonians). *In the notation of the definition above suppose that $m_R(\lambda) > 0$, $\lambda \neq 0$. Then there exists $M_\lambda \leq m_R(\lambda)$ such that*

$$R(\zeta) = -\sum_{k=1}^{M_\lambda} \frac{(P-\lambda^2)^{k-1}}{(\zeta^2-\lambda^2)^k}\Pi_\lambda + A(\zeta,\lambda)\,, \tag{4.2.20}$$

where $\zeta \mapsto A(\zeta,\lambda)$ is holomorphic near λ,

$$\Pi_\lambda := -\frac{1}{2\pi i}\oint_\lambda R(\zeta)\,2\zeta d\zeta\,, \quad (P-\lambda^2)^{M_\lambda}\Pi_\lambda = 0, \tag{4.2.21}$$

and

$$m_R(\lambda) = \operatorname{rank}\Pi_\lambda := \dim \Pi_\lambda(\mathcal{H}_{\mathrm{comp}}). \tag{4.2.22}$$

In addition for any $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\chi \equiv 1$ in a neighbourhood of $B(0,R_0)$, we have

$$m_R(\lambda) = \operatorname{rank}\oint_\lambda R(\zeta)\chi\,2\zeta d\zeta, \tag{4.2.23}$$

where the integral is over a circle containing no other pole of $R(\zeta)$ than λ.

Proof. 1. The first two statements (4.2.20) and (4.2.21) follow from steps 1 and 2 of the proof of Theorem 2.5. These can be read without referring to the material of Chapter 2 so we do not reproduce them here.

2. We now prove (4.2.22). The operator Π_λ has finite rank and hence, for any $k \in \mathbb{N}$,

$$\Pi_\lambda(\mathcal{H}_{\mathrm{comp}}) = \Pi_\lambda(\mathcal{D}^k_{\mathrm{comp}}). \tag{4.2.24}$$

(See the comment after (4.1.11).) If the rank of Π_λ is N_λ then there exist

$$V_\lambda \subset \mathcal{D}^{N_\lambda}_{\mathrm{comp}}, \quad W_\lambda \subset \mathcal{D}^{N_\lambda}, \quad \dim V_\lambda = \dim W_\lambda = N_\lambda,$$
$$W_\lambda = \Pi_\lambda(\mathcal{H}_{\mathrm{comp}}) = \Pi_\lambda(V_\lambda), \quad (P-\lambda^2) : W_\lambda \to W_\lambda.$$

We then put

$$B_\lambda := (\Pi_\lambda|_{V_\lambda})^{-1}(P-\lambda^2)\Pi_\lambda, \quad B_\lambda : \mathcal{H}_{\mathrm{comp}} \to V_\lambda,$$

and note that in the notation of (4.2.21)

$$B_\lambda^{M_\lambda} = 0.$$

The residue theorem and (4.2.20) give

$$\begin{aligned} \frac{1}{2\pi i}\oint_\lambda R(\zeta)d\zeta &= \Pi_\lambda\left(\sum_{k=1}^{M_\lambda}(-1)^{k-1}\frac{(2k-2)!}{(k-1)!}(2\lambda)^{-2k+1}B_\lambda^{k-1}\right) \\ &= \frac{1}{2\lambda}\Pi_\lambda(I_{V_\lambda} + N_\lambda), \end{aligned} \tag{4.2.25}$$

$$N_\lambda : \mathcal{H}_{\mathrm{comp}} \to V_\lambda, \qquad N_\lambda^{M_\lambda} = 0.$$

Since $I_{V_\lambda} + N_\lambda : V_\lambda \to V_\lambda$ is invertible (for instance by a finite Neumann series) it follows that

$$\begin{aligned}\dim \left(\frac{1}{2\pi i} \oint_\lambda R(\zeta) d\zeta \right) (\mathcal{H}_{\mathrm{comp}}) &\geq \dim \Pi_\lambda(V_\lambda) \\ &= \dim \Pi_\lambda(\mathcal{H}_{\mathrm{comp}}) = \operatorname{rank} \Pi_\lambda.\end{aligned}$$

The opposite inequality is clear from (4.2.25). The two give (4.2.22).

3. To obtain (4.2.23) we will use the proof of Theorem 4.4. In the notation of (4.2.10) we put $Q(\zeta) := Q(\zeta, \lambda_0)$ and $K(\zeta) := K(\zeta, \lambda_0)$ (since λ_0 is fixed in (4.2.11)). That gives

$$R(\zeta) = Q(\zeta) - R(\zeta)K(\zeta).$$

For $\chi \in C_c^\infty(\mathbb{R}^n, [0,1])$, $\chi \equiv 1$ in a neighbourhood of $B(0, R_0)$, we can choose χ_j, $j = 0, 1, 2$ in the definition of K – see (4.2.8), (4.2.9) and (4.2.10) – so that $\chi K(\zeta) = K(\zeta)$. Hence

$$R(\zeta) = Q(\zeta) - R(\zeta)\chi K(\zeta).$$

We now use the holomorphy of $\zeta \mapsto Q(\zeta)$ to write

$$\begin{aligned}\frac{1}{2\pi i} \oint_\lambda R(\zeta) 2\zeta d\zeta &= -\frac{1}{2\pi i} \oint_\lambda R(\zeta)\chi K(\zeta) 2\zeta d\zeta \\ &= \Pi_\lambda \chi \left(\frac{1}{2\pi i} \sum_{k=1}^{M_\lambda} \oint_\lambda \frac{(P - \lambda^2)^{k-1} K(\zeta) 2\zeta}{(\zeta^2 - \lambda^2)^k} d\zeta \right) \\ &= \Pi_\lambda \chi K_2(\lambda) = \left(\frac{1}{2\pi i} \oint_\lambda R(\zeta) \chi \, 2\zeta d\zeta \right) K_2(\lambda),\end{aligned}$$

where

$$K_2(\lambda) := -\frac{1}{2\pi i} \sum_{k=1}^{M_\lambda} \oint_\lambda \frac{(P - \lambda^2)^{k-1} K(\zeta) 2\zeta}{(\zeta^2 - \lambda^2)^k} d\zeta.$$

Since $\partial_\lambda^\ell K(\lambda) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{comp}}$, the residue calculus implies that

$$K_2(\lambda) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{H}_{\mathrm{comp}}.$$

Using (4.2.22) it follows that

$$\begin{aligned} m_R(\lambda) &\leq \dim \left(\oint_\lambda R(\zeta) \chi \, 2\zeta d\zeta \right) (\mathcal{H}_{\mathrm{comp}}) \\ &\leq \dim \left(\oint_\lambda R(\zeta) \, 2\zeta d\zeta \right) (\mathcal{H}_{\mathrm{comp}}) = m_R(\lambda),\end{aligned}$$

completing the proof of (4.2.23). □

DEFINITION 4.8 (**Resonant states).** *Let $\lambda \neq 0$ be a resonance of P and let Π_λ be given by* (4.2.21). *Then, in the notation of* (4.2.20), *an element of $\mathcal{D}_{\rm loc}$*

$$u \in \Pi_\lambda(\mathcal{H}_{\rm comp}), \quad (P-\lambda^2)u = 0,$$

is called a resonant state. *Also,*

$$v \in \Pi_\lambda(\mathcal{H}_{\rm comp})$$

is call a generalized resonant state.

We note that in the notation of (4.2.20) $(P-\lambda^2)^{M_\lambda} v = 0$.

THEOREM 4.9 (**Characterization of resonant states).** *A vector $u \in \mathcal{D}_{\rm loc}$ is a resonant state corresponding to $\lambda \in \mathbb{C}\setminus\{0\}$ if and only if $(P-\lambda^2)u = 0$ and there exist $g \in L^2_{\rm comp}(\mathbb{R}^n)$ and $R > 0$ such that*

$$u|_{\mathbb{R}^n\setminus B(0,R)} = R_0(\lambda) g|_{\mathbb{R}^n\setminus B(0,R)}. \tag{4.2.26}$$

INTERPRETATION. The theorem provides a stationary characterization of resonant states as *outgoing* functions. A dynamical interpretation of the outgoing property (4.2.26) in the spirit of Lax–Phillips [**LP68**] is given as follows. Suppose that

$$u_0 := R_0(\lambda) f, \ \ f \in \mathcal{D}'(\mathbb{R}^3), \ \ \operatorname{supp} f \in B(0,R), \ \ \lambda \in \mathbb{C}, \tag{4.2.27}$$

and that $u(t,x) \in C(\mathbb{R}; \mathcal{D}'(\mathbb{R}^n)$, $n \geq 3$, odd satisfies

$$\begin{gathered} \Box u(x,t) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = -i\lambda_0 u_0(x), \quad x \in \mathbb{R}^n \setminus B(0,R), \end{gathered} \tag{4.2.28}$$

Then

$$\operatorname{supp} u \subset \{(t,x) : t < |x| + R\}. \tag{4.2.29}$$

This means that outgoing initial data in the sense of (4.2.26) gives outgoing solutions of the free wave equation in the sense of (4.2.29) – see Exercise 4.3 for an outline of the proof.

Proof of Theorem 4.9. 1. Suppose that $\chi_1 \in C^\infty_{\rm c}(\mathbb{R}^n)$ is equal to 1 in $B(0,R_0)$ and $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$ is equal to 1 on $\operatorname{supp}\chi_1$. Then for $\operatorname{Im}\zeta > 0$,

$$\begin{aligned} (-\Delta - \zeta^2)(1-\chi)R(\zeta)\chi_1 &= (P-\zeta^2)(1-\chi)R(\zeta)\chi_1 \\ &= (1-\chi)\chi_1 - [P,\chi]R(\zeta)\chi_1 \\ &= -[P,\chi]R(\zeta)\chi_1. \end{aligned}$$

This implies that for $\operatorname{Im}\zeta > 0$

$$R_0(\zeta)[P,\chi]R(\zeta)\chi_1 = -(1-\chi)R(\zeta)\chi_1. \tag{4.2.30}$$

By analytic continuation (4.2.30) holds for $\zeta \in \mathbb{C}$.

2. Using (4.2.21) we have from (4.2.30)

$$
\begin{aligned}
(1-\chi)\Pi_\lambda\chi_1 &= \frac{1}{2\pi i}\oint_\lambda R_0(\zeta)[P,\chi]R(\zeta)\chi_1 2\zeta d\zeta \\
&= \frac{1}{2\pi i}\oint_\lambda \sum_{j=1}^{M_\lambda} \frac{R_0(\zeta)}{(\zeta^2-\lambda^2)^j} 2\zeta d\zeta\, [P,\chi](P-\lambda^2)^{j-1}\Pi_\lambda\chi_1 \\
&= \sum_{j=1}^{M_\lambda} T_j(\lambda)\,[P,\chi](P-\lambda^2)^{j-1}\Pi_\lambda\chi_1,
\end{aligned} \tag{4.2.31}
$$

where

$$
T_j(\lambda) := \frac{1}{(j-1)!}((2\zeta)^{-1}\partial_\zeta)^{j-1}R_0(\zeta)|_{\zeta=\lambda}, \quad T_1(\lambda) = R_0(\lambda).
$$

3. Now suppose that $u = \Pi_\lambda v$, $v \in \mathcal{H}_{\mathrm{comp}}$, is a resonant state. Choose χ and χ_1 in Step 1 so that $\chi_1 v = v$. Since $(P-\lambda^2)^{j-1}u = 0$ for $j > 1$, (4.2.31) gives

$$
(1-\chi)u = (1-\chi)\Pi_\lambda v = (1-\chi)\Pi_\lambda\chi_1 v = T_1(\lambda)[P,\chi]\Pi_\lambda\chi_1 v.
$$

Since $T_1(\lambda) = R_0(\lambda)$ we obtain (4.2.26) with $g := [P,\chi]\Pi_\lambda v$.

4. We now assume that

$$
u|_{\mathbb{R}^n\setminus B(0,R)} = (R_0(\lambda)g)|_{\mathbb{R}^n\setminus B(0,R)} \quad \text{for some } g \in L^2_{\mathrm{comp}},
$$

and that $(P-\lambda^2)u = 0$. By applying $-\Delta - \lambda^2$ to both sides we see that $\operatorname{supp} g \subset B(0,R)$.

We have $R(\zeta)(P-\zeta^2)\chi = \chi$, again by meromorphic continuation. Hence, using $(P-\lambda^2)u = 0$,

$$
\begin{aligned}
\chi u &= R(\zeta)(P-\lambda^2+\lambda^2-\zeta^2)\chi u \\
&= R(\zeta)[-\Delta,\chi]u + (\lambda^2-\zeta^2)R(\zeta)\chi u.
\end{aligned} \tag{4.2.32}
$$

To analyse $(1-\chi)u$ we note that by reversing the roles of P and $-\Delta$ in the derivation of (4.2.30) we obtain

$$
R(\zeta)[\Delta,\chi]R_0(\zeta)\chi_1 = (1-\chi)R_0(\zeta)\chi_1.
$$

In particular by choosing $\chi_1 = 1$ on a neighbourhood of $B(0,R)$ we see that

$$
R(\zeta)[\Delta,\chi]R_0(\zeta)g = (1-\chi)R_0(\zeta)g.
$$

Hence,

$$
\begin{aligned}
(1-\chi)R_0(\zeta)g &= R(\zeta)[\Delta,\chi]R_0(\zeta)g \\
&= R(\zeta)[\Delta,\chi]\left(R_0(\lambda)g + R_0(\zeta)g - R_0(\lambda)g\right) \\
&= R(\zeta)[\Delta,\chi]u + R(\zeta)[\Delta,\chi]\left(R_0(\zeta)g - R_0(\lambda)\right)g.
\end{aligned} \tag{4.2.33}
$$

We then define

$$u(\zeta) := (1-\chi)R_0(\zeta)g + \chi u, \quad u(\lambda) = u.$$

Adding (4.2.32) and (4.2.33) gives

$$u(\zeta) = R(\zeta)\left([\Delta,\chi]\left(R_0(\zeta) - R_0(\lambda)\right)g + (\lambda^2 - \zeta^2)\chi u\right).$$

Dividing by $\zeta^2 - \lambda^2$ and integrating over a small positively oriented circle centered at λ we obtain

$$\begin{aligned}
u = u(\lambda) &= \frac{1}{2\pi i}\oint_\lambda \frac{u(\zeta)}{\zeta^2 - \lambda^2} 2\zeta d\zeta \\
&= -\frac{1}{2\pi i}\oint_\lambda R(\zeta)\left(\chi u - [\Delta,\chi]\frac{R_0(\zeta) - R_0(\lambda)}{\zeta^2 - \lambda^2} g\right) 2\zeta d\zeta \\
&= \frac{\Pi_\lambda}{2\pi i}\oint_\lambda \sum_{k=1}^{M_\lambda} \frac{(P-\lambda^2)^{k-1}}{(\zeta^2-\lambda^2)^k}\left(\chi u - [\Delta,\chi]\frac{R_0(\zeta) - R_0(\lambda)}{\zeta^2 - \lambda^2} g\right) 2\zeta d\zeta \\
&= \Pi_\lambda v,
\end{aligned}$$

where

$$v := \chi u - \sum_{k=1}^{M_\lambda} \frac{1}{k!}(-\Delta - \lambda^2)^{k-1}[\Delta,\chi]\left((2\lambda)^{-1}\partial_\lambda\right)^k R_0(\lambda) g \in \mathcal{H}_{\mathrm{comp}}.$$

This proves the claim that u is a resonant state in the sense of Definition 4.8. □

4.3. UPPER BOUNDS ON THE NUMBER OF RESONANCES

In this section we will obtain a far reaching generalization of Theorem 3.27 which gave upper bounds on the number of resonances for $P = -\Delta + V$, $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, n odd.

To formulate the result in the black box setting we introduce a *reference operator* $P^\#(h)$. It is defined as follows. Let

$$\mathbb{T}^n_{R_1} := \mathbb{R}^n / R_1\mathbb{Z}, \quad R_1 > R_0.$$

In the notation of (4.1.1) we then put

$$\mathcal{H}^\#_{R_1} := \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}^n_{R_1} \setminus B(0,R_0)), \tag{4.3.1}$$

where we identified $B(0,R_0) \subset \mathbb{R}^n$ with its image under the projection $\mathbb{R}^n \to \mathbb{T}^n_{R_1}$. (We will use the same convention for $B(0,R_1)$ as well.) The corresponding orthogonal projections are denoted by

$$u \mapsto \mathbb{1}_{B(0,R_0)}\, u = u|_{B(0,R_0)}, \quad u \mapsto \mathbb{1}_{\mathbb{T}^n_{R_1}\setminus B(0,R_0)}\, u = u|_{\mathbb{T}^n_{R_1}\setminus B(0,R_0)}.$$

If $P(h)$ is a black box Hamiltonian in the sense of §4.1 with domain $\mathcal{D}$ we define

$$\mathcal{D}^{\#}_{R_1} := \{u \in \mathcal{H}^{\#}_{R_1} : \chi \in C^{\infty}_{\rm c}(B(0,R_1)),\ \chi = 1 \text{ near } B(0,R_0) \implies \chi u \in \mathcal{D},\ (1-\chi)u \in H^2(\mathbb{T}^n_{R_1})\} \tag{4.3.2}$$

and, for any χ with the property in (4.3.2),

$$\begin{gathered} P^{\#}_{R_1}(h) : \mathcal{D}^{\#}_{R_1} \to \mathcal{H}^{\#}_{R_1}, \\ P^{\#}_{R_1}(h)u := P(h)(\chi u) + (-h^2\Delta)((1-\chi)u). \end{gathered} \tag{4.3.3}$$

Assumptions (4.1.4) and (4.1.5) show that this definition is independent of the choice of χ.

DEFINITION 4.10 (The reference operator). *For a black box Hamiltonian $P(h)$ the operator $P^{\#}_{R_1}(h)$ is called a* reference operator*. Once we fix $R_1 > R_0$ we use notation*

$$P^{\#}(h) : \mathcal{H}^{\#} \to \mathcal{D}^{\#}.$$

The spaces spaces $\mathcal{D}^{\#\alpha}_h$ are defined as in (4.1.8) and (4.1.9):

$$\|u\|_{\mathcal{D}^{\sharp\,\alpha}_h} = \|(P^{\#}(h)+i)^{\alpha}u\|_{\mathcal{H}^{\#}}. \tag{4.3.4}$$

REMARK. There are many possible choices for a reference operator. For instance, instead of (4.3.1), (4.3.2) we can take

$$\begin{aligned} \mathcal{H}^{\#} &:= \mathcal{H}_{R_0} \oplus L^2(B(0,R_1) \setminus B(0,R_0)), \\ \mathcal{D}^{\#} &:= \{u \in \mathcal{H}^{\#} : \chi \in C^{\infty}_{\rm c}(B(0,R_1)),\ \chi = 1 \text{ near } B(0,R_0) \Rightarrow \\ &\qquad \chi u \in \mathcal{D},\ (1-\chi)u \in H^2(B(0,R_1)) \cap H^1_0(B(0,R_1))\}. \end{aligned} \tag{4.3.5}$$

This means that we introduce the Dirichlet boundary condition on at $\partial B(0,R_1)$. This will be useful in Example 2 below.

LEMMA 4.11 (Properties of the reference operator). *Suppose $P^{\#}(h)$ is a reference operator defined by (4.3.3) for some $R_1 > R_0$. Then, with $\mathcal{H}^{\#}$ given by (4.3.1),*

$$P^{\#}(h) : \mathcal{H}^{\#} \longrightarrow \mathcal{H}^{\#}, \tag{4.3.6}$$

is a self-adjoint operator with domain given by $\mathcal{D}^{\#}$ defined in (4.3.2).

The resolvent $(P^{\#}(h)+i)^{-1}$ is compact and hence the spectrum of $P^{\#}(h)$ is discrete.

Proof. In the proof we consider $P = P(1)$ as $h > 0$ is a harmless parameter.

1. The symmetry of $P^\#$ will follow from the definition (4.3.3) and from (4.1.5). To see it, choose $\chi_j \in C_c^\infty(B(0,R_1))$, $\chi_j = 1$ near $B(0,R_0)$, so that, with χ from (4.3.2),

$$\chi_1 \equiv 1 \text{ near } \operatorname{supp}\chi, \quad \chi \equiv 1 \text{ near } \operatorname{supp}\chi_2. \tag{4.3.7}$$

For $u, v \in \mathcal{D}^\#$,

$$\begin{aligned}
\langle P^\# u, v\rangle_{\mathcal{H}^\#} &= \langle P(\chi u), \chi_1 v\rangle_{\mathcal{H}} + \langle -\Delta((1-\chi)u), (1-\chi_2)v\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle \chi u, P(\chi_1 v)\rangle_{\mathcal{H}} + \langle (1-\chi)u, -\Delta((1-\chi_2)v)\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle u, P(\chi_1 v) - \Delta((1-\chi_1)v)\rangle_{\mathcal{H}^\#} - \langle (1-\chi)u, P(\chi_1 v)\rangle_{\mathcal{H}^\#} \\
&\quad - \langle \chi u, -\Delta((1-\chi_2)v)\rangle_{\mathcal{H}^\#} + \langle u, -\Delta(\chi_1 - \chi_2)v\rangle_{\mathcal{H}^\#} \\
&= \langle u, P^\# v\rangle_{\mathcal{H}^\#} + \langle u, Qv\rangle_{\mathcal{H}^\#},
\end{aligned}$$

where, using (4.3.7) (and our convention of multiplication operators),

$$\begin{aligned}
Q &:= (1-\chi)\Delta\chi_1 + \chi\Delta(1-\chi_2) - \Delta(\chi_1 - \chi_2) \\
&= (\chi_1 - \chi)\Delta + [\Delta, \chi_1] + (\chi - \chi_2)\Delta - [\Delta, \chi_2] \\
&\qquad - (\chi_1 - \chi_2)\Delta - [\Delta, \chi_1] + [\Delta, \chi_2] \equiv 0,
\end{aligned}$$

It follows that for $u, v \in \mathcal{D}^\#$, $\langle P^\# u, v\rangle_{\mathcal{H}^\#} = \langle u, P^\# v\rangle_{\mathcal{H}^\#}$, that is $P^\#$, is symmetric.

2. According to Theorem B.6 and the definitions following it, self-adjointness of $P^\#$ will follow from showing that $\mathcal{D}((P^\#)^*) \subset \mathcal{D}^\#$. Hence, suppose that $v \in \mathcal{H}^\#$ and that for all $u \in \mathcal{D}^\#$,

$$\langle P^\# u, v\rangle_{\mathcal{H}^\#} \le C\|u\|_{\mathcal{H}^\#}, \tag{4.3.8}$$

That is a characterization of $v \in \mathcal{D}((P^\#)^*)$ and it implies that there exists $w \in \mathcal{H}^\#$ such that $\langle u, w\rangle_{\mathcal{H}^\#} = \langle P^\# u, v\rangle_{\mathcal{H}^\#}$ for all $u \in \mathcal{D}^\#$. We then have $w := (P^\#)^* v \in \mathcal{H}^\#$.

Taking $u \in C_c^\infty(\mathbb{T}^n \setminus \overline{B(0,R_0)})$, we have $\langle u, w\rangle_{\mathcal{H}^\#} = \langle -\Delta u, v\rangle_{\mathcal{H}^\#}$, and hence

$$L^2(\mathbb{T}^n \setminus B(0,R_0)) \ni w|_{\mathbb{T}^n \setminus B(0,R_0)} = -\Delta(v|_{\mathbb{T}^n \setminus B(0,R_0)}).$$

It follows that $v|_{\mathbb{T}^n \setminus B(0,R_0)} \in H^2(\mathbb{T}^n \setminus B(0,R_0))$. With χ as in step 1 this gives

$$(1-\chi)v \in H^2(\mathbb{T}^n). \tag{4.3.9}$$

3. From (4.3.8) we also see that if $\chi_2 v_1 = 0$ (χ_j are as in (4.3.7)) and $(1-\chi_2)v_1 \in H^2(\mathbb{T}^n)$ then $v_1 \in \mathcal{D}((P^\#)^*)$. As $\chi_2(1-\chi)v = 0$, (4.3.9) shows

$$v \in \mathcal{D}((P^\#)^*) \implies v_1 := \chi v = v - (1-\chi)v \in \mathcal{D}((P^\#)^*).$$

We now apply (4.3.8) with this v_1 (noting that $v_1 \in \mathcal{H}$) and $u \in \chi_1 \mathcal{D} \subset \mathcal{D}^\#$:

$$\langle Pu, v_1 \rangle_{\mathcal{H}} = \langle P^\# u, v_1 \rangle_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}.$$

This implies that $v_1 \in \mathcal{D}(P^*) = \mathcal{D}(P)$ showing that

$$\chi v = v_1 \in \mathcal{D}.$$

Combined with (4.3.9) and the definition (4.3.2) we obtain $v \in \mathcal{D}^\#$. We proved that $\mathcal{D}((P^\#)^*) \subset \mathcal{D}^\#$ and as $P^\#$ is symmetric (step 1), it follows that $P^\#$ is self-adjoint.

4. To prove compactness of $(P^\# + i)^{-1}$ we consider a bounded sequence $u_j \in \mathcal{H}^\#$ and $v_j := (P^\# + i)^{-1} u_j \in \mathcal{D}^\#$,

$$\|v_j\|_{\mathcal{H}^\#} + \|P^\# v_j\|_{\mathcal{H}^\#} \leq C.$$

It follows that $(1-\chi)v_j$ is a bounded sequence in $H^2(\mathbb{T}^n)$ which then (see Theorem B.4) has a convergent subsequence in $L^2(\mathbb{T}^n)$, and hence in $\mathcal{H}^\#$.

On the other hand, $\chi v_j \in \mathcal{D}$ and

$$(P+i)\chi v_j = [-\Delta, \chi] v_j + \chi (P^\# + i)(P^\# + i)^{-1} u_j =: w_j + \chi u_j.$$

The w_j's can be considered as elements of $L^2(B(0,R_1) \setminus B(0,R_0))$ and, since $(1-\chi_2)v_j$'s are bounded in $H^2(\mathbb{T}^n)$, w_j are bounded in $L^2(B(0,R_1) \setminus B(0,R_0))$, and hence in $\mathcal{H}$. Also χu_j form a bounded sequence in $\mathcal{H}$. Hence,

$$\chi v_j = \mathbb{1}_{B(0,R_1)} \chi v_j = \mathbb{1}_{B(0,R_1)} (P+i)^{-1}(w_j + \chi u_j).$$

Lemma 4.2 shows that $\mathbb{1}_{B(0,R_1)}(P+i)^{-1}$ is a compact operator which shows that χv_j has a convergent subsequence in $\mathcal{H}$. It follows that v_j has a convergent subsequence in $\mathcal{H}^\#$. $\square$

The lemma shows that the spectrum of $P^\#(h)$ is discrete. To count resonances we make the following assumption about the counting function for the eigenvalues of $P^\#(h)$:

$$|\operatorname{Spec}(P^\#(h)) \cap [-r^2, r^2]| \leq C_0 r^{n^\#} h^{-n^\#}, \quad r \geq 1, \quad 0 < h < h_0, \tag{4.3.10}$$

for some $n^\# \geq n$.

REMARK. The upper bound in terms $t \mapsto t^{n^\#}$ can be replaced by a more general bound $t \mapsto \Phi(t)$ where Φ is increasing and satisfies some natural conditions. The conclusion (4.3.17) below then holds with $t^{n^\#}$ replaced by $\Phi(t)$ – see [**SZ91**],[**Sj02**], and [**Vo92**].

EXAMPLES. 1. **Elliptic perturbations of the semiclassical Laplacian.** Suppose that

$$P(h)u := \sum_{i,j=1}^{n} hD_{x_j}(a_{ij}(x) hD_{x_i} u) + c(x)u,$$

where $a_{ij} - \delta_{ij}, c_j \in C_c^\infty(B(0, R_0), \mathbb{R})$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geq c_0|\xi|^2, \quad c_0 > 0,$$

and $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\mathcal{D} = H^2(\mathbb{R}^n)$. This operator satisfies all the black box assumptions from §4.1. Theorem 4.4 then provides the meromorphic continuation of the resolvent.

The reference operator $P^\#(h)$ is the same elliptic operator acting on $\mathcal{D}^\# = H^2(\mathbb{T}^\#_{R_1})$, $R_1 > R_0$. Standard estimates for the number of eigenvalues of elliptic operators (see for instance [**Zw12**, Theorem 14.11]) show that (4.3.10) holds with $n^\# = n$.

2. **Pseudo-Laplacian for finite volume surfaces.** In Example 3 of §4.1 (see (4.1.22)) the Hilbert spaces is given by $\mathcal{H} = \mathcal{H}_a \oplus L^2([a, \infty))$ In that case it is more useful to consider the reference operator defined using (4.3.5). For $b > a$ we put

$$\mathcal{H}_b^\# := H_a \oplus L^2([a, b]).$$

The operator $P_b^\#$ is then *pseudo-Laplacian* of Lax–Phillips and Colin de Verdière [**CdV83**]. We claim that (4.3.10) holds with $n^\# = 2$ (note that now $n = 1$ and we do not have a semiclassical parameter):

$$|\operatorname{Spec}(P_b^\#) \cap [0, r^2]| \leq Cr^2, \quad r > 1. \tag{4.3.11}$$

Since $P_b^\# \geq 0$ this is the same as (4.3.10).

REMARK. One can improve (4.3.11) to obtain an asymptotic formula for the number of eigenvalues of $P_b^\#$ – see [**CdV83**, §4]. That proceeds through an improved version of the following lemma – see [**CdV83**, Lemma 4.2].

To prove the bound (4.3.11) we use

LEMMA 4.12. *Let $(C_{\alpha,\beta}, g_0)$ be the cylinder $[\alpha, \beta] \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, $\alpha > 0$, equipped with the metric $g_0 = dr^2 + e^{-2r}d\theta^2$. We say that $\varphi \in H^2(C_{\alpha,\beta})$ is a* Neumann eigenfuction *of the Laplacian on $C_{\alpha,\beta}$ with eigenvalue E if*

$$\begin{gathered}-\Delta\varphi = E\varphi, \quad E \leq k^2, \quad \partial_y\varphi(\alpha, \theta) = \partial_y\varphi(\beta, \theta) = 0, \\ \int_0^{2\pi} \varphi(y, \theta)d\theta = 0, \quad \alpha < y < \beta.\end{gathered} \tag{4.3.12}$$

Let $N_{\alpha,\beta}(k)$ be the number of independent Neumann eigenfunctions with $E \leq k^2$. Then, with C independent of α and β,

$$N_{\alpha,\beta}(k) \leq \frac{e^\beta - e^\alpha}{e^{2\alpha}}k^2 + \frac{C}{e^\alpha}k. \tag{4.3.13}$$

Proof. 1. An explicit calculation gives the result for the counting function $M_\ell(k)$ for the same class of of eigenfunctions but for the cylinder $K_{\ell,A} := [A, A+\ell] \times \mathbb{S}^1$, $A > 0$, with the metric $g_1 = dr^2 + d\theta^2$: the eigenfunctions are given by

$$\varphi(r,\theta) = \cos\left(\frac{m\pi(y-A)}{\ell}\right) e^{in\theta}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{Z} \setminus \{0\},$$
$$E = \left(\frac{m\pi}{\ell}\right)^2 + n^2.$$

The counting function of $E \le k^2$ satisfies

$$M_\ell(k) \le \ell k^2 + Ck. \tag{4.3.14}$$

2. We now introduce a diffeomorphism $\kappa : C_{\alpha,\beta} \to K_{\ell,A}$, $\ell = e^\beta - e^\alpha$, $A = e^\alpha$, given by $\kappa(r,\theta) = (e^r, \theta)$. Then

$$\kappa^* g_1 = e^{2r} g_0, \quad \kappa^* d\,\mathrm{Vol}_{g_1} = e^{2r} d\,\mathrm{Vol}_{g_0}, \tag{4.3.15}$$

and

$$\begin{aligned} \int_{K_{\ell,A}} |df|_{g_1}^2 d\,\mathrm{Vol}_{g_1} &= \int_{C_{\alpha,\beta}} |d\kappa^* f|_{\kappa^* g_0}^2 \kappa^* d\,\mathrm{Vol}_{g_1} \\ &= \int_{C_{\alpha,\beta}} |d\kappa^* f|_{g_0}^2 d\,\mathrm{Vol}_{g_0} . \end{aligned} \tag{4.3.16}$$

(We note that $|df(r,\theta)|_{g_1}^2$ is calculated using the dual metric on $T^*_{(r,\theta)} K_{\ell,M}$ so that the two factors e^{2r} cancel.)

From (4.3.15) we see that

$$d\,\mathrm{Vol}_{g_0}|_{C_{\alpha,\beta}} = e^{-2r} \kappa^* d\,\mathrm{Vol}_{g_1}|_{C_{\alpha,\beta}} \le e^{-2\alpha} \kappa^* d\,\mathrm{Vol}_{g_1}|_{C_{\alpha,\beta}},$$

which combined with (4.3.16) gives

$$\frac{\int_{C_{\alpha,\beta}} |d\kappa^* f|_{g_0}^2 d\,\mathrm{Vol}_{g_0}}{\int_{C_{\alpha,\beta}} |\kappa^* f|^2 d\,\mathrm{Vol}_{g_0}} \ge e^{2\alpha} \frac{\int_{K_{\ell,A}} |df|_{g_1}^2 d\,\mathrm{Vol}_{g_1}}{\int_{K_{\ell,A}} |f|^2 d\,\mathrm{Vol}_{g_1}}.$$

This provides a comparison for Rayleigh quotients used in the min-max characterization of eigenvalues (see Theorem B.12): from (B.1.14) we see that eigenvalues of each metric satisfy $\lambda_k(g_0) \ge e^{2\alpha}\lambda_k(g_1)$. This leads to a comparison of the counting function (see (4.3.12)):

$$N_{\alpha,\beta}(k) \le M_{e^\beta - e^\alpha}(k/e^\alpha).$$

Using (4.3.14) we then obtain (4.3.13). ☐

Proof of (4.3.11). The operator $P_b^\#$ with domain given by (4.3.5) is obtained (just as P was) from the quadratic form (4.1.17) restricted to the domain (in the notation of (4.1.17))

$$\mathcal{D}_Q^\# = \{u \in H^1(X) : a_0(r) = 0 \ \text{ for } r > b\}.$$

Here we used the fact that

$$\{u \in H^1((-\infty,\infty)) : u(r)|_{r>b} = 0\} \ni u \mapsto u|_{r\le b} \in H_0^1((-\infty, b]),$$

is an isomorphism.

We now see that

$$\mathcal{D}_Q^\# \subset H^1(X_0) \oplus \bigoplus_{p=0}^{\infty} \left\{ u \in H^1(C_{b+p,b+p+1}), \int_0^{2\pi} u(r,\theta)d\theta \equiv 0 \right\}.$$

Let $N_{X_0}(k)$ be the counting function for eigenvalues less than k^2 of the Neumann realization of $-\Delta_g$ on X_0. By the standard Weyl law we have

$$N_{X_0}(k) \simeq \frac{\mathrm{Vol}(X_0)}{4\pi} k^2.$$

It now follows from Theorem (B.1.14) and Lemma 4.12 that, for $r \ge 1$,

$$\begin{aligned} |\operatorname{Spec}(P_b^\#) \cap [0, r^2]| &\le N_{X_0}(r) + \sum_{p=0}^{\infty} N_{C_{b+p,b+p+1}}(r) \\ &\le Cr^2 + C \sum_{p=0}^{\infty} r^2 e^{-p} \le C' r^2, \end{aligned}$$

which is (4.3.11). $\square$

We now come to the main result of this section. As remarked after (4.3.10) a more general counting functions are possible.

THEOREM 4.13 (Upper bounds on the number of resonances). *Suppose that $P(h)$ is a semiclassical black box Hamiltonian with $n \ge 1$ odd, and that* (4.3.10) *hold. Then for some constant C_1*

$$\sum \{ m_R(\lambda) : |\lambda| \le r\} \le C_1 r^{n^\#} h^{-n^\#}, \quad r > 1. \tag{4.3.17}$$

REMARK. When n is even or when perturbations have long range, global bounds are more complicated – see §4.7. Semiclassical bounds in compact sets away from 0, for compactly supported perturbations, can be proved by the methods presented here – see Theorem 7.4 for a simple version.

Before starting the proof of Theorem 4.13 we modify some notation from the proof of Theorem 4.4. First, we define the semiclassical free resolvent

$$R_0(\lambda, h) = (-h^2\Delta - \lambda^2)^{-1} = h^{-2} R_0(\lambda/h), \quad \operatorname{Im} \lambda > 0, \tag{4.3.18}$$

and the meromorphic extension of the resolvent of $P(h)$.

$$R(\lambda, h) = (P(h) - \lambda^2)^{-1}, \quad \operatorname{Im} \lambda > 0.$$

Step 4 in the proof Theorem 4.4 shows that

$$R(\lambda, h)\chi = (Q_0(\lambda, h) + Q_1(h))(I + K(\lambda, h))^{-1},$$

where

$$\begin{aligned} Q_0(\lambda, h) &:= (1 - \chi_0)R_0(\lambda, h)(1 - \chi_1), \\ Q_1(h) &:= \chi_2(P(h) - \lambda_0^2)^{-1}\chi_1, \\ K(\lambda, h) &:= K_0(\lambda, h) + K_1(\lambda, h), \\ K_0(\lambda, h) &:= [-h^2\Delta, \chi_0]R_0(\lambda, h)(1 - \chi_1)\chi, \\ K_1(\lambda, h) &:= (\lambda_0^2 - \lambda^2)\chi_2(P(h) - \lambda_0^2)^{-1}\chi_1 \\ &\quad + [-h^2\Delta, \chi_2](P(h) - \lambda_0^2)^{-1}\chi_1 \end{aligned} \tag{4.3.19}$$

We use the cut-off functions (4.2.6) and $\chi = \chi_3$.

Step 2 in the proof of Theorem 4.4 shows that λ_0 can be chosen independently of h. We recall that λ_0 has to be chosen so that $I + K(\lambda_0, h)$ is invertible. With λ_0 given by (4.2.11), $\lambda_0 = e^{i\pi/4}\mu$, we apply Lemma 4.3 to obtain

$$\|[-h^2\Delta, \chi_2](P(h) - \lambda_0^2)^{-1}\chi\|_{\mathcal{H} \to \mathcal{H}} \leq Ch/\mu,$$

and

$$\|[-h^2\Delta, \chi_0]R_0(\lambda_0, h)(1 - \chi_1)\chi\|_{\mathcal{H} \to \mathcal{H}} \leq Ch/\mu.$$

Since $0 < h < 1$, we can take μ fixed and large and conclude that

$$K(\lambda_0, h) \leq \frac{1}{2}, \tag{4.3.20}$$

obtaining invertibility of $I + K(\lambda_0, h)$.

We have two lemmas related to $K(\lambda, h)$:

LEMMA 4.14 (Estimates on singular values). *With the notation of (4.3.19) and for $\lambda_0 = e^{i\pi/4}\mu$ with $\mu \geq 1$ fixed and large, we have the following characteristic value estimates:*

$$\begin{aligned} s_j\left(\chi_2(P(h) - \lambda_0^2)^{-1}\chi\right) &\leq C\left(\mu^2 + (hj^{1/n^\#})^2\right)^{-1}, \\ s_j\left([-h^2\Delta, \chi_2](P(h) - \lambda_0^2)^{-1}\chi\right) &\leq Ch\left(\mu^2 + (hj^{1/n^\#})^2\right)^{-1/2}. \end{aligned} \tag{4.3.21}$$

Proof. 1. For $P(h)$ replaced by $P^\#(h)$ the first estimate in (4.3.21) follows from (4.3.10). To see that let $\{\mu_j^2\}_{j=0}^\infty$, $0 \leq \mu_j \leq \mu_{j+1}$, be the eigenvalues of $(P(h)^\#)^2$. Then (4.3.10) means that for $0 < h < h_0$,

$$\mu_j < r^2 \implies j < C_0(\max(r, 1)/h)^{n^\#}.$$

This implies that

$$j = C_0(r/h)^{n^\#},\ r \geq 1 \implies \mu_j \geq r^2 = ((j/C_0)^{1/n^\#}h)^2.$$

If $((j/C_0)^{1/n^\#}h)^2 < 1$ then we only conclude $\mu_j \geq 0$. Thus for all j,

$$\mu_j \geq ((j/C_0)^{1/n^\#}h)^2 - 1.$$

Since $\mu \geq 1$,

$$\mu_j + \mu^2 \geq \tfrac{1}{2}\mu^2 + (h(j/C_0)^{1/n^\#})^2,$$

and

$$s_j\left((P^\#(h) - \lambda_0^2)^{-1}\right) \leq C/(\mu^2 + (hj^{1/n^\#})^2).$$

Since we are taking singular values of self-adjoint operators we also have

$$s_j\left((P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\right) \leq C/(\mu^2 + (hj^{1/n^\#})^2)^{-\frac{1}{2}}. \tag{4.3.22}$$

We then consider multiplication by χ and χ_2 as bounded function on $L^2(\mathbb{T}^n)$ and obtain the first estimate in (4.3.21) with $P^\#(h)$ in place of $P(h)$.

2. To obtain the second estimate in (4.3.21) we note that

$$\begin{aligned}
&s_j\left([-h^2\Delta, \chi_2](P^\#(h) - \lambda_0^2)^{-1}\right) \\
&\quad \leq Ch\|(1-\chi_0)(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{\mathcal{H}^\# \to H_h^1(\mathbb{T}^n)} s_j\left((P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\right),
\end{aligned}$$

where the last factor on the right is estimated by (4.3.22). The estimate the first term we write

$$\begin{aligned}
\|(1-\chi_0)(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{\mathcal{H}^\# \to H_h^1(\mathbb{T}^n)} &\leq C\|(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{\mathcal{H}^\# \to \mathcal{D}_h^{\#\frac{1}{2}}} \\
&= C\|(P^\#(h) + i)^{\frac{1}{2}}(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{\mathcal{H}^\# \to \mathcal{H}^\#} \\
&= C\|(I + (\lambda_0^2 + i)(P^\#(h) - \lambda_0^2)^{-1})^{\frac{1}{2}}\|_{\mathcal{H}^\# \to \mathcal{H}^\#} \leq C',
\end{aligned}$$

where we used definition (4.3.4) and (4.1.10) and the fact that $\|(P^\#(h) - \lambda_0^2)^{-1}\| = \|(P^\#(h) + \mu^2 i)^{-1}\| = \mu^{-2}$.

We reiterate the conclusions of Steps 1 and 2:

$$\begin{aligned}
&s_j(\chi_2(P^\#(h) - \lambda_0^2)^{-1}\chi) \leq C(\mu^2 + (hj^{1/n^\#})^2)^{-1}, \\
&s_j([-h^2\Delta, \chi_2](P^\#(h) - \lambda_0^2)^{-1}\chi) \leq Ch(\mu^2 + (hj^{1/n^\#})^2)^{-1/2}.
\end{aligned} \tag{4.3.23}$$

3. We now compare the resolvents of $P^\#(h)$ and $P(h)$. We claim that for $\chi \in C_c^\infty(B(0, R_1)$ which is equal to 1 in a neighbourhood of $B(0, R_0)$ we have, for any N,

$$\chi(P^\#(h) - \lambda_0^2)^{-1}\chi - \chi(P(h) - \lambda_0^2)^{-1}\chi = \mathcal{O}_N(h^\infty)_{\mathcal{H} \to \mathcal{D}_h^N}. \tag{4.3.24}$$

We note that although operators $P^\#$ and P act on different spaces, the cut-off functions produce operators acting on $\mathcal{H}$ or $\mathcal{H}^\#$.

To see (4.3.24) we choose $\chi_4 \in C_c^\infty(B(0,R_1))$ such that $\chi_4 = 1$ in $\operatorname{supp}\chi$. The estimate (4.3.24) follows from the same estimate for

$$Q(h) := \chi_4(P^\#(h) - \lambda_0^2)^{-1}\chi - \chi_4(P(h) - \lambda_0^2)^{-1}\chi. \tag{4.3.25}$$

Since $\chi_4 P^\# = \chi_4 P$ (see (4.1.5)) we have

$$(P(h) - \lambda_0^2)Q(h) = [-h^2\Delta, \chi_4]\,(A(h) - B(h))\,,$$
$$A(h) := \psi(P^\#(h) - \lambda_0^2)^{-1}\chi, \quad B(h) := \psi(P(h) - \lambda_0^2)^{-1}\chi,$$

where

$$\psi \in C_c^\infty(B(0,R_1)), \quad [-\Delta, \chi_4]\psi = [-\Delta, \chi_4], \quad \operatorname{supp}\psi \cap \operatorname{supp}\chi = \emptyset.$$

4. Since
$$Q(h) = (P(h) - \lambda_0^2)^{-1}[-h^2\Delta, \chi_4]\,(A(h) - B(h))\,,$$
the estimate for $Q(h)$, and hence (4.3.24), follow from

$$A(h), B(h) = \mathcal{O}_N(h^{2N})_{\mathcal{H}\to\mathcal{D}_h^N}. \tag{4.3.26}$$

To establish this we choose $\varphi_j \in C_c^\infty(B(0,R_1))$, $j = 1, \cdots, N$, with the following properties

$$\varphi_1\chi = 1, \quad \varphi_j|_{\operatorname{supp}\varphi_{j-1}} = 1, \quad \varphi_{2N}\psi = 0.$$

That is possible since the supports of ψ and χ are disjoint. Note that φ_j are equal to 1 near $B(0,R_0)$ as χ is equal to 1 there.

The properties of the supports give

$$\psi(P(h) - \lambda_0^2)^{-1}\varphi_{2N}(P(h) - \lambda_0^2) = \psi(P(h) - \lambda_0^2)^{-1}[\varphi_{2N}, P(h)],$$

and

$$[\varphi_j, P(h)](P(h){-}\lambda_0^2)^{-1}\varphi_{j-1}(P(h){-}\lambda_0^2) = [\varphi_j, P(h)](P(h){-}\lambda_0^2)^{-1}[\varphi_{j-1}, P(h)].$$

These can be re-written as

$$\psi(P(h) - \lambda_0^2)^{-1}\varphi_{2N} = \psi(P(h) - \lambda_0^2)^{-1}[\varphi_{2N}, P(h)](P(h) - \lambda_0^2)^{-1},$$
$$[\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}\varphi_{j-1} = [\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}[\varphi_{j-1}, P(h)](P(h) - \lambda_0^2)^{-1}.$$

Hence

$$\begin{aligned} B(h) &= \psi(P(h) - \lambda_0^2)^{-1}\chi = \psi(P(h) - \lambda_0^2)^{-1}\varphi_{2N}\cdots\varphi_1\chi \\ &= \psi(P(h) - \lambda_0^2)^{-1}[\varphi_{2N}, P(h)](P(h) - \lambda_0^2)^{-1} \\ &\qquad [\varphi_{2N-1}, P(h)](P(h) - \lambda_0^2)^{-1}\cdots[\varphi_1, P(h)](P(h) - \lambda_0^2)^{-1}\chi. \end{aligned}$$

With the norms defined in (4.1.9) we have

$$[\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1} = \mathcal{O}(h) : \mathcal{D}_h^k \to \mathcal{D}_h^{k+\frac12},$$

which gives

$$B(h) = \mathcal{O}(h^{2N}) : \mathcal{H} \to \mathcal{D}_h^N.$$

The proof for $A(h)$ is similar. We note that $\mathcal{D}_h$ can be replaced by $\mathcal{D}_h^\#$ in the estimate. This gives (4.3.26). By going back to (4.3.25) we obtain (4.3.24).

5. We can now prove (4.3.21) using (4.3.23) and (4.3.24): with $Q(h)$ defined by (4.3.25) we have (see (B.3.5))

$$\begin{aligned} s_j(\chi_2(P(h) - \lambda_0^2)^{-1}\chi) &\leq s_{[j/2]}(\chi_2(P^\#(h) - \lambda_0^2)^{-1}\chi) + s_{[j/2]}(Q(h)) \\ &\leq \frac{C}{\mu^2 + (hj^{1/n^\#})^2} \\ &\qquad + s_{[j/2]}((P^\#(h) - \lambda_0^2)^{-N})\|Q(h)\|_{\mathcal{H}\to\mathcal{D}_h^N} \\ &\leq \frac{C}{\mu^2 + (hj^{1/n^\#})^2} + \frac{Ch^{2N}}{(\mu^2 + (hj^{1/n^\#})^2)^N} \\ &\leq \frac{C'}{\mu^2 + (hj^{1/n^\#})^2}. \end{aligned}$$

The argument for the second estimate in (4.3.21) is similar. □

LEMMA 4.15 (**Upper bound on multiplicities of resonances**). *With the notation of* (4.3.19) *and for* $\lambda_0 = e^{i\pi/4}\mu$ *with* $\mu > 0$ *fixed and large,* $K(\lambda, h)^{n^\#+1} \in \mathcal{L}^1(\mathcal{H}, \mathcal{H})$, *and*

$$\begin{aligned} m_R(\lambda) \leq m_H(\lambda) &:= \frac{1}{2\pi i}\oint_\lambda \frac{\partial_\zeta H(\zeta, h)}{H(\zeta, h)} d\zeta, \\ H(\zeta, h) &:= \det(I - (-K(\zeta, h))^{n^\#+1}). \end{aligned} \tag{4.3.27}$$

where $m_R(\lambda)$ *is the multiplicity of the resonance at* λ *given by* (4.2.18) *and the integral is over a positively oriented circle containing no other resonances than* λ.

Proof. 1. The trace class property of $K(\lambda, h)^{n^\#+1}$ follows from the estimates on the singular values: from Proposition B.15 we have

$$\begin{aligned} s_j(K(\lambda, h)^{n^\#+1}) &\leq \Big(s_{[j/(n^\#+1)]}(K(\lambda, h))\Big)^{n^\#+1}, \\ s_\ell(K(\lambda, h)) &\leq (|\lambda|^2 + \mu^2)s_{[\ell/3]}(\chi_2(P(h) - \lambda_0^2)^{-1}\chi) \\ &\qquad + s_{[\ell/3]}([-h^2\Delta, \chi_2](P(h) - \lambda_0^2)^{-1}\chi) \\ &\qquad + s_{[\ell/3]}([-h^2\Delta, \chi_0]R_0(\lambda, h)\chi). \end{aligned} \tag{4.3.28}$$

To estimate the last term we consider

$$[-h^2\Delta, \chi_0]R_0(\lambda, h)\chi = \mathcal{O}_{\lambda,h}(1) : L^2(\mathbb{T}^n) \to H^1(\mathbb{T}^n), \quad \mathbb{T}^n := \mathbb{R}^n/(R_2\mathbb{Z})^n,$$

for some large R_2. Hence, using (B.3.6),

$$s_k([-h^2\Delta, \chi_0]R_0(\lambda,h)\chi) \leq$$
$$s_k((-\Delta_{\mathbb{T}^n}+i)^{-1/2})\|[-h^2\Delta,\chi_0]R_0(\lambda,h)\chi\|_{L^2(\mathbb{T}^n)\to H^1(\mathbb{T}^n)} \leq C_{\lambda,h}k^{-1/n}.$$

Using this and (4.3.21) in (4.3.28) we conclude that for $j \geq 1$,

$$s_j(K(\lambda,h)) \leq C'_{\lambda,h}(j^{-2/n^\#}+j^{-1/n^\#}+j^{-1/n})^{n^\#+1} \leq C''_{\lambda,h}j^{-(n^\#+1)/n^\#}.$$

Thus the singular values are summable and definition (B.4.2) shows that $K(\lambda,h)$ is of trace class.

(The main effort in the proof of Theorem 4.13 will be to improve this rough estimate on characteristic values.)

2. The proof of (4.3.27) is based on the Gohberg-Sigal theory reviewed in §C.4 and the identity based (4.2.13) and $(1-q)^{-1} = (1+q+\cdots+q^{n^\#})(1-q^{n^\#+1})^{-1}$:

$$R(\lambda,h)\chi = Q(\lambda,h)\chi W(\lambda,h)(I-(-K(\lambda,h))^{n^\#+1})^{-1}(I+K(\lambda,h)(1-\chi)).$$
$$W(\lambda,h) := (I-K(\lambda,h)+\cdots+(-1)^{n^\#}K(\lambda,h)^{n^\#}).$$

Theorems 4.7 (specifically (4.7)) and C.11 (apply (C.4.3) with $A(\lambda) = I + (-K(\lambda,h))^{n^\#+1}$) give the estimate on the multiplicities. $\square$

Proof of Theorem 4.13. 1. In view of (4.3.27) we need to estimate the number of zeros of H in the disc of $D(0,r)$, $r>1$. We can assume that r is large enough so that $r \gg |\lambda_0|$. The Jensen formula (D.1.10) gives the estimate (D.1.11):

$$\sum\{m_H(\lambda) : |\lambda| \leq r\} \leq C \max_{|\lambda|\leq 2r} \log|H(\lambda,h)| - C\log|H(\lambda_0,h)|. \tag{4.3.29}$$

2. We start with the upper bound on $H(\lambda,h)$. The estimate (4.3.28) and the Weyl inequalities of Proposition B.25 show that

$$\log|H(\lambda,h)| \leq C(\log P_1 + \log P_2 + \log P_3), \tag{4.3.30}$$

where

$$\begin{aligned} P_1 &:= \prod_{j=0}^{\infty}\left(1+s_j\left(((|\lambda|^2+|\lambda_0|^2)\chi_2(P(h)-\lambda_0^2)^{-1}\chi\right)^{n^\#+1}\right), \\ P_2 &:= \prod_{j=0}^{\infty}\left(1+s_j([-h^2\Delta,\chi_2](P(h)-\lambda_0^2)^{-1}\chi)^{n^\#+1}\right), \\ P_3 &:= \prod_{j=0}^{\infty}\left(1+s_j([-h^2\Delta,\chi_0]R_0(\lambda,h)\chi)^{n^\#+1}\right). \end{aligned} \tag{4.3.31}$$

To estimate P_1 and P_2 we use (4.3.21): since $\lambda_0 = e^{\pi i/4}\mu$ is fixed we drop the dependence on μ:

$$\begin{aligned}
\log P_1 &\le \sum_{j=0}^{\infty} \log\left(1 + \left(\frac{C\langle\lambda\rangle^2}{1+(hj^{1/n^\#})^2}\right)^{n^\#+1}\right) \\
&\le C\log\langle\lambda\rangle + \sum_{j=2}^{\infty}\log\left(1+\left(C\langle\lambda\rangle h^{-1}j^{-1/n^\#}\right)^{2(n^\#+1)}\right) \\
&\le C\log\langle\lambda\rangle + \int_1^\infty \log\left(1+\left(Ch^{-1}\langle\lambda\rangle x^{-1/n^\#}\right)^{2(n^\#+1)}\right)dx \\
&\le C\log\langle\lambda\rangle + C'h^{-n^\#}\langle\lambda\rangle^{n^\#}\int_1^\infty \log\left(1+y^{-2(n^\#+1)/n^\#}\right)dy \\
&\le C''h^{-n^\#}\langle\lambda\rangle^{n^\#}.
\end{aligned}$$

(The integral comparison is justified as $x \mapsto \log(1+\alpha x^{-\beta})$, $\alpha, \beta > 0$, is a decreasing function.) A similar argument shows that $\log P_2 \le Ch^{-n^\#}$: using the second estimate in (4.3.21) and dropping the dependence on the fixed constant μ,

$$\begin{aligned}
\log P_2 &\le \sum_{j=0}^{\infty}\log\left(1 + Ch\left(1+(hj^{1/n^\#})^2\right)^{-\frac{1}{2}(n^\#+1)}\right) \\
&\le C + \int_1^\infty \log\left(1+Ch(h^{-1}x^{-1/n^\#})^{(n^\#+1)}\right)dx \\
&\le C + C'h^{-n^\#}\langle\lambda\rangle^{n^\#}\int_1^\infty \log\left(1+y^{-(n^\#+1)/n^\#}\right)dy \\
&\le C''h^{-n^\#}.
\end{aligned}$$

3. We now need to estimate $\log P_3$ in (4.3.31). We will use the following estimate proved in Steps 6 and 7 below:

$$\begin{aligned}
s_j([-h^2\Delta, \chi_0]R_0(\lambda,h)\chi) &\le C\langle\lambda\rangle h^{-1}j^{-1/n} \\
&\quad + \exp\left(\langle\lambda\rangle/h - j^{1/n-1}/C\right).
\end{aligned} \tag{4.3.32}$$

Assuming (4.3.32) we obtain,

$$\begin{aligned}
\log P_3 &\le C\sum_{j=1}^{\infty}\left(\log\left(1+C(\langle\lambda\rangle h^{-1}j^{-1/n})^{n^\#+1}\right) + \log\left(1+e^{\langle\lambda\rangle/h - j^{1/n-1}/C}\right)\right) \\
&\le C(\langle\lambda\rangle/h)^n + \sum_{j=1}^{C(\langle\lambda\rangle/h)^{n-1}}\langle\lambda\rangle/h + \sum_{C\langle\lambda\rangle/h}^{\infty} e^{-j^{1/n-1}/C'} \\
&\le C(\langle\lambda\rangle/h)^n.
\end{aligned}$$

Here we used the same argument as in Step 2 to estimate the sum of $\log(1+C(\langle\lambda\rangle j^{1/n})^{n^\#+1})$. This estimate and the estimates on $\log P_1$ and $\log P_2$ in Step 2, show that (see (4.3.31)) that

$$\log H(\lambda, h) \leq C(\langle\lambda\rangle/h)^{n^\#} \tag{4.3.33}$$

4. Going back to (4.3.29) we need a lower bound on $\log|H(\lambda_0, h)|$. For that we write

$$H(\lambda_0, h)^{-1} = \det\left((I - (-K(\lambda_0, h))^{n^\#+1})^{-1}\right)$$

Now,

$$(I - (-K(\lambda_0, h))^{n^\#+1})^{-1} = I - (I - (-K(\lambda_0, h))^{n^\#+1})^{-1}K(\lambda_0, h))^{n^\#+1}.$$

From (4.3.20) we see that

$$\|(I - (-K(\lambda_0, h))^{n^\#+1})^{-1}\|_{\mathcal{H}\to\mathcal{H}} \leq 2,$$

and hence, using Weyl inequalities again (see Proposition B.25), we see that

$$\begin{aligned} |\det\left((I - (-K(\lambda_0, h))^{n^\#+1})^{-1}\right)| &\leq \prod_{j=0}^{\infty}\left(1 + \left(s_{[j/(n^\#+1)]}(K(\lambda_0, h))\right)^{n^\#+1}\right) \\ &\leq \prod_{j=0}^{\infty}\left(1 + (s_j(K(\lambda_0, h)))^{n^\#+1}\right)^{n^\#+1}. \end{aligned}$$

From the estimates in Steps 2 and 3 we see that

$$|H(\lambda_0, h)^{-1}| \leq Ch^{-n^\#}.$$

This, (4.3.33) and (4.3.29) show that

$$\sum\{m_H(\lambda) : |\lambda| \leq r\} \leq Cr^{n^\#}h^{-n^\#}, \quad r > 1.$$

In view of (4.3.27) this proves the theorem.

5. It remains to establish (4.3.32). Since $R_0(\lambda, h) = h^{-2}R_0(\lambda/h)$ (see (4.3.18)) it is enough to show that

$$s_j([\Delta, \chi_0]R_0(\lambda)\chi) \leq C\langle\lambda\rangle j^{-1/n} + \exp(C\langle\lambda\rangle - j^{1/n-1}/C). \tag{4.3.34}$$

(We remark that the proof of Theorem 3.28 contains a slightly simpler version of this estimate.)

We start with the estimate for $\operatorname{Im}\lambda \geq 0$. Then considering the operator $[\Delta, \chi_0]R_0(\lambda)\chi$ as acting on $L^2(\mathbb{T}^n)$, $\mathbb{T}^n := \mathbb{R}^n/R_2\mathbb{Z}^n$ for some large R_2,

$$\begin{aligned} s_j([\Delta, \chi_0]R_0(\lambda)\chi) &\leq s_j((-\Delta_{\mathbb{T}^n} + I)^{-1/2})\|\chi R_0(\lambda)\chi\|_{L^2(\mathbb{T}^n)\to H^2(\mathbb{T}^n)} \\ &\leq Cj^{-1/n}\langle\lambda\rangle, \end{aligned} \tag{4.3.35}$$

where we used the estimate (3.1.12) for the norm of $\chi R_0(\lambda)\chi$. This gives the estimate (4.3.34) for $\operatorname{Im}\lambda \geq 0$.

6. To obtain estimates for $\operatorname{Im}\lambda < 0$ we use Stone's formula (3.1.19) and write

$$\begin{aligned} \chi(R_0(\lambda) - R_0(-\lambda))\chi &= a_n \lambda^{n-2} E_\chi(\bar\lambda)^* E_\chi(\lambda)\,, \\ E_\chi(\lambda)u(\omega) &:= \int_{\mathbb{R}^n} e^{-i\lambda\langle\omega,y\rangle}\chi(y)u(y)dy\,, \\ E_\rho(\lambda) &: L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})\,. \end{aligned} \tag{4.3.36}$$

Then (B.3.5) gives

$$\begin{aligned} s_j([\Delta,\chi_0]R_0(\lambda)\chi) &\leq C\langle\lambda\rangle^{n-2}\|[-\Delta,\chi_0]E_\rho(\lambda)\| s_{[j/2]}(E_\chi(\lambda)) \\ &\qquad + s_{[j/2]}([\Delta,\chi_0]R_0(-\lambda)\chi) \\ &\leq C\exp(C\langle\lambda\rangle) s_{[j/2]}(E_\chi(\lambda)) + C\langle\lambda\rangle j^{-1/n}\,. \end{aligned} \tag{4.3.37}$$

7. To estimate $s_j(E_\chi(\lambda))$ we repeat the argument of Step 4 of the proof of Theorem 3.28. If $-\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on $\mathbb{S}^{n-1}$ then (B.3.6) gives

$$\begin{aligned} s_j(E_\chi(\lambda)) &\leq s_j((-\Delta_{\mathbb{S}^{n-1}}+1)^{-\ell})\|(-\Delta_{\mathbb{S}^{n-1}}+1)^\ell E_\chi(\lambda)\| \\ &\leq C^\ell j^{-2\ell/(n-1)}\|(-\Delta_{\mathbb{S}^{n-1}}+1)^\ell E_\chi(\lambda)\| \\ &\leq C_1^\ell j^{-2\ell/(n-1)}\exp(C_1|\lambda|)(2\ell)!\,. \end{aligned} \tag{4.3.38}$$

An optimization of this estimate in ℓ gives

$$s_j(E_\chi(\lambda)) \leq C_2 \exp\left(C_2|\lambda| - j^{\frac{1}{n-1}}/C_2\right)\,.$$

Combined with (4.3.37) this gives (4.3.32) completing the proof of theorem. $\square$

Theorem 4.13 applies to Examples 1 and 2 presented earlier in this section. In the case of Example 2 the fact that $n^\# > n$ gives asymptotics for the number of resonances. We conclude this section with a classical scattering problem in which the first polynomial, and optimal, bound was given by Melrose [**Me84b**]:

EXAMPLE. Consider scattering by an obstacle $\mathcal{O}$ in odd dimensions (Example 2 in §4.1) with any self-adjoint boundary condition. Let $m_{\mathcal{O}}(\lambda)$ be the multiplicity of a resonance λ of the corresponding Laplacian. Eigenvalue counting estimates for the Laplacian on a compact manifold with boundary $\mathbb{T}^n \setminus \mathcal{O}$, $\mathbb{T}^n := \mathbb{R}^n/R_2\mathbb{Z}^n$ show that (4.3.10) holds with $n^\# = n$. Hence, Theorem 4.13 applied with $h = 1$ gives

$$\sum \{m_{\mathcal{O}}(\lambda) : |\lambda| \leq r\} \leq Cr^n, \quad r > 1.$$

This bound is optimal as shown by the example of the sphere – see Stefanov [St06] and references given there.

4.4. PLANE WAVES AND THE SCATTERING MATRIX

In this section we define the scattering matrix for a black box operator. Since we will deal with exact formulas and not asymptotic properties we only consider the case $h = 1$. All the formulas remain valid for operator $P(h)$ with the obvious rescaling:

$$-h^2\Delta \rightsquigarrow -\Delta, \quad P(h) \rightsquigarrow h^{-2}P(h), \quad \lambda \rightsquigarrow \lambda/h.$$

Our presentation will be close to that in §§3.6 and 3.7. As did previous section of this chapter it will also depend on the properties of the free resolvent from §3.1.

4.4.1. Outgoing solutions. The plane waves,

$$e_0(\lambda,\omega) = e_0(\lambda,\omega,x) = e^{-i\lambda\langle x,\omega\rangle}, \quad (-\Delta-\lambda^2)e_0 = 0,$$

are important from both physical and mathematical points of view – see for instance the spectral decomposition of $-\Delta$ in (3.4). Motivated by Figure 3.4 we now want to consider solutions to $(P-\lambda^2)w = 0$ which are sums of plane wave (away from the black box) and of an outgoing wave.

In the notation of §4.1 let $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n, [0,1])$ be equal to 1 near $B(0,R_0)$. Let

$$R(\lambda) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$$

be the extension of $(P-\lambda^2)^{-1}$ from $\operatorname{Im}\lambda > 0$, given in Theorem 4.4.

For $\lambda \in \mathbb{R}\setminus\{0\}$ we then define *distorted plane waves*

$$\begin{aligned} e(\lambda,\omega) &= (1-\chi)e_0(\lambda,\omega) + w, \\ w &:= R(\lambda)\left([-\Delta,\chi]e^{-i\lambda\langle\bullet,\omega\rangle}\right) \in \mathcal{D}^\infty_{\mathrm{loc}}. \end{aligned} \tag{4.4.1}$$

Note that in principle e could have poles in λ at places where $R(\lambda)$ has real poles. As we will see that cannot happen. The definition (4.4.1) should be compared to the definition (3.8.1) in the case of potential scattering.

The regularity of w comes from (4.1.6):

$$\begin{aligned} (P+i)^N R(\lambda)\left([\Delta,\chi]e^{-i\lambda\langle\bullet,\omega\rangle}\right) &= R(\lambda)\left((-\Delta+i)^N[\Delta,\chi]e^{-i\lambda\langle\bullet,\omega\rangle}\right) \\ &\in R(\lambda)(\mathcal{H}_{\mathrm{comp}}). \end{aligned}$$

For $\lambda \in \mathbb{R}\setminus(\{0\}\cup\operatorname{Res}(P))$ (and, as we will see for $\lambda\in\mathbb{R}\setminus\{0\}$) we have

$$(P-\lambda^2)e(\lambda,\omega) = 0, \quad e(\lambda,\omega) - (1-\chi)e_0(\lambda,\omega) \ \text{ is outgoing.} \tag{4.4.2}$$

Here the meaning of outgoing is the same as in definition (3.32) and Theorem 3.37 modified to the black box setting.

DEFINITION 4.16 (**Outgoing solutions**). *Suppose P is a black box Hamiltonian in the sense of Definition (4.1). For $\lambda \in \mathbb{R}\setminus\{0\}$ and $f \in \mathcal{H}_{\mathrm{comp}}$, a solution to*

$$(P - \lambda^2)u = f, \tag{4.4.3}$$

is called outgoing if and only if there exists $g \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $R > R_0$ such that

$$u|_{\mathbb{R}^n\setminus B(0,R)} = (R_0(\lambda)g)\,|_{\mathbb{R}^n\setminus B(0,R)},$$

where $R_0(\lambda)$ is the free outgoing resolvent given in 3.1.

A solution u to (4.4.3) is called incoming *if there exist $g_1 \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $R > 0$ such that,*

$$u|_{\mathbb{R}^n\setminus B(0,R)} = (R_0(-\lambda)g_1)\,|_{\mathbb{R}^n\setminus B(0,R)}.$$

INTERPRETATION. From the asymptotics of the free resolvent given in (3.1.20) if u is outgoing then

$$u(x) = \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}}\left(h\left(\frac{x}{|x|}\right) + \mathcal{O}\left(\frac{1}{|x|}\right)\right), \tag{4.4.4}$$

with a full asymptotic expansion. Theorem 3.37 is easily adapted to the black box case and it gives equivalent conditions for being an outgoing solution.

The next result is the black box version of the Rellich uniqueness theorem. The proof is an adaptation of the proof of Theorem 3.35 (and that proof can be read independently of the rest of §3.6 – see Exercise 4.4).

THEOREM 4.17 (**Rellich's uniqueness theorem**). *Suppose that P is a black box Hamiltonian in the sense of Definition 4.1. Suppose that $\lambda \in \mathbb{R}\setminus\{0\}$ and that $u \in \mathcal{D}_{\mathrm{loc}}$ satisfies*

$$(P - \lambda^2)u = 0, \quad \lim_{R\to\infty}\int_{\partial B(0,R)} |(\partial_r - i\lambda)u|^2 dS = 0. \tag{4.4.5}$$

where the last integral makes sense for $R > R_0$ as $u|_{\mathbb{R}^n\setminus B(0,R_0)} \in H^2_{\mathrm{loc}}(\mathbb{R}^n \setminus B(0,R_0))$.

Then

$$u|_{\mathbb{R}^n\setminus B(0,R_0)} \equiv 0 \tag{4.4.6}$$

We conclude from Theorem 4.17 that outgoing solutions to the homogeneous equation have to compactly supported and that the resonances on $\mathbb{R}\setminus\{0\}$ are given by embedded eigenvalues.

We say that $E > 0$ is an embedded eigenvalue of P if

$$(P - E)v = 0, \quad v \in \mathcal{D}, \quad v \neq 0. \tag{4.4.7}$$

The multiplicity of that eigenvalue is the dimension of the spaces of solutions of (4.4.7).

THEOREM 4.18 (Outgoing solutions at positive energies). *Suppose that P is a black box Hamiltonian and $\lambda \in \mathbb{R} \setminus \{0\}$. We have:*

(i) if u satisfies (4.4.3) with $f = 0$ and is outgoing then $u_{\mathbb{R}^n \setminus B(0,R_0)} \equiv 0$;

(ii) if λ is a pole of $R(\lambda)$ then λ^2 is an embedded eigenvalue of P and $m_R(\lambda)$ is the multiplicity of that eigenvalue;

(iii) the distorted plane waves defined in (4.4.1) are defined for all $\lambda \in \mathbb{R} \setminus \{0\}$ and the the map

$$\mathbb{R} \setminus \{0\} \times \mathbb{S}^{n-1} \ni (\lambda, \omega) \mapsto e(\lambda, \omega) \in \mathcal{D}^\infty_{\mathrm{loc}}$$

is real analytic.

Proof. 1. Part (i) is an immediate consequence of Theorem 4.17: if $u = R_0(\lambda) f$ then (4.4.4) holds and the condition (4.4.5) is satisfied.

2. Self-adjointness of P shows that

$$R(\zeta) = \mathcal{O}(1/\operatorname{Im}\zeta)_{\mathcal{H}\to\mathcal{H}}, \quad \text{for } \operatorname{Im}\zeta > 0, \, |\operatorname{Re}\zeta| > c > 0.$$

Hence the pole at λ must be simple, that is $(P - \lambda^2)\Pi_\lambda = 0$. Theorem 4.9 shows that all resonant states are outgoing and hence compactly supported.

More precisely, for $\chi_0 \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 near $B(0, R_0)$ and $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ equal to 1 on $\operatorname{supp}\chi_0$ we use (4.2.31) to write

$$(1 - \chi)\Pi_\lambda \chi_0 = R_0(\lambda)[P, \chi]\Pi_\lambda \chi_0.$$

This means that every element of $\Pi_\lambda(\mathcal{H}_{\mathrm{comp}})$ is outgoing and hence by part (i) compactly supported. We conclude that

$$\Pi_\lambda(\mathcal{H}_{\mathrm{comp}}) \subset \mathcal{D},$$

and each element is an eigenvector of P. The rank of Π_λ is the the dimension of the eigenspace.

3. Part (*iii*) of the theorem follows from part (ii) and (4.4.1): the singular part $R(\lambda)$ near a pole $\lambda_0 \in \mathbb{R} \setminus \{0\}$ is the projection onto an eigenspace and all eigenvectors vanish in $\mathbb{R}^n \setminus B(0, R_0)$. Hence, if

$$R(\lambda) = \frac{\Pi_{\lambda_0}}{\lambda^2 - \lambda_0^2} + A(\lambda)$$

where $A(\lambda)$ is holomorphic near λ_0. It follows that

$$R(\lambda)\left([-\Delta, \chi]e^{-\lambda\langle \bullet, \omega\rangle}\right) = A(\lambda)\left([-\Delta, \chi]e^{-\lambda\langle \bullet, \omega\rangle}\right)$$

is well behaved in λ. □

EXAMPLES 1. **Obstacle scattering.** Suppose P is the Dirichlet (or Neumann) realization of $-\Delta$ on a connected set $\mathbb{R}^n \setminus \mathcal{O}$, where $\mathcal{O}$ is a bounded open set with a smooth boundary. Theorem 4.18 shows that there are no resonances in $\mathbb{R} \setminus \{0\}$: unique continuation for $-\Delta - \lambda^2$ (see Lemma 3.34 for a proof which can be adapted to this situation) shows that any resonant state would have to vanish on $\mathbb{R}^n \setminus \mathcal{O}$. Unlike in potential scattering (see §3.3) there is also no resonance at 0 but that requires another argument:

THEOREM 4.19 (No zero resonance in obstacle scattering). *Suppose that $P = -\Delta_g$ is the Dirichlet or Neumann Laplacian for a metric g on a connected set $\mathbb{R}^n \setminus \mathcal{O}$, n odd, where $\mathcal{O}$ is bounded and has a smooth boundary. We assume that $g_{ij} - \delta_{ij} \in C^\infty_{\rm c}(\mathbb{R}^n \setminus \mathcal{O})$, that is P is a compact metric perturbation of the Euclidean Laplacian.*

Then the meromorphic extension of

$$R(\lambda) = (P - \lambda^2)^{-1} : L^2_{\rm comp}(\mathbb{R}^n \setminus \mathcal{O}) \to H^2_{\rm loc}(\mathbb{R}^n \setminus \mathcal{O})$$

is holomorphic near 0. *In other words,* 0 *is not a resonance in obstacle scattering.*

Proof. 1. Let $R(\lambda) := (-\Delta_g - \lambda^2)^{-1}$ where $-\Delta_g$ is the Dirichlet or Neumann realization of the Laplacian on $\mathbb{R}^n \setminus \mathcal{O}$. We will consider the Neumann case, the other one being similar.

Since

$$\|R(\lambda)\|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, [0, \infty))}, \quad \operatorname{Im} \lambda > 0,$$

we see that the pole 0 can have at most order 2:

$$R(\lambda) = \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + A(\lambda),$$

where A_j are finite rank operators,

$$A_j : L^2_{\rm comp}(\mathbb{R}^n \setminus \mathcal{O}) \to (H^2_{\rm loc} \cap H^1_{0,{\rm loc}})(\mathbb{R}^n \setminus \mathcal{O}), \quad -\Delta A_j = 0,$$

and

$$A(\lambda) : L^2_{\rm comp}(\mathbb{R}^n \setminus \mathcal{O}) \to (H^2_{\rm loc} \cap H^1_{0,{\rm loc}})(\mathbb{R}^n \setminus \mathcal{O})$$

is holomorphic near 0.

2. We now recall (4.2.30),

$$(1 - \chi) R(\lambda) \chi_0 = R_0(\lambda)[\Delta, \chi] R(\lambda) \chi,$$

where $\chi_0, \chi \in C_c^\infty(\mathbb{R}^n)$, $\chi_0 = 1$ near $\mathcal{O}$ and $\chi = 1$ on $\operatorname{supp}\chi$. Hence comparing the singular parts on each side we obtain

$$\begin{aligned}(1-\chi)A_2\chi_0 &= R_0(0)[\Delta,\chi]A_2\chi_0\\ (1-\chi)A_1\chi_0 &= R_0(0)[\Delta,\chi]A_1\chi_0 + \partial_\lambda R_0(0)[\Delta,\chi]A_2\chi_0.\end{aligned} \tag{4.4.8}$$

We note that if $A_j \neq 0$ then $[\Delta, \chi]A_j\chi_0 \neq 0$ by the unique continuation property of $-\Delta$ (see Lemma 3.34).

We conclude that if 0 is a pole of $R(\lambda)$ then there exists $u \in H^2_{\mathrm{loc}}$, $u \neq 0$, (obtained using the first equation in (4.4.8) if $A_2 \neq 0$, and the second one if $A_2 = 0$) such that

$$\begin{gathered}-\Delta_g u = 0, \quad \partial_\nu u|_{\partial\mathcal{O}} = 0,\\ u(x) = \mathcal{O}(r^{2-n}), \quad \partial_r u(x) = \mathcal{O}(r^{1-n}), \quad r := |x| \to \infty,\end{gathered} \tag{4.4.9}$$

where $\partial_\nu u$ is the outward normal derivative of u for $\mathcal{O}$. The behaviour as $r \to \infty$ comes from the asymptotics of $R_0(0)$ – see Theorem 3.3.

3. We now apply the divergence theorem to $\bar{u}u$: writing $B_R = B(0, R)$,

$$\begin{aligned}\int_{B_R\setminus\mathcal{O}} |\nabla u|^2 dx &= -\int_{B_R\setminus\mathcal{O}} \Delta u\bar{u} - \int_{\partial\mathcal{O}} \partial_\nu u\bar{u} dS + \int_{\partial B_R} \partial_r u\bar{u} dS\\ &= \mathcal{O}(R^{3-2n}) \int_{\partial B_R} dS = \mathcal{O}(R^{2-n}) \to 0, \quad R \to \infty.\end{aligned}$$

Since $n \geq 3$, $\nabla u \equiv 0$ and as $u \to 0$, $r \to \infty$, $u \equiv 0$. (In the case of the Dirichlet boundary condition we could simply invoke the maximum principle.) $\square$

2. **Scattering on finite volume surfaces.** This is the case of scattering on X given by (4.1.22). Then $n = 1$ and outside of the black box $P = -\partial_s^2$ on $L^2([a, \infty))$ – see (4.1.13). It is clear that for $\lambda^2 > 0$ any solution to $(-\partial_s^2 - \lambda^2)a_0(s) = 0$ (we use the notation preceding (4.1.14)), $a_0 \in L^2([a, \infty))$ has to be identically 0. Hence, in agreement with Theorem 4.18, $\mathbb{1}_{[a,\infty)}\, u = 0$.

The next result is a an adaptation of Theorem 3.47 to the black box setting. The proof is left as Exercise 4.5.

For simplicity we make the reality assumption (4.4.11). To formulate it we assume that there exists an involution of $\mathcal{H}$ (see (4.1.1)), $u \mapsto \bar{u}$ such that

$$\overline{zu} = \bar{z}\bar{u}, \quad (\bar{u})|_{\mathbb{R}^n\setminus B(0,R_0)} = \overline{u|_{\mathbb{R}^n\setminus B(0,R_0)}}, \quad \langle \bar{u}, \bar{v}\rangle_{\mathcal{H}} = \langle v, u\rangle_{\mathcal{H}}. \tag{4.4.10}$$

THEOREM 4.20 (**Stone's formula for black box Hamiltonians**). *Suppose that P is a black a box Hamiltonian satisfying*

$$P(\bar{u}) = \overline{Pu}, \tag{4.4.11}$$

where $u \mapsto \bar{u}$ is an involution satisfying (4.4.10).

For $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{S}^{n-1}$ define $e(\lambda, \omega)$ by (4.4.1) (see (iii) of Theorem 4.18). Then $\overline{e(\lambda, \omega)} = e(-\lambda, \omega)$, and for $f \in \mathcal{H}_{\mathrm{comp}}$,

$$(R(\lambda) - R(-\lambda))f = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} e(\lambda, \omega) \langle f, e(\lambda, \omega) \rangle d\omega. \tag{4.4.12}$$

The spectral measure of P corresponding the continuous spectrum is given by

$$\langle dE_\lambda f, g \rangle = \frac{\lambda^{n-1}}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} \langle e(\lambda, \omega), g \rangle \overline{\langle e(\lambda, \omega), f \rangle} d\omega, \tag{4.4.13}$$

and

$$P = \sum_{k=K_-}^{K_+} E_k u_j \langle \bullet, u_j \rangle + \int_0^\infty \lambda^2 dE_\lambda, \quad I = \sum_{k=1}^{K} u_k \langle \bullet, u_k \rangle + \int_0^\infty dE_\lambda,$$

where u_k's are normalized eigenfunctions of P corresponding to eigenvalues E_k, $E_k \leq E_{k+1}$, where $K_\pm$ can take values $\pm\infty$.

4.4.2. The Fermi Golden Rule. We now present a result about perturbation of embedded eigenvalues in the black box setting. Our principal example is that of scattering on finite volume surfaces (4.1.22) which is a black box perturbation of $-\partial_s^2$ on a a half-line. Other examples can be constructed using hypoelliptic operators, see also Exercise 4.1.

We consider a smooth family of black box Hamiltonians acting on a fixed Hilbert space $\mathcal{H}$ and self-adjoint with the same domain $\mathcal{D}$:

$$\begin{gathered} P(s) \in C^\infty((-s_0, s_0); \mathcal{L}(\mathcal{D}, \mathcal{H})), \\ P(s)^* = P(s), \quad P(s)(\bar{u}) = \overline{(P(s)u)}. \end{gathered} \tag{4.4.14}$$

Here $u \mapsto \bar{u}$ is an involution satisfying (4.4.10). We will denote

$$P := P(0), \quad \dot{P} := \partial_s P(s)|_{s=0} : \mathcal{D} \to \mathcal{H}_{R_0}.$$

The fact that the image of the derivative is in $\mathcal{H}_{R_0}$ follows from (4.1.5).

We start with a lemma which will be proved in §4.5.5:

LEMMA 4.21 (Smoothness of simple resonances). *Suppose that $P(s)$ is a family of black box Hamiltonians (4.4.14) and that $\lambda \in \mathbb{R} \setminus \{0\}$ is a simple eigenvalue of $P = P(0)$. Let u be the eigenstate corresponding to λ. Then there exist $s_1 > 0$, $\varepsilon_1 > 0$ and*

$$\begin{gathered} u(s) \in C^\infty((-s_1, s_1); \mathcal{D}_{\mathrm{loc}}), \quad \lambda(s) \in C^\infty((-s_1, s_1); \mathbb{C}), \\ u(0) = u, \quad \lambda(0) = \lambda, \end{gathered} \tag{4.4.15}$$

such that $\lambda(s)$, is the unique resonance of $P(s)$ in $D(\lambda, \varepsilon_1)$ and $u(s)$ is a corresponding resonance state.

REMARK. The statement remains true for resonances $\lambda \neq 0$ – see the proof in §4.5.5 and [**St94**] for a yet more general version. One can also study the perturbation of the zero resonance or zero eigenvalue but that involves more analysis – see §3.3.

We now present a condition which guarantees dissolution of an embedded eigenvalue to a resonance under a perturbation:

THEOREM 4.22 (The Fermi Golden Rule for embedded eigenvalues). *Suppose that $\lambda^2 > 0$ is an embedded eigenvalue of $P = P(0)$ where $s \mapsto P(s)$ satisfies (4.4.14) and $(P - \lambda^2)u = 0$, $\|u\|_{\mathcal{H}} = 1$.*

Then in the notation of (4.4.15),

$$\operatorname{Im} \ddot{\lambda} = -\frac{\lambda^{n-3}}{4(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} |\langle \dot{P}u, e(\lambda,\omega)\rangle|^2 d\omega, \tag{4.4.16}$$

where $e(\lambda,\omega)$ is the distorted plane wave defined in (4.4.1).

Proof. 1. Let $\mathcal{H}_R := \mathcal{H}_{R_0} \oplus L^2(B(0,R) \setminus B(0,R_0))$ for some $R > R_0$. Let $z(s) = \lambda(s)^2$ and $u(s)$ be given by Lemma 4.21. Then

$$\operatorname{Im} z(s)\|u(s)\|^2_{\mathcal{H}_R} = -\operatorname{Im} \int_{\partial B(0,R)} \overline{u(s)}\partial_r u(s) dS. \tag{4.4.17}$$

We have already seen this in (2.8.17) for one dimensional problems. In the black box case, choose $\chi \in C^\infty_{\rm c}(B(0,R))$ such that $\chi = 1$ near $B(0,R_0)$. Dropping the dependence on s and using self-adjointness of P we see that for any $v \in \mathcal{D}_{\rm loc}$,

$$\begin{aligned}
\operatorname{Im}\langle (P-z)v, v|_{B(0,R)}\rangle &= \operatorname{Im}\langle Pv, v|_{B(0,R)}\rangle - \operatorname{Im} z\|v\|^2_{\mathcal{H}_R} \\
&= \operatorname{Im}\langle P\chi v, \chi v\rangle + \operatorname{Im}\left(\langle P\chi v, (1-\chi)v\rangle + \langle P(1-\chi)v, \chi v\rangle\right) \\
&\quad + \operatorname{Im}\langle P(1-\chi)v, (1-\chi)v|_{B(0,R)}\rangle - \operatorname{Im} z\|v\|^2_{\mathcal{H}_R} \\
&= \operatorname{Im}\langle P(1-\chi)v, (1-\chi)v|_{B(0,R)}\rangle - \operatorname{Im} z\|v\|^2_{\mathcal{H}_R} \\
&= -\operatorname{Im}\int_{B(0,R)} \Delta((1-\chi)v)(1-\chi)\bar{v}dx - \operatorname{Im} z\|v\|^2_{\mathcal{H}_R} \\
&= -\operatorname{Im}\int_{\partial B(0,R)} \partial_r v\bar{v} dS - \operatorname{Im} z\|v\|^2_{\mathcal{H}_R}.
\end{aligned} \tag{4.4.18}$$

Putting $v = u$ gives (4.4.17) since the the left hand side vanishes.

2. Since $\operatorname{Im} z(s) \leq 0$ it follows that $\operatorname{Im} \dot{z} = 0$ ($\dot{z} := \partial_s z(s)|_{s=0}$ and $\operatorname{Im} z(0) = 0$). Also, from Theorem 4.18,

$$u(0)|_{\mathbb{R}^n\setminus B(0,R_0)} = 0 \ \text{ and } \|u(0)\|_{\mathcal{H}_R} = \|u(0)\|_{\mathcal{H}} = 1. \tag{4.4.19}$$

We now differentiate (4.4.17) twice with respect to s. Using $\operatorname{Im} z(0) = \operatorname{Im} \dot{z}(0) = 0$, $u(0)|_{\mathbb{R}^n\setminus B(0,R_0)} = 0$ and (4.4.18) with $v = \dot{u}$, we obtain (at

$s = 0$),

$$
\begin{aligned}
\operatorname{Im} \ddot z &= \operatorname{Im} z\, \partial_s^2 \|u(s)\|^2_{\mathcal{H}_R}|_{s=0} + 2 \operatorname{Im} \dot z\, \partial_s \|u(s)\|^2_{\mathcal{H}_R}|_{s=0} \\
&\quad - \operatorname{Im} \int_{\partial B(0,R)} \left(2\partial_r \dot u \bar{\dot u} + \partial_r \ddot u \bar u + \partial_r u \bar{\ddot u} \right) dS \\
&= -2 \operatorname{Im} \int_{\partial B(0,R)} \partial_r \dot u \bar{\dot u} dS \\
&= 2 \operatorname{Im} \langle (P - z)\dot u, \dot u|_{B(0,R)} \rangle .
\end{aligned} \tag{4.4.20}
$$

Also, differentiating $(P(s) - z(s))u(s) = 0$ gives

$$
(P - z)\dot u = \dot z u - \dot P u. \tag{4.4.21}
$$

Since $u \in \mathcal{H}_{R_0}$ and $\dot P : \mathcal{D} \to \mathcal{H}_{R_0}$ we see that the right hand side in (4.4.21) is in $\mathcal{H}_{R_0}$. In particular, we can drop the restriction to $B(0, R)$ on the right hand side of (4.4.20).

3. We now claim that $\dot u$ is outgoing in the sense of Definition 4.16. To see that we observe from (4.2.31) (applied with $M_{\lambda(s)} = 1$ as our resonances are simple) that

$$
(1 - \chi)u(s) = R_0(\lambda)[P, \chi]u(s), \quad z = \lambda^2,
$$

where χ as in (4.4.18). From (4.4.19) we see that $[P, \chi]u(0) = 0$ and $[\dot P, \chi]u(0) = 0$. Hence

$$
(1 - \chi)\dot u = R_0(\lambda)[P, \chi]\dot u.
$$

To find an expression for $\dot u$ we first observe that $\dot z u - \dot P u = 0$ is orthogonal to u. In fact, if χ is as in (4.4.18) then $\chi \dot u \in \mathcal{D}$ and $\chi \equiv 1$ near $\operatorname{supp} u$. Then,

$$
\begin{aligned}
\langle \dot z u - \dot P u, u \rangle &= \langle (P - z)\dot u, u \rangle \\
&= \langle (P - z)(\chi \dot u), u \rangle \\
&= \langle \chi \dot u, (P - z)u \rangle = 0.
\end{aligned} \tag{4.4.22}
$$

In view of this orthogonality property we define $v := R(\lambda)(\dot z u - \dot P u)$ which is another outgoing solution to (4.4.21). Theorem 4.18 shows that $v - \dot u$ must be compactly supported and from the simplicity of the eigenvalue it follows that $v - \dot u = \alpha u$, $\alpha \in \mathbb{C}$. Because of(4.4.22) this means that we can replace $\dot u$ with v in (4.4.20) which gives

$$
\operatorname{Im} \ddot z = 2 \operatorname{Im} \langle \dot z u - \dot P u, R(\lambda)(\dot z u - \dot P u) \rangle, \quad z = \lambda^2.
$$

Noting that

$$
(R(\lambda) - R(-\lambda))u = 0, \quad R(\lambda)^* = R(-\lambda),
$$

we obtain

$$
\operatorname{Im} \langle \dot z u, R(\lambda) \dot z u \rangle = |\dot z|^2 \langle u, (R(\lambda) - R(-\lambda))u \rangle / 2i = 0.
$$

Since $\operatorname{Im} \dot z = 0$ we also have

$$\begin{aligned}\operatorname{Im}(\langle \dot z u, R(\lambda)\dot P u\rangle + \langle \dot P u, \dot z R(\lambda) u\rangle) &= \dot z \operatorname{Im}(-\langle R(\lambda)\dot P u, u\rangle + \langle \dot P u, R(\lambda) u\rangle) \\ &= \dot z \operatorname{Im}\langle \dot P u, (R(\lambda) - R(-\lambda) u\rangle \\ &= 0.\end{aligned}$$

Hence (4.4.12) gives

$$\begin{aligned}2\lambda \operatorname{Im} \ddot\lambda &= 2 \operatorname{Im}\langle \dot P u, R(\lambda)\dot P u\rangle \\ &= \frac{1}{i}\Big(\langle \dot P u, R(\lambda)\dot P u\rangle - \langle \dot P u, R(-\lambda)\dot P u\rangle\Big) \\ &= -\frac{1}{2}\frac{\lambda^{n-2}}{(2\pi)^{n-1}}\int_{\mathbb{S}^{n-1}} |\langle \dot P u, e(\lambda,\omega)\rangle|^2 d\omega.\end{aligned}$$

This proves (4.4.16). $\square$

REMARK. Formally (4.4.20) follows from differentiating $0 = \langle (P(s) - z(s))u(s), u(s)\rangle$ but we have to be careful as for $s \neq 0$, $u(s)$ will not, typically, be in $\mathcal{H}$. The use of (4.4.17) (and of its derivation (4.4.18) based on Green's formula) remedies this problem. The complex scaling method described in §4.5 allows a more direct argument which can be adapted to the case of long range perturbations.

EXAMPLE. Suppose that (X, g) is the manifold given by (4.1.22). As explained before that definition, we put the scattering problem for $-\Delta_g$ in the black box formalism with $n = 1$ (and $\mathbb{R}^n$ replaced by a finite union of half lines). We then define the analogues of $e(\lambda, \omega)$ (the sphere at infinity, $\mathbb{S}^{n-1}$, is replaced by $\mathbb{Z}/N\mathbb{Z}$), $e_j(\lambda, x)$, $j = 1, \cdots, N$,

$$\begin{gathered}(-\Delta_g - \tfrac14 - \lambda^2) e_j(\lambda, x) = 0, \\ \frac{1}{b_\ell}\int_0^{b_\ell} e_j|_{X_\ell} d\theta = e^{r/2}\left(e^{-i\lambda r}\delta_{\ell j} + s_{\ell j}(\lambda) e^{i\lambda r}\right).\end{gathered} \tag{4.4.23}$$

Let us consider a conformal change of the metric which results in the family

$$P(t) = e^{\frac12 tf}(-\Delta_g - \tfrac14) e^{\frac12 tf}, \quad f \in C^\infty_{\rm c}(X_0; \mathbb{R}).$$

The conjugation of the Laplacian, $e^{-\frac12 tf}(e^{tf}\Delta_g)e^{\frac12 tf}$, was introduced to fix the Hilbert space on which the operators $P(t)$ are self-adjoint. We then see that the assumption (4.4.14) are satisfied.

Suppose that $E = \lambda^2 + \frac{1}{4} > \frac{1}{4}$ is an embedded eigenvalue of $-\Delta_g$ and u is the corresponding normalized eigenfunction. Then (4.4.16) gives

$$\begin{aligned} \operatorname{Im} \ddot{\lambda} &= -\frac{1}{4\lambda^2} \sum_{\ell=1}^{N} \left| \langle \tfrac{1}{2}(f(\Delta_g + \tfrac{1}{4}) + (\Delta_g + \tfrac{1}{4})f)u, e_\ell(\lambda) \rangle \right|^2 \\ &= -\tfrac{1}{4}\lambda^2 \sum_{\ell=1}^{N} |\langle fu, e_\ell(\lambda) \rangle|^2. \end{aligned} \tag{4.4.24}$$

This can be used to show that for a generic $f \in C_c^\infty(\Omega; \mathbb{R})$, where $\emptyset \neq \Omega \subset X_0$ is an open set, there are no embedded eigenvalues. First one shows that for a generic f all the embedded eigenvalues are simple and that follows from showing generic simplicity of eigenvalues for the reference operator, that is for the pseudo-Laplacian in Example 2 in §4.3. Then (4.4.24) can be used to show that any finite number of eigenvalues become resonances under a perturbation – see [**CdV83**] for details (a less direct argument than (4.4.24) is used there). See also the proof Theorem 2.25 for an example of the scheme for proving generic results.

4.4.3. Definition of the scattering matrix. We now use the plane waves (4.4.1) to define the scattering matrix and to obtain its representation. This is very similar to what has been done in §3.7 and we leave the proofs as exercises for the reader. The first result is an adaptation of the boundary pairing result of Theorem 3.39. The proof is left as an exercise.

THEOREM 4.23 (**Boundary pairing for black box Hamiltonians).** *Let P be a black box Hamiltonian in the sense of Definition 4.1.*

Suppose that $u_\ell \in \mathcal{D}_{\mathrm{loc}}$, $\ell = 1, 2$ satisfy

$$(P - \lambda^2)u_\ell = F_\ell \in \mathcal{H}_{\mathrm{comp}}, \quad \lambda \in \mathbb{R} \setminus \{0\},$$

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}\, u_\ell(r\theta) = r^{-\frac{n-1}{2}} \left(e^{i\lambda r} f_\ell(\theta) + e^{-i\lambda r} g_\ell(\theta) \right) + \mathcal{O}(r^{-\frac{n+1}{2}}), \quad \theta \in \mathbb{S}^{n-1},$$

with $f_\ell, g_\ell \in C^\infty(\mathbb{S}^{n-1})$, and the expansion valid also for derivatives with respect to ∂_r. Then

$$2i\lambda \int_{\mathbb{S}^{n-1}} \left(g_1 \bar{g}_2 - f_1 \bar{f}_2 \right) d\omega = \langle F_1, u_2 \rangle_{\mathcal{H}} - \langle u_1, F_2 \rangle_{\mathcal{H}}. \tag{4.4.25}$$

The interesting case comes from considering $F_\ell \equiv 0$ in which case the incoming or outgoing data can be prescribed. The proof again follows directly from the proof of Theorem 3.42.

THEOREM 4.24 (**Prescribing incoming data in black box scattering).** *Let P be a black box Hamiltonian in the sense of Definition 4.1. Then* *for* $\lambda \in \mathbb{R} \setminus \{0\}$ *and any* $g \in C^\infty(\mathbb{S}^{n-1})$ *there exist* unique $f \in C^\infty(\mathbb{S}^{n-1})$ *and* $v \in \mathcal{D}_{\mathrm{loc}}$ *such that*

$$(4.4.26)\qquad \begin{gathered} (P-\lambda^2)v = 0, \\ \mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\, v(r\theta) = r^{-\frac{n-1}{2}}\left(e^{i\lambda r} f(\theta) + e^{-i\lambda r} g(\theta)\right) + \mathcal{O}(r^{-\frac{n+1}{2}}). \end{gathered}$$

Using this two theorems we can now defined the absolute and relative scattering matrices for black box scattering – see Definition 3.40 for more motivation.

DEFINITION 4.25 (**Scattering matrix**). *In the notation of Theorem 4.24, the map*

$$(4.4.27)\qquad S_{\mathrm{abs}}(\lambda) : C^\infty(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1}), \quad S_{\mathrm{abs}}(\lambda) : g \longmapsto f,$$

is called the absolute scattering matrix. *By Theorem 4.23 it extends to a unitary transformation*

$$S_{\mathrm{abs}}(\lambda) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}).$$

The scattering matrix *is defined as*

$$(4.4.28)\qquad S(\lambda) = i^{n-1} S_{\mathrm{abs}}(\lambda) J, \quad Jg(\theta) := g(-\theta).$$

Representation of the Schwartz kernel of $S(\lambda)$ in terms of the resolvent is again the same as in §3.7:

THEOREM 4.26 (**Description of the scattering matrix**). *Let P be a black box Hamiltonian. For $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ define*

$$E_\rho : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1}), \quad E_\rho(\lambda)(x,\omega) := \rho(x) e^{-i\lambda\langle x,\omega\rangle}.$$

Choose $\chi_i \in C_{\mathrm{c}}^\infty(\mathbb{R}^n;[0,1])$, $i = 1,2,3$, such that for some $R_1 > R_0$,

$$\chi_i|_{B(0,R_1)} = 1, \quad \chi_{i+1}|_{\mathrm{supp}\,\chi_i} = 1, \quad i = 1,2.$$

Then the scattering matrix is given by

$$(4.4.29)\qquad S(\lambda) = I + a_n \lambda^{n-2} E_{\chi_3}(\lambda)[\Delta,\chi_1] R(\lambda) [\Delta,\chi_2] E_{\chi_3}(\bar\lambda)^*,$$

where $R(\lambda)$ is the extension of $(P-\lambda^2)^{-1}$ and $a_n := (2\pi)^{-n+1}/2i$.

EXAMPLES. 1. **Obstacle scattering.** Suppose $P = -\Delta_{\mathcal{O}}$, the Dirichlet Laplacian on a connected set $\mathbb{R}^n \setminus \mathcal{O}$ where $\mathcal{O}$ is bounded and $\partial\mathcal{O}$ is smooth. Then comparison of (4.4.29) with (4.4.1) (used with $\chi = \chi_2$ and noting that $(1-\chi_2)[\Delta,\chi_1] \equiv 0$ shows that

$$(4.4.30)\qquad \begin{gathered} S_{\mathcal{O}}(\lambda) = I + A_{\mathcal{O}}(\lambda), \\ A_{\mathcal{O}}(\lambda,\omega,\theta) = a_n \lambda^{n-2} \int_{\mathbb{R}^n} e^{-i\lambda\langle x,\omega\rangle} [-\Delta,\chi_1] e(\lambda,-\theta,x)\,dx, \end{gathered}$$

where $e(\lambda, \theta, x)$ is the unique function satisfying

$$(-\Delta - \lambda^2)e(\lambda, \theta, x) = 0, \quad e(\lambda, \theta, \bullet)|_{\partial\mathcal{O}} = 0,$$
$$e(\lambda, \theta, x) = e^{-i\lambda\langle x, \theta\rangle} + \frac{e^{i\lambda|x|}}{|x|^{(n-1)/2}}\left(h(\tfrac{x}{|x|}) + \mathcal{O}(\tfrac{1}{|x|})\right).$$

We now apply Green's formula in (4.4.30) noting that, because of support properties of χ_1, we can change the domain of integration to $\mathbb{R}^n \setminus \mathcal{O}$:

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\mathcal{O}} e^{-i\lambda\langle x,\omega\rangle}[-\Delta, \chi_1]e(\lambda, -\theta, x)dx \\
&\quad = \int_{\mathbb{R}^n\setminus\mathcal{O}} e^{-i\lambda\langle x,\omega\rangle}\left(\chi_1(x)\Delta e(\lambda, -\theta, x) - \Delta(\chi_1(x)e(\lambda, -\theta, x))\right)dx \\
&\quad = \int_{\mathbb{R}^n\setminus\mathcal{O}} \left(-\lambda^2 e^{-i\lambda\langle x,\omega\rangle}\chi_1 e(\lambda, -\theta, x) - e^{-i\lambda\langle x,\omega\rangle}\Delta(\chi_1(x)e(\lambda, -\theta, x)\right)dx \\
&\quad = \int_{\mathbb{R}^n\setminus\mathcal{O}} \left(\Delta e^{-i\lambda\langle x,\omega\rangle}\chi_1 e(\lambda, -\theta, x) - e^{-i\lambda\langle x,\omega\rangle}\Delta(\chi_1(x)e(\lambda, -\theta, x))\right)dx.
\end{aligned}$$

Applying Green's formula gives the following representation of $A_{\mathcal{O}}$ in terms of the normal derivative of the plane wave at the boundary:

$$A_{\mathcal{O}}(\lambda, \omega, \theta) = \frac{i\lambda^{n-2}}{2(2\pi)^{n-1}}\int_{\partial\mathcal{O}} e^{-i\lambda\langle x,\omega\rangle}\partial_\nu e(\lambda, -\theta, x)dS(x).$$

2. **Surfaces with cusps.** We now consider the case of scattering on surfaces (4.1.22). We will also compute the scattering explicitly in the case of scattering on the *modular surface* $X = SL_2(\mathbb{Z})\backslash\mathbb{H}^2$.

In the case of surface with N cusps the scattering matrix appears in (4.4.23). If we compare that expansion with the expansion in Theorem 4.24 we see that we need to replace $\mathbb{S}^{n-1}$ with the discrete set of points, $\mathbb{Z}/N\mathbb{Z}$: the boundaries of the cusps at infinity. Then the scattering matrix acts on $L^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ is in fact a *matrix*:

$$S(\lambda) : \mathbb{C}^N \to \mathbb{C}^N, \quad S(\lambda)u_j = \sum_{\ell=1}^{N} s_{j\ell}(\lambda)u_\ell, \tag{4.4.31}$$

with $s_{j\ell}(\lambda)$'s given by (4.4.23).

3. **The modular surface.** The discrete group

$$SL_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : a, b, c, d \in \mathbb{Z}, \quad ad - cb = 1\right\}$$

acts on

$$(\mathbb{H}^2, g) := \left(\{(x, y) : x, y \in \mathbb{R}, y > 0\}, \frac{dx^2 + dy^2}{y^2}\right),$$

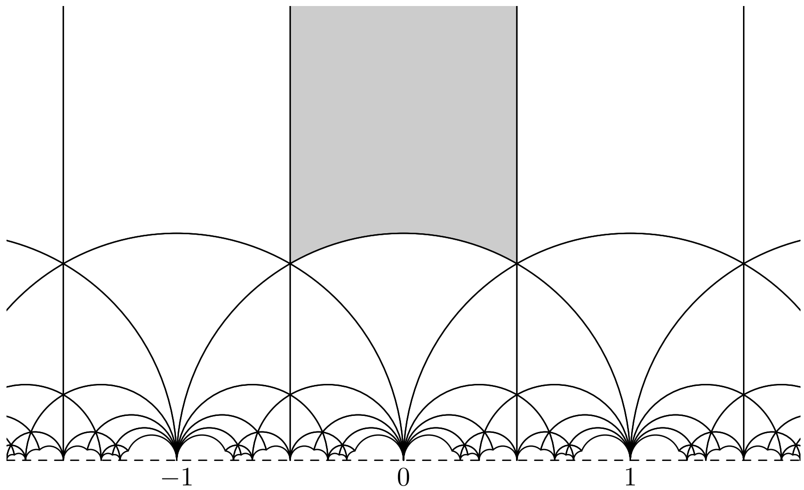


Figure 4.2. The shaded region is the fundamental domain (4.4.33) of $SL_2(\mathbb{Z})$ used in Example 3. Every region determined by circles and lines is a fundamental domain.

by linear fractional transformations

$$x + iy = z \mapsto \frac{az+b}{cz+d}.$$

The action of $SL_2(\mathbb{Z})$ is then generated by two transformations

$$S(z) := -\frac{1}{z}, \quad T(z) := z+1, \tag{4.4.32}$$

and the famous *fundamental domain* of its action is given by

$$\begin{gathered}\{(x,y) : -\tfrac{1}{2} < x < 0, (1-x^2)^{\frac{1}{2}} < y\} \cup \\ \{(x,y) : 0 \le x \le \tfrac{1}{2}, (1-x^2)^{\frac{1}{2}} \le y\},\end{gathered} \tag{4.4.33}$$

see Figure 4.2 and [**Ah78**, §7.2] for an elementary presentation. The surface X is given by $SL_2(\mathbb{Z})\backslash\mathbb{H}^2$. (The compact part X_0 is not smooth as there are two conic singularities but that does not cause trouble in the analysis and we neglect this point.) Functions on X can be identified with functions on $\mathbb{H}^2$ invariant under the action of $SL_2(\mathbb{Z})$. That only needs to be checked for the two generators (4.4.32).

We now want to give a (relatively) explicit construction of the generalized plane waves $e(x,\lambda)$ given by (4.4.1) in general (with no dependence on ω) and (4.4.23) (with $N=1$) for the case of surfaces with cusps. In keeping with traditional notation we will put

$$s = \tfrac{1}{2} - i\lambda, \quad y^s = e^{r/2}e^{-i\lambda r}, \quad y^{1-s} = e^{r/2}e^{i\lambda r}, \quad y = e^r. \tag{4.4.34}$$

We then define, for $c, d \in \mathbb{Z}$,

$$B(z,s) := \sum_{(c,d)\neq(0,0)} \frac{y^s}{|cz+d|^{2s}}, \quad z = x+iy \in \mathbb{H}^2.$$

We first note that the sum converges for $\operatorname{Re} s \gg 1$ and that

$$\frac{y^s}{|cz+d|^{2s}} = \gamma^*(y^s), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $-\Delta_g(y^s) = -y^2\partial_y^2(y^s) = s(s-1)y^s$, it follows that

$$-\Delta_g B(z,s) = s(1-s)B(z,s), \quad \operatorname{Re} s \gg 1.$$

We also see that

$$B(z+1,s) = B(z,s), \quad B(-1/z,s) = B(z,s).$$

In fact, the first identity is obvious and for the second we note that

$$B(-1/z,s) = \sum_{(c,d)\neq 0} \frac{(\operatorname{Im}(-1/z))^s}{|-c/z+d|^{2s}} = \sum_{(c,d)\neq 0} \frac{y^s}{|dz-c|^{2s}} = B(z,s).$$

Hence, $B(z,s)$ defines a function on $X = SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$.

We now look at $B(z,s)$ in

$$X_0 \simeq \{(x,y) : -\tfrac{1}{2} < x \le \tfrac{1}{2}, \quad y > 2\}$$

We write

$$B(z,s) = \sum_{d\neq 0} |d|^{-2s} y^s + \sum_{c\neq 0} \frac{y^s}{|cz+d|^{2s}} =: 2\zeta(2s)y^s + w(x,y,s).$$

As in (4.4.23), we calculate the "scattering" component of w:

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} w(x,y,s)dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{c\neq 0} \frac{y^s}{|cz+d|^{2s}} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{c\neq 0} \sum_{k\in\mathbb{Z}} \sum_{\ell=0}^{|c|-1} \frac{y^s}{((c(x+k)+\ell)^2 + c^2y^2)^s} \\
&= \sum_{c\neq 0} \sum_{\ell=0}^{|c|-1} \int_{\mathbb{R}} \frac{y^s}{c^{2s}((x+\ell)^2+y^2)^s} dx \qquad (4.4.35) \\
&= \sum_{c\neq 0} \frac{1}{|c|^{2s-1}} y^{1-s} \int_{\mathbb{R}} \frac{1}{(1+x^2)^s} dx \\
&= 2\zeta(2s-1) \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)} y^{1-s}.
\end{aligned}$$

Here we used some facts about beta functions: with the substitution $t = x^2/(1+x^2)$, $dx = \frac{1}{2}t^{-1/2}(1-t)^{-3/2}dt$,

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^s}dx = 2\int_0^\infty \frac{1}{(1+x^2)^s}dx = \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{3}{2}+s}dt$$
$$= \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} = \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)},$$

see for instance [**HöI**, (3.4.8),(3.4.9)].

Putting,

$$e(z,s) := \frac{1}{2\zeta(2s)}B(z,s), \quad \operatorname{Re} s \gg 1$$

we have found a function on X such that

$$(-\Delta_g - s(1-s))e(z,s) = 0, \quad \operatorname{Re} s \gg 1,$$
$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e(z,s)|_{X_1}dx = y^s + \frac{\zeta(2s-1)}{\zeta(2s)}\frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-s}. \tag{4.4.36}$$

The second term on the right hand side of the last equality is in $L^2(X_1, d\operatorname{Vol}_g)$, $d\operatorname{Vol}_g = y^{-2}dxdy$ for $\operatorname{Re} s \gg 1$. If $\chi \in C_c^\infty([0,\infty))$ is equal to 1 for $y < 2$ then proceeding as in (4.4.35) we see that

$$e(z,s) - (1-\chi(y))y^s \in L^2(X, d\operatorname{Vol}_g), \quad \operatorname{Re} s \gg 1.$$

But, as in the construction of $e(y,\lambda)$ in (4.4.1) this means that

$$e(z,s) = (1-\chi(y))y^s + (-\Delta_g - s(1-s))^{-1}([-\Delta_g, \chi]y^s).$$

The meromorphic continuation of $(-\Delta_g - s(1-s))^{-1} = (-\Delta_g - \frac{1}{4} - \lambda^2)^{-1}$ shows that $e(z,s)$ is meromorphic in $\mathbb{C}$ with no poles for $\operatorname{Re} s = \frac{1}{2}$, $s \neq \frac{1}{2}$.

Returning to (4.4.23) and (4.4.31) (and keeping in mind the change of convention (4.4.34)) we obtain the scattering matrix for the modular surface – in this case a number:

$$S(\lambda) = \pi^{\frac{1}{2}}\frac{\Gamma(-i\lambda)\zeta(2i\lambda)}{\Gamma(\frac{1}{2}-i\lambda)\zeta(1-2i\lambda)}.$$

The unitarity (modulus one) on the real axis follow from general scattering theory or from the properties of the Riemann zeta function and the Gamma function.

Theorem 4.27 below shows that the resonances of $P = -\Delta_g - \frac{1}{4}$ are given by λ such that

$$\lambda^2 \in \operatorname{Spec}_{\mathrm{pp}}(P) \quad \text{or} \quad \Gamma(\tfrac{1}{2}-i\lambda)\zeta(1-2i\lambda) = 0.$$

The latter condition means that $1 - 2i\lambda$ is a non-trivial zero of the zeta function. Hence, the Riemann hypothesis states that all resonances which do not come from eigenvalues lie on the line

$$\operatorname{Im}\lambda = -\tfrac{1}{4}.$$

4.4.4. Resonance multiplicities. The representation of the scattering matrix given in Theorem 4.26 and the meromorphy of $R(\lambda)$ show that $S(\lambda)$ forms a meromorphic family of operators on $\mathbb{C}$ (we assume here, as elsewhere, that n is odd – otherwise we have to work with the logarithmic plane). The unitarity relation shows that

$$S(\lambda)^{-1} = S(\bar{\lambda})^*, \quad \lambda \in \mathbb{C}.$$

We note that unitarity of $S(\lambda)$ is valid for $\lambda \in \mathbb{R} \setminus \{0\}$ by Theorem 4.23 and remains valid at $\lambda = 0$ by continuity which follows from the fact that $S(\lambda)$ is meromorphic.

Definition 4.25 gives $S_{\mathrm{abs}}(\lambda) = S_{\mathrm{abs}}(-\lambda)^{-1}$ and hence (see Theorem 3.43)

$$S(\lambda)^{-1} = JS(-\lambda)J, \quad Jf(\theta) = f(-\theta), \quad \lambda \in \mathbb{C}.$$

The set of poles is contained in the set of poles of $R(\lambda)$ and the multiplicity is defined using the Gohberg–Sigal theory in (4.4.37) below. The precise relation between the multiplicities of the poles of $S(\lambda)$ and $R(\lambda)$ is given as follows:

THEOREM 4.27 (Equivalence of multiplicities). *Suppose P is a black box Hamiltonian and $S(\lambda)$ is its scattering matrix. Then*

$$\begin{aligned} m_S(\lambda) &= \widetilde{m}_R(\lambda) - \widetilde{m}_R(-\lambda), \\ m_S(\lambda) &:= -\frac{1}{2\pi}\oint_\lambda \operatorname{tr} \partial_\zeta S(\zeta) S(\zeta)^{-1} d\zeta, \\ \widetilde{m}_R(\lambda) &= \operatorname{rank} \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \Pi_\lambda, \end{aligned} \tag{4.4.37}$$

where the integral is over a positively oriented circle enclosing λ and not other pole of $S(\lambda)$ or $S(\lambda)^{-1}$, and Π_λ is given by (4.2.21).

REMARKS. 1. As always high multiplicities cause problems in the analysis. We will prove the theorem under the assumption that $m_R(\lambda) = 1$ when $\lambda \notin \mathbb{R}$. To get the general statement we can use the approach from Theorem 3.14 to perturb resonances or Theorem 4.39 in §4.5. A direct proof using the method of complex scaling can be found in Nedelec [**Ne04**].

2. We note that $\widetilde{m}_R(\lambda) = m_R(\lambda)$ for $\lambda^2 \notin \mathbb{R}$ – any compactly supported resonant states have to be eigenfunctions. (See also Exercise 4.7). Hence

(4.4.37) really means that

$$m_S(\lambda) = \begin{cases} m_R(\lambda) & \operatorname{Im}\lambda < 0 \\ -m_R(\lambda) & \operatorname{Im}\lambda > 0 \end{cases}, \quad \lambda^2 \notin \mathbb{R},$$

and the difference is relevant only on the imaginary axis.

An artificial example in which $\widetilde{m}_R(\lambda) < m_R(\lambda)$, $\lambda \in i(0,\infty)$ is given as follows. Consider the Dirichlet realization of $P = -\Delta + V$ on $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, where we assume that Ω has two connected components, Ω_1, Ω_2, with Ω_1 unbounded, Ω_2 bounded and $\partial\Omega_j$ smooth. We can find $V \in C_c^\infty(\Omega_2)$ such that that $-\Delta + V$ with Dirichlet boundary conditions has an eigenvalue $-t^2$, $t > 0$. Then $0 = \widetilde{m}_R(it) < m_R(it)$.

Proof. 1. Since $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ we have $m_S(\lambda) = 0$ for $\lambda \in \mathbb{R}$. On the other hand for $\lambda^2 > 0$ the only poles of $R(\lambda)$ come from embedded eigenvalues – see part (ii) of Theorem 4.18 and hence $\widetilde{m}_R(\lambda) = \widetilde{m}_R(\lambda) = 0$.

This means that we only need to establish (4.4.37) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

2. We now prove the theorem under the assumption that $m_R(\lambda) = 1$ – see Remark 1 after the statement of the theorem on how to remove that hypothesis.

3. From Theorem C.11 we recall that

$$m_S(\lambda) = N_\lambda(S^{-1}) - N_\lambda(S),$$

where, near λ, with invertible and holomorphic $U(\zeta)$, $V(\zeta)$,

$$\begin{gathered} S(\zeta) = U(\zeta)(P_0 + \sum_{m=1}^{M} (\zeta-\lambda)^{k_m} P_m) V(\zeta), \\ P_m P_\ell = \delta_{km}, \quad \operatorname{rank}(I - P_0) = M, \quad \operatorname{rank} P_m = 1, \ m \neq 0, \\ N_\lambda(S^{-1}) = -\sum_{k_\ell < 0} k_\ell, \quad N_\lambda(S) = \sum_{k_\ell > 0} k_\ell. \end{gathered} \tag{4.4.38}$$

Hence to prove (4.4.37) it is enough to show that

$$N_\lambda(S^{-1}) = \widetilde{m}_R(\lambda). \tag{4.4.39}$$

From the discussion in Step 2, we can assume that $\widetilde{m}_R(\lambda) \leq 1$. From (4.4.29) and (4.4.38) we see that, under the simplicity assumption $\widetilde{m}_R(\lambda) \geq N_\lambda(S^{-1})$.

4. To see that $\widetilde{m}_R(\lambda) = 1$ implies that $S(\zeta)$ has a pole at λ we write

$$\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \Pi_\lambda \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)} \varphi = u|_{\mathbb{R}^n \setminus B(0,R_0)} \langle \varphi, v|_{\mathbb{R}^n \setminus B(0,R_0)} \rangle_{\mathcal{H}}, \quad \varphi \in \mathcal{H}_{\mathrm{comp}},$$

where $(P - \lambda^2)u = 0$, $(P^* - \bar{\lambda}^2)v = 0$. From (4.4.29) we see that

$$S(\zeta) = a_n \zeta^{n-2} \frac{F \otimes G}{\zeta^2 - \lambda^2} + T(\zeta),$$

where T is holomorphic hear λ and

$$F(\omega) = \widehat{f}(\lambda\omega), \qquad f(x) := [\Delta, \chi_1]u(x),$$
$$G(\theta) = \widehat{g}(-\bar{\lambda}\theta), \qquad g(x) := [\Delta, \chi_2]v(x).$$

We also note that

$$(-\Delta - \lambda^2)(1-\chi_1)u = f, \qquad (-\Delta - \bar{\lambda}^2)(1-\chi_2)v = g.$$

5. To prove that $S(\zeta)$ has a pole at $\zeta = \lambda$ we need to show that $F \neq 0$ and $G \neq 0$. We note that $f \in C^\infty_{\rm c}(\mathbb{R}^n)$ and hence $\hat{f}$ is an entire function on $\mathbb{C}^n$. If $F \equiv 0$ then

$$f(e^{i\theta}\eta) \equiv 0 \quad \text{for } \eta \in \Sigma \cap \mathbb{R}^n,$$

where

$$\lambda = e^{i\theta}|\lambda|, \quad \Sigma := \{\eta \cdot \eta = |\lambda|^2\} \subset \mathbb{C}^n, \quad \eta \cdot \eta = \sum_{j=1}^n \eta_j^2, \quad \eta \in \mathbb{C}^n.$$

Since $\Sigma \cap \mathbb{R}^n$ is a totally real submanifold of the complex variety Σ, $f(e^{i\theta}\eta) \equiv 0$ for $\eta \in \Sigma$. This implies that

$$W(\xi) = \frac{f(\xi)}{\xi \cdot \xi - \lambda^2} = \frac{e^{-2i\theta} f(e^{i\theta}\eta)}{\eta \cdot \eta - |\lambda|^2}, \quad \xi = e^{i\theta}\eta, \quad \xi \in \mathbb{C}^n,$$

is an entire function. Paley-Wiener theorem as applied in [**HöI**, Theorem 7.3.2] shows that $W = \hat{w}$, $w \in C^\infty_{\rm c}(\mathbb{R}^n)$. By Theorem 4.9

$$(-\Delta - \lambda^2)((1-\chi_1)u - w) = 0,$$
$$((1-\chi_1)u - w)\,|_{\mathbb{R}^n \setminus B(0,R)} = (R_0(\lambda)g)|_{\mathbb{R}^n \setminus B(0,R)},$$

where $g \in L^2_{\rm comp}(\mathbb{R}^n \setminus B(0,R_0))$. Applying the opposite implication in Theorem 4.9 with $P = -\Delta$ we conlude that $(1-\chi_1)u - w$ would be a resonant state for $-\Delta$. Hence $(1-\chi_1)u = w \in C^\infty_{\rm c}(\mathbb{R}^n)$ which contradicts the fact that u is a nontrivial element of $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}\,\Pi_\lambda(\mathcal{H}_{\rm comp})$. We conclude that $F \neq 0$.

A similar argument shows that $G \neq 0$ and hence $\widetilde{m}_R(\lambda) = 1$ implies that $N_\lambda(S^{-1}) = 1$, completing the proof. □

4.5. COMPLEX SCALING

So far resonances for black box perturbations were defined as poles of the meromorphic continuation of the resolvent. The structure of that continuation near the poles given in Theorem 4.7 shows that it behaves like a resolvent of a non-self-adjoint operator. The method of complex scaling allows an equivalent definition: instead continuing the resolvent, the Hamiltonian P is deformed to a non-self-adjoint Hamiltonian P_θ and the resonances λ^2

with $\arg\lambda^2 > -2\theta$ are the eigenvalues of P_θ: for $-2\theta < \arg z < 2\pi - 2\theta$, $P_\theta - z$ is a Fredholm operator on suitable spaces. The advantage of this method is both practical and theoretical. It allows numerical calculation of resonances by discretizing P_θ and it provides access to method from spectral theory of non-self-adjoint operators. That will become particulary apparent in Part 3.

In the simple but instructive setting of dimension one the method was described in Section 2.7.

4.5.1. The complex scaled operator. For $0 \leq \theta < \pi$, let $\Gamma_\theta \subset \mathbb{C}^n$ be the following deformation of $\mathbb{R}^n \subset \mathbb{C}^n$:

$$\begin{gathered} \Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_1) = B_{\mathbb{R}^n}(0, R_1)\,, \\ \Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, R_2) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, R_2)\,, \\ \Gamma_\theta = f_\theta(\mathbb{R}^n), \quad f_\theta : \mathbb{R}^n \to \mathbb{C}^n \text{ is injective.} \end{gathered} \tag{4.5.1}$$

Here we take $R_0 < R_1 < R_2$ where R_0 is the same as in (4.1.1). Since no deformation is performed in $B(0, R_1)$ we can consider $\Gamma_\theta \setminus B(0, R_1)$ as a deformation of $\mathbb{R}^n \setminus B(0, R_1)$.

The next definition and lemma will allow us to define the deformation of P to an operator on $C^\infty(\Gamma_\theta)$ in a coordinate free way.

DEFINITION 4.28 (Totally real submanifolds). *(i) An n-dimensional smooth submanifold M of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ is called (maximally)* totally real *if for any $m \in M$,*

$$T_m M \cap i T_m M = \{0\}, \tag{4.5.2}$$

where we identify the tangent space $T_m M$ with a subspaces of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.

(ii) If $u \in C^\infty(M)$ then $\tilde{u} \in C^\infty(\mathbb{C}^n)$ is called an almost analytic extension *of u if*

$$\bar{\partial}_{z_j} \tilde{u}(z) = \mathcal{O}(d(z, M)^\infty), \quad z \in \mathbb{C}^n. \tag{4.5.3}$$

We first see that it is easy to find totally real submanifolds of the form (4.5.1):

LEMMA 4.29 (Totally real deformations of $\mathbb{R}^n$). *Suppose that $\Gamma_\theta \subset \mathbb{C}^n$ is given by (4.5.1). Then it is totally real if and only if*

$$\det(\partial_x f_\theta) \neq 0. \tag{4.5.4}$$

In particular, if $0 \leq \theta < \pi/2$ and

$$f_\theta(x) = x + i\partial_x F_\theta : \mathbb{R}^n \to \mathbb{C}^n \tag{4.5.5}$$

where $F_\theta : \mathbb{R}^n \to \mathbb{R}$ is a convex function then Γ_θ is totally real.

Proof. 1. Since (4.5.4) is a statement about derivatives we can assume that $f_\theta =: A \in M_{n\times n}(\mathbb{C})$ is a linear function and $M := \Gamma_\theta$ is a (real) linear subspace of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, $M = A(\mathbb{R}^n)$.

2. The condition that $M \cap iM \neq 0$ is equivalent to the existence of an *invertible* $B \in \mathbb{M}^{n\times n}(\mathbb{C})$ (a complex linear transformation) such that $B(M) = \mathbb{R}^n \subset \mathbb{C}^n$. Indeed, if M is totally real and $e_1, \ldots, e_n$ is a real basis of M, then it is a complex basis of $\mathbb{C}^n$. We then define by B as mapping $e_1, \ldots, e_n$ to the canonical basis of $\mathbb{C}^n$. Since $B(ix) = iB(x)$ the converse is clear.

3. Hence, if $\det A \neq 0$ we can take $B = A^{-1}$ and M is totally real. If M is totally real then $B \circ A : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible (real) matrix, and in particular $\det A \neq 0$. $\square$

For simplicity we will restrict our attention to the case of Γ_θ given by

$$0 \leq \theta < \pi/2, \quad f_\theta \ \text{ given by (4.5.5) where } F_\theta \text{ is convex.} \tag{4.5.6}$$

For scaling for angles greater than $\pi/2$ (which, as we will see, would recover all resonances in odd dimension case) see [**SZ91**].

EXAMPLE. We can construct a convex F_θ so that $\Gamma_\theta = f_\theta(\mathbb{R}^n)$ with f_θ given by (4.5.5) satisfies (4.5.1) as follows. Take $g \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$g(t) = 0 \text{ for } t \leq R_1 \ , \quad g(t) = \tfrac{1}{2}t^2 \text{ for } t \geq 2R_1, \quad g''(t) \geq 0.$$

We note that these assumptions imply that g' is non-decreasing and hence $g' \geq 0$. We then put

$$F_\theta(x) = \tan\theta \, g(|x|), \quad 0 \leq \theta < \tfrac{1}{2}\pi.$$

It follows that

$$\partial_x^2 F_\theta(x) = \tan\theta \left(\frac{g'(|x|)}{|x|^3}(|x|^2 I - x \otimes x) + \frac{g''(|x|)}{|x|^2} x \otimes x \right)$$

is positive definite and $\partial_x^2 F_\theta = \tan\theta I$ for $|x| > 2R_1$.

We now show that smooth functions on totally real submanifolds have almost analytic extensions which can use to define restrictions of holomorphic differential operotors:

LEMMA 4.30 (Totally real submanifolds and almost analytic extensions). *Suppose that M is a totally real submanifold of $\mathbb{C}^n$. Then every $u \in C^\infty(M)$ has an almost analytic extension to $\mathbb{C}^n$ in the sense of* (4.5.3).

If $\widetilde{P} = \sum_{|\alpha|\leq m} a_\alpha(z)\partial_z^\alpha$ is a holomorphic differential operator near M (a_α are holomorphic in $\mathbb{C}^n$ near M) then $\widetilde{P}$ defines a unique *differential operator P_M whose action on $C^\infty(M)$ is given by*

$$P_M u = \widetilde{P}(\tilde{u})|_M. \tag{4.5.7}$$

Proof. 1. We recall that for any $v \in C_c^\infty(\mathbb{R}^n)$ and an open set $\widetilde{V} \subset \mathbb{C}^n$, $\operatorname{supp} v \subset V$, we can find $\widetilde{v} \in C_c^\infty(\mathbb{C}^n)$ such that

$$\begin{aligned} &\widetilde{v}|_{\mathbb{R}^n} = u, \quad \bar{\partial}_{z_j}\widetilde{v}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty), \quad \operatorname{supp}\widetilde{v} \subset \widetilde{V}, \\ &\partial_{\bar{z}_j} := \tfrac{1}{2}(\partial_{x_j} + i\partial_{y_j}), \quad z = x + iy, \quad x, y \in \mathbb{R}^n, \end{aligned} \tag{4.5.8}$$

see for instance [**DS99**, (8.1),(8.2)]. That gives almost analytic extensions in the case of $M = \mathbb{R}^n$.

2. Using a partition of unity we only need to construct extensions of $u \in C_c^\infty(U)$, where $U \subset M$ is given by $U = f(B(0,r))$, $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$, $f : \mathbb{R}^n \to \mathbb{C}^n$. From Lemma 4.29 we see that $\partial_x f(x)$ is non-degenerate for x in a neighbourhood of $B(0,r)$ (we can decrease r). Let $\widetilde{f} : \mathbb{C}^n \to \mathbb{C}^n$ be an almost analytic extension of f. Then, $\partial_z \widetilde{f}|_{B(0,r)} = \partial_x f$ is non-degenerate. Hence for some neighbourhoods $\widetilde{V}, \widetilde{U} \subset \mathbb{C}^n$ of $B(0,r)$ and U respectively, $\widetilde{f} : \widetilde{V} \to \widetilde{U}$ is a diffeomorphism. In addition, $\widetilde{f}^{-1}$ is almost analytic as well. To see this we write $z = x + iy$, $\zeta = \xi + i\eta$, $x, y, \xi, \eta \in \mathbb{R}^n$, $z = \widetilde{f}(\zeta, \bar{\zeta})$ and $\zeta = \widetilde{g}(z, \bar{z})$, $\widetilde{g} := \widetilde{f}^{-1}$. Then, since $d(z, M) \sim |\operatorname{Im}\zeta|$,

$$\begin{aligned} \begin{bmatrix} \partial_z\zeta & \partial_{\bar{z}}\zeta \\ \partial_z\bar{\zeta} & \partial_{\bar{z}}\bar{\zeta} \end{bmatrix} &= \begin{bmatrix} \partial_\zeta z & \partial_{\bar{\zeta}} z \\ \partial_\zeta \bar{z} & \partial_{\bar{\zeta}}\bar{z} \end{bmatrix}^{-1} = \begin{bmatrix} \partial_\zeta z & \mathcal{O}(|\operatorname{Im}\zeta|^\infty) \\ \mathcal{O}(|\operatorname{Im}\zeta|^\infty) & \overline{\partial_\zeta z} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (\partial_\zeta z)^{-1} & \mathcal{O}(|\operatorname{Im}\zeta|^\infty) \\ \mathcal{O}(|\operatorname{Im}\zeta|^\infty) & (\overline{\partial_\zeta z})^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \partial_z\zeta & \mathcal{O}(d(M,z)^\infty) \\ O(d(M,z)^\infty) & \partial_{\bar{z}}\bar{\zeta} \end{bmatrix} \end{aligned}.$$

3. Using (4.5.8) we now define

$$\widetilde{u} = \widetilde{v} \circ \widetilde{f}^{-1}, \quad v := u \circ f \in C_c^\infty(B(0,r)), \quad \operatorname{supp}\widetilde{v} \subset \widetilde{V}. \tag{4.5.9}$$

Using almost analyticity of $\widetilde{v}$ and $\widetilde{f}^{-1}$ we obtain

$$\bar{\partial}_{z_j}\widetilde{u} = \frac{\partial\zeta}{\partial\bar{z}_j} \cdot \frac{\partial\widetilde{v}}{\partial\zeta} + \frac{\partial\bar{\zeta}}{\partial\bar{z}_j} \cdot \frac{\partial\widetilde{v}}{\partial\bar{\zeta}} = \mathcal{O}(d(M,z)^\infty),$$

which shows that $\widetilde{u}$ is an almost analytic continuation of u.

4. Arguing as in Steps 2 and 3, we can assume that $M = \mathbb{R}^n$ and that the coefficients of $\widetilde{P}$ satisfy $\partial_{\bar{z}} a_\alpha(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$. To define $P_{\mathbb{R}^n}$, we then to show that if $\widetilde{u}$ is almost analytic and $u|_{\mathbb{R}^n} = 0$ then $\widetilde{P}\widetilde{u}|_{\mathbb{R}^n} = 0$. But this follows by induction from showing that $\partial_z\widetilde{u}|_{\mathbb{R}^n} = 0$, which is obvious since $\partial_z\widetilde{u} = \partial_x\widetilde{u} + \mathcal{O}(|\operatorname{Im} z|^\infty)$. □

We can now define the complex scaled operator, P_θ:

DEFINITION 4.31 (**The complex scaled operator**). *Suppose that Γ_θ is given by (4.5.1) with f_θ satisfying (4.5.6). Suppose that P is a black box Hamiltonian in the sense of Definition 4.1. With $\chi \in C_c^\infty(B(0,R_1))$ equal to 1 on $B(0,R_0)$, (so that $1-\chi$ is a smooth function on $\mathbb{R}^n$ and Γ_θ), define*

$$
\begin{aligned}
&\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0,R_0)),\\
&\mathcal{D}_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, \quad (1-\chi)u \in H^2(\Gamma_\theta),\\
&P_\theta u = P(\chi u) + (-\Delta_\theta)((1-\chi)u),
\end{aligned}
\tag{4.5.10}
$$

where

$$
\Delta_\theta := \Delta_{\Gamma_\theta}
\tag{4.5.11}
$$

is defined using (4.5.7).

Using F_θ in (4.5.6) we calculate $-\Delta_\theta$. First we note that for $z = x + i\partial_x F_\theta(x)$ we have

$$
\frac{\partial}{\partial z} = \left(\frac{\partial x}{\partial z}\right)^T \frac{\partial}{\partial x} = (I + iF_\theta''(x))^{-1}\frac{\partial}{\partial x},
\tag{4.5.12}
$$

where $F_\theta''(x) := \partial_x^2 F_\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric matrix.

Since $-\Delta_\theta = \partial_z \cdot \partial_z$ (see (4.5.7)) we obtain, in the coordinates $x \in \mathbb{R}^n$ on Γ_θ,

$$
-\Delta_\theta u = \left((I + iF_\theta''(x))^{-1}\partial_x\right)\cdot\left((I + iF_\theta''(x))^{-1}\partial_x u\right), \quad u \in C^\infty(\Gamma_\theta).
\tag{4.5.13}
$$

The symbol of this operator is given by

$$
\sigma(\Delta_\theta)(x,\xi) = ((I + iF_\theta''(x))^{-1}\xi)\cdot((I + iF_\theta''(x))^{-1}\xi).
\tag{4.5.14}
$$

Here we used (x,ξ) as coordinates on $T^*\Gamma_\theta \simeq T^*\mathbb{R}^n$.

We now have

THEOREM 4.32 (**Ellipticity of $\boldsymbol{\Delta_\theta}$**). *The operator Δ_θ defined in (4.5.11) is an elliptic differential operator of order two:*

$$
|\xi|^2/C \leq |\sigma(\Delta_\theta)(x,\xi)| \leq C|\xi|^2
\tag{4.5.15}
$$

Proof. By homogeneity in ξ we need to show that for $\xi \neq 0$,

$$
((I + iF_\theta''(x))^{-1}\xi)\cdot((I + iF_\theta''(x))^{-1}\xi) \neq 0.
$$

Noting that $I + (F_\theta''(x))^2$ is invertible ($F_\theta''(x)$ is real and symmetric) it is enough to show that for $\eta \neq 0$,

$$
((I - iF_\theta''(x))\eta)\cdot((I - iF_\theta''(x))\eta) \neq 0, \quad \eta := (I + (F_\theta''(x))^2)^{-1}\xi \neq 0.
$$

The left hand side is equal to

$$
|\eta|^2 - |F_\theta''(x)\eta|^2 - 2i\langle F_\theta''(x)\eta,\eta\rangle.
\tag{4.5.16}
$$

Since $F''_\theta(x)$ is positive semidefinite (our convexity assumption),

$$\langle F''_\theta(x)\eta, \eta\rangle = 0 \implies F''_\theta(x)\eta = 0.$$

This concludes the proof as then the real part is equal to $|\eta|^2$. $\square$

Using invertibility of

$$-e^{-2i\theta}\Delta - \lambda^2 = e^{-2i\theta}(-\Delta - (e^{i\theta}\lambda)^2) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

for $\operatorname{Im}(e^{i\theta}\lambda) > 0$ and the ellipticity of Δ_θ it is easy to show that

$$-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta) \text{ is a Fredholm operator for } \operatorname{Im}(e^{i\theta}\lambda) > 0,$$

see Exercise 4.8. Instead we will show that $-\Delta_\theta - \lambda^2$ is invertible for this range of λ's and describe the inverse using the kernel of $R_0(\lambda)$.

4.5.2. The resolvent of $-\Delta_\theta$.

We will construct explicitly the inverse of $-\Delta_\theta - \lambda^2$ where Δ_θ is given by (4.5.11):

$$R_\theta(\lambda) := (-\Delta_\theta - \lambda^2)^{-1} : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta), \quad \operatorname{Im}(e^{i\theta}\lambda) > 0.$$

The operators $R_\theta(\lambda)$ will then form a holomorphic family of operators in the half plane $-\theta < \arg\lambda < \pi - \theta$.

The idea is to extend $|x-y| = ((x-y)\cdot(x-y))^{\frac{1}{2}}$ holomorphically from $\mathbb{R}^n \times \mathbb{R}^n$ to a complex neighbourhood of

$$\bigcup_{0\le\theta\le\theta_0} \Gamma_\theta \times \Gamma_\theta \subset \mathbb{C}^n \times \mathbb{C}^n.$$

Hence we start with:

LEMMA 4.33 (Well defined square root). *Suppose that Γ_θ is given by (4.5.1) with f_θ given by (4.5.6), $0 \le \theta < \pi/2$. Then for $z, w \in \Gamma_\theta$,*

$$\operatorname{Im}(z-w)\cdot(z-w) \ge 0, \tag{4.5.17}$$

where for $v, v' \in \mathbb{C}^n$, $v \cdot v' := v_1 v'_1 + \cdots + v_n v'_n$. Moreover,

$$(z-w)\cdot(z-w) = 0 \implies z = w. \tag{4.5.18}$$

In particular, the branch of the square root

$$((z-w)\cdot(z-w))^{\frac{1}{2}}, \quad z, w \in \Gamma_\theta, \tag{4.5.19}$$

which is positive for positive arguments is well defined and

$$\left\{ \begin{array}{c} \operatorname{Im} \\ \operatorname{Re} \end{array} \right\} ((z-w)\cdot(z-w))^{\frac{1}{2}} \ge 0, \quad z, w \in \Gamma_\theta. \tag{4.5.20}$$

Proof. 1. For $z = x + i\partial_x F_\theta(x)$ and $w = y + i\partial_x F(y)$

$$\operatorname{Im}(z-w)\cdot(z-w) = 2\langle F_\theta'(x) - F_\theta'(y), x-y\rangle. \tag{4.5.21}$$

2. Define

$$A(x,y) := \int_0^t F_\theta''(tx + (1-t)y)dt,$$

which, by the convexity assumption on F_θ is a positive semidefinite matrix. Since $F_\theta'(x) - F_\theta'(y) = A(x,y)(x-y)$,

$$\langle F_\theta'(x) - F_\theta'(y), x-y\rangle = \langle A(x,y)(x-y), x-y\rangle \geq 0,$$

proving (4.5.17).

3. To see (4.5.18) we note that $\langle A(x,y)(x-y), x-y\rangle = 0$ implies $A(x,y)(x-y) = 0$, that is $F_\theta'(x) - F_\theta'(y) = 0$. Hence $0 = (z-w)\cdot(z-w) = \langle x-y, x-y\rangle$, implies $x = y$. □

We can now proceed with the following definition:

DEFINITION 4.34 (Complex scaling of the free resolvent). *For $\operatorname{Im}(\lambda e^{i\theta}) > 0$, $0 \leq \theta < \pi/2$, we define the following operator $C_c^\infty(\Gamma_\theta) \to C^\infty(\Gamma_\theta)$:*

$$R_\theta(\lambda)\varphi(z) = \int_{\Gamma_\theta} R_0(\lambda, z-w)\varphi(w)dw, \quad \varphi \in C_c^\infty(\Gamma_\theta), \tag{4.5.22}$$

where, in the notation of (3.1.16) and (4.5.19),

$$R_0(\lambda, w) = \frac{e^{i\lambda(w\cdot w)^{\frac{1}{2}}}}{((w\cdot w))^{\frac{1}{2}})^{n-2}} P_n(\lambda\,(w\cdot w)^{\frac{1}{2}}), \tag{4.5.23}$$
$$dw := \det(I + iF_\theta''(y))dy.$$

INTERPRETATION. The element of integration, dw, is given by

$$\begin{gathered} dw = dw_1 \wedge \cdots dw_n = \det(I + iF_\theta''(y))dy, \\ w = y + i\partial_y F_\theta(y) \in \Gamma_\theta. \end{gathered} \tag{4.5.24}$$

The integral in (4.5.22) should then be considered as a contour integral – see [**Zw12**, §13.2] for a down-to-earth review. For $u, v \in C_c^\infty(\Gamma_\theta)$, let $\widetilde{u}, \widetilde{v} \in C_c^\infty(\mathbb{C}^n)$ be their almost analytic extensions. We have

$$\begin{aligned} \partial_{w_j}\widetilde{u}(w)\,dw_1 \wedge \cdots dw_n = (-1)^{j-1} d(\widetilde{u}(w)\,dw_1 \cdots dw_{j-1} \wedge dw_{j+1} \cdots dz_n) \\ - \sum_{k=1}^n \bar{\partial}_{w_k}\widetilde{u}\,d\bar{w}_k \wedge dw_1 \cdots dw_{j-1} \wedge dw_{j+1} \cdots dw_n, \end{aligned}$$

and the second term on the right vanishes on Γ_θ. Hence by Stokes's theorem

$$\int_{\Gamma_\theta} \partial_{w_j}\widetilde{u}(w)v(w)dw = -\int_{\Gamma_\theta} u(w)\partial_{w_j}\widetilde{v}(w)dw. \tag{4.5.25}$$

The next result shows that $R_\theta(\lambda)$ is in fact the inverse of $-\Delta_\theta - \lambda^2$:

THEOREM 4.35 (Resolvent of $\boldsymbol{\Delta_\theta}$). *For* $\operatorname{Im}(e^{i\theta}\lambda) > 0$, $0 \le \theta < \pi/2$, $R_\theta(\lambda) : C^\infty_{\rm c}(\Gamma_\theta) \to C^\infty(\Gamma_\theta)$ *given by* (4.5.22) *extends to an operator*

$$R_\theta(\lambda) : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta), \tag{4.5.26}$$

which is the two sided inverse of $-\Delta_\theta - \lambda^2$*. Moreover, for* $\delta > 0$

$$\begin{gathered} \operatorname{Im}\lambda > \delta \operatorname{Re}\lambda \ge 0 \implies \\ R_\theta(\lambda) = \mathcal{O}_\delta((\operatorname{Im}\lambda)^{-2}) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta). \end{gathered} \tag{4.5.27}$$

Proof. 1. We first refer to Theorem 3.3 for the properties of P_n appearing in (4.5.23). They give

$$|R_0(\lambda, w)| \le Ce^{\operatorname{Re}(i\lambda(w\cdot w)^{\frac12}}(|(w\cdot w)^{\frac12}|^{2-n} + \lambda^{\frac{n-3}{2}}|(w\cdot w)^{\frac12}|^{-\frac{n-1}{2}}). \tag{4.5.28}$$

In fact,

$$\begin{aligned} |\zeta^{2-n}P_n(\lambda\zeta)| &\le a_n|\zeta|^{2-n}(1 + |\lambda\zeta| + \cdots + |\lambda\zeta|^{\frac{n-3}{2}}) \\ &\le a_n\tfrac{n-1}{2}|\zeta|^{2-n}(1 + |\lambda\zeta|^{\frac{n-3}{2}}) \le b_n(|\zeta|^{2-n} + |\lambda|^{\frac{n-3}{2}}|\zeta|^{-\frac{n-1}{2}}). \end{aligned}$$

2. Let $\delta := \operatorname{Im}(e^{i\theta}\lambda)/|\lambda|$. Since $\Gamma_\theta \cap \mathbb{C}^n \setminus B(0, R_2) = e^{i\theta}\mathbb{R}^n \setminus B(0, R_2)$,

$$\begin{aligned} &\operatorname{Re}\left(i\lambda\left((z-w)\cdot(z-w)\right)^{\frac12}\right) \\ &= \operatorname{Re}\left(i\lambda(e^{i\theta} + \mathcal{O}((1+|z|+|w|)^{-1}))\right)|z-w| \\ &= -\operatorname{Im}(e^{i\theta}\lambda)|z-w| + \mathcal{O}((1+|z|+|w|)^{-1})|\lambda|)|z-w| \\ &= -\operatorname{Im}(e^{i\theta}\lambda)\left(1 + \mathcal{O}(\delta^{-1}(1+|z|+|w|)^{-1})\right)|z-w|, \quad z, w \in \Gamma_\theta. \end{aligned}$$

From (4.5.18) and again from the fact that Γ_θ agrees with $e^{i\theta}\mathbb{R}$ outside of a ball in $\mathbb{C}^n$, we deduce that

$$|((z-w)\cdot(z-w)^{\frac12}| \ge |z-w|/C, \quad z, w \in \Gamma_\theta. \tag{4.5.29}$$

We also trivially have

$$\operatorname{Re}\left(i\lambda\left((z-w)\cdot(z-w)\right)^{\frac12}\right) \le |\lambda||z-w|.$$

Using these three inequalities in (4.5.23) and (4.5.28) gives, with C_j's *depending* on λ and δ,

$$\begin{aligned}\int_{\Gamma_\theta} |R_0(\lambda, z-w)||dw| &\le C_0 \int_{\Gamma_\theta\cap\{|z-w|\ge C_0\}} \frac{e^{-|z-w|/C_0}}{|z-w|^{\frac{n-1}{2}}}|dw| \\ &\qquad + C_0 \int_{\Gamma_\theta\cap\{|z-w|\le C_0\}} |z-w|^{2-n}|dw| \\ &\le C_1 \int_0^\infty e^{-r/C_0} r^{\frac{n-1}{2}} dr + C_1 \le C_2,\end{aligned}$$

where $|dw| = |\det(I + iF_\theta''(y))|dy$. The same estimate holds when we integrated with respect to $|dz|$. Hence the boundedness on $L^2(\Gamma_\theta)$ follows from Schur's criterion (A.5.3). Since Theorem 4.32 gave us ellipticity of Δ_θ, (4.5.26) follows.

3. Now suppose that for $0 < \delta \le 1$, $\operatorname{Im}\lambda > \delta \operatorname{Re}\lambda \ge 0$. Then for $\zeta \in \mathbb{C}$ satisfying $\operatorname{Re}\zeta, \operatorname{Im}\zeta \ge 0$,

$$\begin{aligned}\operatorname{Re}(i\lambda\zeta) &= -\operatorname{Im}\lambda \operatorname{Re}\zeta - \operatorname{Re}\lambda \operatorname{Im}\zeta \\ &\le -\operatorname{Im}\lambda(\operatorname{Re}\zeta + \delta \operatorname{Im}\zeta) \le -\delta \operatorname{Im}\lambda|\zeta|.\end{aligned}$$

In view of (4.5.20) we can apply this inequality with $\zeta := ((z-w)\cdot(z-w))^{\frac{1}{2}}$ to obtain

$$\operatorname{Re}(i\lambda((z-w)\cdot(z-w))^{\frac{1}{2}}) \le -\delta \operatorname{Im}\lambda|((z-w)\cdot(z-w)^{\frac{1}{2}}|, \quad z, w \in \Gamma_\theta.$$

Hence, under our assumption on λ, this and (4.5.29) give

$$\exp(i\lambda((z-w)\cdot(z-w))^{\frac{1}{2}}) \le \exp(-\delta \operatorname{Im}\lambda|z-w|/C), \quad z, w \in \Gamma_\theta.$$

We now proceed as in Step 2 and use (4.5.28). However we keep track of the dependence on λ with the constants depending on δ and changing from line to line:

$$\begin{aligned}\int_{\Gamma_\theta} |R_0(\lambda, z-w)||dw| &\le C|\operatorname{Im}\lambda|^{\frac{n-3}{2}} \int_{\Gamma_\theta} \frac{e^{-\operatorname{Im}\lambda|z-w|/C}}{|z-w|^{\frac{n-1}{2}}}|dw| \\ &\qquad + C\int_{\Gamma_\theta} \frac{e^{-\operatorname{Im}\lambda|z-w|/C}}{|z-w|^{2-n}}|dw| \\ &\le C|\operatorname{Im}\lambda|^{\frac{n-3}{2}} \int_0^\infty e^{-\operatorname{Im}\lambda r} r^{\frac{n-1}{2}} dr + C\int_0^\infty e^{-\operatorname{Im}\lambda r} r dr \\ &= C(\operatorname{Im}\lambda)^{-2}.\end{aligned}$$

This proves (4.5.27).

4. Suppose that $\varphi, \psi \in C_c^\infty(\Gamma_\theta)$, with $\widetilde{\varphi}$ and $\widetilde{\psi}$ being the corresponding almost analytic extensions.

Since

$$\Delta_\theta \psi = \big(\sum_{j=1}^n \partial_{z_j}^2 \widetilde{\psi}\big)|_{\Gamma_\theta},$$

integration by parts using (4.5.25) shows that

$$\int_{\Gamma_\theta} \Delta_\theta \varphi(w)\psi(w)dw := \int_{\Gamma_\theta} \varphi(w)\Delta_\theta\psi(w)dw. \tag{4.5.30}$$

Hence to show that $R_\theta(\lambda)$ given by (4.5.22) is the left and right inverse of $-\Delta_\theta - \lambda^2$ it is enough to show that for $\varphi \in C_c^\infty(\Gamma_\theta)$ we have

$$\begin{aligned} &\int_{\Gamma_\theta} ((-\Delta_\theta)_z - \lambda^2)R_0(\lambda, z-w)\varphi(w)dw = \varphi(z), \\ &\int_{\Gamma_\theta} ((-\Delta_\theta)_w - \lambda^2)R_0(\lambda, z-w)\varphi(w)dw = \varphi(z). \end{aligned} \tag{4.5.31}$$

Since $(\Delta_\theta)_z R_0(\lambda, z-w) = (\Delta_\theta)_w R_0(\lambda, z-w)$, we only need to prove the first identity.

5. We first show that

$$((-\Delta_\theta)_z - \lambda^2)R_0(\lambda, z-w) = 0 \quad z \neq w, \quad z, w \in \Gamma_\theta. \tag{4.5.32}$$

In fact, let Ω, be an open set such that

$$\bigcup_{0\leq\theta'\leq\theta} \Gamma_{\theta'} \times \Gamma_{\theta'} \setminus \Delta(\mathbb{C}^n) \subset \Omega \subset \mathbb{C}^n \times \mathbb{C}^n \setminus \Delta(\mathbb{C}^n),$$

where $\Delta(\mathbb{C}^n) := \{(z,z) : z \in \mathbb{C}^n\}$. By taking Ω small enough we can assume that its connected components intersect $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta(\mathbb{R}^n) \subset \mathbb{C}^n \times \mathbb{C}^n \setminus \Delta(\mathbb{C}^n)$ and that $R_0(\lambda, z-w)$ for $(z,w) \in \Omega$.

We note that $\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta(\mathbb{R}^n) \subset \Omega_\theta \setminus \Delta(\mathbb{C}^n)$ is (maximally) totally real and

$$(-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2)R_0(\lambda, z-w)|_{\mathbb{R}^n\times\mathbb{R}^n\setminus\Delta(\mathbb{R}^n)} = 0.$$

This implies that the Taylor series vanishes on that submanifold and we have

$$(-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2)R_0(\lambda, z-w)|_{\Omega} = 0,$$

Since $((-\Delta_\theta)_z - \lambda^2)R_0(\lambda, z, w) = (-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2)R_0(\lambda, z-w)|_{z,w\in\Gamma_\theta}$, the equations (4.5.32) follows.

6. For a fixed $z \in \Gamma_\theta$, (4.5.23) shows that $R_0(\lambda, z-w) \in H_{\mathrm{loc}}^{-n/2+2-}((\Gamma_\theta)_w)$. On the other hand (4.5.32) shows that the support of

$$((-\Delta_\theta)_z - \lambda^2)R_0(\lambda, z-w) = ((-\Delta_\theta)_w - \lambda^2)R_0(\lambda, z-w) \in H_{\mathrm{loc}}^{-n/2-}((\Gamma_\theta)_w),$$

is contained in $\{z\}$. Since as a function of $z \in \Gamma_\theta$, $R_0(\lambda, z-w)$ is a smooth family of distributions in w, Schwartz's Lemma [**HöI**, Theorem 2.3.4] shows that

$$\begin{gathered}((-\Delta_\theta)_z - \lambda^2)R_0(\lambda, z, w) = c_\theta(z)\delta_z(w), \quad c_\theta \in L^\infty(\Gamma_\theta), \\ \delta_z \in \mathscr{S}'(\Gamma_\theta), \quad \int_{\Gamma_\theta} \varphi(w)\delta_z(w)dw = \varphi(z), \quad \varphi \in \mathscr{S}(\Gamma_\theta).\end{gathered}$$

We need to show that $c_\theta(z) \equiv 1$. For that put

$$\varphi(z) := \exp(-e^{-i2\theta} z \cdot z), \quad z \in \mathbb{C}^n, \quad \varphi_{\theta'} := \varphi|_{\Gamma_{\theta'}}, \quad 0 \le \theta' < \pi/2.$$

Then $\varphi_{\theta'} \in \mathscr{S}(\Gamma_{\theta'})$ for $|\theta' - \theta| < \pi/4$, $0 \le \theta' < \pi/2$.

A contour deformation and the decay and holomorphy of φ show that

$$\begin{aligned}\int_{\Gamma_\theta} R_0(\lambda, z-w)\varphi(w)dw &= \int_{-z+\Gamma_\theta} R_0(\lambda, w)\varphi(z+w)dw \\ &= \int_{\Gamma_\theta} R_0(\lambda, w)\varphi(z+w)dw.\end{aligned}$$

(We use Stokes's theorem – see for instance [**Zw12**, §13.2.1] – and note that Γ_θ can be deformed to $\Gamma_\theta - z$ using totally real submanifolds all passing through $w = 0$ on which $R_0(\lambda, w)$ is holomorphic for $w \neq 0$.) Hence

$$z \mapsto \int_{\Gamma_\theta} R_0(\lambda, z-w)\varphi(w)dw,$$

and

$$z \mapsto G(z) := (-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2) \int_{\Gamma_\theta} R_0(\lambda, z-w)\varphi(w)dw,$$

are holomorphic in a neighbourhood V of $\bigcup_{0 \le \theta' < \pi/2 - \varepsilon} \Gamma_{\theta'}$.

On the other hand $G(z) = c_{\theta'}(z)\varphi(z)$ for $z \in \Gamma_{\theta'}$. It follows that $c_{\theta'} = c|_{\Gamma_{\theta'}}$ where c is a holomorphic function in V. Since $c|_{\Gamma_0} = c_{\mathbb{R}^n} = 1$ it follows that $c \equiv 1$ proving (4.5.31).

7. We have shown that $R_\theta(\lambda) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$ and that $(-\Delta_\theta - \lambda^2)R_\theta(\lambda) = I$. Theorem 4.32 and the fact that

$$-\Delta_\theta = -e^{-2i\theta}\Delta_x, \quad z = x + iF'_\theta(x) \in \Gamma_\theta,$$

show that $R_\theta : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta)$ completing the proof. □

4.5.3. Fredholm properties of P_θ. Using the invertibility of $-\Delta_\theta - \lambda^2$, for $\operatorname{Im}(e^{i\theta}\lambda) > 0$ established in Theorem 4.35 we now show that for the same range of λ's, $P_\theta - \lambda^2$ is a Fredholm operator:

THEOREM 4.36 (Fredholm property of the scaled operator). *Let P_θ, $\mathcal{D}_\theta$ and $\mathcal{H}_\theta$, $0 \le \theta < \pi/2$, be given in Definition 4.31.*

If $\operatorname{Im}(e^{i\theta}\lambda) > 0$ then

$$P_\theta - \lambda^2 : \mathcal{D}_\theta \to \mathcal{H}_\theta,$$

is a Fredholm operator of index 0. In particular the spectrum of P_θ in $\mathbb{C} \setminus e^{-2i\theta}[0,\infty)$ is discrete.

REMARK. The condition for the Fredholm property is formulated in terms of λ though of course in terms of λ^2 it means that $\lambda^2 \in \mathbb{C} \setminus e^{-2i\theta}[0,\infty)$.

Proof. The strategy of the proof is to construct $Q_\theta(\lambda)$ and $S_\theta(\lambda)$ such that for $\lambda^2 \notin e^{-2i\theta}[0,\infty)$,

$$(P_\theta - \lambda^2)Q_\theta = I + K_\theta(\lambda), \quad S_\theta(\lambda)(P_\theta - \lambda^2) = I + L_\theta(\lambda),$$

where $K_\theta(\lambda) : \mathcal{H}_\theta \to H_\theta$, $L_\theta(\lambda) : \mathcal{D}_\theta \to \mathcal{D}_\theta$ are compact operators. That will show the Fredholm property (see Remark 1 at the end of §C.3). To see that the index is 0 we will show that for some λ_0, $I + K_\theta(\lambda_0)$ and $I + L_\theta(\lambda_0)$ are invertible. That means that $P - \lambda_0^2 : \mathcal{D}_\theta \to \mathcal{H}_\theta$ is invertible. Hence, for all $\lambda \notin e^{-2i\theta}[0,\infty)$ the index is 0 – see Theorem C.5.

1. To find Q_θ We follow the proof of Theorem 4.4. Let R_j, $j = 0, 1$ be as in (4.5.1) and let $\chi_j \in C_c^\infty(B(0,R_1))$ satisfy $\chi_0 \equiv 1$ on $B(0, R_0 + \varepsilon))$, $\varepsilon > 0$,

$$\chi_j(x) \equiv 1 \quad \text{for } x \in \operatorname{supp}\chi_{j-1}, \quad j = 1, 2.$$

For λ and λ_0 satisfying $\operatorname{Im}(e^{i\theta}\lambda_0) > 0$, $\operatorname{Im}\lambda_0 > 0$ and $\operatorname{Im}(e^{i\theta}\lambda) > 0$ (λ_0 will be chosen later) we put

$$Q_\theta(\lambda,\lambda_0) := (1-\chi_0)R_\theta(\lambda)(1-\chi_1) + \chi_2(P-\lambda_0^2)^{-1}\chi_1. \tag{4.5.33}$$

Then

$$(P_\theta - \lambda^2)Q_\theta(\lambda,\lambda) = I + K_\theta(\lambda,\lambda_0),$$

$$\begin{aligned} K_\theta(\lambda,\lambda_0) := {} & [\Delta,\chi_0]R_\theta(\lambda)(1-\chi_1) - [\Delta,\chi_2](P-\lambda_0^2)^{-1}\chi_1 \\ & + (\lambda_0^2 - \lambda^2)\chi_2(P-\lambda_0^2)^{-1}\chi_1. \end{aligned} \tag{4.5.34}$$

2. We now show that $K_\theta(\lambda,\lambda_0) : \mathcal{H}_\theta \to \mathcal{H}_\theta$ forms a family of compact operators and we do this by analysing individual terms.

First, compactness of the terms involving $(P-\lambda_0^2)^{-1}$ follows from Lemma 4.14: the singular values of these operators go to 0. Theorem 4.35 shows that

$$[\Delta,\chi_0]R_\theta(\lambda)(1-\chi_1) : \mathcal{H}_\theta \to H^1(B(0,R_1) \setminus B(0,R_0)).$$

Since $H^1(B(0,R_1)\setminus B(0,R_0))$ embeds compactly in $L^2(B(0,R_1)\setminus B(0,R_0)) \subset \mathcal{H}_\theta$ (see Theorem B.4) the term involving R_θ is also compact.

3. It remains to show that we can find λ_0 satisfying $\operatorname{Im}\lambda_0 > 0$ and $\operatorname{Im}(e^{i\theta}\lambda_0) > 0$ for which $I + K_\theta(\lambda_0, \lambda_0)$ is invertible. For that we note that if $\lambda_0 = e^{\pi i/4}\mu$, $\mu \gg 1$, then the conditions are satisfied and in addition $\operatorname{Re}\lambda > 0$. Hence the argument in Step 2 of the proof of Theorem 4.4 applies using (4.5.27) in Theorem 4.35. That completes the analysis of Q_θ.

4. The construction of S_θ is similar and we use the same notation:

$$S_\theta(\lambda, \lambda_0) := (1 - \chi_1) R_\theta(\lambda)(1 - \chi_0) + \chi_1 (P - \lambda_0^2)^{-1}\chi_2, \tag{4.5.35}$$

so that

$$\begin{aligned} S_\theta(\lambda, \lambda)(P_\theta - \lambda^2) &= I + L_\theta(\lambda, \lambda_0), \\ L_\theta(\lambda, \lambda_0) &:= (1 - \chi_1) R_\theta(\lambda)[\Delta, \chi_0] - \chi_1 (P - \lambda_0^2)^{-1}[\Delta, \chi_2] \\ &\quad + (\lambda_0^2 - \lambda^2)\chi_1 (P - \lambda_0^2)^{-1}\chi_2. \end{aligned} \tag{4.5.36}$$

Since $S_\theta(\lambda, \lambda_0) : \mathcal{H}_\theta \to \mathcal{D}_\theta$, we see that $L_\theta(\lambda, \lambda_0) : \mathcal{D}_\theta \to \mathcal{D}_\theta$.

5. Same argument as in Step 2 shows that $L_\theta(\lambda, \lambda_0)$ is compact as an operator from $\mathcal{H}_\theta$ to $\mathcal{H}_\theta$. This implies that it is enough to show that

$$(P_\theta - \lambda_0^2) L_\theta(\lambda, \lambda_0) : \mathcal{D}_\theta \to H_\theta \quad \text{is compact.}$$

We expand the operator on the left (using the support properties of χ_j's given in Step 1):

$$\begin{aligned} (P_\theta - \lambda_0^2) L_\theta(\lambda, \lambda_0) &= \widetilde{L}_\theta(\lambda, \lambda_0) + (\lambda_0^2 - \lambda^2)\chi_1, \\ \widetilde{L}_\theta(\lambda, \lambda_0) &:= [\Delta, \chi_1] R_\theta(\lambda)[\Delta, \chi_0] + [\Delta, \chi_1](P - \lambda_0^2)^{-1}[\Delta, \chi_2] \\ &\quad - (\lambda_0^2 - \lambda^2)[\Delta, \chi_1](P - \lambda_0^2)^{-1}\chi_2. \end{aligned} \tag{4.5.37}$$

Arguing as in Step 2 shows that

$$\widetilde{L}_\theta(\lambda, \lambda_0) : \mathcal{D}_\theta \hookrightarrow \mathcal{H}_\theta \to \mathcal{H}_\theta,$$

is compact. It remains to show that multiplication by χ_1 from $\mathcal{D}_\theta \to \mathcal{H}_\theta$ is compact. But Definition 4.31 gives

$$\chi_1 \mathcal{D}_\theta = \chi_1 \mathcal{D} = \chi_1 (P + i)^{-1}\mathcal{H} \hookrightarrow \chi_1 \mathcal{H} = \chi_1 \mathcal{H}_\theta,$$

and the second inclusion is compact by (4.1.12).

6. It remains to show that for λ_0 chosen in Step 3, $(I + L_\theta(\lambda_0, \lambda_0))^{-1} : \mathcal{D}_\theta \to \mathcal{D}_\theta$ exits. Same argument as in Step 3 shows that the inverse exists as a maps $\mathcal{H}_\theta \to \mathcal{H}_\theta$. Then in the notation of (4.5.37),

$$\begin{aligned} (P_\theta - \lambda_0^2)(I + L_\theta(\lambda_0, \lambda_0))^{-1} &= P_\theta - \lambda_0^2 \\ &\quad + (P_\theta - \lambda_0^2) L_\theta(\lambda_0, \lambda_0)(I + L_\theta(\lambda_0, \lambda_0))^{-1} \\ &= P_\theta - \lambda_0^2 + \widetilde{L}_\theta(\lambda_0, \lambda_0)(I + L_\theta(\lambda_0, \lambda_0))^{-1} \\ &= \mathcal{O}(1)_{\mathcal{D}_\theta \to \mathcal{H}_\theta} + \mathcal{O}(1)_{\mathcal{H}_\theta \to \mathcal{H}_\theta} = \mathcal{O}(1)_{\mathcal{D}_\theta \to \mathcal{H}_\theta}, \end{aligned}$$

which concludes the proof: we have found left and right inverses of $P_\theta - \lambda_0^2 : \mathcal{D}_\theta \to \mathcal{H}_\theta$, and approximate inverses at any λ. $\square$

4.5.4. Resonances as eigenvalues of P_θ. We established the fact that the spectrum of P_θ in $\mathbb{C} \setminus e^{-i\theta}[0,\infty)$ is discrete. It is remarkable that the spectrum does not depend on θ and coincides, with agreement of multiplicities, with the squares of resonances with $-\theta < \arg\lambda < \pi - \theta$ (that is satisfying $\operatorname{Im}(e^{i\theta}\lambda) > 0$). In an equivalent convention we would simply say that the spectrum coincides with the resonances – see §2.1.

For $\operatorname{Im}(e^{i\theta}\lambda) > 0$, Theorems 4.36 and C.9 show that

$$\Pi^\theta_{\lambda^2} := \frac{1}{2\pi i}\oint_\lambda (\zeta^2 - P_\theta)^{-1}\, 2\zeta\, d\zeta,$$

is a finite rank projection. The multiplicity of the eigenvalue of P_θ at λ^2 is given by the trace of this projection:

$$m_\theta(\lambda) := \operatorname{tr}\Pi^\theta_{\lambda^2}, \quad \operatorname{Im}(e^{i\theta}\lambda) > 0. \tag{4.5.38}$$

The agreement with resonances follows from the agreement of resolvents in the interaction region:

THEOREM 4.37 (Agreement of the resolvents away from scaling). *Let $R_0 < R_1$ be as in the definition of the black box Hamiltonian (§4.1) and Γ_θ (4.5.1).*

If $\chi \in C^\infty_{\mathrm{c}}(B(0,R_1))$ is equal to one near $B(0,R_0)$ then

$$\chi(P-\lambda^2)^{-1}\chi = \chi(P_\theta - \lambda^2)^{-1}\chi \quad \textit{for} \;\; \operatorname{Im}(e^{i\theta}\lambda) > 0. \tag{4.5.39}$$

Proof. 1. From the proof of Theorem 4.36 we see that, if we choose χ_3 (in the notation of (4.5.33)) so that $\chi_3 = 1$ on $\operatorname{supp}\chi_j$, $j = 0,1,2$ then,

$$(1-\chi_3)K_\theta(\lambda) = 0, \quad K_\theta(\lambda)\chi_3 = K_0(\lambda)\chi_3, \quad K_\theta(\lambda) := K_\theta(\lambda,\lambda_0),$$

where we used Definition 4.34. Then as in Step 3 of the proof of Theorem 4.4,

$$\begin{aligned}(P_\theta - \lambda^2)^{-1} &= Q_\theta(\lambda)(I + K_\theta(\lambda))^{-1}\\ &= Q_\theta(\lambda)((I + K_\theta(\lambda)(1-\chi_3))(I + K_\theta(\lambda)\chi_3))^{-1}\\ &= Q_\theta(\lambda)(I + K_\theta(\lambda)\chi_3)^{-1}(I + K_\theta(\lambda)(1-\chi_3))\\ &= Q_\theta(\lambda)(I + K_0(\lambda)\chi_3)^{-1}(I + K_\theta(\lambda)(1-\chi_3)).\end{aligned} \tag{4.5.40}$$

We also recall from (4.2.14) in the proof of Theorem 4.4 that

$$(P-\lambda^2)^{-1} = Q_0(\lambda)(I + K_0(\lambda)\chi_3)^{-1}(I + K_0(\lambda)(1-\chi_3)). \tag{4.5.41}$$

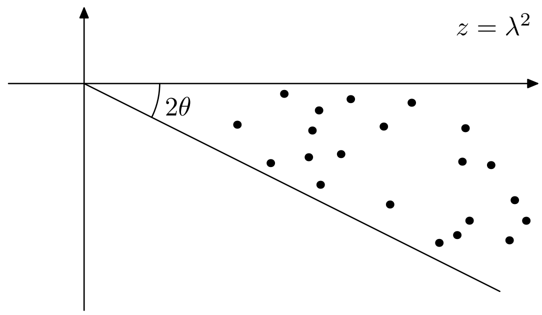


Figure 4.3. The resonances as eigenvalues the scaled operator P_θ, pictured on the z-plane. (On the λ-plane the angle would be θ rather than 2θ.)

2. For χ in the statement of the theorem we choose $\chi_0 \in C_c^\infty(B(0, R_1))$ in the definition of Q_θ and K_θ so that $\chi_0 = 1$ (and hence $\chi_j = 1$, $j = 1, 2, 3$) on $\operatorname{supp} \chi$.

For $|\lambda - \lambda_0| \ll |\lambda_0|^{-2}$ the Neumann series argument shows that

$$\chi_3(I + K_0(\lambda))^{-1}\chi = (I + K_0(\lambda))^{-1}\chi,$$

and hence this holds for all λ. Also,

$$\chi Q_\theta(\lambda)\chi_3 = \chi Q_0(\lambda)\chi_3, \quad (1 - \chi_3)\chi = 0.$$

Applying χ on both sides of (4.5.40) and (4.5.41) and using these two formulas proves (4.5.39). $\square$

It is now easy to prove

THEOREM 4.38. *The spectrum of P_θ in $\mathbb{C} \setminus e^{-2i\theta}[0, \infty)$ agrees with resonances satisfying $\operatorname{Im}(e^{i\theta}\lambda) > 0$. More precisely,*

$$m_R(\lambda) = m_\theta(\lambda), \quad \operatorname{Im}(e^{i\theta}\lambda) > 0, \tag{4.5.42}$$

where $m_R(\lambda)$ is the multiplicity of the resonance at λ given in (4.2.18) and $m_\theta(\lambda)$ is the multiplicity of the eigenvalue of P_θ at λ^2 – see (4.5.38).

Proof. 1. Since $\Pi^\theta_{\lambda^2}$ is a projection we see that

$$m_\theta(\lambda) = \operatorname{tr} \Pi^\theta_{\lambda^2} = \operatorname{rank} \Pi^\theta_{\lambda^2}.$$

Arguing as in the proof of (4.2.23) we see that

$$m_\theta(\lambda) = \operatorname{rank} \Pi^\theta_{\lambda^2}\chi.$$

where $\chi \in C_c^\infty(B(0, R_1))$ is equal to 1 near $B(0, R_0)$.

2. We now claim that

$$m_\theta(\lambda) = \operatorname{rank} \chi\Pi^\theta_{\lambda^2}\chi. \tag{4.5.43}$$

Otherwise there would exist solutions $v \in \mathcal{D}_\theta$ to $(P_\theta - \lambda^2)^k v = 0$, $u := (P_\theta - \lambda^2)^{k-1} v \neq 0$, satisfying $\chi v = 0$. But that would mean that u can be indetified with an element of $H^2(\Gamma_\theta)$ satisfying

$$(-\Delta_\theta - \lambda^2)u = 0, \quad u|_{B(0,R_0)} \equiv 0.$$

Since $-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$ is invertible (Theorem 4.35), $u \equiv 0$.

3. Combining (4.5.43) with (4.5.39) we see that

$$m_\theta(\lambda) = \operatorname{rank} \oint_\lambda \chi(P - \zeta^2)^{-1}\chi 2\zeta d\zeta.$$

We need to show that this is the same as

$$m_R(\lambda) = \operatorname{rank} \oint (P - \zeta^2)^{-1}\chi 2\zeta d\zeta.$$

We now argue as in Step 2, to see that otherwise we would have solutions to $(-\Delta - \lambda^2)u = 0$ equal to 0 in $B(0, R_0)$. But unique continuation results for second order elliptic differential equations show that $u \equiv 0$. $\square$

4.5.5. Applications. As the first application we provide

Proof of Lemma 4.21. 1. Since $\lambda \in \mathbb{R} \setminus \{0\}$ we can take any $\theta > 0$ and use (4.5.38) and Theorem 4.38 to see that

$$\Pi(0) := \frac{1}{2\pi i} \operatorname{tr} \int_\gamma (\zeta^2 - P_\theta(0))^{-1}\, 2\zeta\, d\zeta,$$
$$1 = m_0(\lambda) := \operatorname{tr} \Pi(0), \quad \gamma : t \to \lambda + \varepsilon e^{2\pi i t}, \quad t \in [0, 1),$$

where we write $P_\theta(s) := (P(s))_\theta$, for the complex scaled operators in our family.

2. We now pose a Grushin problem for the operator $P_\theta(0)$ – see §C.1. For that let we write the rank one projection $\Pi(0)$ as

$$\begin{gathered} \Pi(0)f = w\langle f, \widetilde{w}\rangle_{\mathcal{H}_\theta}, \quad w, \widetilde{w} \in \mathcal{D}_\theta, \quad \langle w, \widetilde{w}\rangle_{\mathcal{H}_\theta} = 1, \\ (P_\theta(0) - \lambda^2)w = 0, \quad (P_\theta(0)^* - \bar{\lambda}^2)\widetilde{w} = 0. \end{gathered} \tag{4.5.44}$$

The equation for $\widetilde{w}$ follows from the fact that $\Pi(0)$ commutes with $P_\theta(0)$ and hence for all $f \in \mathcal{D}_\theta$,

$$\begin{aligned} 0 &= (P_\theta(0) - \lambda^2)\Pi(0)f = \Pi(0)(P_\theta(0) - \lambda^2)f = w\langle (P_\theta(0) - \lambda^2)f, \widetilde{w}\rangle \\ &= w\langle f, (P_\theta(0)^* - \bar{\lambda}^2)\widetilde{w}\rangle. \end{aligned}$$

We then define

$$\mathscr{P}_\theta(s, \zeta) := \begin{pmatrix} P_\theta(s) - \zeta^2 & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}_\theta \times \mathbb{C} \to \mathcal{H}_\theta \times \mathbb{C},$$
$$R_- := u_- w, \quad R_+ u := \langle u, \widetilde{w}\rangle.$$

Writing

$$(P_\theta(0)-\zeta^2)^{-1}=\frac{\Pi(0)}{\lambda^2-\zeta^2}+Q(\zeta),$$

where $Q(\zeta)$ is holomorphic near $\zeta=\lambda$, we check that

$$\mathscr{E}_\theta(0,\zeta):=\begin{pmatrix} Q(\zeta) & E_+ \\ E_- & \zeta^2-\lambda^2\end{pmatrix}, \quad E_+v_+:=v_+w, \quad E_-v:=\langle v,\widetilde{w}\rangle,$$

satisfies

$$\mathscr{P}_\theta(0,\zeta)\mathscr{E}(0,\zeta)=I_{\mathcal{H}_\theta\times\mathbb{C}}, \quad \mathscr{E}(0,\zeta)\mathscr{P}_\theta(0,\zeta)=I_{\mathcal{D}_\theta\times\mathbb{C}}.$$

The only computation that requires some reflection is checking that $R_+Q(\zeta)=0_{\mathcal{D}_\theta\to\mathbb{C}}$. But using (4.5.44),

$$\begin{aligned} R_+Q(\zeta)v &= \langle Q(\zeta)v,\widetilde{w}\rangle = \langle (P_\theta(0)-\zeta^2)^{-1}v,\widetilde{w}\rangle - (\lambda^2-\zeta^2)^{-1}\rangle\Pi(0)v,\widetilde{w}\rangle \\ &= \langle v,(P_\theta(0)^*-\bar{\zeta}^2)^{-1}\widetilde{w}\rangle - (\lambda^2-\zeta^2)^{-1}\langle v,\widetilde{w}\rangle = 0. \end{aligned}$$

3. The assumption (4.4.14) shows that for $|s|\le\sigma_0\ll 1$

$$\begin{aligned} &\|(P_\theta(s)-P_\theta(0))Q(\zeta)\|_{\mathcal{H}_\theta\to\mathcal{H}_\theta} \le \|P(s)-P(0)\|_{\mathcal{D}\to\mathcal{H}_{R_0}}\|Q(\zeta)\|_{\mathcal{H}_\theta\to\mathcal{D}_\theta}<1, \\ &\|Q(\zeta)(P_\theta(s)-P_\theta(0))\|_{\mathcal{D}_\theta\to\mathcal{D}_\theta} \le \|P(s)-P(0)\|_{\mathcal{D}\to\mathcal{H}_{R_0}}\|Q(\zeta)\|_{\mathcal{H}_\theta\to\mathcal{D}_\theta}<1, \end{aligned}$$

that is, the assumptions of Lemma C.3 are satisfied. We conclude that we have an inverse depending smoothly on s (and holomorphic in ζ near λ):

$$\mathscr{E}_\theta(s,\zeta)=\begin{pmatrix} Q(s,\zeta) & E_+(s,\zeta) \\ E_-(s,\zeta) & E_{-+}(s,\zeta)\end{pmatrix},$$

$$E_+(s,\zeta)v_+:=v_+w(s,\zeta), \quad E_-v:=\langle v,\widetilde{w}(s,\bar{\zeta})\rangle, \quad E_{-+}(0,\zeta)=\zeta^2-\lambda^2.$$

4. Since $\partial_\zeta E_{-+}(0,\zeta)\neq 0$, we see that for small s, $E_{-+}(s,\zeta)=0$ has a unique smooth solution $\zeta=\lambda(s)$, $\lambda(0)=0$. The Schur complement formula (C.1.1) then shows that $\lambda(s)$ is the unique simple eigenvalue of $P_\theta(s)$ near λ. The eigenfunction is then given by $w(s):=w(s,\lambda(s))$ and it depends smoothly on s. In the notation of (4.5.1) the resonant state of $P(s)$, $u(s)$ satisfies $\mathbb{1}_{B(0,R_1)}u(s)=\mathbb{1}_{B(0,R_1)}w(s,\lambda(s))$. Since we can choose R_1 arbitrarily large, the smoothness of $s\mapsto u(s)\in\mathcal{D}_{\mathrm{loc}}$ follows. This completes the proof of Lemma 4.21. $\square$

The next application is the higher dimensional version of Theorem 2.25. By a generic set we again mean an intersection of open dense sets. The space of perturbations we will consider is

$$\begin{aligned} &\dot{C}^\infty(\overline{B(0,R_1)}\setminus B(0,R_0);\mathbb{R}) := \\ &\qquad \left\{u\in C^\infty(\mathbb{R}^n) : \operatorname{supp} u\subset \overline{B(0,R_1)}\setminus B(0,R_0)\right\}. \end{aligned}$$

(This definition is self explanatory but we mention that the general notation comes from [**HöIII**, §B.2].)

THEOREM 4.39. *Suppose that P is a black box Hamiltonian in the sense of Definition 4.1. Then for any $R_1 > R_0$ there exists a generic set $\mathcal{V} \subset \dot{C}^\infty(\overline{B(0,R_1)} \setminus B(0,R_0); \mathbb{R})$ such that all resonances of $P + V$, $V \in \mathcal{V}$, with*

$$-\frac{\pi}{2} < \arg\lambda < 0 \quad \textit{are } \text{simple.} \tag{4.5.45}$$

REMARKS. 1. The condition $\arg\lambda > -\pi/2$ can be eliminated in odd dimensions using large angle complex scaling of [**SZ91**] (and to $\arg\lambda > -2\pi + \varepsilon$ in odd dimensions.) One can replace complex scaling by Agmon's theory of resonance perturbation [**Ag98**] as in Borthwick–Perry [**BP02**].

2. The reason for demanding that $\arg\lambda < 0$ is the fact that compactly supported embedded eigenvalues cannot be split using perturbations supported away from the black box. The positive eigenvalues are typically unstable and can be perturbed to become resonances – see Theorem 4.22 and the example after its proof.

3. It is not clear how to prove this theorem following the same strategy as that in the proof of Theorem 2.25 as the situation in Step 4 of the proof is more complicated in higher dimensions.

Before proving Theorem 4.39 we state the following lemma.

LEMMA 4.40. *For $0 < \theta < \pi/2$ let P_θ be the complex scaled operator of a black box Hamiltonian (see Definition 4.31). If z_0, $\arg z_0 > -2\theta$ is an eigenvalue of P_θ (that is $z_0 = \lambda^2$ where λ is a resonance of P), then for some K and ε sufficiently small,*

$$\begin{gathered}(P_\theta - z)^{-1} = \sum_{k=1}^{K} \frac{(P_\theta - z_0)^{k-1}\Pi}{(z_0 - z)^k} + G(z), \\ \Pi := \frac{1}{2\pi i}\int_\gamma (z-P)^{-1}dz, \quad \gamma(t) = z_0 + i\varepsilon e^{it}, \ t \in [0, 2\pi),\end{gathered} \tag{4.5.46}$$

where $G(z) : \mathcal{H}_\theta \to \mathcal{D}_\theta$ is a holomorphic family of operators for z near z_0.

In addition there exist $w_j, \widetilde{w}_j \in \mathcal{D}_\theta^\infty$, $j = 1, \cdots, N = \operatorname{tr}\Pi$, such that

$$\begin{gathered}\Pi v = \sum_{j=1}^{N} w_j \langle v, \widetilde{w}_j\rangle, \quad \langle w_k, \widetilde{w}_j\rangle = \delta_{kj}, \\ (P_\theta - z)^K w_j = 0, \quad (P_\theta^* - \bar{z})^K \widetilde{w}_j = 0.\end{gathered} \tag{4.5.47}$$

Proof. 1. The expansion (4.5.46) follows as in the proof of Theorem 4.7: everything is easier now as we deal with a genuine resolvent, $(P_\theta - z)^{-1}$.

2. We only need to check (4.5.47). Since Π is an operator of rank N, there exist a basis $\{w_j\}_{j=1^N}$ of $\Pi\mathcal{H}_\theta$. Also, there exist $\widetilde{w}_j \in \mathcal{H}_\theta$ such that the first formula in (4.5.47) holds. The projection property, $\Pi w_k = w_k$, then shows that $\langle w_k, \widetilde{w}_j\rangle = \delta_{jk}$.

3. Since P_θ commutes with Π it follows that $w_j \in \mathcal{D}_\theta^\infty$. Since P_θ^* commutes with Π^* it also follows that $\widetilde{w}_j$ are in the domain of $(P_\theta^*)^k$ which is the same as the domain of P_θ^k, for any k. Since $(P_\theta - z)^K\Pi = 0$ the last part of (4.5.47) follows. $\square$

Proof of Theorem 4.39. 1. We identify resonances λ with $0 > \arg\lambda > -\theta$ with eigenvalues of $z \in \operatorname{Spec}(P_\theta)$, $0 > \arg z > -2\theta$, $z = \lambda^2$. We recall (and rename) the definition of multiplicity:

$$m_V(z) := \frac{1}{2\pi i}\operatorname{tr}\oint_z (\zeta - (P_\theta + V))^{-1} d\zeta,$$

where the integral is over a sufficiently small positively oriented circle around z. We then define

$$\begin{aligned} E_\theta^r &:= \{W \in \mathcal{C}_{R_0,R_1} : m_W(z) \le 1,\ z \in \Gamma_r\},\\ \mathcal{C}_{R_0,R_1} &:= \dot{C}^\infty(\overline{B(0,R_1)} \setminus B(0,R_0); \mathbb{R}). \qquad (4.5.48)\\ \Gamma_r &:= \{z : -\theta + 1/r \le \arg z \le -1/r,\ 1/r \le |z| \le r\}. \end{aligned}$$

We want to show that for $r > 0$, E_θ^r is open and dense. That will show that the set

$$E_\theta := \{W \in \mathcal{C}_{R_0,R_1} \ :\ m_W(z) \le 1 \text{ for } \arg z > -\theta\} = \bigcap_{n\in\mathbb{N}} E_\theta^n$$

is generic (and in particular, by the Baire category theorem, it has a nowhere dense complement). By taking

$$\mathcal{V} := \bigcap_{n\in\mathbb{N}} E_{\pi/2-1/n},$$

we obtain the generic set in the statement of the theorem.

2. Suppose that $P_\theta + W$ has exactly one resonance z_0 in $D(z_0, 2r)$. For $\Omega := D(z_0, r)$ we then define

$$\Pi_W(\Omega) := \frac{1}{2\pi i}\int_{\partial\Omega} (\zeta - (P_\theta + W))^{-1} d\zeta, \quad m_W(\Omega) := \operatorname{tr}\Pi_W(\Omega). \qquad (4.5.49)$$

If $V \in \mathcal{C}_{B_0,B_1}$ and $\|V\|_\infty$ is sufficiently small then for $\zeta \in \partial\Omega$,

$$(P_\theta + W + V - \zeta)^{-1} = (P_\theta + W - \zeta)^{-1}(I + V(P_\theta + W - \zeta)^{-1})^{-1},$$

exists and we can define Π_{W+V}. This also shows that if $\|V\|_\infty < \varepsilon$ for sufficiently small ε then for $\zeta \in \partial\Omega$,

$$(P_\theta + W - \zeta)^{-1} - (P_\theta + W + V - \zeta)^{-1} = \mathcal{O}_\varepsilon(\|V\|_\infty)_{\mathcal{H}_\theta\to\mathcal{H}_\theta}.$$

Hence,

$$\|\Pi_W(\Omega) - \Pi_{W+V}(\Omega)\|_{\mathcal{H}_\theta \to \mathcal{H}_\theta} \leq C_\varepsilon \|V\|_\infty.$$

If we take $\|V\|_\infty < \min(\varepsilon, 1/C_\varepsilon)$ then $\Pi_W(\Omega)$ and $\Pi_{W+V}(\Omega)$ have the same rank. (If for two finite rank projections P_1 and P_2, $\|P_1 - P_2\| < 1$ then the ranks are the same. Indeed, we have $\|P_1 - P_1 P_2 P_1\| < 1$ and hence $P_1 P_2 P_1$ is invertible on the range of P_1 and $\operatorname{rank} P_1 \leq \operatorname{rank} P_2$.)

We restate this as follows:

$$m_{W+V}(\Omega) \text{ is constant for } \|V\|_\infty \text{ sufficiently small.} \tag{4.5.50}$$

3. The statement (4.5.50) immediately implies that E_θ^r is open: if z is a simple resonance then $m_W(\Omega) = 1$ and it stays constant under small changes in the potential. (We could have also used the argument in the proof of Lemma (4.21) to see that but (4.5.50) will be needed later.)

4. Now we want to show that E_θ^r is dense. This follows from the following statement

$$\begin{gathered} \forall\, W \in \mathcal{C}_{B_0,B_1},\ \varepsilon > 0\ \ \exists\, V \in \mathcal{C}_{B_0,B_1} \\ W + V \in E_\theta^r, \quad \|V\|_\infty < \varepsilon. \end{gathered} \tag{4.5.51}$$

Since the number of eigenvalues of $P_\theta + W$ in Γ_r (see (4.5.48)) is finite it is enough to prove a local statement as it can be applied successively to obtain (4.5.51) (once an eigenvalue is simple it stays simple for sufficiently small perturbations by Step 3).

Hence to obtain (4.5.51) it suffices to prove, in the notation of Step 2,

$$\begin{gathered} \forall\, W \in \mathcal{C}_{B_0,B_1},\ \varepsilon > 0\ \ \exists\, V \in \mathcal{C}_{B_0,B_1}\ \forall\, z \in \Omega \\ m_{W+V}(z) \leq 1, \quad \|V\|_\infty < \varepsilon. \end{gathered} \tag{4.5.52}$$

5. To establish (4.5.52) we proceed by induction. One of the following cases has to occur:

$$\begin{gathered} \forall\, \varepsilon > 0\ \ \exists\, V \in \mathcal{C}_{R_0 R_1},\ z \in \Omega \\ 1 \leq m_{W+V}(z) < m_{W+V}(\Omega), \quad \|V\|_\infty < \varepsilon, \end{gathered} \tag{4.5.53}$$

or

$$\begin{gathered} \exists\, \varepsilon > 0\ \forall\, V \in \mathcal{C}_{R_0 R_1}, \|V\|_\infty < \varepsilon\ \exists\, z = z(V) \in \Omega \\ m_{W+V}(z) = m_{W+V}(\Omega). \end{gathered} \tag{4.5.54}$$

The first possibility means that by adding an arbitrarily small V to W we can obtain at least two distinct eigenvalues of $P_\theta + V + W$. The second possibility means that for any small perturbation the maximal multiplicity persists.

6. We will show that (4.5.54) *cannot occur*. Assuming (4.5.53) we prove (4.5.52) by induction on $m_W(z_0)$ (where z_0 is the unique eigenvalues of $P_\theta + W$ in $D(z_0, 2r)$, $\Omega := D(z_0, r)$ – see Step 2). If $m_W(z_0) = 1$ there is nothing to prove. Assuming that we proved (4.5.52) for $m_W(z_0) < M$ assume that $m_W(z_0) = M$. Using (4.5.53) we see that we can find V_0, $\|V_0\|_\infty < \varepsilon/2$ such that $m_{W+V_0}(\Omega) = m_W(\Omega)$ (using (4.5.50)) and such that all eigenvalues in Ω, $z_1, \cdots, z_k$, satisfy $m_{W+V_0}(z_j) < M$. We now find r_j such that,

$$
\begin{gathered}
D(z_j, 2r_j) \subset \Omega, \quad D(z_j, 2r_j) \cap D(z_k, 2r_k) = \emptyset, \quad j \neq k, \\
\{z_j\} = D(z_j, 2r_j) \cap \operatorname{Spec}(P_\theta + W + V_0).
\end{gathered}
$$

We put $\Omega_j := D(z_j, r_j)$ and apply (4.5.52) successively to $W + V_0 + \cdots V_{j-1}$, $j = 1, \cdots, k$, in Ω_j with $\|V_j\|_\infty < \varepsilon/2^{j+1}$. That gives the desired $V = \sum_{j=0}^k V_j$.

6. It remains to show that (4.5.54) is impossible. Hence, assume that $m_W(z) = M$ and that (4.5.54) holds. For $V \in \mathcal{C}_{R_0 R_1}$, $\|V\|_\infty < \varepsilon$, put

$$
k(V) := \min\{k : (P_\theta + W + V - z(V))^k \Pi_{W+V}(\Omega) = 0\}.
$$

Then $1 \leq k(V) \leq M$ and $B_{C^M}(0, \varepsilon) \ni V \mapsto k(V)$ is a *lower semi-continuous* function. In fact, if $\|V_j - V\|_{C^M} \to 0$ and then, from (4.5.49), we see that $(P_\theta + W + V_j - z(V_j))^k \Pi_{W+V_j}(\Omega) = 0$, then $(P_\theta + W + V - z(V))^k \Pi_{W+V}(\Omega) = 0$.

Defining

$$
k_0 := \max\{k(V) : V \in \mathcal{V}_{R_0 R_1}, \|V\|_\infty < \varepsilon/2\},
$$

we see that if $k(V') = k_0$, $k(V + V') = k_0$ for $\|V\|_{C^M} < \delta$, with a sufficiently small δ. Hence we can replace W by $W + V'$, decrease ε and assume that

$$
\begin{gathered}
(P_\theta + W + V - z(V))^{k_0} \Pi_{V+W}(\Omega) = 0, \\
(P_\theta + W + V - z(V))^{k_0 - 1} \Pi_{V+W}(\Omega) \neq 0, \\
m_{W+V}(z(V)) = \operatorname{tr} \Pi_{V+W} = M > 1, \quad \forall V, \quad \|V\|_{C^M} < \varepsilon.
\end{gathered}
\tag{4.5.55}
$$

7. To see that (4.5.55) is impossible we first assume that $k_0 > 1$. Take $V = V(t) = W + tV$, $\|V\|_{C^M} < \varepsilon$, $t \in [-1, 1]$. For $h, g \in \mathcal{D}_\theta$ we define (dropping Ω in $\Pi_\bullet(\Omega)$)

$$
\begin{aligned}
w(t) &:= (P_\theta + W + tV - z(t))^{k_0 - 1} \Pi_{W+tV} h, \\
\widetilde{w}(t) &:= (P_\theta^* + W + tV - \overline{z(t)})^{k_0 - 1} \Pi_{W+tV}^* g.
\end{aligned}
$$

By our assumption (4.5.55) we can choose g and h so that

$$
w := w(0) \neq 0, \quad \widetilde{w} := \widetilde{w}(0) \neq 0. \tag{4.5.56}
$$

From Step 2 (or arguments presented in the proof of Lemma 4.40) we see that $t \mapsto z(tV), \Pi_{W+tV}, w(t)$ are are smooth functions of t. We then differentiate

$$
\begin{aligned}
0 &= \frac{d}{dt}(P_\theta + W + tV - z(t))^{k_0}\Pi_{W+tV}h \\
&= V(P_\theta + W + tV - z(t))^{k_0-1}\Pi_{W+tV}h + (P_\theta + W + tV - z(t))H(t)
\end{aligned}
$$

where $H(t) \in \mathcal{D}_\theta^\infty$. We now put $t = 0$ and take the $\mathcal{H}_\theta$ inner product with $\widetilde{w}$: the term with $H(0)$ disappears as $(P_\theta^* + W + tV)^{k_0}\Pi_W^* \equiv 0$ and we obtain

$$\forall\, V \in \mathcal{C}_{R_0,R_1} \quad \langle Vw, \widetilde{w}\rangle = 0.$$

This shows that

$$\overline{w|_{B(0,R_0)\setminus B(0,R_1)}}\,\widetilde{w}|_{B(0,R_0)\setminus B(0,R_1)} \equiv 0.$$

Since w and $\widetilde{w}$ solve $(-\Delta_\theta - z)w = 0$ and $(-\Delta_\theta^* - \bar{z})\widetilde{w} = 0$ in $B(0,R_0) \setminus B(0,R_1)$, the unique continuation property of the equations shows that $w = \widetilde{w} = 0$ in $\Gamma_\theta \setminus B(0,R_0)$. But then z has to be an eigenvalue of $P + W$ (there is no scaling near $B(0,R_0)$) and $\arg z = 0$, a contradiction.

8. It remains to consider the case of $k_0 = 1$ in (4.5.55). We then use the notation of Lemma 4.40 and have

$$
\begin{aligned}
0 &= \frac{d}{dt}(P_\theta + W + tV - z(tV))\Pi_{W+tV}w_k \\
&= V\Pi_{W+tV}w_k - \frac{d}{dt}z(tV)\Pi_{W+tV}w_k + (P_\theta + W + tV - z(tV))\frac{d}{dt}\Pi_{W+tV}w_k.
\end{aligned}
$$

We then put $t = 0$ and take an inner product with $\widetilde{w}_j$. That gives:

$$\frac{d}{dt}z(tV)(0)\delta_{kj} = \langle Vw_k, \widetilde{w}_j\rangle, \quad k, j = 1, \cdots, M.$$

Taking $k \neq j$ and arguing as at the end of Step 7 we again obtain a contradiction. $\square$

4.6. SINGULARITIES AND RESONANCE FREE REGIONS

In this section we present an abstract *non-trapping* condition which guarantees the presence of resonance free regions for black box Hamiltonians. This generalizes Theorem 3.10 and provides an expansion of scattered waves for "non-trapping black boxes". In §6.2 we will present a semiclassical version of the non-trapping resolvent estimates, see also §6.6.

To explain the idea of the proof we first present the result in the simpler case of potential scattering. It improves Theorem 3.10 in the case of *smooth* potential since the logarithmic region is now *arbitrarily* large:

THEOREM 4.41 (Non-trapping estimates for smooth potentials). *Suppose that* $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$, $n \geq 3$, *odd, and* $R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1}$, $\operatorname{Im}\lambda > 0$. *For any* $M > 0$ *there exists* C_0 *such that* $R_V(\lambda) : L^2_{\rm comp} \to L^2_{\rm loc}$ *continues* holomorphically *to*

$$\Omega_M := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > -M \log|\lambda|, \ |\lambda| > C_0\}. \tag{4.6.1}$$

Moreover, for any $\chi \in C_c^\infty(\mathbb{R}^n)$ *there exist* C_1 *and* T *such that*

$$\|\chi R_V(\lambda)\chi\|_{L^2\to L^2} \leq C_1|\lambda|^{-1}e^{T(\operatorname{Im}\lambda)_-}, \quad \lambda \in \Omega_M. \tag{4.6.2}$$

Proof. 1. Let

$$U_V(t) := \frac{\sin t\sqrt{-\Delta+V}}{\sqrt{-\Delta+V}},$$

where functions of $-\Delta + V$ are defined using the spectral theorem.

Then $U_V(t) : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$ solves the wave equation

$$\Box_V U_V(t) = 0, \quad U_V(0) = 0, \quad \partial_t U_V(0) = I, \quad \Box_V := \partial_t^2 - \Delta + V.$$

We use the same notation for the Schwartz kernel of the operator $U_V(t)$:

$$U_V(t)f(x) = \int_{\mathbb{R}^n} U_V(t,x,y)f(y)dy, \quad f \in C_c^\infty(\mathbb{R}^n),$$
$$U_V \in C^\infty(\mathbb{R}_t, \mathcal{D}'(\mathbb{R}^n_x \times \mathbb{R}^n_y)).$$

The sharp Huyghens principle in odd dimensions states

$$\operatorname{supp} U_0(t,\bullet) = \{(x,y) \ : \ |x-y| = |t|\}, \tag{4.6.3}$$

and $\operatorname{singsupp} U_0(t) = \operatorname{supp} U_0(t)$.

The Huyghens principle/finite speed of propagation holds for U_V:

$$\operatorname{supp} U_V(t,\bullet) \subset \{(x,y) : |x-y| \leq |t|\}.$$

The key to the proof is the following result about the singular support (see Exercise E.15) of $U_V(t)$:

$$\operatorname{singsupp} U_V(t,\bullet) = \{(x,y) \ : \ |x-y| = |t|\}. \tag{4.6.4}$$

This follows from general results presented in Theorem E.47 but in this simple case can also be deduced from an explicit parametrix construction for $\partial_t^2 - \Delta + V$ – see for instance [**MU**, §§1,4].

2. Property (4.6.4) implies the following statement. Suppose $\psi_a \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies

$$\operatorname{supp}\psi_a \subset \{(t,x) : |x| + T_a \leq t \leq |x| + T_a + 1\}, \quad T_a \geq a,$$

(we can take $T_a = a$ but this generality will be useful in later) with a large enough so that

$$\operatorname{supp} V \subset B(0,a).$$

We claim that for $\chi \in C^\infty_{\rm c}(B(0,a))$

$$\psi_a U_V(t)\chi \in C^\infty(\mathbb{R}; \mathcal{L}(L^2, H^1)). \tag{4.6.5}$$

Since

$$t \geq |x| + T_a, \ |x-y| = t \implies |y| \geq t - |x| \geq T_a \geq a,$$

we see that if $(t,x) \in \operatorname{supp}\psi_a$ and $y \in \operatorname{supp}\chi$ then $|x-y| \neq t$ so that (4.6.4) implies that

$$\psi_a(t,x)U_V(t,x,y)\chi(y) \in C^\infty(\mathbb{R}_t \times \mathbb{R}^{2n}_{(x,y)}),$$

and that implies a stronger statement than (4.6.5).

3. Just as in (3.1.11), we can use spectral theorem to relate the propagator U_V to the resolvent by the formula

$$R_V(\lambda) := \mathcal{F}^*_{t\to\lambda}(H(t)U_V(t)) := \int_0^\infty e^{i\lambda t}U_V(t)dt, \quad \operatorname{Im}\lambda > 0, \tag{4.6.6}$$

where $H(t)$ is the Heaviside function. We will use related formulas to obtain an effective expression for $R_V(\lambda)$.

4. For $\zeta \in C^\infty(\mathbb{R})$ equal to 1 for $s \leq \frac{1}{3}$ and 0 for $s \geq \frac{2}{3}$ we put

$$\zeta_a(x,t) := \zeta(t - T_a - |x|), \tag{4.6.7}$$

so that

$$\zeta_a(x,t) = \begin{cases} 1 & t \leq |x| + T_a \\ 0 & t \geq |x| + T_a + 1. \end{cases}$$

Suppose that

$$\operatorname{supp}\chi_a \subset B(0,a), \quad \chi_a|_{\operatorname{supp} V} \equiv 1$$

and write

$$\begin{aligned} &\Box_V \zeta_a U_V(t)\chi_a = [\Box_V, \zeta_a]U_V(t)\chi_a =: F_a(t), \\ &\zeta_a U_V(t)\chi_a|_{t=0} = 0, \quad \partial_t(\zeta_a U_V(t)\chi_a)|_{t=0} = \chi_a. \end{aligned} \tag{4.6.8}$$

From (4.6.5) we deduce that

$$F_a(t) \in C^\infty(\mathbb{R}; \mathcal{L}(L^2, L^2)). \tag{4.6.9}$$

5. Putting

$$\widetilde{R}_a(\lambda)g := \mathcal{F}^*_{t\to\lambda}(\zeta_a H(t)U_V(t)\chi_a g), \quad g \in L^2 \tag{4.6.10}$$

we obtain

$$(-\Delta + V - \lambda^2)\widetilde{R}_a(\lambda)g = \chi_a g + \mathcal{F}^*_{t\to\lambda}(F_a(t))g). \tag{4.6.11}$$

We note that the Fourier transforms are well defined because of the $\zeta_a(t,x)$ factor and (4.6.7).

6. We will modify $\widetilde{R}_a(\lambda)$ to obtain $R^\#_a(\lambda)$ such that

$$(-\Delta + V - \lambda^2)R^\#_a(\lambda) = \chi_a(I + K_a(\lambda)), \tag{4.6.12}$$

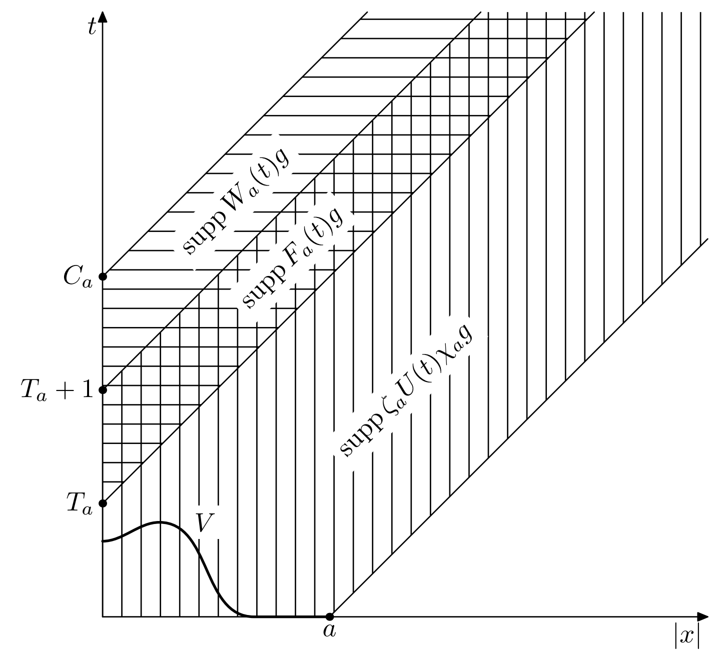


Figure 4.4. Support properties of operators appearing in the proof of Theorem 4.41: $\zeta_a U_V(t)\chi_a$, $F_a(t)$ and $W_a(t)$, applied to $g \in L^2$ – the support of $F_a(t)g$ is contained in the intersection of the supports of $\zeta_a U_V(t)\chi_a$ and $W_a(t)g$. Condition (4.6.2) implies that $F_a(t)g, W_a(t)g \in C^\infty$. An approximation of $R(\lambda)\chi_a$ is obtained by taking $R^\#(\lambda) = \mathcal{F}^*_{t\to\lambda}(\zeta_a U(t)\chi_a + W_a(t))$ which is well defined because of the support properties shown in the figure.

where $K(\lambda)$ has small norm for $\lambda \in \Omega_M$ (see (4.6.1) for the definition of Ω_M) and $R_a^\#(\lambda)$ satisfies the estimate (4.6.2). We will then have

$$R(\lambda)\chi_a = R_a^\#(\lambda)(I + K_a(\lambda))^{-1}, \tag{4.6.13}$$

from which (4.6.2) will follow.

7. The modification is achieved by solving the free wave equation:

$$\Box_0 W_a(t) = -F_a(t), \quad W_a(0) = 0, \quad \partial_t W_a(0) = 0. \tag{4.6.14}$$

We will use $W_a(t)$ to cancel most of the second term on the right in (4.6.11).

We first observe that (4.6.9) shows that for any $g \in L^2$,

$$F_a(t)g(x) \in C^\infty(\mathbb{R}_t, L^2(\mathbb{R}^n_x)), \;\; \operatorname{supp} F_a(t)g(x) \subset \{(t,x) : |x|+T_a \leq |x|+T_a+1\}.$$

Consequently, the Duhamel formula

$$[W_a(t)g](x) = -\int_0^t U_0(t-s,x,y)[F_a(s)g](y)dyds, \tag{4.6.15}$$

and (4.6.3) show that

$$[W_a(t)g](x) \in C^\infty(\mathbb{R}_t, L^2(\mathbb{R}^n)). \tag{4.6.16}$$

(In fact $[W_a(t)g(x)]$ is smooth but that is not important here.) We claim that

$$\operatorname{supp}[W_a(t)g](x) \subset \{(x,t) : |x| + T_a \leq t \leq |x| + C_a\}. \tag{4.6.17}$$

The lower bound on t in the support comes from the support properties of F_a and the finite speed of propagation. Hence we only need to check the upper bound.

8. To establish (4.6.17) we rewrite $W_a(t)$ as follows

$$W_a(t) = \zeta_a H(t) U_V(t) \chi_a - H(t) U_0(t) \chi_a + Q_a(t). \tag{4.6.18}$$

The first two terms satisfy the support property (4.6.17) with $C_a = T_a + 1$ and $C_a = |a|$ respectively.

The last term satisfies $Q_a(t) \equiv 0$ for $t \leq 0$, and

$$\begin{aligned} \Box_0 Q_a(t) &= \Box_0 W_a(t) - \Box_V(\zeta_a H(t) U_V(t) \chi_a) + V \zeta_a H(t) U_V(t) \chi_a \\ &\qquad - \Box_0 (H(t) U_0(t) \chi_a) \\ &= -F_a(t) + [\Box_V, \zeta_a] U_V(t) \chi_a + V \zeta_a H(t) U_V(t) \chi_0 \\ &= V \zeta_a H(t) U_V(t) \chi_a. \end{aligned} \tag{4.6.19}$$

Here we used the definition of $F_a(t)$ in (4.6.8) and the fact that

$$\Box_V(H(t) U_V(t) \chi_a g) = \Box_0(H(t) U_0(t) \chi_a g) = \delta(t) \chi_a g, \quad g \in L^2.$$

In view of (4.6.7) the right hand side of (4.6.19) is compactly supported in both x and t. The sharp Huyghens principle (4.6.3) and the Duhamela formula (see (4.6.15)) show

$$Q_a(t) = -\int_0^t U_0(t-s) V \zeta_a U_V(s) \chi_a ds,$$

then show that for any $g \in L^2$, (recall that $\operatorname{supp} V \subset B(0,a)$)

$$\begin{aligned} \operatorname{supp}[Q_a(t) g(x)] &\subset \{(t,x) : \exists\, |y| \leq a,\ 0 \leq s \leq |y| + T_a + 1,\ |x-y| = t - s\} \\ &\subset \{(t,x) : 0 \leq t \leq 2a + |x| + T_a + 1\}. \end{aligned}$$

Hence (4.6.17) holds with $C_a = 2a + T_a + 1$.

In particular this means that $\mathcal{F}^*_{t\to\lambda}(W_a(t)g)$ is well defined. (This is the crucial part of the argument: the compact support of the perturbation – $V \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ – forces the compact support in time of $W_a(t)$ which compares the truncated perturbed evolution with the free evolution.)

9. We now put

$$R_a^\#(\lambda) g := \widetilde{R}_a(\lambda) g + \mathcal{F}^*_{t\to\lambda}(W_a(t)g), \quad g \in L^2,$$

where $\widetilde{R}_a(\lambda)$ was defined in (4.6.10). If

$$\chi_a = 1 \quad \text{on } \operatorname{supp} V$$

(which is possible since $\operatorname{supp} V \subset B(0, a)$) then

$$\begin{aligned}(-\Delta + V - \lambda^2) R_a^\#(\lambda) g &= \chi_a g + V \mathcal{F}^*_{t\to\lambda}(W_a(t) g) \\ &= \chi_a (I + K_a(\lambda)) g,\end{aligned}$$

where $K_a(\lambda) := V \mathcal{F}^*_{t\to\lambda}(W_a(t))$.

10. In view of the definitions of $\widetilde{R}_a$ and $R_a^\#(\lambda)$, (4.6.2) will follow once we show the following estimates

$$\|\mathcal{F}^*_{t\to\lambda} \zeta_a H(t) U_V(t) \chi_a\|_{L^2\to L^2} \le C_0 \langle\lambda\rangle^{-1} e^{C_0(\operatorname{Im}\lambda)_-}, \quad \chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n), \tag{4.6.20}$$

$$\|\chi \mathcal{F}^*_{t\to\lambda}(W_a(t))\|_{L^2\to L^2} \le C_0 \langle\lambda\rangle^{-1} e^{C_0(\operatorname{Im}\lambda)_-}, \tag{4.6.21}$$

$$\|\mathcal{F}^*_{t\to\lambda}(V W_a(t))\|_{L^2\to L^2} \le C_N \langle\lambda\rangle^{-N} e^{C_1(\operatorname{Im}\lambda)_-}, \tag{4.6.22}$$

where C_0 depends on χ and a and C_1 depends on a, and N is arbitrary.

11. We start with the last two bounds. In fact, a stronger bound than (4.6.21) is valid: for any $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$,

$$\|\chi \mathcal{F}^*_{t\to\lambda}(W_a(t))\|_{L^2\to L^2} \le C_N \langle\lambda\rangle^{-N} e^{C_1(\operatorname{Im}\lambda)_-}.$$

In fact, the regularity property (4.6.16) and the support property (4.6.17) imply that $\chi W_a(t) \in C^\infty_{\mathrm{c}}((0,\infty); \mathcal{L}(L^2, L^2))$, and the bound on the Fourier transform follows. This also implies (4.6.22) as we can take $\chi = V$.

12. To prove (4.6.20) we use an argument similar to that in the proof of Theorem 3.1. We first note that

$$U_V(t), \partial_t U_V(t) = \mathcal{O}(\exp C|t|)_{L^2\to L^2}$$

where the exponential growth is due to the possible presence of negative eigenvalues of $-\Delta + V$. Hence

$$\chi \mathcal{F}^*_{t\to\lambda}(\zeta_a H(t) U_V(t) \chi_a) = \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-})_{L^2\to L^2},$$

where C depends on χ and a. We also have (since $U_V(0) = 0$)

$$\begin{aligned} i\lambda \chi \mathcal{F}^*_{t\to\lambda}(\zeta_a H(t) U_V(t) \chi_a) &= -\mathcal{F}^*_{t\to\lambda}(\zeta_a H(t) \partial_t U_V(t) \chi_a) \\ &= \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-})_{L^2\to L^2},\end{aligned}$$

from which (4.6.20) follows. This completes the proof of (4.41). □

We now move to the general black box case. The proof follows the same strategy but we need more cut-off functions. All of them will be either identically 0 or 1 in a neighbourhood of the black box.

We also require an abstract condition which will replace (4.6.2). Let P be a black box Hamiltonian in the sense of Definition 4.1. Since it is a self-adjoint operator we use the spectral theorem to define

$$U(t) := \frac{\sin t\sqrt{P}}{\sqrt{P}}. \tag{4.6.23}$$

We assume here that $h = 1$.

DEFINITION 4.42 (Non-trapping black box). *Suppose that P is a black box Hamiltonian and that $U(t)$ is given by (4.6.23). We say that P is* non-trapping *if*

$$P \geq -C \tag{4.6.24}$$

for some C and if the following condition holds:

$$\begin{gathered} \forall\, a > R_0 \quad \exists\, T_a \quad \forall\, \chi \in C_c^\infty(B(0,a)),\ \chi|_{B(0,R_0+\varepsilon)} \equiv 1, \\ \chi U(t) \chi|_{t > T_a} \in C^\infty((T_a, \infty); \mathcal{L}(\mathcal{H}, \mathcal{D})), \end{gathered} \tag{4.6.25}$$

where the space $\mathcal{D}$ is the domain of P.

EXAMPLE. Suppose that $P = -\Delta_g$ where g is a Riemannian metric on $\mathbb{R}^n$, n odd, with the property that $g^{ij} - \delta_{ij} \in C_c^\infty(B(0, R_0))$. Suppose that the metric is *classically non-trapping* that is,

$$\begin{gathered} \forall\, (x,\xi) \in T^*\mathbb{R}^n \setminus 0 \quad \pi(\exp tH_p(x,\xi)) \to \infty, \quad t \to \pm\infty, \\ p(x,\xi) = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j. \end{gathered}$$

Here $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the natural projection and H_p is the Hamilton vector field of p – see Proposition E.5.

In particular this implies that for any $a > 0$ there exists T_a such that

$$|x| < a,\ p(x,\xi) = 1,\ |t| > T_a \implies |\pi((\exp tH_p)(x,\xi))| > a.$$

Hence the result on propagation of singularities – see Theorem E.47 – shows that for $\chi \in C_c^\infty(B(0,a))$ and any $N > 0$.

$$\chi(\sin t\sqrt{-\Delta_g}/\sqrt{-\Delta_g})\chi \in C^\infty((T_a,\infty); \mathcal{L}(L^2(\mathbb{R}^n), H^N(\mathbb{R}^n)).$$

This means that classical non-trapping for the metric g implies non-trapping for the propagator in the sense of Definition 4.42.

We can now state a theorem relating propagation of singularities to resonance region for general black box Hamiltonians in odd dimensions. The proof follows the same idea as the proof of Theorem 4.41 but with more cut-offs related to the abstract black box.

THEOREM 4.43 (**Non-trapping estimates for black box Hamiltonians**). *Suppose that P is a black box Hamiltonian in the sense of Defintion 4.1 and $R(\lambda) : \mathcal{H}_{\rm comp} \to \mathcal{D}_{\rm loc}$ is the meromorphically continued resolvent.*

If P is non-trapping *in the sense of Definition 4.42 then for any M there exists C_0 such that $R(\lambda)$ is holomorphic in*

$$\Omega_M := \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > -M \log |\lambda|, \ |\lambda| > C_0\}. \tag{4.6.26}$$

Moreover, for any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near $B(0, R_0)$ (the black box) there exist C_1 and T such that

$$\|\chi R(\lambda)\chi\|_{\mathcal{H}\to\mathcal{H}} \le C_1 |\lambda|^{-1} e^{T(\operatorname{Im}\lambda)_-}, \quad \lambda \in \Omega_M. \tag{4.6.27}$$

REMARKS. 1. The estimate (4.6.27) can be improved to an estimate valid between $\mathcal{H}$ and $\mathcal{D}^\alpha$, $\alpha = 0, \frac{1}{2}, 1$ – see (4.6.44). That is important for obtaining resonance expansions for data in natural spaces.

2. The condition (4.6.25) seems weaker that the condition

$$\chi U(t) \chi \in C^\infty((T_a, \infty); \mathcal{L}(\mathcal{H}, \mathcal{D}^N)), \tag{4.6.28}$$

for all N. However, differentiation of (4.6.23) with respect to t and changing the cut-offs χ shows that (4.6.27) implies the seemingly stronger statement (4.6.28).

3. As observed in [**Sj02**, §3] condition (4.6.25) can be weakened to demanding that

$$\chi U(t) \chi \in C^\infty((T_a, T_a + c); \mathcal{L}(L^2, \mathcal{D}))$$

for some $c > 0$. That can already be seen in the proof of Theorem 4.41: we only use (4.6.5) and for that only smoothness for in $(T_a, T_a + c)$ for some c is needed. See [**Sj02**, §3] for a slightly different argument.

4. As pointed out in [**BW13**] the proof below shows that C^∞ in (4.6.27) can be replaced by C^k for any $k \ge 0$. In the case of $k = 0$ we obtain a resonance free region of the form $\operatorname{Im} \lambda \ge -C$, $|\lambda| > C$ for any C – see Step 6 of the proof: the Fourier transform of a continuous compactly supported function is $o(1) e^{C|\operatorname{Im}\lambda|}$, as $|\operatorname{Re} \lambda| \to \infty$. When $k > 0$ then we obtain a logarithmic strip $\operatorname{Im} \lambda \ge -M_0 \log |\lambda|$, $|\lambda| \ge C_0$ for some fixed M_0.

Proof. 1. The operator $U(t)$ defined in (4.6.23) has the mapping property

$$U(t) : D^\alpha \longrightarrow \mathcal{D}^{\alpha+\frac{1}{2}}, \quad U(t) \in C(\mathbb{R}_t, \mathcal{L}(\mathcal{D}^\alpha \mathcal{D}^{\alpha+\frac{1}{2}})), \tag{4.6.29}$$

and it solves the equation

$$\Box U(t) = 0, \quad U(0) = 0, \quad \partial_t U(0) = I_{\mathcal{H}}, \quad \Box := \partial_t^2 + P.$$

Suppose that a is chosen large enough so that $\operatorname{supp}\chi \subset B(0,a)$ and let $\chi_a \in C_c^\infty(B(0,a))$ be equal to 1 on the support of χ. Let $\psi_a \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\psi_a \cap (\mathbb{R}\times\mathbb{R}^n \setminus B(0,R_0)) \subset \{(t,x) : |x| + T_a \le t \le |x| + T_a + 1\},$$
$$\psi_a|_{\mathbb{R}\times B(0,R_0+\varepsilon)} = \psi_a^0(t), \quad \psi_a^0 \in C_c^\infty(T_a + R_0, T_a + R_0 + 1),$$

see Figure 4.5. In particular, $u \mapsto \psi_a u$ is well defined as an operator from $C^\infty(\mathbb{R}, \mathcal{H}_{\mathrm{loc}})$ to intself.

Using (4.6.25) we see that

$$\psi_a U(t)\chi_a \in C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}, \mathcal{D})). \tag{4.6.30}$$

In fact, let $\chi_0, \chi_1 \in C_c^\infty(B(0,a))$ satisfy $\chi_0 \equiv 1$ on $\operatorname{supp}\chi_a$ and $\chi_1 \equiv 1$ on $\operatorname{supp}\chi_0$. Then (4.6.25) gives

$$\chi_1 U_V(t)\chi_1|_{t>T_a} \in C^\infty((T_a,\infty); \mathcal{L}(L^2, \mathcal{D})). \tag{4.6.31}$$

Also,

$$\Box_0\left((1-\chi_0)U_V(t)\chi_a\right) = -[\Box_0, \chi_0]\chi_1 U_V(t)\chi_a|_{t>T_a} =: G_a(t)$$
$$G_a(t)|_{t>T_a} \in C^\infty((T_a,\infty), \mathcal{L}(\mathcal{H}, H^1_{\mathrm{comp}}(B(0,a)\setminus\overline{B(0,R_0)}\,)),$$

(the support restriction comes from $-[\Box_0,\chi_0] = [\Delta,\chi_0]$and

$$(1-\chi_0)U_V(0)\chi_a = 0, \quad (1-\chi_0)\partial_t U_V(0)\chi_a = 0.$$

The Duhamel formula and the support properties of U_0 (4.6.3) and $G_a(t)$ give

$$\begin{aligned} \psi_a(1-\chi_0)U_V(t)\chi_a &= \psi_a \int_0^t U_0(t-s)\left[G_a(s)|_{s>T_a}\right] ds \\ &\in C^\infty((T_a,\infty), \mathcal{L}(\mathcal{H}, \mathcal{D})), \end{aligned} \tag{4.6.32}$$

see Figure 4.5. (Here we used (4.6.29) with $\alpha = \frac{1}{2}$.)

Since $(1-\chi_0)(1-\chi_1) = (1-\chi_1)$ and $\psi_a\chi_1|_{t>T_a} = \psi_a\chi_1$, (4.6.31) and (4.6.32) show (4.6.30).

2. We now modify the function $\zeta_a \in C^\infty$ in (4.6.7) so that it is independent of x in $B(0,R_0)$:

$$\begin{aligned} \zeta_a(x,t)|_{|x|>R_0+2\varepsilon} &= \begin{cases} 1 & t \le |x| + T_a, \\ 0 & t \ge |x| + T_a + 1, \end{cases} \\ \zeta_a(x,t)|_{|x|<R_0+\varepsilon} = \zeta_a^0(t) &= \begin{cases} 1 & t \le R_0 + \varepsilon + T_a, \\ 0 & t \ge R_0 + \varepsilon + T_a + 1. \end{cases} \end{aligned} \tag{4.6.33}$$

Using ζ_a we put

$$F_a(t) := [\Box, \zeta_a]U(t)\psi_a \in C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}, \mathcal{D}^{\frac{1}{2}})), \tag{4.6.34}$$

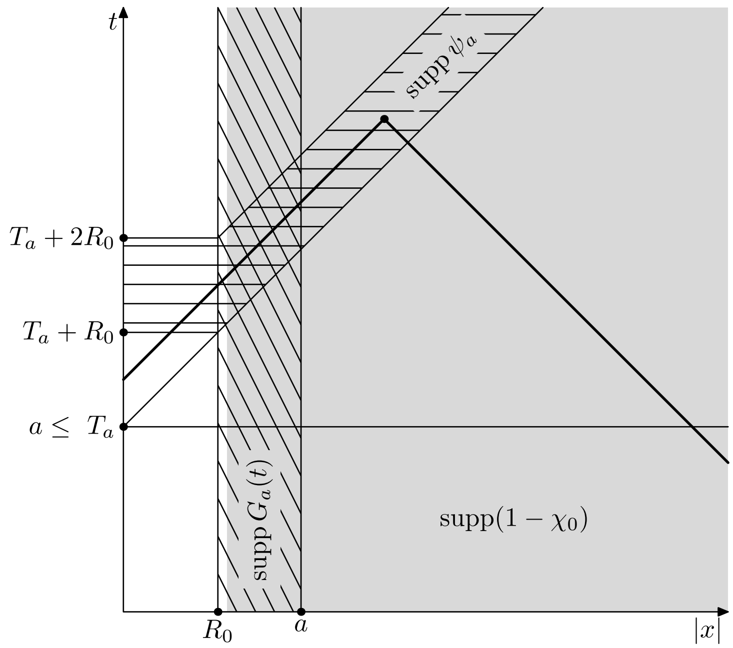


Figure 4.5. Supports of ψ_a, $1 - \chi_0$ and $G_a(t)$ showing the validity of (4.6.32). The cone represents the support of $U(t-s)$ in the Duhamel formula.

where the regularity comes from (4.6.30) and we change from $\mathcal{D}$ to $\mathcal{D}^{\frac{1}{2}}$ (in fact H^1 since the cut-offs move us outside of the "black box") because of the first order operator $[\Box, \zeta_a]$ – see (4.1.9) and (4.1.10).

3. Now choose $\chi_b \in C_c^\infty(B(0,a))$ such that $\chi_b = 1$ near $B(0, R_0)$ and $\chi_a \equiv 1$ on $\operatorname{supp}\chi_b$ and solve

$$\Box_0 W_a(t) = -(1-\chi_b)F_a(t), \quad W_a(t) \equiv 0, \quad t \leq 0. \tag{4.6.35}$$

We then proceed as in Step 7 of Theorem 4.41 to see that for $g \in \mathcal{H}$,

$$\begin{gathered} \operatorname{supp}[W_a(t)g](x) \subset \{(x,t) : |x| + T_a \leq t \leq |x| + C_a\}, \\ [W_a(t)g](x) \in C^\infty(\mathbb{R}_t, \mathcal{D}^{\frac{1}{2}}). \end{gathered} \tag{4.6.36}$$

The smoothness statement comes from (4.6.34) and to see the support property we write

$$W_a(t) = (1-\chi_b)\zeta_a H(t)U(t)\chi_a - H(t)U_0(t)(1-\chi_b)\chi_a + Q_a(t),$$

where $H(t)$ is the Heaviside function and where $Q_a(t)$ solves

$$\begin{aligned}\Box_0 Q_a(t) &= \Box_0 W_a(t) - (1-\chi_b)\Box(\zeta_a H(t)U(t)\chi_a) \\ &\qquad + [\Box_0, \chi_b]\zeta_a H(t)U(t)\chi_a - \Box_0(H(t)U_0(t)(1-\chi_b)\chi_a) \\ &= -(1-\chi_b)F_a(t) + (1-\chi_b)(\chi_a + F_a(t)) \\ &\qquad + [\Box_0, \chi_b]\zeta_a H(t)U(t)\chi_a(1-\chi_b)\chi_a - \chi_a(1-\chi_b)\chi_a \\ &= -[\Delta, \chi_b]\zeta_a H(t)U(t)\chi_a.\end{aligned}$$

For any $g \in \mathcal{H}$,

$$[\Delta, \chi_b]\zeta_a H(t)U(t)\chi_a \in L^2(B(0,a) \setminus B(0,R_0)),$$

that is the support is contained in a fixed compact subset of $\mathbb{R}_t \times \mathbb{R}^n_x$. (The compactness in t comes from ζ_a.) The sharp Huyghens principle (see Step 8 of the proof of Theorem 4.41) then shows that

$$\operatorname{supp}[Q_a(t)g](x) \subset \{(t,x) : 0 \le t \le C_a + |x|\}.$$

The lower bound on t in (4.6.36) follows from the support property of $(1-\chi_b)F_a(t)$ and the final speed of propagation.

5. The support property in (4.6.36) allows us to define the following approximation of the resolvent $R(\lambda)\chi_a$: choose $\chi_c \in C^\infty_{\rm c}(B(0,a))$ equal to 1 near $B(0,R_0)$ and such that $\chi_b = 1$ on $\operatorname{supp}\chi_c$ and put

$$R^\#_a(\lambda) = \mathcal{F}^*_{t\to\lambda}\left(\zeta_a H(t)U(t)\chi_a + (1-\chi_c)W_a(t)\right). \tag{4.6.37}$$

We note that $R^\#_a(\lambda) : \mathcal{H} \to \mathcal{D}$ for $\operatorname{Im}\lambda > 0$ except for poles given by a discrete set of eigenvalues – see Theorem 4.5, (4.6.24) and (4.6.6).

4. We calculate

$$(P-\lambda^2)R^\#_a(\lambda) = \chi_a(I + K_a(\lambda)), \tag{4.6.38}$$

where, using (4.6.35),

$$K_a(\lambda) := \mathcal{F}^*_{t\to\lambda}\left(\chi_b F_a(t) + [\Delta, \chi_c]W_a(t)\right).$$

We prove (4.6.25) by showing that

$$\chi R^\#_a(\lambda) = \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{-1})_{\mathcal{H}\to\mathcal{H}}, \tag{4.6.39}$$

$$K_a(\lambda) = \mathcal{O}_N(e^{C(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{-N})_{\mathcal{H}\to\mathcal{H}}, \tag{4.6.40}$$

for any N.

In fact, for $\operatorname{Im}\lambda > 0$, in view of mapping properties of $R^\#_a(\lambda)$ discussed in Step 3, we have

$$R(\lambda)\chi_a = R^\#_a(\lambda)(I + K_a(\lambda))^{-1}, \tag{4.6.41}$$

and by analytic continuation in holds in Ω_M: the bound (4.6.40) implies invertibility of $I + K_a(\lambda)$ there.

5. To obtain (4.6.39) we proceed as in Step 12 of the proof of Theorem 4.41. Since P is bounded from below (see (4.6.24)) the functional calculus of self-adjoint operators implies that

$$U(t), \partial_t U(t) = \mathcal{O}(\exp C|t|)_{\mathcal{H}\to\mathcal{H}}.$$

Hence

$$\chi \mathcal{F}^*_{t\to\lambda}(\zeta_a H(t) U_V(t)\chi_a) = \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-})_{\mathcal{H}\to\mathcal{H}},$$

where C depends on χ and a. We also have (since $U(0) = 0$)

$$\begin{aligned} i\lambda\chi\mathcal{F}^*_{t\to\lambda}(\zeta_a H(t)U(t)\chi_a) &= -\mathcal{F}^*_{t\to\lambda}(\zeta_a H(t)\partial_t U(t)\chi_a) \\ &= \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-})_{\mathcal{H}\to\mathcal{H}}. \end{aligned}$$

The second term in $R^\#_a(\lambda)$ satisfies an even better bound in view of (4.6.36): $\mathcal{F}^*_{t\to\lambda}(\chi W_a(t)) = \mathcal{O}(e^{C(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{-N})_{\mathcal{H}\to\mathcal{H}}$. This proves (4.6.39).

6. It remains to show (4.6.40). Since the cut-offs χ_b and $[\Delta, \chi_c]$ restrict t to a compact set smoothness in t – from (4.6.34) and (4.6.36) – we obtain the estimates $\mathcal{O}(e^{C(\operatorname{Im}\lambda)_-}\langle\lambda\rangle^{-N})_{\mathcal{H}\to\mathcal{H}}$. (It is here that we need $\mathcal{D}$ rather than $\mathcal{D}^{\frac12}$ in (4.6.25) as we need to apply the first operator $[\Delta, \chi_c]$ to $W_a(t)$.) That completes the proof. $\square$

We can state a theorem about the expansion of scattered waves. The proof follows the same lines as the proof of Theorem 2.9.

THEOREM 4.44 (Resonance expansions for non-trapping black box Hamiltonians.). *Suppose that P is a non-trapping black box Hamiltonian in the sense of Definition 4.42 and that that $w(t)$ is the solution of*

$$(\partial_t^2 + P)w(t) = 0\,, \quad w(0) = w_0 \in \mathcal{D}^{\frac12}_{\text{comp}}\,, \quad \partial_t w(0) = w_1 \in \mathcal{H}_{\text{comp}}\,.$$

Then, for any $A > 0$,

$$w(t) = \sum_{\substack{\lambda_j \in \operatorname{Res}(P) \\ \operatorname{Im}\lambda_j > -A}} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^j e^{-i\lambda_j t} f_{j,\ell} + E_A(t)\,, \tag{4.6.42}$$

where sum is finite,

$$\sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell} = \operatorname{Res}_{\mu=\lambda_j}\left((iR(\mu)w_1 + \lambda R(\mu)w_0)\, e^{-i\mu t}\right)\,, \tag{4.6.43}$$

$$(P - \lambda_j)^{\ell+1} f_{j,\ell} = 0\,,$$

and for any $K > 0$*, such that* $\operatorname{supp} w_j \subset B(0, K)$*, there exist constants* $C_{K,A}$ *and* $T_{K,A}$

$$\|E_A(t)\|_{\mathcal{D}} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{\mathcal{D}^{\frac{1}{2}}} + \|w_1\|_{\mathcal{H}} \right), \quad t \geq T_{K,A}.$$

Proof. To repeat the proof of Theorem 2.9 we need the following improvement of the estimate (4.6.25):

$$\|\chi R(\lambda) \chi\|_{\mathcal{H} \to \mathcal{D}^\alpha} \leq C \langle \lambda \rangle^{\alpha/2 - 1} e^{C (\operatorname{Im} \lambda)_-}, \quad \alpha = 0, 1, 2. \tag{4.6.44}$$

From (4.6.41) we see that it is enough to prove these estimates with $R(\lambda)\chi$, $\operatorname{supp} \chi \subset B(0, a)$, replaced by $R_a^\#(\lambda)$, where $R_a^\#(\lambda)$ is given by (4.6.37) with $\chi_a = \chi$. But this follow from the support and mapping properties of $\zeta_a U(t) \chi_a$ and $W_a(t)$ following the argument in the proof of (3.1.12) in Theorem 3.1 with $-\Delta$ replaced by P. □

4.7. NOTES

The black box formalism was introduced in [**SZ91**]. The presentation in §4.2 comes from that paper with additional improvements, including (4.2.13) and the proof of Theorem 4.7, from Sjöstrand [**Sj02**].

Theorem 4.13 comes essentially from [**SZ91**] but the proof follows Vodev [**Vo92**] and Petkov–Zworski [**PZ01**] and is based on the same ideas as the proof of Theorem 3.27. The case of global bounds in even dimensions is proved in Vodev [**Vo94a**],[**Vo94b**] (see Intissar [**In86**] for an earlier contribution) while semiclassical bounds valid for a large class of operators (including some long range perturbations of $-h^2\Delta$) are given in Sjöstrand [**Sj96a**].

The scattering theoretical interpretation of spectral theory on finite volume hyperbolic quotients goes back to Selberg and was made explicit by Fadeev–Pavlov [**FP72**] and Lax–Phillips [**LP76**]. The pseudo-Laplacian was introduced by Colin de Verdière [**CdV83**] and was used to show generic absence of embedded eigenvalues for variable curvature surfaces with cusps. The Fermi Golden rule in that setting was discovered by Phillips–Sarnak [**PS85**] and led to much work – see Hillairet–Judge [**HJ18**] for recent progress and references and [**LZ16**] for a version in which a boundary condition changes. For an early mathematical treatment of the the Fermi Golden rule see Simon [**Si73**] and for more recent developments, for instance Soffer–Weinstein [**SW98**] and Jensen–Nenciu [**JN06**]. For more general perturbations and continuity statements about resonances see Stefanov [**St94**].

The derivation of the scattering matrix for the modular surfaces combines the classical approach of Titchmarsh and Heath-Brown [**Ti86**, Notes for Chapter II] with the black box approach. That in essence is Colin de Verdière's proof of the meromorphic continuation of Eisenstein series [**CdV81b**].

The method of *complex scaling* originated in the work of Aguilar-Combes [**AC71**], Balslev-Combes [**BC71**] and was developed by Simon [**Si72**],[**Si73**], [**Si79a**], Hunziker [**Hu86**], Helffer-Sjöstrand [**HS86**], Hislop–Sigal [**HS89**] (see also [**HS96**]) and other authors. For compactly supported black box perturbations (and large θ) it was introduced in Sjöstrand–Zworski [**SZ91**] while an adaptation to the case of long range black box perturbations was provided in Sjöstrand [**Sj96a**]. In our presentation we opted for a quick approach which benefits from the precise knowledge of the resolvent of the free Laplacian. A more systematic approach is based on the theory of differential operators with analytic coefficients – see [**SZ91**] and [**Sj02**]. For an adaptation to asymptotically euclidean manifold satisfying certain (strong) analyticity conditions at infinity see Wunsch–Zworski [**WZ00**].

The complex scaling method has been extensively used in computational chemistry – see Reinhardt [**Re07**] for a review. As the method of *perfectly matched layers* it reappeared in numerical analysis – see Berenger [**Be94**].

Another computational technique for scattering resonances is the method of *complex absorbing potentials* (CAP) – see Seideman–Miller [**SM92**], Riss–Meyer [**RM95**] for the original presentation and Jagau at al [**J*14**] for more recent developments. The method is based on replacing P by

$$P_\varepsilon := P - i\varepsilon(1-\chi)x^2, \tag{4.7.1}$$

where $\chi \equiv 1$ near the black box. The potential $(1-\chi)x^2$ in (4.7.1), or another potentials with similar properties is a CAP. The operator P_ε has discrete spectrum and as $\varepsilon \to 0+$ the eigenvalues with $\arg z > -\pi/4$ tend to resonances uniformly on compact sets – see Zworski [**Zw18**]. This definition using "viscosity limits" is similar to the definition of Pollicott–Ruelle resonances obtained by adding $i\varepsilon\Delta$ to the generator of an Anosov flow – see Dyatlov–Zworski [**DZ15**] and Drouot [**Dr17**]. A relation of resonances to eigenvalues of a fixed CAP (P_ε for a fixed ε) was investigated mathematically by Stefanov [**St05**].

Theorem 4.39 is a slight generalization of a result of Klopp–Zworski [**KZ95**]. It extends to the case of resonances the now classical result of Uhlenbeck [**Uh76**] for eigenvalues of the Laplacian on a compact manifold. Instead of complex scaling one could use Agmon's theory of resonance perturbations [**Ag98**] as was done by Borthwick–Perry [**BP02**]. That extends applicability of the method to, for instance, scattering on asymptotically hyperbolic manifolds – see §5.1.

The presentation of resolvent estimates for non-trapping black boxes in §4.6 is based on Tang–Zworski [**TZ00**, §3] but the method is due to Vaĭnberg [**Va73**],[**Va89**]. See these references for the case of even dimensions. For a different presentation see Sjöstrand [**Sj02**, §3] and for some recent applications Baskin–Wunsch [**BW13**], Baskin–Spence–Wunsch [**BSW16**] and Galkowski [**Ga17**]. Another point of view on linking propagation of singularities with resolvent estimates and energy decay is given by the Lax–Phillips theory – see [**LP68**] and for a "black box" presentation [**SZ94**]. One of the main applications is to obstacle problems where propagation of singularities was established by Andersson, Ivrii, Melrose, Lebeau, Sjöstrand and Taylor – see Hörmander [**HöIII**, Chapter 24] and references given there. Earlier results for star shaped obstacle but with geometric bounds on the distance of resonances to the real axis were obtained by Morawetz [**Mo61**], Ralston [**Ra78**], see also Morawetz–Ralston–Strauss [**MRS77**]. That non-trapping condition is *necessary* for uniform energy decay in obstacle scattering is a classical result of Ralston [**Ra69**]. For energy decay of solutions to the (conformal) wave equation for obstacles in hyperbolic space see Hintz–Zworski [**HZ18**].

For non-trapping obstacles with analytic boundaries and for strictly convex obstacles with smooth boundaries larger, $|\operatorname{Im}\lambda| \leq (\operatorname{Re}\lambda)^{\frac{1}{3}}/C - C$, resonance free regions were established by Lebeau [**Le84**], Popov [**Po85**], Bardos–Lebeau–Rauch [**BLR87**], Hargé–Lebeau [**HL94**] and Sjöstrand–Zworski [**SZ95**]. Convex obstacles with smooth boundaries provide one of the rare instances in which asymptotics for the number of resonances are possible [**SZ99**]. That in particular, shows optimality of the cubic resonance free regions. See also Jin [**Ji15**],[**Ji14**] for more general boundary conditions.

For recent advances on resonance free regions for transmission problems see Galkowski [**Ga14**],[**Ga16**],[**Ga19**] and references given there.

4.8. EXERCISES

Section 4.1

1. Suppose $\mathcal{O}$ is bounded region with a smooth boundary $\Gamma := \mathcal{O}$. Suppose that $V : L^2(\Gamma) \to L^2(\Gamma)$ is a bounded operator. Show that

$$P := -\Delta + V \otimes \delta_\Gamma$$

satisfies the black box hypothesis. Construct Γ and V so that P has embedded eigenvalues. (See Galkowski–Smith [**GS15**] for more information and deeper analysis.)

2. Define a Hilbert space $H := \bigoplus_{k=1}^{\infty} L^2([0,\infty))$ with the norm $\|\{b_k\}_{k=1}^{\infty}\|_H^2 := \sum_{k=1}^{\infty} \|b_k\|^2_{L^2([0,\infty))}$. For $G(s)$, a continuous function on $[0,\infty)$, define

$$H_G := \left\{ \{b_k\}_{k=1}^{\infty} \ : \ \sum_{k=1}^{\infty} \int_0^{\infty} (|\partial_s b_k(s)|^2 + k^2 e^{G(s)} |b_k(s)|^2) ds < \infty \right\}.$$

Show that if $G(s) \to \infty$ as $s \to \infty$ then

$$H_G \hookrightarrow H \text{ is a compact inclusion.}$$

In particular this proves the compactness of (4.1.21).

Section 4.2

3. Prove (4.2.29). Here are suggested steps, see also [**HZ17**, Proof of (4.3)].

(a) Show that for a fixed f in (4.2.27), $\lambda \mapsto u \in C(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^n))$ is a holomorphic function of $\lambda \in \mathbb{C}$. Conclude that it is enough to prove (4.2.29) for $\operatorname{Im} \lambda > 0$.

(b) Use the Fourier transform to show that if $\tau \in \mathbb{R}$ (for $f \in C_c^\infty(\mathbb{R}^n)$ and in the distributional sense for $f \in \mathcal{E}'(\mathbb{R}^n)$)

$$\mathcal{F}_{t\to\tau} u(\tau, x) = \frac{1}{2(2\pi)^{n-1}} \sum_{\pm} \int_{\mathbb{S}^{n-1}} e^{\pm i\tau\langle \omega, x\rangle} \frac{\hat{f}(\pm\tau\omega)}{\tau^2 - \lambda^2} (1 - \lambda/\tau)(\pm\tau)_+^{n-1} d\omega,$$

where $\mathcal{F}_{t\to\tau}$ is the Fourier transform in t. This formula holds in any dimension.

(c) Conclude that for $n \geq 3$ and odd

$$\mathcal{F}_{t\to\tau} u(\tau, x) = \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} e^{i\tau\langle \omega, x\rangle} \frac{\hat{f}(\tau\omega)}{\tau + \lambda} \tau^{n-2} d\omega,$$

and hence $\tau \mapsto \mathcal{F}_{t\to\tau} u(\tau, x)$ extends to a holomorphic function in $\operatorname{Im} \tau > -\operatorname{Im} \lambda$.

(d) Use the Schwartz-Paley–Wiener theorem [**HöI**, Theorem 7.3.1] applied to f to prove that

$$|\mathcal{F}_{t\to\tau} u(\tau, x)| \leq C\langle\tau\rangle^M e^{\operatorname{Im}\tau(|x|+R)}, \quad \operatorname{Im}\tau > -\operatorname{Im}\lambda + \varepsilon$$

for some C and M depending on $\varepsilon > 0$. Use the other direction of the Schwartz-Paley–Wiener theorem (applied to $\mathcal{F}_{t\to\tau} u(\tau, x)$) to obtain (4.2.29).

Section 4.4

4. Using the proof of Theorem 3.35 justify Theorem 4.17.

5. Prove Theorem 4.20 using the proof of Theorem 3.47 as a blueprint. That proof can be read independently of the rest of §3.9.

6. Suppose that (4.4.11) holds. For $S(\lambda)$ given in Definition 4.25 and the generalized plane wave given by (4.4.1) show that

$$S(\lambda)e(-\lambda,\bullet) = e(\lambda,\bullet), \tag{4.8.1}$$

in the sense that for $f \in \mathcal{H}_{\text{comp}}$,

$$S(\lambda)\left(\langle e(-\lambda,\bullet), f\rangle\right) = \langle e(\lambda,\bullet), f\rangle, \quad \langle e(-\lambda,\bullet), f\rangle \in C^\infty(\mathbb{S}^{n-1}).$$

7. Show that if $\lambda^2 > 0$ then $m_R(\lambda) = m_R(-\lambda)$.

Section 4.5

8. Use Theorem 4.32 and invertibility of

$$e^{-2i\theta}\Delta - \lambda^2 = e^{-2i\theta}(\Delta - (e^{i\theta}\lambda)^2) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \operatorname{Im}(e^{i\theta}\lambda) > 0,$$

to show directly (that is, without using the explicit calculations of §4.5.2) that

$$-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta),$$

is a Fredholm operator for $\operatorname{Im}(e^{i\theta}\lambda) > 0$.

9. Show (4.5.30) directly using (4.5.13) and the expression $dw = \det(I + F_\theta''(y))dy$.

Chapter 5

SCATTERING ON HYPERBOLIC MANIFOLDS

5.1 Asymptotically hyperbolic manifolds
5.2 A motivating example
5.3 The modified Laplacian
5.4 Phase space dynamics
5.5 Propagation estimates
5.6 Meromorphic continuation
5.7 Applications to general relativity
5.8 Notes
5.9 Exercises

In this chapter, we show meromorphic continuation of the resolvent of the Laplacian on complete asymptotically hyperbolic Riemannian manifolds. A basic example is given by the hyperbolic space $\mathbb{H}^n$, which can be viewed as the open unit ball in $\mathbb{R}^n$ with the metric

$$g = 4\frac{dw^2}{(1-|w|^2)^2}, \quad w \in B_{\mathbb{R}^n}(0,1). \tag{5.0.1}$$

A larger family of examples is provided by *convex co-compact hyperbolic surfaces*, which are complete two-dimensional Riemannian manifolds of constant sectional curvature -1 whose infinite ends are *funnels*, that is they have the form

$$[0,\infty)_v \times \mathbb{S}^1_\theta, \quad \mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z},\ \ell > 0; \quad g = dv^2 + (\cosh v)^2 d\theta^2. \tag{5.0.2}$$

Convex co-compact hyperbolic surfaces can be viewed as the quotients of $\mathbb{H}^2$ by certain discrete subgroups of its isometry group $\mathrm{PSL}(2;\mathbb{R})$, and have applications in algebra and number theory [**Bo16**], [**BGS11**]. From the point of view of this book they give fundamental examples of *hyperbolic trapped sets* and are a model object to study the effects of hyperbolic trapping on distribution of resonances.

We take a geometric approach to scattering on hyperbolic manifolds and consider more general *asymptotically hyperbolic ends*, whose metrics approach (5.0.2) in a certain sense as $v \to +\infty$, and satisfy an additional evenness assumption. We follow a modified version of the microlocal approach of Vasy [**Va12**] – see [**Zw16**] for an introduction to the method in the non-semiclassical setting. Here we emphasize that in addition to establishing the meromorphic continuation of the resolvent, the method works well in the *semiclassical limit*. That makes the methods of Part 3 applicable in scattering on asymptotically hyperbolic manifolds and, as indicated in §5.7, black hole backgrounds.

On an asymptotically hyperbolic manifold of dimension n, the scattering resolvent is the meromorphic continuation of

$$\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \Big)^{-1} : L^2(M) \to L^2(M), \quad \operatorname{Im}\lambda > 0, \tag{5.0.3}$$

to the entire complex plane. Compared to Euclidean scattering, we have an additional $(n-1)^2/4$ term, which is related to the fact that the essential spectrum of $-\Delta_g$ is $[(n-1)^2/4, \infty)$. An explanation of this shift of the spectral parameter is provided below in (5.2.9).

In the Euclidean case, the method of complex scaling described in §4.5 provides meromorphic continuation of the resolvent by relating it to a resolvent of a *non-self-adjoint* operator, P_θ, such that $P_\theta - \lambda^2$ is a Fredholm operator for $\arg\lambda > -\theta$ (see Theorem 4.37). In the case of asymptotically hyperbolic manifolds the resolvent (5.0.3) is related to the inverse $P(\lambda)^{-1}$ for a family of operators $P(\lambda)$ – see (5.2.13) for a model case and Definition 5.11 for the general case. The main point is that $P(\lambda)$ is a family of Fredholm operators on suitably defined spaces – see Proposition 5.9 for the Fredholm property in a model case and Theorem 5.30 for the general case. Unlike the method of complex scaling in which resonances with $\arg\lambda > -\theta$ are identified with eigenvalues, λ^2, of P_θ, the Fredholm property works in strips: $P(\lambda)$ has the Fredholm property in $\{\operatorname{Im}\lambda > -C\}$ on spaces which depend on C.

The operators $P(\lambda)$ are constructed by *extending* the manifold across the boundary at infinity – see §5.2 for an explanation in a model case. The family $P(\lambda)$ then has the property of translating the outgoing condition for solutions to $(-\Delta_g - \lambda^2 - (n-1)^2/4)u = f$, $f \in C_c^\infty(M)$ to smoothness (across the boundary at infinity) of solutions to $P(\lambda)\tilde{u} = \tilde{f}$ – see Proposition 5.8. The price one pays is that P transitions from elliptic to hyperbolic behavior as one moves across the boundary at infinity – see Proposition 5.12. However, propagation results inspired by *radial estimates* of Melrose [**Me94**] (see §5.5) can be used to establish the Fredholm property.

This rather technically complicated chapter is structured as follows:

- in §5.1, we define asymptotically hyperbolic manifolds and study their geometric properties;
- in §5.2, we use an example to motivate the use of the modified spectral family of the Laplacian $P(\lambda)$;
- in §§5.3–5.5, we introduce the operator $P(\lambda)$ and use microlocal techniques to prove estimates on this operator;
- in §5.6, we show that $P(\lambda)$ has a meromorphic inverse and use this to give a meromorphic continuation of (5.0.3);
- in §5.7, we apply the methods of the present chapter to wave decay on certain Lorentzian spacetimes.

Some of the complicated parts of this chapter are only needed for applications to high frequency behavior (see §§5.6.3,6.2.3). A reader who is only interested in the meromorphic continuation of the scattering resolvent (Theorem 5.33) can safely skip Lemma 5.19 and §§5.4.4,5.6.3.

5.1. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

An asymptotically hyperbolic manifold is a complete Riemannian manifold whose infinite ends are locally asymptotic to the infinity of the hyperbolic space (5.0.1). A standard way to treat these manifolds is to define them as interiors of compact manifolds with boundary. For instance, (5.0.1) defines the hyperbolic space as the interior of the closed unit ball in $\mathbb{R}^n$. We will use the following general

DEFINITION 5.1. *Let $\overline{M}$ be a compact manifold with boundary ∂M and interior M. A* ***boundary defining function*** *on $\overline{M}$ is a C^∞ function*

$$y_1 : \overline{M} \to [0, \infty)$$

such that $y_1 = 0$ on ∂M, $dy_1 \neq 0$ on ∂M, and $y_1 > 0$ on M.

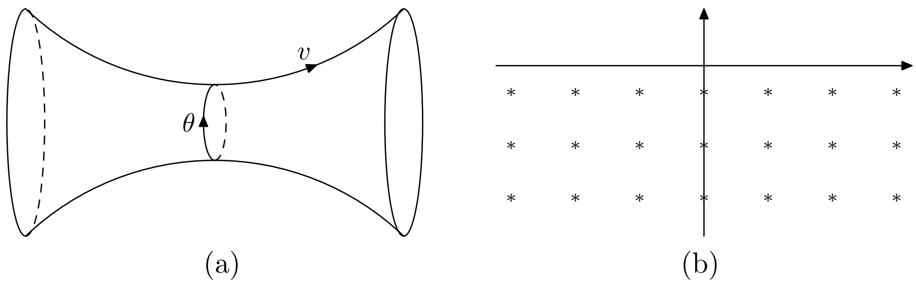


Figure 5.1. (a) The hyperbolic cylinder $\mathbb{R}_v \times \mathbb{S}^1_\theta$. (b) Resonances of the hyperbolic cylinder (see page 355), with imaginary parts given by $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$

Such functions always exist, as can be shown using local charts and a partition of unity. Any two boundary defining functions $y_1, \tilde{y}_1$ are multiples of each other:

$$\tilde{y}_1 = e^f y_1 \quad \text{for some } f \in C^\infty(\overline{M}; \mathbb{R}). \tag{5.1.1}$$

Moreover, a neighborhood of the boundary has a *product structure*: namely, for any boundary defining function y_1 and $\varepsilon_1 > 0$ small enough, there exists a diffeomorphism

$$y_1^{-1}\big([0,\varepsilon_1)\big) \to [0,\varepsilon_1) \times \partial M, \quad x \mapsto (y_1(x), y'(x)), \quad y'|_{\partial M} = I. \tag{5.1.2}$$

We now define asymptotically hyperbolic manifolds:

DEFINITION 5.2. *An* ***asymptotically hyperbolic manifold*** *is a complete Riemannian manifold* (M, g) *such that:*

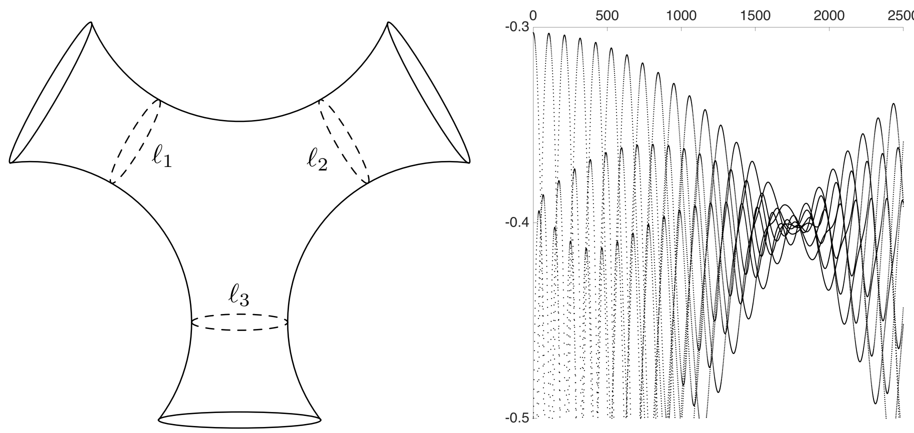
(1) *M is the interior of a compact manifold with boundary, denoted $\overline{M}$;*
(2) *for a boundary defining function y_1, the metric $y_1^2 g$ extends to a smooth Riemannian metric on $\overline{M}$;*
(3) *we have $|dy_1|_{y_1^2 g} = 1$ on $\partial\overline{M}$.*

Note that properties (2) and (3) do not depend on the choice of the boundary defining function, as follows immediately from (5.1.1). In particular, the invariance of property (3) follows from the identity

$$|d(e^f y_1)|_{e^{2f} y_1^2 g} = e^{-f} |e^f dy_1|_{y_1^2 g} = |dy_1|_{y_1^2 g} \quad \text{on } \partial\overline{M}.$$

Manifolds which satisfy properties (1) and (2) from Definition 5.2 are called *conformally compact*. Property (3) has the additional effect that the sectional curvatures on M converge to -1 at infinity; see [**GL91**, §2].

The boundary metric $(y_1^2 g)|_{\partial M}$ depends on the choice of the boundary defining function y_1. However, by (5.1.1), a different choice of y_1 multiplies



(a) (b)

Figure 5.2. (a) An example of a convex co-compact hyperbolic surface: three-funnel surface with geodesic neck lengths ℓ_1, ℓ_2, ℓ_3. (b) Resonances for the three-funnel surface with $\ell_1 = \ell_2 = \ell_3 = 7$, computed numerically by Borthwick and Weich [**Bo14, BW14**]. See [**Bo16**, Chapter 16] for more numerical results.

the boundary metric by a conformal factor. Therefore, to each asymptotically hyperbolic manifold corresponds a conformal class of Riemannian metrics on ∂M:

$$[g]_{\partial M} = \{(y_1^2 g)|_{\partial M} : y_1 \text{ is a boundary defining function}\}. \tag{5.1.3}$$

EXAMPLES. 1. The hyperbolic space (5.0.1) is asymptotically hyperbolic, where we may take for instance $y_1 = 1 - |w|^2$.

2. Another example is the *hyperbolic cylinder*

$$M = \mathbb{R}_v \times \mathbb{S}^1_\theta, \quad \mathbb{S}^1 = \mathbb{R}/(\ell\mathbb{Z}), \quad g = dv^2 + (\cosh v)^2 d\theta^2. \tag{5.1.4}$$

The hyperbolic cylinder is the union of two funnels (5.0.2) glued along a geodesic neck of length ℓ. The two infinite ends are given by $v = \pm\infty$; the compactification $\overline{M}$ is obtained by attaching a circle to each infinite end and using the defining function $y_1 = (\cosh v)^{-1}$. See Figure 5.1. For a detailed presentation of Vasy's method in the setting of the hyperbolic cylinder, see [**DD13**, Appendix B].

3. More generally, one may consider *convex co-compact hyperbolic manifolds*, which are asymptotically hyperbolic manifolds of constant sectional curvature -1. These manifolds can always be written as quotients $\Gamma\backslash\mathbb{H}^n$ of the hyperbolic space by a discrete group of isometries. A detailed study of this setting is outside the scope of this book, however we refer the reader to Borthwick [**Bo16**] for an extensive overview of classical and recent results in the case of surfaces. See Figure 5.2 for an example.

5.1.1. Canonical product structures. We now show existence of canonical product structures near the boundary of an asymptotically hyperbolic manifold, in which the metric takes the form of a stretched product. They are defined as follows:

DEFINITION 5.3 (Canonical product structures). *Let (M,g) be an asymptotically hyperbolic manifold. A boundary defining function y_1 is called* ***canonical*** *if*

$$|dy_1|_{y_1^2 g} = 1 \quad \text{in a neighborhood of } \partial M.$$

For such a function y_1, a product structure (y_1, y') from (5.1.2) is called ***canonical****, if the pushforward of the metric g under (5.1.2) has the form*

$$g = \frac{dy_1^2 + g_1(y_1, y', dy')}{y_1^2} \quad \text{in a neighborhood of } \partial M \tag{5.1.5}$$

where $g_1(y_1, y', dy')$ is a family of Riemannian metrics on ∂M depending smoothly on $y_1 \in [0,\varepsilon)$.

EXAMPLES. 1. On the hyperbolic space (5.0.1), a canonical coordinate system on $\{w \neq 0\}$ is given by

$$y_1 = \frac{1-|w|}{1+|w|} \in [0,1), \quad \theta = \frac{w}{|w|} \in \mathbb{S}^{n-1}, \tag{5.1.6}$$

and the metric in the (y_1, θ) coordinates takes the form

$$g = \frac{dy_1^2 + g_1}{y_1^2}, \quad g_1 = \frac{(1-y_1^2)^2}{4} g_S(\theta, d\theta) \tag{5.1.7}$$

where g_S denotes the standard metric on $\mathbb{S}^{n-1}$.

2. On the hyperbolic cylinder (5.1.4), a canonical coordinate system on $\{\pm v > 0\}$ is given by

$$y_1 = \exp(\mp v) \in [0,1), \quad \theta \in \mathbb{S}^1, \tag{5.1.8}$$

and the metric in the (y_1, θ) coordinates takes the form

$$g = \frac{dy_1^2 + g_1}{y_1^2}, \quad g_1 = \frac{(1+y_1^2)^2}{4} d\theta^2. \tag{5.1.9}$$

The following theorem shows existence and uniqueness of canonical coordinates, if one fixes the representative of the conformal class at the boundary:

THEOREM 5.4 (Existence of canonical product structures). *Let (M,g) be an asymptotically hyperbolic manifold and fix $g_0 \in [g]_{\partial M}$. Then there exists a canonical product structure (y_1, y') on $\overline{M}$ such that*

$$g_0 = (y_1^2 g)|_{\partial M}. \tag{5.1.10}$$

Any two such product structures coincide in a neighborhood of ∂M.

Proof. 1. We first construct y_1. Fix a boundary defining function $\tilde{y}_1$ such that $g_0 = (\tilde{y}_1^2 g)|_{\partial M}$ and denote $\bar{g} := \tilde{y}_1^2 g$. Then

$$y_1 := e^f \tilde{y}_1, \quad f \in C^\infty(\overline{M}; \mathbb{R}), \quad f|_{\partial M} = 0,$$

is a canonical boundary defining function if and only if near ∂M,

$$|d\tilde{y}_1 + \tilde{y}_1 df|_{\bar{g}}^2 = 1.$$

This equation degenerates at ∂M but it is equivalent to the eikonal equation

$$F(x, df(x)) = 0 \quad \text{for all } x \text{ near } \partial M, \tag{5.1.11}$$

with the function $F \in C^\infty(T^*\overline{M}; \mathbb{R})$ given by

$$\begin{aligned} F(x,\xi) &= \frac{|d\tilde{y}_1(x) + \tilde{y}_1(x)\xi|_{\bar{g}(x)}^2 - 1}{\tilde{y}_1(x)} \\ &= \frac{|d\tilde{y}_1(x)|_{\bar{g}(x)}^2 - 1}{\tilde{y}_1(x)} + 2\langle d\tilde{y}_1(x), \xi\rangle_{\bar{g}(x)} + \tilde{y}_1(x)|\xi|_{\bar{g}(x)}^2. \end{aligned} \tag{5.1.12}$$

With H_F denoting the Hamiltonian vector field of F, we compute

$$H_F \tilde{y}_1(x,\xi) = 2|d\tilde{y}_1(x)|_{\bar{g}(x)}^2 = 2 \quad \text{for } x \in \partial M. \tag{5.1.13}$$

Therefore, the equation (5.1.11) is noncharacteristic with respect to ∂M. The restriction $f|_{\partial M} = 0$ together with (5.1.11) lead to the initial condition

$$df(x) = \frac{1 - |d\tilde{y}_1(x)|_{\bar{g}(x)}^2}{2\tilde{y}_1(x)} d\tilde{y}_1(x) \quad \text{for } x \in \partial M. \tag{5.1.14}$$

The existence and local uniqueness of solutions to (5.1.11), (5.1.14) satisfying $f|_{\partial M} = 0$ now follows from the local existence and uniqueness theorem for first order partial differential equations, see for instance [**TaI**, Theorem 1.15.3] or [**Ev98**, Theorem 3.2]. (Here is a sketch of a geometric proof: it suffices to construct the graph Λ of df as a Lagrangian submanifold of $T^*\overline{M}$ contained in $F^{-1}(0)$. The condition (5.1.14) gives $\Lambda \cap T^*\partial M = \Lambda_0$ where $\Lambda_0 \subset F^{-1}(0)$ is an isotropic submanifold. To obtain Λ we take the flowout of Λ_0 by the vector field H_F, which is a graph in the x variables by (5.1.13).)

2. Having obtained y_1, we now construct a canonical product structure (y_1, y'). The gradient vector field $\nabla_{y_1^2 g} y_1$ is inward pointing at the boundary, therefore for $\varepsilon_1 > 0$ small enough the map

$$\Phi : [0, \varepsilon_1) \times \partial M \to \overline{M}, \quad \Phi(s, y') = \exp(s\nabla_{y_1^2 g} y_1)(y') \tag{5.1.15}$$

is a diffeomorphism onto a neighborhood of ∂M. Since y_1 is a canonical boundary defining function, we have $y_1(\Phi(s, y')) = s$. Thus in the product structure $(y_1, y') := \Phi^{-1}$, the gradient $\nabla_{y_1^2 g} y_1$ is equal to ∂_{y_1}. It follows that (y_1, y') is a canonical product structure. Reversing this argument we see that canonical product structures are locally unique. $\square$

5.1.2. Even metrics. We next study the subclass of *even* asymptotically hyperbolic manifolds. These are the manifolds for which meromorphic extension is established in this chapter. Without the evenness assumption, the resolvent (5.0.3) does not admit a meromorphic extension to $\mathbb{C}$ with poles of finite rank as shown by Guillarmou [**Gu05**, Theorem 1.4].

DEFINITION 5.5 (Even asymptotically hyperbolic metric). *An asymptotically hyperbolic manifold (M, g) is called* ***even*** *if there exists a canonical product structure (y_1, y') from Definition 5.3 such that the corresponding metric g_1 satisfies*

$$\partial_{y_1}^{2k+1} g_1(0, y', dy') = 0, \quad k \in \mathbb{N}_0. \tag{5.1.16}$$

In other words, $g_1(y_1, y', dy')$ can be extended as an even function of y_1 past the boundary $\{y_1 = 0\}$.

It is easy to see that the examples (5.1.7) and (5.1.9) are even metrics.

The evenness of a metric does not depend on the choice of the canonical product structure, as shown by

THEOREM 5.6 (Invariance of the evenness condition). *Assume that (M, g) is an asymptotically hyperbolic manifold (y_1, y'), $(\tilde{y}_1, \tilde{y}')$ are two canonical product structures on $\overline{M}$, and $g_1, \tilde{g}_1$ the corresponding metrics. Assume that (5.1.16) holds for $\tilde{g}_1$. Then it also holds for g_1, and*

$$\partial_{\tilde{y}_1}^{2k} y_1 = 0 \quad \text{on } \partial M, \quad k \in \mathbb{N}_0, \tag{5.1.17}$$

$$\partial_{\tilde{y}_1}^{2k+1}(\psi \circ y') = 0 \quad \text{on } \partial M, \quad k \in \mathbb{N}_0, \quad \psi \in C^\infty(\partial M). \tag{5.1.18}$$

Proof. 1. We first show (5.1.17). Since both y_1 and $\tilde{y}_1$ are canonical boundary defining functions, we have $y_1 = e^f \tilde{y}_1$ where $f \in C^\infty(\overline{M}; \mathbb{R})$ solves the eikonal equation (5.1.11). In coordinates $(\tilde{y}_1, \tilde{y}')$ the eikonal equation takes the form

$$2\partial_{\tilde{y}_1} f + \tilde{y}_1\big((\partial_{\tilde{y}_1} f)^2 + |\partial_{\tilde{y}'} f|^2_{\tilde{g}_1(\tilde{y}_1, \tilde{y}')}\big) = 0. \tag{5.1.19}$$

Differentiating (5.1.19) $2k$ times in $\tilde{y}_1$, we obtain for each $k \in \mathbb{N}_0$,

$$\partial_{\tilde{y}_1}^{2k+1} f|_{\tilde{y}_1=0} = -k\partial_{\tilde{y}_1}^{2k-1}\big((\partial_{\tilde{y}_1} f)^2 + |\partial_{\tilde{y}'} f|^2_{\tilde{g}_1(\tilde{y}_1, \tilde{y}')}\big)\big|_{\tilde{y}_1=0}. \tag{5.1.20}$$

By induction using the Leibniz rule and the fact that $\tilde{g}_1$ satisfies (5.1.16), we see that $\partial_{\tilde{y}_1}^{2k+1} f|_{\tilde{y}_1=0} = 0$ for all k, implying (5.1.17).

2. We next show (5.1.18). By (5.1.15) the function $\psi \circ y'$ is characterized by the conditions $(\psi \circ y')|_{\partial M} = \psi$ and

$$\big\langle d(\psi \circ y'), \nabla_{y_1^2 g} y_1 \big\rangle = 0.$$

In coordinates $(\tilde{y}_1, \tilde{y}')$ the latter equation takes the form

$$\partial_{\tilde{y}_1}(\psi \circ y') + \tilde{y}_1\big(\partial_{\tilde{y}_1} f \cdot \partial_{\tilde{y}_1}(\psi \circ y') + \langle \partial_{\tilde{y}'} f, \partial_{\tilde{y}'}(\psi \circ y')\rangle_{\tilde{g}_1(\tilde{y}_1, \tilde{y}')}\big) = 0.$$

Differentiating this expression $2k$ times in $\tilde{y}_1$ at ∂M and arguing by induction as in step 1, we obtain (5.1.18).

3. We finally show that g_1 satisfies (5.1.16). It is enough to prove that for each $\psi \in C^\infty(\partial M)$, we have

$$\partial_{y_1}^{2k+1}\big(|d(\psi \circ y')|^2_{y_1^2 g}\big)\big|_{y_1=0} = 0, \quad k \in \mathbb{N}_0. \tag{5.1.21}$$

We have $\partial_{y_1} = \nabla_{y_1^2 g} y_1 = e^{-f}(\partial_{\tilde{y}_1} + \tilde{y}_1 \nabla_{\tilde{y}_1^2 g} f)$. By (5.1.17) it suffices to prove (5.1.21) with y_1 replaced by $\tilde{y}_1$:

$$\partial_{\tilde{y}_1}^{2k+1}\big(|d(\psi \circ y')|^2_{\tilde{y}_1^2 g}\big)\big|_{\tilde{y}_1=0} = 0, \quad k \in \mathbb{N}_0. \tag{5.1.22}$$

Now (5.1.22) follows from (5.1.18) and the fact that $\tilde{g}_1$ satisfies (5.1.16). $\square$

5.1.3. The even extension. We finally describe an extension of an even asymptotically hyperbolic manifold, which is the underlying manifold for the analysis of the rest of this chapter:

DEFINITION 5.7. *Let (M, g) be an even asymptotically hyperbolic manifold and fix a canonical product structure $(y_1, y') \in [0, \varepsilon_1) \times \partial M$. Consider the diffeomorphism*

$$M \cap \{y_1 < \varepsilon_1\} \to (0, \varepsilon_1^2) \times \partial M, \quad x \mapsto (x_1, x') := (y_1^2, y'). \tag{5.1.23}$$

We define:

- *the **even compactification** $\overline{M}_{\mathrm{even}}$ of M to be the manifold with boundary obtained by gluing M with $[0, \varepsilon_1^2) \times \partial M$ using the map (5.1.23);*
- *the **even extension** $\overline{X} = \overline{X}_\varepsilon$ of M to be the manifold with boundary obtained by gluing M with $[-\varepsilon, \varepsilon_1^2) \times \partial M$ using the map (5.1.23). Here $\varepsilon > 0$ is a given constant.*

See Figure 5.3 on page 326. Note that by (5.1.17) and (5.1.18), the compactifications $\overline{M}_{\mathrm{even}}$ arising from different choices of canonical product structures are diffeomorphically equivalent.

We also note that a function in $C^\infty(\overline{M}_{\mathrm{even}})$ can be thought of as a function in $C^\infty(\overline{M})$ which admits an extension across the boundary which is even in y_1 in the coordinates (y_1, y').

We will suppress the dependence of $\overline{X}$ on ε; we fix ε small enough for the construction to work.

EXAMPLES. 1. For the hyperbolic space (5.0.1), we may take

$$\overline{X} = \overline{B}_{\mathbb{R}^n}(0, 2), \quad \overline{M}_{\mathrm{even}} = \overline{B}_{\mathbb{R}^n}(0, 1), \tag{5.1.24}$$

where, using polar coordinates r, θ on $\overline{M}_{\text{even}}$, and with y_1 defined by (5.1.6),

$$r = \frac{2|w|}{1+|w|^2} = \frac{1-y_1^2}{1+y_1^2}, \quad \theta = \frac{w}{|w|} \in \mathbb{S}^{n-1}.$$

The metric g in the (r, θ) coordinates takes the form

$$g = \frac{dr^2}{(1-r^2)^2} + \frac{r^2 g_S(\theta, d\theta)}{1-r^2}. \tag{5.1.25}$$

2. For the hyperbolic cylinder (5.1.4), we may take

$$\overline{X} = [-2,2]_r \times \mathbb{S}^1_\theta, \quad \overline{M}_{\text{even}} = [-1,1]_r \times \mathbb{S}^1_\theta, \tag{5.1.26}$$

where, with y_1 defined in (5.1.8),

$$r = \tanh v; \quad 1 - r^2 = \frac{4y_1^2}{(1+y_1^2)^2}.$$

The metric g in the (r, θ) coordinates takes the form

$$g = \frac{dr^2}{(1-r^2)^2} + \frac{d\theta^2}{1-r^2}. \tag{5.1.27}$$

5.2. A MOTIVATING EXAMPLE

Before proceeding with the general construction of the scattering resolvent, we present a simple example of an asymptotically hyperbolic manifold, and use it as motivation for introducing the modified Laplacian in §5.3 below. We will not provide proper proofs for most claims made in this section; instead they will follow from the general construction presented in the rest of the present chapter.

Consider an asymptotically hyperbolic manifold (M, g) with a boundary defining function y_1 and a canonical product structure

$$(y_1, y') : \overline{M} \cap \{y_1 < 1\} \to [0, 1) \times N, \tag{5.2.1}$$

in which the metric has the form

$$g = \frac{dy_1^2 + g_0(y', dy')}{y_1^2}.$$

Here $N = \partial M$ has dimension $n-1$ and g_0 is a Riemannian metric on N. The simplification compared to (5.1.5) is that g_0 does not depend on y_1; this automatically makes (M, g) even in the sense of Definition 5.5.

On the domain $\{y_1 < 1\}$ of (5.2.1), we calculate the volume form

$$d\,\mathrm{Vol}_g = y_1^{-n}\, dy_1 d\,\mathrm{Vol}_{g_0}$$

and the Laplacian (here Δ_{g_0} is the Laplacian in the y' variables)

$$\Delta_g = y_1^2\partial_{y_1}^2 + (2-n)y_1\partial_{y_1} + y_1^2\Delta_{g_0}.$$

The scattering resolvent of M is a meromorphic family of operators

$$R(\lambda) : C_c^\infty(M) \to C^\infty(M), \quad \lambda \in \mathbb{C},$$

such that for all $f \in C_c^\infty(M)$,

$$\Big(- \Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big) R(\lambda) f = f; \tag{5.2.2}$$

$$R(\lambda) f \in L^2(M; d\operatorname{Vol}_g) \quad \text{for } \operatorname{Im}\lambda > 0. \tag{5.2.3}$$

The equation

$$\Big(- \Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big) u = f \tag{5.2.4}$$

has infinitely many solutions for every given f. However, the solution $u = R(\lambda)f$ can be identified uniquely (except at resonances) if we know enough information about the asymptotic behavior of u at the infinite end of M, which in our case corresponds to $y_1 \to 0$.

Given (5.2.3), it makes sense to first consider the case $\operatorname{Im}\lambda > 0$ and understand the asymptotic behavior of the L^2 solutions of (5.2.4). In this section, we do it using separation of variables. Let $(\beta_k)_{k\in\mathbb{N}}$ be the spectrum of the Laplacian $-\Delta_{g_0}$ on N, and let $v_k \in C^\infty(N)$ be the corresponding orthonormal basis of eigenfunctions. Assume that

$$u \in C^\infty(M) \cap L^2(M; d\operatorname{Vol}_g) \tag{5.2.5}$$

is a solution to (5.2.4) and write the Fourier series in $\{y_1 < 1\}$,

$$u(y_1, y') = \sum_k u_k(y_1) v_k(y'), \quad u_k \in C^\infty\big((0,1)\big).$$

The right-hand side f is supported in $\{y_1 \geq \delta\}$ for some $\delta > 0$. For simplicity of notation, we assume that $\operatorname{supp} f \subset \{y_1 \geq 1\}$. Then (5.2.4) implies that u_k solves the ordinary differential equation in $(0,1)$

$$\Big(- y_1^2\partial_{y_1}^2 + (n-2)y_1\partial_{y_1} - \lambda^2 - \frac{(n-1)^2}{4} + y_1^2\beta_k\Big) u_k(y_1) = 0. \tag{5.2.6}$$

This is Bessel's equation, and (unless $\lambda \in i\mathbb{Z}$ where a bit of extra care is needed), it has two solutions asymptotic to $y_1^{\alpha_+}$ and $y_1^{\alpha_-}$, where $\alpha_\pm \in \mathbb{C}$, called the *indicial roots*, are solutions to the equation

$$I(\alpha) = 0, \quad I(\alpha) := -\Big(\alpha - \frac{n-1}{2}\Big)^2 - \lambda^2. \tag{5.2.7}$$

Indeed, we have

$$\Big(- y_1^2\partial_{y_1}^2 + (n-2)y_1\partial_{y_1} - \lambda^2 - \frac{(n-1)^2}{4} + y_1^2\beta_k\Big) y_1^\alpha = I(\alpha) y_1^\alpha + \beta_k y_1^{\alpha+2};$$

then (see Exercise 5.2) the two solutions to (5.2.6) can be found in the form of power series

$$y_1^{\alpha_\pm} \sum_{j=0}^{\infty} a_{j,\pm} y_1^{2j} \tag{5.2.8}$$

for some coefficients $a_{j,\pm} \in \mathbb{C}$, $a_{0,\pm} \neq 0$.

In our case, the indicial roots are

$$\alpha_\pm = \frac{n-1}{2} \pm i\lambda. \tag{5.2.9}$$

This explains the shift of the spectral parameter by $\frac{(n-1)^2}{4}$ in (5.0.3) – without this, $\alpha_\pm$ will not be holomorphic functions of λ.

Recall that we are looking for $u \in L^2(M; d\,\mathrm{Vol}_g)$, thus we need

$$y_1^{-n/2} u_k \in L^2((0,1); dy_1). \tag{5.2.10}$$

Since $\operatorname{Im}\lambda > 0$, (5.2.10) holds for $y_1^{\alpha_-}$, but not for $y_1^{\alpha_+}$. Therefore, $y_1^{-\alpha_-} u_k$ has to be the sum of a power series in y_1^2. This leads to the following statement, which is a corollary of Theorem 5.33 below:

PROPOSITION 5.8. *Assume that $u \in C^\infty(M) \cap L^2(M; d\,\mathrm{Vol}_g)$ solves the equation (5.2.4), for some $f \in C_c^\infty(M)$ and $\operatorname{Im}\lambda > 0$. Then*

$$u \in y_1^{\frac{n-1}{2} - i\lambda} C^\infty(\overline{M}_{\mathrm{even}}) \tag{5.2.11}$$

where the even compactification $\overline{M}_{\mathrm{even}}$ was introduced in Definition 5.7.

Returning to meromorphic continuation of the resolvent to $\operatorname{Im}\lambda \leq 0$, the main idea is to define $R(\lambda)f$ as the solution to (5.2.4) satisfying the outgoing condition (5.2.11). For this, the outgoing condition should be strong enough to rule out all other solutions. In practice, we replace $C^\infty(\overline{M}_{\mathrm{even}})$ by the Sobolev space $\bar{H}^s(X)$ (see Definition E.25), where $\overline{X}$ is the even extension introduced in Definition 5.7 and s is large enough depending on λ.

To see why an outgoing solution to (5.2.4) is uniquely defined (for λ not a resonance), we consider the *conjugated operator*

$$y_1^{i\lambda - \frac{n-1}{2}} \Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \Big) y_1^{\frac{n-1}{2} - i\lambda}. \tag{5.2.12}$$

In the product coordinates (5.2.1), it takes the form

$$-y_1^2 \partial_{y_1}^2 - y_1^2 \Delta_{g_0} + (2i\lambda - 1) y_1 \partial_{y_1}.$$

Passing to the product structure $(x_1, x') = (y_1^2, y')$ on the even compactification $\overline{M}_{\mathrm{even}}$ and extending the resulting operator to $\overline{X} = \{x_1 \geq -\varepsilon\}$, we obtain

$$-4x_1^2 \partial_{x_1}^2 - x_1 \Delta_{g_0} + 4(i\lambda - 1) x_1 \partial_{x_1}.$$

The result can be divided on the left by x_1, obtaining the final operator on $\overline{X}$ that will be the focus of this chapter (with the previous calculation showing that the two operators below coincide in $\{0 < x_1 < 1\}$):

$$P(\lambda) = \begin{cases} x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}\big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\big)x_1^{\frac{n-1}{4}-\frac{i\lambda}{2}} & \text{on } \{x_1 > 0\}, \\ -4x_1\partial_{x_1}^2 - \Delta_{g_0} + 4(i\lambda - 1)\partial_{x_1} & \text{on } \{x_1 < 1\}. \end{cases} \tag{5.2.13}$$

The operator $P(\lambda)$ has nondegenerate principal part for $x_1 \neq 0$, and the degeneracy at $x_1 = 0$ is minimal as only the coefficient of $\partial_{x_1}^2$ vanishes. This minimal degeneracy, which is the result of passing to the space $\overline{M}_{\text{even}}$ and dividing the conjugated operator by x_1, makes it possible to prove the following key statement (see Theorem 5.30 below for the general case):

PROPOSITION 5.9. *Consider the space*

$$\mathcal{X}^s := \{u \in \bar{H}^s(X) \mid P(0)u \in \bar{H}^{s-1}(X)\};$$

see Definition E.25 for Sobolev spaces on manifolds with boundary. Then

$$P(\lambda) : \mathcal{X}^s \to \bar{H}^{s-1}(X), \quad \operatorname{Im}\lambda > \frac{1}{2} - s$$

is a Fredholm operator. Moreover, it is invertible for some values of λ.

To show the Fredholm property of $P(\lambda)$, we need to show its invertibility modulo a smoothing operator. In $M = \{x_1 > 0\}$, this property follows from the fact that $P(\lambda)$ is elliptic. In $\overline{X} \setminus \overline{M}_{\text{even}} = \{x_1 < 0\}$, $P(\lambda)$ is hyperbolic and one may use standard energy estimates (see Theorem E.64). Finally, on $\{x_1 = 0\}$ the operator $P(\lambda)$ denegerates and we use radial source/sink estimates, which is where the condition $\operatorname{Im}\lambda > \frac{1}{2} - s$ becomes important.

Given Proposition 5.9, we see by Analytic Fredholm Theory (Theorem C.8) that the inverse

$$P(\lambda)^{-1} : \bar{H}^{s-1}(X) \to \mathcal{X}^s$$

is a meromorphic family of operators. The scattering resolvent is then defined as

$$R(\lambda) = x_1^{\frac{n-1}{4}-\frac{i\lambda}{2}} \mathbb{1}_M\, P(\lambda)^{-1}\, \mathbb{1}_M\, x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}},$$

which coincides with the L^2 resolvent for $\operatorname{Im}\lambda > 0$ since the outgoing condition (5.2.11) implies that $u \in L^2(M; d\operatorname{Vol}_g)$.

5.3. THE MODIFIED LAPLACIAN

In this section we introduce the modified spectral family of the Laplacian, generalizing (5.2.13) – see Definition 5.11 below. As explained in §5.2, the Fredholm property of this operator on appropriately chosen spaces will ultimately give meromorphic continuation of the resolvent (5.0.3).

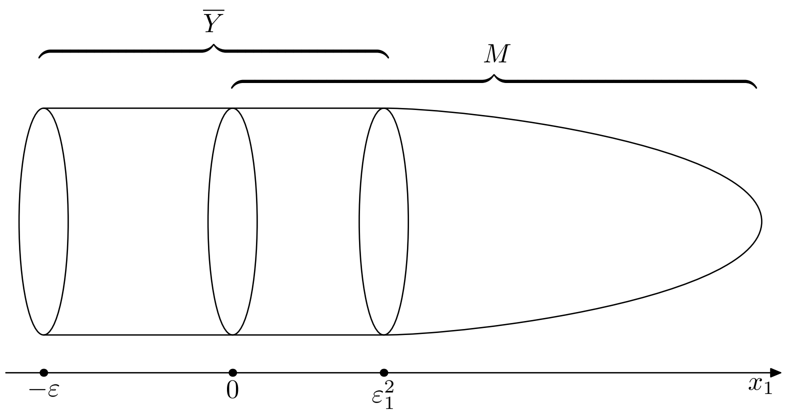


Figure 5.3. The even extension $\overline{X} = M \cup \overline{Y}$.

Let (M, g) be an even asymptotically hyperbolic manifold, see Definition 5.5. Fix a canonical boundary defining function y_1 and let $(y_1, y') \in [0, \varepsilon_1) \times \partial M$ be the corresponding canonical product structure; see Definition 5.3. Let $\overline{M}_{\text{even}}$ be the even compactification of M and $\overline{X} = \overline{X}_\varepsilon$ be its even extension, see Definition 5.7.

We denote by X the interior of the even extension $\overline{X} \supset \overline{M}_{\text{even}}$.

Put

$$x_1 := y_1^2,$$

so that x_1 is a boundary defining function of $\overline{M}_{\text{even}}$ and $x_1 + \varepsilon$ is a boundary defining function of $\overline{X}$. We write (see Figure 5.3)

$$\overline{X} = M \cup \overline{Y}, \quad M = \{x_1 > 0\}, \quad \overline{Y} := \{-\varepsilon \le x_1 < \varepsilon_1^2\}, \tag{5.3.1}$$

where $\{y_1 < \varepsilon_1\}$ is the domain of the product structure (y_1, y'). On $\overline{Y}$, we have the product structure

$$(x_1, x') \in [-\varepsilon, \varepsilon_1^2) \times \partial M, \quad x' := y'. \tag{5.3.2}$$

Since (M, g) is an even metric, recalling (5.1.5) and (5.1.16) we can write on $M \cap Y$ in the coordinates (5.3.2),

$$g = \frac{dx_1^2}{4x_1^2} + \frac{g_1(x_1, x', dx')}{x_1} \tag{5.3.3}$$

where g_1 is smooth in $x_1 \in [0, \varepsilon_1^2)$. We fix an extension of g_1 to $x_1 \in [-\varepsilon, \varepsilon_1^2)$ as a smooth family of Riemannian metrics on ∂M:

$$g_1 \in C^\infty\big([-\varepsilon, \varepsilon_1^2) \times \partial M; \otimes^2 T^* \partial M\big). \tag{5.3.4}$$

Following (5.2.13), consider the differential operator on M,

$$x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}\Big(-\Delta_g-\lambda^2-\frac{(n-1)^2}{4}\Big)x_1^{\frac{n-1}{4}-\frac{i\lambda}{2}}. \tag{5.3.5}$$

LEMMA 5.10. *On $M\cap Y=\{0<x_1<\varepsilon_1^2\}$, the operator (5.3.5) has the following form under (5.3.2):*

$$-4x_1\partial_{x_1}^2+4(i\lambda-1)\partial_{x_1}-\gamma(4x_1\partial_{x_1}+n-1-2i\lambda)-\Delta_{g_1},$$

where Δ_{g_1} is the Laplace–Beltrami operator of g_1 in the x' variables and

$$\gamma(x_1,x'):=J^{-1}\frac{\partial J}{\partial x_1}\in C^\infty(\overline{Y};\mathbb{R}),\quad J:=\sqrt{|\det(g^1_{jk})|} \tag{5.3.6}$$

is the logarithmic derivative of the Jacobian of the metric g_1, which is independent of the choice of local coordinates on ∂M.

Proof. We compute from (5.3.3)

$$\Delta_g=4x_1^{\frac{n+1}{2}}\partial_{x_1}x_1^{\frac{3-n}{2}}\partial_{x_1}+4x_1^2\gamma\partial_{x_1}+x_1\Delta_{g_1}.$$

Using the identity $x_1\partial_{x_1}x_1^\alpha=x_1^\alpha(x_1\partial_{x_1}+\alpha)$, $\alpha\in\mathbb{C}$, we compute (5.3.5) as

$$\begin{aligned}&-\frac{1}{x_1}\Big(2x_1\partial_{x_1}+\frac{1-n}{2}-i\lambda\Big)\Big(2x_1\partial_{x_1}+\frac{n-1}{2}-i\lambda\Big)\\&-\gamma(4x_1\partial_{x_1}+n-1-2i\lambda)-\Delta_{g_1}-\frac{1}{x_1}\Big(\lambda^2+\frac{(n-1)^2}{4}\Big)\\&=-4x_1\partial_{x_1}^2+4(i\lambda-1)\partial_{x_1}-\gamma(4x_1\partial_{x_1}+n-1-2i\lambda)-\Delta_{g_1},\end{aligned}$$

finishing the proof. □

Using Lemma 5.10, we continue the operator (5.3.5) to $\overline{X}$, obtaining the central object of study for the rest of this chapter:

DEFINITION 5.11. *Define the* ***extended modified Laplacian*** *as the second order differential operator $P(\lambda)$ on $\overline{X}$ given by (5.3.5) on M and by the following formula on $\overline{Y}$:*

$$P(\lambda)=-4x_1\partial_{x_1}^2+4(i\lambda-1)\partial_{x_1}-\gamma(4x_1\partial_{x_1}+n-1-2i\lambda)-\Delta_{g_1} \tag{5.3.7}$$

where g_1 is defined in (5.3.4) and $\gamma\in C^\infty(\overline{Y};\mathbb{R})$ is defined in (5.3.6).

We next introduce the semiclassically rescaled version of the operator $P(\lambda)$:

$$P_h(\omega):=h^2P(h^{-1}\omega),\quad 0<h\ll 1,\ \omega\in\mathbb{C}. \tag{5.3.8}$$

Most results of the present chapter do not require this semiclassical rescaling, however it is essential for the study of the high frequency limit $\operatorname{Re}\lambda\to\infty$ in §5.6.3 and Part 3 of the book and is also used in Proposition 5.28 below.

To streamline the exposition we use the rescaled operator $P_h(\omega)$ throughout this chapter.

We calculate

$$P_h(\omega) = x_1^{\frac{i\omega}{2h}-\frac{n+3}{4}}\Big(-h^2\Delta_g - \omega^2 - \frac{(n-1)^2}{4}h^2\Big)x_1^{\frac{n-1}{4}-\frac{i\omega}{2h}} \tag{5.3.9}$$

on M and, denoting $D = \frac{1}{i}\partial$,

$$\begin{aligned} P_h(\omega) = {} & 4x_1(hD_{x_1})^2 - 4(\omega + ih)hD_{x_1} - h^2\Delta_{g_1} \\ & - ih\gamma(4x_1hD_{x_1} + i(1-n)h - 2\omega) \end{aligned} \tag{5.3.10}$$

on $\overline{Y}$. It follows from (5.3.9) and (5.3.10) that $P_h(\omega)$ is a second order semiclassical differential operator on X (see §E.1.1) with coefficients smooth up to the boundary, depending holomorphically on the parameter ω.

Consider the semiclassical principal symbol of $P_h(\omega)$ defined using (E.1.3),

$$p = p(x,\xi;\omega) := \sigma_h(P_h(\omega)) \ \in \ \mathrm{Poly}^2(T^*X). \tag{5.3.11}$$

We compute on $M = \{x_1 > 0\}$, for ω real

$$p(x,\xi;\omega) = \frac{1}{x_1}\Big(\Big|\xi - \omega\frac{dx_1}{2x_1}\Big|^2_{g(x)} - \omega^2\Big) \tag{5.3.12}$$

and on $\overline{Y} = \{-\varepsilon \le x_1 < \varepsilon_1^2\}$,

$$p(x_1,x',\xi_1,\xi';\omega) = 4x_1\xi_1^2 - 4\omega\xi_1 + p_1, \quad p_1 := |\xi'|^2_{g_1(x_1,x')}. \tag{5.3.13}$$

Note that p is a second order polynomial in ω and

$$\partial_\omega p \in \mathrm{Poly}^1(T^*X), \quad \partial_\omega^2 p \in \mathrm{Poly}^0(T^*X). \tag{5.3.14}$$

The behavior of p for any $\omega \in \mathbb{R}$ can be reduced to the cases $\omega = 0$ and $\omega = 1$, as follows from the scaling relation

$$p(x, s\xi; s\omega) = s^2 p(x,\xi;\omega), \quad s \in \mathbb{R}. \tag{5.3.15}$$

REMARK. In (5.3.5), we used the function $x_1 = y_1^2$, where y_1 was a canonical boundary defining function on M. This has the advantage that the metric has the simple form (5.3.3). However, in many examples y_1 is given near ∂M by a formula that does not extend smoothly to the entire M (see e.g. (5.1.6) and (5.1.8)). To remedy this we can consider a more general function $e^{-2\psi}x_1$, where $\psi \in C^\infty(\overline{X};\mathbb{R})$, and study the resulting operators

$$P_\psi(\lambda) := e^{(\frac{n+3}{2}-i\lambda)\psi}P(\lambda)e^{(i\lambda-\frac{n-1}{2})\psi}, \quad P_{\psi,h}(\omega) := h^2P_\psi(h^{-1}\omega). \tag{5.3.16}$$

The operator $P_\psi(\lambda)$ has similar properties to $P(\lambda)$. In particular, Theorem 5.30 (meromorphy of the inverse) applies to this operator as well. This more general form is used in the examples below and for applications to general relativity in §5.7.

EXAMPLES. 1. For the hyperbolic space (see (5.1.24), (5.1.25)), we choose ψ so that $e^{-2\psi}x_1 = 1 - r^2$. Then on M,

$$P_\psi(\lambda) = (1-r^2)^{\frac{i\lambda}{2}-\frac{n+3}{4}}\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)(1-r^2)^{\frac{n-1}{4}-\frac{i\lambda}{2}}, \tag{5.3.17}$$

and on the entire $\overline{X}$ we have

$$\begin{aligned} P_\psi(\lambda) = &-(1-r^2)\partial_r^2 + (n+1-2i\lambda)r\partial_r + \frac{1-n}{r}\partial_r \\ &- \lambda^2 - ni\lambda + \frac{n^2-1}{4} - \frac{1}{r^2}\Delta_S, \end{aligned} \tag{5.3.18}$$

with Δ_S the Laplacian on the sphere. The principal symbol of $P_{\psi,h}(\omega)$ is

$$p_\psi(r,\theta,\xi_r,\xi_\theta;\omega) = (1-r^2)\xi_r^2 + 2\omega r\xi_r - \omega^2 + \frac{|\xi_\theta|_{g_S}^2}{r^2}. \tag{5.3.19}$$

If we instead take $\psi = 0$ and use the defining function $x_1 = y_1^2$ where y_1 is given by (5.1.6), then

$$\begin{aligned} P(\lambda) = &-4x_1\partial_{x_1}^2 + 4(i\lambda-1)\partial_{x_1} \\ &+ \frac{n-1}{1-x_1}(4x_1\partial_{x_1} + n - 1 - 2i\lambda) - \frac{4}{(1-x_1)^2}\Delta_{g_S}. \end{aligned} \tag{5.3.20}$$

2. For the hyperbolic cylinder (see (5.1.26), (5.1.27)), we also choose ψ so that $e^{-2\psi}x_1 = 1 - r^2$. Then on M, $P_\psi(\lambda)$ has the form (5.3.17) and on the entire $\overline{X}$ we have

$$P_\psi(\lambda) = -(1-r^2)\partial_r^2 + 2(1-i\lambda)r\partial_r - \lambda^2 - i\lambda + \tfrac{1}{4} - \partial_\theta^2. \tag{5.3.21}$$

The principal symbol of $P_{\psi,h}(\omega)$ is

$$p_\psi(r,\theta,\xi_r,\xi_\theta;\omega) = (1-r^2)\xi_r^2 + 2\omega r\xi_r - \omega^2 + \xi_\theta^2. \tag{5.3.22}$$

If we instead take $\psi = 0$ and use the defining function $x_1 = y_1^2$ where y_1 is given by (5.1.8), then we have

$$\begin{aligned} P(\lambda) = &-4x_1\partial_{x_1}^2 + 4(i\lambda-1)\partial_{x_1} \\ &- \frac{1}{1+x_1}(4x_1\partial_{x_1} + 1 - 2i\lambda) - \frac{4}{(1+x_1)^2}\partial_\theta^2. \end{aligned} \tag{5.3.23}$$

We conclude this section with two facts about the operator $P(\lambda)$. The first one gives basic properties of the nonsemiclassical principal symbol p_0 of $P(\lambda)$, and is a direct corollary of (5.3.12) and (5.3.13):

PROPOSITION 5.12. *Let $p_0(x,\xi)$ be the quadratic in ξ part of the symbol p, that is the quadratic form obtained by putting $\omega := 0$ in p. Then $P(\lambda)$, considered as a nonsemiclassical differential operator, is*

- ***elliptic** on $M = \{x_1 > 0\}$, in the sense that p_0 is positive definite;*

- ***hyperbolic*** *on* $X \setminus \overline{M}_{\mathrm{even}} = \{x_1 < 0\}$ *with respect to* x_1*, in the sense of Definition E.55.*

The other fact concerns the imaginary part of the operator $P(\lambda)$, formulated in terms of its rescaled version $P_h(\omega)$. It is needed to verify the threshold conditions in radial estimates (Theorems E.52, E.54) which are used to analyse the operator $P(\lambda)$ on the interface $\{x_1 = 0\}$ between the elliptic and the hyperbolic regions. Define a volume form $d\,\mathrm{Vol}$ on $\overline{X}$ as follows:

$$d\,\mathrm{Vol} = \begin{cases} 2x_1^{\frac{n+1}{2}}\,d\,\mathrm{Vol}_g & \text{on } M; \\ dx_1 d\,\mathrm{Vol}_{g_1} & \text{on } \overline{Y}. \end{cases} \tag{5.3.24}$$

A direct calculation using (5.3.9) and (5.3.10) shows that

$$P_h(\omega)^* = P_h(\bar{\omega}) \tag{5.3.25}$$

where $P_h(\omega)^*$ denotes the formal adjoint of $P_h(\omega)$ in $L^2(X; d\,\mathrm{Vol})$. It follows that for $\omega \in \mathbb{R}$, the operator $P_h(\omega)$ is symmetric. For general values of ω, we have the following statement:

PROPOSITION 5.13. *Let* $\omega = \omega_R + ih\omega_I$, $\omega_R, \omega_I \in \mathbb{R}$, *and put*

$$\operatorname{Im} P_h(\omega) = \frac{P_h(\omega) - P_h(\omega)^*}{2i}.$$

Then $h^{-1} \operatorname{Im} P_h(\omega)$ *is a first order semiclassical differential operator (see §E.1.1) and its principal symbol is (with* p *defined in (5.3.11))*

$$\sigma_h\big(h^{-1} \operatorname{Im} P_h(\omega)\big)(x, \xi; \omega_R, \omega_I) = \omega_I \partial_\omega p(x, \xi; \omega_R). \tag{5.3.26}$$

REMARK. Note that here we parametrize $P_h(\omega)$ by $\omega_R = \operatorname{Re}\omega$ and $\omega_I = h^{-1}\operatorname{Im}\omega$ rather than by $\operatorname{Re}\omega$ and $\operatorname{Im}\omega$ as before. This still gives a family of operators in $\mathrm{Diff}_h^2(\overline{X})$ depending smoothly on ω_R, ω_I, and the semiclassical principal symbol of $P_h(\omega)$ does not depend on ω_I.

Proof. By (5.3.25), we have

$$\operatorname{Im} P_h(\omega) = \operatorname{Im}\big(P_h(\omega) - P_h(\omega_R)\big).$$

By (5.3.9) and (5.3.10), $P_h(\omega)$ is a quadratic polynomial in hD_x, ω with coefficients smooth in x. It follows that $h^{-1}(P_h(\omega) - P_h(\omega_R))$ is a first order semiclassical differential operator with principal symbol $i\omega_I \partial_\omega p$; (5.3.26) follows. □

5.4. PHASE SPACE DYNAMICS

We now study the zero set and the Hamiltonian flow of the principal symbol p of the semiclassically rescaled modified Laplacian $P_h(\omega)$ (see (5.3.11)), in preparation for the propagation estimates of §5.5. In this section, we always consider the case of $\omega \in \mathbb{R}$.

To understand the behavior of p both for bounded ξ and for $|\xi|$ going to infinity, we use the *fiber-radially compactified cotangent bundle* $\overline{T}^* X$, see §E.1.3. The values of $|\xi| = \infty$ correspond to the boundary $\partial\overline{T}^* X$ (called the *fiber infinity*), which is diffeomorphic to the cosphere bundle over X.

Since p is a polynomial of order 2 in ξ, the rescaled symbol $\langle\xi\rangle^{-2}p$ and the rescaled Hamiltonian vector field $\langle\xi\rangle^{-1}H_p$ extend smoothly to $\overline{T}^* X$ (see Propositions E.4 and E.5). Therefore, we consider the flow

$$\exp(t\langle\xi\rangle^{-1}H_p) \quad \text{on } \{\langle\xi\rangle^{-2}p = 0\} \subset \overline{T}^* X. \tag{5.4.1}$$

Note that the symbol p depends on the choice of ω.

We will often use the following coordinates on the fibers of $\overline{T}^* Y \setminus 0$, where $Y \subset X$ is defined in (5.3.1):

$$\rho = (\xi_1^2 + |\xi'|^2_{g_1})^{-1/2} \in [0, \infty), \quad \hat{\xi} = (\hat{\xi}_1, \hat{\xi}') = \rho\xi \in S^*_x Y. \tag{5.4.2}$$

Here $S^* Y$ denotes the cosphere bundle:

$$S^* Y = \{(x, \hat{\xi}_1, \hat{\xi}') : \hat{\xi}_1^2 + |\hat{\xi}'|^2_{g_1} = 1\}.$$

Using (5.3.13), we calculate in the coordinates $(x, \rho, \hat{\xi})$,

$$\rho^2 p = 4x_1\hat{\xi}_1^2 - 4\rho\omega\hat{\xi}_1 + |\hat{\xi}'|^2_{g_1}. \tag{5.4.3}$$

5.4.1. Characteristic set. We first study the characteristic set

$$\{\langle\xi\rangle^{-2}p = 0\} \subset \overline{T}^* X.$$

We will show that for $\omega \neq 0$ it splits into two components. To define them, we use an auxiliary function

$$\varphi = \varphi(x_1) \in C^\infty(\overline{X}; \mathbb{R}), \tag{5.4.4}$$

such that, with $\varepsilon_1 > 0$ defined in (5.3.1)

$$\varphi(x_1) = \frac{1}{2}\log x_1 \quad \text{for } x_1 \geq \varepsilon_1^2; \tag{5.4.5}$$

$$\varphi'(x_1) > 0, \ x_1\varphi'(x_1) < 1 \quad \text{for all } x_1. \tag{5.4.6}$$

To construct $\varphi(x_1)$ it suffices to construct φ' which is straightforward, see Figure 5.4. We have

$$\Big| d\varphi - \frac{dx_1}{2x_1}\Big|_g < 1 \quad \text{on } M = \{x_1 > 0\}. \tag{5.4.7}$$

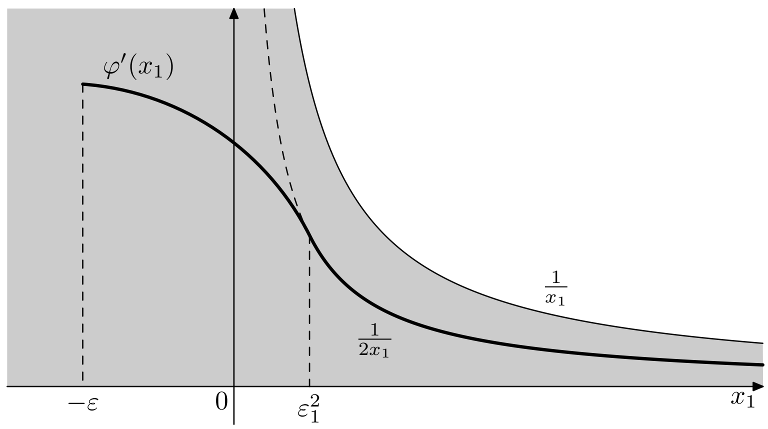


Figure 5.4. The derivative of the function φ from (5.4.4). Here (5.4.5) means that $\varphi' = \frac{1}{2x_1}$ for $x_1 \geq \varepsilon_1^2$ and (5.4.6) means that the graph of φ' lies in the shaded region.

On $\{x_1 \geq \varepsilon_1^2\}$ this follows from (5.4.5) and on $\{0 < x_1 \leq \varepsilon_1^2\}$ this follows from (5.3.3) and (5.4.6).

LEMMA 5.14 (Splitting of the characteristic set). *There exist closed ω-dependent sets $\Sigma_\pm \subset \overline{T}^*X$ such that (see Figure 5.5)*

$$\{\langle\xi\rangle^{-2}p = 0\} = \Sigma_+ \sqcup \Sigma_-, \quad \omega \in \mathbb{R} \setminus \{0\}; \tag{5.4.8}$$

$$\{\langle\xi\rangle^{-2}p = 0\} \cap \partial\overline{T}^*X = \widehat{\Sigma}_+ \sqcup \widehat{\Sigma}_-, \quad \omega \in \mathbb{R}; \tag{5.4.9}$$

here $\widehat{\Sigma}_\pm := \Sigma_\pm \cap \partial\overline{T}^*X$ are independent of ω. Moreover,

$$\pm\,\langle\xi\rangle^{-1}(\partial_\omega p + \partial_\xi p \cdot \partial_x \varphi) > 0 \quad \textit{on } \Sigma_\pm, \quad \omega \in \mathbb{R} \setminus \{0\} \tag{5.4.10}$$

where φ was defined in (5.4.4). Finally,

$$\Sigma_\pm \cap \partial\overline{T}^*M = \emptyset \quad \textit{for } \omega \in \mathbb{R}; \tag{5.4.11}$$

$$\Sigma_\pm \cap \overline{T}^*M = \emptyset \quad \textit{for } \pm\omega > 0. \tag{5.4.12}$$

REMARK. See (5.7.12) below for a general relativistic interpretation of the sets $\Sigma_\pm$ in terms of two halves of the light cone.

Proof. 1. Define the function

$$q := \partial_\omega p + \partial_\xi p \cdot \partial_x \varphi \;\in\; \mathrm{Poly}^1(T^*X).$$

We put

$$\Sigma_\pm := \{\langle\xi\rangle^{-2}p = 0\} \cap \{\pm\langle\xi\rangle^{-1}q \geq 0\}. \tag{5.4.13}$$

Clearly, $\Sigma_\pm$ are closed and their union equals the characteristic set. To show that $\Sigma_+ \cap \Sigma_- = \emptyset$, as well as (5.4.10), we need

$$\{\langle\xi\rangle^{-2}p = 0\} \cap \{\langle\xi\rangle^{-1}q = 0\} = \emptyset \quad \text{for } \omega \in \mathbb{R}\setminus\{0\}. \tag{5.4.14}$$

The splitting (5.4.9) follows from (5.4.8) and the fact that $\Sigma_\pm \cap \partial\overline{T}^*X$ are independent of ω (since $\partial_\omega p \in \mathrm{Poly}^1(T^*X)$, $\partial_\omega q \in \mathrm{Poly}^0(T^*X)$).

2. We first show (5.4.14) on $\overline{T}^*M = \{x_1 > 0\}$. By Proposition 5.12,

$$\{\langle\xi\rangle^{-2}p = 0\} \cap \partial\overline{T}^*M = \emptyset;$$

this proves (5.4.11). In the interior T^*M, we have by (5.3.12),

$$p = 0 \quad\Longrightarrow\quad \Big|\xi - \omega\frac{dx_1}{2x_1}\Big|_g = |\omega|,$$
$$q = \frac{2}{x_1}\Big(\Big\langle \xi - \omega\frac{dx_1}{2x_1}, d\varphi - \frac{dx_1}{2x_1}\Big\rangle_g - \omega\Big).$$

By (5.4.7) and the Cauchy–Schwarz inequality for the metric g, for $\omega \in \mathbb{R}\setminus 0$ the function q has the same sign as $-\omega$ on $\{p=0\}\cap T^*M$. This gives (5.4.12) and finishes the proof of (5.4.14) on $\overline{T}^*M$.

3. It remains to show (5.4.14) on $\overline{T}^*(X\setminus M) = \{x_1 \le 0\}$. Using (5.3.13) we compute on $\overline{Y}$

$$q = -4\xi_1 + 4(2x_1\xi_1 - \omega)\varphi'(x_1). \tag{5.4.15}$$

In the interior $T^*(X\setminus M)\cap\{p=0\}\cap\{q=0\}$, we have by (5.3.13),

$$4(x_1\xi_1 - \omega)\xi_1 + |\xi'|^2_{g_1(x_1,x')} = 0, \tag{5.4.16}$$
$$(2x_1\xi_1 - \omega)\varphi'(x_1) = \xi_1. \tag{5.4.17}$$

Since $\omega \ne 0$, we have $\xi_1 \ne 0$. Solving (5.4.17) for ω and substituting it into (5.4.16), we get a contradiction with (5.4.6).

On the fiber infinity $\partial\overline{T}^*(X\setminus M)\cap\{\langle\xi\rangle^{-2}p=0\}\cap\{\langle\xi\rangle^{-1}q=0\}$, we have in the coordinates (5.4.2),

$$4x_1\hat\xi_1^2 + |\hat\xi'|^2_{g_1(x_1,x')} = 0, \tag{5.4.18}$$
$$\hat\xi_1(2x_1\varphi'(x_1) - 1) = 0. \tag{5.4.19}$$

Since $x_1 \le 0$, (5.4.6) implies that $2x_1\varphi'(x_1) - 1 < 0$. Therefore, (5.4.19) implies that $\hat\xi_1 = 0$, giving a contradiction with (5.4.18) and finishing the proof. $\square$

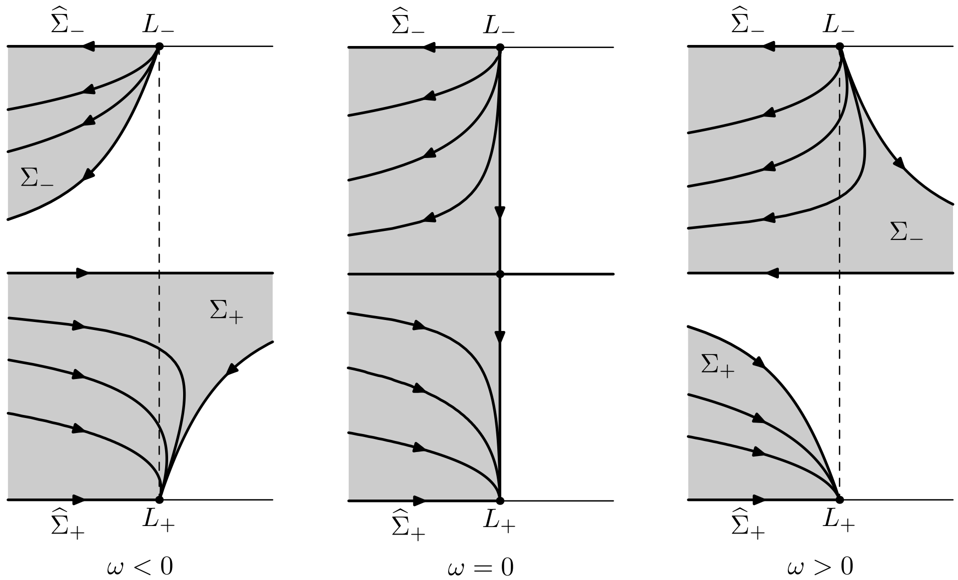


Figure 5.5. The flow of $\langle\xi\rangle^{-1}H_p$, with p given by (5.3.13), projected to the x_1, ξ_1 plane. The horizontal coordinate is x_1; the dashed line is $\{x_1 = 0\}$. The vertical coordinate is $\xi_1/(1+\langle\xi_1\rangle)$, so that the top and bottom lines correspond to the fiber infinity. The characteristic set $\{\langle\xi\rangle^{-2}p = 0\}$ is shaded. In the case $\omega = 0$, the midline $\{\xi = 0\}$ lies in the characteristic set and consists of fixed points of the flow.

5.4.2. Local dynamics of the Hamiltonian flow and the radial sets. We now analyse the Hamiltonian flow (5.4.1), concentrating on its behavior on $\overline{T}^*Y$ (see (5.3.1)). Using (5.3.13), we calculate in the coordinates induced by (5.3.2),

$$H_p = 4(2x_1\xi_1 - \omega)\partial_{x_1} - 4\xi_1^2\partial_{\xi_1} + H_{p_1},$$

where H_{p_1} is the sum of $-(\partial_{x_1}p_1)\partial_{\xi_1}$ and the generator of the geodesic flow of the metric $g_1(x_1, \bullet)$.

In the coordinates (5.4.2) near the fiber infinity $\partial\overline{T}^*Y$, we compute

$$\begin{aligned}\rho H_p = {} & 4(2x_1\hat{\xi}_1 - \rho\omega)\partial_{x_1} \\ & + 4\hat{\xi}_1^2(\hat{\xi}_1\rho\partial_\rho - |\hat{\xi}'|^2_{g_1}\partial_{\hat{\xi}_1} + \hat{\xi}_1\hat{\xi}'\cdot\partial_{\hat{\xi}'}) + \rho H_{p_1};\end{aligned} \tag{5.4.20}$$

the resulting flow is a reparametrization of (5.4.1).

Let $\widehat{\Sigma}_\pm$ be given by Lemma 5.14. Using the coordinates (5.4.2), define the *radial sets* (see Figure 5.5)

$$L_\pm := \{\rho = 0,\ x_1 = 0,\ \hat{\xi}' = 0,\ \hat{\xi}_1 = \mp 1\} = \widehat{\Sigma}_\pm \cap \{x_1 = 0\}; \tag{5.4.21}$$

note that $L_+ \sqcup L_-$ is the intersection of the conormal bundle to $\{x_1 = 0\}$ with the fiber infinity. See (5.7.15) below for a general relativistic interpretation of the sets $L_\pm$ in terms of the red-shift trajectories.

The importance of the sets $L_\pm$ comes from the following

LEMMA 5.15 (**Radial sources/sinks**). *$L_\pm$ are invariant under the Hamiltonian flow (5.4.1). Moreover, L_+ is a radial sink and L_- is a radial source for the flow (5.4.1) in the sense of Definition E.50, for all $\omega \in \mathbb{R}$.*

Proof. 1. We argue in a neighborhood of $L_+ \sqcup L_-$, using the coordinates (5.4.2) and replacing the flow (5.4.1) by its rescaling $\exp(t\rho H_p)$. Using the function $\rho^2 p_1(x, \xi) = |\hat{\xi}'|^2_{g_1}$, we write

$$L_+ \sqcup L_- = \{\rho = 0,\ x_1 = 0,\ \rho^2 p_1 = 0\}.$$

Using (5.4.20), we compute

$$\begin{aligned} \rho H_p \rho &= (4\hat{\xi}_1^3 + H_{p_1}\rho)\rho, \\ \rho H_p x_1 &= 8\hat{\xi}_1 x_1 - 4\omega\rho, \\ \rho H_p(\rho^2 p_1) &= (8\hat{\xi}_1^3 + 2H_{p_1}\rho)\rho^2 p_1 + 4(\rho^2 \partial_{x_1} p_1)(2\hat{\xi}_1 x_1 - \omega\rho). \end{aligned} \tag{5.4.22}$$

Here $H_{p_1}\rho$ is a symbol of order 0 and thus extends smoothly to the fiber infinity. We have $\rho^{-2} = \xi_1^2 + p_1$, thus $H_{p_1}(\rho^{-2}) = -2\xi_1 \partial_{x_1} p_1$. Since p_1 is a quadratic form in ξ', so is $\partial_{x_1} p_1$, and we obtain near $L_+ \sqcup L_-$,

$$\begin{aligned} \rho^2 \partial_{x_1} p_1 &= \mathcal{O}(\rho^2 p_1), \\ H_{p_1}\rho &= \hat{\xi}_1 \rho^2 \partial_{x_1} p_1 = \mathcal{O}(\rho^2 p_1). \end{aligned} \tag{5.4.23}$$

By (5.4.22) and (5.4.23), in a neighborhood of $L_\pm$ we have

$$\pm\rho H_p(x_1^2 + \rho^2 p_1) \le -4(x_1^2 + \rho^2 p_1) + C_0 \rho |x_1|, \tag{5.4.24}$$

$$\pm\rho H_p \rho \le -2\rho \tag{5.4.25}$$

where the constant $C_0 > 0$ depends on ω.

2. It follows from (5.4.24), (5.4.25) that

$$\begin{aligned} \pm\rho H_p f &\le -2f \quad \text{near } L_\pm, \\ f &:= x_1^2 + \rho^2 p_1 + C_0^2 \rho^2, \end{aligned}$$

where we have used the inequality $2C_0 \rho |x_1| \le x_1^2 + C_0^2 \rho^2$.

Since f is a quadratic defining function of $L_\pm$, we see that $L_\pm$ are invariant under the flow (5.4.1). Moreover, we have uniformly in (x, ξ) in a neighborhood of $L_\pm$,

$$\begin{aligned} e^{\pm t\rho H_p}(x, \xi) &\to L_\pm \quad \text{as } t \to \infty, \\ \rho(e^{\pm t\rho H_p}(x, \xi)) &\le e^{-2t}\rho(x, \xi) \quad \text{for } t \ge 0, \end{aligned}$$

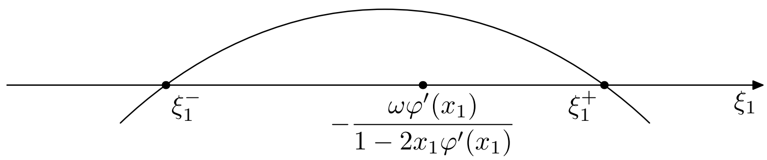


Figure 5.6. The graph of the left-hand side of (5.4.27), as a function of ξ_1.

where for the second inequality we used (5.4.25). This shows that L_+ is a radial sink and L_- is a radial source. □

5.4.3. Global dynamics of the Hamiltonian flow. We next study the global behavior of the flow on $\overline{T}^*X$. We use the following technical

LEMMA 5.16. *For $\omega \in \mathbb{R} \setminus 0$, we have (see Figure 5.5)*

$$\pm\langle\xi\rangle^{-1}H_p x_1 > 0 \quad on \quad \Sigma_\pm \cap \{x_1 \le 0\} \setminus L_\pm. \tag{5.4.26}$$

REMARK. See (5.7.13) for a general relativistic interpretation of (5.4.26) in terms of an observer being pushed farther into the black hole region.

Proof. 1. We first consider the case of $x_1 < 0$ and finite ξ. Fix $x_1 < 0, x', \xi'$. We write the equation $p = 0$ using (5.3.13):

$$4x_1\xi_1^2 - 4\omega\xi_1 + |\xi'|^2_{g_1} = 0. \tag{5.4.27}$$

This is a quadratic equation in ξ_1 with discriminant $16(\omega^2 - x_1|\xi'|^2_{g_1}) > 0$, therefore it has two roots $\xi_1^- < \xi_1^+$. See Figure 5.6.

By (5.4.10), (5.4.15), and (5.4.6), $\Sigma_\pm$ are characterized by the inequalities

$$\pm\Big(\xi_1 + \frac{\omega\varphi'(x_1)}{1-2x_1\varphi'(x_1)}\Big) < 0.$$

Substituting $\xi_1 = -\omega\varphi'(x_1)/(1-2x_1\varphi'(x_1))$ into the left-hand side of (5.4.27), we obtain a positive number. Therefore,

$$(x_1, x', \xi_1^\mp, \xi') \in \Sigma_\pm.$$

On the other hand, $H_p x_1 = \partial_{\xi_1} p$ is positive at ξ_1^- and negative at ξ_1^+; (5.4.26) follows.

2. The case of the fiber infinity $\partial\overline{T}^*X \cap \{x_1 < 0\}$ is considered similarly, writing the equation $\langle\xi\rangle^{-2}p = 0$ in the coordinates (5.4.2) as

$$4x_1\hat{\xi}_1^2 + |\hat{\xi}'|^2_{g_1} = 0$$

and solving it in $\hat{\xi}_1$.

By (5.4.21), the remaining case is $x_1 = 0$ and ξ finite. It is handled similarly to the first case, with (5.4.27) now a linear equation whose only root lies in $\Sigma_\mp$ for $\pm\omega > 0$. □

Since $\overline{X}$ is a manifold with boundary, some Hamiltonian trajectories may exit $\overline{T}^*X$ through the boundary $\overline{X}$. This is formalized in

DEFINITION 5.17. *We say that a trajectory $\gamma(t) = e^{t\langle\xi\rangle^{-1}H_p}(x,\xi)$, $(x,\xi) \in \overline{T}^*X$, **exits** $\overline{T}^*X$ at time $t_0 \in \mathbb{R}$, if the projection of $\gamma(t_0)$ to the base lies in $\{x_1 = -\varepsilon\} = \partial X$, with ε fixed in the beginning of §5.3.*

Combining (5.4.26) with (5.4.11), we see that

$$\pm\langle\xi\rangle H_p x_1 > 0 \quad \text{on } \widehat{\Sigma}_\pm \setminus L_\pm.$$

Together with Lemma 5.15 this gives the structure of the flow on the fiber infinity (see the proof of Lemma 5.19 below for a detailed argument; note that the restriction of $\langle\xi\rangle^{-1}H_p$ to $\partial\overline{T}^*X$ does not depend on ω by (5.3.14)):

LEMMA 5.18 (Global dynamics at fiber infinity). *Let $(x,\xi) \in \widehat{\Sigma}_\pm$ and put $\gamma(t) = e^{t\langle\xi\rangle^{-1}H_p}(x,\xi)$. Then:*

1. *$\gamma(t) \to L_\pm$ as $t \to \pm\infty$.*
2. *If $(x,\xi) \notin L_\pm$, then γ exits $\overline{T}^*X$ at some time t_0, $\pm t_0 \le 0$.*

The situation on the entire $\overline{T}^*X$ is more complicated because trajectories may become trapped inside the manifold M (see also Figure 5.5):

LEMMA 5.19 (Global dynamics in general). *There exists $\delta > 0$ such that for all $\omega \in \mathbb{R}\setminus\{0\}$ and $(x,\xi) \in \overline{T}^*X$, $\gamma(t) := e^{t\langle\xi\rangle^{-1}H_p}(x,\xi)$, the following holds:*

1. *If $(x,\xi) \in \Sigma_\pm$, then either*
   (a) *$\gamma(t) \to L_\pm$ as $t \to \pm\infty$, or*
   (b) *there exists $t_0 \ge 0$ such that $\gamma(t) \in \{x_1 > \delta\}$ for all t, $\pm t \ge t_0$.*
2. *If $(x,\xi) \in \Sigma_\pm \setminus L_\pm$, then there exists $t_0 \ge 0$ such that either*
   (a) *$\gamma(t)$ exits $\overline{T}^*X$ at time $\mp t_0$, or*
   (b) *$\gamma(t) \in \{x_1 > \delta\}$ for all t, $\mp t \ge t_0$.*

Proof. 1. We concentrate on the case $(x,\xi) \in \Sigma_+$; the case $(x,\xi) \in \Sigma_-$ can be handled similarly, reversing the direction of the flow. Using (5.3.15) we reduce to the case $\omega \in \{1,-1\}$.

By Lemma 5.15, there exists a neighborhood U_+ of L_+ such that

$$e^{t\langle\xi\rangle^{-1}H_p}(x,\xi) \to L_+ \quad \text{as } t \to \infty \quad \text{uniformly in } (x,\xi) \in U_+. \tag{5.4.28}$$

By Lemma 5.16, there exists $\delta > 0$ such that

$$\langle\xi\rangle^{-1}H_p x_1 \ge \delta \quad \text{on } \Sigma_+ \cap \{x_1 \le \delta\} \setminus U_+. \tag{5.4.29}$$

Indeed, if (5.4.29) failed for each δ, we could find a sequence of counterexamples; the limit of any its convergent subsequence would give a counterexample to (5.4.26).

2. We first prove part 1. Note that $\gamma(t) \in \Sigma_+$ for all t for which it is well-defined. Put

$$x_1(t) := x_1(\gamma(t)), \quad \dot{x}_1(t) = \langle\xi\rangle^{-1} H_p x_1(\gamma(t)).$$

By (5.4.26) we have $x_1(t) = -\varepsilon \;\Rightarrow\; \dot{x}_1(t) > 0$. Thus the trajectory $\gamma(t)$ cannot exit $\overline{T}^*X$ for positive times.

Next, if $\gamma(t) \in U_+$ for some $t \geq 0$, then case 1(a) holds. On the other hand, if $\gamma(t) \in \Sigma_+ \setminus U_+$ for all $t \geq 0$, then it follows from (5.4.29) that $x_1(t) \leq \delta \;\Rightarrow\; \dot{x}_1(t) \geq \delta$ for all $t \geq 0$. Then $\gamma(t) \in \{x_1 > \delta\}$ for all $t \geq 0$ large enough, thus case 1(b) holds.

3. We now prove part 2. Assume that case 2(a) does not hold; then $\gamma(t) \in \Sigma_+$ is well-defined for all $t \leq 0$. Since $(x,\xi) \notin L_+$, we may take a neighborhood V_+ of L_+ such that $(x,\xi) \notin V_+$. By (5.4.28), there exists $T > 0$ such that

$$e^{t\langle\xi\rangle^{-1}H_p}(U_+) \subset V_+ \quad \text{for all } t \geq T.$$

Since $(x,\xi) \notin V_+$, it follows that

$$\gamma(t) \in \Sigma_+ \setminus U_+ \quad \text{for all } t \leq -T.$$

By (5.4.29), we have $x_1(t) \leq \delta \;\Rightarrow\; \dot{x}_1(t) \geq \delta$ for all $t \leq -T$. Since $x_1(t) \geq -\varepsilon$ for all $t \leq -T$, we have $x_1(t) > \delta$ for all $t \leq -T$, thus case 2(b) holds. □

5.4.4. Hamiltonian flow and geodesics on M. We finally study the relation of Hamiltonian trajectories of p inside T^*M to geodesics on the original Riemannian manifold (M,g). This relation is exploited in §§5.6.3,6.2.3. Define the smooth map

$$j : T^*M \setminus 0 \to T^*M, \quad j(x,\xi) := \Big(x, \xi + |\xi|_g \frac{dx_1}{2x_1}\Big). \tag{5.4.30}$$

Then (5.3.12) implies that for each $\omega > 0$, the characteristic set of p intersected with $\overline{T}^*M$ is the image of a rescaled cosphere bundle under j:

$$\{\langle\xi\rangle^{-2}p = 0\} \cap \overline{T}^*M \;=\; j\big(\{(x,\xi) \in T^*M \colon |\xi|_g = \omega\}\big). \tag{5.4.31}$$

On M, geodesics are the trajectories of the Hamiltonian flow $\exp(tH_{|\xi|_g^2})$, and they give rise to trajectories of H_p on the characteristic set as follows:

LEMMA 5.20 (Geodesics and bicharacteristics of the modified Laplacian). *Fix $\omega > 0$ and let $(x,\xi) \in T^*M$, $|\xi|_g = \omega$. Then we have*

$$j\big(\exp(tH_{|\xi|_g^2})(x,\xi)\big) = \exp(s(t)\langle\xi\rangle^{-1}H_p)\big(j(x,\xi)\big) \tag{5.4.32}$$

for some smooth function $s : \mathbb{R} \to \mathbb{R}$, *depending on* (x,ξ) *and such that* $s(0) = 0$, $s'(t) > 0$ *for all* t. *Moreover,* $\lim_{t\to-\infty} s(t) = -\infty$.

Proof. 1. Put $\tilde\gamma(t) = (x(t),\xi(t)) := \exp(tH_{|\xi|_g^2})(x,\xi)$. Then $|\xi(t)|_g = \omega$, thus

$$j(\tilde\gamma(t)) = j_\omega(\tilde\gamma(t)), \quad t \in \mathbb{R},$$

where $j_\omega : T^*M \to T^*M$ is the symplectomorphism defined by

$$j_\omega(x,\xi) = \Big(x, \xi + \omega\frac{dx_1}{2x_1}\Big).$$

By (5.3.12), we have on T^*M

$$(x_1 p)\circ j_\omega = |\xi|_g^2 - \omega^2.$$

Since symplectomorphisms preserve Hamiltonian flows, we have

$$\partial_t j_\omega(\tilde\gamma(t)) = dj_\omega(\tilde\gamma(t))H_{|\xi|_g^2}(\tilde\gamma(t)) = H_{x_1p}\big(j_\omega(\tilde\gamma(t))\big) = x_1(t)H_p\big(j_\omega(\tilde\gamma(t))\big)$$

where $x_1(t) := x_1(x(t)) > 0$. Now (5.4.32) follows with

$$s(t) := \int_0^t x_1(r)\Big\langle \xi(r) + \omega\frac{dx_1}{2x_1}(r)\Big\rangle\,dr. \tag{5.4.33}$$

2. To show that $s(t) \to -\infty$ as $t \to -\infty$, we argue by contradiction, assuming that

$$s_- := \lim_{t\to-\infty} s(t) \in (-\infty, 0).$$

By (5.4.33) there exists a sequence $t_j \to -\infty$ with $x_1(t_j) \to 0$. Put $\gamma(s) := \exp(s\langle\xi\rangle^{-1}H_p)(j(x,\xi))$. Using the identity

$$x_1(t) = x_1(\gamma(s(t))) \tag{5.4.34}$$

and taking the limit along the subsequence t_j, we get $x_1(\gamma(s_-)) = 0$. Moreover, by (5.4.12) $\gamma(s_-) \in \Sigma_- \cap T^*X$. Finally, $x_1(\gamma(s)) > 0$ for $s \in (s_-, 0]$. Therefore

$$\gamma(s_-) \in \Sigma_- \cap \{x_1 = 0\} \setminus L_-, \quad \langle\xi\rangle^{-1}H_p x_1(\gamma(s_-)) = \partial_s|_{s=s_-} x_1(\gamma(s)) \geq 0.$$

This gives a contradiction with Lemma 5.16. □

REMARK. For a geodesic $\tilde\gamma(t) = \exp(tH_{|\xi|_g^2})(x,\xi)$ on M, $|\xi|_g = \omega > 0$, we have two possibilities as $t \to -\infty$ (see §6.1):

- either $\tilde\gamma$ *escapes* to infinity, that is $x_1(\tilde\gamma(t)) \to 0$ as $t \to -\infty$,
- or $\tilde\gamma$ is *trapped*, that is $x_1(\tilde\gamma(t)) \geq c > 0$ for some c and all $t \leq 0$.

Lemma 5.20 together with part 1 of Lemma 5.19 shows that when $\tilde\gamma(t)$ escapes as $t \to -\infty$, the corresponding trajectory of the flow (5.4.1)

$$\gamma(s) := \exp(s\langle\xi\rangle^{-1}H_p)(j(x,\xi)), \quad j(x,\xi) \in \Sigma_-$$

satisfies $x_1(\gamma(s)) \to 0$ as $s \to -\infty$ and thus converges to L_-.

As $t \to \infty$ we also have two possibilities for $\tilde{\gamma}(t)$, escape and trapping. However if $\tilde{\gamma}(t)$ escapes as $t \to \infty$ then the trajectory $\gamma(s)$ crosses $\{x_1 = 0\}$ for some finite $s_+ := \lim_{t\to\infty} s(t) > 0$ and exits $\overline{T}^*X$ at a later time. Indeed, we have $s_+ < \infty$ since otherwise we get a contradiction with part 2 of Lemma 5.19. Using the identity (5.4.34) we see that $\gamma(s_+) \in \Sigma_- \cap \{x_1 = 0\} \setminus L_-$, thus by (5.4.26) $x_1(\gamma(s)) < 0$ for all $s > s_+$. By part 2 of Lemma 5.19 this implies that γ exits T^*X. See Figure 5.5.

5.5. PROPAGATION ESTIMATES

In this section, we prove estimates for the operator $P_h(\omega)$ introduced in (5.3.8), as well as its formal adjoint $P_h(\omega)^*$ defined using the density (5.3.24). For that we combine the properties of the semiclassical principal symbol $p = \sigma_h(P_h(\omega))$ established in §5.4 with the propagation of singularities and radial source/sink estimates of §E.4, as well as hyperbolic estimates of §E.5. The resulting Propositions 5.27 and 5.28 will be the key components of the proof of meromorphic continuation of the scattering resolvent in §5.6. We will freely use the notation of Appendix E.

We first discuss the functional spaces used. Recall from §5.3 that the even extension $\overline{X}$ is a compact manifold with boundary $\partial X = \{x_1 = -\varepsilon\}$ and interior $X = \{x_1 > -\varepsilon\}$. Consider the semiclassical Sobolev spaces (see Definition E.25)

$$\bar{H}^s_h(X), \quad \dot{H}^s_h(\overline{X}), \quad s \in \mathbb{R}.$$

Recall that $\bar{H}^s_h(X)$ and $\dot{H}^{-s}_h(\overline{X})$ are dual to each other with respect to the standard L^2 pairing. The norms of these spaces depend on h, but the underlying Hilbert spaces $\bar{H}^s(X), \dot{H}^s(\overline{X})$ are h-independent. We will later assume that s is large enough depending on ω – see (5.5.10).

Since $P_h(\omega)$ is a second order semiclassical differential operator with coefficients smooth up to the boundary of X, it defines a bounded operator $\bar{H}^s(X) \to \bar{H}^{s-2}(X)$. However, our propagation estimates bound the norm $\|u\|_{\bar{H}^s_h(X)}$ in terms of $\|P_h(\omega)\|_{\bar{H}^{s-1}_h(X)}$, rather than the weaker norm $\|P_h(\omega)\|_{\bar{H}^{s-2}_h(X)}$. For that reason, we will apply $P_h(\omega)$ to distributions u satisfying

$$u \in \bar{H}^s(X), \quad P_h(\omega)u \in \bar{H}^{s-1}(X). \tag{5.5.1}$$

Similarly, we will apply the adjoint operator $P_h(\omega)^*$ to distributions v satisfying

$$v \in \dot{H}^{1-s}(\overline{X}), \quad P_h(\omega)^*v \in \dot{H}^{-s}(\overline{X}). \tag{5.5.2}$$

Our estimates are uniform in ω as long as

$$|\operatorname{Re}\omega| \le C_0, \quad |\operatorname{Im}\omega| \le C_0 h, \tag{5.5.3}$$

where $C_0 > 0$ is any fixed constant. In terms of the original spectral parameter $\lambda = h^{-1}\omega$, (5.5.3) corresponds to $|\operatorname{Re}\lambda| \le C_0 h^{-1}$, $|\operatorname{Im}\lambda| \le C_0$. In particular, by taking h small enough we can handle arbitrarily large values of λ, as long as $\operatorname{Im}\lambda$ is bounded. The constants in our estimates will be independent on h except where noted otherwise.

Given (5.5.3), we will parametrize $P_h(\omega)$ by $\omega_R = \operatorname{Re}\omega$ and $\omega_I = h^{-1}\operatorname{Im}\omega$, see the remark following Proposition 5.13. In this parametrization the semiclassical principal symbol of $P_h(\omega)$ is given by $p(x,\xi;\operatorname{Re}\omega)$, so that the results of §5.4 apply. The adjoint operator $P_h(\omega)^*$ has semiclassical principal symbol $p(x,\xi;\operatorname{Re}\omega)$ as well. The estimates below will be uniform in ω since the arguments used are stable under perturbations of ω; to simplify the presentation we prove them for fixed values of ω_R, ω_I.

5.5.1. Microlocal estimates. We first prove microlocal propagation estimates for the operators $P_h(\omega), P_h(\omega)^*$. See Figure 5.7 on page 345 for a phase space illustration of these estimates. We use the following notation:

DEFINITION 5.21 (Controlled set). *Assume that $V \subset \overline{T}^*X$ is an open set, fix ω satisfying (5.5.3), and let $p = p(x,\xi;\operatorname{Re}\omega)$ be given by (5.3.11). We say that a point $(x,\xi) \in \overline{T}^*X$ is **controlled** by V, and write $(x,\xi) \in \operatorname{Con}_p(V)$, if either*

(a) $(x,\xi) \notin \{\langle\xi\rangle^{-2}p = 0\}$, *or*

(b) *there exists $t \in \mathbb{R}$ such that $e^{t\langle\xi\rangle^{-1}H_p}(x,\xi) \in V$.*

Note that $\operatorname{Con}_p(V)$ is an open subset of $\overline{T}^*X$.

In this section, we will use pseudodifferential operators in $\Psi_h^0(X)$ which are compactly supported; that is, the supports of their Schwartz kernels do not intersect the boundary of $\overline{X} \times \overline{X}$. Our first estimate is a combination of elliptic bounds and propagation of singularities:

LEMMA 5.22. *Assume that $A, B \in \Psi_h^0(X)$ are compactly supported and*

$$\operatorname{WF}_h(A) \subset \operatorname{Con}_p(\operatorname{ell}_h(B)). \tag{5.5.4}$$

Then for ω satisfying (5.5.3), all s, N, and u, v satisfying (5.5.1), (5.5.2),

$$\begin{aligned}\|Au\|_{H_h^s} \le{}& Ch^{-1}\|P_h(\omega)u\|_{\bar H_h^{s-1}(X)}\\ &+ C\|Bu\|_{H_h^s} + Ch^N\|u\|_{\bar H_h^{-N}(X)},\end{aligned} \tag{5.5.5}$$

$$\begin{aligned}\|Av\|_{H_h^{1-s}} \le{}& Ch^{-1}\|P_h(\omega)^*v\|_{\dot H_h^{-s}(\overline{X})}\\ &+ C\|Bv\|_{H_h^{1-s}} + Ch^N\|v\|_{\dot H_h^{-N}(\overline{X})}.\end{aligned} \tag{5.5.6}$$

Proof. We will show (5.5.5); the estimate (5.5.6) is proved in exactly the same way. We claim that for all $u \in H^s_{\text{loc}}(X)$, $P_h(\omega)u \in H^{s-1}_{\text{loc}}(X)$,

$$\begin{aligned}\|Au\|_{H^s_h} \leq\, & Ch^{-1}\|\chi P_h(\omega)u\|_{H^{s-1}_h} \\ & + C\|Bu\|_{H^s_h} + Ch^N\|\chi u\|_{H^{-N}_h},\end{aligned} \tag{5.5.7}$$

where $\chi \in C^\infty_{\mathrm{c}}(X)$ is some cutoff function depending on A, B. For that we recall that $P_h(\omega) \in \Psi^2_h(X)$ and consider the following cases:

- $\mathrm{WF}_h(A) \cap \{\langle\xi\rangle^{-2}p = 0\} = \emptyset$: (5.5.7) follows by the semiclassical elliptic estimate, Theorem E.33;
- for each $(x,\xi) \in \mathrm{WF}_h(A)$, there exists $t \in \mathbb{R}$ with $e^{t\langle\xi\rangle^{-1}H_p}(x,\xi) \in \mathrm{ell}_h(B)$: (5.5.7) follows by semiclassical propagation of singularities, Theorem E.47 (strictly speaking, we apply Theorem E.47 to the operator $-P_h(\omega)$ in the case $t > 0$);
- a general A satisfying (5.5.4) can be written as a sum of operators falling into the above two cases, by a pseudodifferential partition of unity (Proposition E.30).

Now

$$\begin{aligned}\|\chi P_h(\omega)u\|_{H^{s-1}_h} &\leq C\|P_h(\omega)u\|_{\bar{H}^{s-1}_h(X)}, \\ \|\chi u\|_{H^{-N}_h} &\leq C\|u\|_{\bar{H}^{-N}_h(X)}.\end{aligned} \tag{5.5.8}$$

Therefore, (5.5.7) implies (5.5.5). ☐

The next statement uses the high regularity radial estimate (Theorem E.52) to bound u microlocally near the sets $L_\pm$, assuming s is large enough, and then propagates this bound to a neighborhood of the fiber infinity $\partial\overline{T}^*X$:

LEMMA 5.23 (High regularity microlocal estimate). *Let $A \in \Psi^0_h(X)$ be compactly supported. Then there exists $A_0 \in \Psi^0_h(X)$ compactly supported such that*

$$\mathrm{WF}_h(A_0) \cap \partial\overline{T}^*X = \emptyset \tag{5.5.9}$$

and for ω satisfying (5.5.3), s satisfying

$$s > \frac{1}{2} - \frac{\operatorname{Im}\omega}{h}, \tag{5.5.10}$$

all N, and u satisfying (5.5.1),

$$\begin{aligned}\|Au\|_{H^s_h} \leq\, & Ch^{-1}\|P_h(\omega)u\|_{\bar{H}^{s-1}_h(X)} + C\|A_0u\|_{H^s_h} \\ & + Ch^N\|u\|_{\bar{H}^{-N}_h(X)}.\end{aligned} \tag{5.5.11}$$

Proof. 1. Recall the radial sets $L_\pm$ introduced in (5.4.21); by Lemma 5.15, L_+ is a radial sink and L_- is a radial source for the flow $\exp(t\langle\xi\rangle^{-1}H_p)$. Let ρ be defined in (5.4.2). Put $\langle\xi\rangle := \sqrt{1+\xi_1^2+|\xi'|_{g_1}^2} = \rho^{-1}\sqrt{1+\rho^2}$. Using (5.3.26), (5.4.3), and (5.4.22), we compute

$$\begin{aligned}
&\pm\langle\xi\rangle^{-1}\Big(\sigma_h\big(h^{-1}\operatorname{Im}P_h(\omega)\big)+\Big(s-\frac{1}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}\Big)\Big|_{L_\pm}\\
&\quad=\pm\rho\Big(\sigma_h\big(h^{-1}\operatorname{Im}P_h(\omega)\big)+\Big(\frac{1}{2}-s\Big)\frac{H_p\rho}{\rho}\Big)\Big|_{L_\pm}\\
&\quad=4\Big(\frac{\operatorname{Im}\omega}{h}+s-\frac{1}{2}\Big)>0
\end{aligned}\tag{5.5.12}$$

where the last inequality follows from our assumption (5.5.10).

We apply the high regularity radial estimate, Theorem E.52, to $L_\pm$ and the operator $\mp P_h(\omega)$. Here the threshold regularity condition (E.4.39) follows from (5.5.12) (putting $b:=0$ in (E.4.38)). Using (5.5.8) we deduce that there exist compactly supported $A_\pm\in\Psi_h^0(X)$ such that $L_\pm\subset\operatorname{ell}_h(A_\pm)$ and

$$\|A_\pm u\|_{H_h^s}\le Ch^{-1}\|P_h(\omega)u\|_{\bar H_h^{s-1}(X)}+Ch^N\|u\|_{\bar H_h^{-N}(X)}.\tag{5.5.13}$$

2. By (5.4.9) and part 1 of Lemma 5.18,

$$\partial\overline{T}^*X\ \subset\ \operatorname{Con}_p\big(\operatorname{ell}_h(A_+)\cup\operatorname{ell}_h(A_-)\big).$$

Take compactly supported $A_0\in\Psi_h^0(X)$ satisfying (5.5.9) and elliptic on the compact set $\operatorname{WF}_h(A)\setminus\operatorname{Con}_p(\operatorname{ell}_h(A_+)\cup\operatorname{ell}_h(A_-))\subset T^*X$. Then

$$\operatorname{WF}_h(A)\ \subset\ \operatorname{Con}_p\big(\operatorname{ell}_h(A_0)\cup\operatorname{ell}_h(A_+)\cup\operatorname{ell}_h(A_-)\big).\tag{5.5.14}$$

Applying Lemma 5.22 with $B:=A_0^*A_0+A_+^*A_++A_-^*A_-$, we get

$$\begin{aligned}
\|Au\|_{H_h^s}\le{}&Ch^{-1}\|P_h(\omega)u\|_{\bar H_h^{s-1}(X)}+C\|A_0u\|_{H_h^s}\\
&+C\|A_+u\|_{H_h^s}+C\|A_-u\|_{H_h^s}+Ch^N\|u\|_{\bar H_h^{-N}(X)}.
\end{aligned}\tag{5.5.15}$$

Estimating $\|A_\pm u\|_{H_h^s}$ by (5.5.13), we obtain (5.5.11). □

For the adjoint operator $P_h(\omega)^*$, we cannot obtain an a priori bound microlocally near the radial sets of the form (5.5.13), since Theorem E.52 does not hold in the low regularity space H^{1-s}. We will instead put on the right-hand side the norm of v near the boundary $\overline{X}$. This norm will control v everywhere on $\partial\overline{T}^*X$ except the radial sets $L_\pm$, and the bound is extended to those using the low regularity radial estimate (Theorem E.54). Fix

$$\chi_1\in C_c^\infty(X),\quad \chi_1=1\quad\text{near }\overline{M}_{\text{even}}=\{x_1\ge 0\}.\tag{5.5.16}$$

Then we have

LEMMA 5.24 (Low regularity microlocal estimate). *Assume that $A \in \Psi_h^0(X)$ is compactly supported. Then there exists $A_0' \in \Psi_h^0(X)$ compactly supported such that (5.5.9) holds and for ω satisfying (5.5.3), s satisfying (5.5.10), all N, and v satisfying (5.5.2),*

$$
\begin{aligned}
\|Av\|_{H_h^{1-s}} \le & Ch^{-1}\|P_h(\omega)^*v\|_{\dot H_h^{-s}(\overline{X})} + C\|A_0'v\|_{H_h^{1-s}} \\
& + C\|(1-\chi_1)v\|_{\dot H_h^{1-s}(\overline{X})} + Ch^N\|v\|_{\dot H_h^{-N}(\overline{X})}.
\end{aligned}
\tag{5.5.17}
$$

Proof. 1. We calculate similarly to (5.5.12), under the condition (5.5.10)

$$
\begin{aligned}
\mp\langle\xi\rangle^{-1}\Big(\sigma_h\big(h^{-1}\operatorname{Im}P_h(\omega)^*\big) + \Big(\frac{1}{2}-s\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}\Big)\Big|_{L_\pm} \\
= 4\Big(\frac{\operatorname{Im}\omega}{h} + s - \frac{1}{2}\Big) > 0
\end{aligned}
\tag{5.5.18}
$$

We apply the low regularity radial estimate, Theorem E.54, to $L_\pm$ and the operator $\pm P_h(\omega)^*$, with s replaced by $1-s$ and the threshold regularity condition (E.4.47) following from (5.5.18) (putting $b := 0$ in (E.4.38)). It follows that there exist compactly supported $A_\pm', B_\pm \in \Psi_h^0(X)$ such that

$$
L_\pm \subset \operatorname{ell}_h(A_\pm'), \quad \operatorname{WF}_h(B_\pm) \cap (L_+ \sqcup L_-) = \emptyset,
$$

and the following estimate holds (where we use an analog of (5.5.8)):

$$
\begin{aligned}
\|A_\pm'v\|_{H_h^{1-s}} \le & Ch^{-1}\|P_h(\omega)^*v\|_{\dot H_h^{-s}(\overline{X})} \\
& + C\|B_\pm v\|_{H_h^{1-s}} + Ch^N\|v\|_{\dot H_h^{-N}(\overline{X})},
\end{aligned}
\tag{5.5.19}
$$

2. By (5.4.9) and part 2 of Lemma 5.18,

$$
\partial\overline{T}^*X \setminus (L_+ \sqcup L_-) \ \subset \ \operatorname{Con}_p(\{1-\chi_1 \neq 0\}).
$$

Therefore,

$$
\partial\overline{T}^*X \ \subset \ \operatorname{Con}_p(\operatorname{ell}_h(A_+') \cup \operatorname{ell}_h(A_-') \cup \{1-\chi_1 \neq 0\}).
$$

Arguing as in the proof of Lemma 5.23, we construct A_0' and $\widetilde\chi \in C_c^\infty(X)$ such that (5.5.9) holds and

$$
\begin{aligned}
\operatorname{WF}_h(A) \ &\subset \ \operatorname{Con}_p(\operatorname{ell}_h(A_0') \cup \operatorname{ell}_h(A_+') \cup \operatorname{ell}_h(A_-') \cup \{\widetilde\chi(1-\chi_1) \neq 0\}), \\
\operatorname{WF}_h(B_\pm) \ &\subset \ \operatorname{Con}_p(\operatorname{ell}_h(A_0') \cup \{\widetilde\chi(1-\chi_1) \neq 0\}).
\end{aligned}
$$

By Lemma 5.22, this implies the estimates

$$
\begin{aligned}
\|Av\|_{H_h^{1-s}} \le & Ch^{-1}\|P_h(\omega)^*v\|_{\dot H_h^{-s}(\overline{X})} + C\|A_0'v\|_{H_h^{1-s}} \\
& + C\|A_+'v\|_{H_h^{1-s}} + C\|A_-'v\|_{H_h^{1-s}} \\
& + C\|(1-\chi_1)v\|_{\dot H_h^{1-s}(\overline{X})} + Ch^N\|v\|_{\dot H_h^{-N}(\overline{X})},
\end{aligned}
\tag{5.5.20}
$$

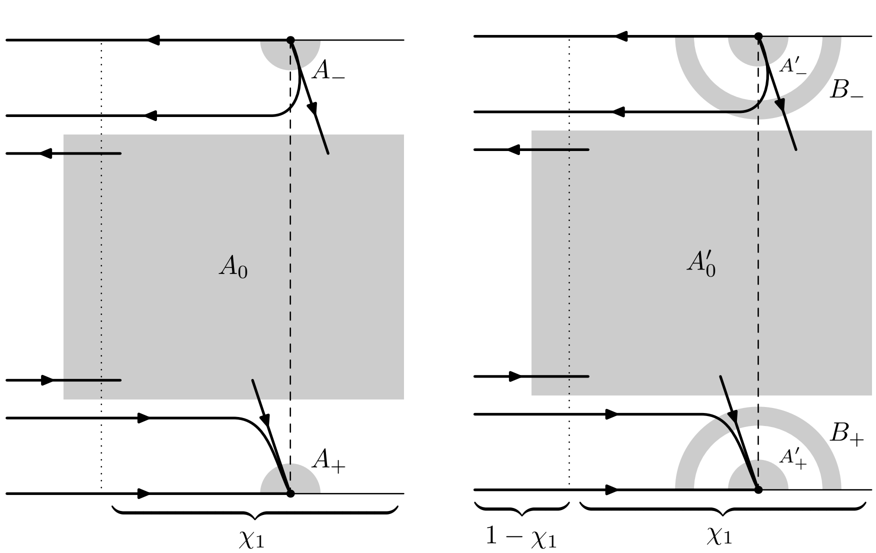


Figure 5.7. A phase space picture of the proof of (5.5.27), on the left, and (5.5.28), on the right. The coordinates are same as in Figure 5.5, in particular the dashed line is $\{x_1 = 0\}$. For (5.5.27), we use hyperbolic estimates to bound u via $\chi_1 u$; χ_1 is controlled by $A_0, A_\pm$; and $A_\pm$ are controlled using high regularity radial estimates. For (5.5.28), we use hyperbolic estimates to bound $(1-\chi_1)v$; χ_1 is controlled by $A'_0, A'_\pm, 1-\chi_1$; $B_\pm$ is controlled by $A'_0, 1-\chi_1$; and $A'_\pm$ is controlled by $B_\pm$ using low regularity radial estimates.

$$\begin{aligned}\|B_\pm v\|_{H_h^{1-s}} \leq\, & Ch^{-1}\|P_h(\omega)^* v\|_{\dot H_h^{-s}(\overline{X})} + C\|A'_0 v\|_{H_h^{1-s}} \\ & + C\|(1-\chi_1)v\|_{\dot H_h^{1-s}(\overline{X})} + Ch^N\|v\|_{\dot H_h^{-N}(\overline{X})}.\end{aligned} \tag{5.5.21}$$

Substituting (5.5.19) into (5.5.20) and combining the result with (5.5.21), we obtain (5.5.17). □

5.5.2. Hyperbolic estimates and global regularity.

The microlocal estimates proved above are only valid away from the boundary $\overline{X}$. To estimate the functions u, v near the boundary, we use the semiclassical hyperbolicity of $P_h(\omega)$ in $\{x_1 < 0\}$ when ω is away from 0:

LEMMA 5.25. *Let χ_1 satisfy (5.5.16) and assume that ω satisfies the following strengthening of (5.5.3) for some fixed $C_0 > 0$:*

$$C_0^{-1} \leq |\operatorname{Re}\omega| \leq C_0, \quad |\operatorname{Im}\omega| \leq C_0 h. \tag{5.5.22}$$

Then for all s and u, v satisfying (5.5.1), (5.5.2),

$$\|u\|_{\bar H_h^s(X)} \leq Ch^{-1}\|P_h(\omega)u\|_{\bar H_h^{s-1}(X)} + C\|\chi_1 u\|_{H_h^s}, \tag{5.5.23}$$

$$\|(1-\chi_1)v\|_{\dot H_h^{1-s}(\overline{X})} \leq Ch^{-1}\|P_h(\omega)^* v\|_{\dot H_h^{-s}(\overline{X})}. \tag{5.5.24}$$

REMARK. The proof below can be interpreted in terms of well-posedness of the Cauchy problem for hyperbolic equations as follows. We view $P_h(\omega)$, $P_h(\omega)^*$ as semiclassically hyperbolic operators on $\{x_1 < 0\}$ where the function x_1 takes the role of time. Then $\|\chi_1 u\|$ in (5.5.23) controls the Cauchy data of u on $\{x_1 = -\delta\}$ for any small $\delta > 0$. This lets us estimate u on $\{x_1 < -\delta\}$, giving (5.5.23). For (5.5.24), the support property for $v \in \dot{H}^{1-s}(\overline{X})$ implies that v vanishes past the boundary $\partial X = \{x_1 = -\varepsilon\}$, thus it has zero Cauchy data on $\{x_1 = -\varepsilon\}$.

Proof. Consider the defining function $t := 1 + x_1/\varepsilon$ on $\overline{X}$ and the product structure (t, x') on $\{t < 1\} = \{x_1 < 0\}$, see (5.3.2). By (5.3.13), the operators $P_h(\omega), P_h(\omega)^*$ are semiclassically hyperbolic with respect to t on $\{t < 1\}$, in the sense of Definition E.55. (See Proposition 5.12 and the paragraph following (5.4.27).) Then (5.5.23), (5.5.24) follow immediately from the semiclassical hyperbolic estimate, Theorem E.57 (with $\chi_2 \equiv 1$). $\square$

In general $P_h(\omega)$ may not be semiclassically hyperbolic (take $\omega = 0$, $\xi' = 0$ in (E.5.8)) so the constants in (5.5.23), (5.5.24) depend on h:

LEMMA 5.26. *Let χ_1 satisfy (5.5.16) and assume that ω satisfies (5.5.3). Then for all s and u, v satisfying (5.5.1), (5.5.2),*

$$\|u\|_{\bar{H}^s(X)} \leq C\|P_h(\omega)u\|_{\bar{H}^{s-1}(X)} + C\|\chi_1 u\|_{H^s}, \tag{5.5.25}$$

$$\|(1-\chi_1)v\|_{\dot{H}^{1-s}(\overline{X})} \leq C\|P_h(\omega)^* v\|_{\dot{H}^{-s}(\overline{X})}, \tag{5.5.26}$$

with the constants in (5.5.25), (5.5.26) depending on h.

Proof. Consider the product structure (t, x') as in the proof of Lemma 5.25. By Proposition 5.12, the operators $P_h(\omega), P_h(\omega)^*$ are hyperbolic with respect to t on $\{t < 1\}$ in the sense of Definition E.55. Then (5.5.25), (5.5.26) follow immediately from the nonsemiclassical hyperbolic estimate, Theorem E.56 (with $\chi_2 \equiv 1$). $\square$

Combining Lemmas 5.23, 5.24, and 5.26, we arrive to the following statement, which is used in §5.6 below to prove the Fredholm property of $P_h(\omega)$. Here we allow the constants depend on h in an unspecified way, thus we replace the norms $\|\bullet\|_{H^s_h}$ by $\|\bullet\|_{H^s}$ and remove the prefactors h^{-1}, h^N from the estimates. See Figure 5.7.

PROPOSITION 5.27. *For ω satisfying (5.5.3), s satisfying (5.5.10), all N, and u, v satisfying (5.5.1), (5.5.2), we have the estimates*

$$\|u\|_{\bar{H}^s(X)} \leq C\|P_h(\omega)u\|_{\bar{H}^{s-1}(X)} + C\|u\|_{\bar{H}^{-N}(X)}, \tag{5.5.27}$$

$$\|v\|_{\dot{H}^{1-s}(\overline{X})} \leq C\|P_h(\omega)^* v\|_{\dot{H}^{-s}(\overline{X})} + C\|v\|_{\dot{H}^{-N}(\overline{X})} \tag{5.5.28}$$

with the constants in (5.5.27), (5.5.28) depending on h.

Proof. Fix χ_1 satisfying (5.5.16). To see (5.5.27), it suffices to substitute (5.5.11) with $A := \chi_1$ into (5.5.25). Here $\mathrm{WF}_h(A_0) \cap \partial T^*M = \emptyset$, thus $A_0 \in \Psi_h^{-s-N}(X)$ for all N which implies $\|A_0 u\|_{H^s} \le C\|u\|_{\bar{H}^{-N}(X)}$.

To see (5.5.28), we first substitute (5.5.26) into (5.5.17) with $A := \chi_1$, to estimate $\|\chi_1 v\|_{H^{1-s}}$. Combining the result with (5.5.26), we obtain an estimate on $\|v\|_{\dot{H}^{1-s}(\overline{X})}$. ☐

5.5.3. Invertibility in the upper half-plane. The Fredholm property of $P_h(\omega)$, following from Proposition 5.27, is not enough to conclude that $P_h(\omega)^{-1}$ is meromorphic. Indeed, it could happen that $P_h(\omega)$ is not invertible for any ω, for instance if it were a family of Fredholm operators of nonzero index. Here we show that this is not the case, by proving that $P_h(\omega)$ is invertible for some ω in the upper half-plane:

PROPOSITION 5.28. *Fix $s \in \mathbb{R}$. Then there exists $\beta > 0$ such that $s > \frac{1}{2} - \beta$ and for*

$$\omega := 1 + ih\beta,$$

small enough h, and u, v satisfying (5.5.1), (5.5.2), we have the estimates

$$\|u\|_{\bar{H}_h^s(X)} \le Ch^{-1}\|P_h(\omega)u\|_{\bar{H}_h^{s-1}(X)}, \tag{5.5.29}$$

$$\|v\|_{\dot{H}_h^{1-s}(\overline{X})} \le Ch^{-1}\|P_h(\omega)^*v\|_{\dot{H}_h^{-s}(\overline{X})}. \tag{5.5.30}$$

Proof. 1. Fix χ_1 satisfying (5.5.16). We first claim that it is enough to prove the following estimates for some choice of $\chi \in C_c^\infty(X)$:

$$\|\chi_1 u\|_{H_h^s} \le Ch^{-1}\|\chi P_h(\omega)u\|_{H_h^{s-1}} + Ch^{1/2}\|\chi u\|_{H_h^{s-1/2}}, \tag{5.5.31}$$

$$\begin{aligned} \|\chi_1 v\|_{H_h^{1-s}} \le\, & Ch^{-1}\|\chi P_h(\omega)^* v\|_{H_h^{-s}} + C\|\chi(1-\chi_1)v\|_{H_h^{1-s}} \\ & + Ch^{1/2}\|\chi v\|_{H_h^{1/2-s}}. \end{aligned} \tag{5.5.32}$$

Indeed, combining these estimates with the hyperbolic bounds (5.5.23), (5.5.24) similarly to the proof of Proposition 5.27, and taking h small enough to remove the $\mathcal{O}(h^{1/2})$ remainder, we obtain (5.5.29), (5.5.30).

2. To show (5.5.31), (5.5.32), we use Lemma E.49, which is a basic positive commutator estimate. Let $\Sigma_\pm$ be defined in Lemma 5.14, where we put $\omega = 1$. Take cutoff functions

$$\psi_\pm \in C_c^\infty(\overline{T}^*X; [0,1]), \quad \operatorname{supp}\psi_\pm \cap \Sigma_\mp = \emptyset$$

with the following properties:

$$\psi_\pm = 1 \quad \text{near } \Sigma_\pm \cap \operatorname{supp}\chi_1, \quad \pm\langle\xi\rangle^{-1}H_p\psi_\pm \ge 0 \quad \text{near } \Sigma_\pm.$$

The existence of such functions follows from (5.4.26), where we make $\psi_\pm$ be increasing functions of x_1 near $\Sigma_\pm$. With $\beta > 0$ to be chosen later, we put

$$f_\pm := e^{\beta\varphi}\psi_\pm, \quad \tilde f_\pm := e^{-\beta\varphi}\psi_\pm,$$

where $\varphi \in C^\infty(\overline{X};\mathbb{R})$ is the function introduced in (5.4.4).

3. Using (5.3.26) and (5.4.10), we compute

$$\begin{aligned} &\mp\langle\xi\rangle^{-1}\big(H_p f_\pm + \sigma_h(h^{-1}\operatorname{Im} P_h(\omega))f_\pm\big)\\ &= \mp e^{\beta\varphi}\langle\xi\rangle^{-1}\big(H_p\psi_\pm + \beta(H_p\varphi + \partial_\omega p)\psi_\pm\big)\\ &\le -\delta\beta f_\pm \quad \text{near } \Sigma_+ \sqcup \Sigma_- \end{aligned} \tag{5.5.33}$$

for some $\delta > 0$ independent of β. Here we used that $\operatorname{supp}\psi_\pm \cap \Sigma_\mp = \emptyset$.

Fix β large enough depending on s such that near $\Sigma_+ \sqcup \Sigma_-$,

$$\mp\langle\xi\rangle^{-1}\Big(H_p f_\pm + \sigma_h\big(h^{-1}\operatorname{Im} P_h(\omega)\big)f_\pm + \Big(s - \frac{1}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}f_\pm\Big) \le -f_\pm. \tag{5.5.34}$$

Using (5.4.8), take compactly supported $\widehat{A}_\pm, \widehat{B} \in \Psi_h^0(X)$ such that

$$\begin{gathered} \operatorname{supp}\chi_1 \ \subset\ \operatorname{ell}_h(\widehat{A}_+) \cup \operatorname{ell}_h(\widehat{A}_-) \cup \operatorname{ell}_h(\widehat{B}),\\ \operatorname{WF}_h(\widehat{A}_\pm) \subset \{f_\pm > 0\}, \quad \operatorname{WF}_h(\widehat{B}) \cap \{\langle\xi\rangle^{-2}p = 0\} = \emptyset, \end{gathered} \tag{5.5.35}$$

and (5.5.34) holds in a neighborhood of $\overline{T}^*X \setminus \operatorname{ell}_h(\widehat{B})$.

By Lemma E.49 applied to $f_\pm$ and the operators $\mp P_h(\omega)$, we get

$$\begin{aligned} \|\widehat{A}_\pm u\|_{H_h^s} \le{}& Ch^{-1}\|\chi P_h(\omega)u\|_{H_h^{s-1}}\\ &+ C\|\widehat{B}u\|_{H_h^s} + Ch^{1/2}\|\chi u\|_{H_h^{s-1/2}}. \end{aligned} \tag{5.5.36}$$

By the elliptic estimate, Theorem E.33, $\|\widehat{B}u\|_{H_h^s}$ is bounded in terms of $\|\chi P_h(\omega)u\|_{H_h^{s-2}}$. By a pseudodifferential partition of unity (Proposition E.30) and Theorem E.33, $\|\chi_1 u\|_{H_h^s}$ is bounded in terms of $\|\widehat{A}_+ u\|_{H_h^s}$, $\|\widehat{A}_- u\|_{H_h^s}$, and $\|\widehat{B}u\|_{H_h^s}$. Combining these estimates with (5.5.36), we obtain (5.5.31).

4. Similarly to (5.5.33), we compute

$$\begin{aligned} &\pm\langle\xi\rangle^{-1}\big(H_p\tilde f_\pm + \sigma_h(h^{-1}\operatorname{Im} P_h(\omega)^*)\tilde f_\pm\big)\\ &= \pm e^{-\beta\varphi}\langle\xi\rangle^{-1}\big(H_p\psi_\pm - \beta(H_p\varphi + \partial_\omega p)\psi_\pm\big)\\ &\le -\delta\beta\tilde f_\pm \quad \text{near } (\Sigma_+ \sqcup \Sigma_-) \cap \operatorname{supp}\chi_1, \end{aligned} \tag{5.5.37}$$

where we used that $H_p\psi_\pm = 0$ near $(\Sigma_+ \sqcup \Sigma_-) \cap \operatorname{supp}\chi_1$. Arguing as in step 3, we see that for $\beta > 0$ large enough, there exist $\widehat{A}_\pm, \widehat{B}$ satisfying (5.5.35) and such that near $\overline{T}^*X \setminus \big(\{\chi_1 \neq 1\} \cup \operatorname{ell}_h(\widehat{B})\big)$,

$$\pm\langle\xi\rangle^{-1}\Big(H_p\tilde{f}_\pm + \sigma_h\big(h^{-1}\operatorname{Im} P_h(\omega)^*\big)\tilde{f}_\pm + \Big(\frac{1}{2}-s\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}\tilde{f}_\pm\Big) \leq -\tilde{f}_\pm.$$

By Lemma E.49 applied to $\tilde{f}_\pm$, the operators $\pm P_h(\omega)^*$, and $B := \widehat{B}^*\widehat{B} + \chi(1-\chi_1)$ with a correct choice of χ, we obtain

$$\begin{aligned}\|\widehat{A}_\pm v\|_{H_h^{1-s}} \leq{}& Ch^{-1}\|\chi P_h(\omega)^* v\|_{H_h^{-s}} + C\|\chi(1-\chi_1)v\|_{H_h^{1-s}}\\ &+ C\|\widehat{B}v\|_{H_h^{1-s}} + Ch^{1/2}\|\chi v\|_{H_h^{1/2-s}}.\end{aligned}$$

Arguing as in step 3, we see that this estimate implies (5.5.32). □

5.6. MEROMORPHIC CONTINUATION

In this section, we use the estimates proved in §5.5 to show that the family of operators $P(\lambda)$ (see Definition 5.11) has a meromorphic inverse $P(\lambda)^{-1}$. We next use $P(\lambda)^{-1}$ to show meromorphic continuation of the scattering resolvent (5.0.3). Finally we give some applications of the estimates of §5.5 to high frequency resolvent bounds.

5.6.1. Meromorphic inverse for the modified Laplacian. The main results of §5.5, Propositions 5.27 and 5.28, estimate $\|u\|_{\bar{H}_h^s(X)}$ in terms of $\|P_h(\omega)u\|_{\bar{H}_h^{s-1}(X)}$, where $P_h(\omega)$ is the semiclassical rescaling of $P(\lambda)$ defined in (5.3.8). However, $P(\lambda)$, as a second order differential operator, is not bounded $\bar{H}^s(X) \to \bar{H}^{s-1}(X)$. To resolve this issue, we consider the domain of $P(\lambda)$,

$$\mathcal{X}^s := \{u \in \bar{H}^s(X) \mid P(0)u \in \bar{H}^{s-1}(X)\}, \tag{5.6.1}$$

endowed with the semiclassical norm

$$\|u\|_{\mathcal{X}_h^s} = \big(\|u\|^2_{\bar{H}_h^s(X)} + \|P_h(0)u\|^2_{\bar{H}_h^{s-1}(X)}\big)^{1/2}.$$

It is easy to see that $\mathcal{X}^s$ is a Hilbert space, by identifying it with the closed subspace

$$\{(u,f) \mid P(0)u = f\} \ \subset\ \bar{H}^s(X) \oplus \bar{H}^{s-1}(X).$$

Moreover, the norms $\|\bullet\|_{\mathcal{X}_h^s}$ for different h are equivalent, with constants depending on h. Since $P(\lambda) - P(0)$ is a first order differential operator, it is bounded $\bar{H}^s(X) \to \bar{H}^{s-1}(X)$; therefore

$$P(\lambda) : \mathcal{X}^s \to \bar{H}^{s-1}(X) \tag{5.6.2}$$

is a family of bounded operators holomorphic in $\lambda \in \mathbb{C}$.

Before we prove meromorphy of $P(\lambda)^{-1}$, let us introduce notation for the kernel, cokernel, and kernel of the adjoint of (5.6.2):

$$\ker^s P(\lambda) = \{u \in \bar{H}^s(X) \mid P(\lambda)u = 0\}, \tag{5.6.3}$$

$$\operatorname{coker}^s P(\lambda) = \{v \in \dot{H}^{1-s}(\overline{X}) \mid \langle P(\lambda)u, v\rangle_{L^2} = 0 \text{ for all } u \in \mathcal{X}^s\}, \tag{5.6.4}$$

$$\ker^{1-s} P(\lambda)^* = \{v \in \dot{H}^{1-s}(\overline{X}) \mid P(\lambda)^* v = 0\}. \tag{5.6.5}$$

Here we recall from §E.1.8 that $\dot{H}^{1-s}(\overline{X})$ is the dual to $\bar{H}^{s-1}(X)$ with respect to the L^2 inner product.

LEMMA 5.29. *For all s, λ, we have* $\operatorname{coker}^s P(\lambda) = \ker^{1-s} P(\lambda)^*$.

Proof. 1. By (5.5.26) (or applying directly Proposition E.62), we have

$$v \in \ker^{1-s} P(\lambda)^* \implies \operatorname{supp} v \subset \overline{M}_{\text{even}} = \{x_1 \geq 0\}. \tag{5.6.6}$$

In particular, v is compactly supported inside X. Therefore, by Lemma E.46, for all $u \in \mathcal{X}^s$ and $v \in \ker^{1-s} P(\lambda)^*$,

$$\langle P(\lambda)u, v\rangle_{L^2} = \langle u, P(\lambda)^* v\rangle_{L^2} = 0.$$

This implies that $\ker^{1-s} P(\lambda)^* \subset \operatorname{coker}^s P(\lambda)$.

2. Assume now that $v \in \operatorname{coker}^s P(\lambda)$. Then we have

$$\langle u, P(\lambda)^* v\rangle_{L^2} = \langle P(\lambda)u, v\rangle_{L^2} = 0, \quad u \in C^\infty(\overline{X}).$$

This implies that $P(\lambda)^* v = 0$ and thus $\operatorname{coker}^s P(\lambda) \subset \ker^{1-s} P(\lambda)^*$. □

The following theorem is the central result of this chapter. It uses the results of §5.5 together with Fredholm theory to show meromorphy of the inverse to $P(\lambda)$:

THEOREM 5.30 (Meromorphic inverse for the modified Laplacian). *Fix $s \in \mathbb{R}$. Then (5.6.2) is a Fredholm operator of index zero for $\operatorname{Im}\lambda > \frac{1}{2} - s$, and it has a meromorphic inverse with poles of finite rank,*

$$P(\lambda)^{-1} : \bar{H}^{s-1}(X) \to \mathcal{X}^s, \quad \operatorname{Im}\lambda > \frac{1}{2} - s. \tag{5.6.7}$$

Proof. 1. By Proposition 5.27, recalling the definition (5.3.8) of the semiclassically rescaled operator $P_h(\omega)$, we have for $\operatorname{Im}\lambda > \frac{1}{2} - s$ and all $u \in \mathcal{X}^s, v \in \dot{H}^{1-s}(\overline{X}), P(0)v \in \dot{H}^{-s}(\overline{X})$,

$$\|u\|_{\mathcal{X}^s} \leq C\|P(\lambda)u\|_{\bar{H}^{s-1}(X)} + C\|u\|_{\bar{H}^{s-1}(X)}, \tag{5.6.8}$$

$$\|v\|_{\dot{H}^{1-s}(\overline{X})} \leq C\|P(\lambda)^* v\|_{\dot{H}^{-s}(\overline{X})} + C\|v\|_{\dot{H}^{-s}(\overline{X})}, \tag{5.6.9}$$

where the constant C depends on s and λ. By Rellich–Kondrachov's theorem [HöIII, Theorem B.2.2] the embeddings $\mathcal{X}^s \to \bar{H}^{s-1}(X)$, $\dot{H}^{1-s}(\overline{X}) \to$

$\dot{H}^{-s}(\overline{X})$ are compact operators. We now follow a standard argument from functional analysis to establish the Fredholm property of $P(\lambda)$.

2. We first show the following statement: if u_j is a bounded sequence in $\mathcal{X}^s$ such that $P(\lambda)u_j$ converges in $\bar{H}^{s-1}(X)$, then u_j has a subsequence which converges in $\mathcal{X}^s$. Indeed, since the embedding $\mathcal{X}^s \to \bar{H}^{s-1}(X)$ is compact and $\|u_j\|_{\mathcal{X}^s}$ is bounded, by passing to a subsequence we may assume that u_j converges in $\bar{H}^{s-1}(X)$. Applying (5.6.8) to the differences $u_j - u_k$, we see that u_j is a Cauchy sequence in $\mathcal{X}^s$; therefore, it converges.

It follows immediately that $\ker^s P(\lambda)$ is finite dimensional. Indeed, otherwise there exists an $\mathcal{X}^s$-orthonormal sequence $u_j \in \ker^s P(\lambda)$; $P(\lambda)u_j = 0$ converges and $\|u_j\|_{\mathcal{X}^s}$ is bounded, yet u_j has no convergent subsequence.

3. We next show that the image of (5.6.2) is a closed subspace of $\bar{H}^{s-1}(X)$. Take a convergent sequence

$$f_j \in P(\lambda)(\mathcal{X}^s), \quad f_j \to f_\infty \quad \text{in } \bar{H}^{s-1}(X).$$

We may write $f_j = P(\lambda)u_j$, where $u_j \in \mathcal{X}^s$ is $\mathcal{X}^s$-orthogonal to $\ker^s P(\lambda)$.

Assume first that $\|u_j\|_{\mathcal{X}^s}$ is bounded. Then by step 2, by passing to a subsequence we can make u_j converge to some u_∞ in $\mathcal{X}^s$. It follows that $f_\infty = P(\lambda)u_\infty$ lies in the image of (5.6.2).

If on the contrary $\|u_j\|_{\mathcal{X}^s}$ is not bounded, then by passing to a subsequence we may assume that $\|u_j\|_{\mathcal{X}^s} \to \infty$. Put

$$\tilde{u}_j := \frac{u_j}{\|u_j\|_{\mathcal{X}^s}}, \quad \tilde{f}_j = P(\lambda)\tilde{u}_j = \frac{f_j}{\|u_j\|_{\mathcal{X}^s}}.$$

Then $\|\tilde{u}_j\|_{\mathcal{X}^s} = 1$ and $\tilde{f}_j \to 0$ in $\bar{H}^{s-1}(X)$. By step 2, passing to a subsequence, we may assume that $\tilde{u}_j$ converges to some $\tilde{u}_\infty$ in $\mathcal{X}^s$. We have $\|\tilde{u}_\infty\|_{\mathcal{X}^s} = 1$, $P(\lambda)\tilde{u}_\infty = 0$, and $\tilde{u}_\infty$ is $\mathcal{X}^s$-orthogonal to $\ker^s P(\lambda)$; this gives a contradiction.

4. To finish the proof of the Fredholm property of (5.6.2), it remains to show that its image has finite codimension. Since this image is closed, by Lemma 5.29 it suffices to show that $\ker^{1-s} P(\lambda)^*$ is finite dimensional. To do this, we may argue as in step 2, using (5.6.9) instead of (5.6.8).

5. To show that (5.6.2) has a meromorphic inverse, we apply Analytic Fredholm Theory, Theorem C.8. For that, we need to show that (5.6.2) is invertible for some choice of λ, $\operatorname{Im}\lambda > \frac{1}{2} - s$. This statement together with continuity of index of Fredholm operators will also imply that $P(\lambda)$ has index zero.

Take $\lambda := h^{-1} + i\beta$, where $h > 0$ is small enough and $\beta > 0$ is fixed in Proposition 5.28. Then by (5.5.29), $\ker^s P(\lambda)$ is trivial; by (5.5.30),

$\operatorname{coker}^s P(\lambda) = \ker^{1-s} P(\lambda)^*$ is trivial. Together with the Fredholm property, these imply that (5.6.2) is invertible as needed. $\square$

The next proposition shows that for each λ, the inverse $P(\lambda)^{-1}$ defined in (5.6.7) does not depend on the choice of s. Moreover, it maps $\bar{H}^{s-1}(X) \to \bar{H}^s(X)$ for all $s > \frac{1}{2} - \operatorname{Im}\lambda$ and thus defines a meromorphic family of continuous operators

$$P(\lambda)^{-1} : C^\infty(\overline{X}) \to C^\infty(\overline{X}), \quad \lambda \in \mathbb{C}. \tag{5.6.10}$$

PROPOSITION 5.31. *Assume that $s < t$, and let $P^{(s)}(\lambda)^{-1}$, $P^{(t)}(\lambda)^{-1}$ be the inverses of $P(\lambda)$ as an operator $\mathcal{X}^s \to \bar{H}^{s-1}(X)$ and $\mathcal{X}^t \to \bar{H}^{t-1}(X)$ respectively. Then we have for $\operatorname{Im}\lambda > \frac{1}{2} - s$,*

$$P^{(s)}(\lambda)^{-1} f = P^{(t)}(\lambda)^{-1} f, \quad f \in \bar{H}^{t-1}(X). \tag{5.6.11}$$

Proof. By analytic continuation, it suffices to prove (5.6.11) when λ is not a pole of either $P^{(s)}(\lambda)^{-1}$ or $P^{(t)}(\lambda)^{-1}$. Then

$$u := P^{(t)}(\lambda)^{-1} f \in \mathcal{X}^t \subset \mathcal{X}^s.$$

Since $P(\lambda)u = f$, we see that $u = P^{(s)}(\lambda)^{-1} f$ as needed. $\square$

We now define resonances and (co)resonant states of $P(\lambda)$:

DEFINITION 5.32. *Let $\lambda \in \mathbb{C}$ and $s > \frac{1}{2} - \operatorname{Im}\lambda$. We say that λ is an **extended resonance** of (M, g), if it is a pole of (5.6.7). In this case we call **extended resonant states** elements of $\ker^s P(\lambda)$ and **extended coresonant states** elements of $\operatorname{coker}^s P(\lambda)$, see (5.6.3), (5.6.4).*

Since (5.6.2) is a Fredholm operator of index zero, λ is an extended resonance if and only if $\dim \ker^s P(\lambda) > 0$. Moreover, $\dim \ker^s P(\lambda) = \dim \operatorname{coker}^s P(\lambda)$. Using Lemma 5.29 we see that for all $t > s > \frac{1}{2} - \operatorname{Im}\lambda$

$$\ker^t P(\lambda) \subset \ker^s P(\lambda), \quad \operatorname{coker}^s P(\lambda) \subset \operatorname{coker}^t P(\lambda).$$

This gives inequalities on dimensions of the above spaces, which show that the spaces of extended (co)resonant states do not depend on s. Combining this with (5.6.6) we see that for $s > \frac{1}{2} - \operatorname{Im}\lambda$

$$\begin{aligned} \ker^s P(\lambda) &= \{u \in C^\infty(\overline{X}) \mid P(\lambda)u = 0\}, \\ \operatorname{coker}^s P(\lambda) &= \{v \in \mathcal{D}'(X) \mid \operatorname{supp} v \subset \overline{M}_{\text{even}},\ P(\lambda)^* v = 0\}. \end{aligned} \tag{5.6.12}$$

We will henceforth drop the superscript s in $\ker^s P(\lambda)$, $\operatorname{coker}^s P(\lambda)$.

5.6.2. Meromorphic continuation of the scattering resolvent. We now return to the original even asymptotically hyperbolic manifold (M, g). Since it is complete, the Laplace–Beltrami operator Δ_g is essentially self-adjoint on $C_c^\infty(M)$ (see for instance [**TaII**, Proposition 8.2.4]). Denote the domain of the self-adjoint extension by

$$H^2(M) = \{u \in L^2(M) \mid \Delta_g u \in L^2(M)\} \tag{5.6.13}$$

where we understand $\Delta_g u$ in the sense of distributions and use the volume form of g to defined the space $L^2(M)$. Following (5.0.3), we define for

$$\operatorname{Im}\lambda > 0, \quad \lambda \notin i\Big[0, \frac{n-1}{2}\Big], \tag{5.6.14}$$

the holomorphic L^2 resolvent

$$\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)^{-1} : L^2(M) \to H^2(M). \tag{5.6.15}$$

As a corollary of Theorem 5.30 we obtain

THEOREM 5.33 (Meromorphic continuation of the scattering resolvent). *The family (5.6.15) admits a meromorphic continuation with poles of finite rank*

$$R(\lambda) : L^2_{\text{comp}}(M) \to H^2_{\text{loc}}(M), \quad \lambda \in \mathbb{C}. \tag{5.6.16}$$

Moreover, for each $s > \frac{1}{2} - \operatorname{Im}\lambda$, $R(\lambda)$ can be extended to

$$R(\lambda) : x_1^{\frac{n+3}{4} - \frac{i\lambda}{2}} \bar{H}^{s-1}(M_{\text{even}}) \to x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} \bar{H}^s(M_{\text{even}}), \tag{5.6.17}$$

where $x_1 = y_1^2$ is a boundary defining function of the even compactification $\overline{M}_{\text{even}}$ introduced in Definition 5.7. In particular,

$$f \in C_c^\infty(M) \quad \Longrightarrow \quad R(\lambda) f \in x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} C^\infty(\overline{M}_{\text{even}}). \tag{5.6.18}$$

REMARKS. 1. Motivated by (5.6.18) and Definition 3.32 we say that a solution $u \in C^\infty(M)$ to

$$\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big) u = f \in C_c^\infty(M) \tag{5.6.19}$$

is *outgoing* if $u \in x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} C^\infty(\overline{M}_{\text{even}})$.

2. We define *resonances* of (M, g) as the poles of (5.6.16). The set of resonances is contained in the set of extended resonances (see Definition 5.32), however there could be extended resonances which are not resonances (that is the singular part of $P(\lambda)^{-1}$ is annihilated by restricting to M). There could also exist an outgoing solution to (5.6.19) with $f \equiv 0$ for some λ which is not a resonance. See Exercises 5.5, 5.6, 5.14. For this reason we will not

define resonant states for the asymptotically hyperbolic case. (We remark without proof that these issues can only arise at $\lambda = -ik$ where $k \in \mathbb{N}$.)

3. An analog of Rellich's theorem on absense of real resonances (Theorem 3.33) holds in the asymptotically hyperbolic setting, see Mazzeo [**Maz**] and Cardoso–Vodev [**CV02**].

Proof. 1. We first consider the case of the right-hand side $f \in C_c^\infty(M)$. Assume λ is not a pole of $P(\lambda)^{-1}$. Using (5.6.10), define

$$R(\lambda)f := x_1^{\frac{n-1}{4}-\frac{i\lambda}{2}}\big(P(\lambda)^{-1}x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}f\big)\big|_M \in C^\infty(M). \tag{5.6.20}$$

Here we make $x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}f$ into an element of $C_c^\infty(X)$ by extending it by zero outside of M, and $R(\lambda)f$ satisfies (5.6.18).

It follows from Definition 5.11 that $R(\lambda)f$ solves the equation

$$\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)R(\lambda)f = f.$$

Moreover, it follows from (5.3.3) and (5.6.18) that when $\operatorname{Im}\lambda > 0$ we have $R(\lambda)f \in L^2(M)$ and thus by (5.6.13) $R(\lambda)f \in H^2(M)$. This gives

$$R(\lambda)f = \Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)^{-1}f, \quad \operatorname{Im}\lambda > 0, \quad \lambda \notin i\Big[0, \frac{n-1}{2}\Big].$$

That is, $R(\lambda)$ does indeed give a meromorphic continuation of (5.6.15).

For $s > \frac{1}{2} - \operatorname{Im}\lambda$ and $f \in x_1^{\frac{n+3}{4}-\frac{i\lambda}{2}}\bar H^{s-1}(M_{\text{even}})$, we can also define $R(\lambda)f$ by (5.6.20), using (5.6.7) and extending $x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}f$ to an element of $H^{s-1}_{\text{comp}}(X)$; the mapping property (5.6.17) follows immediately.

2. It remains to show that $R(\lambda)$ extends to an operator with mapping properties (5.6.16). Such extension will necessarily be unique since $C_c^\infty(M)$ is dense in $L^2_{\text{comp}}(M)$.

We will handle the case of $f \in L^2_{\text{comp}}(M)$ using an elliptic parametrix. For each λ there exist regular properly supported operators $W(\lambda)$, $Z(\lambda)$ on M such that $W(\lambda) : L^2_{\text{comp}}(M) \to H^2_{\text{comp}}(M)$, $Z(\lambda)$ is smoothing, and

$$I = \Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)W(\lambda) + Z(\lambda).$$

The elliptic parametrix statement in this book (Proposition E.32) is semiclassical, so to construct these we put

$$P_1(z;h) := -h^2\Delta_g + 1 - h^2 z \in \Psi_h^2(M).$$

For each z, $P_1(z;h)$ is everywhere elliptic, so by Proposition E.32 there exist

$$W_1(z;h) \in \Psi_h^{-2}(M), \quad Z_1(z;h) = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$$

such that

$$I = P_1(z;h)W_1(z;h) + Z_1(z;h).$$

It remains to put $W(\lambda) := W_1(z;h)$, $Z(\lambda) := Z_1(z;h)$ where $h := 1$, $z := 1 + \lambda^2 + \frac{(n-1)^2}{4}$.

3. By (5.6.20), we have for each λ which is not an extended resonance

$$u = R(\lambda)\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)u, \quad u \in C_c^\infty(M).$$

Applying this to $u := W(\lambda)f$, we get

$$R(\lambda)(I - Z(\lambda))f = W(\lambda)f, \quad f \in C_c^\infty(M).$$

It follows that

$$R(\lambda)f = W(\lambda)f + R(\lambda)Z(\lambda)f, \quad f \in C_c^\infty(M). \tag{5.6.21}$$

We may now define $R(\lambda)f \in H^2_{\mathrm{loc}}(M)$ for $f \in L^2_{\mathrm{comp}}(M)$ using (5.6.21). Here $R(\lambda)Z(\lambda)f$ on the right-hand side is well-defined by (5.6.20), since $Z(\lambda)$ is smoothing and thus $Z(\lambda)f \in C_c^\infty(M)$. □

EXAMPLES. 1. For the hyperbolic space (5.0.1), there are no resonances when n is odd and there are resonances

$$\lambda_k = -i\Big(k + \frac{n-1}{2}\Big), \quad k \in \mathbb{N}_0,$$

when n is even – see [**GZ95b**, §2]. Moreover, for n odd the values λ_k are extended resonances – see Exercises 5.6 and 5.14.

2. For the hyperbolic cylinder (5.1.4), resonances are given by (see [**Bo16**, Proposition 5.2] and Figure 5.1(b))

$$\lambda_{j,k} = \frac{2\pi k}{\ell} - \Big(j + \frac{1}{2}\Big)i, \quad k \in \mathbb{Z}, \ j \in \mathbb{N}_0. \tag{5.6.22}$$

5.6.3. Applications to high frequency asymptotics. We finally give a few applications of the construction of this chapter to high frequency asymptotics, using the semiclassically rescaled operator $P_h(\omega) = h^2 P(h^{-1}\omega)$ from (5.3.8). These will be used in §6.2.3 below. Recall the class of compactly microlocalized pseudodifferential operators Ψ_h^{comp} introduced in Definition E.28, the symbol p defined in (5.3.11), the components $\Sigma_\pm$ of the characteristic set defined in Lemma 5.14, and the radial sets $L_\pm \subset \Sigma_\pm \cap \partial \overline{T}^* X$ defined in (5.4.21). (Recall also that p and $\Sigma_\pm$ depend on ω.) We start with constructing an approximate inverse for $P_h(\omega)$:

THEOREM 5.34 (Approximate inverse with complex absorption). *Fix a compact subset $\Omega \subset (0,\infty)$ and constants $C_0 > 0$, $s > \frac{1}{2} + C_0$. Let the space $\mathcal{X}^s$ be defined in (5.6.1). Assume that*

$$Q \in \Psi_h^{\mathrm{comp}}(M), \quad \sigma_h(Q) \geq 0,$$

controls trapping in the following sense (see Figure 5.5):

$$\begin{gathered} \textit{for all} \quad \omega \in \Omega, \quad (x,\xi) \in \Sigma_-, \quad \gamma(t) := e^{t\langle\xi\rangle^{-1}H_p}(x,\xi), \\ \textit{either} \quad \gamma(t) \to L_- \quad \textit{as } t \to -\infty \\ \textit{or there exists } t_0 \le 0 \quad \textit{such that} \quad \gamma(t_0) \in \operatorname{ell}_h(Q). \end{gathered} \tag{5.6.23}$$

Then for h small enough and $\operatorname{Re}\omega \in \Omega$, $|\operatorname{Im}\omega| \le C_0 h$, the operator $P_h(\omega) - iQ : \mathcal{X}^s \to \bar{H}^{s-1}(X)$ is invertible and

$$\|(P_h(\omega) - iQ)^{-1}\|_{\bar{H}_h^{s-1}(X) \to \mathcal{X}_h^s} \le Ch^{-1}. \tag{5.6.24}$$

REMARKS. 1. An operator Q with above properties always exists. Indeed, by part 1 of Lemma 5.19, it is enough to make Q elliptic on $\Sigma_- \cap \{x_1 \ge \delta\}$, and the latter is a compact subset of T^*M by (5.4.11).

2. Using Lemmas 5.19 and 5.20, the condition (5.6.23) can be reformulated in terms of the geodesic flow $\exp(tH_{|\xi|_g^2})$ on (M,g) (here $\delta > 0$ is defined in Lemma 5.19 and the map j, in (5.4.30)):

$$\begin{gathered} \text{for all} \quad \omega \in \Omega, \quad (\tilde{x},\tilde{\xi}) \in T^*M, \quad |\tilde{\xi}|_g = \omega \\ \text{if } \tilde{\gamma}(t) := \exp(tH_{|\xi|_g^2})(\tilde{x},\tilde{\xi}) \quad \text{lies in} \quad \{x_1 > \delta\} \quad \text{for all} \quad t \le 0 \\ \text{then there exists } t_0 \le 0 \quad \text{such that} \quad j(\tilde{\gamma}(t_0)) \in \operatorname{ell}_h(Q). \end{gathered} \tag{5.6.25}$$

By Proposition 6.4, it is then enough to require that $j(K_{\omega^2}) \subset \operatorname{ell}_h(Q)$ for all $\omega \in \Omega$ where $K_{\omega^2} \subset \{(x,\xi) \in T^*M \colon |\xi|_g = \omega\}$ is the trapped set at energy ω^2, see (6.1.4) and §6.1.2, as well as the proof of Proposition 6.12. In particular if M has no trapped geodesics, we may take $Q = 0$ – see Theorem 6.13.

Proof. 1. We first claim that it suffices to prove the following estimate for all compactly supported $A \in \Psi_h^0(X)$ and all $u \in \mathcal{X}^s$:

$$\|Au\|_{H_h^s} \le Ch^{-1}\|(P_h(\omega) - iQ)u\|_{\bar{H}_h^{s-1}(X)} + Ch^N\|u\|_{\bar{H}_h^{-N}(X)} \tag{5.6.26}$$

Indeed, let χ_1 satisfy (5.5.16). Combining (5.6.26) for $A := \chi_1$ with (5.5.23) (whose proof applies to $P_h(\omega) - iQ$ since Q is supported in $\{x_1 > 0\}$, so we may choose χ_2 in Theorem E.57 such that $\chi_2 Q = 0$), we obtain

$$\|u\|_{\bar{H}_h^s(X)} \le Ch^{-1}\|(P_h(\omega) - iQ)u\|_{\bar{H}_h^{s-1}(X)} + Ch^N\|u\|_{\bar{H}_h^{-N}(X)}. \tag{5.6.27}$$

Since Q is compactly microlocalized, it is a smoothing operator on X, and thus a compact operator $\mathcal{X}^s \to \bar{H}^{s-1}(X)$. Then by Theorem 5.30, $P_h(\omega) - iQ$ is a Fredholm operator of index zero $\mathcal{X}^s \to \bar{H}^{s-1}(X)$. By (5.6.27), for h small enough this operator has trivial kernel and thus is invertible, and (5.6.24) holds. Here to estimate the $\mathcal{X}_h^s$ norm we use the bound

$$\|P_h(0)u\|_{\bar{H}_h^{s-1}(X)} \le \|(P_h(\omega) - iQ)u\|_{\bar{H}_h^{s-1}(X)} + C\|u\|_{\bar{H}_h^s(X)}.$$

2. To show (5.6.26) we follow the proof of Lemma 5.23, putting $A_0 := 0$. The radial estimates (5.5.13) still hold for $P_h(\omega) - iQ$, since $\mathrm{WF}_h(Q) \cap L_\pm = \emptyset$. Therefore, it remains to prove the following version of (5.5.15):

$$
\begin{aligned}
\|Au\|_{H_h^s} \leq & Ch^{-1}\|(P_h(\omega) - iQ)u\|_{\overline{H}_h^{s-1}(X)} \\
& + C\|A_+u\|_{H_h^s} + C\|A_-u\|_{H_h^s} + Ch^N\|u\|_{\overline{H}_h^{-N}(X)}
\end{aligned} \tag{5.6.28}
$$

where the operators $A_\pm \in \Psi_h^0(X)$, chosen in the proof of Lemma 5.23, are elliptic on $L_\pm$.

To show (5.6.28), we follow the proof of Lemma 5.22, using a partition of unity to reduce to the situation when $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some point $(x, \xi) \in \overline{T}^*X$, and considering the following cases for the trajectory $\gamma(t) = e^{t\langle\xi\rangle^{-1}H_p}(x, \xi)$:

(1) $(x, \xi) \in \mathrm{ell}_h(P_h(\omega) - iQ) = \{\langle\xi\rangle^{-2}p \neq 0\} \cup \mathrm{ell}_h(Q)$: then $\|Au\|$ can be bounded using the semiclassical elliptic estimate, Theorem E.33;

(2) $(x, \xi) \in \Sigma_+$: by (5.4.12) and part 1 of Lemma 5.19, $\gamma(t)$ converges to L_+ as $t \to +\infty$ and does not pass through $T^*M \supset \mathrm{WF}_h(Q)$. Therefore, $\|Au\|$ can be bounded by $\|A_+u\|$ using propagation of singularities, Theorem E.47;

(3) $(x, \xi) \in \Sigma_-$ and $\gamma(t)$ converges to L_- as $t \to -\infty$: $\|Au\|$ can be bounded by $\|A_-u\|$ using propagation of singularities;

(4) By (5.4.8) and since Q controls trapping, the remaining case is $(x, \xi) \in \Sigma_-$ and $\gamma(t_0) \in \mathrm{ell}_h(Q)$ for some $t_0 \leq 0$. Then $\|Au\|$ can be bounded by $\|A'u\|$ using propagation of singularities, for some $A' \in \Psi_h^{\mathrm{comp}}(M)$ with $\mathrm{WF}_h(A') \subset \mathrm{ell}_h(Q)$, and $\|A'u\|$ can be bounded using case (1).

The cases (3) and (4) used the fact that $\sigma_h(Q) \geq 0$ to be able to apply propagation of singularities (Theorem E.47) to the operator $P_h(\omega) - iQ$. $\square$

We conclude this section with a semiclassically outgoing property for the operators $P_h(\omega)$ and $P_h(\omega) - iQ$. This property is useful in particular for applying the resolvent gluing method of Datchev–Vasy [**DV12a**] to asymptotically hyperbolic manifolds, with the model resolvent at infinity given by $(P_h(\omega) - iQ)^{-1}$ for Q satisfying (5.6.23).

THEOREM 5.35 (Semiclassically outgoing property). *Fix a compact subset $\Omega \subset (0, \infty)$ and constants $C_0 > 0$, $s > \frac{1}{2} + C_0$, and assume that $Q \in \Psi_h^{\mathrm{comp}}(M)$, $\sigma_h(Q) \geq 0$. Consider compactly supported operators*

$$
A \in \Psi_h^0(M), \quad B \in \Psi_h^0(X), \quad L_- \cup \mathrm{WF}_h(A) \subset \mathrm{ell}_h(B)
$$

satisfying the following control condition:

$$\begin{aligned} &\textit{for all}\quad \omega\in\Omega,\quad (x,\xi)\in \mathrm{WF}_h(A)\cap\Sigma_-,\quad \gamma(t):=e^{t\langle\xi\rangle^{-1}H_p}(x,\xi)\\ &\qquad\qquad \textit{we have}\quad \gamma(t)\to L_-\quad \textit{as}\quad t\to-\infty\\ &\qquad\qquad \textit{and}\quad \gamma(t)\in \mathrm{ell}_h(B)\quad \textit{for all}\quad t\le 0. \end{aligned} \tag{5.6.29}$$

Then for $\operatorname{Re}\omega\in\Omega$, $|\operatorname{Im}\omega|\le C_0h$, *all* N, *and* $u\in\mathcal{X}^s$, *we have*

$$\|Au\|_{H_h^s}\le Ch^{-1}\|B(P_h(\omega)-iQ)u\|_{H_h^{s-1}}+Ch^N\|u\|_{\bar H_h^{-N}(X)}. \tag{5.6.30}$$

REMARK. Using Lemmas 5.19 and 5.20 we see that if

$$\begin{aligned} &\text{for all}\quad \omega\in\Omega,\quad (\tilde x,\tilde\xi)\in T^*M,\quad |\tilde\xi|_g=\omega,\quad j(\tilde x,\tilde \xi)\in \mathrm{WF}_h(A)\\ &\qquad \text{we have}\quad x_1\big(\exp(tH_{|\xi|_g^2})(\tilde x,\tilde\xi)\big)\to 0\quad \text{as}\quad t\to-\infty \end{aligned} \tag{5.6.31}$$

then the condition (5.6.29) holds for any B elliptic on $\{x_1\ge 0\}$. Here the map j is given by (5.4.30) and $\exp(tH_{|\xi|_g^2})$ is the geodesic flow on (M,g). Condition (5.6.31) means that for all $\omega\in\Omega$, we have $\mathrm{WF}_h(A)\cap j(\Gamma^+_{\omega^2})=\emptyset$ where $\Gamma^+_{\omega^2}\subset\{(x,\xi)\in T^*M\colon |\xi|_g=\omega\}$ is the set of backwards trapped geodesics on (M,g) at energy ω^2, see (6.1.4).

Proof. We follow the proof of Lemma 5.23. First of all, since B is elliptic on L_- and $\mathrm{WF}_h(Q)\cap L_-=\emptyset$, Theorem E.52 gives the following strengthening of (5.5.13):

$$\|A_-u\|_{H_h^s}\le Ch^{-1}\|B(P_h(\omega)-iQ)u\|_{H_h^{s-1}}+Ch^N\|u\|_{\bar H_h^{-N}(X)}, \tag{5.6.32}$$

for some $A_-\in\Psi_h^0(X)$ elliptic on L_-.

Since $\mathrm{WF}_h(A)$ lies in $\overline{T}^*M$, by (5.4.12) it does not intersect Σ_+. Thus by (5.4.8) and a pseudodifferential partition of unity we may reduce to the following cases:

(1) $\mathrm{WF}_h(A)\cap\{\langle\xi\rangle^{-2}p=0\}=\emptyset$: by the elliptic estimate, Theorem E.33, we get

$$\|Au\|_{H_h^s}\le C\|B(P_h(\omega)-iQ)u\|_{H_h^{s-2}}+Ch^N\|u\|_{\bar H_h^{-N}(X)}.$$

(2) there exists $T\ge 0$ such that $e^{-T\langle\xi\rangle^{-1}H_p}(\mathrm{WF}_h(A))\subset \mathrm{ell}_h(A_-)$ and $e^{-t\langle\xi\rangle^{-1}H_p}(\mathrm{WF}_h(A))\subset \mathrm{ell}_h(B)$ for all $t\in[0,T]$: by propagation of singularities, Theorem E.47, we get

$$\begin{aligned}\|Au\|_{H_h^s}\le{}& C\|A_-u\|_{H_h^s}+Ch^{-1}\|B(P_h(\omega)-iQ)u\|_{H_h^{s-1}}\\ &+Ch^N\|u\|_{\bar H_h^{-N}(X)},\end{aligned}$$

and the first term on the right-hand side is estimated by (5.6.32). ∎

5.7. APPLICATIONS TO GENERAL RELATIVITY

We now discuss applications of the methods of this chapter to quasi-normal modes and wave decay on black hole spacetimes. To keep the presentation uniform with the rest of the chapter we restrict ourselves to spacetimes which correspond to asymptotically hyperbolic manifolds. Other spacetimes to which the methods developed here apply include the Schwarzchild–de Sitter spacetime, considered in Exercise 5.16, and more general Kerr–de Sitter spacetimes and their stationary perturbations, studied in the original work of Vasy [**Va13**]. The methods of [**Va13**] described here form the basis of the proof by Hintz–Vasy [**HV16**] of global non-linear stability of slowly rotating Kerr–de Sitter black holes under Einstein's equations.

Our starting point is the following procedure, associating a pseudo-Riemannian metric to a family of second order differential operators:

DEFINITION 5.36. *Let X be an n-dimensional manifold and $\mathcal{P}(\lambda)$ a family of second order differential operators on X of the form*

$$\mathcal{P}(\lambda) = -\sum_{j,k=1}^{n} a_{jk}\,\partial_j\partial_k + 2i\sum_{j=1}^{n} a_{0j}\,\lambda\partial_j + a_{00}\,\lambda^2 + B$$

where $(a_{jk})_{j,k=0}^n$ is an invertible real-valued symmetric matrix, $a_{jk}\in C^\infty(X)$, and $B = b_0\lambda + \sum_{j=1}^n b_j\partial_j + c$ for some $b_j, c\in C^\infty(X)$. Then we define the associated pseudo-Riemannian manifold $(\widetilde{X},\tilde g)$ as follows:

$$\widetilde{X} := \mathbb{R}_t\times X,\quad \langle dx_j, dx_k\rangle_{\tilde g} = a_{jk},$$

where $x_0 := t$.

Note that if $\mathcal{P}(i\partial_t)$ is the differential operator on $\widetilde{X}$ obtained by replacing λ by $i\partial_t$, then $\mathcal{P}(i\partial_t)$ differs from the pseudo-Riemannian Laplace–Beltrami operator $-\Box_{\tilde g}$ by a first order differential operator. Note also that $\tilde g$ is *stationary*, that is its components are independent of t.

We apply Definition 5.36 to the modified Laplacian introduced in §5.3. As in that section, let (M,g) be an even asymptotically hyperbolic manifold. We consider the even extension $\overline{X}$, fix a product structure (x_1,x') as in (5.3.2) and extend to $\overline{X}$ the metric $g_1(x_1,x',dx')$ as in (5.3.4). We fix

$$\psi = \psi(x_1)\in C^\infty(\overline{X})$$

and consider the operator $P_\psi(\lambda)$ defined in (5.3.16). Recall that

$$P_\psi(\lambda) = e^{(\frac{n+3}{2}-i\lambda)\psi}P(\lambda)e^{(i\lambda-\frac{n-1}{2})\psi}$$

where $P(\lambda)$ is the extended modified Laplacian, see Definition 5.11.

Let $\tilde{g}$ be the metric on $\widetilde{X} := \mathbb{R}_t \times X$ associated to

$$\mathcal{P}(\lambda) := P_\psi(\lambda)$$

according to Definition 5.36. We can write its action on covectors in terms of the symbol $p(x, \xi; \omega)$ defined in (5.3.11):

$$|(\tau, \xi)|^2_{\tilde{g}(t,x)} = p_\psi(x, \xi; -\tau) = e^{2\psi(x)} p(x, \xi - \tau d\psi; -\tau). \tag{5.7.1}$$

Here we denote elements of $T^*\widetilde{X}$ by (t, x, τ, ξ) where $t, \tau \in \mathbb{R}$, $(x, \xi) \in T^*X$, and $p_\psi(x, \xi; \omega)$ is the semiclassical principal symbol of $P_{\psi,h}(\omega) = h^2 P_\psi(h^{-1}\omega)$, see Exercise 5.7. We use the notation $|(\tau, \xi)|^2_{\tilde{g}(t,x)}$ for the inner product $\langle(\tau, \xi), (\tau, \xi)\rangle_{\tilde{g}(t,x)}$ even though $\tilde{g}$ is not positive definite.

By (5.3.12), we have on $\mathbb{R} \times M$,

$$|(\tau, \xi)|^2_{\tilde{g}(t,x)} = \frac{e^{2\psi}}{x_1}\bigg(\Big|\xi + \tau\Big(\frac{dx_1}{2x_1} - d\psi\Big)\Big|^2_{g(x)} - \tau^2\bigg); \tag{5.7.2}$$

by (5.3.13), we have on $\mathbb{R}\times\overline{Y}$ (here $\overline{Y} = \{-\varepsilon \le x_1 < \varepsilon_1^2\}$ is defined in (5.3.1))

$$\begin{aligned} |(\tau, \xi_1, \xi')|^2_{\tilde{g}(t,x_1,x')} = e^{2\psi}\big(&4(\xi_1 - \psi'(x_1)\tau)\big(x_1\xi_1 + (1 - x_1\psi'(x_1))\tau\big) \\ &+ |\xi'|^2_{g_1(x_1,x')}\big). \end{aligned} \tag{5.7.3}$$

It follows that $\tilde{g}$ is *Lorentzian*, that is it has signature $(n, 1)$. We refer the reader to [**TaI**, §2.7] for an introduction to Lorentzian geometry.

On $\mathbb{R} \times M$, we may write $\tilde{g}$ as conformal to a product metric which is singular at the boundary $\{x_1 = 0\}$, by using the change of variables

$$\Phi : (t, x) \mapsto \Big(t - \psi(x) + \frac{1}{2}\log x_1, x\Big). \tag{5.7.4}$$

Indeed, we compute

$$|(\tau, \xi)|^2_{\Phi^*\tilde{g}(t,x)} = \frac{e^{2\psi}}{x_1}(|\xi|^2_{g(x)} - \tau^2),$$

therefore on $\mathbb{R} \times M$,

$$\Phi^*\tilde{g} = x_1 e^{-2\psi}(-dt^2 + g(x, dx)). \tag{5.7.5}$$

EXAMPLES. 1. For the hyperbolic space (see (5.3.18)), we have

$$\widetilde{X} = \mathbb{R}_t \times X, \quad X = B_{\mathbb{R}^n}(0, 2),$$

and with (r, θ) denoting polar coordinates on X and g_S the standard metric on $\mathbb{S}^{n-1}$,

$$\begin{aligned} |(\tau, \xi)|^2_{\tilde{g}(t,x)} &= (1 - r^2)\xi_r^2 - 2r\tau\xi_r - \tau^2 + \frac{|\xi_\theta|^2_{g_S(\theta)}}{r^2}; \\ \tilde{g} &= -(1 - r^2)dt^2 - 2r\,dr dt + dr^2 + r^2 g_S(\theta, d\theta). \end{aligned} \tag{5.7.6}$$

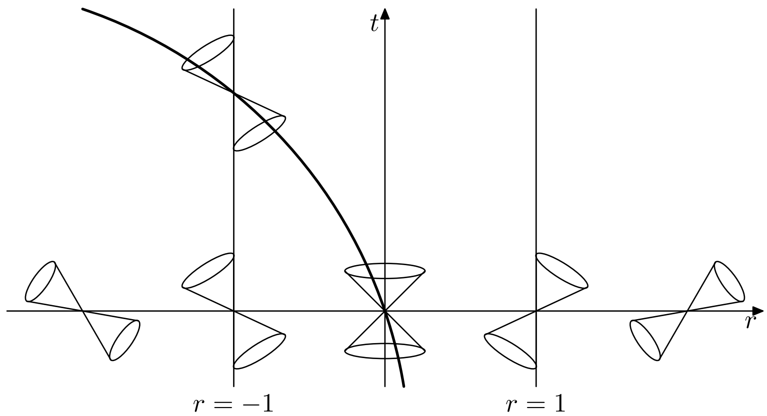


Figure 5.8. The light cones and a timelike curve on the spacetime associated to the hyperbolic cylinder. The event horizons are $\{r = \pm 1\}$.

On $\mathbb{R} \times M$, $M = B_{\mathbb{R}^n}(0,1)$, we use (5.1.25) together with the formula $x_1 e^{-2\psi} = 1 - r^2$ to see that the pullback metric (5.7.5) is

$$\Phi^* \tilde{g} = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2 g_S(\theta, d\theta).$$

Thus $(\widetilde{X}, \tilde{g})$ is a subset of the *de Sitter spacetime* – see [**Va13**, (4.1)].

2. For the hyperbolic cylinder (see (5.3.21)), we have

$$\begin{aligned} \widetilde{X} &= \mathbb{R}_t \times X, \quad X = [-2,2]_r \times \mathbb{S}^1_\theta; \\ |(\tau,\xi)|^2_{\tilde{g}(t,x)} &= (1 - r^2)\xi_r^2 - 2r\tau\xi_r - \tau^2 + \xi_\theta^2, \\ \tilde{g} &= -(1 - r^2)dt^2 - 2r\,dr dt + dr^2 + d\theta^2. \end{aligned} \tag{5.7.7}$$

See Figure 5.8. On $\mathbb{R} \times M$, $M = (-1,1)_r \times \mathbb{S}^1_\theta$, the pullback metric (5.7.5) is (using (5.1.27) and the formula $x_1 e^{-2\psi} = 1 - r^2$)

$$\Phi^* \tilde{g} = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + d\theta^2. \tag{5.7.8}$$

5.7.1. Geometry of spacetime. We now study the properties of the spacetime $(\widetilde{X}, \tilde{g})$. For each $\omega \in \mathbb{R}$, (5.7.1) gives a correspondence

$$\Theta_\omega : (t,x,\tau,\xi) \in \mathscr{C} \cap \{\tau = -\omega\} \quad \mapsto \quad (x, \xi - \tau d\psi) \in \{p = 0\} \tag{5.7.9}$$

between the characteristic set $\{p = 0\} \subset T^*X$ and the intersection of the dual light cone

$$\mathscr{C} := \{(t,x,\tau,\xi) \in T^*\widetilde{X} \colon |(\tau,\xi)|^2_{\tilde{g}(t,x)} = 0\}$$

with the hypersurface $\{\tau = -\omega\}$. (Recall that p depends on ω.)

The geodesic flow on $(\widetilde{X}, \tilde{g})$ is (up to rescaling) the Hamiltonian flow

$$\exp(tH_{\tilde{p}}), \quad \tilde{p}(t, x, \tau, \xi) := |(\tau, \xi)|^2_{\tilde{g}(t,x)}. \tag{5.7.10}$$

The trajectories of the flow (5.7.10) on $\mathscr{C} = \{\tilde{p} = 0\}$ are called *null geodesics*, and τ is constant on each geodesic since $\partial_t \tilde{p} = 0$. Under the map (5.7.9) null geodesics correspond to reparametrized trajectories of the Hamiltonian flow H_p. More precisely, if $\gamma(s) \in \mathscr{C} \cap \{\tau = -\omega\}$ is a trajectory of the flow (5.7.10) and $\gamma_1(s) = \Theta_\omega(\gamma(s))$ then

$$\dot{\gamma}_1(s) = e^{2\psi(\gamma_1(s))} H_p(\gamma_1(s)). \tag{5.7.11}$$

The identity (5.7.11) follows from the fact that $(x, \xi) \mapsto (x, \xi + \omega d\psi)$ is a symplectomorphism for all $\omega \in \mathbb{R}$.

We impose the conditions (5.4.6), (5.4.7) on the function ψ:

$$\begin{gathered} \Big| d\psi - \frac{dx_1}{2x_1} \Big|_g < 1 \quad \text{on } M, \\ \psi'(x_1) > 0, \quad x_1 \psi'(x_1) < 1 \quad \text{for all } x_1. \end{gathered}$$

See also Figure 5.4. Computing from (5.7.2), (5.7.3)

$$|dt|^2_{\tilde{g}(t,x)} = \begin{cases} \frac{e^{2\psi}}{x_1} \big(\big| \frac{dx_1}{2x_1} - d\psi \big|^2_{g(x)} - 1 \big), & \text{on } \mathbb{R} \times M; \\ -4e^{2\psi} \psi'(x_1)(1 - x_1 \psi'(x_1)), & \text{on } \mathbb{R} \times \overline{Y} \end{cases}$$

we see that the hypersurfaces $\{t = \text{const}\}$ are *spacelike*; that is, $|dt|^2_{\tilde{g}} < 0$. This is the reason we could not simply take $\psi \equiv 0$.

The vector field ∂_t is:

- *timelike* on $\mathbb{R} \times M = \{x_1 > 0\}$, namely $|\partial_t|^2_{\tilde{g}} < 0$;
- *null* on $\{x_1 = 0\}$, namely $|\partial_t|^2_{\tilde{g}} = 0$;
- *spacelike* on $\{x_1 < 0\}$, namely $|\partial_t|^2_{\tilde{g}} > 0$.

Rather than inverting (5.7.2), (5.7.3) to obtain the form of the metric on tangent vectors, we can see this by putting $\tau := 0$ and studying the signature of the resulting quadratic form in ξ. Then by (5.7.1) ∂_t being timelike corresponds to $P(\lambda)$ being elliptic and ∂_t being spacelike corresponds to $P(\lambda)$ being hyperbolic (see Proposition 5.12; here hyperbolicity with respect to x_1 corresponds to the hypersurfaces $\{x_1 = c\}$ being spacelike for $c < 0$).

REMARK. The geometry described in this section is very similar to that of the *Schwarzschild–de Sitter* black hole spacetime, see Exercise 5.16. In particular, on that spacetime ∂_t is timelike in $\{x_1 > 0\}$. However, the more general rotating *Kerr–de Sitter* spacetime has an *ergoregion*, namely a region inside $\{x_1 > 0\}$ where ∂_t is spacelike. The corresponding stationary d'Alembert–Beltrami operator $\widetilde{P}(\lambda)$ (see (5.7.20) below) is hyperbolic in the ergoregion, which makes the spectral theory approach to meromorphic continuation for black hole spacetimes (reducing $\widetilde{P}(\lambda)$ to the spectral family of some self-adjoint operator P in $\{x_1 > 0\}$, similar to $-\Delta_g$ for an asymptotically hyperbolic manifold) difficult to apply. The approach of Vasy, presented here, handles the elliptic to hyperbolic transition of $P(\lambda)$ in a systematic way. In particular this approach applies with no essential changes to Kerr–de Sitter spacetimes. See [**Va13**] for details. $\square$

If we remove the zero section of $T^*\widetilde{X}$ then $\mathscr{C}$ splits into two halves

$$\mathscr{C}_\pm := \{(t,x,\tau,\xi) \in \mathscr{C} \mid \pm\langle(\tau,\xi), dt\rangle_{\tilde{g}(t,x)} < 0\}. \tag{5.7.12}$$

The time variable t satisfies the equation $\dot{t} = \partial_\tau \tilde{p} = 2\langle(\tau,\xi), dt\rangle_{\tilde{g}(t,x)}$ under the geodesic flow (5.7.10). Thus the null geodesics on $\mathscr{C}_+$ travel backward in time and those on $\mathscr{C}_-$ travel forward in time. We thus call $\mathscr{C}_+$ the *past-oriented light cone* and $\mathscr{C}_-$ the *future-oriented light cone*.

We see from (5.4.13) (where we put $\varphi := \psi$) and (5.7.1) that under (5.7.9), $\mathscr{C}_\pm$ correspond to the components $\Sigma_\pm$ of the characteristic set when $\omega \neq 0$. Then (5.4.12) corresponds to the following corollary of ∂_t being timelike on $\mathbb{R} \times M$:

$$\mathscr{C}_\pm \cap T^*(\mathbb{R} \times M) \ \subset \ \{\pm\tau > 0\}.$$

Next, we introduce the following subsets of $\widetilde{X}$:

- the *domain of outer communications* is $\{x_1 > 0\} = \mathbb{R} \times M$;
- the *event horizon* is the hypersurface $\{x_1 = 0\}$;
- the *black hole region* is $\{x_1 < 0\}$.

By (5.7.3), the event horizon is a null hypersurface in the sense that $|dx_1|^2_{\tilde{g}} = 0$ on $\{x_1 = 0\}$. Moreover, the hypersurfaces $\{x_1 = c\}$ are spacelike (that is $|dx_1|^2_{\tilde{g}} < 0$) for $c < 0$, and $\langle dx_1, dt\rangle_{\tilde{g}} > 0$ on $\{x_1 < 0\}$. It follows that if $\gamma(s) \in \widetilde{X}$ is a curve which is timelike or lightlike (that is, $|\dot{\gamma}|^2_{\tilde{g}} \leq 0$) and future oriented (that is, $\langle dt, \dot{\gamma}\rangle > 0$), then for all s

$$\gamma(s) \in \{x_1 < 0\} \quad \Longrightarrow \quad \partial_s x_1(\gamma(s)) < 0. \tag{5.7.13}$$

That is, if an observer crosses the event horizon at some point, they can never escape the black hole region and will be pushed farther into this region as time t increases – see Figure 5.8. For null geodesics in $\mathscr{C}_-$ (5.7.13) corresponds to Lemma 5.16, using (5.7.11).

We finally define the *red-shift trajectories*, which are the null geodesics on the event horizon given by (here we assume $\psi = 0$ on $\{x_1 = 0\}$ for simplicity)

$$t = \ln(\mp 4s),\ x_1 = 0,\ x' = \text{const},\ \tau = 0,\ \xi_1 = \mp e^{-t},\ \xi' = 0, \tag{5.7.14}$$

where s, $\mp s > 0$, is the geodesic parameter. We call (5.7.14) red-shift trajectories because the frequency ξ decays exponentially along them as $t \to \infty$.

Writing the trajectories (5.7.14) using the correspondence (5.7.9), we obtain the following Hamiltonian trajectories of e^{sH_p} on $\{p = 0\} \subset T^*X$ for $\omega = 0$:

$$x_1 = 0,\ x' = \text{const},\ \xi_1 = \frac{1}{4s},\ \xi' = 0. \tag{5.7.15}$$

As $s \to 0\mp$, these trajectories converge in $\overline{T}^*X$ to the radial sources/sinks $L_\pm$ defined in (5.4.21). In other words, $L_\pm$ correspond to limits of the red-shift trajectories as $t \to -\infty$.

5.7.2. Resonance expansions. We finally present an application of the results of this chapter to resonance expansions on the spacetime $(\widetilde{X}, \tilde{g})$. We keep the presentation relatively brief, referring to [**Va13**, §§3, 6] or [**Dy12**, §1.1] for more details.

Let $\Box_{\tilde{g}}$ be the d'Alembert–Beltrami operator on $(\widetilde{X}, \tilde{g})$. Consider the future solution to the inhomogeneous wave equation,

$$-\Box_{\tilde{g}} u = f, \quad \operatorname{supp} u \subset \{t > 0\}, \tag{5.7.16}$$

where we assume for simplicity that the right-hand side f satisfies

$$f \in C_c^\infty(\widetilde{X} \cap \{0 < t < 1\}).$$

PROPOSITION 5.37. *The problem (5.7.16) has a unique solution $u \in C^\infty(\widetilde{X})$, which is smooth up to the boundary $\{x_1 = -\varepsilon\}$. Moreover, there exist constants $C, C_1 > 0$ such that for all $t \geq 0$*

$$\|\nabla_{\tilde{g}} u(t)\|_{L^2(X)} \leq C e^{C_1 t} \|f\|_{L^1_t((0,1);L^2(X))}. \tag{5.7.17}$$

REMARK. The estimate (5.7.17) only gives an exponentially growing bound on the energy of solutions to the wave equation. As follows from Theorem 5.40 below, if there is a resonance free strip, then boundedness of energy of u as $t \to \infty$ is equivalent to the following *mode stability* condition: there are no resonances in the upper half-plane and resonances on the real line are algebraically simple. Such conditions are crucial for establishing stability for nonlinear wave equations (such as [**HV16**]). Mode stability is known for slowly rotating Kerr–de Sitter, see [**Dy11a**, Theorem 4]. For general Kerr–de Sitter spacetimes mode stability is an open question.

Proof. 1. To show existence of u we will use well-posedness of the Cauchy problem for hyperbolic equations, Theorem E.61. Since this theorem is stated for the case of a compact space slice, we embed $\overline{X}$ into a compact manifold without boundary X_{ext}, and put $\widetilde{X}_{\text{ext}} := \mathbb{R}_t \times X_{\text{ext}}$. We will extend

$\tilde{g}$ to a stationary Lorentzian metric on $\widetilde{X}_{\mathrm{ext}}$ such that the hypersurfaces $\{t = \mathrm{const}\}$ are spacelike; that is, $|dt|^2_{\tilde{g}} < 0$.

To construct the extension of $\tilde{g}$ we use the following fact: if $\tilde{g}_0, \tilde{g}_1$ are two Lorentzian inner products on $\mathbb{R}^{n+1}_{t,x}$ which are positive definite on $\ker dt \simeq \mathbb{R}^n$, then for each $\alpha \in [0,1]$ there exists unique Lorentzian inner product $\tilde{g}_\alpha = [\tilde{g}_0, \tilde{g}_1]_\alpha$ such that, denoting $\nabla_{\tilde{g}_\alpha} t := \tilde{g}_\alpha^{-1} dt$,

$$\begin{aligned} \tilde{g}_\alpha|_{\ker dt} &= (1-\alpha)\tilde{g}_0|_{\ker dt} + \alpha \tilde{g}_1|_{\ker dt}, \\ \nabla_{\tilde{g}_\alpha} t &= (1-\alpha)\nabla_{\tilde{g}_0} t + \alpha \nabla_{\tilde{g}_1} t. \end{aligned} \tag{5.7.18}$$

We now let X_{ext} be the double space of $\overline{X}$, obtained by gluing $\overline{X}$ along the boundary ∂X to another copy of $\overline{X}$, which we denote $\overline{X}_1$. Let $\tilde{g}_0$ be a smooth stationary extension of $\tilde{g}$ from $\widetilde{X}$ to a neighborhood of it in $\widetilde{X}_{\mathrm{ext}}$ such that $|dt|^2_{\tilde{g}_0} < 0$. Next, let $\tilde{g}_1$ be the mirror image of $\tilde{g}$ on $\widetilde{X}_1$. Take $\chi \in C^\infty(X_{\mathrm{ext}}; [0,1])$ which is supported inside the domain of $\tilde{g}_0$ and equal to 1 on $\overline{X}$. We then define the extension of $\tilde{g}$ to $\widetilde{X}_{\mathrm{ext}}$ using the procedure (5.7.18):

$$\tilde{g}(t,x) := [\tilde{g}_0(t,x), \tilde{g}_1(t,x)]_{1-\chi(x)}.$$

2. Since the hypersurfaces $\{t = \mathrm{const}\}$ are spacelike, the operator $-\Box_{\tilde{g}}$ is hyperbolic on $\widetilde{X}_{\mathrm{ext}}$ with respect to t in the sense of Definition E.55 (where we fix $h := 1$). By Theorem E.61 (where we take zero Cauchy data on $\{t = 0\}$) there exists a function $\tilde{u} \in C^\infty(\widetilde{X}_{\mathrm{ext}})$ such that $-\Box_{\tilde{g}} \tilde{u} = f$ and $\operatorname{supp} \tilde{u} \subset \{t > 0\}$, where we extend f by zero to $\widetilde{X}_{\mathrm{ext}}$. Putting $u := \tilde{u}|_{\widetilde{X}}$ we obtain existence of solutions to the problem (5.7.16).

3. The estimate (5.7.17) is a special case of the following energy estimate for wave equations on Lorentzian manifolds: for any $u \in C^\infty(\widetilde{X})$ smooth up to the boundary $\{x_1 = -\varepsilon\}$ and all $t_1 \geq t_0$

$$E_u(t_1) \leq C e^{C_1(t_1 - t_0)} E_u(t_0) + C \int_{t_0}^{t_1} e^{C_1(t_1 - t)} \|\Box_{\tilde{g}} u(t)\|_{L^2(X)}\, dt \tag{5.7.19}$$

where $E_u(t) := \|\nabla_{\tilde{g}} u(t)\|_{L^2(X)}$. The estimate (5.7.19) is proved by integration by parts in the integral

$$\int_{t_0 \leq t \leq t_1} (\Box_{\tilde{g}} u)(Vu)\, d\operatorname{Vol}_{\tilde{g}}, \quad V := -\nabla_{\tilde{g}} t$$

using that $\tilde{g}$ is stationary, the vector field V is timelike, the hypersurfaces $\{t = \mathrm{const}\}$ and $\{x_1 = -\varepsilon\}$ are spacelike, $\langle dt, V\rangle > 0$ everywhere, and $\langle dx_1, V\rangle < 0$ on $\{x_1 = -\varepsilon\}$. See [**TaI**, Proposition 2.8.1], [**Dy11a**, Proposition 1.1], and [**Dy11b**, §1.1] for details. Finally, uniqueness of u follows immediately from (5.7.17). □

Now, following the strategy of §2.1 define the Fourier–Laplace transform

$$\hat{u}(\lambda) := \int_{\mathbb{R}} e^{i\lambda t} u(t)\, dt \ \in \ \bar{H}^1(X), \quad \lambda \in \mathbb{C}, \quad \operatorname{Im}\lambda > C_1.$$

Here $C_1 > 0$ is the constant in (5.7.17) and the integral converges exponentially fast.

Let $\widetilde{P}(\lambda)$ be the second order differential operator on the space slice X obtained from $-\Box_{\tilde{g}}$ by replacing ∂_t with $-i\lambda$. Fourier transforming the wave equation (5.7.16), we obtain

$$\widetilde{P}(\lambda)\hat{u}(\lambda) = \hat{f}(\lambda), \quad \operatorname{Im}\lambda > C_1. \tag{5.7.20}$$

We now express $\hat{u}(\lambda)$ in terms of the meromorphic inverse of $\widetilde{P}(\lambda)$:

THEOREM 5.38. *Fix $s \in \mathbb{R}$. Then for $\operatorname{Im}\lambda > \frac{1}{2} - s$,*

$$\widetilde{P}(\lambda) : \{w \in \bar{H}^s(X) \mid \widetilde{P}(0)w \in \bar{H}^{s-1}(X)\} \to \bar{H}^{s-1}(X) \tag{5.7.21}$$

is a Fredholm operator of index zero and has a meromorphic inverse with poles of finite rank

$$\widetilde{P}(\lambda)^{-1} : \bar{H}^{s-1}(X) \to \bar{H}^s(X). \tag{5.7.22}$$

The operators (5.7.22) for different values of s coincide on their common domain and define an operator $C^\infty(\overline{X}) \to C^\infty(\overline{X})$, similarly to Proposition 5.31. Moreover, $\widetilde{P}(\lambda)^{-1}$ has no poles in $\{\operatorname{Im}\lambda > C_1\}$ where C_1 is the constant in (5.7.17).

Finally, if u solves the wave equation (5.7.16), then

$$\hat{u}(\lambda) = \widetilde{P}(\lambda)^{-1}\hat{f}(\lambda) \quad \textit{for } \operatorname{Im}\lambda > C_1. \tag{5.7.23}$$

REMARK. The poles of (5.7.22) are called *resonances* or *quasi-normal modes* of the spacetime $(\widetilde{X}, \tilde{g})$. To each resonance λ correspond resonant states $w \in C^\infty(\overline{X})$, $\widetilde{P}(\lambda)w = 0$. These gives rise to solutions $e^{-i\lambda t}w(x)$ of the wave equation which in physics are called *mode solutions*.

Proof. 1. We first establish the Fredholm property and the meromorphy of the inverse of (5.7.21), applying the proof of Theorem 5.30. Recall that we constructed $\tilde{g}$ in the beginning of §5.7 based on the operator $P_\psi(\lambda)$. By the remark following Definition 5.36 the difference $-\Box_{\tilde{g}} - P_\psi(i\partial_t)$ is a first order differential operator on $\widetilde{X}$. From the definition of $\widetilde{P}(\lambda)$ we see that $\widetilde{P}(\lambda) - P_\psi(\lambda)$ is a first order polynomial in λ and ∂_x (with coefficients smooth in x). Recalling the definition (5.3.16) of $P_\psi(\lambda)$, we consider the operator

$$P'(\lambda) := e^{(i\lambda - \frac{n+3}{2})\psi}\widetilde{P}(\lambda)e^{(\frac{n-1}{2} - i\lambda)\psi}$$

and note that $P'(\lambda) - P(\lambda)$ is a first order polynomial in λ and ∂_x. Thus the semiclassically rescaled operator

$$P'_h(\omega) := h^2 P'(h^{-1}\omega) \tag{5.7.24}$$

satisfies (where $P_h(\omega)$ was defined in (5.3.8))

$$P'_h(\omega) = P_h(\omega) + h\,\mathrm{Diff}^1_h(\overline{X}).$$

Now the proof of Theorem 5.30 applies to the operator $P'(\lambda)$. Indeed, most of the analysis only used the principal symbol of $P'_h(\omega)$, which is still equal to p. The only statement that needs to be checked is Proposition 5.13; more precisely we need to show that for $\lambda \in \mathbb{R}$, the operator $P'(\lambda)$ is symmetric. To see this we use that $\Box_{\tilde{g}}$ is symmetric with respect to $d\,\mathrm{Vol}_{\tilde{g}}$ and thus $\widetilde{P}(\lambda)$ is symmetric with respect to the density dS on X defined by $d\,\mathrm{Vol}_{\tilde{g}} = dtdS$. Then $P'(\lambda)$ is symmetric with respect to the density $e^{(n+1)\psi}\,dS$. A direct computation using (5.7.2) and (5.7.3) shows that $2e^{(n+1)\psi}\,dS$ equals the density $d\,\mathrm{Vol}$ defined in (5.3.24).

We have showed the Fredholm property and the meromorphy of the inverse for the operator

$$P'(\lambda) : \{w \in \bar{H}^s(X) \mid P'(0)w \in \bar{H}^{s-1}(X)\} \to \bar{H}^{s-1}(X)$$

which immediately imply these properties for the original operator (5.7.21). The proof of Proposition 5.31 applies without any changes.

2. We next show that if $\operatorname{Im}\lambda > C_1$, then λ is not a pole of $\widetilde{P}(\lambda)^{-1}$. Indeed, assume the contrary. Then there exists a nontrivial solution $w \in C^\infty(\overline{X})$ to the equation $\widetilde{P}(\lambda)w = 0$ (see (5.6.12)). Then $u(t,x) := e^{-i\lambda t}w(x)$ is a solution to the equation $\Box_{\tilde{g}}u = 0$ which violates the bound (5.7.19), giving a contradiction.

Finally, (5.7.23) follows from (5.7.20) and the invertibility of (5.7.21) for $s := 1$. □

Recall that the proof of resonance expansions for potential scattering (Theorem 2.9) used resonance free regions (Theorem 2.10). In the situation studied here a resonance free region may or may not be present depending on the behavior of trapped geodesics (see Chapter 6). Thus we make

DEFINITION 5.39 (Resonance free strip). *Let $\widetilde{P}_h(\omega) := h^2\widetilde{P}(h^{-1}\omega)$ be the semiclassical rescaling of the operator $\widetilde{P}(\lambda)$ defined in the paragraph preceding (5.7.20) and $C_1 > 0$ be the constant in (5.7.17). We say that $(\widetilde{X}, \tilde{g})$ has a* ***resonance free strip*** *of size $\nu > 0$, if there exist $N \geq 0$,*

$C, h_0 > 0$, $C_2 > C_1$, and $s > \frac{1}{2} + \nu$ such that

$$\begin{gathered}\|u\|_{\bar H^s_h(X)} \le Ch^{-1-N}\|\widetilde P_h(\omega)u\|_{\bar H^{s-1}_h(X)}, \quad u \in \bar H^s_h(X),\\ |\operatorname{Re}\omega| = 1, \quad \operatorname{Im}\omega \in [-\nu h, C_2 h], \quad 0 < h \le h_0.\end{gathered} \tag{5.7.25}$$

REMARKS. 1. A resonance free strip is also often called an *essential spectral gap*. In terms of the original operator $\widetilde P(\lambda)$ it implies that there are no resonances with $\operatorname{Im}\lambda \ge -\nu$ and $|\operatorname{Re}\lambda| \ge h_0^{-1}$.

2. It is enough to verify (5.7.25) for the case $\operatorname{Re}\omega = 1$; the case $\operatorname{Re}\omega = -1$ then follows using the identity

$$\overline{\widetilde P_h(\omega)u} = \widetilde P_h(-\bar\omega)\bar u.$$

Finally, we present resonance expansions which are analogues of Theorems 2.9 and 3.11 in the current setting:

THEOREM 5.40 (Resonance expansions for black hole spacetimes). *Let $(\widetilde X, \tilde g)$ be the spacetime constructed in the beginning of §5.7 from an even asymptotically hyperbolic manifold. Assume that $(\widetilde X, \tilde g)$ has a resonance free strip of some size $\nu > 0$. Then for every solution u to the wave equation (5.7.16) we have the resonance expansion as $t \to \infty$*

$$u(t) = -i \sum_{\operatorname{Im}\lambda_j \ge -\nu} \operatorname{Res}_{\lambda=\lambda_j}\big(e^{-i\lambda t}\widetilde P(\lambda)^{-1}\hat f(\lambda)\big) + \mathcal O(e^{-\nu t})_{L^2(X)}. \tag{5.7.26}$$

Here $\widetilde P(\lambda)^{-1} : C^\infty(\overline X) \to C^\infty(\overline X)$ is defined in Theorem 5.38 and the sum is taken over the resonances λ_j of $(\widetilde X, \tilde g)$.

REMARK. To simplify the presentation we give a weak form of the remainder in (5.7.26), in particular we do not speficy the dependence of the constant on the right-hand side f. We refer to [**Va13**, Lemma 3.1] or [**Dy11b**, Proposition 2.1] for a stronger remainder bound in weighted Sobolev spaces $e^{-\nu t}H^s_{t,x}$, which shows that the constant N in Definition 5.39 specifies the number of derivatives lost in the decay bound (compared to (5.7.17)).

Proof. 1. Let $C_2 > C_1, N, s, h_0$ be given in Definition 5.39. By the Fourier inversion formula in t and (5.7.23) we have

$$u(t) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = C_2} e^{-i\lambda t}\widetilde P(\lambda)^{-1}\hat f(\lambda)\, d\lambda. \tag{5.7.27}$$

To justify (5.7.27) we use the wave equation (5.7.16) to write $\partial_t^2 u = P_1\partial_t u + P_0 u$ for $t \ge 1$ where P_j are differential operators of order $2-j$ on X. Differentiating this identity and arguing by induction with the base given by the growth bound (5.7.17), we see that for each $\varphi \in C^\infty_{\mathrm c}(X)$ the function $t \mapsto$

$e^{-C_2 t}\langle u(t,x),\varphi(x)\rangle_{L^2(X)}$ is exponentially decaying with all its derivatives. Applying the Fourier inversion formula to this function we obtain (5.7.27).

2. Rescaling (5.7.25) we get the high frequency resolvent estimate

$$\begin{gathered}\|\widetilde{P}(\lambda)^{-1}\|_{\bar{H}^s(X)\to L^2(X)} \leq C\langle\lambda\rangle^{N-1},\\ |\operatorname{Re}\lambda| \geq h_0^{-1}, \quad \operatorname{Im}\lambda\in[-\nu,C_2].\end{gathered} \tag{5.7.28}$$

Since $f\in C_{\mathrm{c}}^\infty(\widetilde{X}\cap\{0<t<1\})$, we have for all N'

$$\|\hat{f}(\lambda)\|_{\bar{H}^{N'}(X)} \leq C_{N'}\langle\lambda\rangle^{-N'}, \quad \operatorname{Im}\lambda\in[-\nu,C_2]. \tag{5.7.29}$$

Combining (5.7.28) and (5.7.29) we obtain for all N'

$$\begin{gathered}\|\widetilde{P}(\lambda)^{-1}\hat{f}(\lambda)\|_{L^2(X)} \leq C_{N'}\langle\lambda\rangle^{-N'},\\ |\operatorname{Re}\lambda| \geq h_0^{-1}, \quad \operatorname{Im}\lambda\in[-\nu,C_2].\end{gathered} \tag{5.7.30}$$

3. For simplicity we assume that there are no resonances on the line $\{\operatorname{Im}\lambda = -\nu\}$. (Otherwise we deform the contour a tiny bit below the finitely many resonances on that line.) Using (5.7.30) we deform the contour in (5.7.27) to obtain

$$\begin{aligned}u(t) = {}& -i\sum_{\operatorname{Im}\lambda_j\geq-\nu}\operatorname{Res}_{\lambda=\lambda_j}\big(e^{-i\lambda t}\widetilde{P}(\lambda)^{-1}\hat{f}(\lambda)\big)\\ &+\frac{1}{2\pi}\int_{\operatorname{Im}\lambda=-\nu} e^{-i\lambda t}\widetilde{P}(\lambda)^{-1}\hat{f}(\lambda)\,d\lambda.\end{aligned} \tag{5.7.31}$$

Using (5.7.30) again we see that the second line of (5.7.31) is $\mathcal{O}(e^{-\nu t})_{L^2(X)}$, finishing the proof. $\square$

EXAMPLES. 1. For the spacetime (5.7.6) corresponding to the hyperbolic space, we compute (here Δ_S is the Laplacian on the sphere)

$$\begin{aligned}-\Box_{\tilde{g}} = {}& -(1-r^2)\partial_r^2 + 2r\partial_r\partial_t + \partial_t^2 - \frac{1}{r^2}\Delta_S\\ &+(n+1)r\partial_r - \frac{n-1}{r}\partial_r + n\partial_t,\\ \widetilde{P}(\lambda) = {}& -(1-r^2)\partial_r^2 + (n+1-2i\lambda)r\partial_r - \frac{n-1}{r}\partial_r\\ &-\lambda^2 - ni\lambda - \frac{1}{r^2}\Delta_S.\end{aligned}$$

In terms of the operator $P_\psi(\lambda)$ computed in (5.3.18) we have

$$\widetilde{P}(\lambda) = P_\psi(\lambda) + \frac{1-n^2}{4}.$$

Since the hyperbolic space has no trapped trajectories, arguing similarly to Theorem 6.13 we see that $(\widetilde{X}, \tilde{g})$ has a resonance free strip of size ν for any $\nu > 0$ (and $N := 0$ in (5.7.25)).

2. For the spacetime (5.7.7) corresponding to the hyperbolic cylinder, we compute

$$\begin{aligned}-\Box_{\tilde{g}} &= -(1-r^2)\partial_r^2 + 2r\partial_t\partial_r + \partial_t^2 - \partial_\theta^2 + \partial_t + 2r\partial_r,\\ \widetilde{P}(\lambda) &= -(1-r^2)\partial_r^2 + 2(1-i\lambda)r\partial_r - \lambda^2 - i\lambda - \partial_\theta^2.\end{aligned}$$

In terms of the operator $P_\psi(\lambda)$ computed in (5.3.21) we have

$$\widetilde{P}(\lambda) = P_\psi(\lambda) - \frac{1}{4}.$$

The hyperbolic cylinder has normally hyperbolic trapping (see Example 2 on page 403). Arguing similarly to Theorem 6.16 (see also (6.3.28)) we see that $(\widetilde{X}, \tilde{g})$ has a resonance free strip of size ν for any $\nu \in (0, \frac{1}{2})$ (and $N := 1$ in (5.7.25)).

5.8. NOTES

The meromorphic continuation of the resolvent (5.0.3) in the case when $M = \Gamma\backslash\mathbb{H}^2$ is non-compact and has finite volume was proved by Selberg [**Se53**]. That case is an example of black box scattering (with $n = 1$) and was presented in Example 1, §4.1, Example 3, §4.2 and Example 3, §4.4. It is related to many great themes in mathematics, in particular the Selberg trace formula. (See [**GZ99**] for a presentation in the spirit of this book.)

The fundamental example for this chapter is given by quotients $M = \Gamma\backslash\mathbb{H}^2$ of infinite volume which are geometrically finite and have no cusps. Such M are called *convex co-compact* and the study of the resolvent in this case was initiated by Patterson [**Pa75**]. Celebrated results of Patterson [**Pa76**] and Sullivan [**Su79**] on the abscissa of convergence of Poincaré series and the dimension of the limit set for such $\Gamma\backslash\mathbb{H}^2$ have the following scattering theoretical interpretation: they imply existence of a resonance free strip under a "pressure condition" stating that the trapped set (see §6.1) is sufficiently "thin". In obstacle scattering this pressure condition appeared independently in physics in Gaspard–Rice [**GR89**] and in mathematics in Ikawa [**Ik88**] – see §6.6 for a description of recent advances. The book of Borthwick [**Bo16**] provides a comprehensive introduction to scattering and spectral theories on $\Gamma\backslash\mathbb{H}^2$. See also Bourgain–Gamburd–Sarnak [**BGS11**], Jakobson–Naud [**JN10**],[**JN12**],[**JN14**] and Oh–Winter [**OW16**] for related results with arithmetic connections.

For general asymptotically hyperbolic manifolds (and in particular for any convex co-compact $\Gamma\backslash\mathbb{H}^n$) the meromorphic continuation of the resolvent (5.0.3) was established by Mazzeo–Melrose [MM87]. Earlier contributions were made by Agmon [Ag86], Fay [Fa77], Lax–Phillips [LP82], Mandouvalos [Ma88] and Perry [Pe89]. Guillopé–Zworski [GZ95b] provided meromorphic continuation for manifolds with constant curvature in a neighbourhood of infinity and their method was extended to the case of more general constant curvature infinities (mixed rank cusps) by Guillarmou–Mazzeo [GM12] (see Froese–Hislop–Perry [FHP91] for an earlier version of the result obtained by Mourre theory methods).

Guillarmou [Gu05] showed that the evenness condition in Definition 5.5 was needed for a global meromorphic continuation and clarified the construction given in [MM87].

Roughly speaking, the Mazzeo–Melrose method is based on constructing a pseudo-differential calculus which allows solving

$$\Big(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\Big)\widetilde{R}(\lambda) = I + K(\lambda),$$
$$\widetilde{R}(\lambda) \,:\, y_1^N L^2(M) \to y_1^{-N} L^2(M), \quad N > -\operatorname{Im}\lambda + \frac{n-1}{2},$$

where $K(\lambda)$ is compact as in operator $y_1^N L^2(M) \to y_1^N L^2(M)$. That shows meromorphic continuation of (5.0.3) as an operator $y_1^N L^2(M) \to y_1^{-N} L^2(M)$. That is, in spirit, similar to the method used in §§3.2 and 4.2 where the resolvent was continued as an operator $L^2_{\rm comp}(\mathbb{R}^n) \to L^2_{\rm loc}(\mathbb{R}^n)$. This approach does not provide a family of Fredholm operators and hence does not allow for the use of geometric methods to study the distribution of resonances. However, the detailed description of the resolvent in [MM87] and [Gu05] remains essential in the study of the scattering matrix on asymptotically hyperbolic manifods – see Graham–Zworski [GrZw] and references given there. That paper related the scattering matrix to conformal structure at infinity and the work of Fefferman–Graham [FG] which in turn is one of the mathematical foundations of the AdS/CFT correspondence in physics.

Vasy's method which is used in this chapter is dramatically different from earlier approaches and is related to the study of stationary wave equations for Kerr–de Sitter black holes – see [Va13] and §5.7. One of the first applications was Datchev–Dyatlov [DD13] who provided fractal upper bounds on the number of resonances for (even) asymptotically hyperbolic manifolds and in particular for convex co-compact quotients of $\mathbb{H}^n$. Previously only the case of convex co-compact Schottky quotients was known Guillopé–Lin–Zworski [GLZ04] and that was established using transfer operators and zeta function methods. In the context of black holes Vasy's construction has been used to obtain a quantitative version of Hawking radiation by Drouot

[Dr15b], exponential decay of waves in the Kerr–de Sitter case [Va13] (established earlier by Dyatlov [Dy11a, Dy11b] using a different method), the description of quasi-normal modes for perturbations of Kerr–de Sitter black holes by Dyatlov [Dy12] and rigorous definition of quasi-normal modes for Kerr–Anti de Sitter black holes by Gannot [Gan14]. The construction of the Fredholm family also plays a role in the study of linear and non-linear scattering problems – see Baskin–Vasy–Wunsch [BVW15], Hintz–Vasy [HV14b], [HV15], [HV16] and references given there.

A related approach to meromorphic continuation, motivated by the study of Anti-de Sitter black holes, was independently developed by Warnick [Wa15]. It is based on physical space techniques for hyperbolic equations and it also provides meromorphic continuation of resolvents for even asymptotically hyperbolic metrics [Wa15, §7.5].

For a physics perspective on quasi-normal modes of black hole spacetimes (discussed in §5.7) and additional references see Kokkotas–Schmidt [KS99], Dyatlov–Zworski [DZ13] and Cardoso et al [C*18]. In the mathematics literature quasi-normal modes were studied by Bachelot–Motet-Bachelot [BaMB] and Sá Barreto–Zworski [SZ97], who applied the methods of scattering theory and semiclassical analysis to the case of a spherically symmetric black hole. Resonance expansions for Schwarzschild–de Sitter spacetimes were obtained by Bony–Häfner [BH08], see also Melrose–Sá Barreto–Vasy [MSV14]. For Kerr–de Sitter black holes resonance expansions were proved by Dyatlov [Dy12].

5.9. EXERCISES

Section 5.1

1. Let (M,g) be an asymptotically hyperbolic manifold and

$$\varphi_t = \exp\Big(\frac{tH_{|\xi|_g^2}}{2}\Big) : T^*M \to T^*M \tag{5.9.1}$$

the geodesic flow. Fix a canonical product structure (y_1, y') and let (y_1, y', η_1, η') be the corresponding product structure for T^*M. This exercise explores the behaviour of geodesics on M near ∂M.

(a) Denote $\mu_1 := y_1\eta_1$. Show that trajectories of φ_t on S^*M solve the evolution equations

$$\dot y_1 = y_1\mu_1, \quad \dot\mu_1 = \mu_1^2 - 1 - \frac{y_1^3}{2}\langle \eta', \eta'\rangle_{\partial_{y_1} g_1(y_1, y')}, \quad (\dot y', \dot\eta') = y_1^2 \mathcal{G}_{y_1}(y', \eta')$$

where $\mathcal{G}_{y_1}$ is the generator of the geodesic flow of the metric $g_1(y_1, \bullet)$ defined similarly to (5.9.1).

(b) Let $\gamma : \mathbb{R} \to S^*M$ be a geodesic such that $y_1(\gamma(0)) < \varepsilon_1$, $\eta_1(\gamma(0)) \leq 0$. Show that for ε_1 small enough, we have as $t \to \infty$ along the geodesic $\gamma(t)$

$$y_1(t) = \mathcal{O}(e^{-t}), \quad \mu_1(t) = -1 + \mathcal{O}(e^{-2t}),$$
$$\big(y'(t), \eta'(t)\big) = (y'_\infty, \eta'_\infty) + \mathcal{O}(e^{-2t})$$

for some $(y'_\infty, \eta'_\infty) \in T^*\partial M$. (Hint: differentiate the function $e^{2t}\big(y_1(t)^2 + 1 - \mu_1(t)^2\big)$.)

Section 5.2

2. Show that for $\lambda \notin i\mathbb{Z}$, the equation (5.2.6) has two solutions of the form (5.2.8), with

$$a_{j,\pm} := (-\beta_k)^j \prod_{\ell=1}^{j} \frac{1}{I(\alpha_\pm + 2\ell)}$$

and $I(\alpha)$ defined in (5.2.7), $\alpha_\pm$ defined in (5.2.9). Show that the series (5.2.8) converges for all $y_1 > 0$.

3. On the hyperbolic plane $\mathbb{H}^2$ (see (5.0.1)), the Schwartz kernel of the resolvent (with respect to the density induced by the metric)

$$R(\lambda) = \Big(-\Delta_g - \lambda^2 - \frac{1}{4}\Big)^{-1} : L^2(\mathbb{H}^2) \to L^2(\mathbb{H}^2), \quad \operatorname{Im} \lambda > 0,$$

is given by (see e.g. [**Bo16**, Proposition 4.2])

$$R(\lambda)(w,z) = \frac{1}{4\pi} \int_0^1 \big(t(1-t)\big)^{-\frac{1}{2}-i\lambda} (\sigma(w,z) - t)^{-\frac{1}{2}+i\lambda}\, dt,$$

where, with $d_{\mathbb{H}^2}$ denoting the hyperbolic distance,

$$\sigma(w,z) = \cosh^2\Big(\frac{d_{\mathbb{H}^2}(w,z)}{2}\Big) = 1 + \frac{|w-z|^2}{(1-|w|^2)(1-|z|^2)}.$$

Use this formula to prove Proposition 5.8 for $\mathbb{H}^2$. (Hint: to show smoothness on $\overline{M}_{\text{even}}$, rather than just $\overline{M}$, use the change of variables $t \mapsto 1-t$.)

Section 5.3

4. Verify that the formulas (5.3.12) and (5.3.13) for p agree on $M \cap Y$.

Exercises 5.5–5.6 study solutions $v \in \mathcal{D}'(X)$ to the equation $P(\lambda)^* v = 0$ which are supported on $\{x_1 = 0\}$. Such distributions will lie in the cokernel $\operatorname{coker}^s P(\lambda)$ defined in (5.6.5) for sufficiently large s. This gives examples of resonances for $P(\lambda)$ which are not resonances of the original scattering resolvent $R(\lambda)$, see Exercise 5.14.

5. Let $f_0,\dots,f_{k-1}\in\mathcal{D}'(\partial M)$ and $f_{k-1}\not\equiv 0$. Show that

$$P(\lambda)^*\Big(\sum_{j=0}^{k-1}\delta^{(j)}(x_1)f_j(x')\Big)=0 \tag{5.9.2}$$

if and only if $\lambda=-ik$ and f_j solve the equations for $j=0,\dots,k-1$

$$4(j-k)f_{j-1}+\sum_{\ell=j}^{k-1}(-1)^{\ell-j}\binom{\ell}{j}\big(-\Delta_{\ell-j}+(4\ell+5-n-2k)\gamma_{\ell-j}\big)f_\ell=0 \tag{5.9.3}$$

where $f_{-1}:=0$, the second order differential operator Δ_j on ∂M is obtained by differentiating the coefficients of $\Delta_{g_1(x_1,\bullet)}$ j times in x_1 at $x_1=0$, and $\gamma_j:=\partial_{x_1}^j\gamma|_{x_1=0}\in C^\infty(\partial M;\mathbb{R})$.

Note that if f_{k-1} is fixed, then the equations (5.9.3) for $j=1,\dots,k-1$ determine $f_0,\dots,f_{k-2}$ uniquely, and the $j=0$ equation is satisfied if and only if $\mathbf{D}_kf_{k-1}=0$ where $\mathbf{D}_k$ is an order $2k$ differential operator on ∂M with principal part $(-\Delta_0)^k$.

6. (a) Using (5.3.20) multiplied by $(1-x_1)^2$, show that for the hyperbolic space $\mathbb{H}^n$, there exist $f_0,\dots,f_{k-1}$, $f_{k-1}\not\equiv 0$ satisfying (5.9.2) if and only if $\lambda=-ik$ and f_j solve the equations

$$\begin{aligned}(j-k)f_{j-1}+(a_j-\Delta_{g_S})f_j+(j+1)b_{j+1}f_{j+1}&=0,\\ a_j:=2(j+1)(j+1-k)+\frac{(n-1)(n-5-4j+2k)}{4}&,\\ b_j:=(j+1)(j+1-k)+\frac{(n-1)(n-5-4j+2k)}{4}&\end{aligned} \tag{5.9.4}$$

for all $j=0,\dots,k-1$, where we put $f_{-1}=f_k=0$.

(b) Verify the following claim for several small values of k: (5.9.4) has a nontrivial solution if and only if $-\Delta_{g_S}$ has an eigenvalue $(r+\frac{3-n}{2})(r+\frac{n-1}{2})$ for some $r=0,\dots,k-1$. (The claim is true for all k but the proof is algebraically involved, see [**Ba16**, Proposition 2] for the case $n=3$.)

(c) Assuming the claim in part (b) and using the known spectrum of $-\Delta_{g_S}$, conclude that for n even, (5.9.4) never has a nontrivial solution, and for $n\ge 3$ odd, it has a nontrivial solution if and only if $k\ge\frac{n-1}{2}$.

7. Let $P_{\psi,h}(\omega)$ be defined in (5.3.16). Show that $P_{\psi,h}(\omega)$ lies in $\mathrm{Diff}_h^2(X)$ and its principal symbol is

$$p_\psi(x,\xi;\omega)=e^{2\psi}p(x,\xi+\omega d\psi).$$

<u>Section 5.4</u>

8. Show that (5.4.8) and (5.4.26) are no longer true when $\omega=0$.

9. Show that the vector field $\langle\xi\rangle^{-1}H_p$ vanishes on the sets $L_\pm$, that is $L_\pm$ consist of radial points (see Remark 4 after Definition E.50).

Section 5.5

10. Using Exercises E.31, E.35, E.36, and E.41, show the following strengthening of Lemma 5.27:

(a) if $s > m > \frac{1}{2} - h^{-1}\operatorname{Im}\omega$ and $u \in \bar{H}^m(X)$, $P_h(\omega)u \in \bar{H}^{s-1}(X)$, then $u \in \bar{H}^s(X)$ and (5.5.27) holds;

(b) if $s > \frac{1}{2} - h^{-1}\operatorname{Im}\omega$, $v \in \mathcal{D}'(X)$ can be extended beyond $\overline{X}$ to a distribution supported on $\overline{X}$, and $P_h(\omega)^*v \in \dot{H}^{-s}(\overline{X})$, then $v \in \dot{H}^{1-s}(\overline{X})$ and (5.5.28) holds.

Section 5.6

11. Arguing as in the proof of Proposition 5.28, show that (5.5.30) also holds for fixed s and β large enough and negative. Does this work for (5.5.29)? Does this imply that $\lambda = h^{-1} + i\beta$ is not a pole of the operator $P(\lambda)^{-1}$ defined in (5.6.10)?

12. Give a proof of (5.6.12) without using that $P(\lambda)$ has index 0 (which itself relied on Proposition 5.28), instead applying Exercise 5.10.

13. Show that the extended resonances of (M, g) do not depend on the choice of the extension of the metric g_1 in (5.3.4). (Hint: use (5.6.6).)

14. This exercise gives an example of an extended resonance on the three-dimensional hyperbolic space $\mathbb{H}^3$ which is not a scattering resonance. We use the model (5.3.18), (5.3.20).

(a) Arguing as in Exercise 5.6, show that the distribution $v := \delta(x_1)$ is in the cokernel of $P(-i)$ (see (5.6.12)).

(b) Show that the function $u := 1$ is in the kernel of $P_\psi(-i)$.

(c) Why don't parts (a), (b) contradict the fact that $\mathbb{H}^3$ has no scattering resonances?

15. This exercise shows that $R(\lambda)f$ is independent of the way we extend $x_1^{\frac{i\lambda}{2}-\frac{n+3}{4}}f$ from M to X in (5.6.20). Assume that $\lambda \in \mathbb{C}$, $s > \frac{1}{2} - \operatorname{Im}\lambda$, and $f \in \bar{H}^{s-1}(X)$ satisfies $\operatorname{supp} f \cap M = \emptyset$ (that is, $\operatorname{supp} f \subset \{x_1 \leq 0\}$). Define $P(\lambda)^{-1}f$ by (5.6.7). Show that $\operatorname{supp}(P(\lambda)^{-1}f) \cap M = \emptyset$. (Hint: reduce to the case $f \in C^\infty(\overline{X})$, $\operatorname{supp} f \subset \{x_1 < 0\}$. Then use either analytic continuation from the upper half-plane or Theorem E.61.)

Section 5.7

16. This is an advanced exercise relying on all the material of this chapter, as well as on §§6.1–6.3. (See [**Va13**, §6] for a solution in the more general Kerr–de Sitter case and [**Dy15b**, §§3.1–3.3] for more information on the geometry of Kerr–de Sitter spacetimes.) Define the Schwarzschild–de Sitter spacetime as $(\widetilde{M}_{\mathrm{SdS}}, \tilde{g}_{\mathrm{SdS}})$ where $\widetilde{M}_{\mathrm{SdS}} := \mathbb{R}_t \times M_{\mathrm{SdS}}$ and

$$M_{\mathrm{SdS}} := (r_-, r_+)_r \times \mathbb{S}^2_y, \quad \tilde{g}_{\mathrm{SdS}} = -G dt^2 + G^{-1} dr^2 + r^2 g_S(y, dy),$$

g_S is the standard metric on $\mathbb{S}^2$, the function G is given by

$$G(r) = 1 - \frac{\Lambda r^2}{3} - \frac{2M_0}{r}, \quad M_0 > 0, \quad 0 < \Lambda < \frac{1}{9M_0^2},$$

and $r_- < r_+$ are the two positive roots of the equation $G(r) = 0$, with $G > 0$ on (r_-, r_+). Here M_0 is the mass of the black hole and Λ is the cosmological constant.

The metric $\tilde{g}_{\mathrm{SdS}}$ has a form similar to the Lorentzian metric corresponding to the hyperbolic cylinder (5.7.8), replacing $\mathbb{S}^1$ by $\mathbb{S}^2$, $d\theta^2$ by $r^2 g_S(y, dy)$, and the function $1 - r^2$ by $G(r)$. This exercise uses that similarity to obtain meromorphic continuation of the resolvent and resonance expansions for $(\widetilde{M}_{\mathrm{SdS}}, \tilde{g}_{\mathrm{SdS}})$.

(a) Let $F(r) \in C^\infty((r_-, r_+); \mathbb{R})$ be a function such that

$$\partial_r F(r) \pm G(r)^{-1} \quad \text{is smooth at } r = r_\pm.$$

Define $\Phi_F : (t, r, y) \mapsto (t - F(r), r, y)$. Show that the metric $\Phi_F^* \tilde{g}_{\mathrm{SdS}}$ continues smoothly past $\{r = r_\pm\}$. Denote an extension of the spacetime $(\widetilde{M}_{\mathrm{SdS}}, \Phi_F^* \tilde{g}_{\mathrm{SdS}})$ by $(\widetilde{X}, \tilde{g})$, where $\widetilde{X} := \mathbb{R}_t \times X$, and choose F such that the hypersurfaces $\{t = \text{const}\}$ are spacelike with respect to $\tilde{g}$. (In terms of the analogy with the hyperbolic cylinder, we undid the change of variables (5.7.4).)

(b) Arguing as in §5.7.2, show that for each solution u to the wave equation (5.7.16) on $(\widetilde{X}, \tilde{g})$, the Fourier–Laplace transform $\hat{u}(\lambda)$ continues meromorphically to $\lambda \in \mathbb{C}$. (Note: one has to change the regularity threshold condition $s > \frac{1}{2} - \operatorname{Im}\lambda$ to $\mp G'(r_\pm)(s - \frac{1}{2}) + 2\operatorname{Im}\lambda > 0$.)

(c) Show that the trapping in the domain of outer communications is normally hyperbolic, with the trapped set contained in the *photon sphere* $\{r = 3M_0,\ \xi_r = 0\}$, and obtain a resonance expansion similar to Theorem 5.40 with any $\nu < \frac{\sqrt{1 - 9\Lambda M_0^2}}{6\sqrt{3} M_0}$.

Part 3

RESONANCES IN THE SEMICLASSICAL LIMIT

Chapter 6

RESONANCE FREE REGIONS

6.1 Geometry of trapping
6.2 Resonances in strips
6.3 Normally hyperbolic trapping
6.4 Logarithmic resonance free regions
6.5 Lower bounds on resonance widths
6.6 Notes
6.7 Exercises

In this section, we study existence of resonance free regions at high energies or in the semiclassical limit. We have already seen logarithmic resonance free regions in potential scattering in Theorems 2.10, 3.10, 4.41. Theorem 4.43 provided such resonance free region for black box perturbations under an abstract *non-trapping* assumption given in Definition 4.42. A semiclassical version of these results will be discussed in §6.4.

In this chapter we will discuss resonance free regions including resonance free strips in some *trapping* situations.

A basic setting is given by the semiclassical Schrödinger operator on $\mathbb{R}^n$:

$$P = P(h) = -h^2\Delta + V(x), \quad V \in C^\infty_{\rm c}(\mathbb{R}^n;\mathbb{R}). \tag{6.0.1}$$

For technical simplicity, and to be able to refer to §4.5, we present the case of n odd but that restriction is irrelevant for the results of this chapter – see §6.6 for references. Here $h > 0$ is a constant called the *semiclassical parameter*. We consider the semiclassical régime, using the operator $P - z$

where

$$z \in [\alpha, \beta] + i[-\nu(h), \infty), \quad 0 < \alpha \le \beta, \quad \nu(h) \in (0,1), \quad h \to 0.$$

In terms of the λ-plane picture (see §2.1) this means that we consider $P - \lambda^2$ with

$$\lambda \in [a, b] + i[-\mu(h), \infty), \quad 0 < a < b, \quad \mu(h) \in (0,1), \quad h \to 0.$$

(Here the behaviour in $0 \le -\operatorname{Im} z \ll 1$, $0 \le -\operatorname{Im} \lambda \ll 1$ is important.) For the application to the wave equation we also need estimates on $R(\lambda)$ for $\lambda \in [-b, -a] + i[-\mu(h), \infty)$ but these follow from (4.2.19).

The meromorphic continuation of the scattering/outgoing resolvent

$$R(z,h) = (P - z)^{-1} : \begin{cases} L^2 \to H^2, & z \in [\alpha, \beta] + i(0, \infty]; \\ L^2_{\mathrm{loc}} \to H^2_{\mathrm{comp}}, & z \in [\alpha, \beta] + i[-\nu(h), 0] \end{cases} \tag{6.0.2}$$

follows by rescaling from Theorem 3.8. The set of resonances of $P = P(h)$ is denoted by

$$\operatorname{Res}(P) = \operatorname{Res}(P(h)).$$

We say that P has a *resonance free region* of size $\nu(h)$ in the energy range $[\alpha, \beta]$, if there exist $N, h_0 > 0$ such that for all $h \in (0, h_0)$, $\chi \in C_0^\infty(\mathbb{R}^n)$ the following cutoff resolvent estimate holds:

$$\|\chi R(z,h)\chi\|_{L^2 \to L^2} \le C_\chi h^{-N}, \quad z \in [\alpha, \beta] + i[-\nu(h), \infty). \tag{6.0.3}$$

In particular, this implies that there are no resonances in $[\alpha, \beta] + i[-\nu(h), \infty)$, for $h < h_0$. The estimate (6.0.3) implies decay of solutions to the wave equation at high frequency, with the rate depending on $\nu(h)$ – see the table below.

The classical objects associated to the operator P are its semiclassical principal symbol

$$p(x, \xi) = |\xi|^2 + V(x), \quad (x, \xi) \in \mathbb{R}^{2n} = T^*\mathbb{R}^n,$$

and its Hamiltonian flow

$$\exp(tH_p) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n,$$

$$H_p := \sum_{j=1}^n \partial_{\xi_j} p \, \partial_{x_j} - \partial_{x_j} p \, \partial_{\xi_j}. \tag{6.0.4}$$

The size, $\nu(h)$, of the resonance free region depends on the structure of the *trapped set* $K_{[\alpha,\beta]}$, consisting of trajectories of $\exp(tH_p)$ in the energy slab $p^{-1}([\alpha, \beta])$ which do not escape to infinity in either direction – see §6.1.

In case when trapping is present, we treat the interaction region in a way which is decoupled from the analysis at infinity. To illustrate this, and to broaden the class of examples to which the results of this chapter apply, we consider the following two settings, the first of which generalizes (6.0.1):

- $(\mathbb{R}^n, g)$ is a compact metric perturbation of the Euclidean space, $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$, and

$$P(h) = -h^2\Delta_g + V(x); \tag{6.0.5}$$

- (M, g) is an even asymptotically hyperbolic manifold in the sense of Definitions 5.2 and 5.5, and

$$P(h) = -h^2\Delta_g. \tag{6.0.6}$$

Our presentation will be centered around case (6.0.5), indicating what changes are necessary to cover case (6.0.6). The restriction on the supports of $g_{ij} - \delta_{ij}$ and V can be relaxed significantly – see [**Sj96a**] for a general treatment. The key is the structure of the trapped set.

The results of this chapter (see §7.5 for the wave decay statements and §5.7.2 for applications to wave decay on black hole spacetimes) are summarized in the following table:

§	Trapping assumptions	Setting	$\nu(h)$ = size of the gap	Wave decay
§6.2	Nontrapping ($K_{[\alpha,\beta]} = \emptyset$)	(6.0.5), (6.0.6)	Ch, C arbitrary	Exponential, arbitrary rate
§6.4	Nontrapping (Euclidean ∞)	(6.0.5)	$Ch\log(1/h)$	Exponential, arbitrary rate
§6.3	Normally hyperbolic	(6.0.5), (6.0.6)	ch, $c > 0$ fixed	Exponential, fixed rate
§6.5	No assumptions	(6.0.1)	$\exp(-C/h)$	Logarithmic

Remarks about generalizations and improvements, in particular in the case of "no assumptions", will be given in §6.6.

6.1. GEOMETRY OF TRAPPING

In this section, we define and investigate the incoming/outgoing tails and the trapped set associated to the Hamilton flow $\exp(tH_p)$. Here $p(x, \xi)$ is the semiclassical principal symbol of P given by (6.0.5) or (6.0.6), and $\exp(tH_p)$ is the Hamiltonian flow of p on the cotangent bundle – see (6.0.4).

6.1.1. The Euclidean case. We start with the Euclidean case (6.0.5), where

$$p(x, \xi) = |\xi|_g^2 + V(x), \quad (x, \xi) \in T^*\mathbb{R}^n; \quad V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$$

and the Riemannian metric g on $\mathbb{R}^n$ is a compactly supported perturbation of the Euclidean metric. We fix a constant $r_0 > 0$ such that

$$\operatorname{supp} V,\ \operatorname{supp}(g_{ij} - \delta_{ij}) \subset B(0, r_0). \tag{6.1.1}$$

The classical trajectories $(x(t), \xi(t)) = e^{tH_p}(x, \xi)$ solve Hamilton's equations. In $\{|x| \geq r_0\}$, we have $p(x, \xi) = |\xi|^2$ and

$$\dot{x}(t) = 2\xi(t), \quad \dot{\xi}(t) = 0, \tag{6.1.2}$$

so the corresponding trajectories are straight lines.

DEFINITION 6.1. *Let $(x, \xi) \in T^*\mathbb{R}^n$ and $(x(t), \xi(t)) = e^{tH_p}(x, \xi)$ be the corresponding trajectory. We say that (x, ξ) **escapes** as $t \to +\infty$ (respectively as $t \to -\infty$) if*

$$x(t) \to \infty \quad \textit{as } t \to +\infty \textit{ (respectively } t \to -\infty\textit{)}.$$

*We define the **incoming tail** Γ^- and the **outgoing tail** Γ^+ to be the sets of trajectories which do not escape as $t \to +\infty$, respectively as $t \to -\infty$:*

$$\Gamma^{\pm} = \{(x, \xi) \mid x(t) \not\to \infty \textit{ as } t \to \mp\infty\}.$$

*The **trapped set** is defined as the set of points which do not escape in either time direction, namely*

$$K := \Gamma^+ \cap \Gamma^-. \tag{6.1.3}$$

For a set $J \subset \mathbb{R}$, we define the incoming/outgoing tails and the trapped set at energies in J:

$$\Gamma_J^{\pm} := \Gamma^{\pm} \cap p^{-1}(J), \quad K_J := K \cap p^{-1}(J). \tag{6.1.4}$$

For $E \in \mathbb{R}$, we put $\Gamma_E^{\pm} := \Gamma_{\{E\}}^{\pm}$ and $K_E := K_{\{E\}}$.

It follows from the definition that $\Gamma^{\pm}, K, \Gamma_J^{\pm}, K_J$ are invariant under the flow $\exp(tH_p)$. Moreover,

$$K_{(-\infty,0]} = p^{-1}\big((-\infty, 0]\big),$$

that is escaping trajectories only exist at positive energies. See Figure 6.1.

Let

$$r(x) := |x|, \quad x \in \mathbb{R}^n. \tag{6.1.5}$$

We can lift r to a function on $T^*\mathbb{R}^n$, putting $r(x, \xi) := r(x)$. For $r(x) \geq r_0$,

$$H_p r(x, \xi) = \frac{2\langle x, \xi\rangle}{r(x)}, \quad H_p^2 r(x, \xi) = \frac{4(|x|^2|\xi|^2 - \langle x, \xi\rangle^2)}{r(x)^3} \geq 0.$$

This and (6.1.2) imply that every point (x, ξ) satisfying the inequalities

$$r(x) > r_0, \quad \pm H_p r(x, \xi) > 0 \tag{6.1.6}$$

escapes as $t \to \pm\infty$, and conversely every trajectory escaping as $t \to \pm\infty$ eventually satisfies (6.1.6).

We also note that if $H_p r(x_0, \xi_0) = 0$, $r(x_0) \geq r_0$ and $\xi_0 \neq 0$ then $H_p^2 r(x_0, \xi_0) > 0$ which means that $H_p r(x, \xi)$ is strictly increasing on the trajectory through (x_0, ξ_0) and $r(x) \geq r_0$ on that trajectory.

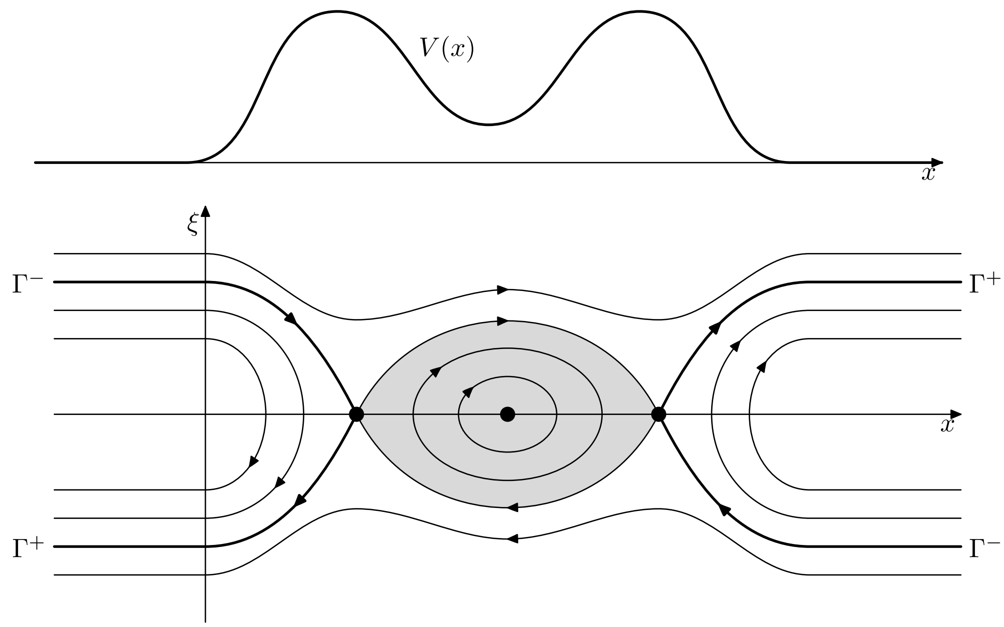


Figure 6.1. Example of Hamiltonian dynamics of the symbol $p(x,\xi) = \xi^2 + V(x)$ on $p^{-1}((0,\infty))$, with the potential V pictured on the top. The three marked points are fixed points of the flow. The shaded region, including the boundary, is the trapped set K, and the four thick trajectories are the components of $\Gamma^\pm \setminus K$.

We record this in

LEMMA 6.2. *Let* $(x,\xi) \in T^*\mathbb{R}^n$ *and* $(x(t),\xi(t)) = e^{tH_p}(x,\xi)$. *Then:*

1. *If* $r(x) \geq r_0$, $\xi \neq 0$, *and* $\pm H_p r(x,\xi) \geq 0$, *then* $(x,\xi) \notin \Gamma^\mp$ *and* $(x(t),\xi(t))$ *satisfies* (6.1.6) *for all* t, $\pm t > 0$.
2. *If* $(x,\xi) \notin \Gamma^\mp$, *then* $(x(t),\xi(t))$ *satisfies* (6.1.6) *for* $\pm t \gg 1$.

Using Lemma 6.2, we establish topological properties of $\Gamma^\pm$ and K:

PROPOSITION 6.3. *The sets* $\Gamma^\pm$ *are closed and for each compact* $J \subset \mathbb{R} \setminus \{0\}$, K_J *is compact. Moreover,*

$$K_{\mathbb{R}\setminus\{0\}} \subset \{r < r_0\}. \tag{6.1.7}$$

Finally, if $K_E = \emptyset$ *for some* $E \in \mathbb{R}$, *then* $K_{[E-\delta,E+\delta]} = \emptyset$ *for some* $\delta > 0$.

Proof. 1. We show that Γ^- is closed with the case of Γ^+ handled similarly. Let $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus \Gamma^-$. By part 2 of Lemma 6.2, there exists $T \geq 0$ such that $e^{TH_p}(x_0,\xi_0)$ satisfies (6.1.6) with a $+$ sign. Since that is an open condition, it is satisfied by $e^{TH_p}(x,\xi)$ for all (x,ξ) in some neighbourhood U of (x_0,ξ_0). Now, by part 1 of Lemma 6.2, $U \cap \Gamma^- = \emptyset$. It follows that $T^*\mathbb{R}^n \setminus \Gamma^-$ is open and thus Γ^- is closed.

2. To see (6.1.7), take $(x,\xi) \in \{r \geq r_0\}$ with $\xi \neq 0$. If $H_p r(x,\xi) \geq 0$, then by part 1 of Lemma 6.2, $(x,\xi) \notin \Gamma^-$. If $H_p r(x,\xi) \leq 0$, then similarly $(x,\xi) \notin \Gamma^+$. In either case $(x,\xi) \notin K$.

3. Since K is an intersection of two closed sets, it is closed. Combining this fact with (6.1.7), we see that $K_J = K \cap p^{-1}(J)$ is bounded, hence compact, for each compact $J \subset \mathbb{R} \setminus \{0\}$.

4. To show the last statement we argue by contradiction, assuming that $K_E = \emptyset$ and there exists a sequence $E_j \to E$ such that $K_{E_j} \neq \emptyset$. Take $(x_j,\xi_j) \in K_{E_j}$. Since $K_E = \emptyset$ we have $E \neq 0$, so we may assume that $E_j \in J$ where $J \subset \mathbb{R} \setminus \{0\}$ is some compact set. Since (x_j,ξ_j) lie in the compact set K_J, we may pass to a subsequence to make (x_j,ξ_j) converge to some point (x_∞,ξ_∞). Since K is closed, $(x_\infty,\xi_\infty) \in K$, moreover $p(x_\infty,\xi_\infty) = E$. Thus $(x_\infty,\xi_\infty) \in K_E$, a contradiction. □

PROPOSITION 6.4. *Let $(x,\xi) \in \Gamma_E^\pm$ for some $E \in \mathbb{R}$ and $(x(t),\xi(t)) = e^{tH_p}(x,\xi)$. Then*

$$(x(t),\xi(t)) \to K_E \quad \text{as } t \to \mp\infty \tag{6.1.8}$$

in the sense that the distance from $(x(t),\xi(t))$ to K_E converges to zero. In particular, if $K_E = \emptyset$, then $\Gamma_E^\pm = \emptyset$ as well. The convergence in (6.1.8) is uniform for (x,ξ) varying in any compact subset of $\Gamma_E^\pm$.

Proof. 1. Again, we consider only the case of $(x,\xi) \in \Gamma_E^-$. We may assume that $E > 0$, since otherwise $\Gamma_E^- = K_E$.

We first show that $(x(t),\xi(t))_{t\geq 0}$ is precompact. More precisely, with $\rho(t) := r(x(t))$,

$$(x(t),\xi(t)) \in p^{-1}(E) \cap \{r \leq \max(r_0,\rho(0))\}, \quad t \geq 0. \tag{6.1.9}$$

Indeed, if (6.1.9) does not hold, then there exists $T > 0$ such that $\rho(T) > r_0$ and $\rho(T) > \rho(0)$. Let $t_0 \in [0,T]$ be the point where the function ρ achieves its maximal value, then $t_0 > 0$, $\rho(t_0) > r_0$, and

$$H_p r(x(t_0),\xi(t_0)) = \dot\rho(t_0) \geq 0.$$

By part 1 of Lemma 6.2, we have $(x(t_0),\xi(t_0)) \notin \Gamma^-$, thus $(x,\xi) \notin \Gamma^-$, a contradiction.

2. We now prove (6.1.8) by contradiction. Indeed, if it does not hold, then there exists a sequence $t_j \to \infty$ and a neighbourhood U of K_E such that $(x(t_j),\xi(t_j)) \notin U$ for all j. By (6.1.9), we may pass to a subsequence to make

$$(x(t_j),\xi(t_j)) \to (x_\infty,\xi_\infty) \quad \text{for some } (x_\infty,\xi_\infty) \notin K_E = \Gamma_E^+ \cap \Gamma_E^-.$$

Since Γ_E^- is closed, we have $(x_\infty, \xi_\infty) \in \Gamma_E^-$. It follows that $(x_\infty, \xi_\infty) \notin \Gamma^+$. Then $r(e^{tH_p}(x_\infty, \xi_\infty)) \to \infty$ as $t \to -\infty$, thus there exists $T > 0$ such that

$$r(e^{-TH_p}(x_\infty, \xi_\infty)) > \max(r_0, \rho(0)).$$

Since $(x(t_j - T), \xi(t_j - T)) = e^{-TH_p}(x(t_j), \xi(t_j))$ converges to $e^{-TH_p}(x_\infty, \xi_\infty)$, we have for j large enough,

$$\rho(t_j - T) > \max(r_0, \rho(0)).$$

But since $t_j \to \infty$ this contradicts (6.1.9).

To show uniformity of convergence we argue as above to prove that for every convergent sequence $(x_j, \xi_j) \in \Gamma_E^-$ and every $t_j \to \infty$ we have $e^{t_j H_p}(x_j, \xi_j) \to K_E$. $\square$

The next result has the flavor of Poincaré's recurrence theorem and to formulate it we recall the definition of the Liouville measure on a non-degenerate energy surface $p^{-1}(E)$. The cotangent bundle $T^*\mathbb{R}^n$ is equipped with a canonical measure obtained from the symplectic form:

$$dm = \omega^n / n!, \quad \omega := d\sigma, \quad \sigma := \xi dx.$$

If $dp|_{p^{-1}(E)} \neq 0$ then $p^{-1}(E)$ is a smooth hypersurface in $T^*\mathbb{R}^n$ and a natural measure, $\mathcal{L}_E$, on $p^{-1}(E)$ is given by

$$dp \wedge d\mathcal{L}_E := dm.$$

Since p and m are preserved by the flow of H_p, so is $\mathcal{L}_E$.

PROPOSITION 6.5. *Denote by m the canonical measure on $T^*\mathbb{R}^n$. Then*

$$m(\Gamma^\pm \setminus K) = 0. \tag{6.1.10}$$

Also, if E satisfies $dp|_{p^{-1}(E)} \neq 0$ and $\mathcal{L}_E$ is the Liouville measure on $p^{-1}(E)$, then

$$\mathcal{L}_E(\Gamma_E^\pm \setminus K_E) = 0. \tag{6.1.11}$$

Proof. We show (6.1.11), with (6.1.10) proved similarly. We assume that $E > 0$, since otherwise $\Gamma_E^\pm = K_E$.

1. By (6.1.9), we have

$$e^{tH_p}(\Gamma_E^- \cap \{r \leq r_0\}) \ \subset \ \Gamma_E^- \cap \{r \leq r_0\}, \quad t \geq 0.$$

For $j \in \mathbb{Z}$, put $A_j := e^{jH_p}(\Gamma_E^- \cap \{r \leq r_0\})$. Then,

$$A_{j+1} \ \subset \ A_j, \quad \bigcup\nolimits_{j \in \mathbb{Z}} A_j = \Gamma_E^-, \quad \bigcap\nolimits_{j \in \mathbb{Z}} A_j = K_E,$$

where the last two statements follow from (6.1.8) and (6.1.7).

2. Since A_j is compact, $\mathcal{L}_E(A_j) < \infty$. It follows that

$$\mathcal{L}_E(K_E) = \lim_{j\to+\infty} \mathcal{L}_E(A_j), \quad \mathcal{L}_E(\Gamma_E^-) = \lim_{j\to-\infty} \mathcal{L}_E(A_j).$$

However, $\mathcal{L}_E$ is invariant under the flow $\exp(tH_p)$, thus $\mathcal{L}_E(A_j) = \mathcal{L}_E(A_0)$ for all j. Therefore

$$\mathcal{L}_E(\Gamma_E^-) = \mathcal{L}_E(K_E) = \mathcal{L}_E(A_0),$$

and it follows that $\mathcal{L}_E(\Gamma_E^- \setminus K_E) = 0$. Similarly $\mathcal{L}_E(\Gamma_E^+ \setminus K_E) = 0$. □

6.1.2. The asymptotically hyperbolic case. We now consider the second case discussed in the introduction, with the Hamiltonian given by (6.0.6). The classical Hamiltonian is[1]

$$p(x,\xi) = |\xi|_g^2, \quad (x,\xi) \in T^*M \tag{6.1.12}$$

and (M,g) in an asymptotically hyperbolic manifold in the sense of Definition 5.2.

The replacement of the function r defined by (6.1.5) is

$$r := y_1^{-1}, \tag{6.1.13}$$

where $y_1 : \overline{M} \to [0,\infty)$ is any fixed canonical boundary defining function in the sense of Definition 5.3. Note that the level sets $r^{-1}([0,C])$ are compact for each C. As in the Euclidean case, we lift r to a function on T^*M. We use Definition 6.1 to introduce the incoming/outgoing tails $\Gamma^\pm$ and the trapped set K, and (6.1.4) to define the sets $\Gamma_J^\pm, K_J$; all of these are subsets of T^*M. We have $K_{(-\infty,0]} = p^{-1}((-\infty,0]) = \{\xi = 0\}$.

The behaviour of the flow $\exp(tH_p)$ near the infinity of M is less straightforward than in the Euclidean case. To establish the properties of $\Gamma^\pm$ and K, we use the following

LEMMA 6.6 (**Convexity near infinity**). *There exists $r_0 > 0$ such that for each $(x,\xi) \in T^*M$,*

$$r(x) \ge r_0 \quad \Longrightarrow \quad H_p^2 r(x,\xi) \ge 2p(x,\xi). \tag{6.1.14}$$

Proof. Using Theorem 5.4, fix a canonical product structure on M,

$$(y_1, y') : y_1^{-1}((0,\varepsilon_1)) \to (0,\varepsilon_1) \times \partial M.$$

Let η_1, η' be the momenta corresponding to y_1, y', then by (5.1.5)

$$p(y_1, y', \eta_1, \eta') = y_1^2(\eta_1^2 + |\eta'|^2_{g_1(y_1,y')}).$$

[1] Note that our use of the symbol p in this chapter differs from Chapter 5 where p was the symbol of the modified Laplacian introduced in (5.3.11). The two are related by (5.3.12).

We compute

$$H_p r = -y_1^{-2} H_p y_1 = -2\eta_1,$$
$$H_p^2 r = 4y_1^{-1} p + 2y_1^2 \langle \eta', \eta' \rangle_{\partial_{y_1} g_1(y_1, y')}.$$

Since $g_1(y_1, y')$ is smooth up to $y_1 = 0$, there exists a constant C such that for all $y_1 \in (0, \varepsilon_1)$,

$$y_1^2 |\langle \eta', \eta' \rangle_{\partial_{y_1} g_1(y_1, y')}| \leq Cp.$$

Therefore,

$$H_p^2 r \geq (4r - 2C)p$$

and (6.1.14) follows for $r \geq r_0$ and r_0 large enough. □

We now show the analogue of Lemma 6.2 in the asymptotically hyperbolic setting:

LEMMA 6.7. *Let $(x, \xi) \in T^*M$ and $(x(t), \xi(t)) = \exp(tH_p)(x, \xi)$. Then:*

1. If $r(x) \geq r_0$, $\xi \neq 0$, and $H_p r(x, \xi) \geq 0$, then $(x, \xi) \notin \Gamma^-$ and

$$r(x(t)) > r_0, \ H_p r(x(t), \xi(t)) > 0 \quad \textit{for all } t > 0. \tag{6.1.15}$$

2. If $(x, \xi) \notin \Gamma^-$, then

$$r(x(t)) > r_0, \ H_p r(x(t), \xi(t)) > 0 \quad \textit{for large enough } t > 0. \tag{6.1.16}$$

Same is true for propagation in the negative time direction, replacing Γ^- by Γ^+ and changing the sign of $H_p r$.

Proof. 1. By rescaling we may assume that $p(x(t), \xi(t)) = 1$. We put $\rho(t) := r(x(t))$ and note that

$$\dot{\rho}(t) = H_p r(x(t), \xi(t)), \quad \ddot{\rho}(t) = H_p^2 r(x(t), \xi(t)).$$

Assume first that $r(x) \geq r_0$ and $H_p r(x, \xi) \geq 0$. In our new notation that means $\rho(0) \geq r_0$ and $\dot{\rho}(0) \geq 0$. By (6.1.14), we have

$$\ddot{\rho}(t) \geq 2 \quad \text{for all } t \text{ such that } \rho(t) \geq r_0. \tag{6.1.17}$$

Let $t > 0$. If the maximal value of ρ on the interval $[0, t]$ is achieved at some $t_0 \in (0, t)$ then $\rho(t_0) \geq r_0$, $\ddot{\rho}(t_0) \leq 0$, a contradiction with (6.1.17). Since ρ is non-decreasing at 0 the maximal value is then achieved at t, showing that $\rho(t) \geq r_0$ for all $t \geq 0$. Then $\ddot{\rho}(t) \geq 2$ for all $t \geq 0$ and

$$\dot{\rho}(t) \geq 2t, \quad \rho(t) \geq r_0 + t^2 \quad \text{for all } t \geq 0,$$

which implies (6.1.15). Moreover, $\rho(t) \to \infty$ as $t \to \infty$, thus $(x, \xi) \notin \Gamma^-$.

2. Assume now that $(x, \xi) \notin \Gamma^-$. Then $\rho(t) \to \infty$ as $t \to \infty$, thus there exists $T_0 > 0$ such that

$$\rho(T_0) > r_0, \quad \dot{\rho}(T_0) > 0.$$

Arguing as in Step 1 of the proof we see that same is true for all $t \geq T_0$, giving (6.1.16). □

With Lemma 6.7 in place Propositions 6.3–6.5 apply to asymptotically hyperbolic manifolds. Indeed, their proofs only used Lemma 6.2, which in the asymptotically hyperbolic case should be replaced by Lemma 6.7. In fact, these statements hold under more general assumptions near infinity – see Exercises 6.1–6.8.

6.2. RESONANCES IN STRIPS

In this section we study the properties of the scattering resolvent of an operator P of the form (6.0.5) or (6.0.6) where the spectral parameter z lies in an h-sized strip

$$\operatorname{Re} z \in [\alpha, \beta] \subset (0, \infty), \quad \operatorname{Im} z \in [-C_0 h, C_0 h], \tag{6.2.1}$$

α, β, C_0 are fixed and the semiclassical parameter h tends to zero. We show that in the nontrapping case, the strip (6.2.1) has no resonances for h small enough – see Theorems 6.10 and 6.13. We also prove existence of semiclassical defect measures associated to sequences of resonant states and show that they are supported on the outgoing tail – see Theorems 6.11 and 6.14. In fact, one way to establish existence of resonance free strips is by using these defect measures and that will be exploited in a situation in which there is trapping – see §6.3.

We assume that $P(h)$ is a second order semiclassical differential operator on a manifold M of the form (6.0.5) or (6.0.6). The L^2 resolvent

$$R(z, h) = (P(h) - z)^{-1} : L^2(M) \to H^2(M), \quad \operatorname{Im} z > 0$$

admits a meromorphic continuation to the region (6.2.1),

$$R(z, h) : L^2_{\mathrm{comp}}(M) \to H^2_{\mathrm{loc}}(M), \quad z \in [\alpha, \beta] + i[-C_0 h, C_0 h]. \tag{6.2.2}$$

Indeed, for the case (6.0.5) this follows from Theorem 4.4 since the operator $P(h)$ satisfies the black box assumptions of §4.1, see Example 1 preceding Lemma 4.12. For the case (6.0.6) this follows from Theorem 5.33 (assuming h is small enough).

Fix h and assume that z is a resonance, that is a pole of $R(z, h)$, in the region (6.2.1). Then

$$R(w, h) = \sum_{j=1}^{J} \frac{B_j}{(w - z)^j} + B(w, z), \tag{6.2.3}$$

where $w \mapsto B(w, z)$ is holomorphic near z and $B_J \neq 0$.

The space of *resonant states* at z is the range of the operator B_J – see Theorems 4.7 and 4.9. It is a finite-dimensional subspace of $C^\infty(M)$ and each resonant state u solves the equation

$$(P(h) - z)u = 0.$$

6.2.1. The Euclidean case. We start with the case (6.0.5), where $P = P(h)$ is a Schrödinger operator on a metric perturbation of $\mathbb{R}^n$, with n odd.

To study the resolvent at high energies, we will use the method of complex scaling as presented in §4.5. More precisely, fix a constant $r_1 > r_0$, with r_0 given by (6.1.1), take a scaling angle $\theta \in (0, \pi/2)$, and define the contour $\Gamma_\theta \subset \mathbb{C}^n$ by

$$\Gamma_\theta = f_\theta(\mathbb{R}^n); \quad f_\theta : \mathbb{R}^n \to \mathbb{C}^n, \quad f_\theta(x) = x + i\partial_x F_\theta(x),$$

where $F_\theta : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function satisfying

$$F_\theta(x) = 0 \quad \text{near } B(0, r_1); \quad F_\theta(x) = \frac{\tan\theta}{2}|x|^2, \quad x \geq 2r_1. \tag{6.2.4}$$

Consider the *complex scaled operator* $P_\theta = P_\theta(h)$, which is a second order semiclassical differential operator on Γ_θ (see §E.1.1) defined as follows:

- on $f_\theta(B(0, r_1)) = B(0, r_1) \subset \mathbb{R}^n$, $P_\theta(h) = P(h) = -h^2\Delta_g + V$;
- on $f_\theta(\mathbb{R}^n \setminus B(0, r_0))$, $P_\theta(h) = -h^2\Delta_{\Gamma_\theta}$, with Δ_{Γ_θ} defined by (4.5.7).

The two definitions agree in the transition region by (6.1.1). We use the map f_θ to identify Γ_θ with $\mathbb{R}^n$; then $P_\theta - z$ defines an operator

$$P_\theta - z : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{6.2.5}$$

By Theorem 4.36, (6.2.5) is a Fredholm operator for z in the region $\mathbb{C} \setminus e^{-2i\theta}[0, \infty)$. We henceforth assume that h is small enough so that this region contains (6.2.1). Moreover, (6.2.5) has a meromorphic inverse

$$(P_\theta - z)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n).$$

By Theorem 4.38, resonances in the region (6.2.1) coincide with the poles of the inverse of (6.2.5).

We record here some basic facts about the semiclassical principal symbol of P_θ:

LEMMA 6.8. *Suppose that $p_\theta = \sigma_h(P_\theta)$ is the semiclassical principal symbol of P_θ. Then*

(1) $\operatorname{Im} p_\theta(x, \xi) \leq 0$ *everywhere.*

(2) For $E \in \mathbb{R}$,

$$\{\langle\xi\rangle^{-2}(p_\theta - E) = 0\} \ \subset \ p^{-1}(E). \tag{6.2.6}$$

Moreover, if $0 < \alpha \le \beta$, *then there exists* $\delta > 0$ *such that for* $E \in [\alpha, \beta]$ *and* $|x| \ge 2r_1$ *(see (6.2.4))*

$$|p_\theta(x,\xi) - E| \ge \delta \langle \xi \rangle^2. \tag{6.2.7}$$

(3) If

$$\varphi_t := \exp(t\langle \xi \rangle^{-1} H_{\operatorname{Re} p_\theta}) : \overline{T}^* \mathbb{R}^n \to \overline{T}^* \mathbb{R}^n, \tag{6.2.8}$$

then for all (x, ξ) *and* $t_0 \le t_1$,

$$\begin{aligned} &\varphi_t(x,\xi) \in \{\langle \xi \rangle^{-2} \operatorname{Im} p_\theta = 0\} \quad \textit{for all } t \in [t_0, t_1] \\ \Longrightarrow \ &\varphi_t(x,\xi) = \exp(t\langle \xi \rangle^{-1} H_p)(x,\xi) \quad \textit{for all } t \in [t_0, t_1]. \end{aligned} \tag{6.2.9}$$

Proof. 1. For $r_1 > r_0$ introduced before (6.2.4)

$$p_\theta(x,\xi) = \begin{cases} \big\langle (1 + i\nabla^2 F_\theta(x))^{-2}\xi, \xi \big\rangle, & |x| \ge r_0; \\ p(x,\xi) = |\xi|_g^2 + V(x), & |x| \le r_1. \end{cases} \tag{6.2.10}$$

Since F_θ is convex, $\nabla^2 F_\theta(x)$ defines a nonnegative real quadratic form. Diagonalizing it, we derive the following properties of p_θ:

$$\operatorname{Im} p_\theta(x,\xi) \le 0 \quad \text{everywhere}; \tag{6.2.11}$$

$$\operatorname{Im} p_\theta(x,\xi) = 0 \implies \nabla^2 F_\theta(x)\xi = 0. \tag{6.2.12}$$

In particular we obtain part (1) or the lemma.

2. Since $\nabla^2 F_\theta(x)$ is nonnegative,

$$\nabla^2 F_\theta(x)\xi = 0 \implies \langle \partial_{x_j} \nabla^2 F_\theta(x)\xi, \xi \rangle = 0.$$

This implies that for each (x, ξ) such that $\operatorname{Im} p_\theta(x,\xi) = 0$,

$$p_\theta(x,\xi) = p(x,\xi), \quad \nabla p_\theta(x,\xi) = \nabla p(x,\xi). \tag{6.2.13}$$

3. From (6.2.13) we obtain (6.2.6). Moreover, for $\alpha > 0$, it follows from (6.2.4) and (6.2.10) that there exists $\delta > 0$ such that

$$\begin{aligned} |p_\theta(x,\xi) - \operatorname{Re} z| &\ge \tfrac{1}{2} \left(|\operatorname{Re} p_\theta(x,\xi) - \operatorname{Re} z| + |\operatorname{Im} p_\theta(x,\xi)| \right) \\ &\ge \delta \langle \xi \rangle^2 \quad \text{for } |x| \ge 2r_1. \end{aligned}$$

That gives (6.2.7).

4. We finally relate the rescaled Hamiltonian flow associated to $\operatorname{Re} p_\theta$ given by (6.2.8) to the corresponding object for the unscaled symbol p. By (6.2.13), φ_t agrees with the flow generated by p as long as $\operatorname{Im} p_\theta = 0$: that gives (6.2.9). $\square$

REMARK. It is convenient to use the rescaled flow (6.2.8) so that it is defined on the compactified co-tangent bundle $\overline{T}^*\mathbb{R}^n$ (see §E.1.3) and we can quote general propagation results of §E.4. Since ξ is bounded on $\Gamma^\pm_{[\alpha,\beta]}$ all the statements from §6.1 remain valid for the flow φ_t.

We now show that at high frequency, L^2 solutions to the equation $(P_\theta - z)u = f$ are controlled away from the outgoing tail. We also show that a positive fraction of their mass is localized near the trapped set. The resulting statement essentially reduces the analysis of resonance free regions inside (6.2.1) to a neighbourhood of the trapped set. We use the notion of semiclassical pseudodifferential operators, their wavefront sets WF_h, and their elliptic sets ell_h, see §§E.1,E.2.

PROPOSITION 6.9 (**Semiclassical outgoing estimates**). *Fix $0 < \alpha \leq \beta$ and $C_0 > 0$. Then the following estimates hold for all N, z satisfying (6.2.1) and $u \in L^2(\mathbb{R}^n)$, $f := (P_\theta - z)u \in L^2(\mathbb{R}^n)$, with constants independent of u, z, h:*

1. Let $A \in \Psi^0_h(\mathbb{R}^n)$ be compactly supported and $\mathrm{WF}_h(A) \cap \Gamma^+_{[\alpha,\beta]} = \emptyset$. Then

$$\|Au\|_{L^2} \leq Ch^{-1}\|f\|_{L^2} + Ch^N\|u\|_{L^2}. \tag{6.2.14}$$

2. Let $B \in \Psi^0_h(\mathbb{R}^n)$ be compactly supported and $K_{[\alpha,\beta]} \subset \mathrm{ell}_h(B)$. Then for h small enough,

$$\|u\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|f\|_{L^2}. \tag{6.2.15}$$

Proof. 1. We start by remarking that $\operatorname{Im} z = \mathcal{O}(h)$ and (6.2.6) imply that

$$\begin{gathered}\sigma_h(P_\theta - z) = p_\theta - \operatorname{Re} z, \quad \operatorname{Re} z \in [\alpha, \beta],\\ \{\langle\xi\rangle^{-2}(p_\theta - z) = 0\} \;\subset\; p^{-1}([\alpha,\beta]).\end{gathered}$$

To prove (6.2.14) we use the following fact: for each $(x,\xi) \in \mathrm{WF}_h(A)$, there exists $T \geq 0$ such that (with φ_t defined in (6.2.8))

$$\varphi_{-T}(x,\xi) \;\in\; \mathrm{ell}_h(P_\theta - z) = \{\langle\xi\rangle^{-2}(p_\theta - \operatorname{Re} z) \neq 0\}. \tag{6.2.16}$$

Indeed, we have $(x,\xi) \notin \Gamma^+_{[\alpha,\beta]}$. If $(x,\xi) \notin p^{-1}([\alpha,\beta])$, then $(x,\xi) \in \mathrm{ell}_h(P_\theta - z)$ by (6.2.6). Otherwise, $(x,\xi) \notin \Gamma^+$, therefore there exists $T_1 \geq 0$ such that $e^{-T_1\langle\xi\rangle^{-1}H_p}(x,\xi) \in \{r \geq 2r_1\}$. By (6.2.7) and (6.2.9), we see that (6.2.16) holds for some $T \in [0, T_1]$.

The estimate (6.2.14) follows from applying propagation of singularities, Theorem E.47, to the operator $\mathbf{P} = P_\theta - z$, with the controlling operator (denoted by B in Theorem E.47) equal to $P_\theta - z$ as well. Here the sign condition (E.4.12) is satisfied by Lemma 6.8 and the control condition (E.4.13) is satisfied by (6.2.16). (See also Remark 1 after Theorem E.47.)

2. We now prove (6.2.15). Fix a cutoff function

$$\chi \in C_{\rm c}^{\infty}(\mathbb{R}^n), \quad \chi = 1 \text{ near } \{|x| \leq 2r_1\}.$$

We first show the estimate

$$\|\chi u\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|f\|_{L^2} + Ch^N\|u\|_{L^2}. \tag{6.2.17}$$

We use the following fact: there exists $T \geq 0$ such that for each $(x, \xi) \in \overline{T}^*\mathbb{R}^n$, $x \in \operatorname{supp}\chi$,

$$\varphi_{-T}(x, \xi) \in \operatorname{ell}_h(P_\theta - z - iB^*B) = \operatorname{ell}_h(P_\theta - z) \cup \operatorname{ell}_h(B). \tag{6.2.18}$$

Indeed, if $(x, \xi) \notin \Gamma^+_{[\alpha,\beta]}$, this follows from (6.2.16). If $(x, \xi) \in \Gamma^+_{[\alpha,\beta]}$, then by Proposition 6.4, and since $K_{[\alpha,\beta]} \subset \operatorname{ell}_h(B)$, there exists $T_1 \geq 0$ such that $e^{-T_1\langle\xi\rangle^{-1}H_p}(x, \xi) \in \operatorname{ell}_h(B)$. By (6.2.9), we see that (6.2.18) holds for some $T \in [0, T_1]$.

To show (6.2.17), it remains to use propagation of singularities, Theorem E.47, with the controlling operator $P_\theta - z - iB^*B$, together with the inequality (here $B^* \in \Psi^0_h$ is uniformly bounded on L^2)

$$\|(P_\theta - z - iB^*B)u\|_{L^2} \leq \|f\|_{L^2} + C\|Bu\|_{L^2}.$$

In fact, we replace A with χ, B with $P_\theta - z - iB^*B \in \Psi^2_h(\mathbb{R}^n)$ and $B_1 = I$ in (E.4.14) to obtain (6.2.17).

3. To conclude the proof of (6.2.15) it remains to show the estimate

$$\|(1 - \chi)u\|_{L^2} \leq C\|f\|_{L^2} + Ch^N\|u\|_{L^2}. \tag{6.2.19}$$

The operator $P_\theta - z$ can be written in terms of the standard quantization (E.1.18) as follows:

$$P_\theta - z = \operatorname{Op}_h(\tilde{p}), \quad \tilde{p} = p_\theta - \operatorname{Re} z + \mathcal{O}(h)_{\overline{S}^1_{1,0}} \in \overline{S}^2_{1,0}$$

where the classes $\overline{S}^k_{1,0} = \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$, defined in (E.1.16), require uniform control on derivatives as $x \to \infty$.

Using (6.2.7) ($1 - \chi$ is supported in the region $|x| > 2r_1$) and the elliptic parametrix construction from the proof of Proposition E.32 for the $\overline{S}^k_{1,0}$ calculus reviewed in §E.1.5, we construct an operator

$$Q \in \operatorname{Op}_h(\overline{S}^{-2}_{1,0}), \quad 1 - \chi = Q(P_\theta - z) + \operatorname{Op}_h(h^\infty \overline{S}^{-\infty}_{1,0}). \tag{6.2.20}$$

By Proposition E.19, $\|Q\|_{L^2 \to L^2} \leq C$, and the remainder in (6.2.20) is $\mathcal{O}(h^\infty)_{L^2 \to L^2}$. Applying (6.2.20) to $u \in L^2$, we get (6.2.19).

Combining (6.2.17) and (6.2.19), and taking h small enough, we obtain (6.2.15). $\square$

An immediate application of Proposition 6.9 is a resonance free strip of size $C_0 h$ for arbitrary C_0 when there is no trapping at the energies in $[\alpha, \beta]$:

THEOREM 6.10 (Nontrapping estimates in strips). *Suppose that P is given by (6.0.5) and P_θ is its complex scaled version (6.2.5). Fix $0 < \alpha \leq \beta$, $C_0 > 0$, $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, and assume that*

$$K_{[\alpha,\beta]} = \emptyset.$$

Then the following estimates hold for h small enough, all $s \geq 0$, and all $z \in [\alpha, \beta] + i[-C_0 h, C_0 h]$:

$$\|(P_\theta - z)^{-1}\|_{H_h^s(\mathbb{R}^n) \to H_h^{s+2}(\mathbb{R}^n)} \leq C h^{-1}, \tag{6.2.21}$$

$$\|\chi R(z,h)\chi\|_{H_h^s(\mathbb{R}^n) \to H_h^{s+2}(\mathbb{R}^n)} \leq C h^{-1}. \tag{6.2.22}$$

In particular we have a resonance free region

$$\operatorname{Res}(P) \cap \big([\alpha, \beta] + i[-C_0 h, C_0 h]\big) = \emptyset. \tag{6.2.23}$$

Proof. 1. Since $C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, the estimate (6.2.21) follows from showing that for each

$$f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n), \quad u := (P_\theta - z)^{-1} f \in H^2(\mathbb{R}^n),$$

we have

$$\|u\|_{H_h^{s+2}(\mathbb{R}^n)} \leq C h^{-1} \|f\|_{H_h^s(\mathbb{R}^n)}. \tag{6.2.24}$$

Next, (6.2.22) follows immediately from (6.2.21) and Theorem 4.37, if we choose the constant r_1 in the construction of the complex scaling contour Γ_θ such that $\operatorname{supp} \chi \subset B(0, r_1)$.

2. The operator $P_\theta - z$ is elliptic near fiber (ξ) infinity, that is, there exists a constant $C_1 > 0$ such that

$$|p_\theta(x,\xi) - \operatorname{Re} z| \geq \langle \xi \rangle^2 / C_1 \quad \text{for } |\xi| \geq C_1.$$

Take $\chi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ with $\chi = 1$ near $B(0, C_1)$. Using an elliptic parametrix as in Step 3 of the proof of Proposition 6.9, we get

$$\|(1 - \chi(hD_x))u\|_{H_h^{s+2}(\mathbb{R}^n)} \leq C \|f\|_{H_h^s(\mathbb{R}^n)} + \mathcal{O}(h^\infty) \|u\|_{L^2(\mathbb{R}^n)}.$$

On the other hand,

$$\|\chi(hD_x)u\|_{H_h^{s+2}(\mathbb{R}^n)} \leq C \|u\|_{L^2(\mathbb{R}^n)}.$$

Adding these estimates, we get

$$\|u\|_{H_h^{s+2}(\mathbb{R}^n)} \leq C \|f\|_{H_h^s(\mathbb{R}^n)} + C \|u\|_{L^2(\mathbb{R}^n)}. \tag{6.2.25}$$

3. We will now use the fact that $K_{[\alpha,\beta]} = \emptyset$. This means that the estimate (6.2.15) holds with $B \equiv 0$, giving

$$\|u\|_{L^2} \le Ch^{-1}\|f\|_{L^2}.$$

Combining this with (6.2.25), we get (6.2.24), finishing the proof. □

6.2.2. Semiclassical defect measures. One way to understand families of functions depending on $h \to 0$ is by associating to them measures on the cotangent bundle. Properties of these measures capture classical properties in a rougher way than propagation estimates – see §E.3. However, results about measures can be sufficient for obtaining, say, resonance free strips. In this section we study semiclassical measures associated to resonant states. We will use them in §6.3 to obtain resonance free regions in the case of *normally hyperbolic trapping*.

We keep working in the Euclidean setting (6.0.5) and use the complex scaled operator $P_\theta(h)$, see §6.2.1. We endow $\mathbb{R}^n$ with the volume form induced by the metric g, so that the operator P is symmetric.

Assume that we have sequences $h_j \to 0$, z_j satisfying (6.2.1), and

$$u_j \in H^2(\mathbb{R}^n)$$

is a family of L^2 normalized $o(h)$-quasimodes to the operator $P_\theta - z$:

$$\|u_j\|_{L^2} = 1, \tag{6.2.26}$$

$$\|(P_\theta(h_j) - z_j)u_j\|_{L^2} = o(h_j). \tag{6.2.27}$$

In particular, one can take z_j to be a sequence of resonances of P and let u_j be elements of the kernel of $P_\theta(h_j) - z_j$; on $\{r \le r_1\}$, u_j coincide with resonant states of $R(z,h)$ by Theorem 4.37.

By Theorem E.42, passing to a subsequence we may assume that u_j converges to some nonnegative Borel measure μ on $T^*\mathbb{R}^n$ in the following sense (see Definition E.28 for the class Ψ_h^{comp}):

$$\langle A(h_j)u_j, u_j\rangle \to \int_{T^*\mathbb{R}^n} \sigma_h(A)\, d\mu \quad \text{as } j \to \infty, \quad A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n). \tag{6.2.28}$$

By (6.2.1), we may moreover pass to a subsequence satisfying

$$\operatorname{Re} z_j \to E \in [\alpha,\beta], \quad \frac{\operatorname{Im} z_j}{h_j} \to \nu \in [-C_0, C_0] \quad \text{as } j \to \infty. \tag{6.2.29}$$

Henceforth we will suppress the dependence of h, z, u on j in notation.

The main result of this section is the following

THEOREM 6.11 (Measures associated to resonant states). *Under the assumptions (6.2.26)–(6.2.29), the measure μ has the following properties:*

1. *μ is supported on Γ_E^+, that is $\mu(T^*\mathbb{R}^n \setminus \Gamma_E^+) = \emptyset$.*
2. *For each neighbourhood U of K_E, we have $\mu(U) > 0$.*
3. *For each $U \subset T^*\mathbb{R}^n \cap \{r \le r_1\}$ and each $t \ge 0$,*

$$\mu(e^{-tH_p}(U)) = e^{2\nu t}\mu(U). \tag{6.2.30}$$

REMARKS. 1. The condition (6.2.30) can also be written in the form

$$\mathcal{L}_{H_p}\mu = -2\nu\mu \quad \text{on } \{r \le r_1\}$$

where $\mathcal{L}_{H_p}$ denotes the Lie derivative, $\frac{d}{dt}\big|_{t=0}(e^{tH_p})^*\mu$.

2. By (6.1.9) (strictly speaking, its analogue for Γ^+), we have

$$e^{-tH_p}(\Gamma_E^+ \cap \{r \le r_1\}) \ \subset\ \Gamma_E^+ \cap \{r \le r_1\}, \quad t \ge 0, \tag{6.2.31}$$

therefore (6.2.30) implies that $\nu \le 0$. This is consistent with the fact that there are no resonances in the upper half z-plane.

3. Theorem 6.11 implies the resonance free strip of Theorem 6.10; indeed, if $K_{[\alpha,\beta]} = \emptyset$ then we may take $U = \emptyset$ in part 2 of Theorem 6.11. Of course, both Theorem 6.10 and Theorem 6.11 rely on Proposition 6.9, where most of the hard work is done.

Proof. 1. By Theorem E.43 applied to the operator $P_\theta - E$ and using (6.2.27) we have

$$\mu(T^*\mathbb{R}^n \setminus p_\theta^{-1}(E)) = 0.$$

By (6.2.6) this implies that $\mu(T^*\mathbb{R}^n \setminus p^{-1}(E)) = 0$. Moreover, for each $A \in \Psi_h^{\mathrm{comp}}(T^*\mathbb{R}^n)$ with $\mathrm{WF}_h(A) \cap \Gamma^+ = \emptyset$, we have by (6.2.14), (6.2.26), and (6.2.27)

$$\|Au\|_{L^2} \le Ch^{-1}\|(P_\theta - z)u\|_{L^2} + Ch^N \to 0 \quad \text{as } j \to \infty.$$

(Recall that we suppressed the dependence on j in notation.) It follows from (6.2.28) that for each $a \in C_c^\infty(T^*\mathbb{R}^n)$,

$$\operatorname{supp} a \cap \Gamma^+ = \emptyset \implies \int_{T^*\mathbb{R}^n} a\, d\mu = 0.$$

Therefore, $\mu(T^*\mathbb{R}^n \setminus \Gamma^+) = 0$, finishing the proof of the first claim.

2. Let U be a neighbourhood of K_E and take $B \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ such that $K_E \subset \mathrm{ell}_h(B)$ and $\mathrm{WF}_h(B) \subset U$. Then by (6.2.15), (6.2.26), and (6.2.27), we have

$$\|Bu\|_{L^2} \ge C^{-1}\|u\|_{L^2} - h^{-1}\|(P_\theta - z)u\|_{L^2} \ge \frac{1}{2C} \quad \text{for large } j.$$

Applying (6.2.28) to the expression $\|Bu\|_{L^2}^2 = \langle B^*Bu, u\rangle$, we see that

$$\int_{T^*\mathbb{R}^n} |\sigma_h(B)|^2\, d\mu > 0,$$

which implies that $\mu(U) > 0$, proving the second claim.

3. To prove the last claim, take a cutoff function

$$\chi \in C_c^\infty(\mathbb{R}^n), \quad \chi = 1 \quad \text{near } \{r \leq r_1\}, \quad \operatorname{supp}\chi \cap \operatorname{supp} F_\theta = \emptyset,$$

where F_θ is the function defining the complex scaling contour, see (6.2.4).

By (6.2.27) we have

$$\|\chi(P_\theta - z)u\|_{L^2} = o(h).$$

Since $\operatorname{supp}\chi \cap \operatorname{supp} F_\theta = \emptyset$ we have

$$\chi P_\theta = \chi P.$$

By Theorem E.44 with P replaced by $\chi(P - \operatorname{Re} z - i\nu h)$ (whose proof still applies despite the fact that $\operatorname{Re} z$ depends on h, since it does not influence the right-hand side of (E.3.12)), we get for each $a \in C_c^\infty(T^*\mathbb{R}^n)$ with $\operatorname{supp} a \subset \{r \leq r_1\}$,

$$\int_{T^*\mathbb{R}^n} H_p a - 2\nu a \, d\mu = 0.$$

Since μ is supported on Γ_E^+ and by (6.2.31), we see that

$$\int_{T^*\mathbb{R}^n} a \circ e^{tH_p} \, d\mu = e^{2\nu t} \int_{T^*\mathbb{R}^n} a \, d\mu \quad \text{when } \operatorname{supp} a \subset \{r \leq r_1\}, \ t \geq 0.$$

This implies (6.2.30). □

6.2.3. The asymptotically hyperbolic case. We now consider the case of asymptotically hyperbolic Laplacian (6.0.6).

We fix a canonical boundary defining function y_1 on M, put $x_1 := y_1^2$, and use the extended modified Laplacian introduced in §5.3, which is a differential operator on the even extension $\overline{X}$ of M. Since that operator is denoted $P(\lambda)$, we first fix some notation to avoid conflicts with Chapter 5. Let $P_h(\omega)$ be the semiclassical version of the extended modified Laplacian introduced in (5.3.8). We assume that h is small. For z in (6.2.1), put

$$\widetilde{P}(z) = \widetilde{P}(z;h) := P_h(\omega) \tag{6.2.32}$$

where $\omega \in \mathbb{C}$ is uniquely defined by

$$z = \omega^2 + \frac{(n-1)^2}{4} h^2, \quad \operatorname{Re}\omega > 0;$$

by (6.2.1), we have

$$\omega = \sqrt{\operatorname{Re} z} + i\frac{\operatorname{Im} z}{2\sqrt{\operatorname{Re} z}} + \mathcal{O}(h^2).$$

Then $\widetilde{P}(z)$ is a second order semiclassical differential operator on $\overline{X}$. It is related to the operator $P = -h^2\Delta_g$ from (6.0.6) by the following corollary of (5.3.9):

$$\widetilde{P}(z) = x_1^{\frac{i\omega}{2h}-\frac{n+3}{4}}(P-z)x_1^{\frac{n-1}{4}-\frac{i\omega}{2h}} \quad \text{on } M \subset X. \tag{6.2.33}$$

Fix

$$s > \frac{1}{2} + \frac{C_0}{2\sqrt{\alpha}}, \tag{6.2.34}$$

where C_0 is the constant from (6.2.1). By Theorem 5.30,

$$\widetilde{P}(z) : \mathcal{X}^s \to \bar{H}^{s-1}(X), \quad z \in [\alpha, \beta] + i[-C_0h, C_0h]$$

is a holomorphic family of Fredholm operators, and it has a meromorphic inverse $\widetilde{P}(z)^{-1}$. Here the space $\mathcal{X}^s \subset \bar{H}^s(X)$ is defined in (5.6.1) and $\bar{H}^s(X)$ denote Sobolev spaces on X as a manifold with boundary (see §E.1.8).

The L^2 resolvent of P continues meromorphically from the upper half plane to (6.2.1) by Theorem 5.33. The resulting resonances are contained in the set of poles of $\widetilde{P}(z)^{-1}$ by (5.6.20):

$$R(z,h)f = x_1^{\frac{n-1}{4}-\frac{i\omega}{2h}}\big(\widetilde{P}(z)^{-1}x_1^{\frac{i\omega}{2h}-\frac{n+3}{4}}f\big)\big|_M, \quad f \in C_c^\infty(M). \tag{6.2.35}$$

The analogue of Proposition 6.9 in the asymptotically hyperbolic case is given by the next result. To formulate it we recall from (5.4.30) the following definition:

$$j : T^*M \setminus 0 \to T^*M, \quad j(x,\xi) := \Big(x, \xi + |\xi|_g \frac{dx_1}{2x_1}\Big).$$

The purpose of j is to map the Hamiltonian dynamics of $p = |\xi|_g^2$ into that of $\sigma_h(\widetilde{P}(z))$, see Lemma 5.20. (Recall from §6.1.2 that the use of the letter p in this chapter is different from Chapter 5.) It is possible to make x_1 equal to 1, and thus j the identity map, on an arbitrary fixed compact subset of M.

PROPOSITION 6.12 (Semiclassical outgoing estimates, asymptotically hyperbolic case). *The following estimates hold for all N, z satisfying (6.2.1), and $u \in \bar{H}^s(X)$, $f := \widetilde{P}(z)u \in \bar{H}^{s-1}(X)$, with constants independent of u, z, h:*

1. Let $A \in \Psi_h^0(M)$ be compactly supported and $\mathrm{WF}_h(A) \cap j(\Gamma^+_{[\alpha,\beta]}) = \emptyset$. Then

$$\|Au\|_{H_h^s} \le Ch^{-1}\|f\|_{\bar{H}_h^{s-1}(X)} + Ch^N\|u\|_{\bar{H}_h^s(X)}. \tag{6.2.36}$$

2. Let $B \in \Psi_h^0(M)$ be compactly supported and $j(K_{[\alpha,\beta]}) \subset \mathrm{ell}_h(B)$. Then for h small enough,

$$\|u\|_{\bar{H}_h^s(X)} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|f\|_{\bar{H}_h^{s-1}(X)}. \tag{6.2.37}$$

Proof. 1. The estimate (6.2.36) follows directly from Theorem 5.35 where we put $Q := 0$. Indeed, the control condition (5.6.29) follows from the fact that $\mathrm{WF}_h(A) \cap j(\Gamma^+_{[\alpha,\beta]}) = \emptyset$, see (5.6.31).

2. For (6.2.37), we use Theorem 5.34. (A more direct approach would be as follows: we first prove Lemma 5.23 with $A_0 := B$ which is possible since the control condition (5.5.14) holds by Lemmas 5.19–5.20 and Proposition 6.4. We next combine it with Lemma 5.25 similarly to the proof of (5.5.27). Here we give a different proof using results already established in §5.6.3.)

Take $Q, Z \in \Psi_h^{\mathrm{comp}}(M)$ such that

$$\sigma_h(Q) \geq 0, \quad \mathrm{WF}_h(Q) \subset \mathrm{ell}_h(B), \quad j(K_{[\alpha,\beta]}) \subset \mathrm{ell}_h(Q);$$
$$\mathrm{WF}_h(Z) \subset \mathrm{ell}_h(B), \quad \mathrm{WF}_h(I - Z) \cap \mathrm{WF}_h(Q) = \emptyset.$$

By Proposition 6.4 and since $j(K_{[\alpha,\beta]}) \subset \mathrm{ell}_h(Q)$, the condition (5.6.25) is satisfied. Thus Q controls trapping in the sense of (5.6.23). Applying Theorem 5.34 to $(I - Z)u$ and recalling (6.2.32) we get

$$\begin{aligned}\|(I - Z)u\|_{\bar{H}_h^s(X)} &\leq Ch^{-1}\|(\widetilde{P}(z) - iQ)(I - Z)u\|_{\bar{H}_h^{s-1}(X)} \\ &\leq Ch^{-1}\|\widetilde{P}(z)(I - Z)u\|_{\bar{H}_h^{s-1}(X)} + Ch^N\|u\|_{\bar{H}_h^s(X)}\end{aligned}$$

where in the last inequality we used that $Q(I - Z) = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. This implies that

$$\begin{aligned}\|u\|_{\bar{H}_h^s(X)} &\leq \|(I - Z)u\|_{\bar{H}_h^s(X)} + \|Zu\|_{H_h^s} \\ &\leq Ch^{-1}\|f\|_{\bar{H}_h^{s-1}(X)} + Ch^{-1}\big\|[\widetilde{P}(z), Z]u\big\|_{H_h^{s-1}} \\ &\quad + \|Zu\|_{H_h^s} + Ch^N\|u\|_{\bar{H}_h^s(X)}.\end{aligned}$$

Now (6.2.37) follows using the elliptic estimate, Theorem E.33, since Z and $h^{-1}[\widetilde{P}(z), Z]$ lie in $\Psi_h^{\mathrm{comp}}(M)$ and have wavefront sets contained in $\mathrm{ell}_h(B)$. Here we take h small enough to remove the $Ch^N\|u\|_{\bar{H}_h^s(X)}$ remainder. □

Armed with Proposition 6.12, we prove the asymptotically hyperbolic version of the nontrapping estimate, Theorem 6.10:

THEOREM 6.13. *Fix $0 < \alpha \leq \beta$, $C_0 > 0$, $\chi \in C_c^\infty(M)$, and assume that*

$$K_{[\alpha,\beta]} = \emptyset.$$

Then the following estimates hold for h small enough, s satisfying (6.2.34), and all $z \in [\alpha, \beta] + i[-C_0 h, C_0 h]$:

$$\|\widetilde{P}(z)^{-1}\|_{\bar{H}_h^{s-1}(X) \to \bar{H}_h^s(X)} \leq Ch^{-1}, \tag{6.2.38}$$

$$\|\chi R(z,h)\chi\|_{H_h^{s-1} \to H_h^s} \leq Ch^{-1}. \tag{6.2.39}$$

Proof. Assume that $u \in \bar{H}^s(X)$ and $f := \widetilde{P}(z)u \in \bar{H}^{s-1}(X)$. By (6.2.37) with $B = 0$, we have

$$\|u\|_{\bar{H}_h^s(X)} \leq Ch^{-1}\|f\|_{\bar{H}_h^{s-1}(X)},$$

implying (6.2.38).

Next, (6.2.39) follows directly from (6.2.38), (6.2.35), and the fact that we may choose the defining function x_1 to be equal to 1 on $\operatorname{supp}\chi$. □

Finally we discuss semiclassical measures associated to $o(h)$ quasimodes for the operator $\widetilde{P}(z)$, see §6.2.2. Namely, assume that we have sequences $h_j \to 0$, z_j satisfying (6.2.1), s satisfies (6.2.34), E, ν are defined by (6.2.29), and $u_j \in \mathcal{X}^s$ satisfy

$$\|u_j\|_{\bar{H}_{h_j}^s(X)} = 1, \tag{6.2.40}$$

$$\|\widetilde{P}(z_j, h_j)u_j\|_{\bar{H}_{h_j}^{s-1}(X)} = o(h_j). \tag{6.2.41}$$

Similarly to (6.2.28) we assume that the restrictions $u_j|_M$ converge to a nonnegative Borel measure μ on T^*M, where we use the volume form induced by the metric g to define the inner product on $L^2(M)$.

The properties of semiclassical measures associated to u_j are given by the following analogue of Theorem 6.11 (whose proof applies to the asymptotically hyperbolic case, substituting Proposition 6.12 in place of Proposition 6.9):

THEOREM 6.14. *Under the assumptions (6.2.34), (6.2.40), (6.2.41), (6.2.28) (for all $A \in \Psi_h^{\mathrm{comp}}(M)$), and (6.2.29), the semiclassical measure μ on T^*M has the following properties:*

1. *μ is supported on $j(\Gamma_E^+)$, that is $\mu(T^*M \setminus j(\Gamma_E^+)) = 0$.*
2. *For each neighbourhood U of $j(K_E)$, we have $\mu(U) > 0$.*
3. *If we choose the function x_1 in (6.2.33) such that[2] $x_1 = 1$ near $\{r \leq r_0\}$, where r is defined in (6.1.13) and r_0 is defined in Lemma 6.6, then for all $U \subset T^*M \cap \{r \leq r_0\}$ and $t \geq 0$ we have*

$$\mu(e^{-tH_p}(U)) = e^{2\nu t}\mu(U). \tag{6.2.42}$$

[2] Strictly speaking this means that we have to use two canonical boundary defining functions on $\overline{M}$: one in (6.1.13) and another one in (6.2.33).

6.3. NORMALLY HYPERBOLIC TRAPPING

In this section, we consider systems whose trapped trajectories form a normally hyperbolic set – see the definition below. This gives a class of examples with nonempty trapped sets where it is typically impossible to describe individual resonances yet one can make precise statements about their distribution. That is due to the fine structure of the trapped set. One setting where normally hyperbolic sets appear are Kerr(–de Sitter) black holes (see Exercise 5.16 for a special case). Another comes from molecular dynamics.

The main result of this section is a resonance free strip given in Theorem 6.16.

We work in the setting of Euclidean Schrödinger operators (6.0.5) or asymptotically hyperbolic Laplacians (6.0.6). To formulate the assumptions on the trapped set, we use the material of §6.1, in particular the sets $\Gamma_J^{\pm}, K_J \subset T^*M$ defined in (6.1.4) using the Hamiltonian flow e^{tH_p} (here $M = \mathbb{R}^n$ in the Euclidean case).

Assume that

$$[\alpha', \beta'] \subset (\alpha, \beta) \subset (0, \infty) \tag{6.3.1}$$

are such that:

(A1) $\Gamma^{\pm}_{(\alpha,\beta)} \subset T^*M$ are C^∞ orientable hypersurfaces intersecting transversely, that is

$$T_{(x,\xi)}(T^*M) = T_{(x,\xi)}\Gamma^+_{(\alpha,\beta)} + T_{(x,\xi)}\Gamma^-_{(\alpha,\beta)} \quad \text{for all } (x,\xi) \in K_{(\alpha,\beta)};$$

(A2) $K_{(\alpha,\beta)}$ is symplectic, that is the restriction $\omega|_{TK_{(\alpha,\beta)}}$ is a nondegenerate 2-form, where ω is the standard symplectic form on T^*M.

We next want to make a *hyperbolicity* assumption on the flow near the trapped set. Roughly speaking, it states that every Hamiltonian trajectory in $p^{-1}([\alpha', \beta'])$ converges exponentially fast to $\Gamma^{\pm}$ as $t \to \pm\infty$. However, it is more convenient to use the linearization of the flow. By (A1), there exist *defining functions* $\varphi_{\pm}$ of $\Gamma^{\pm}$ in some open set

$$U \subset p^{-1}((\alpha, \beta)), \quad K_{[\alpha',\beta']} \subset U,$$

namely

$$\varphi_{\pm} \in C^\infty(U; \mathbb{R}); \tag{6.3.2}$$

$$\{\varphi_{\pm} = 0\} = \Gamma^{\pm} \cap U; \tag{6.3.3}$$

$$d\varphi_{\pm} \neq 0 \quad \text{on } \Gamma^{\pm} \cap U. \tag{6.3.4}$$

Note that for $(x,\xi) \in \Gamma^{\pm} \cap U$, $T_{(x,\xi)}\Gamma^{\pm}$ is the kernel of $d\varphi_{\pm}(x,\xi)$. Therefore, for each

$$(x,\xi) \in K \cap U, \quad v \in T_{(x,\xi)}(T^*M),$$

the quantity $|\langle d\varphi_\pm(x,\xi), v\rangle|$ measures the distance from v to $T_{(x,\xi)}\Gamma^\pm$. The distance from the propagated vector $de^{tH_p}(x,\xi)v$ to $T_{e^{tH_p}(x,\xi)}\Gamma^\pm$ is given by the quantity

$$|\langle d\varphi_\pm(e^{tH_p}(x,\xi)), de^{tH_p}(x,\xi)v\rangle| = |\langle d(\varphi_\pm \circ e^{tH_p})(x,\xi), v\rangle|. \tag{6.3.5}$$

Since $\Gamma^\pm$ are invariant under the flow e^{tH_p} and $\varphi_\pm|_{\Gamma^\pm} = 0$, we also have $\varphi_\pm \circ e^{tH_p}|_{\Gamma^\pm} = 0$. Hence (6.3.4) shows that $d(\varphi_\pm \circ e^{tH_p})(x,\xi)$ is a is a non-zero multiple of $d\varphi_\pm(x,\xi)$ and it makes sense to divide these vectors by each other. At $t = 0$ the multiple is equal to 1 which means that

$$\frac{d(\varphi_\pm \circ e^{\pm tH_p})(x,\xi)}{d\varphi_\pm(x,\xi)} > 0. \tag{6.3.6}$$

The following assumption says that (6.3.5) decays exponentially as $t \to \pm\infty$:

(A3) There exist constants $C, \nu > 0$ such that for all $(x,\xi) \in K \cap U$,

$$\frac{d(\varphi_\pm \circ e^{\pm tH_p})(x,\xi)}{d\varphi_\pm(x,\xi)} \le Ce^{-\nu t}, \quad t \ge 0. \tag{6.3.7}$$

It is easy to see that the constant ν in assumption (A3) is independent of the choice of the defining functions $\varphi_\pm$. Define the *minimal expansion rate*

$$\nu_{\min} > 0$$

as the supremum of all values of ν for which there exists a constant C such that (6.3.7) holds.

DEFINITION 6.15. *We say that the trapping is* ***normally hyperbolic*** *near* $p^{-1}([\alpha', \beta'])$ *if (A1), (A2) and (A3) hold for some* α, β *satisfying* (6.3.1).

REMARKS. 1. For normally hyperbolic trapped sets, we have the following stable/unstable decomposition:

$$T_{(x,\xi)}(T^*M) = T_{(x,\xi)}K \oplus E_+(x,\xi) \oplus E_-(x,\xi), \quad (x,\xi) \in K_{[\alpha',\beta']}, \tag{6.3.8}$$

where $E_\pm(x,\xi)$ are spanned by the Hamiltonian vector fields of $\varphi_\pm$:

$$E_\pm(x,\xi) := \mathbb{R}H_{\varphi_\pm}, \quad T_{(x,\xi)}\Gamma^\pm = T_{(x,\xi)}K \oplus E_\pm(x,\xi).$$

Since the flow e^{tH_p} consists of symplectomorphisms,

$$v = H_{\varphi_\pm}(x,\xi) \implies de^{tH_p}(x,\xi)v = H_{\varphi_\pm \circ e^{-tH_p}}(e^{tH_p}(x,\xi)). \tag{6.3.9}$$

(See [**Zw12**, Theorem 2.10].) Since K is invariant under the flow it follows that the decomposition (6.3.8) is also invariant.

The differential of e^{tH_p} expands vectors in E_+ and contracts vectors in E_-: for some $\nu > 0$

$$|de^{tH_p}(x,\xi)v| \le Ce^{-\nu|t|}|v|, \quad v \in E_\pm(x,\xi), \quad \mp t \ge 0. \tag{6.3.10}$$

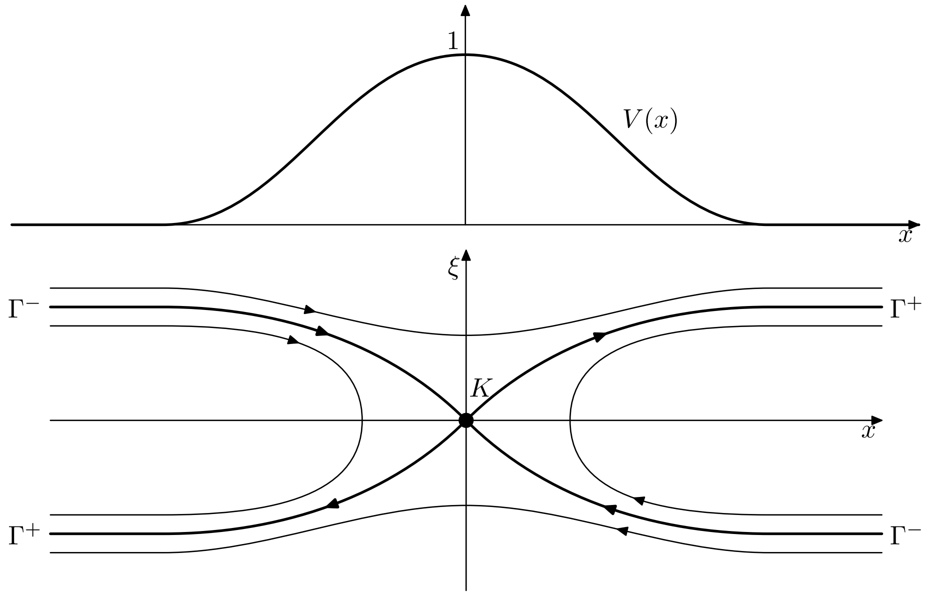


Figure 6.2. A barrier-top potential and the corresponding phase space dynamics, with the sets $\Gamma^\pm_{[1-\delta,1+\delta]}$ shown by solid lines.

In fact, in the notation of (6.3.9), the bound (6.3.7) gives

$$\begin{aligned}\frac{|de^{tH_p}(x,\xi)v|}{|v|} &= \frac{\left|H_{\varphi_\pm\circ e^{-tH_p}}(e^{tH_p}(x,\xi))\right|}{|H_{\varphi_\pm}(e^{tH_p}(x,\xi))|}\cdot\frac{|H_{\varphi_\pm}(e^{tH_p}(x,\xi))|}{|H_{\varphi_\pm}(x,\xi)|}\\ &\le \max_{(y,\eta)\in K\cap U}\left|\frac{d(\varphi_\pm\circ e^{-tH_p})(y,\eta)}{d\varphi_\pm(y,\eta)}\right|\cdot\frac{\max_{K\cap U}|H_{\varphi_\pm}|}{\min_{K\cap U}|H_{\varphi_\pm}|}\\ &\le Ce^{-\nu|t|},\quad \mp t\ge 0.\end{aligned}$$

The properties (6.3.8), (6.3.10), with $E_\pm$ of possibly higher dimensions, define a more general concept of normally hyperbolic trapping where $\Gamma^\pm$ need not be smooth, see Nonnenmacher–Zworski [**NZ15**] and the references there.

2. Under an additional assumption, called *r-normal hyperbolicity* (roughly speaking, the expansion rates in the directions transversal to K are more than r-fold bigger than the expansion rates along K, here $r\ge 1$), normally hyperbolic structures are stable under smooth perturbations of the symbol p, with the resulting submanifolds $\Gamma^\pm$ of class C^r – see Hirsch–Pugh–Shub [**HPS77**]. For r-normally hyperbolic trapped sets under additional pinching conditions there exist bands of resonances with a Weyl law in the first band – see Dyatlov [**Dy15a, Dy16**]. Such band structures are also known for Pollicott–Ruelle resonances of contact Anosov flows where the trapping is also normally hyperbolic – see Faure–Tsujii [**FT13**].

EXAMPLES. 1. Consider the Schrödinger operator (6.0.1) on $\mathbb{R}$ such that the potential $V \in C_c^\infty(\mathbb{R};\mathbb{R})$ has the following properties:

$$V(0) = 1, \quad V'(0) = 0, \quad V''(0) < 0, \quad V(x) < 1 \quad \text{for } x \neq 0.$$

See Figure 6.2. The trapping is normally hyperbolic for energies in $[1-\delta, 1+\delta]$, if $\delta > 0$ is small enough. In fact, the sets $\Gamma^\pm_{[1-\delta,1+\delta]}$ and $K_{[1-\delta,1+\delta]}$ are given by

$$\Gamma^\pm_{[1-\delta,1+\delta]} = \{\xi = \pm\operatorname{sgn}(x)\sqrt{1-V(x)}\}, \quad K_{[1-\delta,1+\delta]} = \{(0,0)\}.$$

Since $V(x) = 1 - \frac{1}{2}|V''(0)|x^2 + \mathcal{O}(x^3)$, the defining functions of $\Gamma^\pm_{[1-\delta,1+\delta]}$ are smooth (we take the positive square root).

Note that $\Gamma^\pm_E = \emptyset$ for $0 < |E-1| \leq \delta$ and that we can take

$$\varphi_\pm(x,\xi) = \xi \mp \operatorname{sgn}(x)\sqrt{1-V(x)}.$$

Since $p = \varphi_+\varphi_- + 1$, we find

$$H_p\varphi_\pm = \mp\{\varphi_+,\varphi_-\}\varphi_\pm; \quad \{\varphi_+,\varphi_-\}|_{(0,0)} = \sqrt{-2V''(0)}.$$

We see that (6.3.7) is satisfied, and

$$\nu_{\min} = \sqrt{-2V''(0)}.$$

2. Consider the operator (6.0.6) where the manifold (M,g) is the hyperbolic cylinder studied in (5.1.4). Denote by ξ,η the momenta corresponding to the coordinates v,θ, then

$$p = \xi^2 + \frac{\eta^2}{\cosh^2 v}.$$

The incoming/outgoing tails and the trapped set at positive energies are

$$\Gamma^\pm_{(0,\infty)} = \{\xi = \pm|\eta|\tanh v,\ \eta \neq 0\}, \quad K_{(0,\infty)} = \{v = \xi = 0, \quad \eta \neq 0\}.$$

Consider the following defining functions of $\Gamma^\pm$:

$$\varphi_\pm = \xi \mp |\eta|\tanh v.$$

Since $p = \varphi_+\varphi_- + \eta^2$ and $\{\eta^2, \varphi_\pm\} = 0$, we have

$$H_p\varphi_\pm = \mp\{\varphi_+,\varphi_-\}\varphi_\pm; \quad \{\varphi_+,\varphi_-\}|_{K_{(0,\infty)}} = 2\sqrt{p}.$$

We see that (6.3.7) is satisfied on $p^{-1}((\alpha,\beta))$ for $\alpha > 0$, and

$$\nu_{\min} = 2\sqrt{\alpha}.$$

We now state the main result of this section, which is a resonance free strip of width just below $\nu_{\min}/2$:

THEOREM 6.16 (Spectral gap for normally hyperbolic trapping). *Assume that the operator P is given by (6.0.5) or (6.0.6) and has normally hyperbolic trapping near $p^{-1}([\alpha',\beta'])$. Fix $\varepsilon, C_0 > 0$, $\chi \in C_c^\infty(M)$. Then for h small enough, the following estimates hold:*

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} = o(h^{-2}), \quad z \in [\alpha',\beta'] + ih\Big[-\frac{\nu_{\min}}{2} + \varepsilon, C_0\Big]; \tag{6.3.11}$$

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le C\frac{\log(1/h)}{h}, \quad z \in [\alpha',\beta']. \tag{6.3.12}$$

REMARKS. 1. The estimate (6.3.12) is optimal as shown by Theorem 7.1 in the following chapter.

2. The proof of Theorem 6.16 in fact gives bounds of the form (6.3.11) on the inverse of the complex scaled operator $P_\theta - z$ (in the Euclidean setting) or the modified Laplacian $\widetilde{P}(z)$ (in the asymptotically hyperbolic setting) – see (6.3.26) and (6.3.28) below.

The starting point of the proof of Theorem 6.16 is the following construction of defining functions of $\Gamma^\pm$ adapted to the flow:

LEMMA 6.17 (Adapted defining functions). *Fix $\varepsilon > 0$ and let the assumptions (A1)–(A3) above hold. Then there exists a neighbourhood $U \subset T^*M$ of $K_{[\alpha',\beta']}$ and functions $\varphi_\pm$ satisfying (6.3.2)–(6.3.4) such that:*

(1) *for $\delta > 0$ small enough*

$$U_\delta := \{|\varphi_+| < \delta,\ |\varphi_-| < \delta,\ p \in (\alpha' - \delta, \beta' + \delta)\} \Subset U; \tag{6.3.13}$$

(2) *there exist $c_\pm \in C^\infty(U;\mathbb{R})$ such that*

$$H_p\varphi_\pm = \mp c_\pm \varphi_\pm, \quad c_\pm \ge \nu_{\min} - \varepsilon \quad \text{on } U; \tag{6.3.14}$$

(3) *$\{\varphi_+, \varphi_-\} \ge 1$ in U.*

Proof. 1. We show existence of φ_+, the case of φ_- being similar. Fix a function $\widetilde{\varphi}_+$ on some neighbourhood U of $K_{[\alpha',\beta']}$ satisfying (6.3.2)–(6.3.4). We can choose U so that $K \cap U$ is invariant under the flow e^{tH_p}.

Since Γ^+ is invariant under the flow, the function $H_p\widetilde{\varphi}_+$ vanishes on Γ^+. Therefore, for some $\tilde{c}_+ \in C^\infty(U;\mathbb{R})$

$$H_p\widetilde{\varphi}_+ = -\tilde{c}_+\widetilde{\varphi}_+ \quad \text{in } U.$$

In $K \cap U$ we have (recall that $\widetilde{\varphi}_+|_{K\cap U} = 0$)

$$\partial_t d(\widetilde{\varphi}_+ \circ e^{tH_p}) = d\big((H_p\widetilde{\varphi}_+) \circ e^{tH_p}\big) = -(\tilde{c}_+ \circ e^{tH_p}) d(\widetilde{\varphi}_+ \circ e^{tH_p}).$$

Therefore, for all $T > 0$ we have in $K \cap U$ (recall (6.3.6))

$$-\frac{1}{T}\log\frac{d(\widetilde{\varphi}_+ \circ e^{TH_p})}{d\widetilde{\varphi}_+} = \langle \tilde{c}_+ \rangle_T := \frac{1}{T}\int_0^T \tilde{c}_+ \circ e^{tH_p}\,dt.$$

By taking any smooth extension of $\tilde{c}_+$ outside of U the right hand side is defined in all of U.

2. By (6.3.7) and for T large enough,

$$\langle \tilde{c}_+ \rangle_T \geq \frac{\nu T - \log C}{T} > \nu_{\min} - \varepsilon \quad \text{in } K \cap U. \tag{6.3.15}$$

We now solve

$$H_p f_+ = \langle \tilde{c}_+ \rangle_T - \tilde{c}_+ \quad \text{in } U$$

by putting

$$f_+ := \frac{1}{T} \int_0^T (T - t)(\tilde{c}_+ \circ e^{tH_p})\, dt, \quad f_+ \in C^\infty(U; \mathbb{R}).$$

3. Now, put

$$\varphi_+ := e^{-f_+} \widetilde{\varphi}_+.$$

Then

$$H_p \varphi_+ = -c_+ \varphi_+, \quad c_+ = \tilde{c}_+ + H_p f_+.$$

From (6.3.15) we obtain

$$c_+ = \langle \tilde{c}_+ \rangle_T > \nu_{\min} - \varepsilon \quad \text{in } K \cap U.$$

Shrinking U, we can make sure that this inequality holds on all of U, and that gives (6.3.14). By taking δ small enough U_δ is contained in any given small neighbourhood of $K_{[\alpha',\beta']}$ and hence (6.3.13) holds as well.

4. By assumption (A2), $K \cap U$ is a symplectic submanifold given by $\varphi_+ = \varphi_- = 0$ and hence for $v \in T_{(x,\xi)}K$, $\omega(v, H_{\varphi_\pm}(x,\xi)) = \langle d\varphi_\pm, v \rangle = 0$. Since $H_{\varphi_\pm}(x,\xi)$ span a complement of $T_{(x,\xi)}K$ (see (6.3.8)) the symplectic form has to be non-degenerate on that span:

$$0 \neq \omega(H_{\varphi_+}, H_{\varphi_-}) = H_{\varphi_+} \varphi_- = \{\varphi_+, \varphi_-\} \ \text{ at } (x,\xi) \in K_{[\alpha',\beta']}.$$

Shrinking U and multiplying φ_+ by a constant, we obtain $\{\varphi_+, \varphi_-\} \geq 1$ in U. $\square$

We now construct an auxiliary pseudodifferential operator. Fix $\delta > 0$ such that the set U_δ defined in (6.3.13) is compactly contained in U. Take an operator

$$\Theta_+ \in \Psi_h^{\mathrm{comp}}(M), \quad \Theta_+^* = \Theta_+, \quad \sigma_h(\Theta_+) = \varphi_+ \quad \text{in } U_\delta.$$

We will henceforth argue in the Euclidean setting (6.0.5), with $M = \mathbb{R}^n$; at the end of this section, we explain what changes are necessary for the asymptotically hyperbolic setting. Let P_θ be the complex scaled operator, see §6.2.1. We choose the constant r_1 in (6.2.4) so that $U \subset \{r < r_1\}$; then

$$P_\theta = P + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \text{microlocally in } U.$$

(See Definition E.29.)

The key component of the proof is a bound on $\Theta_+ u$ for quasimodes u of the operator $P_\theta - z$. That bound implies that $\|\Theta_+ u\| = \mathcal{O}(h)\|u\|$ microlocally in $U_{\delta/4} \supset K_{[\alpha',\beta']}$ when $(P_\theta - z)u = 0$. That refines (6.2.14) in our current setting as we can get closer to Γ^+: the symbol $\sigma_h(\Theta_+)$ vanishes simply on Γ_+ rather than being supported away from it.

LEMMA 6.18 (**Auxiliary pseudodifferential bound**). *Assume that* $u \in L^2(\mathbb{R}^n)$, $(P_\theta - z)u = f \in L^2(\mathbb{R}^n)$, *and*

$$z \in [\alpha', \beta'] + ih\big[-(\nu_{\min} - 2\varepsilon), C_0\big]. \tag{6.3.16}$$

Then for each $A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ *such that* $\mathrm{WF}_h(A) \subset U_{\delta/4}$ *(see (6.3.13)),*

$$\|A\Theta_+ u\|_{L^2} \le Ch^{-1}\|f\|_{L^2} + Ch\|u\|_{L^2}, \tag{6.3.17}$$

where the constant C *is undependent of* u, z, h.

REMARK. The region (6.3.16) is larger than the one in (6.3.11), since $\nu_{\min}$ is not divided by 2. In fact, under additional assumptions one may establish a *second* resonance free strip deeper than the one in (6.3.11) – see [**Dy15a, Dy16**] and [**B*16**]. It can then be used to obtain resonance asymptotics in a band – see [**Dy15a**, Theorem 2].

Proof. 1. Fix an operator

$$Z_+ \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n), \quad Z_+^* = Z_+, \quad \sigma_h(Z_+) = c_+ \quad \text{in } U_\delta. \tag{6.3.18}$$

By (6.3.14), we have for some $R_+ \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$,

$$[P_\theta - z, \Theta_+] = ihZ_+\Theta_+ + h^2 R_+ + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \text{microlocally in } U_\delta.$$

Thus we have microlocally in U_δ

$$(P_\theta - z)\Theta_+ u = \Theta_+ f + ihZ_+\Theta_+ u + h^2 R_+ u + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

Therefore, for each $B_1 \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ such that $\mathrm{WF}_h(B_1) \subset U_\delta$, we have

$$\|B_1(P_\theta - ihZ_+ - z)\Theta_+ u\|_{L^2} \le C\|f\|_{L^2} + Ch^2\|u\|_{L^2}. \tag{6.3.19}$$

2. We will apply Lemma E.49, which is a positive commutator estimate, to the equation (6.3.19) to estimate $\Theta_+ u$. The part of the right-hand side localized away from Γ^+ will be estimated by Proposition 6.9.

3. We first construct the escape function g. Take cutoff functions

$$\chi_1 \in C_c^\infty\big((-\delta,\delta);[0,1]\big), \quad \chi_1 = 1 \quad \text{near } [-\delta/2, \delta/2];$$
$$\chi_2 \in C_c^\infty\big((\alpha'-\delta, \beta'+\delta);[0,1]\big), \quad \chi_2 = 1 \quad \text{near } [\alpha'-\delta/2, \beta'+\delta/2],$$

and define

$$g = \chi_1(\varphi_+)\chi_1(\varphi_-)\chi_2(p) \in C_c^\infty(U_\delta;[0,1]), \quad g = 1 \quad \text{in } U_{\delta/2}.$$

We additionally require that $t\chi_1'(t) \leq 0$. Then

$$H_p g = c_- \varphi_- \chi_1'(\varphi_-)\chi_2(p) \leq 0 \quad \text{on } U \cap \{|\varphi_+| \leq \delta/2\}.$$

Since $U \cap \Gamma^+ = U \cap \{\varphi_+ = 0\}$, the closure of $\{H_p g > 0\}$ is a compact subset of $U_\delta \setminus \Gamma^+$. Thus there exists an operator

$$\begin{gathered} B \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n), \quad \mathrm{WF}_h(B) \subset U_\delta \setminus \Gamma^+, \\ H_p g \leq 0 \quad \text{in a neighbourhood of } T^*\mathbb{R}^n \setminus \mathrm{ell}_h(B). \end{gathered} \tag{6.3.20}$$

We also fix an operator

$$B_1 \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n), \quad \mathrm{WF}_h(B_1) \subset U_\delta, \quad \operatorname{supp} g \subset \mathrm{ell}_h(B_1).$$

4. Put $\mathbf{P} := P_\theta - ihZ_+ - z$. Then by (6.3.14), (6.3.16) and (6.3.18)

$$\sigma_h(h^{-1} \operatorname{Im} \mathbf{P}) = -c_+ - \frac{\operatorname{Im} z}{h} \leq -\varepsilon \quad \text{in } U_\delta.$$

Combining this with (6.3.20), we get

$$H_p g + \sigma_h(h^{-1} \operatorname{Im} \mathbf{P}) g \leq -\varepsilon g \quad \text{in a neighbourhood of } T^*\mathbb{R}^n \setminus \mathrm{ell}_h(B).$$

This implies the condition (E.4.29), where $k = 2$ and we put $s = \frac{1}{2}$ for convenience: the choice of s and the factor $\langle \xi \rangle^{-1}$ do not matter since our operators A, B, B_1 are compactly microlocalized.

Applying Lemma E.49 to (6.3.19), we obtain the following bound for each $A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ with $\mathrm{WF}_h(A) \subset U_{\delta/2}$:

$$\begin{aligned} \|A\Theta_+ u\|_{L^2} \leq{}& C\|B\Theta_+ u\|_{L^2} + Ch^{-1}\|f\|_{L^2} \\ &+ Ch\|u\|_{L^2} + Ch^{1/2}\|B_1\Theta_+ u\|_{L^2} \end{aligned} \tag{6.3.21}$$

5. We now upgrade the $Ch^{1/2}$ term in (6.3.21) to Ch as follows. Repeating the argument in steps 3–4 with $\delta/2$ in place of δ, we see that there exist $B', B_1' \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ such that

$$\mathrm{WF}_h(B') \subset U_{\delta/2} \setminus \Gamma^+, \quad \mathrm{WF}_h(B_1') \subset U_{\delta/2},$$

and (6.3.21) holds with B, B_1 replaced by B', B_1' and each $A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ such that $\mathrm{WF}_h(A) \subset U_{\delta/4}$. Estimating $\|B_1'\Theta_+ u\|_{L^2}$ by the original estimate (6.3.21), we get

$$\begin{aligned} \|A\Theta_+ u\|_{L^2} \leq{}& C\|B'\Theta_+ u\|_{L^2} + Ch^{1/2}\|B\Theta_+ u\|_{L^2} \\ &+ Ch^{-1}\|f\|_{L^2} + Ch\|u\|_{L^2}. \end{aligned}$$

Finally, by (6.2.14) in Proposition 6.9 (here $\mathrm{WF}_h(B\Theta) \cap \Gamma^+ = \emptyset$ and same for B')

$$\|B'\Theta_+ u\|_{L^2} + \|B\Theta_+ u\|_{L^2} \leq Ch^{-1}\|f\|_{L^2} + Ch^N\|u\|_{L^2}.$$

Combining the latter two estimates, we obtain (6.3.17). $\square$

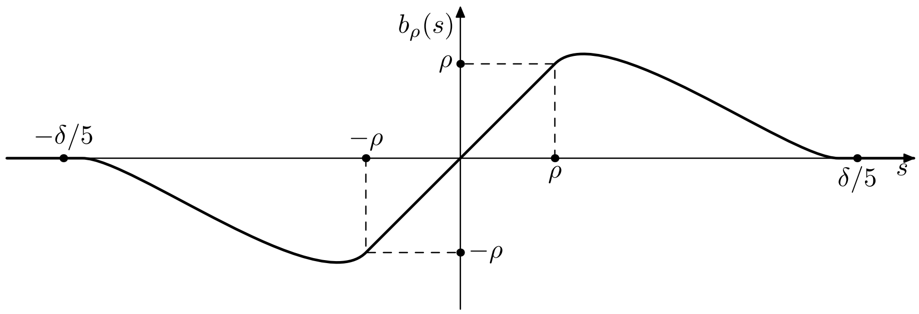


Figure 6.3. The function b_ρ used in the proof of Lemma 6.19.

The next step is to deduce from (6.3.17) partial regularity of semiclassical measures associated to $\mathcal{O}(h^2)$ quasimodes of the operator $P - z$:

LEMMA 6.19. *Consider sequences*

$$
\begin{aligned}
&h_j \to 0, \quad z_j \in [\alpha', \beta'] + ih_j\big[-(\nu_{\min} - 2\varepsilon), C_0\big];\\
&u_j \in L^2(\mathbb{R}^n), \quad f_j := (P_\theta(h_j) - z_j)u_j \in L^2(\mathbb{R}^n);\\
&\|u_j\|_{L^2(\mathbb{R}^n)} = 1, \quad \|f_j\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h_j^2)
\end{aligned}
\tag{6.3.22}
$$

and assume that u_j converges to some measure μ on $T^*\mathbb{R}^n$ in the sense of (6.2.28)*. Then there exists a constant C such that, with U_ρ defined in* (6.3.13)*,*

$$
\mu(U_\rho) \le C\rho, \quad \rho > 0. \tag{6.3.23}
$$

Proof. 1. Fix an operator $A \in \Psi_h^{\mathrm{comp}}(\mathbb{R}^n)$ with $\mathrm{WF}_h(A) \subset U_{\delta/4}$ and $A = I + \mathcal{O}(h^\infty)$ microlocally near $\overline{U_{\delta/5}}$. By Lemma 6.18, we have

$$
\|A\Theta_+ u_j\|_{L^2} = \mathcal{O}(h_j).
$$

Applying (E.3.10) from Theorem E.44 to the operator $P := A\Theta_+$, where $\sigma_h(A\Theta_+) = \varphi_+$ near $\overline{U_{\delta/5}}$, we see that there exists a constant C such that

$$
\Big| \int_{T^*\mathbb{R}^n} \{\varphi_+, b\}\, d\mu \Big| \le C \sup |b| \quad \text{for all } b \in C_c^\infty(U_{\delta/5}). \tag{6.3.24}
$$

By (6.2.14) (see Step 1 of the proof of Theorem 6.11), we also have

$$
\mu(T^*\mathbb{R}^n \setminus \Gamma^+_{[\alpha',\beta']}) = 0. \tag{6.3.25}
$$

2. Since $\|u_j\|_{L^2} = 1$, we have $\mu(U_\rho) \le 1$ for all ρ (as follows from (6.2.28) and the operator norm bound (E.1.55)). Therefore, it suffices to prove (6.3.23) for ρ small enough, in particular for $\rho < \delta/10$.

Take a function (see Figure 6.3)

$$b_\rho \in C_c^\infty\big((-\delta/5,\delta/5);\mathbb{R}\big),\quad b_\rho(s) = s \text{ for } |s| \le \rho;$$
$$\sup |b_\rho| \le 2\rho,\quad b'_\rho \ge -\frac{20\rho}{\delta}.$$

We apply (6.3.24) to

$$b := b_\rho(\varphi_-)\chi_3(\varphi_+)\chi_4(p) \quad \text{where}$$
$$\chi_3 \in C_c^\infty((-\delta/5,\delta/5);[0,1]),\quad \chi_3 = 1 \quad \text{near } 0;$$
$$\chi_4 \in C_c^\infty((\alpha'-\delta/5,\beta'+\delta/5);[0,1]),\quad \chi_4 = 1 \quad \text{near } [\alpha',\beta'].$$

By (6.3.25) and since $b = b_\rho(\varphi_-)$ near $\Gamma^+_{[\alpha',\beta']}$ and $\{\varphi_+,\varphi_-\} \ge 1$, we have for some ρ-independent constant C

$$\begin{aligned} C\rho &\ge \int_{T^*\mathbb{R}^n} \{\varphi_+, b_\rho(\varphi_-)\}\, d\mu \\ &= \int_{U_\rho} b'_\rho(\varphi_-)\{\varphi_+,\varphi_-\}\, d\mu + \int_{T^*\mathbb{R}^n\setminus U_\rho} b'_\rho(\varphi_-)\{\varphi_+,\varphi_-\}\, d\mu \\ &\ge \mu(U_\rho) - C\rho, \end{aligned}$$

proving (6.3.23). □

We are now ready to prove Theorem 6.16 in the Euclidean setting:

Proof of Theorem 6.16. 1. Without loss of generality we assume that ε is small. Choose the complex scaled operator P_θ as explained before Lemma 6.18. We first show the bound for small enough h

$$\|(P_\theta - z)^{-1}\|_{L^2\to L^2} = o(h^{-2}),\quad z \in [\alpha',\beta'] + ih\Big[-\frac{\nu_{\min}}{2}+\varepsilon, C_0\Big] \tag{6.3.26}$$

and to do this we argue by contradiction.

Hence, assume that (6.3.26) does not hold. Then there exist sequences h_j, z_j, u_j, f_j satisfying (6.3.22) and $\operatorname{Im} z_j \ge h_j(-\nu_{\min}/2+\varepsilon)$. By Theorem E.42, we may pass to a subsequence to make u_j converge to some measure μ in the sense of (6.2.28). Passing to a further subsequence, we can also make sure that (6.2.29) holds, that is,

$$\operatorname{Re} z_j \to E \in [\alpha',\beta'],\quad \frac{\operatorname{Im} z_j}{h_j} \to \nu \in \Big[-\frac{\nu_{\min}}{2}+\varepsilon, C_0\Big].$$

2. By Theorem 6.11, we have for each $t \ge 0$,

$$\mu(T^*\mathbb{R}^n \setminus \Gamma_E^+) = 0,\quad \mu(e^{-tH_p}(U_\delta)) = e^{2\nu t}\mu(U_\delta) > 0. \tag{6.3.27}$$

On the other hand, by (6.3.14) (recall the definition (6.3.13) of U_δ)

$$e^{-tH_p}(U_\delta \cap \Gamma_E^+) \;\subset\; U_{\delta\exp(-(\nu_{\min}-\varepsilon)t)},\quad t \ge 0.$$

Therefore, by Lemma 6.19

$$\mu(e^{-tH_p}(U_\delta)) \le Ce^{-(\nu_{\min}-\varepsilon)t}, \quad t \ge 0.$$

Since $\nu \ge -\nu_{\min}/2+\varepsilon$, this contradicts (6.3.27) for sufficiently large $t>0$, finishing the proof of (6.3.26).

3. The bound (6.3.11) follows immediately from (6.3.26) as explained in the proof of Theorem 6.10. Finally, to prove (6.3.12) we use the upper half-plane bound, $\|R(z,h)\|_{L^2\to L^2} = d(z, \mathrm{Spec}(P))^{-1} \le 1/\operatorname{Im} z$, to obtain

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le \frac{C}{\gamma h}, \quad \operatorname{Im} z = \gamma h > 0.$$

By Lemma D.1 applied to

$$\Omega = [\alpha', \beta'] + ih\Big[-\frac{\nu_{\min}}{2}+\varepsilon, \gamma\Big]$$

we get for $\gamma < \nu_{\min}/2-\varepsilon$, a γ-independent constant C, and $z \in [\alpha', \beta']$,

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le Ch^{\theta-2}\gamma^{-\theta}, \quad \theta = \frac{\nu_{\min}/2-\varepsilon}{\nu_{\min}/2-\varepsilon+\gamma}.$$

Since $\theta \ge 1 - C\gamma$, we get

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le Ch^{-1-C\gamma}\gamma^{-1}.$$

Putting

$$\gamma := \frac{1}{\log(1/h)},$$

we obtain $\|\chi R(z,h)\chi\| \le Ch^{-1}\log(1/h)$ for $z \in [\alpha', \beta']$, proving (6.3.12). □

REMARK. For the asymptotically hyperbolic case (6.0.6), the proof of Theorem 6.16 should be modified as follows. Instead of the complex scaled operator $P_\theta - z$, we use the modified Laplacian $\widetilde{P}(z)$ introduced in (6.2.32). We choose the defining function x_1 so that $x_1 = 1$ near U, so that by (6.2.33)

$$\widetilde{P}(z) = P - z + \mathcal{O}(h^\infty) \quad \text{microlocally on } U.$$

We replace Proposition 6.9 by Proposition 6.12 and Theorem 6.11 by Theorem 6.14. Repeating the argument of this section, we obtain the following bound for s satisfying (6.2.34) with $C_0 := \nu_{\min}/2$ and z satisfying the condition in (6.3.11):

$$\|\widetilde{P}(z)^{-1}\|_{\bar{H}_h^{s-1}(X)\to\bar{H}_h^s(X)} = o(h^{-2}). \tag{6.3.28}$$

As in the proof of Theorem 6.13, this implies the cutoff bound

$$\|\chi R(z,h)\chi\|_{H_h^{s-1}\to H_h^s} = o(h^{-2}). \tag{6.3.29}$$

Using the fact that $P - z$ is elliptic at fiber infinity similarly to (6.2.25), we can upgrade (6.3.29) to a bound $H_h^{s-2} \to H_h^s$ for all $s \in \mathbb{R}$, in particular giving the L^2 bound (6.3.11). The bound (6.3.12) follows from (6.3.11) as in the Euclidean case.

6.4. LOGARITHMIC RESONANCE FREE REGIONS

In the Euclidean case we have a semiclassical analogue of the results of §4.6 which provide logarithmic resonance free regions. These are not available in the asymptotically hyperbolic setting since meromorphic continuation presented in §5.6 proceeds strip-by-strip. It is an interesting question if the same results hold in that case, possibly under some stronger assumptions about the structure of infinity.

The proof of existence of logarithmic resonance free regions uses the concept of an *escape function*, that is a function G such that $H_p G_0 > 0$ on the energy surface $p^{-1}(E)$. In §6.2 we used such functions implicitly when citing propagation of singularities (Theorem E.47) – see Lemma E.48. In §6.3 the use was more explicit in the use of Lemma E.49.

The simplest example of an escape function is given by the free Laplacian in which case

$$p = |\xi|^2, \quad G_0(x,\xi) = \chi(|\xi|/E)\langle x, \xi\rangle, \quad E > 0,$$
$$\chi \in C_c^\infty((\tfrac{1}{2}, 2); [0,1]), \quad \chi(1) = 1.$$

Then clearly

$$H_p G_0(x,\xi) = 2\chi(|\xi|/E)|\xi|^2 = 2E^2 > 0 \ \text{ on } p^{-1}(E).$$

We now show that we can construct such G under the non-trapping condition:

LEMMA 6.20 (Construction of an escape function). *Suppose that $p(x,\xi) = |\xi|_g^2 + V(x)$, where $V, g_{ij} - \delta_{ij} \in C_c^\infty$ and g_{ij} is positive definite. Suppose that at $E > 0$, $K_E = \emptyset$. Then for any $R > 0$ there exists $G \in C_c^\infty(T^*\mathbb{R}^n; \mathbb{R})$ and $\delta > 0$ such that*

$$H_p G(x,\xi) \geq 1 \ \text{ for } \ |p(x,\xi) - E| \leq \delta, \quad |x| \leq R. \tag{6.4.1}$$

REMARK. It is convenient to take compactly supported G in which case $H_p G \geq 1$ can only hold in a compact set. The proof can be modified to show that we can construct $G \in S^0_{1,0}(T^*\mathbb{R}^n; \mathbb{R})$ such that $H_p G \geq 1$ is valid for all (x,ξ) satisfying $|p(x,\xi) - E| \leq \delta$.

Proof. 1. From Proposition 6.3 we know that $K_{[E-3\delta, E+3\delta]} = \emptyset$, for some $\delta > 0$. Proposition 6.4 then shows that $\Gamma^{\pm}_{[E-3\delta, E+3\delta]} = \emptyset$ as well.

Define

$$\Sigma_\gamma := \{(x,\xi) : |p(x,\xi) - E| \leq 2\gamma, \ |x| \leq R\}. \tag{6.4.2}$$

Since the set $\Sigma_{2\delta}$, is compact this implies that for any R there exists $T = T(R)$ such for

$$|x(\exp(tH_p(x,\xi)))| > R, \quad |t| \geq T, \quad (x,\xi) \in \Sigma_{2\delta}. \tag{6.4.3}$$

2. For $\rho = (x,\xi) \in \Sigma_\delta$ we first define $G_\rho \in C^\infty_{\rm c}(T^*\mathbb{R})$, a local escape function supported in a neighbourhood of the segment

$$I_\rho = \{\exp(tH_p)(\rho) : \ t \in [-T,T]\},$$

and which satisfies $H_p G_\rho \geq 1$ on the part of I_ρ lying over

$$\pi^* \overline{B(0,R)} = \{(x,\xi) \in T^*\mathbb{R}^n \ : \ |x| \leq R\}$$

3. For that, let Γ be a hypersurface through ρ which is transversal to H_p. For a neighbourhood U_ρ of ρ we define

$$\begin{aligned} V_\rho &= \{\exp(t(U_\rho \cap \Gamma)) : \ t \in (-T-1, T+1)\} \\ &\subset p^{-1}((E-2\delta, E+2\delta)). \end{aligned}$$

If U_ρ is sufficiently small, V_ρ is a neighbourhood of I_ρ and we can identify it with the product

$$\begin{gathered} V_\rho \simeq (-T-1, T+1) \times (U_\rho \cap \Gamma), \quad (x,\xi) := \exp(tH_p)(m), \\ (x,\xi) \in V_\rho, \quad t \in (-T-1,T+1), \quad m \in U_\rho \cap \Gamma. \end{gathered}$$

In view of (6.4.3)

$$(((-T-1,-T) \cup (T,T+1)) \times (U_\rho \cap \Gamma))) \cap \pi^* \overline{B(0,R)} = \emptyset\,.$$

Now let $\varphi_\rho \in C^\infty_{\rm c}(U_\rho \cap \Gamma)$ be identically 1 near ρ, and let

$$\chi_T \in C^\infty_{\rm c}((-T-1,T+1);[0,1]), \quad \chi_T(t) = t, \quad t \in [-T,T].$$

Let also

$$\psi \in C^\infty_{\rm c}((E-2\delta, E+2\delta),[0,1])\,, \quad \psi|_{[E-\delta,E+\delta]} \equiv 1$$

and, using the product coordinates, put

$$G_\rho(\exp(tH_p)(m)) := \chi_T(t)\varphi_\rho(m)\psi(p(m))\,, \quad G_\rho \in C^\infty_{\rm c}(V_\rho). \tag{6.4.4}$$

It follows that

$$H_p G_\rho(\exp(tH_p)(m)) = \chi_T'(t)\varphi_\rho(m)\psi(p(m)), \tag{6.4.5}$$

satisfies

$$H_p G_\rho = 1 \text{ on } V_\rho \cap \{|x| < R\}.$$

4. Since Σ_δ (see (6.4.2)) is compact, applying the previous argument for every $\rho \in \Sigma_\delta$ gives a U_ρ, and a $V_\rho \subset U_\rho$ on which $\varphi_\rho = 1$. Since $\{V_\rho : \ \rho \in \Sigma_\delta\}$

covers the compact set Σ_δ, we can pass to a finite subcover, $\{V_{\rho_j} : j = 1, \dots, N\}$. We let

$$G = \sum_{j=1}^{N} G_{\rho_j}. \tag{6.4.6}$$

The construction of G_{ρ_j}'s now shows that

$$H_p G(x,\xi) \geq 1\,, \quad (x,\xi) \in \Sigma_\delta, \quad G \in C_{\rm c}^\infty(T^*\mathbb{R}^n), \tag{6.4.7}$$

concluding the proof. □

We will now use the function G constructed in Lemma 6.20 to prove

THEOREM 6.21 (Logarithmic resonance free regions for non-trapping perturbations). *Suppose that P is given by (6.0.5) and that for some $E > 0$ the trapped set at energy E defined by in Definition 6.1 is empty:*

$$K_E = \emptyset.$$

Then there exists $\delta > 0$ such that for each $M > 0$ there exists $h_M > 0$ so that for $0 < h < h_M$,

$$\operatorname{Res}(P) \cap ([E-\delta, E+\delta] - i[0, Mh\log(1/h)]) = \emptyset. \tag{6.4.8}$$

In addition, we have a bound on the truncated resolvent: for $\chi \in C_{\rm c}^\infty(\mathbb{R}^n)$,

$$\|\chi R(z,h)\chi\|_{L^2 \to L^2} \leq \frac{C\exp(C|\operatorname{Im} z|/h)}{h}\,, \tag{6.4.9}$$

$z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)]$, $0 < h < h_M$.

REMARKS. 1. The idea of the proof is to conjugate the operator P_θ by an exponential weight to define

$$\begin{aligned} e^{-t\operatorname{Op}_h(G)/h} P_\theta e^{t\operatorname{Op}_h(G)/h} &= P_\theta - it\operatorname{Op}_h(H_{\operatorname{Re} p_\theta} G) \\ &\qquad + t\operatorname{Op}_h(H_{\operatorname{Im} p_\theta} G) + \mathcal{O}(t^2)_{H_h^2 \to L^2}, \end{aligned}$$

with $t = 2Mh\log(1/h)$. For $z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)]$ the properties of G constructed in Lemma 6.20 show that

$$\operatorname{Im}\sigma\left(e^{-t\operatorname{Op}_h(G)/h}(P_\theta - z)e^{t\operatorname{Op}_h(G)/h}\right) \leq 0.$$

Hence we can apply propagation of singularities as in the proof of Proposition 6.12 to obtain a bound on the resolvent. For a toy model case of this argument see Exercise E.28.

2. One can show invertibility of $e^{-t\operatorname{Op}_h(G)/h}(P_\theta - z)e^{t\operatorname{Op}_h(G)/h}$ by estimating the imaginary and real parts directly (analogously to the positive commutator method used in the proof of Theorem E.47) – see [**SZ07a**, §4]. Here we opted for a quicker hybrid approach.

Proof of Theorem 6.21. 1. Let $G \in C_c^\infty(T^*\mathbb{R}^n;\mathbb{R})$ be given by Lemma 6.20. For $t \in \mathbb{R}$ define the operator

$$P_\theta(t) := e^{-t\,\mathrm{Op}_h(G)/h} P_\theta e^{t\,\mathrm{Op}_h(G)/h} : H_h^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

The operator $\mathrm{Op}_h(G)$ is bounded (uniformly in h) on $H_h^s(\mathbb{R}^n)$ for any s and hence

$$e^{t\,\mathrm{Op}_h(G)/h} = \mathcal{O}(e^{C|t|/h}) : H_h^s(\mathbb{R}^n) \to H_h^s(\mathbb{R}^n). \tag{6.4.10}$$

Using the notation $\mathrm{ad}_A B := [A, B]$ and the identity $e^{\mathrm{ad}_A} B = e^A B e^{-A}$, Taylor's formula shows that for any $N \geq 1$,

$$\begin{aligned} P_\theta(t) &= \sum_{k=1}^{N} \frac{(-1)^k}{k!}(t/h)^k \,\mathrm{ad}^k_{\mathrm{Op}_h(G)} P_\theta \\ &+ \frac{(-1)^{N+1} t^{N+1}}{N! h^{N+1}} \int_0^1 (1-\tau)^N \left(e^{-t\tau\,\mathrm{Op}_h(G)/h}\,\mathrm{ad}^{N+1}_{\mathrm{Op}_h(G)} P_\theta e^{t\tau\,\mathrm{Op}_h(G)/h} \right) d\tau. \end{aligned}$$

Symbolic calculus gives,

$$\mathrm{ad}^k_{\mathrm{Op}_h(G)} P_\theta \in h^k \Psi_h^{\mathrm{comp}}(\mathbb{R}^n), \quad k \geq 1.$$

Also, in view of (6.4.10), for $\tau \in [0,1]$

$$e^{-t\tau\,\mathrm{Op}_h(G)/h} \left(\mathrm{ad}^{N+1}_{\mathrm{Op}_h(G)} P_\theta \right) e^{t\tau\,\mathrm{Op}_h(G)/h} = \mathcal{O}(h^{N+1} e^{2C|t|/h})_{H_h^r \to H_h^s},$$

for any r and s.

Since we can take N arbitrarily large, we see that for t satisfying

$$|t| \leq M_1 h \log(1/h) \tag{6.4.11}$$

and any r and s we have

$$P_\theta(t) = P_\theta - (t/h)\,\mathrm{ad}_{\mathrm{Op}_h(G)} P_\theta + e(t), \quad e(t) = \mathcal{O}(t^2)_{H_h^s \to H_h^r}. \tag{6.4.12}$$

2. We define the family of operators depending smoothly on $t \in \mathbb{R}$

$$\begin{gathered} \widetilde{P}_\theta(t) := P_\theta - (t/h)\,\mathrm{ad}_{\mathrm{Op}_h(G)} P_\theta \in \Psi_h^2(\mathbb{R}^n), \\ P_\theta(t) = \widetilde{P}_\theta(t) + e(t). \end{gathered}$$

Let $\widetilde{p}_\theta(t)$ be the principal symbol of $\widetilde{P}_\theta(t)$. Then for $z \in \mathbb{C}$

$$\begin{aligned} \mathrm{Re}(\widetilde{p}_\theta(t) - z) &= \mathrm{Re}\, p_\theta + t H_{\mathrm{Im}\, p_\theta} G - \mathrm{Re}\, z, \\ \mathrm{Im}(\widetilde{p}_\theta(t) - z) &= \mathrm{Im}\, p_\theta - t H_{\mathrm{Re}\, p_\theta} G - \mathrm{Im}\, z. \end{aligned} \tag{6.4.13}$$

We now relate the parameters in the constructions of P_θ and G: we take $R := 2r_1$ in (6.4.1). Then there exists $\varepsilon > 0$ such that for

$$\mathrm{Re}\, z \in [E - \delta, E + \delta], \quad 0 \leq -2\,\mathrm{Im}\, z \leq t \leq \varepsilon \tag{6.4.14}$$

the symbol $\widetilde{p}_\theta(t) - z$ has the following properties (see Definition E.31 for $\mathrm{ell}_h(\bullet)$):

(1) $\operatorname{Im}(\widetilde{p}_\theta(t) - z) \leq 0$ near the characteristic set

$$\{\langle\xi\rangle^{-2}(\widetilde{p}_\theta(t) - z) = 0\} = \overline{T}^*\mathbb{R}^n \setminus \operatorname{ell}_h(\widetilde{P}_\theta(t) - z);$$

(2) for each $(x, \xi) \in \overline{T}^*\mathbb{R}^n$, there exists $T \geq 0$ such that $\varphi_{-T}(x, \xi) \in \operatorname{ell}_h(\widetilde{P}_\theta(t) - z)$, where $\varphi_{-T} := \exp(-T\langle\xi\rangle^{-1}H_{\operatorname{Re}\widetilde{p}_\theta(t)})$.

Indeed, for $t = 0$ (and thus $\operatorname{Im} z = 0$) these follow from Lemma 6.8 and (6.2.16). In the latter statement we used that $\Gamma^+_{[E-\delta,E+\delta]} = \emptyset$ recalled in Step 1. To deduce from here the case $t > 0$ we first note that by Lemma 6.8 for $\operatorname{Re} z \in [E - \delta, E + \delta]$

$$\{\langle\xi\rangle^{-2}(p_\theta - \operatorname{Re} z) = 0\} \subset \{|x| < 2r_1\} \cap p^{-1}(\operatorname{Re} z),$$
$$H_{\operatorname{Re} p_\theta} G = H_p G \quad \text{on} \quad \{\langle\xi\rangle^{-2}(p_\theta - \operatorname{Re} z) = 0\}.$$

Thus by Lemma 6.20 we have

$$H_{\operatorname{Re} p_\theta} G \geq 1 \quad \text{on} \quad \{\langle\xi\rangle^{-2}(p_\theta - \operatorname{Re} z) = 0\}.$$

Since $\widetilde{p}_\theta - z = p_\theta - \operatorname{Re} z + \mathcal{O}(\varepsilon)$, for ε small enough each point (x, ξ) in $\{\langle\xi\rangle^{-2}(\widetilde{p}_\theta - z) = 0\}$ is close to the set $\{\langle\xi\rangle^{-2}(p_\theta - \operatorname{Re} z) = 0\}$, which implies $H_{\operatorname{Re} p_\theta} G(x, \xi) \geq \frac{1}{2}$. Since $\frac{t}{2} + \operatorname{Im} z \geq 0$ we obtain property (1) above.

For property (2) it suffices to use that $H_{\operatorname{Re}\widetilde{p}_\theta(t)} = H_{\operatorname{Re} p_\theta} + \mathcal{O}(\varepsilon)$ and thus

$$\varphi_{-T}(x, \xi) = \exp(-T\langle\xi\rangle^{-1}H_{\operatorname{Re} p_\theta})(x, \xi) + \mathcal{O}(\varepsilon).$$

3. Using properties (1)–(2) above, the elliptic estimate (Theorem E.33), and propagation of singularities (Theorem E.47) similarly to Steps 2–3 of the proof of Proposition 6.9 we get the following: for z, t satisfying (6.4.14) and all $u \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$\|u\|_{L^2} \leq Ch^{-1}\|(\widetilde{P}_\theta(t) - z)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2} \tag{6.4.15}$$

where the constants are uniform in z, t.

We now put $t = \bar{t} := 2Mh\log(1/h)$ and assume that

$$z \in [E - \delta, E + \delta] - i[0, Mh\log(1/h)].$$

Applying (6.4.15) and using (6.4.12) we see that

$$\|u\|_{L^2} \leq Ch^{-1}\|(P_\theta(\bar{t}) - z)u\|_{L^2} + \mathcal{O}(h\log^2(1/h))\|u\|_{L^2}. \tag{6.4.16}$$

For h small enough we can absorb the $\mathcal{O}(h\log^2(1/h))$ term into the left-hand side, which gives invertivibility of

$$P_\theta(\bar{t}) - z : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

4. The bound in (6.4.10) and (6.4.16) show that (with $\|\bullet\|$ denoting the $L^2 \to L^2$ operator norm)

$$
\begin{aligned}
\|(P_\theta - z)^{-1}\| &= \|e^{\bar{t}\operatorname{Op}_h(G)/h}(P_\theta(\bar{t}) - z)^{-1}e^{-\bar{t}\operatorname{Op}_h(G)/h}\| \\
&\leq \|e^{\bar{t}\operatorname{Op}_h(G)/h}\| \cdot \|e^{-\bar{t}\operatorname{Op}_h(G)/h}\| \cdot \|(P_\theta(\bar{t}) - z)^{-1}\| \\
&= \mathcal{O}(h^{-4CM-1}).
\end{aligned}
\tag{6.4.17}
$$

In particular this implies the absence of resonances. We recall from Theorem 4.37 that $\chi R(z,h)\chi = \chi(P_\theta - z)^{-1}\chi$ provided that $\operatorname{supp}\chi \subset B(0, r_1)$, where in the construction of P_χ we can take r_1 arbitrarily large. Hence the bounds (6.4.17) holds for the truncated resolvent.

5. To prove (6.4.9) we take $\varphi, \psi \in L^2(\mathbb{R}^n)$, $\|\varphi\|_{L^2} = \|\psi\|_{L^2} = 1$ we apply Lemma D.1 to

$$
\begin{gathered}
w \longmapsto f(w) := C^{-1}he^{-iC(w+z)/h}\langle \chi R(z+w,h)\chi\varphi, \psi\rangle, \\
\delta_- = Mh\log(1/h) - |\operatorname{Im} z|, \quad \delta_+ = |\operatorname{Im} z|,
\end{gathered}
$$

where $z \in [E - \delta/4, E + \delta/4] - i[0, Mh\log(1/h)]$ is fixed.

Then, in the notation of Lemma D.1, (6.4.17) gives $M = h^{-L}$ for some L and $M_- = 1$, if we take a large enough C in the definition of f. Increasing that C if necessary, Theorem 6.10 gives $M_+ = 1$. We can take $R = \delta/4$. The assumptions of Lemma D.1 are satisfied and hence,

$$
C^{-1}he^{-C|\operatorname{Im} z|/h}\langle \chi R(z,h)\chi\varphi, \psi\rangle = |f(0)| \leq C.
$$

Since φ and ψ were arbitrarily we obtain (6.4.9) (by changing δ). $\square$

REMARK. The bound (6.4.9) can be proved without invoking Theorem 6.10. In fact, we have

$$
\chi R(z,h)\chi = \begin{cases} \mathcal{O}(1/\operatorname{Im} z), & \operatorname{Im} z \geq 0, \\ \mathcal{O}(h^{-4CM-1}), & z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)], \end{cases}
$$

and a different application of Lemma D.1 gives (6.4.9).

6.5. LOWER BOUNDS ON RESONANCE WIDTHS

We will now show that the cut-off resolvent of a semiclassical Schrödinger operator with a compactly supported potential is bounded by $\exp(C/h)$ on the real axis. When the cut-offs are supported outside of a ball containing the support of the potential the estimate is the same as the non-trapping estimate C/h. From this we deduce that the resonance width (imaginary parts) are bounded from below by $\exp(-C/h)$. In Theorem 2.32 we saw an elementary version of this result in the case of one dimension. As in indicated in §6.6 many generalizations are available.

We start with a uniform estimate of the resolvent in the upper half-plane. At this stage it is important to have weights rather than compactly supported cut-offs.

For $x \in \mathbb{R}^n$ we denote $r = |x|$ and $\mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}(x)$ the characteristic function of $\{x \in \mathbb{R}^n : |x| \geq R_0\}$.

THEOREM 6.22 (**Weighted resolvent estimates).** *Suppose that for* $E > 0$,

$$P = P_E := -h^2\Delta + V - E, \quad V, \partial_r V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R}), \quad n \geq 3.$$

For any $s > 1/2$ *there exist* $C, R_0, h_0 > 0$ *such that*

$$\|(1+r)^{-s}(P - i\varepsilon)^{-1}(1+r)^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C/h}, \tag{6.5.1}$$

$$\|(1+r)^{-s}\, \mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}(P - i\varepsilon)^{-1}\, \mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}(1+r)^{-s}\|_{L^2\to L^2} \leq \frac{C}{h}, \tag{6.5.2}$$

for all $\varepsilon > 0$, $h \in (0, h_0)$.

The proof is based on two lemmas. The first constructs a nondecreasing Carleman weight for P which is constant outside of a compact set. (See Lemma 3.34 for a simpler version which case serve as an introduction to Carleman estimates.) To formulate it we put

$$\delta := 2s - 1 > 0, \qquad w = w_\delta(r) := 1 - (1+r)^{-\delta} \geq 0. \tag{6.5.3}$$

LEMMA 6.23 (**Construction of Carleman weights).** *If* $\delta > 0$ *is small enough, there exist* $R_0 > 0$ *and* $\varphi = \varphi(r) \in C^\infty([0,\infty))$ *with* $\varphi > 0$, $\varphi' \geq 0$ *and* $\operatorname{supp}\varphi' \subset [0, R_0)$, *such that*

$$\partial_r \left(w(r)\big(E - V(x) + (\varphi'(r))^2\big)\right) > \frac{Ew'(r)}{4}, \quad r > 0. \tag{6.5.4}$$

Proof. 1. Fix $R > 0$ such that $\operatorname{supp} V \subset B(0,R)$. We first construct a function $\psi = \psi_\delta(r)$ on $\mathbb{R}$ of the form

$$\psi := \begin{cases} A & r \leq R, \\ \dfrac{B}{w(r)} - \dfrac{E}{4} & R < r < R_0, \\ 0 & r \geq R_0, \end{cases}$$

where A, $B > 0$, $R_0 > R$ will be chosen below so that ψ is continuous and

$$\psi - V + (\psi' - \partial_r V)\frac{w}{w'} \geq -\frac{E}{2}, \qquad r > 0,\ r \neq R,\ r \neq R_0. \tag{6.5.5}$$

We should think of ψ as a prototype for $(\varphi')^2$.

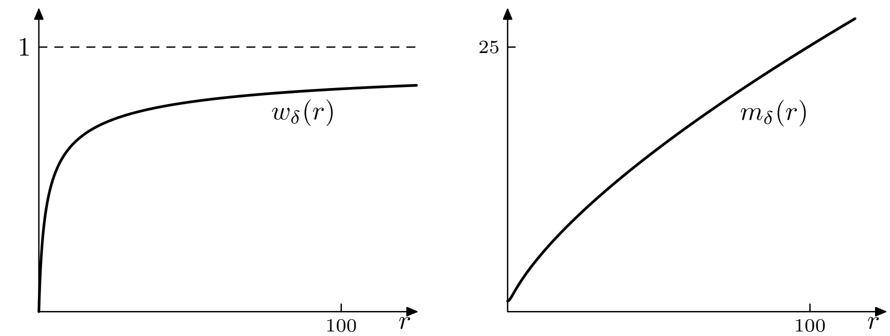


Figure 6.4. Plots of $w = w_\delta(r)$ and $m = m_\delta(r)$ defined in (6.5.3) and (6.5.7) respectively for $\delta = 0.4$.

2. We first arrange it so that ψ satisfies (6.5.5). For $R > R_0$, this is immediate as the left-hand side of (6.5.5) equals 0. For $R < r < R_0$ we compute this left-hand side as

$$\psi + \psi' \frac{w}{w'} = -\frac{E}{4}.$$

Finally for $0 < r < R$ we observe that uniformly in $r \in [0, R]$

$$\frac{w}{w'} = (1+r)\frac{(1+r)^\delta - 1}{\delta} \to (1+r)\log(1+r),$$

as $\delta \to 0^+$. Putting

$$A := \max |V| + 2 \max |\partial_r V|(1+R)\log(1+R),$$

we obtain (6.5.5), provided δ is sufficiently small.

3. To make ψ continuous at $r = R$, put

$$B := w(R)(A + E/4).$$

It remains to choose R_0 such that ψ is continuous at R_0. For that we need

$$w(R_0) = 4B/E = w(R)(1 + 4A/E).$$

Since w takes values in $[0, 1)$, this is possible only if

$$w(R) < \frac{1}{1 + 4A/E}.$$

But since $w(R) \to 0$ as $\delta \to 0^+$, it is enough to take δ sufficiently small.

4. To obtain a smooth φ satisfying (6.5.4) we fix $\rho \in C_0^\infty((0,\infty))$ with $\rho \geq 0$, $\int \rho = 1$. Take $\eta > 0$ and put

$$\varphi(r) := \int_0^r \chi(t)dt, \qquad \chi := \rho_\eta * \sqrt{\psi}, \qquad \rho_\eta(r) := \frac{1}{\eta}\rho\Big(\frac{r}{\eta}\Big).$$

The inequality (6.5.5) gives

$$\frac{\partial_r(w(\chi^2 - V))}{w'} = \chi^2 - V + \frac{w(2\chi\chi' - \partial_r V)}{w'} > -\frac{E}{2} + \mathcal{O}(\eta).$$

Since $\varphi' = \chi$ this concludes the proof of (6.5.4) once we take η small enough. □

We note that if h_0 is small enough (6.5.4) gives the following inequality which will be used in our argument:

$$\partial_r \left(w(E - V + (\varphi')^2 - h\varphi'') \right) \geq Ew'/4 \quad \text{for } h \in (0, h_0]. \tag{6.5.6}$$

The next lemma uses the weight constructed in Lemma 6.23 to prove a global Carleman estimate. Define $m = m_\delta(r)$ by

$$m := (1 + r^2)^{(1+\delta)/4} \sim (1 + r)^s = \delta^{\frac{1}{2}} (w'(r))^{-\frac{1}{2}}, \tag{6.5.7}$$

where $\sim$ is a short hand for $2^{-s} \leq m(r)/(1 + r)^s \leq 1$, $r \geq 0$.

LEMMA 6.24 (Weighted Carleman estimate). *Let δ and $\varphi = \varphi(r)$ be as in Lemma 6.23 and h_0 as in (6.5.6). Then there exists $C > 0$ such that*

$$\begin{aligned} \|m^{-1} e^{\varphi/h} v\|^2_{L^2(\mathbb{R}^n)} \leq & \frac{C}{h^2} \|m e^{\varphi/h} (P - i\varepsilon) v\|^2_{L^2(\mathbb{R}^n)} \\ & + \frac{C\varepsilon}{h} \|e^{\varphi/h} v\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \tag{6.5.8}$$

for all $v \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon \geq 0$, and $h \in (0, h_0]$.

Proof. 1. Let

$$\begin{aligned} P_\varphi :=& e^{\varphi/h} r^{(n-1)/2} (P - i\varepsilon) r^{-(n-1)/2} e^{-\varphi/h} \\ =& - h^2 \partial_r^2 + 2h\varphi' \partial_r + \Lambda + V_\varphi - E - i\varepsilon, \end{aligned}$$

where $\Lambda \geq 0$ a semi-definite operator

$$\Lambda := h^2 r^{-2} \left(-\Delta_{\mathbb{S}^{n-1}} + (n-1)(n-3)/4 \right),$$

and

$$V_\varphi := V - \varphi'^2 + h\varphi''.$$

2. Recalling the relation between m and w' in (6.5.7), we see that (6.5.8) is equivalent to

$$\begin{aligned} \int_0^\infty \int_{\mathbf{S}^{n-1}} w' |u|^2 d\omega dr \leq & \frac{C}{h^2} \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{|P_\varphi u|^2}{w'} d\omega dr \\ & + \frac{C\varepsilon}{h} \int_0^\infty \int_{\mathbb{S}^{n-1}} |u|^2 dr d\omega, \end{aligned} \tag{6.5.9}$$

for

$$u = e^{\varphi/h} r^{(n-1)/2} v \in e^{\varphi/h} r^{(n-1)/2} C_c^\infty(\mathbb{R}^n). \tag{6.5.10}$$

We may assume $\varepsilon \leq h$, since $w' \leq 1$ makes (6.5.9) trivial for $\varepsilon > h$. We will prove

$$\begin{aligned}\int_0^\infty\int_{\mathbb{S}^{n-1}} \partial_r\left(w(E-V_\varphi)\right)|u|^2 d\omega dr \leq& \frac{2}{h^2}\int_0^\infty\int_{\mathbb{S}^{n-1}} \frac{|P_\varphi u|^2}{w'} d\omega dr \\ &+ \frac{C\varepsilon}{h}\int_0^\infty\int_{\mathbb{S}^{n-1}} |u|^2 d\omega dr,\end{aligned} \tag{6.5.11}$$

which, together with (6.5.6), implies (6.5.9).

3. For brevity, we write

$$\|u\|_S := \|u\|_{L^2(\mathbb{S}^{n-1})}, \quad \langle u, v\rangle_S := \langle u, v\rangle_{L^2(\mathbb{S}^{n-1})}.$$

Then for $r > 0$ we put

$$F(r) := \|h\partial_r u(r,\omega)\|_S^2 - \langle(\Lambda + V_\varphi(r,\omega) - E)u(r,\omega), u(r,\omega)\rangle_S. \tag{6.5.12}$$

Note that the compact support of u (see (6.5.10)) shows that

$$\int_0^\infty (w(r)F(r))' dr \leq -\lim_{r\to 0} w(r) \liminf_{r\to 0} F(r) = 0. \tag{6.5.13}$$

Using selfadjointness of $\Lambda + V_\varphi - E$ we compute the derivative of F in terms of P_φ. With the notation $f' := \partial_r f$ we first note that the definition of Λ gives

$$(\Lambda u)' = \Lambda u' - 2r^{-1}\Lambda u.$$

This yields

$$\begin{aligned} F' &= 2\operatorname{Re}\langle h^2 u'', u'\rangle_S - 2\operatorname{Re}\langle(\Lambda + V_\varphi - E)u, u'\rangle_S \\ &\quad + 2r^{-1}\langle \Lambda u, u\rangle_S - \langle V_\varphi' u, u\rangle_S \\ &= -2\operatorname{Re}\langle P_\varphi u, u'\rangle_S + 4h\varphi'\|u'\|_S^2 + 2\varepsilon \operatorname{Im}\langle u, u'\rangle_S \\ &\quad + 2r^{-1}\langle \Lambda u, u\rangle_S - \langle V_\varphi' u, u\rangle_S. \end{aligned}$$

Since $w\varphi' \geq 0$, $2wr^{-1} - w' > 0$ and $\Lambda \geq 0$ (as an operator on $C^\infty(\mathbb{S}^{n-1})$)

$$\begin{aligned} wF' + w'F =& -2w\operatorname{Re}\langle P_\varphi u, u'\rangle_S + \left(4h^{-1}w\varphi' + w'\right)\|hu'\|_S^2 \\ &+ 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S + \left(2wr^{-1} - w'\right)\langle \Lambda u, u\rangle_S \\ &+ \langle (w(E-V_\varphi))' u, u\rangle_S \\ \geq& -2w\operatorname{Re}\langle P_\varphi u, u'\rangle + w'\|hu'\|_S^2 + 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S \\ &+ \langle (w(E-V_\varphi))' u, u\rangle_S. \end{aligned} \tag{6.5.14}$$

4. The ineuality in (6.5.14) and $-2\operatorname{Re}\langle a, b\rangle + \|b\|^2 \geq -\|a\|^2$, applied with $a = (w/hw')P_\varphi u$ and $b = w'hu'$ give the crucial inequality satisfied by F:

$$(wF)' \geq -\frac{w^2}{h^2 w'}\|P_\varphi u\|_S^2 + 2w\varepsilon\operatorname{Im}\langle u, u'\rangle_S + \langle (w(E-V_\varphi))' u, u\rangle_S. \tag{6.5.15}$$

5. We can now return to the proof of (6.5.11). For that we combine (6.5.15) with (6.5.13) and use $w \leq 1$ we obtain

$$\begin{aligned}\int_0^\infty\int_{\mathbb{S}^{n-1}} (w(E-V_\varphi))'\,|u|^2 d\omega dr \leq \frac{1}{h^2}\int_0^\infty\int_{\mathbb{S}^{n-1}} \frac{|P_\varphi u|^2}{w'} d\omega dr \\ + 2\varepsilon \int_0^\infty\int_{\mathbb{S}^{n-1}} |uu'| d\omega dr.\end{aligned} \tag{6.5.16}$$

Using again that $\Lambda \geq 0$, we see that, for all $\gamma > 0$ there is C_γ such that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{S}^{n-1}} |hu'|^2 d\omega dr &= \mathrm{Re}\int_0^\infty\int_{\mathbb{S}^{n-1}} \bar{u}(P_\varphi - 2h\varphi'\partial_r - \Lambda - V_\varphi + E + i\varepsilon)u d\omega dr \\ &\leq \int_0^\infty\int_{\mathbb{S}^{n-1}} |P_\varphi u||u| d\omega dr + 2\int_0^\infty\int_{\mathbb{S}^{n-1}} \varphi'|hu'||u| d\omega dr \\ &\qquad + \int_0^\infty\int_{\mathbb{S}^{n-1}} |E - V_\varphi||u|^2 d\omega dr \\ &\leq \int_0^\infty\int_{\mathbb{S}^{n-1}} |P_\varphi u|^2 d\omega dr + C_\gamma\int_0^\infty\int_{\mathbb{S}^{n-1}} |u|^2 d\omega dr \\ &\qquad + \gamma\int_0^\infty\int_{\mathbb{S}^{n-1}} \varphi'|hu'|^2 d\omega dr.\end{aligned}$$

Choosing $\gamma = 1/(2\max\varphi')$ gives

$$\begin{aligned}\int_0^\infty\int_{\mathbb{S}^{n-1}} |hu'|^2 d\omega dr \leq & 2\int_0^\infty\int_{\mathbb{S}^{n-1}} |P_\varphi u|^2 d\omega dr \\ & + C\int_0^\infty\int_{\mathbb{S}^{n-1}} |u|^2 d\omega dr.\end{aligned} \tag{6.5.17}$$

Applying the ineqaulity

$$2\int_0^\infty\int_{\mathbb{S}^{n-1}} |uu'| d\omega dr \leq h^{-1}\int_0^\infty\int_{\mathbb{S}^{n-1}} |u|^2 + h^{-1}\int_0^\infty\int_{\mathbb{S}^{n-1}} |hu'|^2 d\omega dr$$

to (6.5.16), and using (6.5.17) and $\varepsilon \leq h$, gives (6.5.11). $\square$

Proof of Theorem 6.22. 1. Put $C_0 = 2\max\varphi$. Since $\varphi(r) = \frac{1}{2}C_0$ for $r \geq R_0$, Lemma 6.24 and $\varphi > 0$ give

$$\begin{aligned}e^{-C_0/h}\|m^{-1}\,\mathbb{1}_{B(0,R_0)}\,v\|_{L^2}^2 + \|m^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\,v\|_{L^2}^2 \\ \leq e^{-C_0/h}\|m^{-1}e^{\varphi/h}v\|_{L^2}^2 \\ \leq \frac{C}{h^2}\|m(P-i\varepsilon)v\|_{L^2}^2 + \frac{C_1\varepsilon}{h}\|v\|_{L^2}^2.\end{aligned} \tag{6.5.18}$$

Applying

$$\begin{aligned}2\varepsilon\|v\|_{L^2}^2 &= -2\operatorname{Im}\langle (P-i\varepsilon)v, v\rangle_{L^2}\\ &\le \gamma^{-1}\|m\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}(P-i\varepsilon)v\|_{L^2}^2 + \gamma\|m^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\,v\|_{L^2}^2\\ &\qquad + \gamma_0^{-1}\|m\,\mathbb{1}_{B(0,R_0)}(P-i\varepsilon)v\|_{L^2}^2 + \gamma_0\|m^{-1}\,\mathbb{1}_{B(0,R_0)}\,v\|_{L^2}^2,\end{aligned}$$

with $\gamma = \frac{1}{2}e^{-C_0/h}$ and $\gamma_0 = h/2C_1$ allows us to eliminate the $C_1\varepsilon\|v\|_{L^2}^2/h$ term on the right hand of (6.5.18).

We conclude that for $C_2 > C_0$ and h sufficiently small,

$$\begin{aligned}&e^{-C_0/h}\|m^{-1}\,\mathbb{1}_{B(0,R_0)}\,v\|_{L^2}^2 + \|m^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\,v\|_{L^2}^2 \le\\ &\quad e^{C_2/h}\|m\,\mathbb{1}_{B(0,R_0)}(P-i\varepsilon)v\|_{L^2}^2 + \frac{C}{h^2}\|m\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}(P-i\varepsilon)v\|_{L^2}^2,\end{aligned}\tag{6.5.19}$$

for all $v \in C_0^\infty(\mathbb{R}^n)$.

2. We will deduce from (6.5.19) that, for any $f \in L^2$, we have

$$\begin{aligned}&e^{-C/h}\|\,\mathbb{1}_{B(0,R_0)}(P-i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2\\ &\qquad + \|m^{-1}\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}(P-i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 \le\\ &\qquad\qquad e^{C/h}\|\,\mathbb{1}_{B(0,R_0)}\,f\|_{L^2}^2 + \frac{C}{h^2}\|\,\mathbb{1}_{\mathbb{R}^n\setminus B(0,R_0)}\,f\|_{L^2}^2,\end{aligned}\tag{6.5.20}$$

from which Theorem 6.22 follows.

3. To prove (6.5.20) we need the fact that, for fixed $\varepsilon, h > 0$, and $mv \in H^2$,

$$\frac{1}{C_{\varepsilon,h}}\|mv\|_{H^2} \le \|m(P-i\varepsilon)v\|_{L^2} \le C_{\varepsilon,h}\|mv\|_{H^2}.\tag{6.5.21}$$

For $f \in L^2$ we apply (6.5.21) to $v := (P-i\varepsilon)^{-1}m^{-1}f$ to obtain

$$m(P-i\varepsilon)^{-1}m^{-1}f \in H^2.$$

We then choose $v_k \in C_0^\infty$ satisfying

$$\|mv_k - m(P-i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0 \text{ as } k \to \infty.$$

In particular,

$$\|m^{-1}v_k - m^{-1}(P-i\varepsilon)^{-1}m^{-1}f\|_{L^2} \to 0,$$

and, by (6.5.21) again,

$$\begin{aligned}\|m(P-i\varepsilon)v_k - f\|_{L^2} &\le C_{\varepsilon,h}\|mv_k - m(P-i\varepsilon)^{-1}m^{-1}f\|_{H^2}\\ &\to 0, \quad \text{as } k \to \infty.\end{aligned}$$

Consequently (6.5.20) follows by applying (6.5.19) with v_k in place of v, and letting $k \to \infty$.

4. It remains to prove (6.5.21). We have

$$\|mv\|_{H_h^2} \le (C/\varepsilon)\|(P-i\varepsilon)mv\|_{L^2} \le (C'/\varepsilon)\|mv\|_{H_h^2},\tag{6.5.22}$$

for all v with $mv \in H_h^2$. On the other hand,

$$[P, m] = -2h^2 m' \partial_r - h^2 m'' - h^2(n-1)m'/r = \mathcal{O}(h)_{H_h^2 \to L^2}.$$

Hence the second inequality in (6.5.21) follows from the second inequality in (6.5.22):

$$\|m(P - i\varepsilon)v\|_{L^2} \le (C'/C)\|mv\|_{H_h^2} + \|[P, m]v\|_{L^2} \le C_{\varepsilon,h}\|mv\|_{H^2}.$$

Similarly we deduce the first of (6.5.21) from the first of (6.5.22):

$$\begin{aligned}\|mv\|_{H^2} &\le C_h \|mv\|_{H_h^2} \le C'_{h,\varepsilon}\left(\|m(P - i\varepsilon)v\|_{L^2} + \|[P, m]v\|_{L^2}\right) \\ &\le C_{\varepsilon,h}\|m(P - i\varepsilon)v\|_{L^2}.\end{aligned}$$

This proves (6.5.21) concluding the proof of the theorem. ☐

The weighted estimate immediately estimates for the cut-off resolvent on the real axis:

THEOREM 6.25 (Estimates of the cut-off resolvent). *Suppose that $V, \partial_r V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n; \mathbb{R})$, $n \ge 3$ and that*

$$R(z, h) := (-h^2\Delta + V - z)^{-1}, \quad \operatorname{Im} z > 0.$$

Fix $0 < a < b$ and assume that $E \in [a, b]$. Then there exists $C_0 > 0$ such that for any $R > 0$ and $\chi \in C^\infty_{\mathrm{c}}(B(0, R))$ there exists C_1 and

$$\|\chi R(E, h)\chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C_1 \exp \frac{C_0}{h}.$$

In addition, there exist R_0 such that for $\chi \in C^\infty_{\mathrm{c}}(B(0, R) \setminus B(0, R_0))$,

$$\|\chi R(E, h)\chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le \frac{C_1}{h}.$$

Proof. We only need to remark that the holomorphy of $\chi R(z, h)\chi$, $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, on $(0, \infty)$ implies that

$$\|\chi R(E, h)\chi\|_{L^2 \to L^2} = \lim_{\varepsilon \to 0+} \|\chi R(E - i\varepsilon, h)\chi\|_{L^2 \to L^2}.$$

The uniformity of the constants in (6.5.1) and (6.5.2) with respect to $\varepsilon > 0$ gives the estimates in the theorem. ☐

We now have an important conclusion about the minimal width of scattering resonances:

THEOREM 6.26 (Lower bounds on resonance width). *Suppose that V satisfies the assumptions of Theorem 6.25 and that $0 < a < b$ are fixed. Then there exist constants C_0, h_0 such that for $0 < h < h_0$,*

$$z \in \operatorname{Res}(-h^2\Delta + V), \quad \operatorname{Re} z \in [a, b] \implies -\operatorname{Im} z > e^{-C_0/h}.$$

To prove this we will need the following lemma which provides a useful resolvent identity similar to the identities used in §4.2:

LEMMA 6.27 (A resolvent identity). *For* $V \in L^\infty_{\rm c}(\mathbb{R}^n, \mathbb{R})$ *and* $\operatorname{Im} z > 0$ *put*

$$\begin{aligned} R(z) = R(z,h) &:= (-h^2\Delta + V - z)^{-1}, \\ R_0(z) = R_0(z,h) &:= (-h^2\Delta - z)^{-1}. \end{aligned}$$

Suppose that $\chi, \chi_0 \in C^\infty_{\rm c}(\mathbb{R}^n)$*, have the property that* $\chi_0 = 1$ *on* $\operatorname{supp} V$ *and* $\chi = 1$ *on* $\operatorname{supp} \chi_0$*,*

$$Q = Q(h) := [h^2\Delta, \chi_0] = \mathcal{O}(h) : H^s_h(\mathbb{R}^n) \to H^{s-1}_h(\mathbb{R}^n). \tag{6.5.23}$$

and

$$R_\chi(z) := \chi R(z)\chi, \quad R_{0,\chi}(z) = \chi R_0(z)\chi. \tag{6.5.24}$$

Then, for $z, z_0 \in (0,\infty) + i\mathbb{R}$*,*

$$\begin{aligned} &R_\chi(z) - R_\chi(z_0) = (z - z_0) R_\chi(z)\chi_0(2-\chi_0)R_\chi(z_0) \\ &+ (1 - \chi_0 - R_\chi(z)Q)\left(R_{0,\chi}(z) - R_{0,\chi}(z_0)\right)(1 - \chi_0 + QR_\chi(z_0)). \end{aligned} \tag{6.5.25}$$

Proof. It suffices to prove (6.5.24) for $\operatorname{Im} z > 0$, $\operatorname{Im} z_0 > 0$ as the general case follows by analytic continuation.

1. Since $\chi_0 = 1$ on the support of V,

$$(-h^2\Delta + V)(1-\chi_0) = (-h^2\Delta)(1-\chi_0).$$

Hence, in the notation of (6.5.23),

$$\begin{aligned} R(z)(1-\chi_0) - (1-\chi_0)R_0(z) &= R(z)(1-\chi_0)(-h^2\Delta - z)R_0(z) \\ &\quad - R(z)(-h^2\Delta - z)(1-\chi_0)R_0(z) \\ &= R(z)[1-\chi_0, -h^2\Delta]R_0(z) \\ &= -R(z)QR_0(z) \end{aligned} \tag{6.5.26}$$

and similarly

$$(1-\chi_0)R(z_0) - R_0(z_0)(1-\chi_0) = R_0(z_0)QR(z_0). \tag{6.5.27}$$

2. For $\operatorname{Im} z > 0$ we also have

$$\begin{aligned} R(z) - R(z_0) &= (z - z_0)R(z)R(z_0) \\ &= (z-z_0)\left(R(z)\chi_0(2-\chi_0)R(z_0) + R(z)(1-\chi_0)^2 R(z_0)\right). \end{aligned}$$

We now use (6.5.26) and (6.5.27) to rewrite the second term on the right hand side. With the notation (6.5.23) this gives

$$\begin{aligned} R(z)(1-\chi_0)^2 R(z_0) &= \left(-R(z)QR_0(z) + (1-\chi_0)R_0(z)\right)(1-\chi_0)R(z_0) \\ &= -R(z)QR_0(z)R_0(z_0)(1-\chi_0) \\ &\quad - R(z)QR_0(z)R_0(z_0)QR(z_0) \\ &\quad + (1-\chi_0)R_0(z)R_0(z_0)(1-\chi_0) \\ &\quad + (1-\chi_0)R_0(z)R_0(z_0)QR(z) \\ &= (1-\chi_0 - R(z)Q)R_0(z)R_0(z_0)(1-\chi_0 + QR(z_0)). \end{aligned}$$

3. Since

$$R_0(z)R_0(z_0) = (z-z_0)^{-1}(R_0(z) - R_0(z_0))$$

and $\chi\chi_0 = \chi_0$, $\chi Q = Q$ the combination of the two identities in Step 2 gives (6.5.25). $\square$

Proof of Theorem 6.26. 1. We use the notation introduced in Lemma 6.27 and take $z_0 \in [a,b]$ and z with $\operatorname{Re} z = z_0$, $|\operatorname{Im} z| \leq h$. Theorem 6.26 shows that

$$\|R_\chi(z_0)\|_{L^2\to L^2} \leq C_1 \exp(C_2/h), \tag{6.5.28}$$

for some constants C_1 and C_2 depending only on χ and a, b. From now on $\| \bullet \|$ will denote the operator norm $L^2 \to L^2$.

2. The bound (3.1.12) for the free resolvent rescales to a semiclassical bound

$$R_{0,\chi}(w) = \mathcal{O}(h^{-1}e^{|\operatorname{Im} w|/h}) : H_h^{-k}(\mathbb{R}^n) \to H_h^j(\mathbb{R}^n),$$

for $k, j = 0, 1$, $w \in [a,b] + i[-1,1]$. Cauchy inequalities (or re-examination of the proof of Theorem 3.1) show that we also have

$$\partial_w R_{0,\chi}(w) = \mathcal{O}(h^{-2}) : H_h^{-k}(\mathbb{R}^n) \to H_h^j(\mathbb{R}^n),$$

for $k, j = 0, 1$, $w \in [a,b] + i[-h,h]$. As $z - z_0 = i \operatorname{Im} z = \mathcal{O}(h)$ this gives

$$\|R_{0,\chi}(z) - R_{0,\chi}(z_0)\| = (h^{-2}|\operatorname{Im} z|) : H_h^{-k} \to H_h^j, \quad k, j = 0, 1. \tag{6.5.29}$$

3. We now use the resolvent identity (6.5.25), (6.5.29) and (6.5.28):

$$\begin{aligned}
\|R_\chi(z)\| &\leq \|R_\chi(z_0)\| + C|\operatorname{Im} z|\|R_\chi(z)\|\|R_\chi(z_0)\| \\
&\quad + \sum_{j,k=0}^{1} \|R_\chi(z)\|^j \|Q^j(R_{0,\chi}(z) - R_{0,\chi}(z_0))Q^k\|\|R_\chi(z_0)\|^k \\
&\leq \|R_\chi(z_0)\| + |\operatorname{Im} z|\|R_\chi(z)\|\|R_\chi(z_0)\| \\
&\quad + \sum_{j,k=0}^{1} h^{k+j}\|R_\chi(z)\|^j\|R_\chi(z_0)\|^k\|R_{0,\chi}(z) - R_{0,\chi}(z_0)\|_{H_h^{-k}\to H_h^j} \\
&\leq C_3 e^{C_2/h} + |\operatorname{Im} z| C_3 h^{-2} e^{C_2/h}\|R_\chi(z)\|.
\end{aligned}$$

The meromorphy of $R_\chi(z)$ implies that $R_\chi(z)$ is finite except on a discrete set and hence for

$$|\operatorname{Im} z| < e^{-C_0/h} \leq (2C_3)^{-1} h^2 e^{-C_2/h},$$

we have $\|R_\chi(z)\| \leq 2C_3 e^{C_2/h}$, which completes the proof. □

REMARKS. 1. A more direct proof of Theorem 6.26 can be given using a (much more complicated) version of the proof we presented in one dimension – see Theorem 2.32. The key element is the following inequality [**Bu98**, Proposition 2.2]: suppose that u is outgoing in the sense that

$$u(h) := R_0(z,h)f(h), \quad f \in L^\infty_{\mathrm{comp}}(B(0,R_0)),$$

for R_0 fixed. Then $R_2 > R_1 > R_0$, $|\operatorname{Im} z| \leq Ch$, $\operatorname{Re} z \in (a,b)$, $0 < h < h_0$,

(6.5.30)

$$\begin{aligned}
-\int_{\partial B(0,R_2)} \operatorname{Im} h\partial_r u\bar{u} dS &\geq c\int_{\partial B(0,R_2)} (|u|^2 + |hDu|^2) dS \\
&\quad - Ce^{-c/h}\int_{\partial B(0,R_1)} (|u|^2 + |hDu|^2) dS.
\end{aligned}$$

2. Using (6.5.30) on can show that existence of resonance very close to real axis implies existence of localized quasimodes – see Stefanov [**St00**]. Then Theorem 6.26 can be proved using an argument by contradiction and the results of §7.3.

6.6. NOTES

For a presentation of classical scattering in a more general setting see [**GS87**, Appendix].

Semiclassical defect measures were used to show existence of resonance free strips (to which we will refer to as *spectral gaps*) by Burq [**Bu02b**] for obstacle problems. That provided a weaker result than logarithmic resonance free strips obtained using propagation of singularities (see §§4.6,4.7) but was much simpler technically. Here we followed Dyatlov [Dy16] and applied the semiclassical defect measures to get spectral gaps and optimal resolvent bounds for normally hyperbolic trapping. That improved earlier results of Gérard–Sjöstrand [GS88] in the analytic case and of Wunsch–Zworski [WZ11] in the smooth case and with resolvent bounds. Under additional dynamical assumptions, existence of resonances and a counting law was given in Dyatlov [Dy15a]. One example of normally hyperbolic trapping with the assumptions of §6.3 satisfied is given by photon spheres of black holes – see [WZ11],[Dy11b],[Dy12],[Dy15b], [HV14a] and [HV16]. (For a review see [DZ13].) This kind of trapping occurs also in molecular dynamics [G*10]. For a mathematical treatment under more general assumptions see Nonnenmacher–Zworski [NZ15].

A special case of normally hyperbolic trapping is given by a single hyperbolic trajectory and in that case resonances can be described with great precision. In the obstacle case that was done by Ikawa [Ik83] and Gérard [Gé88]. The semiclassical case was first analysed by Gérard–Sjöstrand [GS87] and that led to many developments. Resonances generated by nondegenerate critical points were described by Sjöstrand [Sj87]. References to more recent results can be found in [B*16].

Another interesting class of trapped sets generating resonances is given by *homoclinic trapped sets.* Although *not* stable under perturbations these trapped sets occur in many situations each with its own rich structure in distribution of resonances. An impressively precise study of this has been made by Bony–Fujiie–Ramond–Zerzeri [B*16].

When the trapped set is hyperbolic in the sense of dynamical systems (see the references below) spectral gaps exist under a classical "topological pressure condition". For obstacle problems that was initiated by Ikawa [Ik88] in mathematics and Gaspard–Rice [GR89] in physics, see also [PS10]. A semiclassical version was proved by Nonnenmacher–Zworski [NZ09a],[NZ09b]. One application is to no-loss Strichartz estimates by Burq–Guillarmou–Hassell [BGH10]. It is now suspected that quantum effects will produce spectral gaps for any hyperbolic trapped set. Evidence for that is provided by a recent result of Bourgain–Dyatlov [BD18] who proved a presence of a spectral gap for *any* convex co-compact Riemann surface. The proof was based on a *fractal uncertainly principle* and microlocal methods developed by Dyatlov–Zahl [DZa16]. See also [DJ16],[DZ18] and, for applications of the fractal uncertainty principle to closed systems, Dyatlov–Jin [DJ18].

A strategy from "gluing" resolvent estimates between different geometries at infinity was developed by Datchev–Vasy [DV12a],[DV12b]. As in [NZ15] it sometimes allows to prove resolvent estimates in the easier setting of complex absorbing potentials (see (4.7.1)) and then glue them to non-trapping estimates near infinity.

Existence of logarithmic resonance free regions for more general semiclassical operators on Euclidean spaces was proved by Martinez [Ma02b] following a long tradition of works in scattering theory – see §4.7. Here we mention the seminal work of Lax and Phillips [LP68] and of Vainberg [Va73] providing an abstract framework for obtaining resonance free regions (see §4.6), and the work of Helffer and Sjöstrand [HS86], [Sj90] on large resonance free regions,

$$K_E = \emptyset \implies \operatorname{Res}(P(h)) \cap B(E, \delta) = \emptyset \tag{6.6.1}$$

for large classes of operators $P(h)$ with analytic coefficients. In the "intermediate" Gevrey case, polynomial resonance free regions were shown to exist by Goodhue [Go73] and Rouleux [Ro01].

The proof of Theorem 6.21 presented here, which uses microlocal exponential weights, is a hybrid of the proof in Sjöstrand–Zworski [SZ07a, §4] and the arguments in §6.2. In the analytic case stronger weights can be used and that results in (6.6.1) – see Sjöstrand [Sj90],[Sj02, §12.5] and references given there.

The review [Zw17, §3.3] can also be consulted about resonance free regions and related open problems.

Theorem 6.22 was first proved by Burq [Bu98],[Bu02a] in more general settings. Different proofs were found by Sjöstrand [Sj02] and Vodev [Vo00]. Cardoso and Vodev [CV02] gave a version for manifolds with asymptotically conic or hyperbolic ends, and, most recently, Rodnianski and Tao [RT15] considered Schrödinger operators on asymptotically conic manifolds, obtaining also bounds for low energies and other refinements. Optimality of the bounds with one cut-off localized outside of the support of the perturbation was discussed in [DDZ15]. An early approach to lower bounds in potential scattering was developed by Fernández–Lavine [FL90].

The proof of Theorem 6.22 was provided by Kiril Datchev [Da14] who used the same methods to establish more general results, in particular relaxing the decay conditions at infinity. It is close in spirit to the earlier proofs of Cardoso and Vodev [CV02], see also [Vo13, Vo14]. In particular, the functional (6.5.12) comes from those papers. For more recent progress in the case of rougher potentials, and in dimension two, see Klopp–Vogel [KV19], Shapiro [Sh19], Vodev [Vo19], and [Sh16] respectively.

The proof of Theorem 6.26 comes from Vodev [Vo14, §5].

6.7. EXERCISES

Section 6.1

Exercises 6.1–6.8 explore a more general set of assumptions on the Hamiltonian p under which the results of §6.1 hold. We assume the following:

(1) M is a manifold, $p \in C^\infty(T^*M;\mathbb{R})$, and the Hamiltonian flow $\exp(tH_p)$ is defined on T^*M for all times;

(2) $r \in C^\infty(T^*M;\mathbb{R})$ is a function and $\alpha \leq \beta$ are numbers such that the sets

$$U_R = r^{-1}\big((-\infty, R]\big) \cap p^{-1}\big([\alpha,\beta]\big) \tag{6.7.1}$$

are compact for each R;

(3) there exists a constant r_0 such that the following convexity assumption holds:

$$p(x,\xi) \in [\alpha,\beta],\ r(x,\xi) \geq r_0,\ H_p r(x,\xi) = 0 \implies H_p^2 r(x,\xi) > 0. \tag{6.7.2}$$

We define the sets $\Gamma^\pm_{[\alpha,\beta]}$ as follows: $(x,\xi) \in \Gamma^\pm_{[\alpha,\beta]}$ if $p(x,\xi) \in [\alpha,\beta]$ and $r(e^{tH_p}(x,\xi))$ does not converge to infinity as $t \to \mp\infty$. We then put $K_{[\alpha,\beta]} := \Gamma^+_{[\alpha,\beta]} \cap \Gamma^-_{[\alpha,\beta]}$.

1. Show that if assumptions (1)–(3) above hold, then they also hold with r replaced by $e^f F(r)$, where $F : \mathbb{R} \to \mathbb{R}$ is any smooth function such that $F' > 0$ everywhere and $\lim_{r\to\infty} F(r) = \infty$, and $f \in C^\infty(M;\mathbb{R})$ is bounded.

2. Show that assumptions (1)–(3) above hold in each of the following cases:
(a) $M = \mathbb{R}^n$ and $p(x,\xi) = |\xi|^2 + V(x)$, where $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ satisfies

$$\limsup_{x\to\infty} V(x) < \alpha, \quad \limsup_{x\to\infty} \langle x, \nabla V(x)\rangle \leq 0,$$

and $r(x,\xi) := |x|$ for $|x|$ large enough.
(b) $M = \mathbb{R}$, $p(x,\xi) = x\xi$, and $r(x,\xi) = \sqrt{x^2+\xi^2}$ for large enough (x,ξ). What are $\Gamma^\pm_{[\alpha,\beta]}$ in this case?

3. Under assumptions (1)–(3) above, show that for each $R \geq r_0$, the set U_R defined in (6.7.1) is convex with respect to the flow $\exp(tH_p)$: that is, if $e^{-t^- H_p}(x,\xi) \in U_R, e^{t^+ H_p}(x,\xi) \in U_R$ for some $t^\pm \geq 0$, then $(x,\xi) \in U_R$. (**Hint**: find the maximal value of $r(e^{tH_p})(x,\xi)$ on the interval $t \in [-t^-, t^+]$.)

4. Under assumptions (1)–(3) above, take (x,ξ) such that

$$p(x,\xi) \in [\alpha,\beta], \quad r(x,\xi) \geq r_0, \quad \pm H_p r(x,\xi) \geq 0.$$

(a) Show that

$$r(e^{tH_p}(x,\xi)) > r_0, \quad \pm H_p r(e^{tH_p}(x,\xi)) > 0 \quad \text{for all } t,\ \pm t > 0.$$

(**Hint**: for the case $H_p r(x,\xi) \geq 0$ and each $T > 0$, find the maximum of the function $r(e^{tH_p}(x,\xi))$ on the interval $[0,T]$.)

(b) Show that $(x,\xi) \notin \Gamma^{\mp}_{[\alpha,\beta]}$, that is $r(e^{tH_p}(x,\xi)) \to \infty$ as $t \to \pm\infty$. (**Hint**: argue by contradiction, taking for the case $H_p r(x,\xi) \geq 0$ a sequence of $t_j \to \infty$ such that $r(e^{t_j H_p}(x,\xi))$ is bounded and extracting a convergent subsequence. Apply assumption (3) to the limiting point of this subsequence.)

5. Using the previous exercise and the proof of Proposition 6.3, show that the sets $\Gamma^{\pm}_{[\alpha,\beta]}$ are closed and $K_{[\alpha,\beta]} \subset \{r < r_0\}$ is compact.

6. Under assumptions (1)–(3) above, show that for each R and each neighbourhood U of $K_{[\alpha,\beta]}$, there exists $T > 0$ such that for all $t^{\pm} \geq T$,

$$p(x,\xi) \in [\alpha,\beta],\ r(e^{-t^- H_p}(x,\xi)) \leq R,\ r(e^{t^+ H_p}(x,\xi)) \leq R \implies (x,\xi) \in U.$$

In other words, every trajectory that passes a long time in a bounded set has to have many points close to the trapped set. (**Hint**: argue by contradiction, taking a sequence $t_j^{\pm} \to \infty$ and $(x_j,\xi_j) \in p^{-1}([\alpha,\beta]) \setminus U$ such that $r(e^{-t_j^- H_p}(x,\xi)), r(e^{t_j^+ H_p}(x,\xi)) \leq R$. Using Exercise 6.3, take a subsequence of (x_j,ξ_j) converging to some $(x_\infty,\xi_\infty) \notin K_{[\alpha,\beta]}$. Use the fact that $e^{tH_p}(x_\infty,\xi_\infty)$ escapes in at least one time direction together with Exercise 6.4 to arrive to a contradiction.)

7. Use the previous exercise to show that for each $(x,\xi) \in \Gamma^{\mp}_{[\alpha,\beta]}$, the trajectory $e^{tH_p}(x,\xi)$ converges to $K_{[\alpha,\beta]}$ as $t \to \pm\infty$, and the convergence is uniform for (x,ξ) in a compact set.

8. Arguing as in the proof of Proposition 6.5, show that the sets $\Gamma^{\pm}_{[\alpha,\beta]} \setminus K_{[\alpha,\beta]}$ have measure zero in T^*M. Show that this is false with respect to the one-dimensional Lebesgue measure on $p^{-1}(0)$ in the case of Exercise 6.2(b), and explain why this does not give a contradiction.

9. This exercise gives an example of a situation where the trapped set is not closed. Consider a Riemannian surface (M,g) with one infinite end which is a cusp $[0,\infty)_r \times \mathbb{S}^1_\theta$ with the metric $g = dr^2 + e^{-2r}d\theta^2$ in the cusp. Let $p(x,\xi) = |\xi|_g^2 = \xi_r^2 + e^{2r}\xi_t^2$ and take $0 < \alpha \leq \beta$. (We denote by $\xi_\bullet$ conjugate variable to $\bullet$, that is, the corresponding momentum variable.)

(a) Show that the function r does not satisfy (6.7.2).

(b) Show that a point (x, ξ) with $r = 0$, $\xi_r > 0$, $\xi_\theta = 0$ does not lie in $\Gamma^-_{[\alpha,\beta]}$, but it lies in the closure of $K_{[\alpha,\beta]}$. (**Hint**: for the latter part, show that nontrapped trajectories form a set of zero measure.)

REMARK. For an example of surfaces with a cusp and no trapping, and a presence of resonance free region see Datchev [**Da16**].

Section 6.2

10. Deduce from Theorem 6.11 that:

(a) $\nu \leq 0$;

(b) if $\nu = 0$, then μ is supported on K; (**Hint:** use the proof of Lemma 6.5.)

(c) if $\nu < 0$, then $\mu(K) = 0$.

11. In the case of (6.0.5) let u be a resonant state with $z \in [\alpha, \beta] - i[0, Ch]$, $\alpha > 0$. Let $\chi_0 \in C^\infty_{\mathrm{c}}(M)$ be equal to 1 on a sufficiently large ball. Prove that for h small enough

$$\|\chi u\|_{L^2} \leq C(\chi, \chi_0)\|\chi_0 u\|_{L^2} \quad \text{for all } \chi \in C^\infty_{\mathrm{c}}(M) \tag{6.7.3}$$

Hint: Consider first the simpler case of $-h^2\Delta + V$. Then (see (3.2.3)) $u = -R_0(z, h)Vu$. Choose χ_1 such that $\chi_1 = 1$ on $\operatorname{supp} V$ and $\operatorname{supp} \chi_1 \subset \{\chi_0 = 1\}$. We only need to estimate $\chi(1 - \chi_1)u$ and for that we observe that $(1-\chi_1)R_0(z, h)Vu = -[\chi_1, R_0(z, h)]Vu = R_0(z, h)[\chi_1, h^2\Delta]\chi_0 u$. Using the rescaled version of (3.1.12) and the gain of h from the commutator we obtain (6.7.3). For the general case, use Theorem 4.9. See [**NZ09a**, (8.12)] for the case of non-compactly supported perturbations.

12. Let $P = -h^2\Delta_g$ where (M, g) has Euclidean infinite ends. Assume that P has an essential spectral gap of size $\beta > 0$ with a polynomial resolvent bound: that is, there exist $h_0 > 0$ and $N > 0$ such that for all $\chi \in C^\infty_{\mathrm{c}}(M)$

$$\begin{gathered} \|\chi R(z, h)\chi\|_{L^2 \to L^2} \leq C_\chi h^{-1-N}, \\ \operatorname{Re} z = 1, \quad -\beta h \leq \operatorname{Im} z \leq h, \quad 0 < h < h_0. \end{gathered}$$

Show that the wave equation on $\mathbb{R}_t \times M$ has a resonance expansion with loss of N derivatives. What changes need to be made when (M, g) is instead asymptotically hyperbolic?

13. Show that each measure μ from Theorem 6.11 is supported in $\{r \leq 2r_1\}$ where r_1 satisfies (6.2.4).

Section 6.3

14. For each $\varphi_\pm$ satisfying (6.3.2)–(6.3.3), assumptions (A1)–(A2) imply that $\{\varphi_+, \varphi_-\} \neq 0$ on $K_{[\alpha',\beta']}$. Use this to show that if (A3) holds for one of the functions $\varphi_{\pm}$, then it holds for the other one as well, and the constants $\nu_{\min}$ coming from φ_{+} and φ_{-} are the same.

Chapter 7

RESONANCES AND TRAPPING

7.1 Lower bounds on the resolvent
7.2 Semiclassical growth estimates
7.3 From quasimodes to resonances
7.4 The Sjöstrand trace formula
7.5 Resonance expansions for strong trapping
7.6 Notes
7.7 Exercises

In this chapter we will discuss effects of trapped classical trajectories on the distribution of resonances. We first present a result of Bony–Burq–Rammond which shows that having trapped trajectories implies a lower bound on the (cut-off) resolvent on the real axis. That lower bound differs from the non-trapping bound by a logarithmic factor. It is optimal as shown by Theorem 6.16.

We then discuss general bounds on the number of scattering poles and on the resolvent. These are applied to show how localized quasimodes (approximate eigenfunctions) imply existence of resonances close to the real axis. The general results of Tang–Zworski and Stefanov are presented in the special case of semiclassical Schrödinger operators but the method applies in great generality, for instance to obstacle problems.

We continue with Sjöstrand's local trace formula and with his lower bounds on the number of resonances. In that case trapping is not explicitly discussed but comes from the presence of certain analytic singularities.

Finally, we present a result of Burq–Zworski on the expansion of solutions of evolution equations in terms of resonances close to the real axis, that is, resonances generated by strong trapping.

7.1. LOWER BOUNDS ON THE RESOLVENT

In §6.10 we have shown that the truncated resolvent satisfies

$$\begin{gathered} K_E = \emptyset \implies \chi R(E,h)\chi = \mathcal{O}_{L^2\to L^2}(1/h), \\ R(E,h) := (P - E - i0)^{-1}, \quad P = -h^2\Delta_g + V, \\ V, g_{ij} - \delta_{ij} \in C_{\rm c}^\infty(\mathbb{R}^n;\mathbb{R}), \end{gathered}$$

and this notation will be used throughout this section.

In this section we consider a *lower bound* on the norm of the resolvent in the case of arbitrary trapping.

THEOREM 7.1 (**Lower bounds on resolvent for trapping perturbations).** *Suppose that $E_0 > 0$ and that $K_{E_0} \neq \emptyset$, and that $\chi \in C_{\rm c}^\infty(\mathbb{R}^n)$ is equal to* 1 *near $\pi(K_{E_0})$.*

Then there exists $C_0 = C_0(E_0)$ such that for any $\delta > 0$ there exists $h_0 = h_0(\delta)$ so that

$$\sup_{|E-E_0|<\delta} \|\chi R(E,h)\chi\|_{L^2\to L^2} \geq \frac{\log(1/h)}{C_0 h}, \tag{7.1.1}$$

for $0 < h < h_0$.

REMARK. Theorem 6.16 shows that this estimate is optimal. The point here is that for any trapping situation we cannot do better than (6.3.12).

Before giving the proof of Theorem 7.1 we need to present an older result, essentially due to Kato, relating resolvent estimates to *local smoothing* in Schrödinger propagation.

THEOREM 7.2 (**Kato's local smoothing).** *Let $E_0 > 0$ and let $K(h) \geq$ 1 be a function on $(0,1)$.*

Suppose that for $|E - E_0| < \delta$ and $\chi \in L^\infty_{\rm comp}(\mathbb{R}^n)$ we have

$$\|\chi R(E,h)\chi\|_{L^2\to L^2} \leq \frac{K(h)}{h}, \tag{7.1.2}$$

Then for $\varphi \in C_{\rm c}^\infty((E-\delta, E+\delta);[0,1])$ and $u \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}} \|\chi\varphi(P)\exp(-itP/h)u\|_{L^2}^2 dt \leq CK(h)\|u\|_{L^2}^2, \tag{7.1.3}$$

with C independent of h.

INTERPRETATION. 1. If the integration in (7.1.3) takes place over a finite interval in time, $[0, T]$, then the estimate is obvious with $CK(h)$ replaced by T. The localization in space, $\chi(x)$ and in energy, $\varphi(P)$ are also not needed. Hence the point lies in having the integral over $\mathbb{R}$. For that χ for which (7.1.2) holds is needed. In our presentation we take $\chi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ but finer weights, such as $\langle x\rangle^{-\frac{1}{2}-\varepsilon}$ also work – see [**VZ00**] and references given there.

2. When $P = -h^2\Delta_g$, where g is a metric, we can change variables in the the t integration in (7.1.3) to obtain

$$\int_{\mathbb{R}} \|\chi\varphi(-h^2\Delta_g)\exp(it\Delta_g)u\|_{L^2}^2 dt \leq ChK(h)\|u\|_{L^2}\,,$$

where $\varphi \in C_{\mathrm{c}}^{\infty}((0,\infty))$.

If $K(h) = 1$, as is the case in (6.2.22) under non-trapping assumption, then

$$\int_{\mathbb{R}} \|\chi\varphi_1(-h^2\Delta_g)(-\Delta_g)^{\frac{1}{4}}\exp(it\Delta_g)u\|_{L^2}^2 dt \leq C\|u\|_{L^2}\,,$$

where $\varphi_1(\lambda) := \varphi(\lambda)/\lambda^{\frac{1}{4}} \in C_{\mathrm{c}}^{\infty}((0,\infty))$.

A dyadic decomposition (see for instance [**Zw12**, Section 7.5] for a presentation in a semiclassical spirit) then shows that

$$\int_{\mathbb{R}} \|\chi(1-\psi)(-\Delta_g)\exp(it\Delta_g)u\|_{H^{\frac{1}{2}}}^2 dt \leq C\|u\|_{L^2}\,, \tag{7.1.4}$$

where $\psi \in C_{\mathrm{c}}^{\infty}(\mathbb{R};[0,1])$, $\psi \equiv 1$ near 0. To control the term with $\psi(-\Delta_g)$ one needs finer analysis of the bottom of the spectrum of $-\Delta_g$ but a crude bound gives

$$\int_{-T}^{T} \|\chi\exp(it\Delta_g)u\|_{H^{\frac{1}{2}}}^2 dt \leq CT\|u\|_{L^2}\,, \tag{7.1.5}$$

This is the local smoothing estimate for non-trapping perturbations. In this formulation the smoothing character is clear: we gain 1/2 derivative when localizing in space and averaging in time.

3. Doi [**Do96**] showed that any trapping produces a loss in the $H^{\frac{1}{2}}$ regularity. The proof of Theorem 7.1 uses Theorem 7.2 and a semiclassical and quantitative version of his argument to obtain the lower bound $K(h) \geq \log(1/h)/C$.

Proof of Theorem 7.2. 1. We apply a TT^* argument which starts with defining

$$T : u \longmapsto \chi\varphi(P)e^{-itP/h}\,,$$

$$T : L^2(\mathbb{R}^n) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^n)\,,$$

so that (7.1.3) can be rewritten as

$$\|Tu\|^2_{L^2_{tx}} \le CK(h)\|u\|^2_{L^2_x},$$

which in turn is equivalent to

$$\|T^*f\|^2_{L^2_x} \le CK(h)\|f\|^2_{L^2_{tx}}.$$

This last inequality follows from showing that

$$TT^* = \mathcal{O}(K(h)) \ : \ L^2(\mathbb{R} \times \mathbb{R}^n) \longrightarrow L^2(\mathbb{R} \times \mathbb{R}^n)\,. \tag{7.1.6}$$

2. To obtain (7.1.6) we start by calculating the adjoint:

$$T^*f = \int_{\mathbb{R}} e^{isP/h}\varphi(P)\chi f(s)ds\,,$$

first defined for $f \in C^\infty_{\mathrm{c}}(\mathbb{R} \times \mathbb{R}^n)$. We have

$$TT^*f = \int_{\mathbb{R}} \chi e^{-i(t-s)P/h}\varphi(P)^2\chi f(s)ds\,.$$

This we can rewrite as

$$TT^*f = \left(\chi e^{-i\bullet P/h}\varphi(P)^2\chi\right) * (f(\bullet))\,(t)\,, \tag{7.1.7}$$

where $*$ denotes the convolution in the t variable.

3. We recall the semiclassical inverse Fourier transform:

$$\mathcal{F}^{-1}_{t\mapsto\lambda}\psi(\lambda) := \frac{1}{2\pi}\int e^{it\lambda/h}\psi(t)dt.$$

Then $\mathcal{F}^{-1}$ of $t \mapsto e^{-itP/h}\varphi(P)$ is formally equal equal to

$$h\delta(P-\lambda)\varphi(P)^2 = h\delta(P-\lambda)\varphi(\lambda)^2,$$

which can then be expressed using the Stone formula (B.1.11):

$$h\delta(P-\lambda)\varphi(\lambda) = \frac{1}{2\pi i}\sum_{\pm} \pm(P-\lambda \mp i0)^{-1}\varphi(\lambda)^2.$$

Returning to (7.1.7) and using the relation between the Fourier transforms and convolution (paying attention to the factor of $\sqrt{h}$ because of the unitarity of $\mathcal{F}$) we see that

$$TT^*f = (h/2\pi i)\mathcal{F}_{\lambda\mapsto t}\left(\left(\sum_{\pm} \pm\chi(P-\lambda\mp i0)^{-1}\varphi(\lambda)^2\chi\right)\mathcal{F}^{-1}_{t\mapsto\lambda}\,(f(t))\right)\,,$$

4. To conclude the proof we apply Plancherel's formula:

$$\begin{aligned}
&\|TT^* f\|_{L^2_{tx}} \\
&\leq h \left\| \left(\sum_{\pm} \pm \chi (P - \lambda \mp i0)^{-1} \chi \right) \varphi(\lambda)^2 \mathcal{F}^*_{t \mapsto \lambda} (f(t)) \right\|_{L^2_{\lambda x}} \\
&\leq 2h \sup_{\lambda} \|\varphi(\lambda)^2 \chi (P - \lambda - i0)^{-1} \chi \|_{L^2_x \to L^2_x} \|f\|_{L^2_{tx}} \\
&\leq 2K(h) \|f\|_{L^2_{tx}} ,
\end{aligned}$$

Here we used hypothesis (7.1.2), the assumptions on φ, and the basic fact that the norms of $\chi(P - \lambda \pm i0)^{-1}\chi$ are the same. This proves (7.1.6) and consequently (7.1.3). $\square$

Proof of Theorem 7.1. 1. We proceed by contraction using Theorem 7.2. That theorem shows that if for some nontrivial $u_0 \in L^2(\mathbb{R}^n)$ and

$$\varphi \in C^\infty_{\mathrm{c}}((E_0 - \delta, E_0 + \delta); [0,1]) , \quad \varphi(E_0) = 1 ,$$

$$\|\chi \varphi(P) \exp(-itP/h) u_0\|^2_{L^2_{tx}} \geq K(h) \|u_0\|_{L^2_x} , \tag{7.1.8}$$

then

$$\sup_{|E - E_0| < \delta} \|\chi (P - E - i0)^{-1} \chi\|_{L^2 \to L^2} \geq \frac{K(h)}{Ch} .$$

Hence we need to show that for χ satisfying

$$\chi \in C^\infty_{\mathrm{c}}(T^*\mathbb{R}^n) , \quad \chi \equiv 1 \ \text{ near } \pi(K_{E_0}) \ , \tag{7.1.9}$$

(7.1.8) holds with

$$K(h) = c \log \frac{1}{h} ,$$

where c is independent of δ and $0 < h < h_0(\delta)$.

2. Functional calculus for pseudodifferential operators (see [**DS99**, Chapter 8] or [**Zw12**, §14.3]) shows that

$$\begin{aligned}
&\varphi(P(h)) \chi(x)^2 \varphi(P(h)) = a^w(x, hD) , \quad a \in \mathcal{S}(T^*\mathbb{R}^n) , \\
&a(x,\xi) = \chi(x)^2 \varphi(p(x,\xi))^2 + \mathcal{O}(h \langle x \rangle^{-\infty} \langle \xi \rangle^{-\infty}) .
\end{aligned} \tag{7.1.10}$$

We put

$$a^w_t(x, hD) := e^{itP/h} a^w(x, hD) e^{-itP/h} .$$

Egorov's Theorem [**Zw12**, Theorem 11.12] shows that for

$$0 < t < \alpha \log \frac{1}{h} , \tag{7.1.11}$$

with α sufficiently small, independent of δ,

$$\begin{aligned} & a_t \in S_\gamma(T^*\mathbb{R}^n)\,,\ 0 < \gamma < \tfrac{1}{2}\,, \\ & a_t - (\exp tH_p)^* a \in h^{2-3\gamma} S_\gamma(T^*\mathbb{R}^n)\,, \end{aligned} \tag{7.1.12}$$

with all the symbol estimates uniform for t satisfying (7.1.11). See Remark 1 at the end of §E.1.7 for a discussion of such symbol classes.

3. With this notation we have

$$\begin{aligned} \|\chi\varphi(P)\exp(-itP/h)u_0\|^2_{L^2_{tx}} &= \int_{\mathbb{R}} \|\chi\varphi(P)e^{-itP/h}u_0\|_{L^2_x}dt \\ &\geq \int_0^{\alpha\log(1/h)} \|\chi\varphi(P)e^{-itP/h}u_0\|_{L^2_x}dt \\ &= \int_0^{\alpha\log(1/h)} \langle a_t^w(x,hD)u_0,u_0\rangle_{L^2_x}dt\,. \end{aligned} \tag{7.1.13}$$

Hence, it remains to find u_0 such that

$$\langle a_t^w(x,hD)u_0,u_0\rangle \geq \tfrac{1}{2}\,, \quad \|u_0\|_{L^2(\mathbb{R}^n)} = 1\,, \tag{7.1.14}$$

uniformly for

$$0 < h < h_0\,, \quad 0 < t < \alpha\log\frac{1}{h}\,.$$

4. To find u_0 satisfying (7.1.14) we choose $(x_0,\xi_0) \in K_{E_0}$ and take for u_0 a coherent state concentrated at (x_0,ξ_0):

$$u_0(x) = (2\pi h)^{-n/4}\exp\left(\frac{i}{h}\left(\langle x-x_0,\xi_0\rangle + \frac{i}{2}|x-x_0|^2\right)\right)\,. \tag{7.1.15}$$

Since K_{E_0} is invariant under the flow

$$\exp(tH_p)(x_0,\xi_0) \in K_{E_0}\,.$$

The assumption (7.1.9) and the fact that $\varphi(E_0) = 1$ show that

$$(\exp tH_p)^*[\chi^2\varphi(p)](x_0,\xi_0) = 1\,,$$

for all time. Consequently (7.1.10) and (7.1.12) give

$$a_t(x_0,\xi_0) = 1 + \mathcal{O}(h^{\frac{1}{2}}), \tag{7.1.16}$$

uniformly for $0 < t < \alpha\log 1/h$.

The properties of $\langle a_t^w(x,hD)u_0,u_0\rangle$ follow from

LEMMA 7.3. *Suppose that u_0 is given by (7.1.15) and that $b \in S_\gamma$, $0 < \gamma < \frac{1}{2}$. Then*

$$\begin{aligned} & \langle b^w(x,hD)u_0,u_0\rangle = b(x_0,\xi_0) + e(h)\,, \\ & |e(h)| \leq C_n h^{\frac{1}{2}} \max_{|\alpha|=1}\sup_{T^*\mathbb{R}^n} |\partial^\alpha b| \leq C_n(b) h^{\frac{1}{2}-\gamma}\,. \end{aligned} \tag{7.1.17}$$

Proof. 1. Using the definition of u_0 (7.1.15) and making a change of variables $x = z + w$, $y = z - w$ we obtain

$$
\begin{aligned}
&\langle b^w(x, hD)u_0, u_0\rangle \\
&\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b((x+y)/2, \xi) e^{\frac{i}{h}\langle x-y, \xi\rangle} u_0(y)\overline{u_0}(x)\, dyd\xi dx \\
&\quad = \frac{2^n}{(2\pi h)^{\frac{3n}{2}}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(z, \xi) e^{\frac{2i}{h}\langle w, \xi-\xi_0\rangle} e^{-\frac{1}{h}(|z-x_0|^2+|w|^2)}\, dwd\xi dz.
\end{aligned}
$$

2. For each fixed z and ξ, the integral in w is

$$
\begin{aligned}
\int_{\mathbb{R}^n} e^{\frac{2i}{h}\langle w, \xi-\xi_0\rangle} e^{-\frac{1}{h}|w|^2}\, dw &= e^{-\frac{1}{h}|\xi-\xi_0|^2} \int_{\mathbb{R}^n} e^{-\frac{1}{h}|w+i(\xi-\xi_0)|^2}\, dw \\
&= 2^{-\frac{n}{2}}(2\pi h)^{\frac{n}{2}} e^{-\frac{1}{h}|\xi-\xi_0|^2}
\end{aligned}
$$

3. Therefore

$$
\begin{aligned}
\langle b^w(x, hD)u_0, u_0\rangle &= \frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} b(z, \xi) e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)}\, dzd\xi \\
&= b_0(x_0, \xi_0)\frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)}\, dzd\xi + e(h) \\
&= a_n b(x_0, \xi_0) + e(h)\,,
\end{aligned}
$$

where

$$
e(h) := \frac{2^{\frac{n}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (b(z, \xi) - b(x_0, \xi_0)) e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)}\, dzd\xi,
$$

and

$$
a_n := \frac{2^{\frac{n}{2}}}{(2\pi)^n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-(|z|^2+|\xi|^2)}\, dzd\xi.
$$

Taking $b \equiv 1$ and recalling that $\|u_0\|_{L^2} = 1$, we deduce that $a_n(h) = 1$.

4. To see that $e(h)$ satisfies the estimate of (7.1.17) we note that

$$
|b(z, \xi) - b(x_0, \xi_0)| \leq \sup_{|\alpha|=1}\sup_{T^*\mathbb{R}^n} |\partial^\alpha b|(|z - x_0| + |\xi - \xi_0|),
$$

and that (since $|x|e^{-|x|^2/2h} \leq h^{\frac{1}{2}}$)

$$
(|z - x_0| + |\xi - \xi_0|)e^{-\frac{1}{h}(|z-x_0|^2+|\xi-\xi_0|^2)} \leq h^{\frac{1}{2}} e^{-\frac{1}{2h}(|z-x_0|^2+|\xi-\xi_0|^2)}.
$$

Definition of $e(h)$ then shows that

$$
|e(h)| \leq h^{\frac{1}{2}} \sup_{|\alpha|=1}\sup_{T^*\mathbb{R}^n} |\partial^\alpha b| 2^n a_n
$$

which gives (7.1.17). ∎

End of proof of Theorem 7.1. 5. We apply the lemma to $b = a_t$ which together with (7.1.16) gives

$$\langle a_t^w(x, hD)u_0, u_0\rangle = a_t(x_0, \xi_0) + \mathcal{O}(h^{\frac{1}{2}}) = 1 + \mathcal{O}(h^{\frac{1}{2}}),$$

again uniformly for $0 < t < \alpha \log(1/h)$. Hence (7.1.14) holds. Using (7.1.13) we obtain

$$\|\chi\varphi(P)\exp(-itP/h)u_0\|^2_{L^2_{tx}} \geq \frac{\alpha}{2}\log\frac{1}{h},$$

which is (7.1.8) with $K(h) = c\log(1/h)$, as needed for (7.1.1). $\square$

7.2. SEMICLASSICAL GROWTH ESTIMATES

In this section we present *local* bounds on the resolvent and on the number of resonances for operators of the form

$$P = P(h) = -h^2\Delta_g + V, \tag{7.2.1}$$

where g is a smooth Riemannian metric on $\mathbb{R}^n$ satisfying $g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n)$ and $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$. If n is odd global bounds for compact black box perturbations were already presented in §4.3. The advantage of the argument here is that it applies to more general operators and to even dimensions – see §7.6 for pointers to the literature. We will also use some methods of the proof in §7.4 devoted to a *local* semiclassical trace formula for resonances.

THEOREM 7.4. *Suppose that $P(h)$ is given by (7.2.1) and that the set of resonances of P_V is denoted $\operatorname{Res}(P(h))$.*

If $\Omega \Subset \{\operatorname{Re} z > 0\}$ then

$$|\operatorname{Res}(P(h)) \cap \Omega| \leq C_\Omega h^{-n}. \tag{7.2.2}$$

Proof. The proof is based on the characterization of resonances using the complex scaling method and then comparing the scaled operator P_θ with an operator $\widetilde{P}_\theta$ such that $\widetilde{P}_\theta - z$ is invertible for $z \in \Omega$.

1. We first choose $0 < \theta < \pi/4$ such that $\Omega \Subset \{\arg z > -2\theta\}$. By Theorem 4.38 the resonances of P coincide with the eigenvalues of the scaled operator P_θ (we recall that $P_\theta - z$ is a Fredholm operator for z in a neighbourhood of Ω). Let $\chi \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$ be equal to 1 on $B(0, R_2)$, where R_2 will be chosen sufficiently large. We then define

$$\widetilde{P}_\theta := P_\theta - iM\chi(hD)\chi(x)^2\chi(hD), \tag{7.2.3}$$

where to define $\chi(hD)$ and $\chi(x)$ we identified Γ_θ with $\mathbb{R}^n$ using (4.5.5).

2. We claim that if M and R_2 in the definition of $\widetilde{P}_\theta$ are large enough then

$$(\widetilde{P}_\theta - z)^{-1} = \mathcal{O}(1) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta). \tag{7.2.4}$$

To see this we calculate the semiclassical symbol of $\widetilde{P}_\theta$ (see §E.1.1) using the identification of Γ_θ with $\mathbb{R}^n$:

$$\mathbb{R}^n \ni x \longmapsto x + i\partial_x F_\theta(x),$$

where $F_\theta : \mathbb{R}^n \to \mathbb{R}$ is a convex function in (4.5.6). We choose F_θ so that $F_\theta(x) = 0$ for $|x| \leq R_1$ and that $\Gamma_\theta \setminus B_{\mathbb{C}^n}(0, 2R_1) = e^{2i\theta}\mathbb{R}^n$. With this notation the symbol of $\widetilde{P}_\theta$, $\sigma_h(\widetilde{P}_\theta)$, is given by

$$\begin{cases} |\xi|_g^2 + V(x) - iM\chi(x)^2\chi(\xi)^2, & |x| \leq R_1, \\ ((I + iF_\theta''(x))^{-1}\xi) \cdot ((I + iF_\theta''(x))^{-1}\xi) - iM\chi(x)^2\chi(\xi)^2, & R_1 \leq |x| \leq 2R_1, \\ e^{-2i\theta}|\xi|^2 - iM\chi(x)^2\chi(\xi)^2, & |x| \geq 2R_1. \end{cases}$$

3. For $z \in \Omega \Subset \{\arg z > -2\theta\}$, $0 < \theta < \pi/4$, we have

$$|e^{-2i\theta}|\xi|^2 - z - iM\chi(x)^2\chi(\xi)^2| \geq (1 + |\xi|^2)/C. \tag{7.2.5}$$

In fact, the left hand size cannot vanish since $\cos 2\theta\, |\xi|^2 = \operatorname{Re} z > 0$ implies

$$\sin 2\theta\, |\xi|^2 + \operatorname{Im} z + M\chi(x)^2\chi(\xi)^2 > \sin 2\theta\, |\xi|^2 - \operatorname{Re} z \tan 2\theta = 0.$$

Hence the left hand side of (7.2.5) is bounded from below for ξ in compact sets. Since for $|\xi| \to \infty$ the asympotic behaviour is given by $|\xi|^2$, (7.2.5) holds.

To control $\sigma_h(\widetilde{P}_h)$ for $|x| \leq R_1$ we choose $R_2 > R_1$ so that

$$|\xi| > R_2/2 \implies |\xi|_g^2 + V(x) > 1 + \max_{z \in \Omega} \operatorname{Re} z.$$

If we also choose $M > \max_{z\in\Omega}(-\operatorname{Im} z)$ then for $z \in \Omega$ and $|x| \leq R_1$,

$$||\xi|_g^2 + V(x) - iM\chi(x)^2\chi(\xi)^2 - z| \geq (1 + |\xi|^2)/C.$$

It remains to consider the symbol for $R_1 \leq |x| \leq 2R_1$, where we assume that $R_2 > 2R_1$. For that we go back to (4.5.16) in the proof of Theorem 4.32. It shows that with $\eta = (I + (F_\theta'')^2)^{-1}\xi$, and for $R_1 \leq |x| \leq R_2$ (so that $\chi(x) = 1$),

$$\sigma_h(\widetilde{P}_h - z) = |\eta|^2 - |F_\theta''(x)\eta|^2 - i(2\langle F_\theta''(x)\eta, \eta\rangle + M\chi(\xi)) - z$$

For $|\xi| \leq R_2$ the term $M\chi(\xi)^2 = M$ provides a lower bound. For $|\xi| \geq R_2$ have (note that since $F_\theta''(x)$ positive semidefinite $|2\langle F_\theta''(x)\eta, \eta\rangle + M\chi(\xi)| \geq |2\langle F_\theta''(x)\eta, \eta\rangle|$)

$$\begin{aligned} |\sigma_h(\widetilde{P}_h - z)| &\geq ||\eta|^2 - |F_\theta''(x)\eta|^2 - i(2\langle F_\theta''(x)\eta, \eta\rangle| - |z| \\ &\geq |\xi|^2/C - |z| \geq (1 + |\xi|^2)/C, \end{aligned}$$

if R_2 is large enough.

We conclude that $\widetilde{P}_\theta - z$ is elliptic as an element of $\Psi_h^2(\Gamma_\theta)$. Theorem E.32 then shows that for h sufficiently small (7.2.4) holds.

4. For $z \in \Omega$ we can now write

$$
\begin{aligned}
P_\theta - z &= (\widetilde{P}_\theta - z)(I + iM(\widetilde{P}_\theta - z)^{-1}\chi(hD)\chi(x)^2\chi(hD)) \\
&=: (\widetilde{P}_\theta - z)(I + K(z)),
\end{aligned}
\tag{7.2.6}
$$

where $K(z) \in \Psi_h^{\mathrm{comp}}$.

Using Theorem C.11 we conclude

$$
\frac{1}{2\pi i} \operatorname{tr} \oint_z (\zeta - P_\theta)^{-1} d\zeta = \frac{1}{2\pi i} \operatorname{tr} \oint_z (I + K(\zeta))^{-1} K'(\zeta) d\zeta =: m_K(z),
$$

where the integral is over a small circle around z, not containing any eigenvalues of P_θ other than possibly z. This means that the eigenvalues of P_θ coincide with multiplicities with the zeros of $\det(I + K(z))$. (The operator $\chi(hD)\chi(x)^2\chi(hD)$ is of trace class as $\chi \in C_c^\infty(\mathbb{R}^n)$).

5. Hence the estimate (7.2.2) follows from the estimate on the number of zeros of

$$
k(z) := \det(I + K(z)),
$$

and for that we use Jensen's formula applied as in (D.1.11):

$$
\sum_{z \in \Omega} m_K(z) \le C \sup_{z \in \Omega'} \log |k(z)| - C \log |k(z_0)|,
\tag{7.2.7}
$$

where $\Omega \Subset \Omega' \Subset \operatorname{Re} z > 0$ where Ω' is a simply connected open set (for instance a rectangle) and $z_0 \in \Omega'$. We take $\Omega' \subset \{-2\theta < \arg z < 2\pi - 2\theta\}$ with the property that (7.2.4) holds for $z \in \Omega'$ and that we can find $z_0 \in \Omega'$ with $\operatorname{Im} z_0 > \delta > 0$. That is certainly possible following the argument in step 2.

6. To apply (7.2.7) we first estimates $k(z)$ from above on Ω' using (B.5.11):

$$
\begin{aligned}
|\log k(z)| &\le \|K(z)\|_{\mathcal{L}^1} \\
&\le \|(\widetilde{P}_\theta - z)^{-1}\| \|\chi(hD)\chi(x)^2\chi(hD)\|_{\mathcal{L}^1} \\
&\le C \operatorname{tr}\left[\chi(hD)\chi(x)^2\chi(hD)\right],
\end{aligned}
\tag{7.2.8}
$$

where the trace class norm equals to the trace as the operator is positive semidefinite. Pseudodifferential calculus [**Zw12**, Theorem 4.11] shows that $\chi(hD)\chi(x)^2\chi(hD) = c(x, hD)$, with $c \in \mathscr{S}(\mathbb{R}^{2n})$. Hence,

$$
\operatorname{tr}\left[\chi(hD)\chi(x)^2\chi(hD)\right] = \frac{1}{(2\pi h)^n} \iint c(x,\xi) dx d\xi = \mathcal{O}(h^{-n}).
$$

We conclude that

$$
\log |k(z)| = \mathcal{O}(h^{-n}), \quad z \in \Omega'.
\tag{7.2.9}
$$

7. It remains to obtain a lower bound at some $z_0 \in \Omega'$ and we take z_0 with $\operatorname{Im} z_0 > \delta > 0$. Arguing as in Step 2 we see that $P_\theta - z_0$ is elliptic as an element of $\Psi_h^2(\Gamma_\theta)$. Hence for h small enough

$$(P_\theta - z_0)^{-1} = \mathcal{O}(1) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta). \tag{7.2.10}$$

Using (7.2.6) we get

$$\begin{aligned}(I + K(z_0))^{-1} &= (P_\theta - z_0)^{-1}(\widetilde{P}_\theta - z_0) \\ &= I - iM(P_\theta - z_0)^{-1}(\chi(hD)\chi(x)^2\chi(hD)) \\ &:= I + \widetilde{K}(z_0).\end{aligned}$$

Using (7.2.10) and arguing as in Step 5 we see that

$$\log|\det(I + \widetilde{K}(z_0))| \le Ch^{-n}.$$

But that gives,

$$\log|k(z_0)| = -\log|\det(I + \widetilde{K}(z_0))| \ge -Ch^{-n}. \tag{7.2.11}$$

Inserting this and (7.2.9) into (7.2.7) gives the bound (7.2.2). □

The next result provides a bound on the cut-off resolvent away from resonances. It will be crucial in showing that existence of a localized quasimode implies existence of a resonance nearby – see §7.3.

THEOREM 7.5 (Exponential resolvent bounds). *Suppose that* $P = P(h) = -h^2\Delta_g + V$ *where* $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ *and* $g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$. *Let* $R(z,h) := (P(h) - z)^{-1}$, $\operatorname{Im} z > 0$ *be the outgoing resolvent which continues meromorphically to* $\operatorname{Im} z < 0$.

Suppose that $\Omega \Subset \{\operatorname{Re} z > 0\}$, $\chi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ *and that* $h \mapsto \delta(h)$ *is a positive function. Then there exist constants* $A = A(\Omega)$ *and* $h_0 = h_0(\Omega)$ *such that for* $0 < h < h_0$,

$$\begin{gathered}\|\chi R(z,h)\chi\|_{L^2\to L^2} \le A\exp\left(Ah^{-n}\log\frac{1}{\delta(h)}\right), \\ \forall\, z \in \Omega \setminus \bigcup_{w \in \operatorname{Res}(P(h))} D(w,\delta(h)).\end{gathered} \tag{7.2.12}$$

REMARKS. 1. The proof of Theorem 7.5 combined with the methods of §4.3 gives the same result for black box Hamiltonians. The assumptions on $P(h)$ can be weakened further and the bound holds in great generality – see [**TZ98**].

2. If $P = -\Delta_g + V$ (or P is a general black box Hamiltonian independent of h) then considering $P(h) := h^2 P$ and rescaling provides the bound on the

meromorphic continuation of the cut-off resolvent ($R(\lambda) = (P - \lambda^2)$):

$$
\begin{aligned}
&\|\chi R(\lambda)\chi|_{L^2 \to L^2} \leq A \left(\exp A|\lambda|^2 \log \frac{1}{\delta(\lambda)} \right), \\
&|\operatorname{Re} \lambda| > 1, \quad \lambda \notin \bigcup_{\zeta^2 \in \operatorname{Res}(P)} D(\zeta, \delta(\zeta)).
\end{aligned}
\tag{7.2.13}
$$

3. The bound on the number of resonances (7.2.2) shows that the sum of radii of $D(w, \delta(h))$, $w \in \Omega' \cap \operatorname{Res}(P(h))$, where $\Omega \Subset \Omega' \Subset \{\operatorname{Re} z > 0\}$, is bounded by $\delta(h)h^{-n}$. Hence if $\delta(h) = o(h^n)$ we can always find Ω', (h-independently) close to Ω, such that $d(\partial\Omega', \operatorname{Res}(P(h))) > \delta(h)$. In particular, the estimate (7.2.12) holds for $z \in \partial\Omega'$.

Proof. 1. Theorem 4.37 shows that for $\chi \in C_c^\infty(B(0, R_1))$ (where R_1 is as in (4.5.1))

$$
\chi R(z, h)\chi = \chi(P_\theta - z)^{-1}\chi,
$$

and hence the estimate above follows from the estimate on the norm of $(P_\theta - z)^{-1}$, $0 < \theta < \pi/4$, $z \in \Omega \Subset \{\arg z > -2\theta\}$.

2. Using (7.2.4) and (7.2.6) we write

$$
\begin{aligned}
\|(P_\theta - z)^{-1}\|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} &= \|(I + K(z))^{-1}(\widetilde{P}_\theta - z)^{-1}\|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} \\
&\leq C\|(I + K(z))^{-1}\|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)}.
\end{aligned}
$$

To estimate the last norm we recall that $K(z)$ is of trace class and hence we can apply (B.5.21):

$$
\|(I + K(z))^{-1}\|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} \leq \det(I + K(z))^{-1} \det(I + [K(z)^* K(z)]^{\frac{1}{2}}).
$$

As in (7.2.8) we see that

$$
\det(I + [K(z)^* K(z)]^{\frac{1}{2}}) \leq C \exp(Ch^{-n}).
$$

Hence the estimate (7.2.12) will follow from the corresponding estimate on $\det(I + K(z))^{-1}$.

3. From (7.2.8), (7.2.11) and (D.1.13) we now obtain that for

$$
z \in \Omega \setminus \bigcup_{w \in \operatorname{Res}(P(h))} D(w, \delta)
$$

we have the bound

$$
\| \log|\det(I + K(z))| \geq -Ch^{-n} \log \frac{1}{\delta},
$$

and, as we indicated above, that proves (7.2.12). $\square$

7.3. FROM QUASIMODES TO RESONANCES

In this section we will again consider operators of the form (7.2.1), $P(h) = -h^2\Delta_g + V$, with $V \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ and $g^{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$. The results hold in greater generality – see §6.6 for references but the ideas behind the proofs are the same as in the arguments presented in this section.

In the presence of classical trapping it is sometimes easy to construct approximate solutions to $(P(h) - E(h))u(h) = 0$, where $E(h)$ is an energy level. That means finding $u(h) \in C_c^\infty(\mathbb{R}^n)$ such that

$$(P(h) - E(h))u(h) = \varepsilon(h), \quad \|u\|_{L^2} = 1, \tag{7.3.1}$$

where $\varepsilon(h) = \mathcal{O}(h^\infty)$ or $\varepsilon(h) = \mathcal{O}(e^{-S_0/h})$, $S_0 > 0$.

EXAMPLE. Suppose that for $E > 0$ the energy surface of $P(h) = -h^2\Delta + V$,

$$\Sigma_E = \{(x,\xi) : |\xi|^2 + V(x) = E\},$$

satisfies

$$\begin{gathered} \Sigma_E = \Sigma_0 \cup \Sigma_\infty, \quad \Sigma_0, \Sigma_\infty \text{ are closed in } T^*\mathbb{R}^n, \quad \Sigma_0 \cap \Sigma_\infty = \emptyset, \\ \Sigma_0 \neq \emptyset, \quad \Sigma_0 \Subset \pi^{-1}(B(0,R)), \quad \operatorname{supp} V \subset B(0,R), \\ (x,\xi) \in \Sigma_\infty \implies |\exp(tH_p)(x_0,\xi_0)| \to \infty, \end{gathered} \tag{7.3.2}$$

see Fig. 7.1 for an example. (Here, as always, $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the natural projection, $\pi(x,\xi) = x$.)

Since Σ_0 and Σ_∞ are disjoint, there exist open sets $\Omega_0 \Subset \Omega_1 \Subset \mathbb{R}^n$ such that

$$\pi(\Sigma_0) \subset \Omega_0, \quad V|_{\Omega_1 \setminus \Omega_0} > E.$$

We can then find $V_1 \in C^\infty(\mathbb{R}^n;\mathbb{R})$ with the following properties:

$$V_1(x) = V(x),\ x \in \Omega_1, \quad V_1(x) > E,\ x \notin \Omega_0, \quad V_1(x) = \alpha|x|^2,\ |x| > R,$$

for some $\alpha > 0$. The operator

$$P_1(h) = -h^2\Delta_g + V_1$$

is essentially self-adjoint and has discrete spectrum – see for instance [**Zw12**, 6.3]:

$$(P_1(h) + V_1)u_j(h) = E_j(h)), \quad \|u_j(h)\|_{L^2} = 1, \quad j \in \mathbb{N},$$

and

$$|\{E_j(h)\}_{j=0}^\infty \cap [E-\delta, E+\delta]| = \frac{1+o(1)}{(2\pi h)^n} \int_{||\xi|_g^2 + V(x) - E| \leq \delta} dx d\xi, \tag{7.3.3}$$

see [**Zw12**, Theorems 6.8, 14.11].

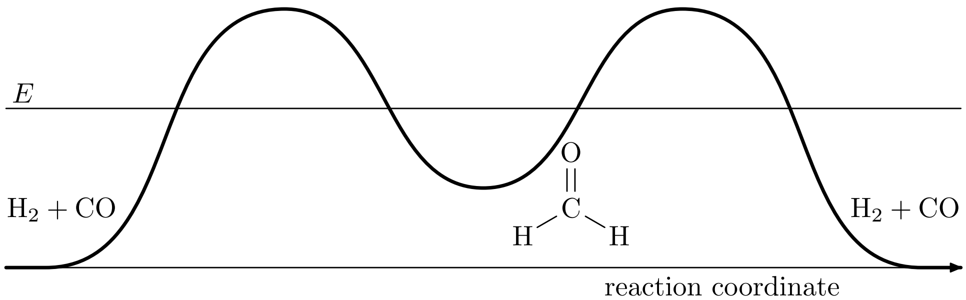


Figure 7.1. An example of a potential for which the assumption of the example in §7.3 are satisfied. It is a classical example from molecular dynamics: a cross-section of the energy surface of formaldehyde, H_2CO. That molecule, considered as a resonant state, has a very long lifetime at energy E due to the strong barrier. That is not surprising considering the well known properties of formaldehyde.

The eigenfunctions of $P_1(h)$ are *exponentially small* in the classically forbidden region $V_1(x) > E + \varepsilon$, $\varepsilon > 0$ thanks to *Agmon estimates* – see [**Zw12**, Theorem 7.4]. In particular, δ small enough,

$$\|u_j(h)\|_{H_h^2(\mathbb{R}^n \setminus \Omega_0)} = \mathcal{O}(e^{-S_0/h}), \quad \text{for } E_j(h) \in [E - \delta, E + \delta]. \tag{7.3.4}$$

Let $\chi \in C_c^\infty(\mathbb{R}^n; [0,1])$ satisfy

$$\chi(x) = 1, \ x \in \Omega_0, \quad \operatorname{supp} \chi \subset \Omega_1.$$

We then see from (7.3.4) and the fact that V coincides with V_1 in Ω_1,

$$(P(h) - E_j(h))(\chi u_j(h)) = \mathcal{O}(e^{-S_0/h})_{L^2}, \tag{7.3.5}$$

that is, (7.3.1) is satisfied with

$$u(h) := \chi u_j(h) / \|\chi u_j(h)\|, \quad \varepsilon(h) = \mathcal{O}(e^{-S_0/h}).$$

Here we notice that (7.3.4) implies that $\|\chi u_j\|_{L^2} = 1 + \mathcal{O}(e^{-S_0/h})$.

From (7.3.3) we see that we have obtained $\sim h^{-n}$ quasimodes. □

If $P_1(h) = -h^2 \Delta_g + V$ is an operator on a *compact* manifold, or $P_1(h)$ is the operator in the example above, then the spectrum of $P_1(h)$ is discrete. Existence of a quasimode (7.3.1) and the spectral theorem immediately imply that there exists an eigenvalue of $P_1(h)$, $E_0(h)$, such that $|E(h) - E_0(h)| \leq \|\varepsilon(h)\|_{L^2}$.

If $E_0(h) > 0$ then we know Theorem 4.18 that we cannot have L^2 solutions to $(P(h) - E_0(h))u_0(h) = 0$ yet existence of a quasimode (7.3.1) should imply existence of a long living quantum state. That is indeed the case as shown in the next result:

THEOREM 7.6 (From quasimodes to resonances). *Suppose that there exists a family $u(h) \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp} u \subset U \Subset \mathbb{R}^n$ where U is independent of h and that for some $E(h) = E_0 + o(1)$, $E_0 > 0$,*

$$(P(h) - E(h))u(h) = \varepsilon_0(h), \qquad \|u(h)\|_{L^2} = 1,$$
$$\varepsilon_0(h) = \mathcal{O}(h^\infty)_{L^2} \quad or \quad \varepsilon_0(h) = \mathcal{O}(e^{-S_0/h})_{L^2}, \;\; S_0 > 0..$$

Then for $0 < h < h_0$ there exists $z(h) \in \operatorname{Res}(P(h))$ such that

$$\begin{gathered} |z(h) - E(h)| \leq \varepsilon(h), \\ \varepsilon(h) = \mathcal{O}(h^\infty) \quad or \quad \varepsilon(h) = \mathcal{O}(e^{-S/h}), \;\; \forall\, S < S_0, \end{gathered} \tag{7.3.6}$$

respectively.

EXAMPLE. Suppose that $P(h) = -h^2\Delta + V$ where V satisfies the assumption (7.3.2) of the example in the beginning of this section. The construction of quasimodes (7.3.5) in that example, Theorem 7.6 and (7.3.3), show that there exist $E(h) = E + o(1)$, and

$$z(h) \in \operatorname{Res}(P(h)), \quad |E(h) - z(h)| \leq e^{-S/h}, \quad S > 0.$$

In many cases very precise form of $E(h)$ can be given by using semiclassical spectral theory for $P_1(h)$ – see for instance [**DS99**, Chapter 3] and references given there. Resonances obtained this way are sometimes called *shape resonances* and, as we indicated already, they can be analyzed more precisely under stronger assumptions – see [**FLM11**] and references given there.

Other constructions can be use to obtain quasimodes and consequently resonances. For instance we can take $P(h) = -h^2\Delta_g$ and construct quasimodes associated to *elliptic* geodesics – see Fig. 7.2 for an illustration and [**TZ98**],[**St99**] for references. In particular, the same construction can be used for obstacle problems. $\square$

REMARKS. 1. Under additional assumptions much finer estimates on the location of $z(h)$ are possible. One can already see this in the simple example presented in Theorem 2.27. For the analysis of the general "well-in-the-island" case see Helffer–Sjöstrand [**HS86**] and for recent advances and references see Fujiie–Lahmar-Benbernou–Martinez [**FLM11**], Grigis–Martinez [**GM14**] and Dalla Venezia–Martinez [**DM17**].

2. The estimate in (7.3.6) can be improved and the localization of the imaginary part is much better than the localization of the real part [**TZ98**],[**St99**]. That can be seen from the proof below.

3. Theorem 7.6 does not address the issue of multiplicities and hence we cannot immediately deduce from (7.3.3) that we have $\sim h^{-n}$ resonances close to the real axis. That was remedied by Stefanov in [**St99**] and we outline his argument in Exercise 7.1.

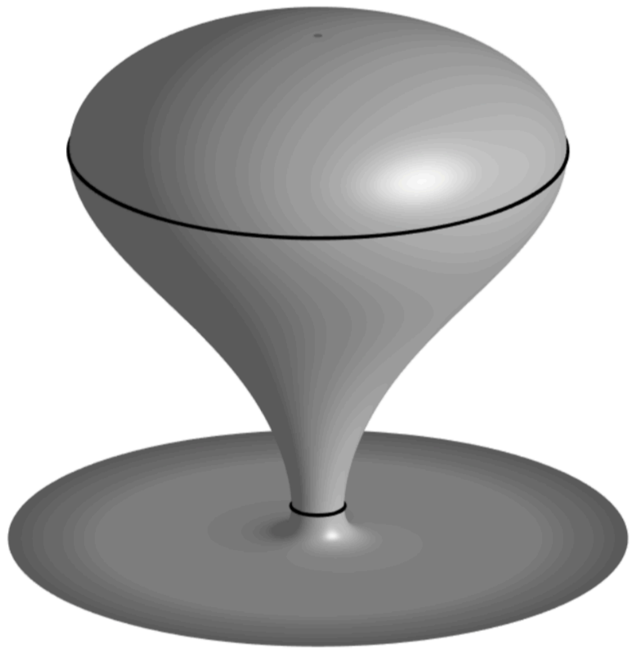


Figure 7.2. A surface with a Euclidean end (the "stand"). The elliptic trajectory around the largest cross-section generates real quasimodes. The hyperbolic trajectory around the "neck" does not produce resonances near the real axis – that is related to the results of §6.3.

The following lemma will play a crucial role in the proof of Theorem 7.6:

LEMMA 7.7 (Semiclassical maximum principle). *Suppose H is a Hilbert space and that $z \mapsto Q(z,h) \in \mathcal{L}(H)$, $0 < h < 1$, is a holomorphic family of operators in a neighbourhood of*

$$
\begin{gathered}
z \in \Omega(h) := (2a(h) - b(h), 2b(h) - a(h)) + i(-\delta(h)h^{-L}, \delta(h)), \\
a(h) < b(h), \quad 0 < \delta(h) < 1, \quad (b(h) - a(h))^2 \geq Ch^{-3L}\delta(h)^2,
\end{gathered}
\tag{7.3.7}
$$

for some $L > 0$ and $C > 0$. Then,

$$
\begin{gathered}
\|Q(z,h)\| \leq \exp(Ch^{-L}), \quad z \in \Omega, \\
\text{and} \\
\|Q(z,h)\| \leq 1/\operatorname{Im} z, \quad \operatorname{Im} z > 0, \ z \in \Omega,
\end{gathered}
\tag{7.3.8}
$$

implies

$$\|Q(E,h)\| \leq e^{C+1}\delta(h)^{-1}, \quad a(h) < E < b(h). \tag{7.3.9}$$

Proof. We apply Lemma D.1 to the holomorphic family

$$F(z,h) := F_{f,g}(z,h) = \langle Q(z - \tfrac{a(h)+b(h)}{2}, h)g, f\rangle_H, \quad \|f\|_H = \|g\|_H = 1,$$

with

$$R = b(h) - a(h), \quad \delta_+ = \delta(h), \quad \delta_- = \delta(h)h^{-L},$$
$$M = M_- = \exp(Ch^{-L}), \quad M_+ = 1/\delta(h).$$

The assumption (D.1.3) is verified as

$$R^2\delta_-^{-2} = (b(h) - a(h))^2\delta(h)^{-2}h^{2L} \geq Ch^{-L} = \log M.$$

Hence (D.1.4) shows that for $\operatorname{Im} z = 0$, $|z| \leq R$

$$\begin{aligned} |F(z,h)| &\leq e e^{Ch^{-L}\delta_+/(\delta_+ + \delta_-)}\delta_+^{-\delta_-/(\delta_+ + \delta_-)} \\ &= e e^{C/(1+h^L)}\delta(h)^{-1/(1+h^L)} \leq e^{C+1}\delta(h)^{-1}. \end{aligned} \tag{7.3.10}$$

Since $\|Q(z,h)\| = \sup_{f,g}|F_{f,g}(z,h)|$, the lemma follow. $\square$

REMARK. The proof shows that we also have

$$\|Q(z,h)\| \leq e^{2C+2}\delta(h)^{-1} \ \text{ for } |\operatorname{Re} z| \leq R \text{ and } |\operatorname{Im} z| \leq \delta(h). \tag{7.3.11}$$

That follows from using (D.1.5) applied with $|y| \leq \delta_+ = \delta(h)$. This remark will be useful in §7.5.

Proof of Theorem 7.6. 1. We argue by contradiction. Suppose

$$\chi \in C_c^\infty(\mathbb{R}^n; [0,1]), \quad \chi(x) = 1, \ x \in U.$$

Then for complex scaling starting outside of U,

$$\begin{aligned} \chi(P(h) - E(h))^{-1}\chi\varepsilon_0(h) &= \chi(P_\theta(h) - E(h))^{-1}\chi(P(h) - E(h))u(h) \\ &= \chi(P_\theta(h) - E(h))^{-1}(P_\theta(h) - E(h))u(h) \\ &= \chi u(h) = u(h). \end{aligned}$$

Hence if we show that

$$\operatorname{Res}(P(h)) \cap D(E(h), \varepsilon(h)) = \emptyset \tag{7.3.12}$$

implies

$$\|\chi(P(h) - E(h))^{-1}\chi\|_{L^2 \to L^2} < \tfrac{1}{2}\|\varepsilon_0(h)\|^{-1}, \tag{7.3.13}$$

we obtain a contradiction to $\|u(h)\| = 1$.

2. We can assume that for some c_0

$$\|\varepsilon_0(h)\| \geq e^{-c_0/h}$$

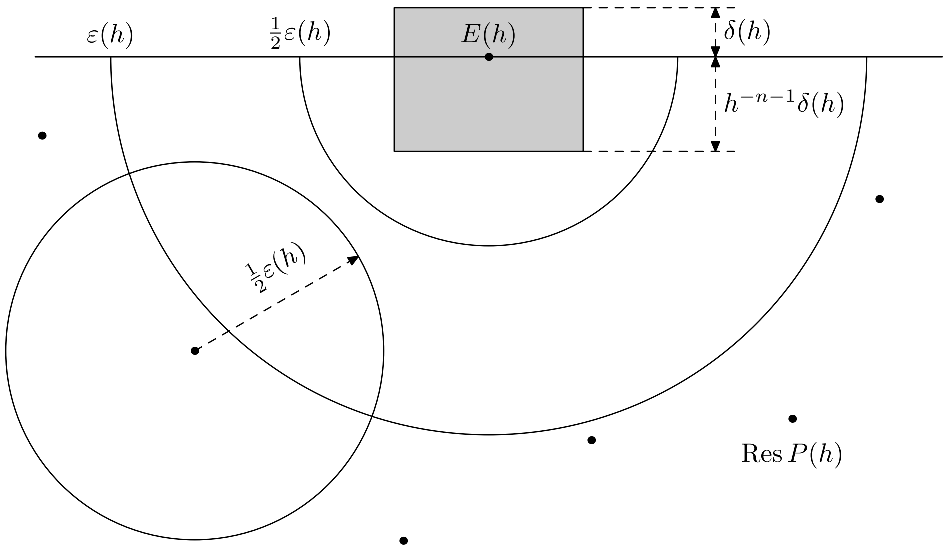


Figure 7.3. The two discs $D(E(h), 2^{-j}\varepsilon(h))$, $j = 0, 1$ used in the proof of Theorem 7.6: absence of resonances in $D(E(h), \varepsilon(h))$ and Theorem 7.5 imply bounds on the resolvent in $D((E(h), \frac{1}{2}\varepsilon(h))$. We can then place a rectangle $[E(h) - \frac{1}{4}\varepsilon(h), E(h) + \frac{1}{4}\varepsilon(h)] + i[-h^{-n-1}\delta(h), \delta(h)]$ inside of that disc and apply Lemma 7.7.

and then put

$$\varepsilon(h) = h^{-2(n+1)}\|\varepsilon_0(h)\|.$$

We note that (7.3.6) holds and that

$$\log(1/\varepsilon(h)) \le C_1/h. \tag{7.3.14}$$

To deduce (7.3.13) from (7.3.12) with this choice of $\varepsilon(h)$ we will use Lemma 7.7.

Theorem 7.5 and (7.3.14) show that for for some constant C,

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le Ae^{Ah^{-n}\log(2/\varepsilon(h))} \le Ce^{Ch^{-n-1}},$$

for

$$z \in D(E(h), \varepsilon(h)) \setminus \bigcup_{w\in \mathrm{Res}(P(h))} D(w, \tfrac{1}{2}\varepsilon(h)).$$

The assumption (7.3.12) then gives

$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le Ce^{Ch^{-(n+1)}}, \quad z \in D(E(h), \tfrac{1}{2}\varepsilon(h)),$$
$$\|\chi R(z,h)\chi\|_{L^2\to L^2} \le 1/\operatorname{Im} z, \quad \operatorname{Im} z > 0,$$

where the last inequality comes from self-adjointness of $P(h)$:

$$\|(P(h) - z)^{-1}\|_{L^2\to L^2} = 1/\operatorname{Im} z, \quad \operatorname{Im} z > 0.$$

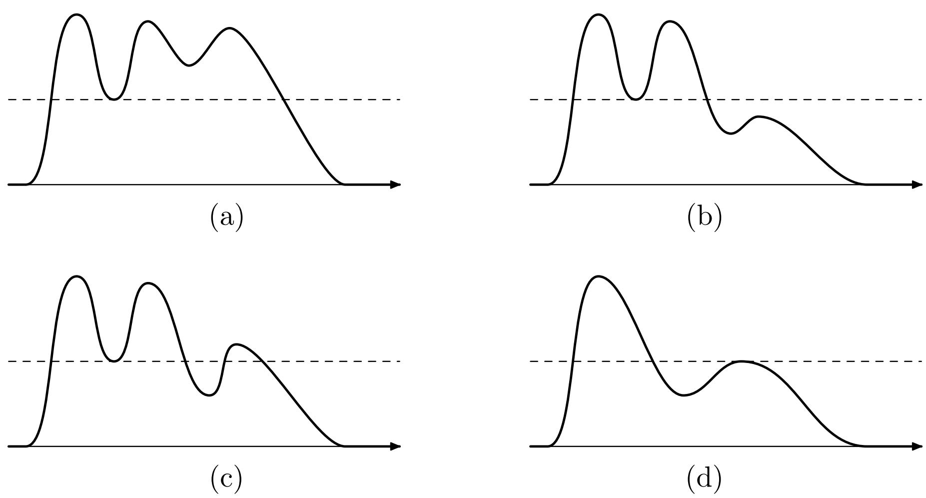


Figure 7.4. Examples of one-dimensional potentials illustrating condition (7.3.2): the component of infinity is non trapping (that is (7.3.2) holds) in cases (a),(b),(c) but *not* (d).

3. We can now apply Lemma 7.7 with

$$Q(z,h) = \chi R(z,h)\chi, \quad \delta(h) = M\|\varepsilon_0(h)\|, \quad M \gg 1,$$
$$a(h) = E(h) - \tfrac{1}{4}\varepsilon(h), \ b(h) = E(h) + \tfrac{1}{4}\varepsilon(h),$$

The assumptions (7.3.7) are satisfied as

$$\begin{aligned}(\tfrac{1}{2}\varepsilon(h))^2 &= \tfrac{1}{4}h^{-4(n+1)}\|\varepsilon_0(h)\|^2 \geq Ch^{-3(n+1)}M^2\|\varepsilon_0(h)\|^2 \\ &= Ch^{-3(n+1)}\delta(h)^2,\end{aligned}$$

if h is small enough. (There is no push for optimality here.) Thus (7.3.9) gives (7.3.13) if M is chosen large enough:

$$\begin{aligned}\|\chi(P(h)-E(h))^{-1}\chi\| = \|\chi R(E(h),h)\chi\| &\leq e^{C+1}\delta(h)^{-1} \\ &= e^{C+1}M^{-1}\|\varepsilon_0(h)\|^{-1} \leq \tfrac{1}{2}\|\varepsilon_0(h)\|^{-1}.\end{aligned}$$

Returning to Step 1 of the proof we have obtained the needed contradiction to the assumption (7.3.12). □

We conclude this section by describing a dichotomy for imaginary parts of resonances in the case a potential barrier for which the component of infinity is non-trapping. This phenomenon of resonances splitting into those close to the real axis and those far away occurs in many other settings, for instance in transmission problems, see Sjöstrand–Vodev [**SV97**] and Galkowski [**Ga19**].

THEOREM 7.8 (**Dichotomy for resonance widths**). *Suppose that* $P(h) = -h^2\Delta + V$ *satisfies* (7.3.2) *for some* $E > 0$.

Then there exist $\delta > 0$, $S > 0$ *such that for every* M *there exists* $h_0 > 0$, *so that for* $0 < h < h_0$,

$$z \in \operatorname{Res}(P(h)), \quad |\operatorname{Re} z - E| < \delta \implies \begin{cases} \operatorname{Im} z > -e^{-S/h} \\ \quad \textit{or} \\ \operatorname{Im} z < -Mh\log(1/h). \end{cases} \tag{7.3.15}$$

for $0 < h < h_0$.

REMARKS. 1. Condition (7.3.2) means that the infinity component of the energy surface $p = E$ is non trapping – see Fig. 7.4.

2. We have a stronger conclusions than (7.3.15): all resonances with $\operatorname{Im} z > -e^{-S/h}$ come from quasimodes localized in $p^{-1}(E) \setminus \Sigma_\infty$ and hence can be related to eigenvalues of the reference problem $P_1(h)$ presented in the first example – see Exercise 7.3.

Proof. 1. Let $P_\theta(h)$ be the complex scaled operator obtained by considering $P(h)$ as a black box Hamiltonian – see §4.5. In particular, $P_\theta(h)$ coincides with $P(h)$ near $\operatorname{supp} V$ and $\operatorname{supp}(g^{ij} - \delta_{ij})$. If $z \in \operatorname{Res}(P(h))$ we consider the corresponding resonant state

$$(P_\theta(h) - z)u_\theta = 0, \quad \|u_\theta\|_{L^2(\Gamma_\theta)} = 1. \tag{7.3.16}$$

2. Let $\chi \in C^\infty_{\rm c}(\mathbb{R}^n; [0,1])$ satisfy

$$\chi(x) = 1 \text{ near } \pi(\Sigma_0) \text{ and } \chi(x) = 0 \text{ near } \pi(\Sigma_\infty). \tag{7.3.17}$$

That is possible as Σ_0 and Σ_∞ are closed and disjoint. Condition (7.3.17) implies that

$$V(x) > E \quad \text{for } x \in \operatorname{supp} d\chi \ . \tag{7.3.18}$$

We first claim that for z and u_θ satisfying (7.3.16) and some $S > 0$ we have the following dichotomy for small enough h:

$$\|\chi u_\theta\| < e^{-S/h} \quad \text{or} \quad \operatorname{Im} z > -e^{-S/h}. \tag{7.3.19}$$

3. To prove (7.3.19) use self-adjointness of $P(h)$ to write

$$\begin{aligned} -2i \operatorname{Im} z \|\chi u_\theta\|^2 &= \langle (P(h) - z)\chi u_\theta, \chi u_\theta\rangle - \langle \chi u_\theta, (P\theta(h) - z)\chi u_\theta\rangle \\ &= \langle [P(h), \chi] u_\theta, \chi u_\theta\rangle. \end{aligned}$$

In view of (7.3.18), Agmon estimates (see [**Zw12**, Theorem 7.4]) show that

$$\|[P(h), \chi]u_\theta\| < e^{-2S/h}\|u_\theta\|$$

for some $S > 0$ and h small enough. Hence

$$|\operatorname{Im} z| \|\chi u_\theta\| < e^{-2S/h},$$

and (7.3.19) follows.

4. We now show that for any $M > 1$ here exists h_0 such that for $0 < h < h_0$,

$$\|\chi u_\theta\| < e^{-S/h} \implies \operatorname{Im} z < -Mh \log(1/h). \tag{7.3.20}$$

In view of (7.3.19) that will prove the theorem.

To establish (7.3.20) we use (7.3.2) to find $V_0 \in C_{\mathrm{c}}^\infty(\mathbb{R}^n; \mathbb{R})$ such that

$$\operatorname{supp}(V - V_0) \subset \operatorname{supp} \chi, \quad V_0(x) > E, \quad x \notin \pi(\Sigma_\infty), \tag{7.3.21}$$

where χ satisfies (7.3.17). That means that V_0 "fills in" the finite components of the energy surface of V. (Its role is the opposite to that of V_1 in the first example of this section.) Then

$$\{(x, \xi) : |\xi|_g^2 + V_0(x) = E\} = \Sigma_\infty$$

and the assumption (7.3.2) shows that

$$|\xi|_g^2 + V_0(x) = E \implies |\pi \exp(tH_{p_0})(x, \xi)| \to \infty, \quad t \to \pm\infty,$$

where $p_0(x, \xi) = |\xi|_g^2 + V_0(x)$. That means that the energy level E is non-trapping for $P_0(h)$. If $P_{0,\theta}(h)$ is the corresponding scaled operator. From (6.4.17) we see that

$$\begin{gathered} \|(P_{0,\theta}(h) - z)^{-1}\| \leq h^{-C_0 M}, \\ z \in [E - \delta, E + \delta] - i[0, Mh \log(1/h)], \end{gathered} \tag{7.3.22}$$

for $0 < h < h_0$.

5. Suppose that (7.3.20) is false that is, $\|\chi u_\theta\| < e^{-S/h}$ and $\operatorname{Im} z \geq -Mh \log(1/h)$. Because of (7.3.21) we can choose $\widetilde{\chi} \in C_{\mathrm{c}}^\infty(\mathbb{R}^n; [0, 1])$ such that $\chi = 1$ on a neighbourhood of $\operatorname{supp} \widetilde{\chi}$ and

$$(P_\theta(h) - z)(1 - \widetilde{\chi}) = (P_{0,\theta}(h) - z)(1 - \widetilde{\chi}).$$

Hence

$$\begin{aligned} (1 - \widetilde{\chi}) u_\theta &= (P_{0,\theta}(h) - z)^{-1} (P_{0,\theta}(h) - z)(1 - \widetilde{\chi}) u_\theta \\ &= (P_{0,\theta}(h) - z)^{-1} (P_\theta(h) - z)(1 - \widetilde{\chi}) u_\theta \\ &= (P_{0,\theta}(h) - z)^{-1} [P_\theta(h), \widetilde{\chi}] u_\theta. \end{aligned} \tag{7.3.23}$$

We can find open sets $U \subset W$ such that $\operatorname{supp} d\widetilde{\chi} \subset U$ and $W \Subset \{\chi = 1\}$. Semiclassical elliptic estimates (see [Zw12, Theorem 7.1]) and $\|\chi u_\theta\| <$

$e^{-S/h}$ then give

$$\begin{aligned}\|[P_\theta(h),\widetilde{\chi}]u_\theta\|_{L^2} &\leq Ch\|u_\theta\|_{H^1_h(U)}\\ &\leq Ch\|(P_\theta(h)-z)u_\theta\|_{L^2(W)} + Ch\|u_\theta\|_{L^2(W)}\\ &\leq Ch\|\chi u_\theta\| \leq Ce^{-S/h}.\end{aligned}$$

Combined with (7.3.22) and (7.3.23) this gives

$$\begin{aligned}1 = \|u_\theta\| &\leq \|(1-\widetilde{\chi})u_\theta\| + \|\widetilde{\chi}u_\theta\|\\ &\leq \|(1-\widetilde{\chi})u_\theta\| + e^{-S/h} \leq 2e^{-S/h}\end{aligned}$$

which provides an obvious contradiction for h small enough. □

7.4. THE SJÖSTRAND TRACE FORMULA

In this section we present a semiclassical *local* trace formula for resonances. It is different from the formula in §3.10 by involving only a finite number of resonances of a semiclassical black box operator.

We provide a complete proof in the simplified situation of $-h^2\Delta + V$, and $V \in C^\infty_{\rm c}(\mathbb{R}^3;\mathbb{R})$. We then give an application by showing that for any potential $V \in C^\infty_{\rm c}(\mathbb{R}^3;\mathbb{R})$ there exist many energy levels E such that for any $r > 0$,

$$|\operatorname{Res}(-h^2\Delta_g + V) \cap D(E,r)| \geq h^{-n}/C(r), \quad 0 < h < h_0(r). \tag{7.4.1}$$

To formulate the theorem we introduce the following subsets of $\mathbb{C}$:

$$\begin{gathered}\Omega := (a,b) + i(c,d), \quad W := (a',b') + i(c',d),\\ 0 < a < a' < b' < b, \quad c < c' < 0 < d,\\ \Omega_- := \Omega \cap \{\operatorname{Im} z \leq 0\}, \quad W_- := W \cap \{\operatorname{Im} z \leq 0\},\\ \Omega_{\mathbb{R}} = \Omega \cap \mathbb{R}, \quad W_{\mathbb{R}} = W \cap \mathbb{R}.\end{gathered} \tag{7.4.2}$$

The complex scaling method described in §4.5 will be crucial in the proof. Hence we choose c, d small enough so that $\Omega \subset \{\arg z > -\theta\}$ for some $\theta < \pi/2$. The regions are illustrated in Fig. 7.5

THEOREM 7.9 (The Sjöstrand trace formula). *For $n = 3$ let*

$$P_V(h) := -h^2\Delta + V(x),$$

where $V \in C^\infty_{\rm c}(\mathbb{R}^3;\mathbb{R})$, $P_0(h) = -h^2\Delta$, and suppose that Ω and W are given by (7.4.2).

Let f be a holomorphic function in a neighbourhood of Ω and satisfies

$$|f(z)| \leq 1, \quad z \in \Omega \setminus W, \tag{7.4.3}$$

and let $\psi \in C^\infty_{\rm c}(\Omega_{\mathbb{R}})$ be equal to 1 on $W_{\mathbb{R}}$.

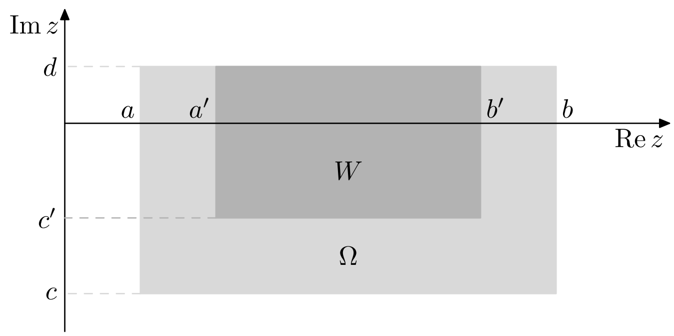


Figure 7.5. The regions Ω and W in Theorem 7.9

Then

$$(\psi f)(P_V(h)) - (\psi f)(P_0(h)) \in \mathcal{L}_1(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)), \tag{7.4.4}$$

and

$$\begin{aligned} &\operatorname{tr}[(\psi f)(P_V(h)) - (\psi f)(P_0(h))] = \\ &\qquad \sum_{z \in \operatorname{Res}(P_V(h)) \cap W_-} f(z) + \mathcal{O}(h^{-n}) + \mathcal{O}_f(h^\infty). \end{aligned} \tag{7.4.5}$$

REMARKS. 1. In the application of (7.4.5) we vary f with a fixed ψ – hence we are not concerned about the dependence of the constants on ψ – only on f. The term $\mathcal{O}_f(h^\infty)$ (see Remark 3 below) might seem disturbing at first but in the applications we choose f (depending on some parameters) so that information is drawn from the first two terms on the right hand side of (7.4.5). One then fixes that f and considers the semiclassical limit $h \to 0$.

2. The restriction to dimension $n = 3$ is due to better trace class properties used in the proof, just as in the proof of Theorem 3.51 for $n = 3$. The proof works also for $n = 1$. The statement remains valid for all n but the proof needs to be modified: complex scaling needs to be presented for odd and even dimensions and we need additional arguments to deal with trace class properties. Roughly, it amounts to replacing $(P_\bullet - z)^{-1}$ in the proof by $(P_\bullet - z)^{-1}(P_\bullet - z_0)^{-m}$, $m \geq (n-1)/2$ but that leads to (minor) algebraic complications.

3. Much weaker assumptions on $P(h)$ are needed and in particular the trace formula works for long range black box Hamiltonians. The formulation is also stronger: the error term $\mathcal{O}_f(h^\infty)$ can be dropped and in (7.4.3) the bound is needed only in $\Omega_- \setminus W$ – see §7.6 for references.

Before the proof we present two lemmas which provide quantitative estimates on trace class norms of resolvent differences.

LEMMA 7.10. *Suppose that $n = 3$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we have*

$$(P_V - z)^{-1} - (P_0 - z)^{-1} = \mathcal{O}(h^{-n} |\operatorname{Im} z|^{-M_0})_{\mathcal{L}^1(L^2(\mathbb{R}^n))}, \tag{7.4.6}$$

for some constant M_0. If $\chi \in C_c^\infty(\mathbb{R}^n)$ and $\chi \equiv 1$ on $\operatorname{supp} V$ then, for any N,

$$\begin{aligned} &(1-\chi)\left((P_V - z)^{-1} - (P_0 - z)^{-1}\right) \\ &\qquad = \mathcal{O}(h^N |\operatorname{Im} z|^{-C_N}) : \langle x \rangle^N H^{-N}(\mathbb{R}^n) \to \langle x \rangle^{-N} H^N(\mathbb{R}^n). \end{aligned} \tag{7.4.7}$$

Proof. 1. For $z \in \mathbb{C} \setminus \mathbb{R}$, $P_V - z$ is a quantization of an *elliptic* symbol in $S(\langle \xi \rangle^2)$ where the symbol class associated to the weight $m(x, \xi) = \langle \xi \rangle^2$ is described in [**Zw12**, §4.4]. Hence the semiclassical version of Beals's Lemma [**Zw12**, Theorem 8.3] shows that

$$(P_V - z)^{-1} = a_V(x, hD, z),$$

where, for some $M_1 > 0$, a_V satisfies

$$\partial_x^\alpha \partial_\xi^\beta a_V(x, \xi, z) = \mathcal{O}_{\alpha,\beta}(\langle \xi \rangle^{-2} |\operatorname{Im} z|^{-M_1 - 2(|\alpha| + |\beta|)}).$$

(We find it more convenient here to refer to the calculus based on weight functions – see [**DS99**, Chapter 7] and [**Zw12**, Chapter 4] rather than the calculus build for our application for scattering on manifolds in Appendix E. See [**DS99**, Chapter 8] and the proof [**Zw12**, Theorem 14.9] for similar arguments.)

2. Since $V \in S(\langle x \rangle^{-N})$ for any N,

$$(P_V - z)^{-1} - (P_0 - z)^{-1} = -(P_V - z)^{-1} V (P_0 - z)^{-1} = b(x, hD, z),$$

where for $z \in \mathbb{C} \setminus \mathbb{R}$, $b \in S(\langle \xi \rangle^{-4} \langle x \rangle^{-N})$ for any N. Moreover, for some $M_2 > 0$,

$$\partial_x^\alpha \partial_\xi^\beta b(x, \xi, z) = \mathcal{O}_{\alpha,\beta,N}(\langle x \rangle^{-N} \langle \xi \rangle^{-4} |\operatorname{Im} z|^{-M_2 - 2(|\alpha| + |\beta|)}).$$

This follows from the composition formula in [**Zw12**, Theorem 4.18] and the remainder estimates – see for instance [**Zw12**, (9.3.17)]. The trace class norm is bounded by (see for instance [**DS99**, Chapter 9])

$$\begin{aligned} \| b(x, hD, z) \|_{\mathcal{L}^1} &\le C h^{-n} \sum_{|\alpha| + |\beta| \le 2n+1} \int_{\mathbb{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta b(x, \xi, h)| dx d\xi \\ &\le C' h^{-n} |\operatorname{Im} z|^{-M_0}, \end{aligned}$$

which is (7.4.6). (It is here that we use $n = 3$ as $\int_{\mathbb{R}^n} \langle \xi \rangle^{-4} d\xi < \infty$ only for $n \le 3$.)

3. To see (7.4.7) we proceed similarly: for $z \in \mathbb{C} \setminus \mathbb{R}$ the pseudodifferential calculus shows that $(1 - \chi) b(x, hD, z) = \mathcal{O}(h^\infty)_{\mathscr{S} \to \mathscr{S}'}$. In fact, all the terms

in the expansion vanish and the remainder is in the residual class. Expanding the remainder explicitly as in [**Zw12**, (9.3.17)] shows a quantitative estimate:

$$(1-\chi)b(x,hD,z) = c(x,hD,z),$$
$$\partial_x^\alpha \partial_\xi^\beta c(x,\xi,z) = \mathcal{O}_{\alpha,\beta,N}(h^N \langle x\rangle^{-N}\langle \xi\rangle^{-N}|\operatorname{Im} z|^{-M_2-2(|\alpha|+|\beta|)}).$$

This and [**Zw12**, Theorem 8.10] prove (7.4.7). □

LEMMA 7.11. *Suppose that $n=3$ and that $\chi_0 \in C_c^\infty(B(0,R_1))$ is equal to* 1 *near* $\operatorname{supp} V$*. (Here R_1 is the same as in* (4.5.1)*). Let $P_{V,\theta}$ and $P_{0,\theta}$ be the complex scaled operators P_V and P_0, in the sense of Definition 4.31. Then for* $\operatorname{Im} z \geq \delta > 0$*,*

$$(1-\chi_0)\left((P_{V,\theta}-z)^{-1} - (P_{0,\theta}-z)^{-1}\right) = \mathcal{O}_\delta(h^\infty)_{\mathcal{L}^1(L^2(\Gamma_\theta))}.$$

Proof. As in the proof of Theorem 7.4 we see that $\operatorname{Im} z > -\delta$, $P_{V,\theta} - z \in \Psi_h^2(\Gamma_\theta)$ is elliptic. Hence the estimate in the Lemma follows from the pseudodifferential calculus similarly to (7.4.7). □

Proof of Theorem 7.9. 1. The starting point is the Helffer-Sjöstrand formula: for $g \in C_c^\infty(\mathbb{R})$ we write $g(P_V)$ is given by

$$g(P_V) := \frac{1}{\pi}\int_{\mathbb{C}} (P_V - z)^{-1}\bar{\partial}_z \tilde{g}(z) dm(z), \tag{7.4.8}$$

where $\tilde{g} \in C_c^\infty(\mathbb{C})$ is an almost analytic continuation of g – see §B.2 and [**DS99**, (8.1) and (8.2)]. We stress that here and below the resolvent $(P_\bullet - z)^{-1} : L^2 \to L^2$, $z \in \mathbb{C}\backslash\mathbb{R}$, $\bullet = V, 0$, is meant in the usual spectral theoretical sense and *not* in the sense of meromorphic continuation across $\mathbb{R}$.

Using the short hand notation,

$$[F(P_\bullet)]_0^V := F(P_V) - F(P_0),$$

we then have, with $g = f\psi$,

$$[(f\psi)(P_\bullet)]_0^V = \frac{1}{\pi}\int_{\mathbb{C}} [(P_\bullet - z)^{-1}]_0^V \bar{\partial}_z \widetilde{\psi}(z) f(z) dm(z).$$

Lemma 7.10 shows that $\|[(P_\bullet - z)^{-1}]_0^V\|_{\mathcal{L}^1} = \mathcal{O}(h^{-n}|\operatorname{Im} z|^{-M_0})$ and this gives (7.4.4).

2. We can arrange that

$$\operatorname{supp} \bar{\partial}_z \widetilde{\psi} \cap \{\operatorname{Im} z < \delta\} \subset \Omega \setminus W.$$

In particular

$$\bar{\partial}_z \widetilde{\psi}(z) f(z) = \mathcal{O}(1), \quad \operatorname{Im} z < \delta, \tag{7.4.9}$$

where the constant is independent of f. This is where the assumption (7.4.3) is used. The Cauchy–Green formula (D.1.1),

$$2i \int_U \bar{\partial}_z \varphi(z) dm(z) = \int_{\partial U} \varphi(z) dz, \quad \varphi \in C_c^\infty(\mathbb{C}),$$

(U is an open set with a C^1 positively oriented boundary) applied with the operator valued function

$$\varphi(z) = \left[P_\bullet - z)^{-1}\right]_0^V f(z)\widetilde{\psi}(z),$$

and $U = \{\operatorname{Im} z > \delta\}$ shows that

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{C}} \left[(P_\bullet - z)^{-1}\right]_0^V \bar{\partial}_z \widetilde{\psi} f dm(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} &\left[(P_\bullet - z)^{-1}\right]_0^V \widetilde{\psi} f dz \\ &+ \frac{1}{\pi}\int_{\operatorname{Im} z<\delta} \left[(P_\bullet - z)^{-1}\right]_0^V \bar{\partial}_z \widetilde{\psi} f dm(z),\end{aligned}$$

where the line $\Gamma_\delta := \mathbb{R} + i\delta$ is oriented from right to left.

From (7.4.6) and (7.4.9) we see that the trace class norm of the second term on the right hand side is bounded by Ch^{-n} with a constant independent of f. Hence,

$$[(f\psi)(P_\bullet)|_0^V = \frac{1}{2\pi i}\int_{\Gamma_\delta} \left[(P_\bullet - z)^{-1}\right]_0^V \widetilde{\psi} f dz + \mathcal{O}(h^{-n})_{\mathcal{L}^1}. \tag{7.4.10}$$

3. For $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near $\operatorname{supp} V$ we apply insert $1 = \chi + (1-\chi)$ on the left and right of the right hand of (7.4.10) and apply (7.4.7). The terms involving $(1-\chi)$ give $\mathcal{O}_f(h^\infty)_{\mathcal{L}^1})$ contributions and we obtain

$$\begin{aligned}[(\psi f)(P_\bullet)]_0^V = \frac{1}{2\pi i}\int_{\Gamma_\delta} &\chi \left[(P_\bullet - z)^{-1}\right]_0^V \chi\widetilde{\psi} f(z) dz \\ &+ \mathcal{O}(h^{-n})_{\mathcal{L}^1} + \mathcal{O}_f(h^\infty)_{\mathcal{L}^1}.\end{aligned}$$

Because of the cut-off χ we can now use Theorem 4.37 and replace $P_\bullet$ by the complex scaled operators $P_{\bullet,\theta}$. Since $\operatorname{Im} z = \delta > 0$ in the integral, Lemma 7.11 shows that we can remove the cut-off function χ at the expense of another error $\mathcal{O}_f(h^\infty)$ error term:

$$\begin{aligned}\operatorname{tr}[(\psi f)(P_\bullet)]_0^V = \frac{1}{2\pi i}\operatorname{tr}\int_{\Gamma_\delta} &\left[(P_{\bullet,\theta} - z)^{-1}\right]_0^V \widetilde{\psi} f(z) dz \\ &+ \mathcal{O}(h^{-n}) + \mathcal{O}_f(h^\infty)\end{aligned}$$

We now bring back the operators $\widetilde{P}_{\bullet,\theta}$ introduced in the proof Theorem 7.4 – see (7.2.3). Invertibility of $\widetilde{P}_{\bullet,\theta} - z$ for z near Ω shows that

$$\int_{\mathbb{C}} [(\widetilde{P}_{\bullet,\theta} - z)^{-1}]_0^V \bar{\partial}_z \widetilde{\psi}(z) f(z) dm(z) = 0.$$

Arguing as in Step 2 with $(\widetilde{P}_{\bullet,\theta} - z)^{-1}$ in place of $(P_\bullet - z)^{-1}$ we obtain

$$\begin{aligned}\operatorname{tr}[(f\psi)(P_\bullet)|_0^V = \frac{1}{2\pi i} \operatorname{tr} \int_{\Gamma_\delta} \Big[(P_{\bullet,\theta} - z)^{-1} - (\widetilde{P}_{\bullet,\theta} - z)^{-1})\Big]_0^V \widetilde{\psi}(z) f(z) dz \\ + \mathcal{O}(h^{-n}) + \mathcal{O}_f(h^\infty),\end{aligned}$$

4. The terms $\bullet = 0, V$ can be treated separately as the differences are of trace class. Applying the Cauchy–Green formula (D.1.1) with $U = \{\operatorname{Im} z < \delta\}$, again and taking the trace we obtain, noting that resonances are the poles of $(P_{\bullet,\theta} - z)^{-1} - (\widetilde{P}_{\bullet,\theta} - z)^{-1}$ (the second term is holomorphic):

$$\begin{aligned}\operatorname{tr}[(f\psi)(P_\bullet)|_0^V &= \sum_{z \in \operatorname{Res}(P_V)} (f\widetilde{\psi})(z) + \mathcal{O}(h^{-n}) + \mathcal{O}_f(h^\infty) \\ &+ \frac{1}{\pi} \int_{\operatorname{Im} z < \delta} \operatorname{tr} \Big[(P_{\bullet,\theta} - z)^{-1} - (\widetilde{P}_{\bullet,\theta} - z)^{-1}\Big]_0^V \bar{\partial}_z \widetilde{\psi}(z) f(z) dm(z).\end{aligned}$$

(Note that the negative orientation of Γ_δ produces the correct sign of the terms $(f\widetilde{\psi})(z)$.)

5. We now use the operators $K_\bullet$ defined by (7.2.6) with P_θ replaced by $P_{\bullet,\theta}$:

$$K_\bullet(x) = iM(\widetilde{P}_{\bullet,\theta} - z)^{-1} \chi(hD) \chi(x)^2 \chi(hD).$$

We note that

$$(P_{\bullet,\theta} - z)^{-1} - (\widetilde{P}_{\bullet,\theta} - z)^{-1} = -K_\bullet(z)(I + K_\bullet(z))^{-1}(\widetilde{P}_{\bullet,\theta} - z)^{-1}, \tag{7.4.11}$$

and that

$$\partial_z K_\bullet(z) = (\widetilde{P}_{\bullet,\theta} - z)^{-1} K_\bullet(z). \tag{7.4.12}$$

From these two identities we deduce that

$$\begin{aligned}\int_{\operatorname{Im} z < \delta} \operatorname{tr} \Big[(P_{\bullet,\theta} - z)^{-1} - (\widetilde{P}_{\bullet,\theta} - z)^{-1}\Big]_0^V \bar{\partial}_z \widetilde{\psi}(z) f(z) dm(z) = \\ \int_{\operatorname{Im} z < \delta} [\partial_z k_\bullet(z) k_\bullet(z)^{-1}]_0^V \bar{\partial}_z \widetilde{\psi}(z) f(z) dm(z),\end{aligned} \tag{7.4.13}$$

where $k_\bullet(z) := \det(I + K_\bullet(z))$.

6. We recall the estimates on the determinant, (7.2.9) and (7.2.11), from the proof of Theorem 7.4:

$$\log |k_\bullet(z)| \leq Ch^{-n}, \ z \in \Omega' \quad \log |k_\bullet(z_0)| \geq -Ch^{-n}.$$

The bound (D.1.15) then shows that for z near Ω (and with $\Omega' \Subset \{\operatorname{Re} z > 0\}$ an h independent neighbourhood of Ω)

$$\begin{aligned} \frac{\partial_z k_\bullet(z)}{k_\bullet(z)} &= H'_\bullet(z) + \sum_{\zeta \in \operatorname{Res}(P(h)) \cap \Omega'} \frac{m_{k_\bullet}(\zeta)}{z-\zeta}, \quad H'_\bullet(z) = \mathcal{O}(h^{-n}), \\ & \sum_{\zeta \in \Omega} m_{k_V}(\zeta) = \mathcal{O}(h^{-n}), \quad m_{k_0}(\zeta) \equiv 0. \end{aligned} \tag{7.4.14}$$

In view of (7.4.9) $\bar\partial_z \widetilde{\psi}(z) f(z)$ is bounded independently of f for $\operatorname{Im} z < \delta$. Since for $U \Subset \mathbb{C}$,

$$\int_U |z-\zeta|^{-1} dm(z) = \mathcal{O}_U(1),$$

(7.4.14) gives the bound $\mathcal{O}(h^{-n})$ for the right hand side of (7.4.13). $\square$

To state an application of Theorem 7.9 we need to review some basic facts about analytic singular support and wave front set. The standard references are [**De92**],[**HöI**, §9.3],[**Ma02a**, §3.3] and [**Sj82**].

DEFINITION 7.12. *Suppose that* $u \in \mathscr{S}'(\mathbb{R}^n)$. *Then the* analytic singular support *of* u *is the closed set* $\operatorname{singsupp}_a u$, *defined by the condition that* $x \notin \operatorname{singsupp}_a u$ *if and only if there exists a neighbourhood* U *of* x *such that* $u|_U$ *is a real analytic function.*

A useful characterization of the analytic singular support is given in terms of the *analytic wave front set* which is the analytic version of the wave front set given in §E.2. One way to define the analytic wave front set is using the FBI (Fourier–Bros–Iagolnitzer) transform: for $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$T_\lambda u(x,\xi) = \int_{\mathbb{R}^n} e^{-\lambda(x-y)^2/2 + i\lambda\langle \xi, x-y\rangle} u(y)dy, \quad \lambda > 0, \tag{7.4.15}$$

where the integral is meant in the sense of a distributional pairing. We note $|T_\lambda u(x,\xi)| \le C\lambda^N$, for some N. (The inequality follows from using a semi-norm of $y \mapsto e^{-\lambda(x-y)^2/2 + i\xi(x-y)}$ in $\mathscr{S}$.)

We then have

DEFINITION 7.13. *For* $u \in \mathscr{S}'(\mathbb{R}^n)$ *the* analytic wave front set *of* u, $\operatorname{WF}_a(u) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, *is defined as follows:*

$$\begin{aligned} (x,\xi) \notin \operatorname{WF_a}(u) \iff & \exists\, U \subset \mathbb{R}^{2n} \text{ a neighbourhood of } (x,\xi) \text{ and } \delta > 0 \\ & \text{such that } |T_\lambda u(y,\eta)| \le Ce^{-\delta\lambda} \text{ for } (y,\eta) \in U \text{ as } \lambda \to \infty. \end{aligned}$$

The result connecting the two objects is

$$\operatorname{singsupp}_a u = \pi(\operatorname{WF}_a(u)), \quad \pi(x,\xi) := x, \quad \pi : T^*\mathbb{R}^n \setminus 0 \to \mathbb{R}^n, \tag{7.4.16}$$

see [**HöI**, Theorems 8.4.5 and 9.6.3].

A lower bound on the number of resonances is now given in any neighbourhood of an energy level lying in the analytic singular support of the distribution function of the potential:

THEOREM 7.14 (**Lower bounds for the number of resonances).** *Suppose that* $n = 3$ *and* $P_V = -h^2\Delta + V$, $V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$. *Let* λ_V *be the distribution function of* V*:*

$$\lambda_V(t) := m(\{x : V(x) > t\}).$$

If $0 < E \in \operatorname{singsupp}_a \lambda_V$, *then for any* $r > 0$ *there exists* $h_0 > 0$ *and* $C > 0$ *such that for* $0 < h < h_0$,

$$\sum_{z \in D(E,r)} m_V(z) \geq h^{-n}/C. \tag{7.4.17}$$

REMARKS. 1. For any potential V, any critical value of V is in the C^∞ singular support of λ_V and hence in $\operatorname{singsupp}_a \lambda_V$.

2. The theorem is valid in all dimensions and for the same class potentials for which (7.4.5) holds – see [**Sj96a**] and [**Sj96b**]. We present the proof for $\mathbb{R}^n$ assuming the validity of Theorem 7.9 for all n.

Proof. 1. Suppose $g \in C_c^\infty(\mathbb{R})$. We then have $g(P_V) = a_V(x, hD; h)$, where

$$\begin{aligned} a_\bullet(x, \xi; h) &= g(\xi^2 + V(x)) + h a_{1,V}(x, \xi, h), \\ \partial_x^\alpha \partial_\xi^\beta a_{1,V}(x, \xi, h) &= \mathcal{O}_{\alpha,\beta}(\langle x \rangle^{-\infty} \langle \xi \rangle^{-\infty}). \end{aligned}$$

(Note that $a_{1,0} \equiv 0$.) This follows from the functional calculus based on the Helffer–Sjöstrand formula with Lemma 7.10 providing the needed resolvent estimates – see [**DS99**, Chapter 8] or [**Zw12**, Theorem 14.9]. It follows that

$$\begin{aligned} \operatorname{tr}(g(P_V) - g(P_0)) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} (g(\xi^2 + V(x)) - g(\xi^2)) d\xi dx \\ + \mathcal{O}_g(h^{-n+1}). \end{aligned} \tag{7.4.18}$$

2. We want to express the first term on the right hand side of (7.4.18) using the distribution function of V, λ_V, or more precisely the distribution functions for $V_\pm$:

$$\lambda_{V_\pm}(t) = \begin{cases} m(\{x : \pm V(x) > t\}), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

For that we put $\tilde{g}(t) := g(-t)$ so that

$$\begin{aligned} \int_{\mathbb{R}^n} (g(\xi^2 + a) - g(\xi^2)) d\xi &= 2^{-1} \operatorname{Vol}(\mathbb{S}^{n-1}) \int_0^\infty (g(r + a) - g(r)) r^{\frac{n-2}{2}} dr \\ &= \pi^{n/2} (\tilde{g} * \chi_+^{\frac{n-2}{2}}(-a) - \tilde{g} * \chi_+^{\frac{n-2}{2}}(0)), \end{aligned}$$

where

$$\chi_+^s = x_+^s/\Gamma(s+1), \qquad x_+^s = \begin{cases} x^s, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

first defined for $\operatorname{Re} s > -1$ and then by analytic continuation for all $s \in \mathbb{C}$, see [**HöI**, §3.2]. (We assume that $n = 3$ but as the argument works for any dimension we proceed in that generality. The constant comes from the calculation of the volume of the sphere: $\operatorname{Vol}(\mathbb{S}^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$.)

We now note that for $F \in C^1(\mathbb{R})$ with $F(0) = 0$,

$$\int_{\mathbb{R}^n} F(V(x))dx = \int_0^\infty \big(F'(t)\lambda_{V_+}(t) - F'(-t)\lambda_{V_-}(t)\big)\, dt.$$

The integration can be changed to integration over $\mathbb{R}$ in view of the support properties of $\lambda_{V_\pm}$. Applying this with

$$F(t) := \tilde{g} * \chi_+^{\frac{n-2}{2}}(-t) - \tilde{g} * \chi_+^{\frac{n-2}{2}}(0),$$

we see that

$$\begin{aligned} &\int_{\mathbb{R}^{2n}} (g(\xi^2 + V(x)) - g(\xi^2))d\xi dx \\ &\quad = \pi^{n/2} \int_{\mathbb{R}} \left(-\tilde{g}' * \chi_+^{\frac{n-2}{2}}(-t)\lambda_{V_+}(t) + \tilde{g}' * \chi_+^{\frac{n-2}{2}}(t)\lambda_{V_-}(t) \right) dt \\ &\quad = \pi^{n/2} \int_{\mathbb{R}} g'(t) \left(\chi_+^{\frac{n-2}{2}} * \lambda_{V_+}(t) - \chi_+^{\frac{n-2}{2}} * \lambda_{V_-}(-t) \right) dt = \\ &\quad = \pi^{n/2} \langle u, g \rangle, \end{aligned} \tag{7.4.19}$$

where $\langle \bullet, \bullet \rangle$ denotes distributional pairing and we used the fact that $(\chi_+^s)' = \chi_+^{s-1}$ and

$$u := -\chi_+^{\frac{n-4}{2}} * (\lambda_{V_+}(\bullet) - \lambda_{V_-}(-\bullet)). \tag{7.4.20}$$

3. Since $\operatorname{singsupp}_a(\chi_+^s) = \{0\}$ and $\chi_+^{-s-2} * \chi_+^s = \delta_0$ we see that u defined in (7.4.20) satisfies

$$\operatorname{singsupp}_a(u) = \operatorname{singsupp}_a(\lambda_{V_+}(\bullet) - \lambda_{V_-}(-\bullet))$$

and for $E > 0$,

$$E \in \operatorname{singsupp}_a(u) \iff E \ \in \operatorname{singsupp}_a(\lambda_V).$$

Since u is real valued (which implies that the wave front set is symmetric with respect to $(x, \xi) \mapsto (x, -\xi)$) we conclude from (7.4.16) that

$$E \in \operatorname{singsupp}_a(u) \iff \forall\, \xi \in \mathbb{R} \setminus 0, \quad (E, \xi) \in \operatorname{WF}_a(u).$$

Hence suppose that $0 < E \in \operatorname{singsupp}_a u$. Then $(E, 1) \in \operatorname{WF}_a(u)$. Suppose that $\psi \in C_c^\infty(\mathbb{R}; [0, 1])$ is equal to 1 near E. Then Definition (7.13) of the analytic wave front set in terms of the FBI transform shows that there exist $t_j \to E$, $\tau_j \to 1$, $\varepsilon_j \to 0$ and $\lambda_j \to +\infty$ such that

$$\left| \int_{\mathbb{R}} e^{-\lambda_j (t-t_j)^2/2 + i\lambda_j \tau_j (t_j - t))} \chi(t) u(t) dt \right| \geq e^{-\varepsilon_j \lambda_j}. \tag{7.4.21}$$

4. We will apply Theorem 7.9 with

$$W = (E - b/2, E + b/2) + i(-a/2, a), \quad \Omega = (E - b, E + b) + i(-a, a),$$

$0 < b \ll 1$, $0 < a/b^2 \ll 1$, and

$$f_j(z) = M_j^{-1} e^{\lambda_j (i\tau_j (t_j - z) - (t_j - z)^2/2)},$$

where

$$\begin{aligned} M_j &:= \max_{z \in \overline{\Omega} \setminus W} |e^{\lambda_j (i\tau_j (t_j - z) - (t_j - z)^2/2)}| \\ &= \max_{z \in \overline{\Omega} \setminus W} e^{\lambda_j (\tau_j \operatorname{Im} z - (t_j - \operatorname{Re} z)^2/2 + \operatorname{Im} z^2/2)} \\ &\leq e^{-c_0 \lambda_j}, \qquad c_0 > 0, \end{aligned} \tag{7.4.22}$$

for j large enough.

In fact, since $t_j \to E$ and $\tau_j \to 1$ we only need to check this for $t_j = E$ and $\tau_j = 1$:

$$\begin{aligned} \max_{\substack{\frac{b}{2} \leq |\operatorname{Re} z - E| \leq b \\ -a \leq \operatorname{Im} z \leq a}} \operatorname{Im} z - \tfrac{1}{2}(E - \operatorname{Re} z)^2 + \tfrac{1}{2} \operatorname{Im} z^2 &= a + \tfrac{1}{2} a^2 - \tfrac{1}{8} b^2, \\ \max_{\substack{|\operatorname{Re} z - E| \leq \frac{b}{2} \\ -a \leq \operatorname{Im} z \leq -\frac{a}{2}}} \operatorname{Im} z - \tfrac{1}{2}(E - \operatorname{Re} z)^2 + \tfrac{1}{2} \operatorname{Im} z^2 &= -\tfrac{1}{2} a + \tfrac{1}{2} a^2. \end{aligned}$$

Our assumptions on a and b in the definition of Ω and W guarantee that both quantities are bounded from above by $-2c_0$, $c_0 > 0$.

5. Applying (7.4.18) and (7.4.19) with $g = \psi f_j$ we obtain

$$\begin{aligned} &\operatorname{tr} [(\psi f_j)(P_V) - (\psi f_j)(P_0)] \\ &\quad = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \big((\psi f_j)(\xi^2 + V(x)) - (\psi f_j)(\xi^2) \big) d\xi dx + \mathcal{O}_j(h^{1-n}) \\ &\quad = \frac{1}{(2\pi h)^n} \langle u, \psi f_j \rangle + \mathcal{O}_j(h^{1-n}) \\ &\quad = \frac{1}{(2\pi h)^n} M_j^{-1} T_{\lambda_j}(\psi u)(t_j, \tau_j) + \mathcal{O}_j(h^{1-n}). \end{aligned}$$

The trace formula (7.4.5) (with the term $\mathcal{O}_{f_j}(h^\infty)$ absorbed into $C_j h^{1-n}$) and the bounds (7.4.21) (7.4.22) give

$$\begin{aligned} |\sum_{z\in W} f_j(z)| &= \left| \frac{1}{(2\pi h)^n} M_j^{-1} T_{\lambda_j}(\psi u)(t_j,\tau_j) + \mathcal{O}_j(h^{1-n}) + \mathcal{O}(h^{-n}) \right| \\ &\geq c_1 h^{-n} M_j^{-1} e^{-\varepsilon_j \lambda_j} - C_j h^{1-n} - C_0 h^{-n} \\ &\geq c_1 h^{-n} e^{(c_0-\varepsilon_j)\lambda_j} - C_j h^{1-n} - C_0 h^{-n}. \end{aligned}$$

We now fix j large enough so that $\varepsilon_j < c_0/2$ and $e^{\lambda_j c_0/2} > 2 + C_0$. If $h_0 := 1/C_j$ then for $0 < h < h_0$,

$$|\sum_{z\in W} f_j(z)| \geq h^{-n}.$$

But that implies that

$$\max_{z\in W} |f_j(z)| \sum_{z\in W} m_V(z) \geq h^{-n},$$

that is the number of resonances in W is bounded from below by h^{-n}/C. Since for any r we can choose W so that $D(E,r) \subset W$ the estimates (7.4.17) follows. $\square$

7.5. RESONANCE EXPANSIONS FOR STRONG TRAPPING

In §§2.3 and 3.2.2 we saw that solutions of the wave equation $(-\partial_t^2 u - \Delta + V)u = 0$, $V \in L^\infty_{\rm comp}(\mathbb{R}^n;\mathbb{R})$, n odd, with compactly supported initial data can be expanded in terms of resonances – see (3.2.12). The same arguments combined with Theorem 6.21 show that the same result holds for non-trapping metrics: if

$$\begin{gathered} P := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C^\infty_{\rm c}(\mathbb{R}^n), \quad V \in C^\infty_{\rm c}(\mathbb{R}^n), \\ \exp tH_p(x,\xi) \to \infty, \quad t \to \pm\infty, \quad p = |\xi|^2_g, \end{gathered}$$

then as $t \to +\infty$,

$$\begin{gathered} u(x,t) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-i\lambda_j t} u_j(x) + E_A(t,x), \\ \|E_A(t)\|_{H^N(K)} \leq C_K e^{-At} \left(\|u(x,0)\|_{H^1} + \|\partial_t u(x,0)\|_{L^2} \right), \end{gathered} \tag{7.5.1}$$

for solutions of the wave equation,

$$(\partial_t^2 - \Delta_g + V)u = 0, \quad u|_{t=0} \in H^1(U), \quad \partial_t u|_{t=0} \in L^2(U),$$

where $U \Subset \mathbb{R}^n$ is a fixed compact set. Here $\lambda_j^2 \in \operatorname{Res}(P)$, $\operatorname{Im}\lambda_j < 0$. To make the statement clear we assumed for simplicity that resonances are semi-simple (algebraic and geometric multiplicities coincide) – see (3.2.12) for the general statement.

In this section we will consider the case of semiclassical Hamiltonians $P(h) = -h^2\Delta_g + V$ with trapped sets which imply existence of resonances close to the real axis – see §7.3 for examples of that. In that case it is natural to expand $u(t)$ given by the Schrödinger evolution $u(t) = \exp(-itP(h)/h)u_0$. See Theorem 7.20 and Exercise 7.4 for applications of the same methods to the wave equation in odd and even dimensions respectively.

We start with some results in some cases where resonances are *not* close to the real axis. The analysis is based on results from §§6.3 and 6.4.

7.5.1. Schrödinger propagator in the case of resonance free regions. Suppose that for $E > 0$,

$$\begin{gathered} P(h) := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C^\infty_{\rm c}(\mathbb{R}^n), \quad V \in C^\infty_{\rm c}(\mathbb{R}^n), \\ p(x,\xi) := |\xi|^2_g + V = E \implies \exp tH_p(x,\xi) \to \infty, \ t \to \pm\infty. \end{gathered} \tag{7.5.2}$$

Theorem 6.21 shows that there exists $\delta > 0$ such that for any M,

$$\operatorname{Res}(P(h)) \cap [E-\delta, E+\delta] - i[0, Mh\log(1/h)] = \emptyset, \quad 0 < h < h_0(M). \tag{7.5.3}$$

Moreover, for any $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$ we have the following bound on the meromorphically continued resolvent:

$$\begin{gathered} \|\chi R(z,h)\chi\|_{L^2\to L^2} \leq C\frac{\exp C \operatorname{Im} z/h}{h}, \\ z \in [E-\delta, E+\delta] - i[0, Mh\log(1/h)] = \emptyset, \quad 0 < h < h_0(M). \end{gathered} \tag{7.5.4}$$

This has the following consequence for the truncated Schrödinger propagator:

THEOREM 7.15 (**Schrödinger propagator at nontrapping energies**). *Suppose that $P(h)$ satisfies (7.5.2), $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$ and $\psi \in C^\infty_{\rm c}((E-\delta, E+\delta))$ where δ is the same as in (7.5.3) and $E > 0$.*

Then there exists T_0 such that

$$\chi \exp(-tP(h)/h)\psi(P(h))\chi = \mathcal{O}\left(\left(\frac{(t-T_0)_+}{h}\right)^{-\infty}\right)_{L^2\to L^2} \tag{7.5.5}$$

REMARK. When the order of χ and $\psi(P(h))$ is reversed on the left hand side of (7.5.5) we obtain an additional terms $\mathcal{O}(h^\infty)$ on the right hand side. That follows from the pseudodifferential calculus: if $\chi_1 = 1$ on $\operatorname{supp}\chi$ then $(1-\chi_1)\psi(P(h))\chi = \mathcal{O}(h^\infty)_{L^2\to L^2}$ – see for instance [**Zw12**, §14.3].

Proof. 1. Let us write

$$R_\pm(z,h) = (P(h) - z)^{-1}\,, \quad \text{analytic for} \quad \pm \operatorname{Im} z > 0\,, \tag{7.5.6}$$

using the same notation for the meromorphic continuation (this means that, in our standard notationt, $R(z,h) = R_+(z,h)$). Using Stone's formula (Theorem B.10) we write the spectral measure as

$$dE_\lambda = (2\pi i)^{-1}(R_-(\lambda,h) - R_+(\lambda,h))d\lambda\,, \quad \lambda \in \mathbb{R}.$$

Using this, the left hand side of (7.5.5) can be rewritten as

$$\chi e^{-itP(h)/h}\psi(P(h))\chi = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-it\lambda/h}\chi(R_-(\lambda,h) - R_+(\lambda,h))\chi\psi(\lambda)d\lambda\,.$$

(This is were the spectral theory helps in handling $\psi(P(h))$ left to the χ.)
2. Let $\tilde\psi \in C_{\rm c}^\infty(\mathbb{C})$ be an almost analytic extension of ψ – see §B.2. We can choose it so that

$$\operatorname{supp}\tilde\psi \subset \{z : \operatorname{Re} z \in (E-\delta, E+\delta)\}.$$

Green's formula (D.1.1) then gives

$$\chi e^{-itP(h)/h}\psi(P(h))\chi = A(h) + B(h),$$

$$A(h) := \frac{1}{2\pi i}\int_{\operatorname{Im} z = -Mh\log\frac{1}{h}} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\tilde\psi(z)dz$$

$$B(h) := \frac{1}{\pi}\iint_{-Mh\log\frac{1}{h}<\operatorname{Im} z<0} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\bar\partial_z\tilde\psi(z)dm(z).$$

Using bound (7.5.4) on the analytic continuation of the resolvent we see that

$$\begin{aligned}\|A(h)\|_{L^2\to L^2} &\le C'h^{-CM}e^{-tM\log(1/h)} = C'h^{-M(C-t)}\\ &= \mathcal{O}((h/t)^M), \quad t > C+2, \quad 0<h<h_0.\end{aligned}$$

To estimate $B(h)$ we use the property of almost analytic extensions:

$$\bar\partial_z\tilde\psi(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$$

which conbined with (7.5.4) gives

(7.5.7)

$$\begin{aligned}\|B(h)\|_{L^2\to L^2} &\le C_N h^{-1}\int_0^{Mh\log\frac{1}{h}} e^{-st/h}e^{Cs/h}s^N ds\\ &= C_N h^N\int_0^{M\log\frac{1}{h}} e^{-r(t-C)}r^N dr\\ &\le C_N h^N e^{CM\log\frac{1}{h}}\min\left(\int_0^\infty e^{-rt}r^N dr, (M\log(1/h))^{N+1}\right)\\ &\le C_N' h^{N-MC-1}\langle t\rangle^{-N-1}.\end{aligned}$$

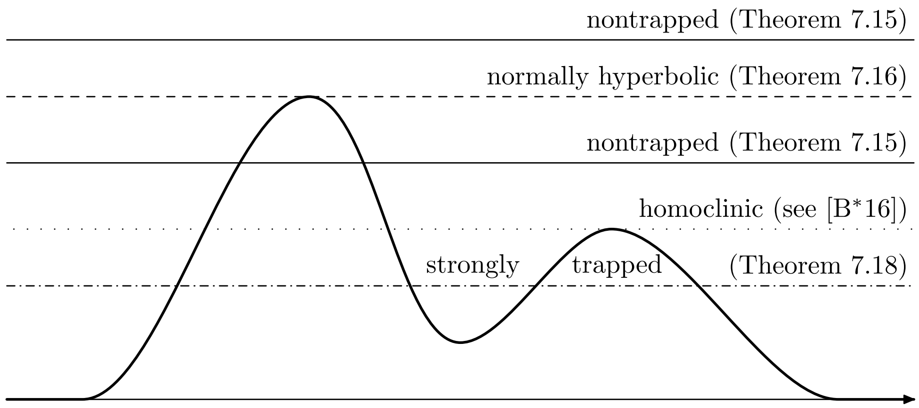


Figure 7.6. An example of energy levels to which Theorems 7.15, 7.16 and 7.18 apply. In the case of a homoclinic orbit we can still use Theorem 7.18 but there is no leading term contribution – see Bony–Fujiie–Ramond–Zerzeri [**B*16**] for a detailed discussion of many interesting cases and for references.

Since N is arbitrary we conclude that

$$B(h) = \mathcal{O}(h^\infty \langle t\rangle^{-\infty}). \tag{7.5.8}$$

Since for all times the propagator is bounded this proves the theorem. □

We now move to the case of *normally hyperbolic trapping* presented in §6.3. We assume that $P(h) = -h^2\Delta_g + V(x)$ and that at $E > 0$, the trapped set $K_E \neq \emptyset$ and that it satisfies the conditions (A1)–(A3) of §6.3. In that case Theorem 6.16 shows that

$$\chi R(z)\chi = o_\varepsilon(h^{-2})_{L^2\to L^2}, \quad z \in [E-\delta, E+\delta] - ih[0, \tfrac{\nu_{\min}-\varepsilon}{2}, 0], \tag{7.5.9}$$

for some δ and any $\varepsilon > 0$.

THEOREM 7.16 (Schrödinger propagator for normally hyperbolic trapping). *Suppose that $P(h)$ is as in (7.5.2) but that at $E > 0$ the trapping is normally hyperbolic in the sense of Definition 6.15.*

If $\chi \in C_c^\infty(\mathbb{R}^n)$ and $\psi \in C^\infty((E-\delta, E+\delta))$ where δ is the same as in (7.5.9), then for any $\nu < \nu_{\min}$,

$$\begin{aligned}\chi \exp(-tP(h)/h)\psi(P(h))\chi = e^{-t\nu/2}o(h^{-2})_{L^2\to L^2}\\ + \mathcal{O}(h^\infty\langle t\rangle^{-\infty})_{L^2\to L^2}.\end{aligned} \tag{7.5.10}$$

Proof. 1. We proceed as in the proof of Theorem 7.15 and use the same notation. For $\nu < \nu_{\min}$ (7.5.9) and Green's formula now give

$$\chi e^{-itP(h)/h}\psi(P(h))\chi = A(h) + B(h),$$

$$A(h) := \frac{1}{2\pi i} \int_{\operatorname{Im} z = -\nu h/2} e^{-itz/h} \chi (R_-(z,h) - R_+(z,h)) \chi \tilde{\psi}(z) dz$$

$$B(h) := \frac{1}{\pi} \iint_{-\nu h/2 < \operatorname{Im} z < 0} e^{-itz/h} \chi (R_-(z,h) - R_+(z,h)) \chi \bar{\partial}_z \tilde{\psi}(z) dm(z).$$

The $B(h)$ term can be estimating in the same way as in (7.5.7) resulting in (7.5.8).

2. Using (7.5.9) we immediately see that $A(h) = e^{-\nu t/2} o(h^{-2})$ which concludes the proof. □

INTERPRETATION. The localization in energy using the cut-off function $\psi(P(h))$ is necessary as different estimates are valid in different energy regimes – see Fig.7.6. At non-trapping energies, once time is large enough the localized propagator (in space *and* energy) is of size $\mathcal{O}(h^\infty t^{-\infty})$, that is negligible.

In the case of normally hyperbolic trapping the localized propagator is semiclasically negligible (that is $\mathcal{O}(h^\infty \langle t \rangle^{-\infty})$) once $t \gg \log(1/h)$. In the intermediate region the the exponentially decaying estimate provides a quantitative decay rate for the propagator.

7.5.2. Schrödinger propagator in the case of resonances converging to the real axis. We now consider the general case of

$$P(h) := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C^\infty_{\mathrm{c}}(\mathbb{R}^n), \quad V \in C^\infty_{\mathrm{c}}(\mathbb{R}^n), \tag{7.5.11}$$

but the results are non-trivial only in the case of existence of resonances with $\operatorname{Im} z = \mathcal{O}(h^\infty)$.

The difficulty in stating a general result is overlap of resonances. Since the results for the Schrödinger equation are semiclassical, Theorems 7.6 (see also Theorem 7.14) show that resonances appear in dense ($\sim h^{-n}$) clouds and the possibility of resonances being very close to each other is inevitable. However, the upper bound (7.2.2) shows that there are always some gaps between clouds of resonances. That motivates the following

DEFINITION 7.17. *A family of rectangles*

$$h \mapsto W(h) := (a(h), b(h)) - i[0, c(h)), \quad 0 < c_0 < a(h) < b(h) < 1/c_0,$$

is called semiclassically admissible *if*

$$d(\partial W(h) \setminus \mathbb{R}, \operatorname{Res}(P(h))) > 2\, c_1 h^n, \quad c_1 h^n < c(h) < 1/c_0. \tag{7.5.12}$$

for some fixed constants $c_0, c_1 > 0$.

REMARK. The power n can be replaced by any larger power. Theorem 7.4 shows that near any fixed a, b and c we can find $a(h), b(h)$ and $c(h)$ so that (7.5.12) holds once c_1 is sufficently small.

THEOREM 7.18 (**Resonance expansion for strong trapping**). *Suppose that $P(h)$ is given by (7.5.11), $\psi \in C^\infty_{\rm c}((0,\infty))$, $\operatorname{supp}\psi = [a,b]$, $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$.*

There exist a constant L_n depending only on the dimension such for any admissible $W(h)$ with $[a,b] \Subset W(h) \cap \mathbb{R}$,

$$t > h^{-L_n} \tag{7.5.13}$$

implies

$$\begin{gathered}\chi \exp(-tP(h)/h)\psi(P(h))\chi = \\ \sum_{z\in W(h)} \chi \operatorname{Res}_{w=z}\left(e^{-itw/h}R(w,h)\right)\chi\psi(P(h)) + \mathcal{O}(h^\infty)_{L^2\to L^2}.\end{gathered} \tag{7.5.14}$$

REMARKS. 1. The result states that for sufficiently large times the localized propagator can be expanded in terms of resonances close to the real axis with a semiclassical negligible error. From Theorem 6.26 (and its generalizations – see [**CV02**],[**Da14**] and references given there) we know the resonances in $W(h)$ satisfy $\operatorname{Im} z > e^{-C/h}$. This means that the expansion is relevant for $h^{-2n-2} < t < e^{C/h}$.

2. The need for admissible rectangles comes from the difficulties in estimating individual terms in the sum over resonances. Unless some, possibly very weak, separation between resonances is imposed we do not know how to estimate the residues.

3. If (7.5.12) $c_1 h^n$ is replaced by h^M for some N then (7.5.14) still holds once L_n in (7.5.13) is changed into an M-dependent constant.

4. The constant L_n in (7.5.13) comes from the proof and is given by $\frac{5}{2}n+\varepsilon$, $\varepsilon > 0$. However, it is not clear what is the optimal general lower bound on t guaranteeing validity of an expansion. In the case of a barrier (see Theorem 7.8) we only need $t \geq T_0$ for some fixed T_0 [**NSZ14**, Main Theorem].

5. The order of χ and $\psi(P(h))$ on the left hand side of (7.5.14) does not matter – see the remark following Theorem 7.15.

Proof. 1. Let $W(h)$ be an admissible rectangle in the sense of Definition 7.17. Since $[a,b] \Subset W(h) \cap \mathbb{R}$ we have

$$a(h) < a < b < b(h).$$

We choose $\psi_h \in C^\infty_{\rm c}(\mathbb{R})$ with satisfying

$$\begin{gathered}\operatorname{supp}\psi_h \subset (a(h) - c_1h^n, b(h) + c_1h^n), \\ \psi_h(s) = 1 \ \text{ for } s \in [a(h), b(h)].\end{gathered}$$

In particular, $\psi_h \equiv 1$ on the support of ψ. Functional calculus (see the remark following Theorem 7.15) then shows that

$$\begin{aligned}\chi \exp(-tP(h)/h)\psi(P(h))\chi &= \chi \exp(-tP(h)/h)\psi_h(P(h))\psi(P(h))\chi \\ &= \chi \exp(-tP(h)/h)\psi_h(P(h))\chi\psi(P(h)) + \mathcal{O}(h^\infty)_{L^2\to L^2}.\end{aligned}$$

This means that we can replace the left hand side of (7.5.14) by the first term in the last line.

3. To apply the same procedure as in Step 2 of the proof of Theorems 7.15 we consider an almost analytic extension of ψ_h. By taking an almost extensions of $s \mapsto \psi_h(h^n(s-\alpha(h))$, $\alpha(h) = a(h), b(h)$, we obtain $\tilde\psi_h \in C^\infty(\mathbb{C})$ satisfying

$$\operatorname{supp}\tilde\psi_h \subset (a(h) - c_1h^n, b(h) + c_1h^n) + i\mathbb{R}, \quad \partial_{x,y}\tilde\psi_h = \mathcal{O}(h^{-n}),$$
$$\bar\partial_z\tilde\psi_h(z) = \begin{cases} \mathcal{O}(|\operatorname{Im} z|/h^n)^\infty, \\ 0 \ \text{ if } |\operatorname{Re} z - a(h)| > c_1h^n, \\ 0 \ \text{ if } |\operatorname{Re} z + b(h)| > c_1h^n. \end{cases} \tag{7.5.15}$$

Applying Green's formula (D.1.1) and using the notation (7.5.6) we obtain

$$\begin{aligned} \chi e^{-itP(h)/h}\psi_h(P(h))\chi &= A(h) + B(h) + C(h), \\ A(h) &:= \frac{1}{2\pi i}\int_{\operatorname{Im} z = c(h)} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\tilde\psi(z)dz \\ B(h) &:= \frac{1}{\pi}\iint_{W(h)} e^{-itz/h}\chi(R_-(z,h) - R_+(z,h))\chi\bar\partial_z\tilde\psi(z)dm(z), \\ C(h) &:= \sum_{w\in W(h)} \chi\operatorname{Res}_{z=w}\left(e^{-itz/h}R(z,h)\right)\chi. \end{aligned} \tag{7.5.16}$$

We note that (7.5.15) shows that $\tilde\psi_h(w) = 1$ for $w \in W(h)\cap\operatorname{Res}(P(h))$ and hence there is no $\tilde\psi_h$ in the formula for $C(h)$.

4. To estimate $A(h)$ we note that (7.5.12)

$$d(\{\operatorname{Im} z = c(h)\}\cap\operatorname{supp}\tilde\psi_h, \operatorname{Res}(P(h))) > c_1h^n.$$

The estimate (7.2.12) in Theorem 7.5 then shows that

$$\|\chi R_+(z,h)\chi\|_{L^2\to L^2} \le Ae^{Ah^{-n}\log\frac{1}{h}}, \tag{7.5.17}$$

for $z \in \{\operatorname{Im} z = c(h)\}\cap\operatorname{supp}\tilde\psi_h$. Since (7.5.6) shows that the norm of $R_-(z,h)$ is bounded by $1/c(h) = \mathcal{O}(h^{-n})$ on for $\operatorname{Im} z = c(h)$, we obtain

$$\begin{aligned}\|A(h)\|_{L^2\to L^2} &\le Ce^{-tc(h)/2h}e^{Ah^{-n}\log\frac{1}{h}} \\ &= \mathcal{O}(h^\infty), \qquad \text{for } t \ge h^{-n+1-\varepsilon},\ \varepsilon > 0.\end{aligned} \tag{7.5.18}$$

5. To estimate $B(h)$ we split it into two terms:

$$B(h) = B_1(h) + B_2(h)$$

where the integration is over $\operatorname{Im} z > -h^M$ and $\operatorname{Im} z < -h^M$ respectively, where $M = M_n > n$ depending only on dimension n will be chosen later. We start with B_2 and we use the estimate (7.5.17) for both $R_\pm$ (since we are now near the real axis and get no benefit from the sign of $\operatorname{Im} z$) and the properties of $\bar{\partial}_z \tilde{\psi}_h$ (see (7.5.15)):

$$\begin{aligned} \|B_2(h)\|_{L^2\to L^2} &\leq Ch^{-n} \int_{s=h^M}^{2c_0} e^{-ts/h} e^{Ah^{-n}\log\frac{1}{h}} ds \\ &\leq Ch^{-n} e^{-th^{M-1}+Ah^{-n}\log\frac{1}{h}} \\ &= \mathcal{O}(h^\infty), \quad \text{for } t \geq h^{-n+1-\varepsilon},\ \varepsilon > 0. \end{aligned} \tag{7.5.19}$$

6. To analyse $B_1(h)$ we apply the maximum principle in the form presented in Lemma 7.7 using the estimates (7.5.17) and $\|R_\pm(z,h)\|_{L^2\to L^2} \leq 1/|\operatorname{Im} z|$, $\pm \operatorname{Im} z > 0$. We take $R = c_1 h^n$ and $L = n+\varepsilon$ (see (7.5.17): $\varepsilon > 0$ is arbitrary and will change from line to line). Hence we need $h^{2n} > Ch^{-3n-\varepsilon}\delta(h)^2$, which means that $\delta(h) < h^{\frac{5}{2}n+\varepsilon}$. Then the estimate (7.3.11) in the remark after the proof of Lemma 7.7 gives

$$\|\chi R_\pm(z,h)\chi\|_{L^2\to L^2} \leq Ch^{-\frac{5}{2}n-\varepsilon}, \quad |\operatorname{Im} z| \leq h^{\frac{5}{2}n+\varepsilon}, \tag{7.5.20}$$

for any $\varepsilon > 0$. We now take $M := \frac{5}{2}n + \varepsilon$ in the splitting of B into $B_1 + B_2$.

The bound (7.5.20) and (7.5.15) give the following estimate

$$\begin{aligned} \|B_2(h)\|_{L^2\to L^2} &\leq C_N \int_0^{h^M} e^{-st/h} h^{-Q} (s/h^n)^N ds \\ &= C_N h^{n-Q} \int_0^{h^{M-n}} e^{-rth^{n-1}} r^N dr \\ &\leq C_N h^{n-Q+N(M-n)}. \end{aligned} \tag{7.5.21}$$

Combining (7.5.18), (7.5.19) and (7.5.21) with (7.5.16) gives (7.5.14). □

7.5.3. Expansions of scattered waves for strong trapping. We now adapt the results of this chapter to study the wave equation. The results are valid for black box Hamiltonians of §4.1 and in particular for obstacle problems or manifolds which have Euclidean ends – see [**BZ01**]. To keep the presentation simple we restrict ourselves to the case of $P = -\Delta_g \geq 0$ where g is a Riemannian metric on $\mathbb{R}^n$ satisfying $g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n)$.

We obtain a resonance expansion of scattered waves in the case of resonances converging to the real axis. It is much weaker than the expansion (7.5.1) valid in the non-trapping case but it addresses a more realistic situation.

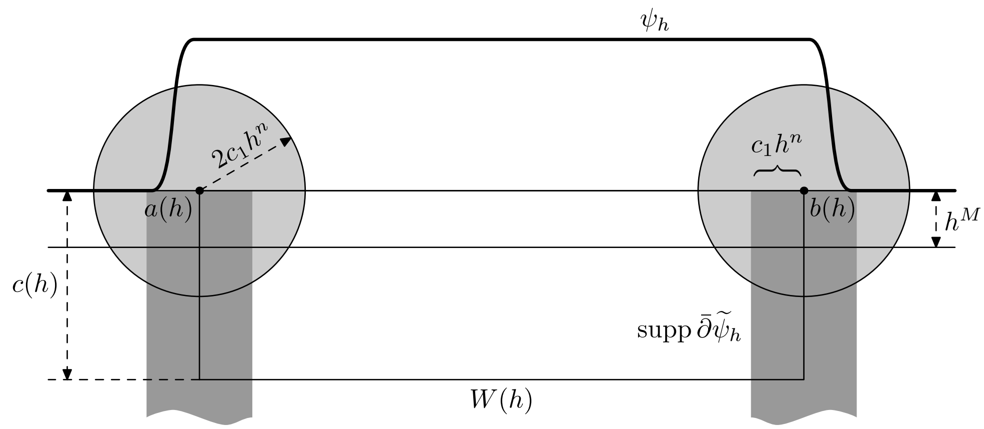


Figure 7.7. The almost analytic continuation and the contour deformation for the semiclassical expansion.

In the case of the wave equation we replace admissible rectangles of Definition 7.17 by admissible contours:

DEFINITION 7.19. *An admissible contour is the posively oriented contour given by*

$$\Gamma := \{z = x - i\gamma(x), x \in \mathbb{R}\}$$

where $\gamma(x) > 0$ *and for some* M, N *and* $c > 0$,

$$\begin{gathered} c\langle x\rangle^{-M} < \gamma(x) < 1/c, \quad |\gamma'(x)| \le 1/c, \\ d((x+i\gamma(x))^2, \operatorname{Res}(P)) > c\langle x\rangle^{-N}. \end{gathered} \tag{7.5.22}$$

REMARK. The global bound on the number of resonances in Theorem 4.13 (or rescaling of the local semiclassical bound (7.2.2)) show that we can find admissible contours for any M and for $N \ge n$.

THEOREM 7.20 (Resonance expansions of scattered waves). *Suppose that* n *is odd and* $P = -\Delta_g$, $g_{ij} - \delta_{ij} \in C^\infty_{\rm c}(\mathbb{R}^n, \mathbb{R})$ *and put*

$$U(t) := \sin t\sqrt{P}/\sqrt{P}, \quad R(\lambda) := (P-\lambda^2)^{-1},$$

with $R(\lambda)$ *first defined for* $\operatorname{Im}\lambda > 0$ *and the continued meromorphically. Let* $\chi \in C^\infty_{\rm c}(\mathbb{R}^n)$ *and let* Γ *be an admissible contour in the sense of Definition 7.19. Then for any* $M > M_0$ *there exists* $\varepsilon > 0$ *and a function* $t \mapsto c(t)$, $|c(t) - t^\varepsilon| \le C$ *such that*

$$\chi U(t)\chi = \sum_{\substack{\operatorname{Im}\lambda > -\gamma(\operatorname{Re}\lambda) \\ 1 < |\operatorname{Re}\lambda| < c(t)}} \chi i \operatorname{Res}_{\zeta=\lambda}\left(e^{-i\zeta t}R(\zeta)\right)\chi + E_\chi(t), \tag{7.5.23}$$

where

$$\|E_\chi(t)\|_{H^L\to L^2} =\le Ct^{-\varepsilon(L-K_0)} \tag{7.5.24}$$

with $L \geq 0$ arbitrary and K_0 a fixed constant.

INTERPRETATION. As in Theorems 2.9 and 3.11 the residues give expansions of solutions to $(\partial_t^2 - \Delta_g)w = 0$ with compact initial data: for $f \in C_{\rm c}^\infty(\mathbb{R}^n)$,

$$\operatorname{Res}_{\zeta=\lambda_j}\left(e^{-i\zeta t}R(\zeta)\right)f(x) = \sum_{\ell=0}^{m_j} t^\ell e^{-i\lambda_j t} f_{\ell,j}(x). \tag{7.5.25}$$

The error term $E_\chi(t)f$ is then $\mathcal{O}(t^\infty)$ but each term in (7.5.25) eventually decays much faster. However, we include more terms as $t \to +\infty$ ($|\operatorname{Re}\lambda_j| \leq c(t) \sim t^\varepsilon$). Hence resonances with $\operatorname{Im}\lambda_j = \mathcal{O}(\langle\lambda_j\rangle^{-\infty})$ provide non-trivial contributions to the expansion.

Proof. 1. Theorem 4.19 shows that there is no eigenvalue or resonance at 0. We start as in the proof of Theorem 2.9 and obtain the analogue of the formula displayed after (2.3.11):

$$\chi U(t)\chi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}\chi(R(\zeta) - R(-\zeta))\chi d\zeta. \tag{7.5.26}$$

2. Let $c(t) \sim t^\varepsilon$ be a function of t yet to be chosen and let Γ be an admissible contour. We define

$$\Gamma_1 := \Gamma \cap \{|\operatorname{Re}\lambda| \leq c(t)\}, \quad \Gamma_3 := \mathbb{R} \setminus (-c(t), c(t)),$$
$$\Gamma_2 := \{-c(t) - i\tau : \tau \in (0, \gamma(c(t)))\} \cup \{c(t) - i\tau : \tau \in (0, \gamma(c(t)))\},$$

with the natural orientation agreeing with the positive orientation of $\mathbb{R}$. By deforming the contour in (7.5.26) to $\Gamma_1 + \Gamma_2 + \Gamma_3$ (with the natural orientations)

$$\chi U(t)\chi = \sum_{\substack{\operatorname{Im}\lambda > -\gamma(\operatorname{Re}\lambda) \\ 1 < |\operatorname{Re}\lambda| < c(t)}} \chi i \operatorname{Res}_{\zeta=\lambda}\left(e^{-i\zeta t}R(\zeta)\right)\chi + V_1(t) + V_2(t) + V_3(t),$$
$$V_j(t) := \frac{1}{2\pi}\int_{\Gamma_j} e^{-i\zeta t}\chi(R(\zeta) - R(-\zeta))\chi d\zeta.$$

We need to show that we can choose $\varepsilon > 0$ and $c(t) \sim t^\varepsilon$ so that

$$V_j(t) = \mathcal{O}(t^{-\varepsilon(L-K_0)})_{H^L \to L^2}. \tag{7.5.27}$$

3. We start with V_1. The separation of Γ_1 from the set of resonances given in (7.5.22) and the estimate (7.2.13) show that

$$\chi(R(\zeta) - R(-\zeta)\chi = \mathcal{O}(e^{A|\zeta|^n \log|\zeta|}), \quad \zeta \in \Gamma_1.$$

Hence

$$\begin{aligned}
\|V_1(t)\|_{L^2\to L^2} &\le Ce^{-at} + C\int_1^{c(t)} e^{-t\gamma(x)}e^{Ax^n\log x}(1+|\gamma'(x)|)dx \\
&\le Ce^{-at} + C'\int_1^{t^\varepsilon} e^{-tx^{-M}+x^n\log x}dx \\
&\le Ce^{-at} + C't^\varepsilon e^{-t^{1-M\varepsilon}+t^{\varepsilon n}\log t} \\
&= \mathcal{O}(t^\infty), \quad \text{if } \varepsilon < \tfrac{1}{M+n}.
\end{aligned} \tag{7.5.28}$$

Hence for ε small enough depending on M (that is, on the admissible contour) we have (7.5.27) for $j = 1$.

4. To estimate $V_3(t)$ we note that Stone's formula (Theorem B.10) shows that

$$(R(\lambda) - R(-\lambda))(I - \Delta_g)^{L/2} = (R(\lambda) - R(-\lambda))\langle\lambda\rangle^L, \quad |\lambda| > 1.$$

Hence for $f \in H^L$

$$V_3(t)f = \int_{\mathbb{R}\setminus[-c(t),c(t)]} e^{-i\zeta t}\chi(R(\zeta) - R(-\zeta))\langle\zeta\rangle^{-L}(I - \Delta_g)^{L/2}\chi f d\zeta$$

Another application of Stone's formula (see (2.3.11)) gives for an odd function φ

$$\begin{aligned}
\frac{1}{\pi i}\int_{c(t)}^\infty (R(\zeta) - R(-\zeta))\varphi(\zeta)d\zeta &= \int_{c(t)}^\infty \frac{\varphi(\lambda)}{\lambda}dE_\lambda \\
&= \frac{\varphi(\sqrt{-\Delta_g})}{\sqrt{-\Delta_g}}\,\mathbb{1}_{[c(t),\infty)}(\sqrt{-\Delta_g}) \\
&= \mathcal{O}(\sup|\varphi(\zeta)/\zeta|)_{L^2\to L^2}.
\end{aligned}$$

Applied with

$$\varphi(\zeta) := \langle\zeta\rangle^{-L}\,\mathbb{1}_{[c(t),\infty)}(\zeta)e^{-i\zeta t},$$

we obtain

$$\|V_3(t)f\|_{L^2} \le \langle c(t)\rangle^{-L-1} \le Ct^{-\varepsilon L-\varepsilon},$$

which gives (7.5.27) for $j = 3$.

5. We now come to estimating $V_2(t)$ and this is where a choice of $c(t)$ is essential. We choose $c(t)$ so that for some fixed N

$$|c(t) - t^\varepsilon| \le 1, \quad D((\pm c(t) + i[0, 1/c])^2, \mathrm{Res}(P)) > t^{-N}. \tag{7.5.29}$$

This can always be accomplished for $N > n$ because of the upper bound on the number of resonances. As in Step 5 of the proof of Theorem 7.18, the bound (7.2.13) and Lemma 7.7 imply that there exist M and Q such that

$$\|\chi R(\pm c(t) + iy)\chi\|_{L^2\to L^2} \le Cc(t)^M, \quad |y| \le c(t)^{-Q}.$$

Using this bound and the bound (7.2.13) (valid on Γ_2 because of (7.5.29)) we estimate V_2 as follows. Let $\chi_0 \in C_c^\infty$ be equal to 1 on $\operatorname{supp}\chi$. Then for for $f \in H^L$,

$$\begin{aligned}\|V_2(t)f\|_{L^2} &\leq Ct^{-\varepsilon L}\int_{-\gamma(c(t))}^{\gamma(c(t))} \|\chi_1 R(c(t)+iy)\chi_1\|_{L^2\to L^2} e^{-yt} dy \|\chi f\|_{H^L} \\ &\leq C' t^{-\varepsilon L}\int_0^{t^{-\varepsilon Q}} t^{\varepsilon M} e^{-ty} dy + C' t^{-\varepsilon L}\int_{t^{-\varepsilon Q}}^{c} e^{-ty + At^{\varepsilon n}\log t} dy \\ &\leq C' t^{-\varepsilon(L+M-Q)} + C'' t^{-\varepsilon L}, \ \text{ if } \varepsilon < \tfrac{1}{n+1}.\end{aligned}$$

This (7.5.27) for $j = 2$ with $K_0 = Q - M$. □

7.6. NOTES

Theorem 7.1 was proved by Bony, Burq and Ramond [**BBR10**]. The comment that C is independent of δ was made by Jean-François Bony. For more connections between resolvent estimates and local smoothing for Schrödinger propagators see [**Bu04**],[**Da09**], and references given there.

The local upper bounds on the number resonances were first established by Sjöstrand [**Sj90**]. That paper also introduced geometric bounds in which the number of resonances was estimated using the dimension of the trapped set (6.1.3) and that started develompment of *fractal Weyl laws*. For more recent developments and references see [**SZ07a**], Datchev–Dyatlov [**DD13**], Nonnenmacher–Sjöstrand–Zworski [**NSZ14**], and also Naud [**Na14**], [**Na15**] and Dyatlov [**Dy19**]. The only fractal *lower bound* was obtained in the setting of quantum maps by Nonnenmacher–Zworski [**NZ07**], see also [**No11**] and [**NSZ11**].

For a physics perspective on counting resonances see Lin [**Li02**], Lu–Sridhar–Zworski [**LSZ03**], Potzuweit et al [**P*12**], Körber et al [**KMBK13**] and references given there. A somewhat distant offshoot of these developments was the study of fractal Weyl laws for networks – see the survey [**EFS15**] by Ermann, Frahm and Shepelyansky.

The observation that bounds on the number of resonances imply the bound on the resolvent was made in [**Zw90**]. It was used by Stefanov–Vodev [**SV96**] to show that a sequence of quasimodes with energies converging to infinity implies existence of resonances converging fast to the real axis. The argument is related classical works on the completeness of sets of eigenfunctions going back to Carleman [**Ca36**]. A quantitative generalization was given by Tang–Zworski [**TZ98**] and that was refined further by Stefanov [**St99**] – Theorem 7.6 comes from [**TZ98**] and Stefanov's more precise version is presented in Exercise 7.1. For exponentially decaying potentials (in which case resonances are defined in a strip and complex scaling may not be available), Theorem 7.6 was proved by Gannot [**Gan15**].

The local trace formula for resonances (7.4.5) was proved by Sjöstrand [**Sj96a**] in much greater generality and without the $\mathcal{O}_f(h^\infty)$ error term. That term is irrelevant to our application to lower bounds which comes from Sjöstrand [**Sj96b**]: §7.4 is meant as an introduction to these two papers in a simpler setting. Refinements of the trace formula were given in Petkov–Zworski [**PZ01**] and in Bruneau–Petkov [**BP03**]. In particular it was shown there that the bound (7.4.3) only needs to hold in $\Omega_- \cap W$. The two papers were motivated by the closely related semiclassical version of the Breit–Wigner approximation for the derivative of the scattering phase (scattering shift). Trace formulas can also be used to obtain lower bounds on the number of resonances using a singularity (in the wave trace) generated by a closed orbit – see Sjöstrand–Zworski [**SZ93**], Guillopé–Zworski [**GZ99**]. The local trace formula allows for the same strategy in semiclassical problems – see Bony [**Bo02**], Bony–Sjöstrand [**BS01**] and Dimassi–Zerzeri [**DZ03**]. Related ideas have also been used in the setting of hyperbolic dynamical systems in Jin–Naud–Zworski [**JZ16**] (see [**Zw17**, Chapter 4] for a general introduction and references).

Under certain assumptions, Weyl asymptotics $\sim h^{-n}$ for the number of resonances of $-h^2\Delta + V$, $V \in C_{\rm c}^\infty(\mathbb{R}^n;\mathbb{R})$ with a small (with respect to h) random perturbation, were obtained by Sjöstrand [**Sj14**].

The presentation of resonance expansions in §7.5 comes from Burq–Zworski [**BZ01**] with some corrections and slight generalizations. Under specific assumptions which imply isolation of resonances better results are possible – see Gérard–Martinez [**GM89**], Merkli–Sigal [**MS99**] Nakamura–Stefanov–Zworski [**NSZ14**], Soffer–Weinstein [**SW98**] and Stefanov [**St01**]. When some global conditions on separation of resonances are imposed stronger expansions for the wave equation were given by Tang–Zworski [**TZ00**]. The complex analytic techniques used for the resonance expansions have an older tradition in the study of other non-self-adjoint problems such as damped wave equations – see Markus [**Mar88**].

7.7. EXERCISES

Section 7.3

1. Prove Stefanov's [**St99**] refinement of Theorem 7.6: suppose that we have $u_j(h)$, $j = 1, \cdots, N(h)$ each satisfying the assumptions of Theorem 7.6 with $E_j(h) \in [a(h), b(h)]$, $0 < a_0 < a(h) \le b(h) < b_0$:

$$(P(h) - E_j(h))u_j(h) = \varepsilon_0(h), \quad \|u_j(h)\| = 1, \quad \operatorname{supp} u_j(h) \subset \Omega \Subset \mathbb{R}^n$$

where Ω is independent of h and $\varepsilon_0(h) = \mathcal{O}(h^\infty)$ or all $\varepsilon_0(h) = \mathcal{O}(e^{-S_0/h})$, $S_0 > 0$. Suppose in addition that $u_j(h)$ are approximately orthogonal in the sense that

$$|\langle u_j(h), u_k(h)\rangle - \delta_{kj}| \leq \delta(h), \tag{7.7.1}$$

where $\delta(h) = \mathcal{O}(h^\infty)$. Then there exists $\varepsilon(h)$ satisfying (7.3.6)

$$|\operatorname{Res}(P(h)) \cap [a(h) - \varepsilon(h), b(h) + \varepsilon(h)] - [0, \varepsilon(h)]| \geq N(h). \tag{7.7.2}$$

We outline the steps of the proof (see [**St99**] for more details):

1. Show that

$$N(h) = \mathcal{O}(h^{-n}). \tag{7.7.3}$$

(**Hint:** construct a self-adjoint operator with a discrete spectrum for which u_j's are quasimodes; use (7.7.1) and the spectral theorem to show that the number of eigenvalues close to $E_j(h)$'s is at least $N(h)$. Then use the bound (7.3.3).)

2. With $\varepsilon(h)$ to be chosen (and satisfying (7.3.6)) let z_j, $j = 1, \cdots, M_0(h)$ be the resonances of $P(h)$ in $\Omega(h) := (a(h) - \varepsilon(h), b(h) + \varepsilon(h)) - (0, \varepsilon(h))$. Let

$$\Pi_j := \frac{1}{2\pi i}\oint_{z_j} \chi R(z,h)\chi dz,$$

where the integral is over a circle containing only z_j. Let Π be the orthogonal projection of L^2 onto

$$\Pi_1 L^2 + \Pi_2 L^2 + \cdots + \Pi_{M_0(h)} L^2 \subset L^2.$$

Then $(1 - \Pi)\chi R(z,h)\chi$ is holomorphic in $\Omega(h)$. (**Hint:** use Theorem 4.7).

3. Choosing $\varepsilon(h)$ suitably apply Lemma 7.7 to

$$Q(z,h) := (1 - \Pi)\chi R(z,h)\chi$$

to obtain

$$\|(1 - \Pi)\chi R(E,h)\chi\| \ll h^n / \max \|\varepsilon_0(h)\|, \quad E \in [a(h), b(h)]. \tag{7.7.4}$$

4. As in Step 2 of the proof of Theorem 7.6 use (7.7.4) to show that

$$\|(1 - \Pi)u_j(h)\| \ll h^n.$$

From this and (7.7.1),(7.7.3) deduce that

$$|\langle \Pi u_j(h), \Pi u_k(h)\rangle - \delta_{jk}| \ll 1/N(h). \tag{7.7.5}$$

5. Show that if $f_j \in L^2$, $j = 1, \cdots, N$ and $|\langle f_j, f_k\rangle - \delta_{jk}| < 1/N$ then the set of f_j's is linearly independent. Deduce from (7.7.5) that the rank of Π is at least $N(h)$. This proves (7.7.2).

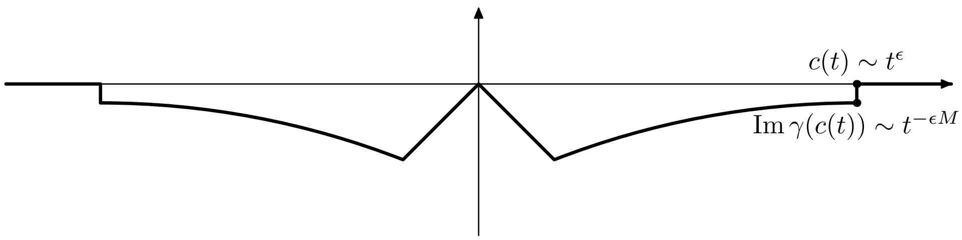


Figure 7.8. An admissible contour for even dimensions.

2. Use (7.7.2) to show that for $P(h) = -h^2\Delta_g + V$ with V and E satisfying (7.3.2), we have for some $S, \delta > 0$,

$$|\operatorname{Res}(P(h)) \cap [E - \delta, E + \delta] - i[0, e^{-S/h}]| \geq h^{-n}/C.$$

(In fact, using Exercise 7.3 one can show the asymptotic formula (7.3.3) for resonances – see [**NSZ03**, Corollary, §5].)

3. Show the following stronger version of Theorem 7.8: under the same assumptions, suppose that $u_\theta(h)$ satisfies $(P_\theta(h) - z(h))u_\theta = 0$, $\|u_\theta\|_{L^2(\Gamma_\theta)} = 1$, that is $u_\theta(h)$ is a resonant state. (Here, as in the proof of Theorem 7.8, $P_\theta(h)$ is a complex scaled operator.) Let $\chi \in C_c^\infty(\mathbb{R}^n, [0, 1])$ satisfy (7.3.17). If $|\operatorname{Re} z(h) - E| < \delta$ then either

$$\operatorname{Im} z > -e^{-S/h}, \quad \|\chi u_\theta(h)\| = 1 + \mathcal{O}(e^{-S/h})$$

or

$$\operatorname{Im} z < -Mh\log(1/h), \quad \|\chi u_\theta(h)\| = \mathcal{O}(e^{-S/h}),$$

for some $S > 0$ and arbitrarily large M arbitrarily large, provided that h is small enough. (**Hint.** From the proof of Theorem 7.8 we see that the only new result is the statement that $\|\chi u_\theta\| = 1 + \mathcal{O}(e^{-S/h})$ when the resonance is close to the real axis. Hence suppose that $\|(1 - \chi)u_\theta\| \geq e^{-S/h}$ and define $v := u_\theta/\|(1-\chi)u_\theta\|$. Then $1 \leq \|v\| \leq e^{S/h}$ and if $S > 0$ small enough Agmon estimates (see [**Zw12**, §7.1]) show that $\|[P(h), \chi_1]v\| \leq e^{-\delta/h}$, where, $\chi_1 \in C_c^\infty$, $(1-\chi_1)(1-\chi) = 1-\chi$, and $P_\theta(h)$ and $P(h)$ coincide on the support of $1 - \chi_1$. Applying the argument in Step 5 of the proof of Theorem 7.8 provides a contradiction.)

Section 7.5

4. This problem gives an analogue of (7.5.23) for n even – see [**BZ01**, Theorem 2].

1. Assume that P is as in Theorem 7.20 and that n is even and $n \geq 4$. The same argument as in the proof of Theorem 4.19 shows that there cannot be

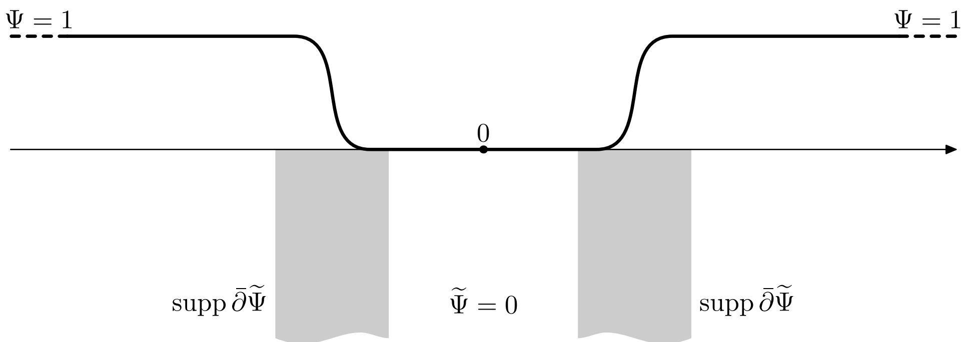


Figure 7.9. The almost analytic continuation and the contour deformation for part 2 of Exercise 7.4.

a resonance or an eigenvalue at 0. The cut-off resolvent then satisfies

$$\chi R(\zeta)\chi = F(\zeta, \zeta^{n-2}\log\zeta), \zeta \in \mathbb{C} \setminus i(-\infty, 0], \tag{7.7.6}$$

where $F(\zeta, \omega)$ is holomorphic near $(0, 0)$. (See Vodev [**Vo94a**].) Use (7.7.6) to show that (7.5.23) holds with

$$\|E_\chi(t)\|_{H^L \to L^2} \leq Ct^{-n+1}, \quad t \to +\infty$$

provided that L is large enough. (**Hint:** use the contour shown in Fig. 7.8 and then deform it near 0 to intervals along the negative imaginary axis.)

2. Let $\Psi \in C^\infty(\mathbb{R}; [0, 1]$ be an even function equal to 0 near 0 and 1 for $s > 1$. With the same assumptions as in part one , show that (7.5.23) and (7.5.24) hold for $\chi U(t)\chi$ replaced by $\chi U(t)\Psi(\sqrt{P})\chi$. (**Hint:** use the contour deformation with the almost analytic extension of Ψ shown in Fig. 7.9.)

Part 4

APPENDICES

Appendix A

NOTATION

A.1 Basic notation
A.2 Functions
A.3 Spaces of functions
A.4 Operators
A.5 Estimates
A.6 Tempered distributions
A.7 Distributions on manifolds and Schwartz kernels

A.1. BASIC NOTATION

We list here some basic notational conventions used in the book. The index provides pointers to the more specialized items.

We stress that unless confusion is likely, or specific values are important, C denotes a constant the value of which may change from line to line. This convention is often assume implicitely.

$\mathbb{R}_+ = (0, \infty)$

$\mathbb{R}^n =$ n-dimensional Euclidean space

x, y denote typical points in $\mathbb{R}^n$: $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

$z = (x, \xi), w = (y, \eta)$ denote typical points in $\mathbb{R}^n \times \mathbb{R}^n$:
$z = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$, $w = (y_1, \dots, y_n, \eta_1, \dots, \eta_n)$

$\mathbb{C}$ = complex plane

$\mathbb{C}^n$ = n-dimensional complex space

$U \Subset V$ means $\bar{U}$ is a compact subset of V

$\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i =$ inner product on $\mathbb{C}^n$

$|x| = \langle x, x\rangle^{1/2}$

$\langle x\rangle = (1 + |x|^2)^{1/2}$

A^T = transpose of the matrix A

I denotes both the identity matrix and the identity mapping.

$|S|$ = cardinality of a finite set S

A.2. FUNCTIONS

The support of a function is denoted "supp", and a subscript "c" on a space of functions means those with compact support.

• Partial derivatives:

$$\partial_{x_j} := \frac{\partial}{\partial x_j}, \quad D_{x_j} := \frac{1}{i}\frac{\partial}{\partial x_j}, \quad \varphi_{x_j} = \partial_{x_j}\varphi.$$

• Multiindex notation: A multiindex is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$, the entries of which are nonnegative integers. The size of α is

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

We then write for $x \in \mathbb{R}^n$:

$$x^\alpha := {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n},$$

where $x = (x_1, \dots, x_n)$. Also

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}, \quad D^\alpha := \frac{1}{i^{|\alpha|}}\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}.$$

If $\varphi : \mathbb{R}^n \to \mathbb{R}$, then we write

$$\nabla\varphi = \partial\varphi := (\varphi_{x_1}, \dots, \varphi_{x_n}) = \text{gradient}.$$

• Poisson bracket: If $f, g : \mathbb{R}^{2n} \to \mathbb{R}$ are C^1 functions,

$$\{f, g\} := \langle \partial_\xi f, \partial_x g\rangle - \langle \partial_x f, \partial_\xi g\rangle = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}. \tag{A.2.1}$$

More generally the Poisson bracket can be defined for functions on the cotangent bundle T^*M of any manifold M, see §E.1.3.

• Fourier transform:

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^n} \varphi(x) e^{-i\langle x,\xi\rangle} dx. \tag{A.2.2}$$

(See also §A.6.)

A.3. SPACES OF FUNCTIONS

Here we present basic notation for functions and distributions on $\mathbb{R}^n$: tempered distributions are discussed in §A.6 the case of manifolds is discussed in §A.7 and §E.1.8.

• $C^\infty(\mathbb{R}^n)$ denotes the space of smooth functions on $\mathbb{R}^n$ and $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions of compact support, that is of smooth functions which vanish outside of a compact set.

• $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions on $\mathbb{R}^n$ (dual space of $C_c^\infty(\mathbb{R}^n)$) and $\mathcal{E}'(\mathbb{R}^n)$ the space of distributions with compact supports (dual space of $C^\infty(\mathbb{R}^n)$ – see [**HöI**, §2.1, §2.3] respectively.

• $L^2(\mathbb{R}^n)$ denotes the space of square integrable functions with respect to the Lebesgue measure on $\mathbb{R}^n$. We also write

$$L^2_{\rm comp}(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : \exists\, R > 0 \ |x| > R \Longrightarrow |u(x)| = 0\}.$$

$$L^2_{\rm loc}(\mathbb{R}^n) := \{u \in \mathcal{D}'(\mathbb{R}^n) : \forall\, \chi \in C_c^\infty(\mathbb{R}^n) \ \chi u \in L^2(\mathbb{R}^n)\}.$$

The same notation is used in the case of Sobolev spaces, $H^s(\mathbb{R}^n)$.

A.4. OPERATORS

We present our notational conventions for operators. Spectral and Fredholm theories of operators are presented in Appendices B and C.

• A^* = adjoint of the operator A

$[A, B] = \mathrm{ad}_A B = AB - BA$ = commutator of A and B

$\sigma(A)$ = symbol of the pseudodifferential operator A

$\mathrm{Spec}(A)$ = spectrum of A.

$\mathrm{tr}(A)$ = trace of A.

• If $A : X \to Y$ is a bounded linear operator, we define the operator norm

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\}.$$

We will often write this norm as

$$\|A\|_{X\to Y}$$

when we want to emphasize the spaces between which A maps.

The space of bounded linear operators from X to Y is denoted $\mathcal{L}(X, Y)$; and the space of bounded linear operators from X to itself is denoted $\mathcal{L}(X)$.

A.5. ESTIMATES

We group here some basic estimates and conventions.

• If we want to specify boundedness in the space X, we write

$$f = \mathcal{O}(F)_X$$

to mean that there exists a constant C such that

$$\|f\|_X \leq CF.$$

If f_t depends on a parameter t and bound depends on the parameter we write

$$f_t = \mathcal{O}_t(F)_X$$

to mean that that for each t there exists $C(t)$ such that

$$\|f\|_X \leq C(t)F.$$

• If A is a bounded linear operator between the spaces X, Y, we will often write

$$A = \mathcal{O}(F)_{X\to Y}$$

to mean that for some constant C,

$$\|A\|_{X\to Y} \leq CF.$$

• We write

$$f = \mathcal{O}(h^\infty) \quad \text{as } h \to 0$$

if for each positive integer N there exists a constant C_N such that

$$|f| \leq C_N h^N \quad \text{for all } 0 < h \leq 1.$$

• Grönwall's inequality states that if $B(s) \geq 0$ and

$$F(s) \leq A(s) + \int_0^s B(\sigma)F(\sigma)d\sigma, \quad 0 \leq s \leq t,$$

then

$$F(t) \leq A(t) + \int_0^t B(s)A(s)e^{\int_s^t B(\sigma)d\sigma}ds. \tag{A.5.1}$$

- Young's Inequality states that if

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p, q, r \leq \infty.$$

then for $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$,

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \|h\|_{L^m} := \left(\int_{\mathbb{R}^n} |h(x)|^m dx\right)^{\frac{1}{m}} \tag{A.5.2}$$

- Schur's estimate states that if

$$Ku(x) := \int_Y K(x,y)u(y)dy, \quad x \in X$$

then

$$\|K\|^2_{L^2 \to L^2} \leq \sup_{x \in X} \int |K(x,y)|dy \times \sup_{y \in Y} \int |K(x,y)|dx. \tag{A.5.3}$$

when the right hand side is finite.

- Interpolation estimate:

For $m \leq \ell \leq p$, there exists C such that

$$\sup_{|\alpha|=\ell} \|\partial^\alpha f\|_{L^\infty} \leq C \left(\sup_{|\alpha|=m} \|\partial^\alpha f\|_{L^\infty}\right)^{\frac{p-\ell}{p-m}} \left(\sup_{|\alpha|=p} \|\partial^\alpha f\|_{L^\infty}\right)^{\frac{\ell-m}{p-m}}. \tag{A.5.4}$$

A.6. TEMPERED DISTRIBUTIONS

A standard reference for this section is [**HöI**, §7.1].

- The *Schwartz space* is defined as

$$\mathscr{S} = \mathscr{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta\}.$$

We say that

$$\varphi_j \to \varphi \quad \text{in } \mathscr{S}$$

if

$$\sup_{\mathbb{R}^n} |x^\alpha \partial^\beta (\varphi_j - \varphi)| \to 0$$

for all multiindices α, β

• We write $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ for the space of *tempered distributions*, which is the dual of $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$. That is, $u \in \mathscr{S}'$ if $u : \mathscr{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathscr{S}$ implies $u(\varphi_j) \to u(\varphi)$.

Sometimes the distributional pairing is denoted by

$$u(\varphi) = \langle u, \varphi \rangle = \langle u(x), \varphi(x) \rangle. \tag{A.6.1}$$

We say

$$u_j \to u \quad \text{in } \mathscr{S}'$$

if

$$u_j(\varphi) \to u(\varphi) \quad \text{for all } \varphi \in \mathscr{S}.$$

• Fourier transform defined by (A.2.2) is an continuous invertible operator on $\mathscr{S}$ and

$$\mathcal{F}^{-1}\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi) e^{i\langle x, \xi \rangle} d\xi = \frac{1}{(2\pi)^n} \mathcal{F}\varphi(-x).$$

The Fourier transform is defined on $\mathscr{S}$ by duality:

$$\text{for } u \in \mathscr{S}' \text{ we define } \mathcal{F}u \in \mathscr{S} \text{ by } \mathcal{F}u(\varphi) := u(\mathcal{F}\varphi).$$

A.7. DISTRIBUTIONS ON MANIFOLDS AND SCHWARTZ KERNELS

• Homogeneous distribution on $\mathbb{R}$: for $s \in \mathbb{C}$, $\operatorname{Re} s > -1$,

$$x_\pm^s := \begin{cases} |x|^s & \pm x > 0\,, \\ 0 & \pm x \le 0\,. \end{cases}$$

This family of distributions continues meromorphically to $s \in \mathbb{C}$ – see [**HöI**, §3.2].

• Smooth functions on a manifold M:

We denote by $C^\infty(M)$ the space of smooth functions and by $C^\infty_{\mathrm{c}}(M)$ the space of smooth compactly supported functions. The topology on C^∞ is given by seminorms $\sup_K |\partial^\alpha \varphi|$, $\alpha \in \mathbb{N}^n$ and $K \Subset M$. In other words

$$C^\infty(M) \ni \varphi_j \to 0 \iff \forall\, \alpha \in \mathbb{N}^n \; \forall\, K \Subset M \quad \max_K |\partial^\alpha \varphi_j| \to 0.$$

The topology on $C^\infty_{\mathrm{c}}(M)$ is determine by demanding that

$$C^\infty_{\mathrm{c}}(M) \ni \varphi_j \to 0 \iff$$
$$\exists\, K \Subset M \; \exists\, J \in \mathbb{N} \; \forall \alpha \in \mathbb{N}^n \text{ and } j > J \quad \operatorname{supp} \varphi_j \Subset K, \quad \max |\partial^\alpha \varphi_j| \to 0.$$

• Distributions on a manifold M:

We denote by $\mathcal{D}'(M)$ the space of distributions on M, that is the dual space to $C_c^\infty(M)$, and by $\mathcal{E}'(M)$ the space of compactly supported distributions.

We use the notation (A.6.1) for the distributional pairing. When there is possibility of confusion with inner products on a Hilbert space $\mathcal{H}$, we denote that inner produce by $\langle u, v\rangle_{\mathcal{H}}$.

• Schwartz kernels:

Let M_1, M_2 be two manifolds and fix some smooth density dy on M_2. Each sequentially continuous operator

$$A : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$$

is given using a *Schwartz kernel*:

$$\mathcal{K}_A \in \mathcal{D}'(M_1 \times M_2), \quad Af(x) = \int_{M_2} \mathcal{K}_A(x, y) f(y)\, dy. \tag{A.7.1}$$

Formally speaking, (A.7.1) means that

$$\langle Af, g\rangle = \langle \mathcal{K}_A(x, y), g(x) \otimes f(y)\rangle$$

for each $f \in C_c^\infty(M_2), g \in C_c^\infty(M_1)$ – see [HöI, Theorem 5.2.1 and §6.3].

• Adjoint:

Fix smooth densities on M_1, M_2; this induces inner products on $L^2(M_1)$, $L^2(M_2)$. If $A : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$, then the adjoint $A^* : C_c^\infty(M_1) \to \mathcal{D}'(M_2)$ is defined by the identity

$$\langle Au, v\rangle_{L^2(M_1)} = \langle u, A^*v\rangle_{L^2(M_2)} \quad \text{for all} \quad u \in C_c^\infty(M_2),\ v \in C_c^\infty(M_1).$$

The Schwartz kernel of A^* is given by $\mathcal{K}_{A^*}(y, x) = \overline{\mathcal{K}_A(x, y)}$.

• Smoothing operators:

We say that A is *smoothing* if it is sequentially continuous $\mathcal{E}'(M_2) \to C^\infty(M_1)$. This is equivalent to

$$\mathcal{K}_A \in C^\infty(M_1 \times M_2).$$

• Regular operators:

We say that A is *regular* if it is sequentially continuous $C_c^\infty(M_2) \to C^\infty(M_1)$ and the adjoint A^* is sequentially continuous $C_c^\infty(M_1) \to C^\infty(M_2)$; note that such A maps $\mathcal{E}'(M_2) \to \mathcal{D}'(M_1)$.

• Compactly supported operators:

We say that A is *compactly supported* if $\mathcal{K}_A$ is compactly supported, that is A maps $C^\infty(M_2) \to \mathcal{E}'(M_1)$.

• Properly supported operators:

We say that A is *properly supported* if for each $\chi_1 \in C_c^\infty(M_1)$, there exists $\chi_2 \in C_c^\infty(M_2)$ such that

$$\chi_1 A = \chi_1 A \chi_2$$

and same property holds for A^*.

If A is properly supported then A maps $C^\infty(M_2) \to \mathcal{D}'(M_1)$ and $C_c^\infty(M_2) \to \mathcal{E}'(M_1)$.

Being properly supported can be formulated in terms of the support of $\mathcal{K}_A$: if $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $x \in M_1$, $y \in M_2$ then A is properly supported if

$$\pi_j : \operatorname{supp} \mathcal{K}_A \to M_j,\ j = 1, 2, \text{ are proper maps.}$$

(A map is proper if a pre-image of any compact set is compact.)

If $A = A(h)$ is a family of operators depending on some parameter h, then the support properties is understood in the sense independent of h.

If $A : C_c^\infty(M_2) \to \mathcal{D}'(M_1), B : C_c^\infty(M_3) \to \mathcal{D}'(M_2)$ are two operators and at least one of them is properly supported and regular, then the product $AB : C_c^\infty(M_3) \to \mathcal{D}'(M_1)$ is well-defined. In particular, regular properly supported operators on some manifold M form an algebra.

• Locally finite collections of sets

We say that a collection of open subsets $\{U_j \subset M\}$ is *locally finite* if for each compact set $K \subset M$, we have $U_j \cap K = \emptyset$ for all but finitely many indices j. We only work with paracompact manifolds, which implies that any locally finite collection is at most countable.

Appendix B

SPECTRAL THEORY

B.1 Spectral theory of self-adjoint operators
B.2 Functional calculus
B.3 Singular values
B.4 The trace class
B.5 Weyl inequalities and Fredholm determinants
B.6 Lidskiĭ's theorem
B.7 Notes
B.8 Exercises

B.1. SPECTRAL THEORY OF SELF-ADJOINT OPERATORS

B.1.1. Bounded operators. Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a *bounded operator*, $A : H \to H$, we define the adjoint $A^* : H \to H$ using the inner product:

$$\langle Au, v \rangle = \langle u, A^* v \rangle .$$

An operator A is *self-adjoint* if $A^* = A$.

THEOREM B.1 (Spectral theorem for bounded operators). *Let A be a bounded self-adjoint operator on H. Then there exist a measure space $(X, \mathcal{M}, \mu)$, a real-valued funtion $f \in L^\infty(X, \mu)$ and a unitary operator $U : H \to L^2(X, \mu)$ such that*

$$U^* M_f U = A ,$$

where M_f is the multiplication operator:

$$[M_f u](x) = f(x)u(x)\,, \quad u \in L^2(X, \mu)\,.$$

The same theorem applies to *normal operators*, that is, operators satisfying

$$[A, A^*] = AA^* - A^*A = 0\,.$$

In that case f can be complex valued but otherwise the statement is the same.

DEFINITION B.2. *Suppose that A is a bounded operator on H. Then the* spectrum of A, $\mathrm{Spec}(A) \subset \mathbb{C}$, *is defined by*

$$\mathrm{Spec}(A) = \complement\{\lambda \in \mathbb{C} \;:\; (A-\lambda)^{-1} : H \to H \text{ exists}\}\,.$$

We say that $\lambda \in \mathrm{Spec}(A)$ is an eigenvalue *of A, if there exists $u \in H$ such that*

$$Au = \lambda u\,. \tag{B.1.1}$$

Theorem B.1 implies that for a self-ajoint bounded operator A,

$$\mathrm{Spec}(A) = \overline{\mathrm{image}\,(f)} \Subset \mathbb{R}\,.$$

The following important result concerns spectrum of compact operators: $A : H \to H$ is called compact if the image of $\{u : \|u\| \leq 1\}$ under A is a pre-compact subset of H.

THEOREM B.3 (**Spectra of compact operators**). *Suppose A is a compact operator on H. Then*

(i) Every $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$ is an eigenvalue of A.

(ii) For all nonzero $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$, there exist N such that

$$\ker(A-\lambda)^N = \ker(A-\lambda)^{N+1}\,.$$

(iii) The eigenvalues can only accumulate at 0.

(iv) $\mathrm{Spec}(A)$ *is countable.*

(v) Every $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$ is a finite rank pole of the resolvent operator $\lambda \longmapsto (A-\lambda)^{-1}$ (see §C.3).

(vi) Suppose in addition that A is self-adjoint. Then there exists an orthonormal set $\{u_k\}_{k\in K} \subset H$, *$K = \{0, 1, 2, \cdots, N\}$ or $K = \mathbb{N}$, such that*

$$Au(x) = \sum_{k\in K} \lambda_k u_k(x) \langle u, u_k\rangle\,, \tag{B.1.2}$$

where $\lambda_0 \geq \lambda_1 \geq \cdots$ are the non-zero eigenvalues of A.

(vii) Conversely, if (B.1.2) *holds with $\lambda_j \to 0$ then A is compact.*

One of the most frequently encountered classes of compact operators are inclusions between Hilbert spaces. Here is one which is used in this book:

THEOREM B.4 (Rellich–Kondrachov theorem for unbounded domains). *Suppose that the Hilbert $H \subset L^2(\mathbb{R}^n)$ is defined by the norm*

$$\|u\|_H^2 = \|\langle\xi\rangle^\alpha \widehat{u}\|_{L^2(\mathbb{R}^n)}^2 + \|a(x)^{-1}u\|_{L^2(\mathbb{R}^n)}^2 ,$$
$$\alpha > 0 , \quad a(x) > 0 , \quad \lim_{|x|\to\infty} a(x) = 0 ,$$

where $\widehat{u}$ is the Fourier transform of u and a is continuous.

Then the inclusion

$$H \hookrightarrow L^2 \ \text{ is compact } .$$

B.1.2. Unbounded operators. We next review the more complicated theory for unbounded operators.

DEFINITION B.5. *(i) An unbounded operator $A : H \to H$ is given by a subspace $\mathcal{D}(A) \subset H$ and a linear operator $A : \mathcal{D}(A) \to H$. We call $\mathcal{D}(A)$ the* domain *of A, and say that A is* densely defined *if $\mathcal{D}(A)$ is dense in H.*

(ii) The graph *of A is*

$$\operatorname{graph}(A) := \{(u, Au) \mid u \in \mathcal{D}(A)\} \subset H \times H.$$

(iii) If A, B are unbounded operators on H, we say that $A \subseteq B$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Au = Bu$ for all $u \in \mathcal{D}(A)$.

(iv) The operator A is closed *if* $\operatorname{graph}(A)$ *is a closed subspace of $H \times H$ equipped with the norm $\|(u, v)\|^2 = \|u\|^2 + \|v\|^2$.*

(v) An unbounded operator A is closable *if there exists a closed unbounded operator $\bar{A}$ such that $A \subseteq \bar{A}$. The operator $\bar{A}$ is unique and is called the* closure *of A.*

THEOREM B.6 (Adjoint operator). *Suppose $A : H \to H$ is an unbounded, densely defined operator. Then there exists an unbounded operator $A^* : H \to H$ defined by the rule*

$$\langle A^* v, u\rangle := \langle v, Au\rangle \tag{B.1.3}$$

for all $v \in \mathcal{D}(A^*), u \in \mathcal{D}(A)$, *where*

$$\mathcal{D}(A^*) := \{v \in H \mid |\langle Au, v\rangle| \leq C(v)\|u\| \text{ for all } u \in \mathcal{D}(A)\}. \tag{B.1.4}$$

Here $C(v)$ *is a constant depending on* v.

The unbounded operator A^* is always closed. If A^* is densely defined, then A is closable and $\bar{A} = (A^*)^*$, $\bar{A}^* = A^*$.

DEFINITION B.7. *(i) An unbounded densely defined operator* A *is called* symmetric *if*

$$A \subseteq A^*. \tag{B.1.5}$$

Equivalently, $\langle Au, v\rangle = \langle u, Av\rangle$ *for all* $u, v \in \mathcal{D}(A)$.

(ii) An unbounded densely defined operator A *is called* self-adjoint *if*

$$A = A^*. \tag{B.1.6}$$

(iii) A symmetric operator is called essentially self-adjoint *if*

$$\bar{A} = A^*. \tag{B.1.7}$$

THEOREM B.8 (**Spectral Theorem for unbounded operators**). *Let* A *be an unbounded self-adjoint operator on* H. *Then there exist a measure space* $(X, \mathcal{M}, \mu)$, *a real-valued measurable function* f *and a unitary operator*

$$U : H \to L^2(X, \mu)$$

such that

$$x \in \mathcal{D}(A) \text{ if and only if } M_f(Ux) \in L^2(X, \mu) \tag{B.1.8}$$

and

$$x \in \mathcal{D}(A) \text{ implies } U(Ax) = M_f(Ux). \tag{B.1.9}$$

Here $M_f : x \mapsto fx$ denotes the unbounded multiplication operator on X.

As an immediate consequence we obtain this useful result valid for both bounded and unbounded operators:

THEOREM B.9 (**Distance to spectrum**).

(i) *If* A *is a self-adjoint operator, then*

$$\operatorname{Spec}(A) = \text{ess-image}\,(f) \subset \mathbb{R},$$

where f *is given in the Spectral Theorems B.8, B.1 and*

$$\text{ess-image}\,(f) := \{t \mid \mu(f^{-1}((t-\epsilon, t+\epsilon)) > 0 \text{ for all } \epsilon > 0\}.$$

(ii) *Furthermore, if* $\lambda \in \mathbb{C} \setminus \mathrm{Spec}(A)$, *then*

$$\|(A-\lambda)^{-1}\| = \frac{1}{\mathrm{dist}\,(\lambda, \mathrm{Spec}(A))}. \tag{B.1.10}$$

The spectral projector is given by boundary values of the resolvent in the following way:

THEOREM B.10 (Stone's formula). *Let* $E = E(P)$ *be the spectral measure of a self-adjoint operator* P. *For* $a < b$

$$\begin{aligned} &\tfrac{1}{2}\left(E((a,b)) + E([a,b])\right) = \\ &\qquad \lim_{\epsilon\to 0+} \frac{1}{2\pi i}\int_a^b \left((P-t-i\epsilon)^{-1} - (P-t+i\epsilon)^{-1}\right) dt. \end{aligned} \tag{B.1.11}$$

If there exists a dense *subspace* V *such that for* $f \in V$ *the limit*

$$\lim_{\epsilon\to 0+} \langle (P-t-i\epsilon)^{-1} f, f\rangle =: \langle (P-t-i0)^{-1} f, f\rangle, \quad a < t < b$$

exists, then $\mathrm{Spec}(P) \cap (a,b)$ *is absolutely continuous and on* (a,b)

$$dE_t(P) = \frac{1}{2\pi i}\left((P-t-i0)^{-1} - (P-t+i0)^{-1}\right) dt\,. \tag{B.1.12}$$

REMARK. An informal but instructive way of writing (B.1.12) is

$$\delta(P-\lambda) = \frac{1}{2\pi i}\left((P-\lambda-i0)^{-1} - (P-\lambda+i0)^{-1}\right) \tag{B.1.13}$$

There are many criteria determining if an operator is essentially self-adjoint and there are many subtleties in the subject. Here we only need the simplest one:

THEOREM B.11 (Criteria for essential self-adjointness). *Suppose that* $A : H \to H$ *is symmetric. Then the following conditions are equivalent:*

(i) A *is essentially self-adjoint.*

(ii) *For both signs,* $(A^* \pm i)x = 0$, $x \in \mathcal{D}(A^*)$, *implies* $x = 0$.

(iii) *For both signs,* $\{(A \pm i)x \mid x \in \mathcal{D}(A)\}$ *is dense in* H.

THEOREM B.12 (Maximin and minimax principles). *Suppose that* $A : H \to H$ *is self-adjoint and semibounded, meaning* $A \geq -c_0$. *Assume also that* $(A + 2c_0)^{-1} : H \to H$ *is a compact operator. Then the spectrum of* A *is discrete:* $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ *and*

$$\lambda_j = \max_{\substack{V\subset H \\ \mathrm{codim} V < j}} \min_{\substack{v\in V \\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2}, \tag{B.1.14}$$

and

$$\lambda_j = \min_{\substack{V\subset H \\ \dim V \leq j}} \max_{\substack{v\in V \\ v\neq 0}} \frac{\langle Av, v\rangle}{\|v\|^2}. \tag{B.1.15}$$

(In these formulas, V denotes a linear subspace of H.)

B.2. FUNCTIONAL CALCULUS

If A is a self-adjoint operator on a Hilbert space H and $\varphi \in L^\infty(\mathbb{R})$, the spectral theorem (see Theorem B.8 for the statement and notation used here) shows that we can define $\varphi(A)x$ for $x \in \mathcal{D}(A)$ by

$$\varphi(A)x := U^* M_{\varphi(f)} U x.$$

Since $\varphi(A)$ is bounded and $\mathcal{D}(A)$ is dense this defines a bounded operator

$$\varphi(A) \in \mathcal{L}(H).$$

If $\varphi \in C_c^\infty(\mathbb{R})$ (or even $\varphi \in C_c^2(\mathbb{R})$) an elegant and useful formula for $\varphi(A)$ was given by Helffer–Sjöstrand – see Dimassi–Sjöstrand [**DS99**, Chapter 8] for proofs and for references.

To present the Helffer–Sjöstrand formula we recall the notion of an almost analytic extension: we say that $\widetilde{\varphi} \in C_c^\infty(\mathbb{C})$ is an *almost analytic extension* of $\varphi \in C_c^\infty(\mathbb{R})$ if

$$\widetilde{\varphi}|_{\mathbb{R}} = \varphi, \quad \bar{\partial}_z \widetilde{\varphi}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty). \tag{B.2.1}$$

(Here $\bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y)$, $z = x + iy$, is the Cauchy–Riemann operator.) Such extensions can always be found – see [**DS99**, Chapter 8] or [**HöI**, after the proof of Theorem 3.1.11].

Using this concept, for $\varphi \in C_c^\infty(\mathbb{R})$,

$$\varphi(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial}_z \widetilde{\varphi}(z)(A - z)^{-1} dm(z), \tag{B.2.2}$$

where $dm(z) = dxdy$ – see [**DS99**, Theorem 8.1]. In view of (B.2.1), the integral converges in the sense of Riemann integration of operators in $\mathcal{L}(H)$.

B.3. SINGULAR VALUES

Let H, H_1, H_2 be separable Hilbert spaces and $\mathcal{L}(H_1, H_2)$ denotes the space of bounded linear operators $H_1 \to H_2$.

For a compact self-adjoint operator $A : H \to H$, the Hilbert–Schmidt theorem (part (iv) of Theorem B.3 says that there exists a complete orthonormal basis of eigenvectors $e_0, e_1, \ldots$ of A with eigenvalues $\lambda_j(A) \to 0$.

We can write

$$A = \sum_{j=0}^{\infty} \lambda_j(A)(e_j \otimes e_j), \tag{B.3.1}$$

where for $e \in H_1, f \in H_2$, we define their tensor product $e \otimes f \in L(H_1; H_2)$ by

$$(e \otimes f)u := \langle u, e\rangle f, \quad u \in H_1. \tag{B.3.2}$$

(Note that we use here a different convention than in Chapters 2 and 3.) If A is nonnegative, then we order $\lambda_j(A)$ so that $\lambda_j(A) \geq \lambda_{j+1}(A)$, and note that $\|A\| = \lambda_0(A)$.

For the case of a general (not necessarily self-adjoint) operator, the following decomposition holds:

PROPOSITION B.13. *Assume that $A : H_1 \to H_2$ is a compact operator. Then we can write*

$$A = \sum_{j=0}^{\infty} s_j(e_j \otimes f_j), \tag{B.3.3}$$

where $s_0 \geq s_1 \geq \dots$ is a sequence converging to 0 and $\{e_j \mid s_j \neq 0\} \subset H_1$ and $\{f_j \mid s_j \neq 0\} \subset H_2$ are orthonormal systems.

Moreover, the numbers $s_j = s_j(A)$, called the singular values *of A, do not depend on the choice of the decomposition* (B.3.3)*, and in fact $s_j(A) = \lambda_j(A^*A)^{1/2}$.*

Proof. 1. We first show the existence of the decomposition (B.3.3). The operator $A^*A : H_1 \to H_1$ is compact and self-adjoint, and nonnegative. Let e_j be an orthonormal basis of H_1 composed of eigenvectors of A^*A with eigenvalues $s_j(A)^2 := \lambda_j(A^*A)$. Put $f_j := s_j^{-1}Ae_j$ for $s_j \neq 0$ and $f_j = 0$ for $s_j = 0$, so that (B.3.3) holds. It remains to note that for $s_j \neq 0, s_k \neq 0$, we have

$$\langle f_j, f_k\rangle = \frac{\langle Ae_j, Ae_k\rangle}{s_j s_k} = \frac{\langle A^*Ae_j, e_k\rangle}{s_j s_k} = \delta_{jk}.$$

2. If A admits the decomposition (B.3.3) for some sequence non-increasing sequence $s_j \geq 0$ and orthonormal sets $\{e_j\}$, $\{f_j\}$, then

$$A^*A = \sum_{j=0}^{\infty} s_j^2 e_j \otimes e_j,$$

and it follows immediately that $s_j^2 = \lambda_j(A^*A)$. $\square$

We note that

$$\sup_{\|u\|=1} \frac{\|Au\|^2}{\|u\|^2} = \sup_{\|u\|=1} \frac{\langle A^*Au, u\rangle}{\|u\|^2} = \lambda_0(A^*A),$$

in other words,

$$\|A\|_{H_1 \to H_2} = s_0(A). \tag{B.3.4}$$

We also have $s_j(A) = s_j(A^*)$ since the form of the decomposition (B.3.3) persists under taking adjoints.

The singular values can also be characterized as follows:

PROPOSITION B.14. *We have for each n,*

$$s_n(A) = \min\{\|A - K\|_{H_1 \to H_2} : K \in \mathcal{L}(H_1; H_2),\ \operatorname{rank} K \le n\}.$$

Moreover, the minimum is achieved by an operator K such that $s_j(K) = s_j(A)$ for $0 \le j < n$ and $s_{j-n}(A - K) = s_j(A)$ for $j \ge n$.

Proof. 1. Assume that $K \in \mathcal{L}(H_1; H_2)$ and $\operatorname{rank} K \le n$. We will show that $\|A - K\|_{H_1 \to H_2} \ge s_n(A)$ and for that take the decomposition (B.3.3) of A. Since $n + 1$ vectors $e_0, \dots, e_n$ are linearly independent, there exists a nontrivial linear combination u of these vectors such that $K(u) = 0$. We then have

$$\|(A - K)u\|^2 = \|Au\|^2 = \langle A^*Au, u\rangle \ge s_n^2 \|u\|^2$$

and thus $\|A - K\| \ge s_n(A)$ as needed.

2. Using the decomposition (B.3.3), for

$$K := \sum_{j=0}^{n-1} s_j (e_j \otimes f_j),$$

we have $s_j(K) = s_j(A)$ for $0 \le j < n$, $s_{j-n}(A - K) = s_j(A)$ for $j \ge n$, and in particular $\|A - K\| = s_0(A - K) = s_n(A)$. $\square$

Proposition B.14 lets us prove the following inequalities:

PROPOSITION B.15. *For compact operators A, B,*

$$s_{j+k}(A + B) \le s_j(A) + s_k(B), \tag{B.3.5}$$

$$s_{j+k}(AB) \le s_j(A) s_k(B). \tag{B.3.6}$$

If A is compact and B is bounded then

$$s_j(AB), s_j(BA) \le \|B\| s_j(A). \tag{B.3.7}$$

Proof. Using Proposition B.14, we write $A = K_A + R_A$, $B = K_B + R_B$, where $\operatorname{rank} K_A \le j$, $\operatorname{rank} K_B \le k$, $\|R_A\| \le s_j(A)$, $\|R_B\| \le s_k(B)$. Then

$$\begin{aligned} A + B &= (K_A + K_B) + (R_A + R_B), \\ AB &= (K_A B + R_A K_B) + R_A R_B, \end{aligned}$$

and $\operatorname{rank}(K_A + K_B), \operatorname{rank}(K_A B + R_A K_B) \le j + k$, $\|R_A + R_B\| \le s_j(A) + s_k(B)$, $\|R_A R_B\| \le s_j(A) s_k(B)$, so it remains to apply Proposition B.14 one more time. □

The singular values of an operator are continuous in the norm topology:

PROPOSITION B.16. *For compact operators $A, B : H_1 \to H_2$,*

$$|s_j(A) - s_j(B)| \le \|A - B\|_{H_1 \to H_2}.$$

Proof. We have $s_j(A) \le s_j(B) + \|A - B\|$ and $s_j(B) \le s_j(A) + \|A - B\|$ by Proposition B.15, with $k = 0$. □

EXAMPLE. Suppose that (M, g) is compact manifold n dimensional Riemannian manifold and that $-\Delta_M$ is the Laplace-Beltrami operator on M. Then the Weyl law for eigenvalue asymptotics states that

$$\begin{gathered} |\{\lambda \ge 0 : \lambda^2 \in \operatorname{Spec}(-\Delta_M)\,,\ |\lambda| \le r\}| = c_n \operatorname{vol}_g(M) r^n (1 + o(1))\,, \\ c_n = \operatorname{vol}(B_{\mathbb{R}^n}(0,1))/(2\pi)^n\,, \end{gathered}$$

see for instance [**Zw12**, Theorem 14.11, part (ii)]. If we order the eigenvalues of $-\Delta_M$ as $0 = \lambda_0^2 < \lambda_1^2 \le \lambda_2^2 \le \cdots$, it then follows that

$$\lambda_j \ge (c_n \operatorname{vol}_g(M))^{-\frac{1}{n}} j^{\frac{1}{n}} (1 - o(1))\,, \quad j \to \infty\,, \tag{B.3.8}$$

and

$$s_j((-\Delta_M + 1)^{-s/2}) \le C_M j^{-\frac{s}{n}}\,, \quad s > 0. \tag{B.3.9}$$

Suppose now that $A : L^2(M) \to H^s(M)$, $s > 0$. Then

$$\begin{aligned} s_j(A) &\le s_j((-\Delta_M + 1)^{-s/2}) \|(-\Delta_M - 1)^{s/2} A\|_{L^2 \to L^2} \\ &\le s_j((-\Delta_M + 1)^{-s/2}) \|A\|_{L^2 \to H^s} \\ &\le C_A j^{-\frac{s}{n}}\,. \end{aligned} \tag{B.3.10}$$

It follows that if $s > n$ then $\sum_{j=0}^{\infty} s_j(A) < \infty$. In the notation of the next section, A is of trace class, $A \in \mathcal{L}_1(L^2(M))$ – see (B.4.2).

B.4. THE TRACE CLASS

We now discuss operators of trace class. First of all, if $A : H \to H$ is bounded and has finite rank, then the trace, $\operatorname{tr} A \in \mathbb{C}$, can be defined as the trace of the restriction of A to any finite-dimensional subspace $V \subset H$ containing the range of A. In fact, if

$$A = \sum_{j=0}^{N-1} a_j (u_j \otimes v_j),$$

where $a_j \in \mathbb{C}$, $u_j, v_j \in H$, then

$$\operatorname{tr} A = \sum_{j=0}^{N-1} a_j \langle v_j, u_j \rangle. \tag{B.4.1}$$

This gives a linear functional on the space of all finite-rank operators, and moreover, as seen directly from (B.4.1),

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad A : H_1 \to H_2,\ B : H_2 \to H_1, \quad \operatorname{rank} A < \infty.$$

To extend the notion of trace to infinite rank operators, we give the following

DEFINITION B.17. *Let $A : H_1 \to H_2$ be a compact operator. We say that A is of trace class, and write $A \in \mathcal{L}_1(H_1; H_2)$, if the* trace norm

$$\|A\|_1 := \sum_{j=0}^{\infty} s_j(A) \tag{B.4.2}$$

is finite. When $H_1 = H_2 = H$ we write $A \in \mathcal{L}_1(H)$.

Proposition B.13 gives the following alternative expression for the trace class norm:

$$\|A\|_1 = \max_{\{e_k\},\{f_\ell\}} \sum_{k,\ell} \langle A e_k, f_\ell \rangle, \tag{B.4.3}$$

where the maximum is taken over all pairs of orthonormal bases of H_1 and H_2.

We note that $\|A\|_{H_1 \to H_2} = s_0(A) \leq \|A\|_1$. For finite rank operators (B.4.1) applied to (B.3.3) gives

$$|\operatorname{tr} A| \leq \|A\|_1. \tag{B.4.4}$$

The partial sums in the definition of $\|A\|_1$ can be characterized by the following result of Ky Fan:

PROPOSITION B.18. *Let $A : H_1 \to H_2$ be compact. Then for each n,*

$$\sum_{j=0}^{n-1} s_j(A) = \max\{|\operatorname{tr}(QA)| \;\mid\; \|Q\|_{H_2\to H_1} \le 1,\ \operatorname{rank} Q \le n\}.$$

Proof. 1. If $\|Q\|_{H_2\to H_1} \le 1$, then by (B.3.6), $s_j(QA) \le s_j(A)$ for all j. On the other hand, since $\operatorname{rank}(QA) \le n$, we see that $s_j(QA) = 0$ for $j \ge n$. Therefore, by (B.4.4) we have

$$|\operatorname{tr}(QA)| \le \sum_{j=0}^{n-1} s_j(QA) \le \sum_{j=0}^{n-1} s_j(A).$$

2. If we consider the decomposition (B.3.3) for A and put $Q := \sum_{j=0}^{n-1} f_j \otimes e_j$, then $\|Q\|_{H_2\to H_1} \le 1$, $\operatorname{rank} Q \le n$, and by (B.4.1),

$$\operatorname{tr}(QA) = \operatorname{tr}\sum_{j=0}^{n-1} s_j(A)(e_j \otimes e_j) = \sum_{j=0}^{n-1} s_j(A). \quad \square$$

To see that $\|\cdot\|_1$ is in fact a norm, it suffices to prove

PROPOSITION B.19. *For $A, B : H_1 \to H_2$ compact operators, we have*

$$\|A + B\|_1 \le \|A\|_1 + \|B\|_1.$$

Proof. It suffices to prove that for each n,

$$\sum_{j=0}^{n-1} s_j(A+B) \le \sum_{j=0}^{n-1} s_j(A) + \sum_{j=0}^{n-1} s_j(B).$$

This follows immediately from Proposition B.18, as for each $Q : H_2 \to H_1$ with $\|Q\|_{H_2\to H_1} \le 1$ and $\operatorname{rank} Q \le n$, we have $\operatorname{tr}(Q(A+B)) = \operatorname{tr}(QA) + \operatorname{tr}(QB)$. $\square$

The space $\mathcal{L}_1(H_1; H_2)$ equipped with the norm $\|\bullet\|_1$ is a Banach space, but we do not prove or use this fact here.

Finite rank operators are dense in $\mathcal{L}_1(H_1; H_2)$ since for each $A \in \mathcal{L}_1(H_1; H_2)$, using (B.3.3) we have

$$\sum_{j=0}^{n-1} s_j(e_j \otimes f_j) \to A \quad \text{in } \mathcal{L}_1(H_1; H_2). \tag{B.4.5}$$

Another way of demonstrating this fact, which will be more convenient for families of operators later, is given by

PROPOSITION B.20. *Assume that $A \in \mathcal{L}_1(H_1; H_2)$, H_2 is infinite dimensional, and $(u_j)_{j=1}^{\infty}$ is a Hilbert basis of H_2 respectively. Let $\Pi_N : H_2 \to H_2$ be the orthogonal projection onto the subspace spanned by $u_1, \dots, u_N$. Then*

$$\|A - \Pi_N A\|_1 \to 0 \quad \textit{as } N \to \infty. \tag{B.4.6}$$

Proof. 1. Since A is compact and $\Pi_N \to I$ in the strong operator topology ($\|(\Pi_N - I)u\|_{H_2} \to 0$ for all $u \in H_1$),

$$f(N) := \|(I - \Pi_N)A\|_{H_1 \to H_2} \to 0, \quad N \to \infty.$$

(We can for instance use (B.3.3) and $s_j \to 0$.)

2. From $s_j((I - \Pi_N)A) \leq f(N)$, (B.3.6) and the dominated convergence theorem we see that

$$\sum_j s_j((I - \Pi_N)A) \leq \sum_j \min(s_j(A), f(N)) \to 0, \quad N \to \infty$$

which proves (B.4.6). □

By (B.3.6) and the fact that $s_j(A^*) = s_j(A)$, we see that for $A \in \mathcal{L}_1$.

$$\|AB\|_1 \leq \|A\|_1 \|B\|_{H_1 \to H_2}, \quad \|A\|_1 = \|A^*\|_1. \tag{B.4.7}$$

In particular, $\mathcal{L}_1(H; H)$ is a two-sided ideal in the algebra of closed operators on H.

By (B.4.4), the trace functional on finite-dimensional operators extends uniquely to a bounded linear functional $\operatorname{tr} : \mathcal{L}_1(H; H) \to \mathbb{C}$, and for each Hilbert basis u_j of H,

$$\operatorname{tr} A = \sum_j \langle A u_j, u_j \rangle, \quad A \in \mathcal{L}_1(H; H). \tag{B.4.8}$$

Indeed, this follows immediately from Proposition B.20, since $\operatorname{tr}(\Pi_N A) = \sum_{j=1}^N \langle A u_j, u_j \rangle$.

By approximation by finite rank operators, we see that for each $A \in \mathcal{L}_1(H_1; H_2)$ and bounded $B : H_2 \to H_1$, we have

$$\operatorname{tr}(AB) = \operatorname{tr}(BA), \quad A \in \mathcal{L}_1(H_1; H_2),\ B \in \mathcal{L}(H_2; H_1). \tag{B.4.9}$$

The fundamental example of trace class operators is given by the following proposition (see also the example at the end of §B.3).

PROPOSITION B.21. *Let X be a manifold of dimension m and $A : H^s(X) \to H^{s'}(X)$ be bounded, where $s' > s + m$. Assume also that the Schwartz kernel of A has compact support. Then $A \in \mathcal{L}_1(H^s(X); H^s(X))$.*

Proof. 1. Using coordinate charts and a partition of unity, we can reduce to the case when X is the m-dimensional torus: $X = (\mathbb{R}/2\pi\mathbb{Z})^m$. By (B.4.7), it is enough to show that the inclusion operator $\iota : H^{s'}(X) \to H^s(X)$ is of trace class.

2. An orthogonal basis of $H^s(X)$ and $H^{s'}(X)$ is given by e^{ikx}, where $k \in \mathbb{Z}^m$; we have $\iota(e^{ikx}) = e^{ikx}$ and

$$\|e^{ikx}\|_{H^s} \le C\langle k\rangle^{s-s'}\|e^{ikx}\|_{H^{s'}}.$$

This gives a decomposition (B.3.3) of ι, and we see that the singular value corresponding to e^{ikx} is bounded by $C\langle k\rangle^{s-s'}$. It remains to note that

$$\sum_{k\in\mathbb{Z}^m} \langle k\rangle^{s-s'} < \infty \quad \text{for } s' > s + m. \quad \square$$

To compute the trace of an operator on L^2, the following formula is particularly useful:

PROPOSITION B.22. *Let X be a manifold with a fixed volume form $d\,\mathrm{Vol}$ (so that Schwartz kernels – see §A.7 – can be regarded as functions) and $A : \mathcal{D}'(X) \to \mathcal{C}_c^\infty(X)$ be an operator with Schwartz kernel $K_A(x, y) \in C_c^\infty(X \times X)$. Then $A : L^2(X) \to L^2(X)$ is of trace class and*

$$\operatorname{tr} A = \int K_A(x, x)\, d\,\mathrm{Vol}(x). \tag{B.4.10}$$

Proof. 1. We start with the case when A has the form

$$A = \sum_{j=1}^N a_j(u_j \otimes v_j) \tag{B.4.11}$$

where $u_j, v_j \in C_c^\infty(X)$ and $u_j \otimes v_j$ is defined by (B.3.2). Note that

$$K_A(x, y) = \sum_{j=1}^N a_j v_j(x)\,\overline{u_j(y)}.$$

Then by (B.4.1),

$$\begin{aligned}\operatorname{tr} A = \sum_{j=1}^N a_j\langle v_j, u_j\rangle &= \sum_{j=1}^N a_j \int_X v_j(x)\,\overline{u_j(x)}\, d\,\mathrm{Vol}(x)\\ &= \int_X K_A(x, x)\, d\,\mathrm{Vol}(x)\end{aligned}$$

which proves (B.4.10).

2. For general A, using coordinate charts and a partition of unity, we reduce to the case when $X = (\mathbb{R}/2\pi\mathbb{Z})^m$. We write the Fourier series of K_A,

$$K_A(x,y) = \sum_{\ell,r\in\mathbb{Z}^m} a_{\ell r} e^{i\ell x + iry}$$

and the series converges in C^∞. If $e_\ell(x) = e^{i\ell x}$, then we can write

$$A = \sum_{\ell,r\in\mathbb{Z}^m} a_{\ell r} e_{-r} \otimes e_\ell,$$

and since the coefficients $a_{\ell r}$ are rapidly decreasing, the series converges in the trace class norm $\| \bullet \|_1$. Since the partial sum of the series has the form (B.4.11), the proof is finished by the continuity of the trace with respect to the trace class norm. $\square$

EXAMPLE. Suppose that (M, dx), and $(N, d\omega)$ are measure spaces and that $e_j(x,\omega) \in L^2(M \times N)$. Then

$$Ku(x) := \int_M \int_N e_1(x,\omega)\overline{e_2(y,\omega)}u(y)d\omega dy,$$

defines

$$K \in \mathcal{L}_1(L^2(M)), \quad \operatorname{tr} K = \int_{M\times N} e_1(x,\omega)\overline{e_2(x,\omega)}dx d\omega. \tag{B.4.12}$$

In fact, $K = E_1 E_2^*$ where $E_j : L^2(N) \to L^2(M)$ are defined by $E_j g(x) = \int_N e_j(x,\omega)g(\omega)d\omega$. If $\{\psi_k\}$ is an orthonormal basis of $L^2(M, dx)$, then

$$\sum_k \|E_j^* \psi_k\|^2_{L^2(N)} = \int_{M\times N} |e_j(x,\omega)|^2 dx d\omega.$$

Using (B.4.3),

$$\begin{aligned}
\|K\|_{\mathcal{L}^1} &= \max_{\{\varphi_\ell\},\{\psi_k\}} \sum_{k,\ell} \langle K\varphi_\ell, \psi_k\rangle_{L^2(M)} \\
&\leq \max_{\{\varphi_\ell\},\{\psi_k\}} \sum_{k,\ell} \|E_1^*\psi_k\|^2_{L^2(N)} \|E_2^*\varphi_\ell\|^2_{L^2(N)} \\
&= \|e_1\|^2_{L^2(M\times N)} \|e_2\|^2_{L^2(M\times N)},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{tr} K &= \sum_k \langle E_2^*\psi_k, E_1^*\psi_k\rangle \\
&= \int_N \sum_k \langle \psi_k, e_2(\bullet,\omega)\rangle_{L^2(M)} \langle e_1(\bullet,\omega), \psi_k\rangle_{L^2(M)} d\omega \\
&= \langle e_1, e_2\rangle_{L^2(M\times N)},
\end{aligned}$$

which gives (B.4.12).

REMARK. The example above can be generalized. It leads to another characterization of the trace class: $K \in \mathcal{L}_1(H)$ if and only if $K = E_1 E_2^*$ where $E_j : H_0 \to H$ and

$$\|E_j^*\|_{\mathcal{L}_2(H,H_0)} := \sum_k \|E_j^* \psi_k\|_{H_0}^2 < \infty.$$

for any (or one) orthormal basis of H. The operators in $\mathcal{L}_2(H, H_0)$ are called *Hilbert–Schmidt operators.*

B.5. WEYL INEQUALITIES AND FREDHOLM DETERMINANTS

We now discuss the relation of singular values and the trace with the spectrum. Let $A : H \to H$ be a compact operator and $\lambda_0(A), \lambda_1(A), \dots$ be its eigenvalues, listed according to multiplicity and ordered so that

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq |\lambda_2(A)| \geq \dots \tag{B.5.1}$$

If A has only finitely many eigenvalues, we put the rest of $\lambda_j(A)$ equal to zero. We always have $\lambda_j(A) \to 0$ and $A - \lambda I$ is invertible unless $\lambda = 0$ or λ is an eigenvalue. The spectral projection corresponding to an eigenvalue $\lambda \neq 0$ of A is defined by

$$\Pi_\lambda := \frac{1}{2\pi i} \oint_\lambda (zI - A)^{-1}\, dz, \quad \lambda \neq 0, \tag{B.5.2}$$

where the integral is taken over a contour enclosing λ but no other eigenvalues or zero. By the Cauchy formula and the identity

$$(zI - A)^{-1}(wI - A)^{-1} = (w - z)^{-1}((zI - A)^{-1} - (wI - A)^{-1}), \quad z \neq w$$

we see that $\Pi_\lambda^2 = \Pi_\lambda$ (see Theorem C.9 for a detailed argument). Moreover, Π_λ, is compact since A is compact and

$$\Pi_\lambda = \frac{A}{2\pi i} \oint_\lambda z^{-1}(zI - A)^{-1}\, dz.$$

therefore, Π_λ has finite rank. The (algebraic) multiplicity of λ, $m_A(\lambda)$, is defined as

$$m_A(\lambda) := \operatorname{rank} \Pi_\lambda. \tag{B.5.3}$$

Since A commutes with Π_λ and thus $A(\operatorname{Ran}\Pi_\lambda) \subset \operatorname{Ran}\Pi_\lambda$, and moreover λ is the only eigenvalue of $A|_{\operatorname{Ran}\Pi_\lambda}$, therefore

$$(A - \lambda I)^{m_A(\lambda)}\Pi_\lambda = 0.$$

B.5.1. Weyl Inequalities. We start with

PROPOSITION B.23. *In the notation of* (B.5.1) *we have, for each* n,

$$\prod_{j=0}^{n-1} |\lambda_j(A)| \leq \prod_{j=0}^{n-1} s_j(A). \tag{B.5.4}$$

Proof. 1. We may assume that $\lambda_{n-1}(A) \neq 0$. Let $H_1 \subset H$ be the finite dimensional Hilbert space spanned by some linearly independent set $u_0, \ldots, u_{n-1}$ of eigenvectors of A corresponding to $\lambda_0(A), \ldots, \lambda_{n-1}(A)$. If

$$m_A(\lambda_j) > \dim \ker(A - \lambda_j I),$$

for some λ_j, then, once we are out of eigenvectors of A, we add to the list vectors in $\ker((A-\lambda_j I)^2)$, then vectors in $\ker((A-\lambda_j I)^3)$, and so on. Then $A(H_1) \subset H_1$ and the restriction $A|_{H_1}$ has eigenvalues $\lambda_0(A), \ldots, \lambda_{n-1}(A)$, counted with multiplicity.

2. Denote by $\iota : H_1 \to H$ the inclusion map and by $\pi : H \to H_1$, the orthogonal projector. Then the operator $A_1 := \pi A \iota : H_1 \to H_1$ has eigenvalues $\lambda_0(A), \ldots, \lambda_{n-1}(A)$ and by (B.3.6), we find $s_j(A_1) \leq s_j(A)$ for all j. Now,

$$\left| \prod_{j=0}^{n-1} \lambda_j(A) \right| = |\det(A_1)| = |\det(A_1^* A_1)|^{1/2} = \prod_{j=0}^{n-1} s_j(A_1) \leq \prod_{j=0}^{n-1} s_j(A)$$

which proves (B.5.4). □

We next present the following inequality of Hardy–Littlewood–Pólya. It holds under weaker assumptions but we only need the following version:

LEMMA B.24. *Assume that* $\Phi \in C^\infty(\mathbb{R};\mathbb{R})$, $\Phi' \geq 0$, $\Phi'' \geq 0$, *and* $\Phi(x) \to 0$, $x\Phi'(x) \to 0$ *as* $x \to -\infty$. *Then for all* $a_1 \geq \cdots \geq a_n$, $b_1 \geq \cdots \geq b_n$ *such that*

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, \quad 1 \leq k \leq n,$$

we have

$$\sum_{j=1}^{k} \Phi(a_j) \leq \sum_{j=1}^{k} \Phi(b_j), \quad 1 \leq k \leq n.$$

Proof. 1. By Taylor's formula, we have

$$\Phi(x) = \Phi(y) + (x-y)\Phi'(y) + \int_y^x (x-t)\Phi''(t)\, dt.$$

Letting $y \to -\infty$, we see that

$$\Phi(x) = \int_{-\infty}^{x} (x-t)\Phi''(t)\, dt = \int_{-\infty}^{\infty} (x-t)_+ \Phi''(t)\, dt, \tag{B.5.5}$$

where the integral converges absolutely as $\Phi'' \geq 0$.

2. From (B.5.5) we see that

$$\sum_{j=1}^{k} \Phi(a_j) = \int_{-\infty}^{\infty} \Big(\sum_{j=1}^{k} (a_j - t)_+ \Big) \Phi''(t)\, dt.$$

Therefore, it suffices to prove that for each $t \in \mathbb{R}$,

$$\sum_{j=1}^{k} (a_j - t)_+ \leq \sum_{j=1}^{k} (b_j - t)_+.$$

Fix t and choose the largest $\ell \leq k$ such that $a_\ell - t > 0$. Then

$$\sum_{j=1}^{k} (a_j - t)_+ = \sum_{j=1}^{\ell} (a_j - t) \leq \sum_{j=1}^{\ell} (b_j - t) \leq \sum_{j=1}^{k} (b_j - t)_+$$

and the proof is finished. □

Combining Proposition B.23 and Lemma B.24, we immediately get the following general *Weyl inequality*:

PROPOSITION B.25. *Assume that $f \in C^\infty(0,\infty)$ is real-valued, $f' \geq 0$, $\lim_{x\to+0} f(x) = 0$, $\lim_{x\to+0} f'(x) x \log x = 0$, and $t \mapsto f(e^t)$ is convex. Then for each compact operator A and each n, we have*

$$\sum_{j=0}^{n-1} f(|\lambda_j(A)|) \leq \sum_{j=0}^{n-1} f(s_j(A)). \tag{B.5.6}$$

Proof. Define $\Phi(t) := f(e^t)$ and note that Φ satisfies the assumptions of Lemma B.24. Proposition B.23 shows that

$$\sum_{j=0}^{k-1} \log |\lambda_j(A)| \leq \sum_{j=0}^{k-1} \log s_j(A)$$

for all k and (B.5.6) follows. □

We write down two particularly useful special cases. Taking $f(x) = x$ in Proposition B.25, we obtain for all n,

$$\sum_{j=0}^{n-1} |\lambda_j(A)| \leq \sum_{j=0}^{n-1} s_j(A), \tag{B.5.7}$$

while taking $f(x) = \log(1 + x)$, we get

$$\prod_{j=0}^{n-1} (1 + |\lambda_j(A)|) \leq \prod_{j=0}^{n-1} (1 + s_j(A)). \tag{B.5.8}$$

Note that (B.5.7) in particular implies

$$\sum_{j=0}^{\infty} |\lambda_j(A)| \leq \|A\|_1. \tag{B.5.9}$$

B.5.2. Fredholm determinants. Assume that $A : H \to H$ is a finite rank operator and $\lambda_0(A), \ldots, \lambda_{n-1}(A)$ are its non-zero eigenvalues ordered as in (B.5.1). Then the determinant of $I - A$ is defined as

$$\det(I - A) = \prod_{j=0}^{n-1}(1 - \lambda_j(A)). \tag{B.5.10}$$

This is of course consistent with the definition for finite dimensional H. If for some finite dimensional space V, $\operatorname{Ran}(A) \subset V$ if $\pi : H \to V$ is the orthogonal projection onto V then the determinant is equal to the determinant of $\pi(I - A) : V \to V$. (To see this note if $(A - \lambda)^\ell v = 0$, $\lambda \neq 0$ then $v \in \operatorname{Ran} A \subset V$. Hence, the non-zero eigenvalues of A are the same as non-zero eigenvalues of $A|_V$.)

By (B.5.8), we find

$$|\det(I - A)| \leq \prod_{j=0}^{\infty}(1 + s_j(A)) \leq e^{\|A\|_1}. \tag{B.5.11}$$

From the properties of the determinant on finite dimensional spaces, we have for finite rank operators A, B

$$\det((I - A)(I - B)) = \det(I - A)\det(I - B), \tag{B.5.12}$$

when $\operatorname{rank} A, \operatorname{rank} B < \infty$, and

$$\det(I - AB) = \det(I - BA), \tag{B.5.13}$$

when $\operatorname{rank} A < \infty$.

For the last identity we use the fact that the nonzero eigenvalues of AB and BA coincide with multplicities – to see it, note that for each $\lambda \neq 0$ and each j, the maps

$$\ker((BA - \lambda)^j) \xrightarrow{A} \ker((AB - \lambda)^j) \xrightarrow{B} \ker((BA - \lambda)^j)$$

are injective. We can also see it by noting that for $\lambda \neq 0$, $(AB - \lambda)^{-1} = \lambda^{-1}A(BA - \lambda)^{-1}B - \lambda^{-1}$.

We also see from (B.5.10) that $I - A$ is invertible if and only if $\det(I - A) \neq 0$.

Another useful identity is given in the following lemma:

LEMMA B.26. *Suppose that $t \mapsto A_t$, $t \in [0,1]$, is a C^1 family of finite rank operators such that for some fixed finite dimensional subspace $V \subset H$,*

$$\mathrm{Ran}(A_t) \subset V, \quad I - A_t \quad \textit{is invertible for } t \in [0,1].$$

Then

$$\partial_t \log \det(I - A_t) = -\operatorname{tr}((I - A_t)^{-1} \partial_t A_t). \tag{B.5.14}$$

Proof. 1. Let $\pi : H \to V$ be the orthogonal projection then $B_t := \pi(I - A_t) : V \to V$ and $\det(I - A_t) = \det B_t$ where $\det B_t$ is defined as the determinant on the finite dimensional space V. Similarly, $\operatorname{tr}_H(I - A_t)^{-1}\partial_t A_t = \operatorname{tr}_V B_t^{-1} \partial B_t$. Hence we need to prove that for a family, B_t, of invertible operators on a finite dimensional Hilbert space,

$$\partial \log \det B_t = \operatorname{tr} B_t^{-1} \partial B_t. \tag{B.5.15}$$

2. We first prove a version of *Jacobi's formula*: suppose $\tau \mapsto C(\tau) : \mathbb{C}^n \to \mathbb{C}^n$ is a C^1 family of matrices such that $C(0) = I_{\mathbb{C}^n}$. Then

$$\partial_\tau \det C(\tau)|_{\tau=0} = \operatorname{tr} \partial_\tau C(0). \tag{B.5.16}$$

In fact, let $C_j(\tau)$ be columns of $C(\tau) = [C_1(\tau), \cdots, C_n(\tau)]$ and e_j be the standard basis of $\mathbb{C}^n$. Then

$$\begin{aligned}
\partial_\tau \det C(\tau)|_{\tau=0} &= \sum_{j=1}^n \det[C_1(0), \cdots, \partial_\tau C_j(0), \cdots, C_n(0)] \\
&= \sum_{j=1}^n \det[e_1, \cdots, \partial_\tau C_j(0), \cdots, e_n] \\
&= \sum_{j=1}^n \partial_\tau C_{jj}(0) = \operatorname{tr} \partial_\tau C(0).
\end{aligned}$$

3. We now identify V with $\mathbb{C}^n$ and apply (B.5.16) to $C(\tau) := B_t^{-1} B_{t+\tau}$ so that

$$\partial_t \log \det B_t = \partial_\tau C(\tau)|_{\tau=0} = \operatorname{tr} \partial_\tau C(0) = \operatorname{tr} B_t^{-1} \partial_t B_t,$$

which is (B.5.16). $\square$

We can now define the determinant for operator A of trace class.

PROPOSITION B.27. *The the map $A \mapsto \det(I - A)$ defined for finite rank operators extends uniquely to a continuous (nonlinear) functional on $\mathcal{L}_1(H; H)$.*

Proof. 1. We first consider the case when $I - A$ is invertible. Let B_k be a family of finite rank operators such that $\|A - B_k\|_1 \to 0$. If k and ℓ are large enough than the there exists C such that for all $t \in [0,1]$,

$$\|(I - (tB_\ell + (1-t)B_k))^{-1}\|_{H \to H} \leq C.$$

We then define $B(t) := tB_\ell + (1-t)B_k$. By (B.5.14), for all $t \in [0,1]$

$$\begin{aligned}|\partial_t \log\det(I - B(t))| &= |\operatorname{tr}((I - B(t))^{-1}(B_\ell - B_k))| \\ &\leq \|(I - B(t))^{-1}(B_\ell - B_k)\|_1 \leq C\|B_\ell - B_k\|_1.\end{aligned}$$

Hence, as B_k is a Cauchy sequence in $\mathcal{L}_1$,

$$|\log\det(I - B_\ell) - \log(\det(I - B_k))| \leq C\|B_\ell - B_k\|_{\mathcal{L}_1} \to 0, \quad k, \ell \to \infty.$$

In particular, for m sufficiently large, the limit

$$\nu_m := \lim_{k \to \infty} (\log\det(I - B_k) - \log\det(I - B_m))$$

exists. We then put

$$\det(I - A) = \det(I - B_m)e^{\nu_m}$$

and the estimates above show that $A \mapsto \det(I - A)$ is continuous on the open subset of $\mathcal{L}_1$ consisting of operators for which $I - A$ is invertible.

2. Suppose now that $I - A$ is not invertible. If B_k are as in Step 1, we claim that $\det(I - B_k) \to 0$. Since for each j,

$$|\det(I - B_k)| \leq |1 - \lambda_j(B_k)| \prod_r (1 + |\lambda_r(B_k)|) \leq |1 - \lambda_j(B_k)| e^{\|B_k\|_1},$$

it suffices to show that for each $\varepsilon > 0$ and each k large enough depending on ε, there exists j such that $|1 - \lambda_j(B_k)| \leq \varepsilon$.

3. Suppose the opposite holds. Then, by passing to a subsequence, we may assume that exists $\varepsilon > 0$ such that $|1 - \lambda_j(B_k)| \geq \varepsilon$ for all j, k. We may also choose ε so that 1 is the only eigenvalue of A such that $|1 - \lambda| \leq \varepsilon$. Then as $k \to \infty$,

$$0 = \oint_{|1-\lambda|=\varepsilon} (z - B_k)^{-1}\, dz \to \oint_{|1-\lambda|=\varepsilon} (z - A)^{-1}\, dz,$$

however the right-hand side is nonzero applied to any eigenvector of A with eigenvalue 1, a contradiction. ∎

This finishes the verification that $\det(I - A)$ extends to a continuous functional on operators of trace class, A. The identities (B.5.12) and (B.5.13) then extend to

$$\det((I - A)(I - B)) = \det(I - A)\det(I - B), \tag{B.5.17}$$

for $A, B \in \mathcal{L}_1(H; H)$ and

$$\begin{gathered}\det(I - AB) = \det(I - BA), \\ A \in \mathcal{L}_1(H_1; H_2), \quad B \in \mathcal{L}(H_2; H_1).\end{gathered} \tag{B.5.18}$$

PROPOSITION B.28. *If $A \in \mathcal{L}_1(H; H)$, then $I - A$ is invertible if and only if* $\det(I - A) \neq 0$.

Proof. If $I - A$ is not invertible, then $\det(I - A) = 0$ by Step 2 of the proof of Proposition B.27. If $I - A$ is invertible, then $(I - A)^{-1} = (I - B)$, where $B = -A(I - A)^{-1}$ is of trace class. Then by (B.5.12), $1 = \det(I - A)\det(I - B)$ and hence $\det(I - A) \neq 0$. ☐

The following estimates involving determinants will also be useful

PROPOSITION B.29. *Suppose that $A, B \in \mathcal{L}^1$. Then*

$$|\det(I + A)| \leq e^{\|A\|_1} \tag{B.5.19}$$

and

$$|\det(I + A) - \det(I + B)| \leq \|A - B\|_1 e^{1 + \|A\|_1 + \|B\|_1}. \tag{B.5.20}$$

In the case of matrices we know that $(I - A)^{-1}$ can expressed using Cramer's rule and hence its norm can be estimated using $|\det(I - A)|^{-1}$. There is also an infinite dimensional version of this result:

$$\|(I - A)^{-1}\| \leq \frac{\det(I + (A^*A)^{\frac{1}{2}})}{|\det(I - A)|}, \quad A \in \mathcal{L}_1, \tag{B.5.21}$$

see [**GK69**, Theorem 5.1, Chapter 5.1].

We finally discuss determinants of holomorphic families of operators. Assume that $A(z)$ is a holomorphic family of operators in trace class for z in some domain $\Omega \subset \mathbb{C}$. Using Proposition B.20, we write

$$\det(I - A(z)) = \lim_{N \to \infty} \det(I - \Pi_N A(z)), \tag{B.5.22}$$

and the limit so far is pointwise in z. However, $|\det(I - \Pi_N A(z))| \leq e^{\|\Pi_N A(z)\|_1} \leq e^{\|A(z)\|_1}$ is bounded locally uniformly in z as $N \to \infty$. These uniform boundedness shows equicontinuity (using the Cauchy formula for the derivative) and hence the limit (B.5.22) is actually uniform on compact sets. As a corollary, the determinant $\det(I - A(z))$ is a holomorphic function of z, and for z such that $I - A(z)$ is invertible, we find using (B.5.14) and (B.5.22),

$$\partial_z \log \det(I - A(z)) = -\operatorname{tr}\big((I - A(z))^{-1} \partial_z A(z)\big). \tag{B.5.23}$$

B.6. LIDSKIĬ'S THEOREM

Let $A \in \mathcal{L}_1(H;H)$. In this section, we further explore the relation between the spectrum of A, the trace $\operatorname{tr} A$, and the determinant $\det(I-A)$. Consider the holomorphic function

$$D(z) := \det(I - zA), \quad z \in \mathbb{C}.$$

By (B.5.23), away from the zeroes of D we have

$$\partial_z \log D(z) = -\operatorname{tr}((I - zA)^{-1}A).$$

In this section, we prove

PROPOSITION B.30. *For $A \in \mathcal{L}_1(H;H)$,*

$$\det(I - zA) = \prod_j (1 - z\lambda_j(A)).$$

Before we start the proof, let us note that by taking the derivative at $z = 0$, we get the following theorem of Lidskiĭ:

PROPOSITION B.31. *For $A \in \mathcal{L}_1(H;H)$, we have*

$$\operatorname{tr} A = \sum_j \lambda_j(A).$$

The proof of Proposition B.30 starts with analyzing the zeroes of $D(z)$:

LEMMA B.32. *The zeroes of $D(z)$ are given by $\lambda_j(A)^{-1}$ for $\lambda_j(A) \neq 0$, and the multiplicity of $\lambda_j(A)^{-1}$ as a zero of $D(z)$ is equal to the (algebraic) multiplicity of $\lambda_j(A)$ as an eigenvalue of A.*

Proof. 1.The fact that the zeroes of $D(z)$ are exactly $\lambda_j(A)^{-1}$ follows immediately from Proposition B.28.

2. To see that the multiplicities coincide, we fix $\lambda = \lambda_j(A)$ and note that the multiplicity of λ^{-1} as a zero of $D(z)$ is equal to

$$\frac{1}{2\pi i}\oint_{\lambda^{-1}} \partial_z \log D(z)\, dz = -\frac{1}{2\pi i} \operatorname{tr} \oint_{\lambda^{-1}} (I - zA)^{-1} A\, dz \tag{B.6.1}$$

where integration is over a contour containing λ^{-1} but no other zeroes of $D(z)$. Let Π_λ be defined in (B.5.2), then using the change of variables $z \mapsto z^{-1}$ we compute

$$-\frac{1}{2\pi i}\oint_{\lambda^{-1}} (I - zA)^{-1} A\, dz = \frac{1}{2\pi i}\oint_{\lambda} z^{-1}(z - A)^{-1} A\, dz = \Pi_\lambda,$$

thus (B.6.1) is equal to $\operatorname{tr} \Pi_\lambda$, that is, the multiplicity of λ as an eigenvalue of A. $\square$

Proof of Proposition B.30. 1. First we note that $D(z)$ is of subexponential growth, namely for each $\varepsilon > 0$, there exists C_ε such that

$$|D(z)| \leq C_\varepsilon e^{\varepsilon|z|}. \tag{B.6.2}$$

Indeed, by (B.5.11) and approximation by finite rank operators, for each n

$$|D(z)| \leq \prod_{j=0}^{\infty}(1+|z|s_j(A)) \leq e^{\sum_{j=n}^{\infty} s_j(A)|z|} \prod_{j=0}^{n-1}(1+|z|s_j(A)),$$

and we have $\sum_{j=n}^{\infty} s_j(A) \leq \varepsilon$ for n large enough.

2. We now note that by (B.5.9), $\sum_{j=0}^{\infty} |\lambda_j(A)| < \infty$. It follows that the function $W(z) := \prod_{j=0}^{\infty}(1 - z\lambda_j(A))$ is entire, satisfies the bound (B.6.2), and has the same zeros as $D(z)$, counted with multiplicities. Therefore $D(z) = e^{g(z)}W(z)$, where g is an entire function. In terminology of entire functions we have shown that $D(z)$ is an entire of function of type 0. Hence, see §D.2, $g = 0$, proving Proposition B.30. □

As an example of an application we record the following lemma which is useful in the text:

LEMMA B.33. *Suppose that*

$$H_s := \langle x\rangle^s L^2(\mathbb{R}^n), \quad \|u\|_{H_s}^2 := \int_{\mathbb{R}^n} \langle x\rangle^{-2s}|u(x)|^2 dx.$$

and that a linear operator $A : H_0 \to H_0$ extends to an operator $\widetilde{A} : H_1 \to H_1$. If $A \in \mathcal{L}_1(H_0)$ and $\widetilde{A} \in \mathcal{L}_1(H_1)$ then

$$\operatorname{tr}_{H_0} A = \operatorname{tr}_{H_1} \widetilde{A}. \tag{B.6.3}$$

Proof. 1. Lidskiĭ's theorem shows that the trace is the same for *topologically equivalent* inner products: the eigenvalues are independent of a specific choice of the inner product.

2. Let $B_0 = B(0,1)$ and $B_j = B(0,2^j) \setminus B(0,2^{j-1})$, $j \geq 1$, and define the following equivalent inner product on H_s:

$$\langle u, v\rangle_s := \sum_{j=0}^{\infty} 2^{-2sj}\langle u,v\rangle_{L^2(B_j)}. \tag{B.6.4}$$

Let $\{e_{j,\ell}\}_{\ell=0}^{\infty}$, $\operatorname{supp} e_{j,\ell} \subset B_j$ be an orthonormal basis of $L^2(B_j)$. Then $\{e_{j,\ell}\}_{j,\ell=0}^{\infty}$ is an orthonormal basis of H_0 and $\{f_{j,\ell}\}_{j,\ell=0}^{\infty}$, $f_{j,\ell} := 2^{js}e_{j,\ell}$ is an orthonormal basis of H_s (with respect to the innner product $\langle \bullet, \bullet\rangle_s$ in (B.6.4)).

3. We now have

$$\begin{aligned}\operatorname{tr}_{H_0} A &= \sum_{j,\ell=0}^{\infty} \langle A e_{j,\ell}, e_{j,\ell}\rangle_0 = \sum_{j,\ell=0}^{\infty} \int_{B_j} A e_{j,\ell}(x)\overline{e_{j,\ell}(x)}dx \\ &= \sum_{j,\ell=0}^{\infty} \int_{B_j} \widetilde{A} e_{j,\ell}(x)\overline{e_{j,\ell}(x)}dx = \sum_{j=0}^{\infty} 2^{-2sj} \sum_{\ell=0}^{\infty} \int_{B_j} \widetilde{A} f_{j,\ell}(x)\overline{f_{j,\ell}(x)}dx \\ &= \sum_{j,\ell=0}^{\infty} \langle \widetilde{A} f_{j,\ell}, f_{j,\ell}\rangle_1 = \operatorname{tr}_{H_1} \widetilde{A},\end{aligned}$$

completing the proof. □

B.7. NOTES

Standard references for the results presented in this appendix include Davies [**Da95**], Helffer [**He13**], Reed–Simon [**RS80**] (spectral theory) and Gohberg–Krein [**GK69**], Simon [**Si79b**] (singular values, trace class). Some of our presention also follows [**Sj02**, Chapter 5].

B.8. EXERCISES

1. Show that if the operator $f \otimes g$ is defined by $f \otimes g(h) := g\langle h, f\rangle_H$ then

$$\det(I + f \otimes g) = 1 + \langle g, f\rangle.$$

2. Show that if $A : H \to H$ satisfies $\|A\| < 1$ and $A \in \mathcal{L}_1(H)$ then

$$\det(I - A) = \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} A^k \right).$$

Hint: Use the facts that the operator $\log(I - A)$ is defined using the Taylor series and that $\|A^k\|_{\mathcal{L}_1} \leq \|A\|^{k-1}\|A\|_{\mathcal{L}_1}$.

Appendix C

FREDHOLM THEORY

C.1 Grushin problems
C.2 Fredholm operators
C.3 Meromorphic continuation of operators
C.4 Gohberg–Sigal theory
C.5 Notes
C.6 Exercises

In this appendix we will describe the role of the Schur complement formula in spectral theory, in particular in analytic Fredholm theory.

C.1. GRUSHIN PROBLEMS

Linear algebra. The *Schur complement formula* for two-by-two systems of matrices states that if

$$\begin{pmatrix} P & R_- \\ R_+ & R_{+-} \end{pmatrix}^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix},$$

then P is invertible if and only if E_{-+} is invertible, with

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{-+}^{-1} = R_{+-} - R_+ P^{-1} R_-. \tag{C.1.1}$$

Generalization. We can generalize to problems of the form

$$\begin{pmatrix} P & R_- \\ R_+ & R_{+-} \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{C.1.2}$$

where

$$P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2, \quad R_{+-} : X_- \to X_+.$$

are bounded operators on Banach spaces X_1, X_2, X_+, X_-. If the operator (C.1.2) has a bounded inverse from $X_2 \oplus X_+ \to X_1 \oplus X_-$ then, just as for matrices, invertibility of P is equivalent to the invertibility of E_{-+} and (C.1.1) holds.

DEFINITION C.1. *When $R_{-+} = 0$ we call (C.1.2) a* Grushin problem*:*

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}, \tag{C.1.3}$$

$$P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2,$$

If the Grushin problem (C.1.3) is invertible, we call it well-posed *and we write its inverse as follows:*

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{C.1.4}$$

for operators

$$E : X_2 \to X_1, \quad E_{-+} : X_+ \to X_-, \quad E_+ : X_+ \to X_1, \quad E_- : X_2 \to X_-.$$

In practice, we start with an operator P and build a Grushin problem by choosing $R_\pm$ so taht (C.1.3) is invertible.

The following lemma is immediate:

LEMMA C.2 (The operators in a Grushin problem). *If (C.1.3) is well-posed, then the operators R_+, E_- are surjective, and the operators E_+, R_- are injective.*

The next lemma is a result of a Neumann series calculation:

LEMMA C.3 (Perturbation of a Grushin problem). *Suppose that (C.1.3) is well posed with the inverse given by (C.1.4). If $A : X_1 \to X_2$ is a bounded operator satisfying*

$$\max\left(\|EA\|_{X_1 \to X_1}, \|AE\|_{X_2 \to X_2}\right) < 1. \tag{C.1.5}$$

Then the Grushin problem

$$\begin{pmatrix} P + A & R_- \\ R_+ & 0 \end{pmatrix}, \tag{C.1.6}$$

is well posed with the inverse

$$\begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix},$$

where

$$F_{-+} = E_{-+} + \sum_{k=1}^{\infty} (-1)^k E_- A (EA)^{k-1} E_+. \tag{C.1.7}$$

Proof. Since

$$\begin{pmatrix} P+A & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \left(\begin{pmatrix} I_{X_1} & 0 \\ 0 & I_{X_-} \end{pmatrix} + \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right)$$
$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} \left(\begin{pmatrix} I_{X_1} & 0 \\ 0 & I_{X_-} \end{pmatrix} + \begin{pmatrix} EA & 0 \\ E_-A & 0 \end{pmatrix} \right),$$

and

$$\begin{pmatrix} EA & 0 \\ E_-A & 0 \end{pmatrix}^k = \begin{pmatrix} (EA)^k & 0 \\ E_-A(EA)^{k-1} & 0 \end{pmatrix}, \quad k \geq 1, \tag{C.1.8}$$

we find the right inverse, and similarly the left inverse, of (C.1.6) by a Neumann series if (C.1.5) holds. A simple calculation then gives (C.1.7). □

C.2. FREDHOLM OPERATORS

DEFINITIONS. (i) A bounded linear operator $P : X_1 \to X_2$ is called a *Fredholm operator* if the kernel of P,

$$\ker P := \{u \in X_1 \mid Pu = 0\},$$

and the cokernel of P,

$$\operatorname{coker} P := X_2/PX_1, \text{ where } PX_1 := \{Pu \mid u \in X_1\},$$

are both finite dimensional. Here the cokernel of P is defined algebraically, that is a vector space of cosets, $u + PX_1$, $u \in X_2$.

(ii) The *index* of a Fredholm operator is

$$\operatorname{ind} P := \dim\ker P - \dim\operatorname{coker} P.$$

EXAMPLE. Many important Fredholm operators have the form

$$P = I + K, \tag{C.2.1}$$

where K a compact operator mapping a Banach space X to itself. See Example 2 later in this section for a simple proof when X is a Hilbert space.

Theorem C.5 below shows that the index does not change under continuous deformations of Fredholm operators (with respect to operator norm topology). Hence for operators of the form (C.2.1) the index is 0:

$$\operatorname{ind} P = \operatorname{ind}(I + tK) = \operatorname{ind} I = 0 \quad (0 \leq t \leq 1). \tag{C.2.2}$$

The connection between Grushin problems and Fredholm operators is this:

THEOREM C.4 (Grushin problem for Fredholm operators). (i) *Suppose that $P : X_1 \to X_2$ is a Fredholm operator.*

Then there exist finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$, $R_+ : X_1 \to X_+$, for which the Grushin problem (C.1.3) is well posed. In particular, $PX_1 \subset X_2$ is closed.

(ii) *Conversely, suppose that that for some choice of spaces $X_\pm$ and operators $R_\pm$, the Grushin problem (C.1.3) is well posed.*

Then $P : X_1 \to X_2$ is a Fredholm operator if and only if $E_{-+} : X_+ \to X_-$ is a Fredholm operator. In that case we have

$$\operatorname{ind} P = \operatorname{ind} E_{-+}. \tag{C.2.3}$$

Assertion (ii) is particularly useful when the spaces $X_\pm$ are finite dimensional.

Proof. 1. Assume $P : X_1 \to X_2$ is a Fredholm operator. Let $n_+ := \dim\ker P$ and $n_- := \dim\operatorname{coker} P$, and write $X_+ := \mathbb{C}^{n_+}$, $X_- := \mathbb{C}^{n_-}$. If $\ker P$ is spanned by $x_j \in X_1$, $j = 1, \cdots, n_+$ then, by the Hahn-Banach theorem we can find $x_j^* : X_1 \to \mathbb{R}$, such that $x_j^*(x_i) = \delta_{ij}$ and $\|x_j^*\| \leq 1$. It follows that

$$R_+ : X_1 \to \mathbb{C}^{n_+}, \qquad R_+(x) := (x_1^*(x), \cdots, x_{n_+}^*(x)),$$

has maximal rank, that is $\ker(R_+|_{\ker P}) = \{0\}$.

Now choose $y_j \in X_2$, $j = 1, \cdots, n_-$ so that $y_j + PX_1$ form a basis of X_2/PX_1. Then define

$$R_- : \mathbb{C}^{n_+} \to X_2, \qquad R_-(a_1, \cdots a_{n_-}) := \sum_{j=1}^{n_-} a_j y_j.$$

The operator R_- has maximal rank and $R_- X_- \cap PX_1 = \{0\}$.

We conclude that the operator

$$\begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : X_1 \oplus \mathbb{C}^{n_-} \to X_2 \oplus \mathbb{C}^{n_+}$$

has a trivial kernel and is onto. Hence it is invertible, and by the Open Mapping Theorem the inverse is continuous.

In particular, consider P acting on the quotient space $X_1/\ker P$, which is a Banach space since $\ker P$ is closed. We have $n_+ = 0$, and

$$PX_1 = P(X_1/\ker P) = \begin{pmatrix} P & R_- \end{pmatrix} \begin{pmatrix} X_1/\ker P \\ \{0\} \end{pmatrix}$$

is a closed subspace (the image of a closed subspace by the invertible operator $(P \quad R_-)$).

2. Conversely, suppose that Grushin problem (C.1.3) is well-posed. According to Lemma C.2, the operators R_+, E_- are surjective, and the operators E_+, R_- are injective. If we take $u_- = 0$ then

$$\text{(C.2.4)} \qquad \begin{cases} \text{the equation } Pu = v \ \text{ is equivalent to} \\ u = Ev + E_+v_+, \ 0 = E_-v + E_{-+}v_+. \end{cases}$$

This means that

$$E_- : PX_1 \to E_{-+}X_+,$$

and we can define the induced map

$$E_-^{\#} : X_2/PX_1 \to X_-/E_{-+}X_+.$$

Since E_- is surjective, so is $E_-^{\#}$. We also see that $\ker E_-^{\#} = \{0\}$. In fact, if $E_-v \in E_{-+}X_+$, we can use (C.2.4) to obtain $u \in X_1$ such that $v = Pu$. Hence $E_-^{\#}$ is a bijection of the cokernels X_2/PX_1 and $X_-/E_{-+}X_+$.

3. Next, we claim that

$$E_+ : \ker E_{-+} \to \ker P$$

is a bijection. Indeed, if $u \in \ker P$, then for $v_+ := R_+u$, $u = E_+v_+$ and $E_{-+}v_+ = 0$. Therefore E_+ is onto; and this is all we need check, since E_+ injective.

We conclude that

$$\dim \ker P = \dim \ker E_{-+}, \quad \dim \operatorname{coker} P = \dim \operatorname{coker} E_{-+}.$$

In particular, the indices of P and E_{-+} are equal. □

EXAMPLES. 1. Suppose X is a Banach space and $K : X \to X$. Then

$$\text{(C.2.5)} \qquad \dim KX < \infty \implies I + K \text{ is a Fredholm operator.}$$

Proof of (C.2.5). 1. A finite rank operator can be written as $K = \sum_{j=1}^{J} v_j w_j^*$ where $v_j \in X$, $w_j^* \in X^*$ (the dual space to X) and $J = \operatorname{rank} K := \dim KX$. In particular the sets

$$\{v_j\}_{j=1}^J \subset X, \quad \{w_j^*\}_{j=1}^J \subset X^*,$$

are linearly independent. This shows that $\ker K = \bigcap_{j=1}^{j} \ker w_j^*$ is a closed subspace of X of codimension J.

2. Independence of w_j^*'s shows that can find $w_k \in X$, $k = 1, \cdots, J$ such that $w_j^*(w_k) = \delta_{jk}$. Then $\Pi := \sum_{j=1} w_j w_j^* : X \to X$ is a projection on a finite dimensional space and $\ker K = (I - \Pi)X$.

3. We write

$$\begin{aligned} I + K &= I + K\Pi = I + \Pi K \Pi + (I - \Pi) K \Pi \\ &= (I + \Pi K \Pi)(I + (I - \Pi) K \Pi), \end{aligned}$$

where the second factor is invertible:

$$(I + (I - \Pi) K \Pi)^{-1} = I - (I - \Pi) K \Pi.$$

4. Hence we only need to check that the finite dimensionality of the kernel and cokernel of $I + \Pi K \Pi$ and that follows from the properties of the finite dimensional operator $I_{\Pi X} + \Pi K \Pi$. ☐

2. Suppose now that X is a Hilbert space. Then

(C.2.6) K is a compact operator $\implies$ $I + K$ is a Fredholm operator.

The compactness of K means that $\overline{K B_X(0,1)} \Subset X$, where $B_X(0,1)$ is the unit ball in X. (Property (C.2.6) holds in Banach spaces as well but a little more work is needed and in this book only Hilbert spaces are considered.)

Proof of (C.2.6). 1. Any compact operator $K : X \to X$, where X is a Hilbert space can be approximated in norm by finite rank operators. This implies that there exists $K_0 : X \to X$, $\dim K_0 X < \infty$ such that $\|K - K_0\| < \frac{1}{2}$. In particular $I + K - K_0$ is invertible,

2. We write

$$I + K = (I + K - K_0)(I + (I + (K - K_0))^{-1} K_0).$$

The first factor is invertible and $(I + K - K_0)^{-1} K_0$ is a finite rank operator. Hence (C.2.6) follows from (C.2.5). ☐

THEOREM C.5 (Invariance of the index under deformations). *The set of Fredholm operators is open in $L(X_1, X_2)$, and the index is constant in each component of that set.*

Proof. When P is a Fredholm operator, we can use Theorem C.4 to obtain $E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}$, with

(C.2.7) $$\operatorname{ind} E_{-+} = n_+ - n_-.$$

by the Rank-Nullity Theorem of linear algebra. The Grushin problem remains well-posed (with the same operators $R_\pm$) if P is replaced by P', provided $\|P - P'\| < \varepsilon$ for some sufficiently small $\varepsilon > 0$. Hence the set of Fredholm operators is open.

Using (C.2.7) we see that the index of P' is the same as the index of P. Consequently it remains constant in each connected component of the set of Fredholm operators. $\square$

REMARKS. 1. A bounded linear operator $P : X_1 \to X_2$ is a Fredholm operator if and only if there exists a bounded linear operator $E : X_2 \to X_1$ such that

$$PE = I_{X_2} + K_2, \quad EP = I_{X_1} + K_1, \qquad \text{(C.2.8)}$$
$$\text{where } K_j : X_j \to X_j \text{ are finite rank operators.}$$

In fact, if P is a Fredholm operator then we use Theorem C.4 and obtain (C.2.8) with $K_2 = -R_- E_-$ and $K_1 = -E_+ R_+$. On the other hand, if (C.2.8) holds then $\ker P \subset \ker(I + K_1)$ and $\operatorname{coker} P \subset \operatorname{coker}(I + K_2)$, hence both spaces are finite dimensional.

In particular (C.2.8) shows that adding a finite rank operator to a Fredholm operator, maintain the Fredholm property. Theorem C.5 shows that the index does not change.

2. If $P : X_1 \to X_2$ has index 0, Theorem C.4 shows that we can take $X_- = X_+ = \mathbb{C}^n$, for some n. In that case, we check easily that

$$(P - R_-(I_{\mathbb{C}^n} - E_{-+})R_+)(E - E_+E_-) = I_{X_2}. \qquad \text{(C.2.9)}$$

That means putting

$$K := -R_-(I - E_{-+})R_+, \quad \operatorname{rank} K \le n,$$

makes $P + K$ invertible.

C.3. MEROMORPHIC CONTINUATION OF OPERATORS

DEFINITION C.6. *Let $\Omega \subset \mathbb{C}$ be a connected open set. If X and Y are Banach spaces then, $z \mapsto B(z) \in \mathcal{L}(X, Y)$ is* holomorphic *in Ω if for any $x \in X$ and $y^* \in Y^*$ (the dual of Y), $z \mapsto y^*(B(z)x)$ is a holomorphic function in Ω.*

We recall that the notion of differentiability used in Definition C.6 is equivalent to the existence the holomorphic derivate of $B(z)$ in the norm topology – see [**Ka80**, Chapter III, Theorem 3.12].

DEFINITION C.7. *We say that $z \mapsto B(z)$ is a meromorphic family of operators in Ω if for any $z_0 \in \Omega$ there exist operators B_j, $1 \le j \le J$, of* finite rank *and a family of operators* $z \mapsto B_0(z)$*, holomorphic near* z*, such that*

$$B(z) = B_0(z) + \frac{B_1}{z - z_0} + \cdots \frac{B_J}{(z - z_0)^J}, \quad \textit{near } z_0.$$

We say that $B(z)$ *is a* meromorphic family of Fredholm operators *if for every* z_0*,* $B_0(z)$ *is a Fredholm operator for* z *near* z_0*. For nonsingular* z_0*,* $B_0(z) = B(z)$*.*

REMARK. The Cauchy formula is valid for holomorphic families of operators:

$$B(\mu) = \frac{1}{2\pi i} \oint_\gamma \frac{B(\lambda)}{\lambda - \mu} d\lambda,$$

the integral is over a positively oriented curve enclosing μ. Consequently, the Cauchy estimates hold:

$$\|\partial_\lambda B(\lambda)\|_{X \to Y} \leq \frac{1}{R} \max_{|\lambda - \zeta| \leq R} \|B(\zeta)\|_{X \to Y}. \tag{C.3.1}$$

The Grushin problem framework provides a proof of the following standard result:

THEOREM C.8 (Analytic Fredholm Theory). *Suppose* $\Omega \subset \mathbb{C}$ *is a connected open set and* $\{A(z)\}_{z \in \Omega}$ *is a holomorphic family of Fredholm operators.*

If $A(z_0)^{-1}$ *exists at some point* $z_0 \in \Omega$*, then the family* $z \mapsto A(z)^{-1}$*,* $z \in \Omega$*, is a meromorphic family of operators with poles of finite rank.*

REMARK. The result also holds when the family $z \mapsto A(z)$ is meromorphic. We present the simpler holomorphic case here with the more general result following from the finer analysis in §C.4

Proof. 1. For any $w \in \Omega$ we produce a Grushin problem for $P = A(w)$, as described in the proof of Theorem C.4. The same operators $R_\pm^w$ also provide a well-posed Grushin problem for $P = A(z)$ for z in some sufficiently small neighborhood $V(w)$ of w. According to Theorem C.5

$$\operatorname{ind} A(z) = \operatorname{ind} A(z_0) = 0.$$

Consequently

$$n_+ = n_- = n,$$

and $E_{-+}^w(z)$ is an $n \times n$ matrix with holomorphic coefficients. The invertibility of $E_{-+}^w(z)$ is equivalent to the invertibility of $A(z)$.

2. It follows that for any $w \in \Omega$ there exists a function $f_w(z) := \det E_{-+}^w(z)$, holomorphic in a neighbourhood of w such that $A(z)$ is invertible if and only

if $f_w(z) \neq 0$. Since Ω is connected and since $A(z_0)$ is invertible for at least one $z_0 \in \Omega$, none of f_w's can be identically zero.

3. Since $\det E_{-+}^w(z)$ is not identically 0, $E_{-+}^w(z)^{-1}$ is a meromorphic family of matrices in a neighbourhood of w. Applying (C.1.1), we conclude that

$$A(z)^{-1} = E(z) - E_+(z)E_{-+}^w(z)^{-1}E_-(z)$$

is a meromorphic family of operators in the neighborhood w. As w was arbitrary, $A(z)^{-1}$ is meromorphic in all of Ω. □

As a simple consequence we present

THEOREM C.9. *Let $\Omega \subset \mathbb{C}$ be a connected open set. Suppose that X_1, X_2 are Banach spaces and $X_1 \subset X_2$ is a continuous inclusion. If for $z \in \Omega$,*

$$P - z : X_1 \longrightarrow X_2,$$

is a Fredholm operator and for some $z_0 \in \Omega$, $P - z_0$ is invertible then $z \mapsto (P-z)^{-1} : X_2 \to X_1$ is a meromorphic family of operators on Ω.

For $z \in \Omega$ define

$$\Pi_z := \frac{1}{2\pi i} \oint_z (w-P)^{-1} dw, \tag{C.3.2}$$

where the integral is over a positively oriented circle centered at z and including no poles of $(w-P)^{-1}$ except possibly z Then Π_z is a bounded projection of finite rank:

$$\Pi_z^2 = \Pi_z, \quad \Pi_z : X_2 \to X_1 \subset X_2.$$

Proof. In view of Theorem C.8 we only need to prove that $\Pi_z^2 = \Pi_z$. For that we choose two positively oriented circles $\gamma_j : t \mapsto z + r_j e^{it}$, $0 \le t \le 2\pi$, $0 < r_1 < r_2 \ll 1$. Then

$$\Pi_z = \frac{1}{2\pi i} \int_{\gamma_j} (w_j - P)^{-1} dw_j, \quad j = 1, 2,$$

and, using the resolvent identity,

$$\begin{aligned} \Pi_z^2 &= \frac{1}{2\pi i}\frac{1}{2\pi i} \int_{\gamma_2}\int_{\gamma_1} (w_1 - P)^{-1}(w_2 - P)^{-1} dw_1 dw_2 \\ &= \frac{1}{2\pi i}\frac{1}{2\pi i} \int_{\gamma_2}\int_{\gamma_1} \left((w_1 - P)^{-1} - (w_2 - P)^{-1}\right) \frac{dw_1 dw_2}{w_2 - w_1} \end{aligned}$$

Since for $w_2 \in \gamma_2$, $\int_{\gamma_1} dw_1/(w_2 - w_1) = 0$ and for $w_1 \in \gamma_1$, $\int_{\gamma_2} dw_2/(w_2 - w_1) = 2\pi i$, we see that

$$\Pi_z^2 = \frac{1}{2\pi i}\frac{1}{2\pi i} \int_{\gamma_2}\int_{\gamma_1} (w_1 - P)^{-1} \frac{dw_1 dw_2}{w_2 - w_1} = \frac{1}{2\pi i} \int_{\gamma_1} (w_1 - P)^{-1} dw_1 = \Pi_z,$$

completing the proof. □

C.4. GOHBERG–SIGAL THEORY

Suppose $A(\lambda) : X \to X, \ , \ \lambda \in \Omega$, is a meromorphic family of Fredholm operators with poles of finite rank acting on a Banach space X. Here Ω a connected open subset of $\mathbb{C}$. From the definition in §C.3 this means that near any $\mu \in \Omega$, we have

$$A(\lambda) = \sum_{j=1}^{J} \frac{A_j}{(\lambda - \mu)^k} + A_0(\lambda)\,, \tag{C.4.1}$$

where $\lambda \mapsto A_0(\lambda)$ is holomorphic near μ. If $K = 0$ that means that $A(\lambda) = A_0(\lambda)$ is holomorphic near μ.

The main result of this section is the following factorization theorem.

THEOREM C.10. *Suppose that*

$$\lambda \mapsto A(\lambda), \quad \lambda \in \Omega,$$

is a meromorphic family of Fredholm operators (see Definition C.7). If $A_0(\mu)$ in (C.4.1) has index 0 then there exist families of operators $\lambda \mapsto U_j(\lambda)$, $j = 1, 2$, holomorphic and invertible near μ, and operators P_m, $1 \leq m \leq M$, such that, near μ,

$$\begin{gathered} A(\lambda) = U_1(\lambda)(P_0 + \sum_{m=1}^{M} (\lambda - \mu)^{k_m} P_m) U_2(\lambda)\,, \quad k_\ell \in \mathbb{Z} \setminus \{0\}\,, \\ P_\ell P_m = \delta_{\ell m} P_m\,, \quad \operatorname{rank} P_\ell = 1\,, \quad \ell > 0\,, \quad \operatorname{rank}(I - P_0) < \infty\,. \end{gathered} \tag{C.4.2}$$

INTERPRETATION. 1. The inverse, $A(\lambda)^{-1}$ exists, near μ, as a meromorphic family of operators if and only if $P_0 + \sum_{m=1}^{M} P_m = I$, in which case

$$A(\lambda)^{-1} = U_2(\lambda)^{-1}(P_0 + \sum_{m=1}^{M} (\lambda - \mu)^{-k_m} P_m) U_1(\lambda)^{-1}\,. \tag{C.4.3}$$

This shows that if $A(\lambda_0)^{-1}$ exists at some $\lambda_0 \in \Omega$ then, as Ω is connected, $A(\lambda)^{-1}$ is a meromorphic family of operators in Ω. Hence Theorem C.10 implies the stronger version of Theorem C.8 in which we allow $z \mapsto A(z)$ to be meromorphic.

2. The factorization of $A(\lambda)$ provides a definition of a null multiplicity of A at μ: in the notation of (C.4.2),

$$N_\mu(A) = \begin{cases} \sum_{k_\ell > 0} k_\ell\,, & \text{if } M = \operatorname{rank}(I - P_0), \\[2ex] \infty\,, & \text{if } M < \operatorname{rank}(I - P_0). \end{cases} \tag{C.4.4}$$

When $N_\mu(A) < \infty$ then $A(\lambda)^{-1}$ is meromorphic and

$$N_\mu(A^{-1}) = -\sum_{k_\ell<0} k_\ell \,. \tag{C.4.5}$$

Theorem C.10 and the definitions (C.4.4), (C.4.5) give the following result about multiplicities of poles and zeros of operators:

THEOREM C.11. *Suppose that $A(\lambda)$ and $A(\lambda)^{-1}$, $\lambda \in \Omega$ are meromorphic families of Fredholm operators on a Hilbert space X.*

Then the operator $\oint_\mu \partial_\lambda A(\lambda)A(\lambda)^{-1}d\lambda$ has finite rank and

$$\frac{1}{2\pi i}\operatorname{tr}\oint_\mu \partial_\lambda A(\lambda)A(\lambda)^{-1}d\lambda = N_\mu(A) - N_\mu(A^{-1}). \tag{C.4.6}$$

Here the integral is over a positively oriented circle which includes μ and no other pole of $\partial_\lambda A(\lambda)A(\lambda)^{-1}$.

REMARKS. 1. We stated the theorem in the special case of Hilbert spaces. However, since in (C.4.6) we are taking a trace of a finite rank operator, that trace can be defined in the Banach space case and the result remains valid – see [**GS71**, §2.2].

2. When $A(\lambda) = I + K(\lambda)$ where $K(\lambda)$ is a meromorphic family of trace class operators then we obtain a formula for the multiplicity of zeros and poles of $\det(I + K(\lambda))$ given by the right hand side of (C.4.6):

$$\begin{gathered}\frac{1}{2\pi i}\operatorname{tr}\oint_\mu \frac{D'(\lambda)}{D(\lambda)}d\lambda = n_+(\mu) - n_-(\mu)\,, \\ D(\lambda) := \det(I + K(\lambda))\,, \quad n_\pm(\mu) := N_\mu((I+K)^{\pm 1})\,.\end{gathered} \tag{C.4.7}$$

Proof of Theorem C.11 assuming Theorem C.10. 1. We use (C.4.2), put

$$P(\lambda) := P_0 + P_+(\lambda), \quad P_\pm(\lambda) := \sum_{m=1}^{M}(\lambda-\mu)^{\pm k_m}P_m$$

and calculate:

$$\begin{aligned}(\partial_\lambda A)A^{-1} = \partial_\lambda U_1 U_1^{-1} &+ U_1(\partial_\lambda P)P^{-1}U_1^{-1} \\ &+ U_1 P(\partial_\lambda U_2)U_2^{-1}P^{-1}U_1^{-1}.\end{aligned} \tag{C.4.8}$$

2. The first term is holomorphic near μ and hence the integral vanishes. The last term we rewrite as

$$\begin{aligned}&U_1P_0(\partial_\lambda U_2)U_2^{-1}P_0U_1^{-1} + U_1P_+(\partial_\lambda U_2)U_2^{-1}P_0U_1^{-1} \\ &\quad + U_1^{-1}P_0(\partial_\lambda U_2)U_2^{-1}P_-U_1^{-1} + U_1P_+(\partial_\lambda U_2)U_2^{-1}P_-U_1^{-1}.\end{aligned}$$

The first term is holomorphic and the integral vanishes. Because $P_\pm$ factors have finite rank we can apply cyclicity of the trace to see that the traces of the first two terms vanish and the trace of the last one is holomorphic.

3. Hence the only contribution to $\operatorname{tr} \oint_\mu \partial_\lambda A A^{-1}$ comes from the middle term on the right hand side of (C.4.8). Since $\partial_\lambda P$ has finite rank, we again apply the cyclicity of the trace to conclude that

$$\operatorname{tr} \oint_\mu \partial_\lambda A(\lambda) A(\lambda)^{-1} d\lambda = \oint_\mu \operatorname{tr} \partial_\lambda P(\lambda) P(\lambda)^{-1} d\lambda = \sum_{k_\ell > 0} k_\ell - \sum_{k_\ell < 0} k_\ell,$$

which is (C.4.6). □

Another consequence is an operator valued version of Rouché's theorem:

THEOREM C.12 (Rouché's Theorem for operator valued functions). *Suppose that $A(\lambda)$ and $B(\lambda)$ satisfy the assumptions of Theorem C.11 and that $U \Subset \Omega$ is a simply connected open set with a C^1 boundary ∂U on which A and B have not zeros or poles. If $A(\lambda)^{-1}(A(\lambda) - B(\lambda))$ is of trace class and*

$$\|A(\lambda)^{-1}(A(\lambda) - B(\lambda))\|_{X \to X} < 1, \quad \lambda \in U, \tag{C.4.9}$$

then

$$\sum_{\mu \in U} N_\mu(A) - N_\mu(A^{-1}) = \sum_{\mu \in U} N_\mu(B) - N_\mu(B^{-1}). \tag{C.4.10}$$

EXAMPLE. Suppose that $A(\lambda)$ and $B(\lambda)$ are holomorphic families of matrices. Then $\|A(\lambda)^{-1}(A(\lambda) - B(\lambda)\| < 1$ on ∂U implies that the number of zeros (counted with multiplicities) of $\det A(\lambda)$ in U is the same as the number of zeros of $\det B(\lambda)$ in U.

Proof of Theorem C.12 assuming Theorem C.11. 1. Theorem C.11 and the Cauchy formula show that to prove (C.4.10) we need to show that

$$\operatorname{tr} \int_{\partial U} \partial_\lambda B(\lambda) B(\lambda)^{-1} d\lambda = \operatorname{tr} \int_{\partial U} \partial_\lambda A(\lambda) A(\lambda)^{-1} d\lambda. \tag{C.4.11}$$

2. If we put $C(\lambda) := A(\lambda)^{-1}(A(\lambda) - B(\lambda))$ then (C.4.9) gives

$$C(\lambda) \in \mathcal{L}_1(X), \quad \|C(\lambda)\|_{\mathcal{L}(X)} < 1, \quad B(\lambda) = A(\lambda)(I - C(\lambda)).$$

In addition $I - C(\lambda)$ satisfies the assumptions of Theorem C.11. We write,

$$(\partial_\lambda B) B^{-1} = (\partial_\lambda A) A^{-1} - A(\partial_\lambda C)(I - C)^{-1} A^{-1}. \tag{C.4.12}$$

We claim that the trace of the integral of the last term over ∂U vanishes. That will prove (C.4.11) and hence (C.4.10).

3. We first note that

$$A(\partial_\lambda C)(I - C)^{-1}A^{-1} = A(\partial_\lambda C)A^{-1} + \sum_{k=1}^{\infty} A(\partial_\lambda C)C^k A^{-1}. \tag{C.4.13}$$

Since

$$\|A(\partial_\lambda C)C^k A^{-1}\|_{\mathcal{L}_1(X)} \leq \|A\|_{\mathcal{L}(X)}\|\partial_\lambda C\|_{\mathcal{L}(X)}\|A^{-1}\|_{\mathcal{L}(X)}\|C\|_{\mathcal{L}_1(X)}\|C\|^{k-1}_{\mathcal{L}(X)},$$

the infinite sum converges in $\mathcal{L}_1(X)$. (The first term in (C.4.13) does *not* need to be in $\mathcal{L}_1(X)$; however Theorem C.11 guarantees that its integral over ∂U is of trace class.)

4. We first show that the trace of the integral of the first term on the right hand side of (C.4.13) is zero. In fact,

$$A(\partial_\lambda C)A^{-1} = \partial_\lambda(ACA^{-1}) - (\partial_\lambda A)CA^{-1} + ACA^{-1}(\partial_\lambda A)A^{-1},$$

where the last two terms on the right hand side are of trace class and using the cyclicity of the trace,

$$\operatorname{tr}\left((\partial_\lambda A)CA^{-1}\right) - \operatorname{tr}\left(ACA^{-1}(\partial_\lambda A)A^{-1}\right) = 0.$$

Hence

$$\operatorname{tr}\int_{\partial U} A(\partial_\lambda C)A^{-1}d\lambda = \operatorname{tr}\int_{\partial U} \partial_\lambda\left(ACA^{-1}\right)d\lambda = 0.$$

5. It remains to handle the second term on the right hand side of (C.4.13):

$$\begin{aligned}\operatorname{tr}\int_{\partial U}\sum_{k=1}^{\infty} A(\partial_\lambda C)C^k A^{-1}d\lambda &= \sum_{k=1}^{\infty}\int_{\partial U}\operatorname{tr}\left(A(\partial_\lambda C)C^k A^{-1}\right)d\lambda \\ &= \sum_{k=1}^{\infty}\frac{1}{k+1}\operatorname{tr}\int_{\partial U}\partial_\lambda(C^{k+1})d\lambda = 0,\end{aligned}$$

where we used cyclicity of the trace. (Note that for $k \geq 1$, $\lambda \mapsto \partial_\lambda(C^{k+1}) = \sum_{\ell=0}^{k} C^\ell(\partial_\lambda C)C^{k-\ell}$ is a continuous famuly of trace class operators on ∂U.)

6. Steps 4 and 5 showed that the trace of the integral of the left hand side of (C.4.13) over ∂U is equal to 0. Going back to (C.4.12) and (C.4.11) we obtain (C.4.10). $\square$

The first step in the proof of Theorem C.10 is the following Lemma concerning matrix valued meromorphic functions. It is the finite dimensional version of Theorem C.10

LEMMA C.13. *Suppose that $\lambda \mapsto M(\lambda)$, $\lambda \in D(0,r)$, is a family of $n \times n$ matrices with meromorphic entries. Then there exist families of $n \times n$*

matrices, $\lambda \mapsto E(\lambda), F(\lambda)$, holomorphic and invertible in $D(0,\rho)$ for some $\rho \leq r$ and such that

$$M(\lambda) = E(\lambda) \begin{pmatrix} \lambda^{k_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda^{k_2} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \dots & \lambda^{k_N} & 0 & \dots & \vdots \\ \vdots & \vdots & \dots & 0 & 0 & \dots & \vdots \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} F(\lambda), \tag{C.4.14}$$

where $N \leq n$ and $k_j \in \mathbb{Z}$.

Proof. 1. Since the entries of $M(\lambda)$ are meromorphic near 0 we have

$$M(\lambda) = (\lambda^{p_{ij}} a_{ij}(\lambda))_{1 \leq i,j \leq n},,$$

We can choose ρ small enough so that

$$|a_{ij}(\lambda)| > \varepsilon > 0 \text{ in } D(0,\rho) \quad \text{or} \quad a_{ij} \equiv 0\ .$$

2. By row and column operations, that is by multliplying $M(\lambda)$ by invertible matrices on the left and on the right respectively, we can transform $A(\lambda)$ to a matrix with $p_{11} = \min_{1 \leq i,j \leq n} p_{ij}$, $|a_{11}(\lambda)| > \varepsilon$ in $D(0,\rho)$. Then

$$\frac{\lambda^{p_{ij}} a_{p_{ij}}(\lambda)}{\lambda^{p_{11}} a_{11}(\lambda)}, \quad 1 \leq i,j \leq q,$$

are holomorphic in $D(0,\rho)$. Hence, further row and column operations depending holomorphically on λ produce

$$M(\lambda) = E_1(\lambda) \begin{pmatrix} \lambda^{p_{11}} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M_1(\lambda) & \\ 0 & & & \end{pmatrix} F_1(\lambda),$$

where $M_1(\lambda)$ is now an $(n-1) \times (n-1)$ matrix with meromorphic coefficients, and $E_1(\lambda)$ and $F_1(\lambda)$ are invertible $n \times n$ matrices with holomorphic coefficients.

3. We can apply the same procedure to $M_1(\lambda)$ until we reach $M_n(\lambda) \neq 0$ or $M_N(\lambda) = \mathbf{0}_{(n-N)\times(n-N)}$. □

We also need two general facts:

LEMMA C.14. *Suppose that X_0 is a Banach space and $Y_0 \subset X$ is a finite dimensional subspace of X_0. Then there exists a* closed *subspace $X_0 \subset X$ such that*

$$X_0 \cap Y_0 = \{0\}, \quad \textit{and} \quad Z_0 + Y_0 = X.$$

Proof. 1. Let $y_1, \cdots y_N$ be a basis of Y_0, and let $\tilde{y}_j^* : Y_0 \to \mathbb{R}$ be defined by $\tilde{y}_j^*(y_i) = \delta_{ij}$. By the Hahn-Banach Theorem we can extend $\tilde{y}_j^*$ to $y_j^* : X_0 \to \mathbb{R}$ so that $\|y_j^*\| = \|\tilde{y}_j^*\| = 1$.

2. We then define a continuous linear transformation $\Pi : X_0 \to X_0$ by $\Pi(x) = \sum_{j=1}^N y_j y_j^*$, so that $\Pi X_0 = Y_0$, $\Pi^2 = \Pi$. Since $\ker \Pi = (I - \Pi)X_0$, putting $Z_0 := \ker \Pi$ provides a closed subspace complementing Y_0 in X. □

LEMMA C.15. *Suppose X is a Banach space and $X_1 \subset X$ is a closed subspace of X satisfying* $\dim X/X_1 < \infty$*. Suppose also that a finite dimensional subspace $Y_0 \subset X$ satisfies $Y_0 \cap X_1 = \{0\}$. Then there exists a finite dimensional subspace Y_1 such that $Y_0 \subset Y_1$, $Y_1 + X_1 = X$, $Y_1 \cap X_1 = \{0\}$.*

Proof. 1. Let Π be the projection constructed for Y_0 as in step 2 of the proof of Lemma C.14. The subspace $X_2 = X_1 + Y_0 \subset X$ has finite codimension as $X/(X_1 + Y_0) \to X/X_1$, $x + X_1 + Y_0 \mapsto (I - \Pi)x + X_1$ is injective. (Recall that $X_1 \cap Y_0 = \{0\}$.)

2. If $X_2 \neq X$ we now need to find a complement of the subspace X_2. For that we find a set $x_1, \cdots x_J$ such that $x_j + X_2$ are a basis of X/X_2 and put $Y_2 := \mathrm{Span}\{x_1, \cdots x_J\}$. The desired space is then $Y_1 := Y_2 + Y_0$. □

Proof of Theorem C.10. 1. Without loss of generality we can assume that $\mu = 0$. Since we assumed that $A_0(0)$ is a Fredholm operator of index 0, Remark 2 in §C.2 shows that there exists a finite rank operator C such that $A_0(0) + C$ is invertible. Consequently for λ in a small neighourhood of 0,

$$B(\lambda) := A_0(\lambda) + C$$

is also invertible and

$$A(\lambda) = B(\lambda)(I + K(\lambda)), \quad K(\lambda) := B(\lambda)^{-1}\left(\sum_{j=1}^{J} A_j \lambda^{-j} - C\right).$$

We can now consider the Laurent series of $K(\lambda)$,

$$K(\lambda) = \sum_{j=1}^{J} K_j \lambda^{-j} + K_0(\lambda),$$

where K_j' s have finite rank and $K_0(\lambda)$ is holomorphic in $D(0, \rho)$.

2. We define

$$X_0 = \ker C \cap \bigcap_{j=1}^{J} \ker A_j \subset X,$$

which is a closed subspace of finite codimension. We note that

$$K(\lambda)v = 0, \quad v \in X_0, \quad \lambda \in D(0,\rho). \tag{C.4.15}$$

Applying Lemma C.14 with

$$X_0 = \bigcap_{j=1}^{J} \ker K_j, \quad Y_0 = \bigcap_{j=1}^{J} \ker K_j \cap \sum_{i=1}^{J} K_i X,$$

shows that there exists $Z_0 \subset X_0$, a closed subspace of finite codimension satisfying

$$\bigcap_{j=1}^{J} \ker K_j = Z_0 + \bigcap_{j=1}^{J} \ker K_j \cap \sum_{i=1}^{J} K_i X, \quad Z_0 \cap \sum_{i=1}^{J} K_i X = \{0\}.$$

We then put

$$X_1 = X_0 \cap Z_0 \subset \bigcap_{j=1}^{J} \ker K_j,$$

which is a closed subspace of finite codimension. Because of the construction of Z_0,

$$X_1 \cap \sum_{j=1}^{J} K_j X = \{0\}.$$

Lemma C.15 used with $Y_0 = \sum_j K_j X$ shows that there exists a finite dimensional complement of X_1, Y_1, invariant under K_j's (since $Y_0 \subset Y_1$):

$$\begin{gathered} X_1 + Y_1 = X, \quad X_1 \cap Y_1 = \{0\}, \quad \dim Y_1 < \infty, \\ K_j|_{X_1} = 0, \quad K_j Y_1 \subset Y_1, \quad j = 1, \cdots J, \quad K(\lambda)|_{X_1} = 0. \end{gathered} \tag{C.4.16}$$

3. We define $P : X \to Y_1$ as the projection onto Y_1 with $\ker P = X_1$. Since X_1 is closed the Closed Graph Theorem implies that P is continuous. (If $x_j \to x$ and $Px_j =: y_j \to y$ then $x_j = \tilde{x}_j + y_j$, $X_1 \ni \tilde{x}_j \to x - y$. That implies that $x - y \in X_1$ and that $Px = y$. That means that the graph of P is closed.)

The properties $(I-P)K_jP = 0$ (invariance of Y_1 under K_j's), $X_1 \subset X_0$, and (C.4.15) show that

$$\begin{aligned} I + K(\lambda) &= I + K(\lambda)P \\ &= I + PK(\lambda)P + (I-P)K_0(\lambda)P \\ &= (I + PK(\lambda)P)(I + (I-P)K_0(\lambda)P). \end{aligned}$$

The projection property, $P(I-P)=0$, shows that the last factor is invertible:

$$(I+(I-P)K_0(\lambda)P)^{-1} = I-(I-P)K_0(\lambda)P.$$

Hence,

$$A(\lambda) = B(\lambda)(I+PK(\lambda)P)C(\lambda), \quad C(\lambda) := I+(I-P)K_0(\lambda)P, \tag{C.4.17}$$

and both $B(\lambda)$ and $C(\lambda)$ are invertible and holomorphic in $D(0,\rho)$.

4. The operator $P(I+PK(\lambda)P)P$ acts on the finite dimensional space Y_1 and hence we can apply Lemma C.13 to it:

$$P(I+PK(\lambda)P)P = E(\lambda)\left(\sum_{j=1}^{N}\lambda^{k_j}P'_j\right)F(\lambda)P, \tag{C.4.18}$$

where $P'_j : Y_1 \to Y_1$ are one dimensional projections satisftying $P'_jP'_i = \delta_{ij}P'_j$, $N \le \dim Y_0$, and

$$E(\lambda), F(\lambda) : PX \to PX \quad \text{are holomorphic and invertible,}$$

for $\lambda \in D(0,\rho_1)$, where $0<\rho_1<\rho$.

5. Let us put denote $\iota_P : PX \hookrightarrow X$ the inclusion map, and

$$P_0 := I-P, \quad P_j = \iota_P P'_j P : X \to X, \quad P_j^2 = P_j, \quad \dim P_jX = 1.$$

From (C.4.18) we get

$$\begin{aligned} I+PK(\lambda)P &= P_0 + P(I+PK(\lambda)P)P \\ &= P_0 + \iota_P E(\lambda)\left(\sum_{j=1}^{N}\lambda^{k_j}P'_j\right)F(\lambda)P \\ &= (P_0+\iota_P E(\lambda))\left(P_0+\sum_{j=1}^{N}\lambda^{k_j}P_j\right)(P_0+\iota_P F(\lambda)P). \end{aligned}$$

The outside factors are invertible:

$$\begin{aligned} (P_0+\iota_P E(\lambda)P)^{-1} &= P_0+\iota_P E(\lambda)^{-1}P, \\ (P_0+\iota_P F(\lambda)P)^{-1} &= P_0+\iota_P F(\lambda)^{-1}P, \end{aligned}$$

and hence (C.4.17) shows that (C.4.2) holds with

$$U_1(\lambda) = B(\lambda)(P_0+\iota_P E(\lambda)P), \quad U_2(\lambda) = (P_0+\iota_P F(\lambda)P)C(\lambda).$$

This completes the proof of Theorem C.10. $\square$

C.5. NOTES

For more about Grushin problems and connection to Feshbach reduction and other linear algebra constructions useful in spectral theory see Sjöstrand–Zworski [**SZ07b**] and references given there. We refer to Hörmander [**HöII**, Sect.19.1] for an introduction to Fredholm operators.

The Gohberg–Sigal generalization of residue theory to operator valued meromorphic functions comes from the classical paper [**GS71**] which can be consulted for additional results, in particular for a stronger version of Theorem C.12. The proof of Lemma C.13 comes from Vodev [**Vo94a**, Appendix].

C.6. EXERCISES

1. Find the decomposition of Lemma C.13 for the following matrices:

$$A(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad A(\lambda) = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}.$$

2. Assume that $A(\lambda) : X \to X$ is a holomorphic family of Fredholm operators of index zero on a Banach space X depending on $\lambda \in \Omega \subset \mathbb{C}$, $0 \in \Omega$. Assume also that there exist $u_1, \dots, u_N \in X, v_1, \dots, v_N \in X^*$ such that

$$\ker A(0) = \operatorname{span}\{u_j\}_{j=1}^N, \quad A(0)^* v_j = 0, \quad v_j(A'(0)u_k) = \delta_{jk}.$$

Show that near $\lambda = 0$, $A(\lambda)^{-1}$ has the expansion

$$A(\lambda)^{-1} = \sum_{j=1}^N \frac{u_j \otimes v_j}{\lambda} + A_0(\lambda), \quad (f \otimes g)(u) := g(u)f, \quad f \in X, \ g \in X^*.$$

where $A_0(\lambda)$ is holomorphic at 0. **Hint:** Set up a Grushin problem for $A(0)$ (use the procedure in Step 1 of the proof of Theorem C.4 with $x_j^*(u_i) = \delta_{ij}$ and $v_j(y_i) = \delta_{ij}$) and then apply Lemma C.3 with $P = A(0)$ and $A = A(\lambda) - A(0)$.

3. Assume that $A(\lambda) : X \to X$ is a holomorphic family of Fredholm operators of index zero on a Banach space X depending on $\lambda \in \Omega$, $0 \in \Omega$. For $\ell \in \mathbb{N}_0$, define the space V_ℓ of polynomials $p(\lambda) : \mathbb{C} \to X$ in λ of order no more than ℓ such that $A(\lambda)p(\lambda) = \mathcal{O}(\lambda^{\ell+1})_X$ near $\lambda = 0$. In particular, V_0 is the kernel of $A(0)$.

(a) Let $T_\ell : V_\ell \to V_{\ell-1}$ be the homomorphism erasing the λ^ℓ term; here $V_{-1} := 0$. Show that the kernel of T_ℓ is isomorphic to V_0. Consider also the injective homomorphism $S_\ell : V_\ell \to V_{\ell+1}$ defined by multiplication by λ.

(b) Show that all the spaces V_ℓ are finite dimensional and S_ℓ is an isomorphism for ℓ large enough. **Hint:** Use Theorem C.10.

(c) Show that the algebraic multiplicity of 0 as a pole of $A(\lambda)^{-1}$ is equal to the limit $\lim_{\ell\to\infty} \dim V_\ell$.

Appendix D

COMPLEX ANALYSIS

D.1 General facts
D.2 Entire functions

D.1. GENERAL FACTS

For a function f of two variables, (x,y), $z = x+iy$, we write

$$\partial_z f = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \bar\partial_z f = \partial_{\bar z} f = \frac{\partial f}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

A function is holomorphic in an open set $\Omega \subset \mathbb{C}$ if and only if $\partial_{\bar z} f \equiv 0$, where the derivatives are taken in the sense of distributions.

If U has a C^1 boundary ∂U, positively oriented in the sense U is always to the left of the direction on γ, we have the following consequence of Green's formula: for $f \in C^1(\overline{U})$,

(D.1.1)
$$\int_{\partial U} f(w)dw = 2i \int_U \frac{\partial f}{\partial \bar w}(w)dm(w), \quad w = x+iy, \quad dm(w) := dxdy$$
$$f(z) = \frac{1}{2\pi i}\int_{\partial U} \frac{f(w)}{w-z}dw - \frac{1}{\pi}\int_U \frac{1}{w-z}\frac{\partial f}{\partial \bar w}(w)dm(w),$$

see for instance [**HöI**, (3.1.9),(3.1.11)]. This is sometimes referred to as the *Cauchy–Green formula.*

D.1.1. Maximum principle. The following quantitative application of the maximum principle is useful in the study of resonances.

LEMMA D.1 (Three line theorem in a rectangle). *Suppose that* $f(z)$ *is holomorphic in a neighbourhood of* $\Omega := [-2R, 2R] + i[-\delta_-, \delta_+]$. *Suppose also that for* $M, M_+, M_- > 0$, *and* $0 \leq \delta_+ < \delta_- < 1$,

$$\begin{gathered} |f(z)| \leq M_\pm, \quad \operatorname{Im} z = \pm\delta_\pm, \quad |\operatorname{Re} z| \leq 2R, \\ |f(z)| \leq M, \quad z \in \Omega. \end{gathered} \tag{D.1.2}$$

and that

$$R^2 \delta_-^{-2} \geq \log\left(\frac{M}{\min_\pm M_\pm}\right). \tag{D.1.3}$$

Then for $\operatorname{Im} z = 0$ *and* $|\operatorname{Re} z| \leq R$,

$$|f(z)| \leq e M_+^\theta M_-^{1-\theta}, \quad \theta := \frac{\delta_-}{\delta_+ + \delta_-}. \tag{D.1.4}$$

Proof. 1. If we replace Ω by $\Omega' := [-R, R] + i[-\delta_-, \delta_+]$ then it is enough to prove that $|f(0)| \leq e M_+^\theta M_-^{1-\theta}$ under the assumption (D.1.2) with $2R$ replaced by R and Ω by Ω'.

2. Putting $m := \log M$, $m_\pm := \log M_\pm$ and $z = x + iy$, we consider the following subharmonic function defined in a neighbourhood of Ω':

$$u(z) := \log|f(x+iy)| - \frac{(\delta_- + y)m_+ + (\delta_+ - y)m_-}{\delta_+ + \delta_-} - \delta_-^{-2}(x^2 - y^2).$$

We now use (D.1.2) for $\operatorname{Im} z = \pm\delta_\pm$, $|\operatorname{Re} z| \leq R$:

$$u(z) \leq m_\pm - m_\pm - \delta_-^{-2}(x^2 - \delta_\pm^2) \leq \delta_-^{-2}\delta_\pm^2 \leq 1.$$

Also, (D.1.3) gives for $|\operatorname{Re} z| = R$, $-\delta_- \leq \operatorname{Im} z \leq \delta_+$,

$$u(z) \leq m - \min m_\pm - \delta_-^{-2}(R^2 - \delta_-^2) \leq 1.$$

The maximum principle for subharmonic functions now shows that

$$\log|f(iy)| \leq 1 + \frac{\delta_- + y}{\delta_- + \delta_+} m_+ + \frac{\delta_+ - y}{\delta_- + \delta_+} m_- + \delta_-^{-2} y^2, \tag{D.1.5}$$

and in particular,

$$|f(0)| \leq e M_+^\theta M_-^{1-\theta}, \quad \theta = \frac{\delta_-}{\delta_+ + \delta_-}.$$

That concludes the proof. □

We also use the Borel–Carathéodory theorem: for f holomorphic in the closed disc $\overline{D(0,R)}$ and $0 < r < R$ we have

$$\max_{|z|\leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z|\leq R} \operatorname{Re} f(z) + \frac{R+r}{R-r} |f(0)| \,, \tag{D.1.6}$$

see [**Ti39**, §5.5].

A more general version of (D.1.6) can be given as follows: suppose that for two open sets Ω_j, $j = 1, 2$,

$$\begin{gathered} \Omega_0 \Subset \Omega_1 \Subset \mathbb{C}, \quad \Omega_1 \text{ is simply connected,} \\ f \text{ is holomorphic in a neighbourhood of } \Omega_1, \ z_0 \in \Omega_1. \end{gathered} \tag{D.1.7}$$

Then there exists $C_0 = C_0(\Omega_0, \Omega_1, z_0)$ such that

$$\sup_{z\in\Omega_0} |f(z)| \leq C_0 (\sup_{z\in\Omega_1} \operatorname{Re} f(z) + |f(z_0)|). \tag{D.1.8}$$

This follows applying the Riemman mapping theorem to obtain a biholomorphic mapping, F, of Ω_1 onto $D(0,1)$ such that $F(z_0) = 0$. We then find $r < 1$ such that $F(\Omega_0) \subset D(0,r)$ and apply (D.1.6) with $R = 1$. (We can easily do away with the simple connectedness assumption.)

D.1.2. Estimates on the number of zeros. The basic result relating the growth of a holomorphic function f to the growth of the number of its zeros is the *Jensen formula*:

Suppose that $f(0) \neq 0$. Then

$$\int_0^r \frac{n(t)}{t} dt + \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta} r)| d\theta \,, \tag{D.1.9}$$

where $n(t)$ is the number of zeros of $f(z)$ with $|z| < t$, see [**Ti39**, §3.6].

From this we get an estimate on the number of zeros of f in a disc of radius r:

$$\begin{aligned} n(r) &\leq \frac{1}{\log 2} \int_r^{2r} \frac{n(t)}{t} dt \\ &\leq \frac{1}{\log 2} \left(\log \max_{|z|=2r} |f(z)| - \log |f(0)| \right) . \end{aligned} \tag{D.1.10}$$

If $f(0) = 0$ we apply the formula to $f(z)/z^p$ where p is the order of vanishing of f at 0.

A more general version of the upper bound on the number of zeros, which follows from the version for discs, can be stated as follows: if (D.1.7) holds then then there exists a constant $C_1 = C_1(\Omega_0, \Omega_1, z_0)$ such that the number of zero of f in Ω_0, n_{Ω_0}, satisfies

$$n_{\Omega_0} \leq C_1 (\max_{z\in\Omega_1} \log |f(z)| - \log |f(z_0)|). \tag{D.1.11}$$

An estimate due to Carleman gives information about zeros of functions bounded in a halfplane [Ti39, §3.71]: suppose that f is holomorphic in $\operatorname{Im} z \geq 0$ and that

$$|f(z)| \leq C, \quad \operatorname{Im} z \geq 0.$$

If $\{z_j\}_{j=0}^{\infty}$ are the zeros of f in $\operatorname{Im} z > 0$ (included according to their multiplicities) then

$$\sum_{j=0}^{\infty} \frac{\operatorname{Im} z_j}{|z_j|^2} < \infty. \tag{D.1.12}$$

The estimate (D.1.12) is a consequence of Carleman's Theorem [Ti39, §3.7] which is a version of Jensen's formula (D.1.9) for a half-plane.

D.1.3. Lower bounds on moduli. An upper bound in Ω_1 and a lower bound at a point z_0 also give lower bounds for the function away from zeros: suppose z_j, $j = 1, 2, \ldots$, are the zeros of f in Ω_1. The simplest version can be stated as follows: there exists $C_2 = C_2(\Omega_0, \Omega_1, z_0)$ such that for any sufficiently small $\delta > 0$

$$\begin{gathered} \log|f(z)| \geq -C_2 \log \frac{1}{\delta} \left(\max_{z \in \Omega_1} \log|f(z)| - \log|f(z_0)| \right), \\ z \in \Omega_0 \setminus \bigcup_j D(z_j, \delta). \end{gathered} \tag{D.1.13}$$

To prove (D.1.13) we proceed as follows: choose an open simply connected set Ω so that $\Omega_0 \Subset \Omega \Subset \Omega_1$. Then

$$f(z) = e^{i\alpha} e^{g(z)} P(z), \quad P(z) := \prod_{z_j \in \overline{\Omega}} (z - z_j), \quad z \in \Omega, \tag{D.1.14}$$

where g is holomorphic in a neighbourhood of Ω, $\operatorname{Im} g(z_0) = 0$, and $\alpha \in \mathbb{R}$. We will prove that we can choose g (by changing its value by an imaginary constant) so that for some constant $C_4 = C_4(\Omega_0, \Omega_1, z)$,

$$|g(z)| \leq C_4 M, \quad z \in \Omega_0, \qquad M := \max_{z \in \Omega_1} \log|f(z)| - \log|f(z_0)|. \tag{D.1.15}$$

Since the bound on the number of zeros (D.1.11) shows that

$$\log|P(z)| \geq -\log \frac{1}{\delta} C_1 M, \quad z \in \mathbb{C} \setminus \bigcup_j D(z_j, \delta) \tag{D.1.16}$$

the estimate (D.1.13) follows from (D.1.15). (That estimate is stronger than what is needed for (D.1.13) but we use it in an essential way in §7.4.)

Proof of (D.1.15). 1. The key component of the following estimate due to Cartan: given arbitrary numbers $z_n \in \mathbb{C}$, $n = 1, \ldots, N$ and any $\eta > 0$ there exists a set, $\bigcup_{l=1}^{L} D(a_l, r_l)$, formed by the union of $L \leq N$ discs, $D(a_l, r_l)$, centered at some points $a_l \in \mathbb{C}$, such that $\sum_{\ell=1}^{L} r_\ell < 2e\eta$ and

$$\prod_{n=1}^{N} |z - z_n| > \eta^M , \quad z \in \mathbb{C} \setminus \bigcup_{\ell=1}^{L} D(a_\ell, r_\ell) . \tag{D.1.17}$$

For the proof see for instance [**Ha89**, Lemma 6.17].

2. Returning to (D.1.14) we see from (D.1.11) and (D.1.17) that

$$\begin{gathered} \operatorname{Re} g(z) = \log |f(z)| - \log |P(z)| \leq (1 + \log \frac{1}{\eta} C_1) M, \\ z \in \Omega \setminus \bigcup_{\ell=1}^{L} D(a_\ell, r_\ell). \end{gathered} \tag{D.1.18}$$

If η is small enough, there exists an open, simply connected Ω' such that $\Omega_0 \Subset \Omega' \Subset \Omega$ and $\partial\Omega' \cap D(a_\ell, r_\ell) = \emptyset$, $1 \leq \ell \leq L$. (We again use the Riemann mapping theorem to obtain a bi-holomorphic F from Ω to $D(0,1)$ and note that the sum of diameters $F(D(a_l, r_\ell))$ is bounded by $C\eta$. Hence there exists $r > 1 - C\eta$ such that $\partial D(0, r)$ is disjoint from the union of these images. We then choose η small enough so that $F(\Omega_0) \Subset D(0, r)$.)

The maximum principle and (D.1.18) now give

$$\operatorname{Re} g(z) \leq C_5 M, \quad z \in \Omega'. \tag{D.1.19}$$

Since $\log |P(z)| \leq C_5 M$,

$$\operatorname{Re} g(z_0) \geq \log |f(z_0)| - \log |P(z_0)| \geq -C_6 M,$$

and $\operatorname{Im} g(z_0) = 0$ (by assumption) we get $|g(z_0)| \leq C_7 M$. This and (D.1.19) now show that we can apply (D.1.8) with Ω_1 replaced by Ω' to obtain (D.1.15). □

D.2. ENTIRE FUNCTIONS

We recall some facts about entire referring to [**Le64**] for proofs and more details.

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function. We then say that f is an *entire function*. We can use canonical products to factorize f and that is particularly nice for functions with a polynomial growth of the number of zeros. For that we recall the definition

$$E_p(z) = (1 - z) \exp \left(z + \frac{z^2}{2} + \cdots \frac{z^p}{p} \right) .$$

If a sequence $\{z_k\}_{k=1}^{\infty}$, $z_k \in \mathbb{C}$, satisfies

$$\sum \frac{1}{|z_n|^{p+1}} < \infty \tag{D.2.1}$$

then the infinite *Weierstrass product*

$$P(z) := \prod_{k=1}^{\infty} E_p(z/z_k) \tag{D.2.2}$$

conveges and

$$m_P(z) := \frac{1}{2\pi i} \lim_{\epsilon \to 0} \oint_{\gamma_\epsilon(z)} \frac{P'(w)}{P(w)} dw = |\{k : z_k = z\}| \,.$$

where $\gamma_\epsilon(z)$ is the positively oriented circle $[0, 2\pi) \ni t \mapsto z + \epsilon e^{it}$.

Using the notation $n(r)$ above we have the following estimate:

$$\max_{|z| \le r} \log |P(z)| \le k_p r^p \left(\int_0^r \frac{n(t)}{t^{p+1}} dt + r \int_r^\infty \frac{n(t)}{t^{p+2}} dt \right) \,. \tag{D.2.3}$$

In particular, when

$$n(r) \le C r^p \,, \tag{D.2.4}$$

we have

$$\log |P(z)| \le C |z|^p \log |z|. \tag{D.2.5}$$

A lower bound also holds and here is the case we use. When (D.2.4) is satisfied then for any $\epsilon > 0$ there existst r_0 such that

$$\log |P(z)| \ge -|z|^{p+\epsilon} \,, \quad z \notin \bigcup_{m_P(w) > 0} D(w, \langle w \rangle^{-p-\epsilon}) \,, \quad |z| \ge r_0 \,. \tag{D.2.6}$$

One consequence of these two bounds and of (D.1.6) is a version of Hadarmard's factorization theorem: suppose that f is entire and that

$$|f(z)| \le C e^{C|z|^p}.$$

If $\{z_k\}_{k=1}^{\infty}$ are the zeros of f (included according to their multiplicities) then

$$f(z) = e^{g(z)} P(z), \tag{D.2.7}$$

where $P(z)$ is given by (D.2.2) and g is a polynomial of degree less than or equal to p.

We say that f is of *exponential type* $\tau \in [0, \infty]$ if

$$\limsup_{r \to \infty} \frac{\log \sup_{\lambda \le r} |f(r)|}{r} = \tau \,.$$

When the type satisfies $0 < \tau < \infty$ the function is of *normal type*. The indicator function h gives a more precise notion of order:

$$h(\theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r} .$$

It turns out that the function h is an indicator function of a convex set $K \subset \mathbb{C}$:

$$h(\theta) = \sup_{z\in K} (\cos\theta \, \mathrm{Re}\, z + \sin\theta \, \mathrm{Im}\, z) .$$

The set K is called the indicator diagram of f.

When $h(\theta)$ is a limit along a density one sequence of r's (not just lim sup) and the convergence is uniform in θ, the function f is said to have *completely regular growth*. In that case we can describe the distribution of zeros in sectors using the indicator function – see [**Le64**]. Here we quote a specific result which is used in Section 2.5:

THEOREM D.2 (Asymptotics of zeros). *If f is of exponential type in $\mathbb{C}$ and if*

$$\text{(D.2.8)} \qquad \int_{\mathbb{R}} \frac{\log^+|f(x)|}{1+x^2} dx < \infty , \quad \log^+ r := \max(\log r, 0).$$

then f has completely regular growth and the indicator diagram of f is given by an interval $I_f \subset i\mathbb{R}$.

For $\epsilon > 0$ define $\Lambda_\epsilon := \{z = e^{i\theta}x : x \in \mathbb{R}, |\theta| < \epsilon\}$. Then, writing $m_f(z)$ for the multiplicity of a zero of f,

$$\lim_{r\to\infty} \frac{1}{r} \sum_{z\in \complement\Lambda_\epsilon\cap D(0,r)} m_f(z) = 0, \qquad \lim_{r\to\infty} \frac{1}{r} \sum_{z\in \Lambda_\epsilon\cap D(0,r)} m_f(z) = \frac{|I_f|}{2\pi}.$$

It is not difficult to check that if f satisfies (D.2.8) and it has normal type τ then

$$\text{(D.2.9)} \qquad |f(z)| \le (1+|z|)^N e^{\tau(\mathrm{Im}\, z)_-} \implies I_f = [-i\tau, 0] .$$

Appendix E

SEMICLASSICAL ANALYSIS

E.1 Pseudodifferential operators
E.2 Wavefront sets and ellipticity
E.3 Semiclassical defect measures
E.4 Propagation estimates
E.5 Hyperbolic estimates
E.6 Notes
E.7 Exercises

In this appendix we present results from microlocal and semiclassical analyses. They use the notion of a *semiclassical pseudodifferential operator*. Roughly speaking, these operators have the form

$$\mathrm{Op}_h(a) = a(x, hD_x), \quad D_x := \frac{1}{i}\partial_x$$

where $a(x, \xi)$ is a smooth function called the *symbol* of the operator. The expression $\mathrm{Op}_h(a)$ is called a *quantization* of a. The small parameter $h > 0$ corresponds to the expected wave length of the functions that we study. In applications to high frequency behaviour of resonances, we will often have $h \sim (\mathrm{Re}\,\lambda)^{-1}$ where λ is a resonance.

The emphasis is on aspects of the theory not easily accessible in recent texts. When easy references to [**DS99**] or [**Zw12**] are available we use them instead of referring to this appendix. Its most important application is to the material in Chapter 5.

E.1. PSEUDODIFFERENTIAL OPERATORS

The class of pseudodifferential operators includes all differential operators, however it is considerably more versatile. For instance, if a is nonzero (or, more precisely, *elliptic* – see §E.2.2 below), then we can consider the operator $\mathrm{Op}_h(a^{-1})$ which will be an approximate inverse to $\mathrm{Op}_h(a)$.

One can also quantize symbols which are compactly supported in some set in the (x, ξ) space. By applying the resulting operator to a function, we can *microlocalize* this function to the corresponding set, and define the notion of *wavefront set* – see §E.2. Other advantages of pseudodifferential calculus, such as propagation of singularities, will become apparent later in this appendix.

The price to pay is that semiclassical calculus will always yield errors that are smoothing operators of norm $\mathcal{O}(h^\infty)$. This means that the $h \to 0$ semiclassical calculus is best suited to analysis at high frequencies, while the fixed h calculus only specifies the location of C^∞ singularities.

E.1.1. Differential operators. To motivate the construction that follows, we first introduce the algebra of *semiclassical differential operators* $\mathrm{Diff}_h^k(M)$ of order k on a manifold M. In local coordinates, these have the form

$$A = \sum_{|\alpha| \le k} \sum_{j=0}^{k-|\alpha|} h^j a_{\alpha j}(x)(hD_x)^\alpha : C^\infty(M) \to C^\infty(M) \tag{E.1.1}$$

where α is a multiindex, $a_{\alpha j}$ are smooth functions on M, and $D_x = \frac{1}{i}\partial_x$. In this book, we will often consider the semiclassical Helmholtz operator

$$h^2(-\Delta_g - \lambda^2) \in \mathrm{Diff}_h^2(M) \tag{E.1.2}$$

on a Riemannian manifold (M, g), where h is chosen small enough so that $\omega := h\lambda$ is bounded. The $h \to 0$ limit corresponds to $\lambda \to \infty$, and semiclassical calculus is particularly suited to analysing high energy behaviour of resonances. Another example of a semiclassical differential operator is $hX + V \in \mathrm{Diff}_h^1(M)$ where X is a C^∞ vector field and V is a C^∞ potential.

The restriction $|\alpha| + j \le k$ in (E.1.1) means that terms which are higher order in h also have to be lower order as differential operators. (For instance, $h^3\Delta_g \notin \mathrm{Diff}_h^2(M)$.) This implies that the principal symbol defined in (E.1.3) below is independent of h and determines the operator modulo $h\,\mathrm{Diff}_h^{k-1}(M)$. This also corresponds to the class of symbols of pseudodifferential operators that we introduce in Definition E.3 below.

Note that if A is a (nonsemiclassical) differential operator of order k on M, then $h^k A$ belongs to $\mathrm{Diff}_h^k(M)$. Moreover, if $A \in \mathrm{Diff}_h^k(M)$ and $B \in \mathrm{Diff}_h^\ell(M)$, then the composition AB lies in $\mathrm{Diff}_h^{k+\ell}(M)$.

The class $\mathrm{Diff}^k_h(M)$ is independent of the choice of coordinates on M, but the individual coefficients $a_\alpha(x)$ are not. However, one can invariantly define the (semiclassical) *principal symbol* of $A \in \mathrm{Diff}^k_h(M)$

$$\sigma_h(A)(x,\xi) = \sum_{|\alpha|\le k} a_{\alpha 0}(x)\xi^\alpha \in \mathrm{Poly}^k(T^*M), \tag{E.1.3}$$

where T^*M is the cotangent bundle of M, x is a coordinate system on M, (x,ξ) is the induced coordinate system on T^*M, and $\mathrm{Poly}^k(T^*M)$ stands for the class of smooth functions on T^*M which are polynomials of degree at most k on each cotangent space. Note that the kernel of the map $A \mapsto \sigma_h(A)$ on $\mathrm{Diff}^k_h(M)$ is equal to $h\,\mathrm{Diff}^{k-1}_h(M)$.

To justify the use of the cotangent bundle in the definition of $\sigma_h(A)$ we consider the case when $A = \frac{h}{i}X$, where X is a vector field on M; then

$$\sigma_h(A)(x,\xi) = \langle \xi, X_x\rangle, \quad x \in M,\ \xi \in T^*_xM,$$

where $\langle\cdot,\cdot\rangle$ stands for the natural pairing between covectors and vectors. The coordinate invariance of $\sigma_h(A)$ for a general $A \in \mathrm{Diff}^k_h(M)$ can be proved by writing A as a polynomial in operators of the form $\frac{h}{i}X$ and using the multiplicativity property (E.1.4) below.

As an example, the principal symbol of the operator (E.1.2) is

$$\sigma_h\big(h^2(-\Delta_g - \lambda^2)\big)(x,\xi) = \langle \xi,\xi\rangle_{g_x} - \omega^2, \quad \omega := h\lambda.$$

A direct calculation shows that the symbol map is multiplicative:

$$\sigma_h(AB) = \sigma_h(A)\sigma_h(B), \quad A \in \mathrm{Diff}^k_h(M),\ B \in \mathrm{Diff}^\ell_h(M). \tag{E.1.4}$$

Since multiplication of functions is commutative, this implies that the commutator $[A,B]$ lies in $h\,\mathrm{Diff}^{k+\ell-1}_h(M)$. The principal symbol of this commutator is computed by the formula

$$\sigma_h(h^{-1}[A,B]) = \frac{1}{i}\{\sigma_h(A), \sigma_h(B)\}, \tag{E.1.5}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined in (A.2.1).

E.1.2. Symbols. We start the construction of pseudodifferential calculus by specifying which functions on T^*M can be quantized. In general, one can quantize a function $a(x,\xi;h)$ of $(x,\xi) \in T^*M$ and $h \in (0,h_0)$, where $h_0 > 0$ is a fixed constant, satisfying the following derivative bounds:

$$\sup_{h\in(0,h_0)} \sup_{\substack{x\in K\\ \xi\in T^*_xM}} \langle\xi\rangle^{|\beta|-k} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| < \infty \tag{E.1.6}$$

for some $k \in \mathbb{R}$ (called the order of the symbol), all multiindices α,β, and all compact subsets $K \subset M$. Here $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$ and $|\xi|$ denotes the length of the covector ξ with respect to some Riemannian metric on M (whose choice does not matter in the definition). The left-hand sides of (E.1.6) define a Fréchet space of symbols, which we denote

$$S^k_{1,0}(T^*M).$$

The subscript '1, 0' corresponds to gaining 1 power of ξ when differentiating in ξ and 0 powers of ξ when differentiating in x. The class $S^k_{1,0}(T^*M)$ is independent of the choice of coordinates – see Exercise E.2.

In order to simplify exposition (in particular to avoid using quotient spaces for principal symbols – see the end of §E.1.7), we further restrict ourselves to the class $S^k_h(T^*M)$ of *polyhomogeneous* symbols, which have an asymptotic expansion in powers of h and ξ. These symbols may have complex order, so for notational convenience we define $S^k_{1,0} := S^{\mathrm{Re}\,k}_{1,0}$ for $k \in \mathbb{C}$. The building blocks of polyhomogeneous symbols are given by

DEFINITION E.1. *We say that $a \in C^\infty(T^*M)$ is **positively homogeneous** of order $k \in \mathbb{C}$, if there exists a continuous function $F : M \to (0, \infty)$ such that*

$$a(x, s\xi) = s^k a(x, \xi) \quad \text{for all} \quad s \geq 1,\ |\xi| \geq F(x).$$

If a is positively homogeneous of order k, then $a \in S^k_{1,0}(T^*M)$. To obtain general h-independent polyhomogeneous symbols, we use the following asymptotic expansion as $|\xi| \to \infty$:

DEFINITION E.2. *Let $b(x, \xi) \in S^k_{1,0}(T^*M)$ be h-independent. We write*

$$b(x, \xi) \sim \sum_{\ell=0}^{\infty} b_\ell(x, \xi) \tag{E.1.7}$$

for some h-independent $b_\ell \in S^{k-\ell}_{1,0}(T^*M)$, $\ell = 0, 1, \dots$ if

$$b - \sum_{\ell=0}^{N-1} b_\ell \ \in\ S^{k-N}_{1,0}(T^*M) \quad \text{for all } N \in \mathbb{N}_0.$$

*If (E.1.7) holds for some b_ℓ which are positively homogeneous of order $k - \ell$, then we say that b is a **polyhomogeneous symbol** of order k and denote $b \in S^k(T^*M)$.*

As an example, if k is a nonnegative integer then the class of polynomial symbols $\mathrm{Poly}^k(T^*M)$ lies inside $S^k(T^*M)$, with (E.1.7) having only finitely many nonzero terms. An example of a function in $S^0_{1,0}(T^*M)$ which is not polyhomogeneous is $b(x, \xi) = 1 + \langle\xi\rangle^{-1/2}$.

The class of h-dependent polyhomogeneous symbols is given by

DEFINITION E.3. *Let $a(x,\xi;h) \in S^k_{1,0}(T^*M)$. We write*

$$a(x,\xi;h) \sim \sum_{j=0}^{\infty} h^j a_j(x,\xi;h), \tag{E.1.8}$$

for some $a_j \in S^{k-j}_{1,0}(T^*M)$, $j = 0,1,\dots$ if

$$a - \sum_{j=0}^{N-1} h^j a_j \;\in\; h^N S^{k-N}_{1,0}(T^*M) \quad \text{for all } N \in \mathbb{N}_0.$$

If (E.1.8) holds for some polyhomogeneous $a_j(x,\xi) \in S^{k-j}(T^*M)$, then we say that a is a ***semiclassical polyhomogeneous symbol*** *of order k and denote $a \in S^k_h(T^*M)$.*

As an example, in a fixed coordinate system the full symbol of a differential operator of the form (E.1.1),

$$a(x,\xi;h) = \sum_{|\alpha|\le k} \sum_{j=0}^{k-|\alpha|} h^j a_{\alpha j}(x)\xi^\alpha \tag{E.1.9}$$

lies in $S^k_h(T^*M)$. Here $a_j(x,\xi) := \sum_{|\alpha|\le k-j} a_{\alpha j}(x)\xi^\alpha \in \mathrm{Poly}^{k-j}(T^*M)$ is equal to zero for $j > k$.

For each sequence $a_j \in S^{k-j}_{1,0}(T^*M)$, $j = 0,1,\dots$ there exists $a \in S^k_h(T^*M)$ such that (E.1.8) holds. In fact we can take

$$a(x,\xi;h) := \sum_{j=0}^{\infty} \chi\Big(\frac{\lambda_j h}{\langle\xi\rangle}\Big) h^j a_j(x,\xi;h)$$

where we fix $\chi \in C^\infty_{\mathrm{c}}((-2,2);[0,1])$ such that $\chi \equiv 1$ on $[-1,1]$ and choose an increasing sequence of positive numbers $\{\lambda_j\}_{j\ge 0}$ which converges to ∞ fast enough depending on the symbols a_j. This is a version of Borel's Theorem, see [**Zw12**, Theorem 4.15]. A similar statement is true for (E.1.7), see [**HöIII**, Proposition 18.1.3].

Given the sequence $b_0, b_1, \dots$, the choice of b satisfying the expansion (E.1.7) is unique modulo an element of the *residual class*

$$S^{-\infty}(T^*M) := \bigcap_{k\in\mathbb{R}} S^k_{1,0}(T^*M).$$

Similarly the choice of a satisfying (E.1.8) is unique modulo an element of the class $h^\infty S^{-\infty}(T^*M)$ defined as follows: $a \in h^\infty S^{-\infty}(T^*M)$ if

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h) = \mathcal{O}(h^N \langle\xi\rangle^{-N}) \quad \text{for all} \quad N, \tag{E.1.10}$$

uniformly when x varies in any compact subset of M.

E.1.3. Fiber-radial compactification. To better understand the behaviour of symbols as $\xi \to \infty$, we consider them as functions on the *fiber-radially compactified* cotangent bundle $\overline{T}^*M$. This bundle is a manifold with interior T^*M and boundary diffeomorphic to the sphere bundle

$$\partial\overline{T}^*M \simeq S^*M = (T^*M \setminus 0)\,/\,\mathbb{R}^+,$$

where the group $\mathbb{R}^+$ acts on $T^*M \setminus 0 := \{(x,\xi) \in T^*M : \xi \neq 0\}$ by setting $s.(x,\xi) = (x, s\xi)$, $s \in \mathbb{R}^+$. We call $\partial\overline{T}^*M$ the *fiber infinity*. Denote by

$$\kappa : T^*M \setminus 0 \to \partial\overline{T}^*M \tag{E.1.11}$$

the natural projection map. Then for each $(x,\xi) \in T^*M \setminus 0$ the ray $(x, s\xi)$ converges to $\kappa(x,\xi)$ in $\overline{T}^*M$ as $s \to \infty$.

More precisely, if g is a smooth Riemannian metric on M, then we can model $\overline{T}^*M$ by the coball bundle

$$B^*M = \{(x,\xi) \in T^*M : |\xi|_g \leq 1\}.$$

An embedding $T^*M \to B^*M$ is given by

$$(x,\xi) \mapsto \Big(x, \frac{\xi}{1+\langle\xi\rangle}\Big), \quad \langle\xi\rangle := \sqrt{1+|\xi|_g^2}$$

and the map $\kappa : T^*M \setminus 0 \to \partial B^*M$ is given by

$$(x,\xi) \mapsto \Big(x, \frac{\xi}{|\xi|_g}\Big).$$

The smooth structure of $\overline{T}^*M$ does not depend on the metric g. Moreover, the function $\rho(x,\xi) = \langle\xi\rangle^{-1}$ extends to a *boundary defining function* on $\overline{T}^*M$ in the sense that $\rho = 0$ and $d\rho \neq 0$ on $\partial\overline{T}^*M$, and $\rho > 0$ on T^*M. Note that we cannot use the simpler embedding $(x,\xi) \mapsto (x, \frac{\xi}{\langle\xi\rangle})$ since the resulting boundary defining function behaves like $|\xi|^{-2}$ rather than $|\xi|^{-1}$.

If $x_1, \dots, x_n$ are local coordinates on M and $j \in \{1, \dots, n\}$, then

$$x_1, \dots, x_n, \frac{\xi_1}{\xi_j}, \dots, \frac{\xi_{j-1}}{\xi_j}, \frac{1}{\xi_j}, \frac{\xi_{j+1}}{\xi_j}, \dots, \frac{\xi_n}{\xi_j}$$

give a coordinate system on $\overline{T}^*M$ in the cone $\{|\xi_j| > c|\xi|\}$ for any $c > 0$. Moreover, $\frac{1}{\xi_j}$ is a boundary defining function. It follows that $\partial_{x_1}, \dots, \partial_{x_n}$, $\xi_j\partial_{\xi_1}, \dots, \xi_j\partial_{\xi_n}$ form a frame for smooth vector fields on $\overline{T}^*M$ tangent to the boundary $\partial\overline{T}^*M$. Therefore we have $a \in S^k_{1,0}(T^*M)$ if and only if the function $\langle\xi\rangle^{-k}X_1 \dots X_m a(x,\xi)$ is bounded uniformly for x in compact sets for any choice of vector fields $X_1, \dots, X_m$ on $\overline{T}^*M$ which are tangent to the boundary. This implies a characterization of polyhomogeneous symbols:

PROPOSITION E.4. *Let $a(x,\xi) \in C^\infty(T^*M)$. Then $a \in S^k(T^*M)$ if and only if $\langle\xi\rangle^{-k}a$ extends to a smooth function on $\overline{T}^*M$.*

We finally study Hamiltonian vector fields. Define the *canonical 1-form*

$$\xi \cdot dx := \sum_{j=1}^n \xi_j \, dx_j \ \in \ C^\infty(T^*M; T^*(T^*M)).$$

This form is independent of the choice of coordinates, since for each vector field W on T^*M we have $(\xi \cdot dx)(W) = \langle \xi, d\pi \cdot W\rangle$ where $\pi : T^*M \to M$ is the canonical projection map. Define the *symplectic 2-form* on T^*M by

$$\omega := d(\xi \cdot dx). \tag{E.1.12}$$

For $p(x,\xi) \in C^\infty(T^*M;\mathbb{R})$, define the *Hamiltonian vector field* H_p on T^*M by the formula

$$\omega(W, H_p) = dp(W) \quad \text{for all vector fields } W \text{ on } T^*M. \tag{E.1.13}$$

In local coordinates we have

$$\omega = \sum_{j=1}^n d\xi_j \wedge dx_j, \quad H_p = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

In particular, if $a \in C^\infty(T^*M)$ then $H_p a = \{p, a\}$ where $\{\bullet, \bullet\}$ is the Poisson bracket defined in (A.2.1).

A direct calculation shows that an appropriate rescaling of the Hamiltonian vector field of a polyhomogeneous symbol can be extended to $\overline{T}^*M$:

PROPOSITION E.5. *Let $p \in S^k(T^*M;\mathbb{R})$. Then $\langle\xi\rangle^{1-k}H_p$ extends to a smooth vector field on $\overline{T}^*M$ which is tangent to $\partial\overline{T}^*M$.*

E.1.4. Method of stationary phase. The proofs of properties of pseudodifferential calculus rely on asymptotic expansions as $h \to 0$ of integrals of the form

$$I_{\Phi,a}(h) = \int_M e^{\frac{i}{h}\Phi(x)} a(x)\, dx, \quad h > 0.$$

Here M is an n-dimensional manifold, dx is some smooth density on M, $a \in C_c^\infty(M)$, and $\Phi \in C^\infty(M;\mathbb{R})$ is a Morse function, as defined below:

DEFINITION E.6. *Let $\Phi \in C^\infty(M;\mathbb{R})$. We say that $x \in M$ is a* ***critical point*** *of Φ, if $d\Phi(x) = 0$. For a critical point x, denote by*

$$\nabla^2\Phi(x) \in T_x^*M \otimes T_x^*M$$

the Hessian of Φ at x. We say that Φ is a ***Morse function*** *if $\nabla^2\Phi(x)$ is nondegenerate for each critical point x.*

REMARKS. 1. At a critical point x, the Hessian is well defined by the formula $\nabla^2\Phi(x)(V,W) = VW\Phi(x) = WV\Phi(x)$ for all vector fields V,W on M. Here we use that the commutator $[V,W]$ is a vector field and thus $[V,W]\Phi(x) = 0$, which implies that $VW\Phi(x) = WV\Phi(x)$ depends only on V_x, W_x. This also shows that $\nabla^2\Phi(x)$ is symmetric.

2. The critical points of Morse functions are necessarily isolated. If x is a critical point, we denote by $\operatorname{sgn}\nabla^2\Phi(x)$ the signature of the corresponding Hessian, equal to $\delta_+ - \delta_-$, where $\delta_\pm$ are the maximal dimensions of subspaces on which $\pm\nabla^2\Phi(x)$ is positive definite. Also, let $\det\nabla^2\Phi(x)$ be the determinant of the matrix of $\nabla^2\Phi(x)$ in any basis of T_xM which has unit volume with respect to the density dx.

PROPOSITION E.7 (Method of stationary phase). *Assume that Φ is a Morse function and let $x_1,\dots,x_R$ be the critical points of Φ lying in* $\operatorname{supp} a$. *Then for each $N\in\mathbb{N}_0$, we have*

$$I_{\Phi,a}(h) = \sum_{j=0}^{N-1}\sum_{k=1}^{R} e^{\frac{i}{h}\Phi(x_k)}h^{j+n/2}L_j a(x_k) + \mathcal{O}(h^{N+n/2})\|a\|_{C^{2N+n+1}} \tag{E.1.14}$$

for some differential operators L_j of order $2j$. The operators L_j and the constants in $\mathcal{O}(\cdot)$ depend on Φ, but not on a. Moreover,

$$L_0a(x_k) = (2\pi)^{n/2}\exp\Big(\frac{i\pi}{4}\operatorname{sgn}\nabla^2\Phi(x_k)\Big)|\det\nabla^2\Phi(x_k)|^{-1/2}a(x_k).$$

A proof can be found for instance in [**HöI**, Theorem 7.7.5] or [**Zw12**, Theorem 3.16]. As a special case we have

$$I_{\Phi,a}(h) = \mathcal{O}(h^\infty)\quad\text{if}\quad d\Phi\neq 0\quad\text{on }\operatorname{supp} a. \tag{E.1.15}$$

This fact, sometimes known as the *method of nonstationary phase*, can also be proved directly using repeated integration by parts.

We often apply the method of stationary phase in a situation when Φ, a depend smoothly on some parameter y, in which case (E.1.14) is locally uniform in y – see [**HöI**, Theorem 7.7.6]. One can also differentiate the expansion (E.1.14) any number of times in y.

E.1.5. Quantization on the Euclidean space. We now define pseudodifferential operators on $\mathbb{R}^n$. For that, consider the class of symbols

$$\overline{S}^k_{1,0}(T^*\mathbb{R}^n) \ \subset\ S^k_{1,0}(T^*\mathbb{R}^n) \tag{E.1.16}$$

defined using (E.1.6) but with $K := \mathbb{R}^n$. In other words, the corresponding derivatives are bounded uniformly as $x\to\infty$. We define the classes $\overline{S}^k(T^*\mathbb{R}^n)$ and $\overline{S}^k_h(T^*\mathbb{R}^n)$ similarly to §E.1.2, where in Definition E.1 we require F to be constant and use the Euclidean metric.

REMARK ON NOTATION. Since in scattering theory we consider operators on non-compact manifolds the class $S^k_{1,0}(T^*M)$ cannot be defined using uniform growth estimates unless additional structure is introduced. In the case of $\mathbb{R}^n$ we use affine structure and that leads to the definition of $\overline{S}^k_{1,0}(T^*\mathbb{R}^n)$: the bounds in (E.1.6) are invariant under symplectic lifts of affine transformations of $\mathbb{R}^n$ but not under lifts of arbitrary diffeomorphisms.

For $a \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$, define the operator

$$\operatorname{Op}_h(a) = a(x, hD_x) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n),\ \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \tag{E.1.17}$$

quantizing a by the following formula (known as *standard quantization*):

$$\operatorname{Op}_h(a)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) f(y)\, dy d\xi. \tag{E.1.18}$$

The integral in (E.1.18) does not converge, however it can be calculated by first integrating in y and then using that the Fourier transform acts on $\mathscr{S}(\mathbb{R}^n)$ to integrate in ξ. It can also be defined using the concept of an oscillatory integral, see [**HöI**, §7.8] or [**Zw12**, §3.6].

We remark that [**Zw12**] uses the Weyl quantization which is different from the standard quantization (E.1.17) used here. The two quantizations give the same class of operators and we will use the results of [**Zw12**] here: the proofs can be easily adapted to standard quantization or one can use the change of quantization formula [**Zw12**, Theorems 4.13 and 4.17].

For the mapping property (E.1.17) we refer to [**Zw12**, Theorem 4.16]. In fact, each $\mathscr{S} \to \mathscr{S}$ seminorm of $\operatorname{Op}_h(a)$ and of its adjoint is bounded polynomially in h. More precisely, for each N_1 there exists N_2 (depending only on N_1, n, k) and C (depending also on a) such that for all $u \in \mathscr{S}(\mathbb{R}^n)$

$$\sum_{|\alpha|+|\beta|\le N_1} \|x^\alpha \partial_x^\beta \operatorname{Op}_h(a)u\|_{L^\infty} \le Ch^{-N_2} \sum_{|\alpha|+|\beta|\le N_2} \|x^\alpha \partial_x^\beta u\|_{L^\infty}. \tag{E.1.19}$$

It also follows from (E.1.18) that if $a \in h^\infty \overline{S}^{-\infty}(T^*\mathbb{R}^n)$ (defined by requiring (E.1.10) uniformly in x) then $\operatorname{Op}_h(a)$ has Schwartz kernel in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and each C^∞ seminorm of this kernel is $\mathcal{O}(h^\infty)$.

By the Fourier inversion formula, differential operators are quantizations of polynomials in ξ:

$$\operatorname{Op}_h\Big(\sum_\alpha a_\alpha(x)\xi^\alpha\Big) = \sum_\alpha a_\alpha(x)(hD_x)^\alpha. \tag{E.1.20}$$

In particular, $\operatorname{Op}_h(1)$ is the identity operator.

The algebraic properties of (E.1.18) are given by

PROPOSITION E.8. *Let $a \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$, $b \in \overline{S}^\ell_{1,0}(T^*\mathbb{R}^n)$. Then:*

1. *We have* $\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b)$*, where* $a\#b \in \overline{S}^{k+\ell}_{1,0}(T^*\mathbb{R}^n)$ *and*

$$a\#b(x,\xi;h) \sim \sum_{j=0}^{\infty} \frac{(-ih)^j}{j!} \langle \partial_\xi, \partial_y\rangle^j \big(a(x,\xi;h)b(y,\eta;h)\big)\Big|_{\substack{y=x\\ \eta=\xi}}. \tag{E.1.21}$$

2. *We have* $\mathrm{Op}_h(a)^* = \mathrm{Op}_h(a^*)$*, where* $a^* \in \overline{S}^{k}_{1,0}(T^*\mathbb{R}^n)$ *and*

$$a^*(x,\xi;h) \sim \sum_{j=0}^{\infty} \frac{(-ih)^j}{j!} \langle \partial_\xi, \partial_x\rangle^j\, \overline{a(x,\xi;h)}. \tag{E.1.22}$$

The asymptotic expansions are understood in the sense of (E.1.8) *in the classes* $\overline{S}_{1,0}$.

For the proofs the reader is referred to [**Zw12**, Theorems 9.5 and 4.14].

We note the following corollaries of the expansions (E.1.21) and (E.1.22), the first two of which generalize (E.1.4) and (E.1.5):

$$a\#b = ab + \mathcal{O}(h)_{\overline{S}^{k+\ell-1}_{1,0}(T^*\mathbb{R}^n)}, \tag{E.1.23}$$

$$a\#b - b\#a = \frac{h}{i}\{a,b\} + \mathcal{O}(h^2)_{\overline{S}^{k+\ell-2}_{1,0}(T^*\mathbb{R}^n)}, \tag{E.1.24}$$

$$a^* = \bar{a} + \mathcal{O}(h)_{\overline{S}^{k-1}_{1,0}(T^*\mathbb{R}^n)}, \tag{E.1.25}$$

$$a\#b = \mathcal{O}(h^\infty)_{\overline{S}^{-\infty}(T^*\mathbb{R}^n)}, \quad \text{if } \operatorname{supp} a \cap \operatorname{supp} b = \emptyset. \tag{E.1.26}$$

We also remark that the class of semiclassical polyhomogeneous symbols (see Definition E.3) is preserved under the operations in Proposition E.8. That is, if $a \in \overline{S}^k_h(T^*\mathbb{R}^n)$, $b \in \overline{S}^\ell_h(T^*\mathbb{R}^n)$, then $a\#b \in \overline{S}^{k+\ell}_h(T^*\mathbb{R}^n)$ and $a^* \in \overline{S}^k_h(T^*\mathbb{R}^n)$.

E.1.6. Change of variables for pseudodifferential operators.

E.1.6. Change of variables for pseudodifferential operators. The independence of the class of pseudodifferential operators on a manifold on the choice of local charts follows from a change of variables statement for quantization on $\mathbb{R}^n$, see Proposition E.10 below. To state it we make the following definition which will also be useful in §E.1.7 below.

DEFINITION E.9. *Let M be a manifold. A **cutoff chart** on M is a pair (φ,χ) where $\varphi : U \to V$ is a C^∞ diffeomorphism, $U \subset M$ and $V \subset \mathbb{R}^n$ are open sets, and $\chi \in C^\infty_{\mathrm{c}}(U)$. We define the **lifted diffeomorphism***

$$\widetilde{\varphi} : T^*U \to T^*V, \quad (x,\xi) \mapsto \big(\varphi(x), (d\varphi(x))^{-T}\xi\big). \tag{E.1.27}$$

Here $(d\varphi(x))^{-T}$ denotes the inverse of the adjoint of the map $d\varphi(x)$.

REMARKS. 1. The operators $\chi\varphi^* : u \mapsto \chi(u \circ \varphi)$ and $(\varphi^{-1})^*\chi : u \mapsto (\chi u) \circ \varphi^{-1}$ map

$$\chi\varphi^* : C^\infty(V) \to C^\infty_{\mathrm{c}}(U), \quad (\varphi^{-1})^*\chi : C^\infty(U) \to C^\infty_{\mathrm{c}}(V)$$

and thus naturally extend to operators

$$\chi\varphi^* : C^\infty(\mathbb{R}^n) \to C^\infty_{\mathrm{c}}(M), \quad (\varphi^{-1})^*\chi : C^\infty(M) \to C^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{E.1.28}$$

2. The lifted diffeomorphism in (E.1.27) $\widetilde{\varphi}$ is a *symplectomorphism* in the sense that $\widetilde{\varphi}^*\omega_V = \omega_U$ where ω_U, ω_V are the symplectic forms on T^*U and T^*V respectively (see (E.1.12)).

PROPOSITION E.10 (Change of variables). *Let (φ, χ) be a cutoff chart on $\mathbb{R}^n$. Then for each $a \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$, there exists $b \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$ such that*

$$\chi\varphi^* \operatorname{Op}_h(a)(\varphi^{-1})^*\chi = \operatorname{Op}_h(b). \tag{E.1.29}$$

Moreover we have the following asymptotic expansion in the sense of (E.1.8) *in the classes $\overline{S}_{1,0}$:*

$$b \sim \sum_{j=0}^{\infty} h^j L_j(a \circ \widetilde{\varphi}) \tag{E.1.30}$$

where L_j are h-independent differential operators of order $2j$ on T^*U with coefficients compactly supported in x, mapping $S^k(T^*U) \to S^{k-j}(T^*U)$, and $L_0 = \chi(x)^2$.

REMARK. If a is a semiclassical polyhomogeneous symbol in $\overline{S}^k_h(T^*\mathbb{R}^n)$, then by (E.1.30) b lies in $\overline{S}^k_h(T^*\mathbb{R}^n)$ as well.

Proof. We argue similarly to [**Zw12**, Theorem 9.9], with the difference being that [**Zw12**] avoided the use of the cutoff χ by requiring that φ be a global diffeomorphism on $\mathbb{R}^n$.

We recover the symbol b by oscillatory testing [**Zw12**, Theorem 4.19]: we have (E.1.29) where

$$\begin{aligned}
b(x,\xi) &= e^{-\frac{i}{h}\langle x,\xi\rangle}\chi\varphi^* \operatorname{Op}_h(a)(\varphi^{-1})^*\chi(e^{\frac{i}{h}\langle \bullet,\xi\rangle}) \\
&= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle\varphi(x)-y',\eta\rangle + \langle\varphi^{-1}(y')-x,\xi\rangle)}\chi(x)\chi(\varphi^{-1}(y'))a(\varphi(x),\eta)\,dy'd\eta \\
&= (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\langle\varphi(x)-\varphi(y),\eta\rangle + \langle y-x,\xi\rangle)}\chi(x)\chi(y)a(\varphi(x),\eta)J(y)\,dyd\eta
\end{aligned}$$

where we make the change of variables $y' = \varphi(y)$ and $J(y) = |\det d\varphi(y)|$ is the Jacobian. The integrals above converge absolutely when $a \in \mathscr{S}(T^*\mathbb{R}^n)$. For general $a \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$ we can define them as oscillatory integrals, using integration by parts in y' or y – see [HöI, §7.8] or [Zw12, §3.6]. Equivalently we can integrate in y' or y first and then use the method of nonstationary phase (E.1.15) to show that the resulting integral decays like $\mathcal{O}(\langle\eta\rangle^{-\infty})$ for each fixed x,ξ,h.

It remains to show that $b\in\overline{S}^k_{1,0}(T^*\mathbb{R}^n)$ and prove the expansion (E.1.30). Note that due to the factors $\chi(x)\chi(y)$ we may restrict to the case $x,y\in U$.

To handle the case of large values of ξ, we put $\xi=r\xi'$ where $r=\langle\xi\rangle\geq 1$, so that $|\xi'|\leq 1$. Making the change of variables $\eta=r\eta'$, we get

$$b(x,r\xi')=(2\pi h')^{-n}\int_{\mathbb{R}^{2n}}e^{i\Phi/h'}\chi(x)\chi(y)a(\varphi(x),r\eta')J(y)\,dyd\eta',$$
$$\Phi=\langle\varphi(x)-\varphi(y),\eta'\rangle+\langle y-x,\xi'\rangle,\quad h':=h/r.$$

To obtain the expansion (E.1.30), we use the method of stationary phase (Proposition E.7). The critical points of the phase are given by the equations

$$\varphi(x)=\varphi(y),\quad \xi'=(d\varphi(y))^T\eta'.$$

Therefore for each $x\in U,\xi'\in\mathbb{R}^n$ there exists unique critical point given by

$$y=x,\quad \eta'=(d\varphi(x))^{-T}\xi'.$$

At the critical point we have $\Phi=0$ and

$$\nabla^2\Phi=-\langle\nabla^2\varphi(y),\eta'\rangle-\langle d\varphi(y),d\eta'\rangle.$$

Therefore, Φ is a Morse function and $\operatorname{sgn}\nabla^2\Phi=0$, $|\det\nabla^2\Phi|=J(y)^2$ at the critical point.

The amplitude $\chi(x)\chi(y)a(\varphi(x),r\eta')J(y)$ is compactly supported in $y\in U$, but not necessarily in η'. We thus write it as a sum a_1+a_2, where

$$a_1\in C^\infty_{\mathrm{c}}(U\times\mathbb{R}^n),\quad \operatorname{supp}a_2\cap\{\xi'=(d\varphi(y))^T\eta'\}=\emptyset.$$

We expand the integral featuring a_1 by Proposition E.7, obtaining (E.1.30); note that since the asymptotic parameter is $h'=h/r$, each next term in the expansion gains one power of h and $\langle\xi\rangle^{-1}$. As for the integral featuring a_2, it is $\mathcal{O}((h')^\infty)=\mathcal{O}(h^\infty\langle\xi\rangle^{-\infty})$ by repeated integration by parts in y, as $|\partial_y\Phi|^{-1}\leq C\langle\eta'\rangle^{-1}$ on $\operatorname{supp}a_2$. □

E.1.7. Quantization on general manifolds. We are now ready to define pseudodifferential operators on a manifold M, by piecing together pullbacks of pseudodifferential operators on $\mathbb{R}^n$ by cutoff charts. We use the following general residual class:

DEFINITION E.11. *Let $A=A(h):C^\infty_{\mathrm{c}}(M_2)\to\mathcal{D}'(M_1)$ be an operator. We write that $A\in h^\infty\Psi^{-\infty}$, or $A=\mathcal{O}(h^\infty)_{\Psi^{-\infty}}$, if A is smoothing and each $C^\infty(M_1\times M_2)$ seminorm of the Schwartz kernel of A is $\mathcal{O}(h^\infty)$.*

We also use the notion of properly supported, compactly supported, and regular operators, see §A.7. For h-dependent families of properly supported or compactly supported operators, we require that the support property hold uniformly in h.

DEFINITION E.12. *Let M be a manifold and $k \in \mathbb{C}$. Define the class of **semiclassical pseudodifferential operators** $\Psi_h^k(M)$ as follows: a family of operators $A = A(h) : C_c^\infty(M) \to C^\infty(M)$ lies in $\Psi_h^k(M)$ if and only if it can be written as*

$$A = \sum_j \chi_j \varphi_j^* \operatorname{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}, \tag{E.1.31}$$

for some cutoff charts (φ_j, χ_j) and symbols $a_j \in \overline{S}_h^k(T^*\mathbb{R}^n)$, where the domains of φ_j form a locally finite collection (see §A.7).

Note that all pseudodifferential operators are necessarily regular, and each $C_c^\infty \to C^\infty$ seminorm of A and A^* is bounded polynomially in h similarly to (E.1.19). Moreover

$$\bigcap_{j \in \mathbb{N}_0} h^j \Psi_h^{k-j}(M) = \bigcap_{N \in \mathbb{R}} h^N \Psi_h^{-N}(M) = h^\infty \Psi^{-\infty}.$$

Alternatively we can define pseudodifferential operators by requiring that the Schwartz kernel of A be $\mathcal{O}(h^\infty)_{C^\infty}$ away from the diagonal and the localization of A to each cutoff chart be a pseudodifferential operator on $\mathbb{R}^n$:

PROPOSITION E.13. *An operator $A : C_c^\infty(M) \to \mathcal{D}'(M)$ lies in $\Psi_h^k(M)$ if and only if both of the following conditions hold:*

1. For each $\psi, \psi' \in C^\infty(M)$ such that $\operatorname{supp}\psi \cap \operatorname{supp}\psi' = \emptyset$, we have $\psi A \psi' \in h^\infty \Psi^{-\infty}$.

2. For each cutoff chart (φ, χ), there exists $a_{\varphi,\chi} \in \overline{S}_h^k(T^*\mathbb{R}^n)$ such that

$$(\varphi^{-1})^* \chi A \chi \varphi^* = \operatorname{Op}_h(a_{\varphi,\chi}). \tag{E.1.32}$$

Proof. 1. Assume first that $A \in \Psi_h^k(M)$. To verify property 1, it suffices to show that $\widetilde{\chi}\psi A \psi' \widetilde{\chi} \in h^\infty \Psi^{-\infty}$ for all $\widetilde{\chi} \in C_c^\infty(M)$, thus we may assume that $\psi, \psi' \in C_c^\infty(M)$. We then write by (E.1.31)

$$\begin{aligned} \psi A \psi' &= \sum_j \psi \chi_j \varphi_j^* \operatorname{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j \psi' + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \\ &= \sum_j \varphi_j^* \big((\psi\chi_j) \circ \varphi_j^{-1}\big) \operatorname{Op}_h(a_j) \big((\psi'\chi_j) \circ \varphi_j^{-1}\big)(\varphi_j^{-1})^* + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \end{aligned}$$

and the sum above has finitely many nonzero terms due to the local finiteness condition. The supports of the functions $(\psi\chi_j) \circ \varphi_j^{-1}, (\psi'\chi_j) \circ \varphi_j^{-1} \in C_c^\infty(\mathbb{R}^n)$ do not intersect, therefore by (E.1.26) we have $\psi A \psi' \in h^\infty \Psi^{-\infty}$.

To verify property 2, we write by (E.1.31)

$$(\varphi^{-1})^*\chi A\chi\varphi^* = \sum_j \chi_j'(\varphi_j')^* \operatorname{Op}_h(a_j)((\varphi_j')^{-1})^*\chi_j' + \mathcal{O}(h^\infty)_{\mathscr{S}'\to\mathscr{S}} \tag{E.1.33}$$

where (φ_j', χ_j') are the following cutoff charts on $\mathbb{R}^n$:

$$\varphi_j' = \varphi_j \circ \varphi^{-1}, \quad \chi_j' = (\chi\chi_j)\circ\varphi^{-1}.$$

By Proposition E.10, and since operators in $\mathcal{O}(h^\infty)_{\mathscr{S}'\to\mathscr{S}}$ are pseudodifferential with symbols in the class $h^\infty\mathscr{S}(\mathbb{R}^{2n})$, we have (E.1.32) with some $a_{\varphi,\chi} \in \overline{S}_h^k(T^*\mathbb{R}^n)$.

2. Now, assume that A satisfies properties 1 and 2 in the statement of this proposition. We will write it in the form (E.1.31). Take a collection of cutoff charts (φ_j, χ_j) on M such that the domains U_j of φ_j form a locally finite covering of M and $\sum_j \chi_j = 1$. Take also $\chi_j', \chi_j'' \in C_c^\infty(U_j)$ such that $\chi_j' = 1$ near $\operatorname{supp}\chi_j$ and $\chi_j'' = 1$ near $\operatorname{supp}\chi_j'$. We write

$$A = \sum_j \chi_j A = \sum_j \chi_j A\chi_j' + \sum_j \chi_j A(1-\chi_j'). \tag{E.1.34}$$

By property 1, we see that the second term on the right-hand side is in $h^\infty\Psi^{-\infty}$. As for the first term, we write it as

$$\begin{gathered}\sum_j \chi_j A\chi_j' = \sum_j \chi_j'\varphi_j^* A_j(\varphi_j^{-1})^*\chi_j',\\ A_j := (\chi_j\circ\varphi_j^{-1})(\varphi_j^{-1})^*\chi_j'' A\chi_j''\varphi_j^*.\end{gathered} \tag{E.1.35}$$

This has the form (E.1.31) as (φ_j, χ_j'') are cutoff charts and thus $A_j = \operatorname{Op}_h(a_j)$ for some $a_j \in \overline{S}_h^k(T^*\mathbb{R}^n)$ by property 2. □

Proposition E.13, together with (E.1.9) and (E.1.20), implies that the class of semiclassical differential operators $\operatorname{Diff}_h^k(M)$ defined in §E.1.1 is contained in $\Psi_h^k(M)$.

We now define the principal symbol of a pseudodifferential operator on a manifold, extending the definition (E.1.3) given for differential operators. For $a \in \overline{S}_h^k(T^*\mathbb{R}^n)$, we say that $a^0 \in \overline{S}^k(T^*\mathbb{R}^n)$ is the *principal part* of a if it is the leading term in the expansion (E.1.8) of a.

PROPOSITION E.14. *Let $A \in \Psi_h^k(M)$. Then there exists a unique*

$$\sigma_h(A) \in S^k(T^*M),$$

*called the **principal symbol** of A, with the following properties:*

1. For each representation (E.1.31) of A, we have

$$\sigma_h(A) = \textstyle\sum_j \chi_j(x)^2(a_j^0 \circ \widetilde{\varphi}_j) \tag{E.1.36}$$

with $\widetilde{\varphi}_j$ *defined in* (E.1.27) *and* $a_j^0 \in \overline{S}^k(T^*\mathbb{R}^n)$ *the principal part of* a_j.

2. *For each cutoff chart* (φ, χ) *and* $a_{\varphi,\chi}$ *defined in* (E.1.32),

$$\chi(x)^2\sigma_h(A) = a^0_{\varphi,\chi} \circ \widetilde{\varphi} \quad \text{on } T^*U \tag{E.1.37}$$

where $a^0_{\varphi,\chi} \in \overline{S}^k(T^*\mathbb{R}^n)$ *is the principal part of* $a_{\varphi,\chi}$, *supported inside* T^*U.

Proof. Take a representation (E.1.31) of A and define $\sigma_h(A)$ by (E.1.36). It follows from (E.1.33) and Proposition E.10 that (E.1.37) holds for each cutoff chart (φ, χ) and thus $\sigma_h(A)$ is independent of the choice of the representation (E.1.31). □

We also define a (non-canonical) quantization procedure:

PROPOSITION E.15. *Let* (φ_j, χ_j) *be cutoff charts and* χ'_j *functions satisfying the conditions of the paragraph preceding* (E.1.34), *and* $\widetilde{\varphi}_j$ *be defined by* (E.1.27). *For* $a \in S^k_h(T^*M)$, *consider the operator*

$$\operatorname{Op}_h^M(a) := \sum_j \chi'_j \varphi_j^* \operatorname{Op}_h\big((\chi_j a) \circ \widetilde{\varphi}_j^{-1}\big)(\varphi_j^{-1})^* \chi'_j. \tag{E.1.38}$$

Then $A \in \Psi^k_h(M)$ *is properly supported and*

$$\sigma_h(\operatorname{Op}_h^M(a)) = a^0 \tag{E.1.39}$$

where $a^0 \in S^k(T^*M)$ *is the principal part of* a. *If the projection of* $\operatorname{supp} a$ *onto* M *is compact, then* $\operatorname{Op}_h^M(a)$ *is compactly supported. Moreover, if* $a = a(x;h)$ *is independent of* ξ, *then* $\operatorname{Op}_h^M(a)u = au$ *for all* u. *In particular,* $\operatorname{Op}_h^M(1)$ *is the identity operator.*

Proof. The fact that $A \in \Psi^k_h(M)$ follows immediately from Definition E.12, and (E.1.39) follows from (E.1.36). The support properties of $\operatorname{Op}_h^M(a)$ follow from the fact that $\operatorname{supp}\chi_j \subset U_j$ and U_j form a locally finite collection. The formula for $\operatorname{Op}_h^M(a)$ when $a = a(x;h)$ follows directly from (E.1.20). □

When there is no risk of confusing the quantization map Op_h^M defined above with the map Op_h defined in (E.1.17), we denote Op_h^M by Op_h.

PROPOSITION E.16 (Basic properties of σ_h, Op_h^M). *The map*

$$A \in \Psi^k_h(M) \;\mapsto\; \sigma_h(A) \in S^k(T^*M) \tag{E.1.40}$$

is onto and its kernel is given by $h\Psi_h^{k-1}(M)$. *Moreover, for each* $A \in \Psi^k_h(M)$ *there exists* $a \in S^k_h(T^*M)$ *such that*

$$A = \operatorname{Op}_h^M(a) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}. \tag{E.1.41}$$

Proof. The fact that (E.1.40) is onto follows from (E.1.39) and the fact that the kernel of (E.1.40) contains $h\Psi_h^{k-1}(M)$ follows from (E.1.36). Now, assume that $A \in \Psi_h^k(M)$ and $\sigma_h(A) = 0$. By (E.1.37) the operator $h^{-1}A$ satisfies conditions 1 and 2 of Proposition E.13 with k replaced by $k-1$, therefore $h^{-1}A \in \Psi_h^{k-1}(M)$.

Finally, let $A \in \Psi_h^k(M)$; we construct $a \in S_h^k(M)$ such that (E.1.41) holds. Let $a_0 := \sigma_h(A)$. Then by (E.1.39) and the first part of the current proposition, we have $A - \operatorname{Op}_h^M(a_0) \in h\Psi_h^{k-1}(M)$. Repeating this process, we construct symbols $a_j \in S^{k-j}(T^*M)$ by the formula

$$a_j = \sigma_h\Big(h^{-j}\Big(A - \sum_{\ell=0}^{j-1} h^\ell \operatorname{Op}_h^M(a_\ell)\Big)\Big). \tag{E.1.42}$$

It remains to take a in the form (E.1.8): $a \sim \sum_{j=0}^\infty h^j a_j$. □

We now prove basic algebraic properties of the class $\Psi_h^k(M)$:

PROPOSITION E.17. *1. For all properly supported $A \in \Psi_h^k(M)$, $B \in \Psi_h^\ell(M)$, we have $AB \in \Psi_h^{k+\ell}(M)$ and*

$$\sigma_h(AB) = \sigma_h(A)\sigma_h(B), \tag{E.1.43}$$

$$\sigma_h(h^{-1}[A,B]) = \frac{1}{i}\{\sigma_h(A), \sigma_h(B)\}. \tag{E.1.44}$$

2. For $A \in \Psi_h^k(M)$, we have $A^* \in \Psi_h^k(M)$ (where the adjoint is taken with respect to any fixed smooth density on M) and

$$\sigma_h(A^*) = \overline{\sigma_h(A)}. \tag{E.1.45}$$

Proof. 1. We first note that if either A or B lies in $h^\infty\Psi^{-\infty}$ then AB lies in $h^\infty\Psi^{-\infty}$ as well, as follows from the fact that A, B, A^*, B^* are bounded $C_c^\infty(M) \to C_c^\infty(M)$ and $C^\infty(M) \to C^\infty(M)$ polynomially in h.

Now, to show that $AB \in \Psi_h^{k+\ell}(M)$ in general we use Proposition E.13. First of all, assume that $\psi, \psi' \in C^\infty(M)$ and $\operatorname{supp}\psi \cap \operatorname{supp}\psi' = \emptyset$; we will show that $\psi AB\psi' = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. As in the beginning of the proof of Proposition E.13 we may assume that ψ, ψ' are compactly supported. Take $\psi'' \in C_c^\infty(M)$ with $\operatorname{supp}\psi \cap \operatorname{supp}(1-\psi'') = \operatorname{supp}\psi'' \cap \operatorname{supp}\psi' = \emptyset$ and write

$$\psi AB\psi' = \psi A(1-\psi'')B\psi' + \psi A\psi'' B\psi'.$$

Since $\psi A(1-\psi''), \psi'' B\psi' = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$, we get $\psi AB\psi' = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$.

Next, let (φ, χ) be a cutoff chart on M and $U \subset M$ be the domain of φ. Choose $\chi' \in C_c^\infty(U)$ such that $\chi' = 1$ near $\operatorname{supp}\chi$, then by property 1 in Proposition E.13 we have $\chi AB\chi = \chi A(\chi')^2 B\chi + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. We write

$$(\varphi^{-1})^*\chi AB\chi\varphi^* = (\chi\circ\varphi^{-1})\widetilde{A}\widetilde{B}(\chi\circ\varphi^{-1}) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}},$$
$$\widetilde{A} := (\varphi^{-1})^*\chi' A\chi'\varphi^*, \quad \widetilde{B} := (\varphi^{-1})^*\chi' B\chi'\varphi^*.$$

We have $\widetilde{A} = \mathrm{Op}_h(a)$, $\widetilde{B} = \mathrm{Op}_h(b)$ for some $a \in \overline{S}^k_h(T^*\mathbb{R}^n)$, $b \in \overline{S}^\ell_h(T^*\mathbb{R}^n)$, therefore by Proposition E.8 we get $(\varphi^{-1})^*\chi AB\chi\varphi^* = \mathrm{Op}_h(c)$ for some $c \in \overline{S}^{k+\ell}_h(T^*\mathbb{R}^n)$. This shows that $AB \in \Psi^{k+\ell}_h(M)$.

Next, by (E.1.23) and (E.1.37) we have $c \circ \widetilde{\varphi} = \chi(x)^2\sigma_h(A)\sigma_h(B) + \mathcal{O}(h)_{S^{k+\ell-1}_h(T^*U)}$. This shows (E.1.43). Moreover,

$$(\varphi^{-1})^*\chi[A,B]\chi\varphi^* = (\chi\circ\varphi^{-1})[\widetilde{A},\widetilde{B}](\chi\circ\varphi^{-1}) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$$

which together with (E.1.24) gives (E.1.44).

2. To show that $A^* \in \Psi^k_h(M)$, we again use Proposition E.13. First of all, if $\psi,\psi' \in C^\infty(M)$ and $\operatorname{supp}\psi \cap \operatorname{supp}\psi' = \emptyset$, then $\psi A^*\psi' = (\overline{\psi'}A\overline{\psi})^* = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. Next, let (φ,χ) be a cutoff chart, $\varphi : U \to V$, then

$$(\varphi^{-1})^*\overline{\chi}A\overline{\chi}\varphi^* = \mathrm{Op}_h(a) \quad \text{for some} \quad a \in \overline{S}^k_h(T^*\mathbb{R}^n). \tag{E.1.46}$$

Let $J \in C^\infty(V)$ be the Jacobian of φ^{-1} with respect to the density fixed on M and the standard density on $\mathbb{R}^n$. By the change of variables formula we see that the adjoints of the operators (E.1.28) are given by

$$(\overline{\chi}\varphi^*)^* = J(\varphi^{-1})^*\chi, \quad ((\varphi^{-1})^*\overline{\chi})^* = \chi\varphi^* J^{-1}.$$

Fix $\widetilde{\chi} \in C^\infty_c(V)$ with $\widetilde{\chi} = 1$ on $\varphi(\operatorname{supp}\chi)$. The adjoint of (E.1.46) implies

$$(\varphi^{-1})^*\chi A^*\chi\varphi^* = \widetilde{\chi}J^{-1}\,\mathrm{Op}_h(a)^* J\widetilde{\chi}. \tag{E.1.47}$$

By Proposition E.8 the right-hand side of (E.1.47) is equal to $\mathrm{Op}_h(a^\star)$ for some $a^\star \in \overline{S}^k_h(T^*\mathbb{R}^n)$, showing that $A^* \in \Psi^k_h(M)$. Moreover by (E.1.23), (E.1.25), and (E.1.37) we have $a^\star \circ \widetilde{\varphi} = \chi(x)^2\overline{\sigma_h(A)} + \mathcal{O}(h)_{S^{k-1}_h(T^*U)}$, which shows (E.1.45). □

REMARKS ON GENERALIZATIONS. 1. Instead of the semiclassical polyhomogeneous class $S^k_h(T^*M)$ one can consider pseudodifferential operators with symbols in the larger class $S^k_{1,0}(T^*M)$. In fact, we can use an even larger class $S^k_\delta(T^*M)$, $\delta \in [0,\frac{1}{2})$ consisting of symbols satisfying the derivative bounds

$$\sup_{h\in(0,h_0)} h^{\delta(|\alpha|+|\beta|)} \sup_{\substack{x\in K\\ \xi\in T^*_xM}} \langle\xi\rangle^{|\beta|-k}|\partial^\alpha_x\partial^\beta_\xi a(x,\xi;h)| < \infty \tag{E.1.48}$$

That is, the derivative $\partial^\alpha_x\partial^\beta_\xi a$ is allowed to grow like $h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{k-|\beta|}$. See [**Zw12**, §4.4.1]. We denote the corresponding pseudodifferential class by $\Psi^k_{\delta,h}(M)$. The principal symbol now lies in a quotient space:

$$A \in \Psi^k_{\delta,h}(M) \quad \longmapsto \quad \sigma_h(A) \in S^k_\delta(T^*M)/h^{1-2\delta}S^{k-1}_\delta(T^*M)$$

and the kernel of the map σ_h is equal to $h^{1-2\delta}\Psi^{k-1}_{\delta,h}(M)$. The product and adjoint formulas (E.1.43) and (E.1.45) are still valid and the commutator formula (E.1.44) takes the form $\sigma_h(h^{2\delta-1}[A,B]) = \frac{h^{2\delta}}{i}\{\sigma_h(A),\sigma_h(B)\}$ where $\{\sigma_h(A),\sigma_h(B)\} \in h^{-2\delta}S^{k+\ell-1}_\delta(T^*M)/h^{1-4\delta}S^{k+\ell-2}_\delta(T^*M)$.

2. An invariant definition of the principal symbol modulo $h^2 S^{k-2}_h(T^*M)$ can be obtained by using Weyl quantization for operators acting on half-densities, see [**Zw12**, Theorem 14.3].

3. If $A = A(h) \in \Psi^k_h(M)$ then for $h := 1$, $A(h)$ is a nonsemiclassical pseudodifferential operator with principal symbol $\sigma_h(A)$, see [**HöIII**, §18.1]. Note that even for fixed h the properties of pseudodifferential calculus are nontrivial since the asymptotic expansions give an improvement in the differential order k in addition to h.

E.1.8. Sobolev spaces. We now introduce Hilbert spaces on which semiclassical pseudodifferential operators act naturally. We start with the case of $\mathbb{R}^n$:

DEFINITION E.18. *For $s \in \mathbb{R}$, the semiclassical Sobolev space*

$$H^s_h(\mathbb{R}^n), \quad \mathscr{S}(\mathbb{R}^n) \subset H^s_h(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n),$$

is defined as the Sobolev space $H^s(\mathbb{R}^n)$ with the h-dependent norm

$$\|u\|_{H^s_h(\mathbb{R}^n)} := \|\langle h\xi\rangle^s \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}. \tag{E.1.49}$$

For notational convenience, we put $H^s_h(\mathbb{R}^n) := H^{\operatorname{Re} s}_h(\mathbb{R}^n)$ for $s \in \mathbb{C}$.

We have $\|u\|_{H^s_h} = \|\langle hD_x\rangle^s u\|_{L^2}$ where $\langle hD_x\rangle^s = \operatorname{Op}_h(\langle\xi\rangle^s)$ is defined by (E.1.18). Thus for any operator $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$

$$\|A\|_{H^s_h(\mathbb{R}^n)\to H^t_h(\mathbb{R}^n)} = \|\langle hD_x\rangle^t A \langle hD_x\rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}. \tag{E.1.50}$$

Therefore pseudodifferential operators act on semiclassical Sobolev spaces:

PROPOSITION E.19 (**Boundedness of pseudodifferential operators on $\mathbb{R}^n$**)**.** *Let $a \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$. Then for each s the operator*

$$\operatorname{Op}_h(a) : H^s_h(\mathbb{R}^n) \to H^{s-k}_h(\mathbb{R}^n)$$

is bounded uniformly in h, and its operator norm is bounded above by some fixed $\overline{S}^k_{1,0}$ seminorm of a.

Proof. By (E.1.50) it suffices to show boundedness of $\langle hD_x\rangle^{s-k}\operatorname{Op}_h(a)\langle hD_x\rangle^{-s}$ on $L^2(\mathbb{R}^n)$. From the pseudodifferential calculus in Proposition E.8 we see that the latter operator is given by $\operatorname{Op}_h(b)$ where $b\in\overline{S}^0_{1,0}(T^*\mathbb{R}^n)$. The operator $\operatorname{Op}_h(b)$ is uniformly bounded on $L^2(\mathbb{R}^n)$ – see for instance [**Zw12**, Theorem 4.23] and for Hörmander's simple proof for the class $\overline{S}^0_{1,0}$, Exercise E.6. ☐

For each cutoff chart (φ,χ) on $\mathbb{R}^n$ and all s we have $\|\chi\varphi^*\|_{H^s_h\to H^s_h}\leq C$, where $\chi\varphi^*: C^\infty(\mathbb{R}^n)\to C^\infty_{\rm c}(\mathbb{R}^n)$ is defined by (E.1.28). Indeed, by (E.1.50) it suffices to show L^2 boundedness of the operator

$$
\begin{gathered}
\langle hD_x\rangle^s\chi\varphi^*\langle hD_x\rangle^{-s}=B_1+B_2,\\
B_1:=\langle hD_x\rangle^s\chi(\chi'\varphi^*\langle hD_x\rangle^{-s}(\varphi^{-1})^*\chi')\chi'\varphi^*,\\
B_2:=\langle hD_x\rangle^s\chi'\varphi^*(\chi\circ\varphi^{-1})\langle hD_x\rangle^{-s}\big(1-(\chi'\circ\varphi^{-1})^2\big)
\end{gathered}
$$

where $\varphi:U\to V$ and we fix $\chi'\in C^\infty_{\rm c}(U)$ such that $\chi'=1$ near $\operatorname{supp}\chi$. By Propositions E.8 and E.10 we have $B_1=\operatorname{Op}_h(b_1)\chi'\varphi^*$ for some $b_1\in\overline{S}^0_{1,0}(T^*\mathbb{R}^n)$, thus by Proposition E.19 we get $\|B_1\|_{L^2\to L^2}\leq C$. On the other hand by (E.1.26) we have $B_2=\langle hD_x\rangle^s\chi'\varphi^*\operatorname{Op}_h(b_2)$ for some $b_2\in h^\infty\overline{S}^{-\infty}_{1,0}(T^*\mathbb{R}^n)$, which implies that $\|B_2\|_{L^2\to L^2}=\mathcal{O}(h^\infty)$.

We can now define Sobolev spaces on manifolds:

DEFINITION E.20. *Let M be a manifold. For $s\in\mathbb{R}$, define the local semiclassical Sobolev space $H^s_{h,\rm loc}(M)$, $C^\infty(M)\subset H^s_{h,\rm loc}(M)\subset\mathcal{D}'(M)$, as the Fréchet space with inner product seminorms*

$$\|(\varphi^{-1})^*\chi u\|_{H^s_h(\mathbb{R}^n)}$$

for all cutoff charts (φ,χ). Let $H^s_{h,\rm comp}(M)$ consist of all compactly supported elements of $H^s_{h,\rm loc}(M)$.

For each cutoff chart (φ,χ) the operators (E.1.28) are continuous

$$\chi\varphi^*: H^s_h(\mathbb{R}^n)\to H^s_{h,\rm comp}(M),\quad (\varphi^{-1})^*\chi: H^s_{h,\rm loc}(M)\to H^s_h(\mathbb{R}^n)$$

with operator seminorms bounded uniformly in h. It follows that $C^\infty_{\rm c}(M)$ is dense in $H^s_{h,\rm comp}(M)$ and $C^\infty(M)$ is dense in $H^s_{h,\rm loc}(M)$.

For each $u\in H^s_{h,\rm comp}(M)$, we can define the norm $\|u\|_{H^s_h}$ by

$$\|u\|^2_{H^s_h}:=\sum_j\|(\varphi_j^{-1})^*\chi_j u\|^2_{H^s_h(\mathbb{R}^n)} \tag{E.1.51}$$

where (φ_j,χ_j) is a collection of cutoff charts such that the domains of φ_j form a locally finite covering of M and $\sum_j|\chi_j|>0$ everywhere on M. Moreover, the norms resulting from two different choices of (φ_j,χ_j) are equivalent with constants uniform in h as long as we require that u is supported in some fixed compact subset of M. In particular, when the manifold M is compact, the norm (E.1.51) produces a Hilbert space $H_h^s(M)$.

If M is compact, then $H_h^s(M)$ is dual to $H_h^{-s}(M)$. More precisely, if we fix a smooth density on M then the L^2 inner product $\langle u,v\rangle_{L^2}$, $u,v\in C^\infty(M)$, extends continuously to $u\in H_h^s(M)$, $v\in H_h^{-s}(M)$. Moreover, if $u\in\mathcal{D}'(M)$ and we have the norm bound $|\langle u,v\rangle_{L^2}|\le L\|v\|_{H_h^{-s}(M)}$ for all $v\in C^\infty(M)$, then $u\in H_h^s(M)$ and $\|u\|_{H_h^s}\le CL$. The constants in the above bounds are independent of h. For noncompact manifolds M, $H^s_{h,\mathrm{comp}}(M)$ is dual to $H^{-s}_{h,\mathrm{loc}}(M)$.

The spaces $H^s_{h,\mathrm{loc}}(M)$ for different values of h consist of the same functions and the norms $\|u\|_{H_h^s}$ for different choices of h are equivalent, with constants depending on h. We thus may use the h-independent notation (with M compact in the third case below)

$$H^s_{\mathrm{loc}}(M),\quad H^s_{\mathrm{comp}}(M),\quad H^s(M) \tag{E.1.52}$$

for these spaces, where to define the norm we put $h:=1$. However, the h-dependent norms $\|\cdot\|_{H_h^s}$ will be used in the semiclassical estimates below, to ensure that the constants in these estimates are uniform in h.

PROPOSITION E.21 (Interpolation inequality in Sobolev spaces). *Let M be a manifold, $V\subset M$ a compact set. Fix real numbers $s_1<r<s_2$ and the corresponding norms (E.1.51). Then there exists a constant C such that for each $\alpha>0$ and each $u\in H^{s_2}_{\mathrm{comp}}(M)$ with $\operatorname{supp} u\subset V$, we have*

$$\|u\|_{H_h^r}\le\alpha\|u\|_{H_h^{s_2}}+C\alpha^{(s_1-r)/(s_2-r)}\|u\|_{H_h^{s_1}}. \tag{E.1.53}$$

Proof. By (E.1.51), we reduce to the classes $H_h^s(\mathbb{R}^n)$. Then (E.1.53) follows immediately from (E.1.49) and the following inequality:

$$\langle h\xi\rangle^r\le\alpha\langle h\xi\rangle^{s_2}+\alpha^{(s_1-r)/(s_2-r)}\langle h\xi\rangle^{s_1}.$$

The latter can be verified directly by multiplying both sides by $\alpha^{r/(s_2-r)}$ and using the inequality $a^r\le a^{s_2}+a^{s_1}$ for $a:=\alpha^{1/(s_2-r)}\langle h\xi\rangle$. □

We now study the action of pseudodifferential operators on Sobolev spaces. The following is a direct corollary of (E.1.31) and Proposition E.19:

PROPOSITION E.22 (Boundedness of pseudodifferential operators on manifolds). *Each $A\in\Psi_h^k(M)$ is bounded uniformly in h on compact sets as an operator*

$$A: H^s_{h,\mathrm{comp}}(M)\to H^{s-k}_{h,\mathrm{loc}}(M).$$

We also use the following version of the sharp Gårding inequality:

PROPOSITION E.23. *Let $A \in \Psi_h^{2k+1}(M)$ be compactly supported, fix some smooth density on M, and assume that $\operatorname{Re}\sigma_h(A) \geq 0$ everywhere. Then there exists a constant C such that for each $u \in H^{k+1/2}_{\mathrm{comp}}(M)$,*

$$\operatorname{Re}\langle Au, u\rangle_{L^2} \geq -Ch\|u\|^2_{H^k_h}. \tag{E.1.54}$$

Proof. By Proposition E.16 we have $A = \operatorname{Op}_h^M(a_0) + \mathcal{O}(h)_{\Psi_h^{2k}(M)}$ where $a_0 := \sigma_h(A)$. It suffices to prove (E.1.54) for the operator $\operatorname{Op}_h^M(a_0)$; recalling the definition (E.1.38) of Op_h^M we reduce to the case when $A = \operatorname{Op}_h(a)$ for $a \in \overline{S}^{2k+1}_{1,0}(T^*\mathbb{R}^n)$, $a \geq 0$, and Op_h is the standard quantization (E.1.18). Then (E.1.54) follows from the sharp Gårding inequality on $\mathbb{R}^n$, see [**Zw12**, Theorem 9.11]. □

As an application of Proposition E.23, we give the following improved bound on norms of pseudodifferential operators:

PROPOSITION E.24. *Let $A \in \Psi_h^0(M)$ be compactly supported and fix some smooth density on M. Then there is a constant C depending on A such that*

$$\|A\|_{L^2\to L^2} \leq \sup_{(x,\xi)\in T^*M} |\sigma_h(A)(x,\xi)| + Ch^{1/2}. \tag{E.1.55}$$

REMARK. In fact, we have (see [**Zw12**, Theorem 13.13])

$$\|A\|_{L^2\to L^2} = \sup_{(x,\xi)\in T^*M} |\sigma_h(A)(x,\xi)| + \mathcal{O}(h).$$

Proof. Let $C_0 := \sup_{(x,\xi)\in T^*M} |\sigma_h(A)(x,\xi)|$. Take $\chi \in C_c^\infty(M)$ such that $|\chi| \leq 1$ everywhere and $\chi = 1$ near $\operatorname{supp}\sigma_h(A)$. Then

$$\sigma_h(|C_0\chi|^2 - A^*A) = |C_0\chi(x)|^2 - |\sigma_h(A)|^2 \geq 0.$$

By Proposition E.23, we have for all $u \in L^2(M)$,

$$C_0^2\|\chi u\|^2_{L^2} - \|Au\|^2_{L^2} = \langle(|C_0\chi|^2 - A^*A)u, u\rangle_{L^2} \geq -Ch\|u\|^2_{L^2}$$

and thus, using the fact that $\|\chi u\|_{L^2} \leq \|u\|_{L^2}$,

$$\|Au\|^2_{L^2} \leq C_0^2\|u\|^2_{L^2} + Ch\|u\|^2_{L^2},$$

which implies (E.1.55). □

We finally review Sobolev spaces on manifolds with boundary. We refer the reader to [**HöIII**, Appendix B.2] and [**TaI**, §§4.4,4.5] for a comprehensive treatment. Let $\overline{M}$ be a compact manifold with boundary ∂M and interior M. We embed $\overline{M}$ into a compact manifold *without boundary*, denoted M_{ext}. One way to do this is to let M_{ext} be the double space of $\overline{M}$, obtained by gluing together two copies of $\overline{M}$ along the boundary.

DEFINITION E.25 (Sobolev spaces on manifolds with boundary). *Let $M \subset \overline{M} \subset M_{\rm ext}$ be as above. For $s \in \mathbb{R}$, define the spaces*

$$\bar{H}_h^s(M) \subset \mathcal{D}'(M), \quad \dot{H}_h^s(\overline{M}) \subset \mathcal{D}'(M_{\rm ext})$$

as follows:

- *$\bar{H}_h^s(M)$ consists of restrictions to M of elements of $H_h^s(M_{\rm ext})$;*
- *$\dot{H}_h^s(\overline{M})$ consists of elements of $H_h^s(M_{\rm ext})$ whose supports are contained in $\overline{M}$.*

The space $\dot{H}_h^s(\overline{M})$ is a closed subspace of the Hilbert space $H_h^s(M_{\rm ext})$ and inherits the norm of this ambient space. As for $\bar{H}_h^s(M)$, we make it into a Hilbert space by identifying it with the orthogonal complement of $\dot{H}_h^s(M_{\rm ext} \setminus M)$ in $H_h^s(M_{\rm ext})$. We have the inclusions

$$H^s_{h,\rm comp}(M) \subset \dot{H}_h^s(\overline{M}), \quad \bar{H}_h^s(M) \subset H^s_{h,\rm loc}(M).$$

Similarly to (E.1.52), we use the notation

$$\bar{H}^s(M), \quad \dot{H}^s(\overline{M})$$

for the spaces $\bar{H}_h^s(M), \dot{H}_h^s(\overline{M})$ when the h-dependence of the norm is irrelevant. The space $C^\infty(\overline{M})$ of functions smooth up to the boundary is dense in $\bar{H}^s(M)$ and $C_{\rm c}^\infty(M)$ is dense in $\dot{H}^s(\overline{M})$ – see [**HöIII**, Theorem B.2.1]. The spaces $\bar{H}_h^s(M)$ and $\dot{H}_h^{-s}(\overline{M})$ are dual to each other with respect to the natural L^2 pairing.

E.2. WAVEFRONT SETS AND ELLIPTICITY

We now define semiclassical wavefront sets, which consist of points in the phase space T^*M where a pseudodifferential operator or a family of distributions is *not* $\mathcal{O}(h^\infty)$. To handle in a uniform way the case of large values of ξ, we embed wavefront sets into the fiber-radially compactified cotangent bundle $\overline{T}^*M$ introduced in §E.1.3.

E.2.1. Wavefront sets of pseudodifferential operators. The wavefront set of a pseudodifferential operator $A \in \Psi_h^k(M)$ is the essential support of its full symbol. Here the essential support is the set of points in $\overline{T}^*M$ near which the symbol is *not* $\mathcal{O}(h^\infty \langle\xi\rangle^{-\infty})$: To control the behavior of the symbol as $|\xi| \to \infty$ we make the essential support a subset of the fiber-radially compactified cotangent bundle $\overline{T}^*M$ defined in §E.1.3.

DEFINITION E.26 (Essential support of a symbol). *Let $a \in S^k_{1,0}(T^*M)$. Define* $\operatorname{ess-supp} a \subset \overline{T}^*M$ *as follows: a point $(x_0, \xi_0) \in \overline{T}^*M$ does **not** lie* *in* ess-supp a *if there exists a neighbourhood* W *of* (x_0, ξ_0) *in* $\overline{T}^*M$ *such that for all multiindices* α, β *and all* N *there exists* $C_{\alpha\beta N}$ *such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta N} h^N \langle\xi\rangle^{-N}, \quad (x,\xi) \in W \cap T^*M.$$

It is clear from the definition that ess-supp a is a closed subset of $\overline{T}^*M$. Moreover, it follows from the compactness of the fibers of $\overline{T}^*M$ that

$$\text{ess-supp}\, a = \emptyset \quad \iff \quad a \in h^\infty S^{-\infty}(T^*M)$$

where the residual class $h^\infty S^{-\infty}(T^*M)$ is defined in (E.1.10).

DEFINITION E.27 (**Wavefront set of a pseudodifferential operator).** *Let* $A \in \Psi_h^k(M)$*. Define the set*

$$\mathrm{WF}_h(A) \subset \overline{T}^*M$$

as follows: a point $(x_0, \xi_0) \in \overline{T}^*M$ *does* ***not*** *lie in* $\mathrm{WF}_h(A)$ *if and only if for each cutoff chart* (φ, χ) *such that* x_0 *lies in the domain of* φ*, if* $a_{\varphi,\chi}$ *is defined by* (E.1.32) *and* $\widetilde{\varphi}$ *is defined by* (E.1.27)*, then* $\widetilde{\varphi}(x_0, \xi_0) \notin \text{ess-supp}\, a_{\varphi,\chi}$.

In other words, $\mathrm{WF}_h(A)$ is the union of the sets $\widetilde{\varphi}^{-1}(\text{ess-supp}\, a_{\varphi,\chi})$ over all cutoff charts (φ, χ).

Recalling that $\langle\xi\rangle^{-k}\sigma_h(A)$ extends to $\overline{T}^*M$ by Proposition E.4, we have

$$\operatorname{supp}\big(\langle\xi\rangle^{-k}\sigma_h(A)\big) \ \subset\ \mathrm{WF}_h(A). \tag{E.2.1}$$

If $a, b \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$ are the symbols in the change of variables formula, Proposition E.10, then by (E.1.30) we have ess-supp $b \subset \widetilde{\varphi}^{-1}(\text{ess-supp}\, a)$. Thus by (E.1.33), if $A \in \Psi_h^k$ has the form (E.1.31) for some symbols $a_j \in \overline{S}^k_h(T^*\mathbb{R}^n)$ then

$$\mathrm{WF}_h(A) \ \subset\ \bigcup_j \widetilde{\varphi}_j^{-1}(\text{ess-supp}\, a_j).$$

Therefore, if Op_h^M is a quantization procedure from Proposition E.15 then for every $a \in S_h^k(M)$ we have

$$\mathrm{WF}_h(\mathrm{Op}_h^M(a)) \ \subset\ \text{ess-supp}\, a. \tag{E.2.2}$$

Conversely, if $A \in \Psi_h^k(M)$ and $a \in S_h^k(T^*M)$ is the symbol constructed in Proposition E.16 such that $A = \mathrm{Op}_h^M(a) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$, then ess-supp $a \subset \mathrm{WF}_h(A)$. In particular, $\mathrm{WF}_h(A) \subset \overline{T}^*M$ is closed and

$$\mathrm{WF}_h(A) = \emptyset \quad \iff \quad A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}. \tag{E.2.3}$$

Moreover, the expansions in Proposition E.8 and the proof of Proposition E.17 imply the following properties:

$$\mathrm{WF}_h(A + B) \ \subset\ \mathrm{WF}_h(A) \cup \mathrm{WF}_h(B), \tag{E.2.4}$$

$$\mathrm{WF}_h(AB) \subset \mathrm{WF}_h(A) \cap \mathrm{WF}_h(B), \tag{E.2.5}$$

$$\mathrm{WF}_h(A^*) = \mathrm{WF}_h(A). \tag{E.2.6}$$

We give two more useful definitions involving wavefront sets:

DEFINITION E.28 (Compactly microlocalized pseudodifferential operators). *A compactly supported operator $A \in \Psi_h^k(M)$ is called **compactly microlocalized** if $\mathrm{WF}_h(A)$ is a compact subset of T^*M. We denote the class of such operators by $\Psi_h^{\mathrm{comp}}(M)$.*

That is, $A \in \Psi_h^{\mathrm{comp}}(M)$ if A is compactly supported and $\mathrm{WF}_h(A)$ does not intersect the fiber infinity $\partial\overline{T}^*M$. Note that $\Psi_h^{\mathrm{comp}}(M) \subset \Psi_h^{\ell}(M)$ for all ℓ, thus by Proposition E.22 every $A \in \Psi_h^{\mathrm{comp}}(M)$ is bounded uniformly in h as an operator $H_{h,\mathrm{loc}}^{-N}(M) \to H_{h,\mathrm{comp}}^{N}(M)$ for all N. In particular A is smoothing, more precisely it maps $\mathcal{D}'(M) \to C_{\mathrm{c}}^\infty(M)$.

DEFINITION E.29 (Microlocal equivalence). *Let $A, B \in \Psi_h^k(M)$ and $U \subset \overline{T}^*M$ be open or closed. We say that*

$$A = B + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \textit{microlocally on } U,$$

in the case when U is open, or

$$A = B + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \textit{microlocally near } U,$$

in the case when U is closed, if $\mathrm{WF}_h(A-B) \cap U = \emptyset$.

We conclude with the construction of a partition of unity made of pseudodifferential operators:

PROPOSITION E.30 (Microlocal partition of unity). *Assume that $V \subset \overline{T}^*M$ is compact and*

$$U_1, \dots, U_m \subset \overline{T}^*M$$

is an open cover of V. Then there exist compactly supported $X_1, \dots, X_m \in \Psi_h^0(M)$ such that $\mathrm{WF}_h(X_j) \subset U_j$ and

$$\sum_{j=1}^m X_j = I + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \textit{microlocally near } V.$$

Proof. Take a partition of unity

$$\chi_j \in C_{\mathrm{c}}^\infty(U_j),\ j = 1, \dots, m; \quad \sum_{j=1}^m \chi_j = 1 \quad \text{near} \quad V.$$

By Proposition E.4 we have $\chi_j \in S^0(T^*M)$. Put $X_j := \mathrm{Op}_h^M(\chi_j)$ where Op_h^M is a quantization procedure defined in Proposition E.15. Then we have $\mathrm{WF}_h(X_j) \subset U_j$ by (E.2.2). Moreover, since $\mathrm{Op}_h^M(1) = I$ we have $\mathrm{WF}_h(I - X_1 - \dots - X_m) \cap V = \emptyset$. $\square$

E.2.2. Ellipticity. We next define the elliptic set of a pseudodifferential operator, consisting of points where its principal symbol does not vanish. Recall that by Proposition E.4 for every $a \in S^k(T^*M)$ the function $\langle\xi\rangle^{-k}a$ extends smoothly to $\overline{T}^*M$.

DEFINITION E.31 (Elliptic set). *Let $A \in \Psi_h^k(M)$. Define the set*

$$\operatorname{ell}_h(A) := \{(x_0,\xi_0) \in \overline{T}^*M \mid \big(\langle\xi\rangle^{-k}\sigma_h(A)\big)(x_0,\xi_0) \neq 0\}.$$

If $(x_0,\xi_0) \in \operatorname{ell}_h(A)$, then we say that A is elliptic at (x_0,ξ_0).

REMARK. For more general operator classes $\Psi^k_{\delta,h}$ discussed at the end of §E.1.7, ellipticity is defined by requiring that $|\sigma_h(A)| \geq c\langle\xi\rangle^k$ in a neighborhood of (x_0,ξ_0) for some constant $c > 0$.

Note that $\operatorname{ell}_h(A)$ is an open subset of $\overline{T}^*M$. The significance of ellipticity comes from our ability to invert A microlocally near its elliptic points:

PROPOSITION E.32 (Elliptic parametrix). *Let $A \in \Psi_h^\ell(M)$, $B \in \Psi_h^k(M)$ be properly supported and satisfy $\operatorname{WF}_h(A) \subset \operatorname{ell}_h(B)$. Then there exist properly supported $Q, Q' \in \Psi_h^{\ell-k}(M)$ such that*

$$A = BQ + \mathcal{O}(h^\infty)_{\Psi^{-\infty}} = Q'B + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}. \tag{E.2.7}$$

Moreover, $\operatorname{WF}_h(Q) \cup \operatorname{WF}_h(Q') \subset \operatorname{WF}_h(A)$.

Proof. We construct Q; the operator Q' is constructed similarly. By considering $\langle\xi\rangle^{-\ell}\sigma_h(A)$ and $\langle\xi\rangle^{-k}\sigma_h(B)$ as smooth functions on $\overline{T}^*M$ and using that $\operatorname{supp}(\langle\xi\rangle^{-\ell}\sigma_h(A)) \subset \operatorname{WF}_h(A) \subset \operatorname{ell}_h(B)$, we construct the symbol

$$q_0 := \frac{\sigma_h(A)}{\sigma_h(B)} \in S^{\ell-k}(M), \quad \operatorname{supp}(\langle\xi\rangle^{k-\ell}q_0) \subset \operatorname{WF}_h(A).$$

Fix a quantization procedure $\operatorname{Op}_h = \operatorname{Op}_h^M$ from Proposition E.15. Then by (E.1.43) we have $\sigma_h(A - B\operatorname{Op}_h(q_0)) = 0$, therefore

$$A = B\operatorname{Op}_h(q_0) + hA_1, \quad A_1 \in \Psi_h^{\ell-1}(M), \quad \operatorname{WF}_h(A_1) \subset \operatorname{WF}_h(A).$$

Arguing by induction, we construct a sequence of operators $A_j \in \Psi_h^{\ell-j}(M)$ and symbols $q_j \in S^{\ell-k-j}(M)$, $j \in \mathbb{N}_0$, such that $A_0 = A$ and

$$A_j = B\operatorname{Op}_h(q_j) + hA_{j+1}, \quad \operatorname{WF}_h(A_j) \subset \operatorname{WF}_h(A).$$

To obtain (E.2.7), it remains to take

$$Q := \operatorname{Op}_h(q), \quad q \sim \sum_{j=0}^\infty h^j q_j$$

where the asymptotic sum is understood in the sense of (E.1.8). $\square$

An immediate corollary of the elliptic parametrix is the following

THEOREM E.33 (Elliptic estimate). *Let $P \in \Psi_h^k(M)$ be properly supported. Assume that $A, B_1 \in \Psi_h^0(M)$ are compactly supported and*

$$\mathrm{WF}_h(A) \subset \mathrm{ell}_h(P) \cap \mathrm{ell}_h(B_1).$$

Fix $s, N \in \mathbb{R}$. Then for each $u \in \mathcal{D}'(M)$, if $B_1 P u \in H^{s-k}_{\mathrm{comp}}(M)$, then $Au \in H^s_{\mathrm{comp}}(M)$ and

$$\|Au\|_{H_h^s} \leq C\|B_1 P u\|_{H_h^{s-k}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}} \tag{E.2.8}$$

where the constant C and the function $\chi \in C_{\mathrm{c}}^\infty(M)$ do not depend on u or h.

REMARKS. 1. The $\mathcal{O}(h^\infty)$ remainder term cannot be removed – see Exercise E.9.

2. We do not need to assume that A, B_1 are operators of order 0. If instead $A \in \Psi_h^m(M)$, $B_1 \in \Psi_h^\ell(M)$, then (E.2.8) becomes

$$\|Au\|_{H_h^{s-m}} \leq C\|B_1 P u\|_{H_h^{s-k-\ell}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}. \tag{E.2.9}$$

Proof. We have $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(B_1 P)$. By Proposition E.32, there exists $Q \in \Psi_h^{-k}(M)$ such that

$$A = QB_1P + R, \quad R = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

Since Q, P are properly supported and A, B_1 are compactly supported, R is compactly supported; therefore, $R = R\chi$ for some $\chi \in C_{\mathrm{c}}^\infty(M)$.

Let $u \in \mathcal{D}'(M)$ and assume that $B_1 P u \in H^{s-k}_{\mathrm{comp}}(M)$. By Proposition E.22, $QB_1Pu \in H^s_{\mathrm{comp}}(M)$ and $\|QB_1Pu\|_{H_h^s} \leq C\|B_1 P u\|_{H_h^{s-k}}$. We also have $Ru \in H^s_{\mathrm{comp}}(M)$ and $\|Ru\|_{H_h^s} = \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}$ for all N. Adding these estimates together, we get (E.2.8). □

Using the elliptic estimate we prove a microlocal version of sharp Gårding inequality which will be used in the positive commutator estimates of §E.4:

PROPOSITION E.34 (Microlocal Gårding inequality). *Assume that $A \in \Psi_h^{2s}(M)$, $B, B_1 \in \Psi_h^0(M)$ are compactly supported and*

$$\langle\xi\rangle^{-2s}\,\mathrm{Re}\,\sigma_h(A) \geq 0 \quad \textit{in a neighbourhood of} \quad \overline{T}^*M \setminus \mathrm{ell}_h(B),$$
$$\mathrm{WF}_h(A) \subset \mathrm{ell}_h(B_1).$$

Then there exists a constant C and $\chi \in C_{\mathrm{c}}^\infty(M)$ such that for each N and all $u \in H^s_{\mathrm{loc}}(M)$,

$$\mathrm{Re}\langle Au, u\rangle \geq -C\|Bu\|^2_{H_h^s} - Ch\|B_1 u\|^2_{H_h^{s-1/2}} - \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.$$

Proof. Take compactly supported $B_2 \in \Psi_h^s(M)$ such that

$$\mathrm{WF}_h(A) \subset \mathrm{ell}_h(B_2), \quad \mathrm{WF}_h(B_2) \subset \mathrm{ell}_h(B_1).$$

The closure of the set $\{\langle\xi\rangle^{-2s}\operatorname{Re}\sigma_h(A) < 0\} \subset \overline{T}^*M$ is compact and contained in $\operatorname{ell}_h(B)$. Therefore there exists a constant $C_0 > 0$ such that

$$\langle\xi\rangle^{-2s}(\operatorname{Re}\sigma_h(A) + C_0|\sigma_h(B_2B)|^2) \geq 0 \quad \text{everywhere.}$$

Consequently for the compactly supported operator

$$Y := A + C_0(B_2B)^*(B_2B) \in \Psi_h^{2s}(M)$$

we have

$$\langle\xi\rangle^{-2s}\operatorname{Re}\sigma_h(Y) \geq 0, \quad \operatorname{WF}_h(Y) \subset \operatorname{ell}_h(B_1).$$

By Proposition E.30 there exists compactly supported $X \in \Psi_h^0(M)$ with

$$\operatorname{WF}_h(Y) \cap \operatorname{WF}_h(I - X) = \emptyset, \quad \operatorname{WF}_h(X) \subset \operatorname{ell}_h(B_1).$$

Now the standard Gårding inequality, Proposition E.23, gives

$$\begin{aligned}\operatorname{Re}\langle Yu, u\rangle &= \operatorname{Re}\langle YXu, Xu\rangle + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}\\ &\geq -Ch\|Xu\|^2_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.\end{aligned}$$

To finish the proof, it remains to note that

$$\operatorname{Re}\langle Yu, u\rangle = \operatorname{Re}\langle Au, u\rangle + C_0\|B_2Bu\|^2_{L^2}, \quad \|B_2Bu\|_{L^2} \leq C\|Bu\|_{H_h^s},$$
$$\|Xu\|_{H_h^{s-1/2}} \leq C\|B_1u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}$$

where the last statement follows by Theorem E.33 with $P := I$. □

E.2.3. Wavefront sets of distributions. We now define semiclassical wavefront sets of h-dependent families of distributions and operators. For that, we need to impose polynomial growth assumptions:

DEFINITION E.35 (**h-tempered distributions and operators**).

1. *A family of distributions $u = u(h) \in \mathcal{D}'(M)$, $h \in (0, h_0)$, is called h-tempered, if for each $\chi \in C_c^\infty(M)$, there exist constants C and N such that $\|\chi u\|_{H_h^{-N}} \leq Ch^{-N}$.*

2. *A family of operators $B(h) : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$ is called h-tempered, if the Schwartz kernels $\mathcal{K}_{B(h)}$ form an h-tempered family in $\mathcal{D}'(M_1 \times M_2)$.*

The definition of an h-tempered distribution is motivated by the following property: if $u \in \mathcal{D}'(M_2)$ is h-tempered then for any properly supported family of operators $A = A(h) : \mathcal{D}'(M_2) \to C^\infty(M_1)$ we have

$$A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad \Longrightarrow \quad Au = \mathcal{O}(h^\infty)_{C^\infty}. \tag{E.2.10}$$

Here the notation $\mathcal{O}(h^\infty)_{C^\infty}$ means that each C^∞ seminorm is $\mathcal{O}(h^\infty)$.

The definition of wavefront set below is motivated by the following definition of the support of a distribution $u \in \mathcal{D}'(M)$: a point $x \in M$ does not lie in $\operatorname{supp} u$ if and only if there exists a neighbourhood U of x such that for each $\chi \in C^\infty_{\mathrm{c}}(M)$ with $\operatorname{supp}\chi \subset U$, we have $\chi u = 0$.

DEFINITION E.36 (Semiclassical wavefront set).

1. Let $u = u(h) \in \mathcal{D}'(M)$ be h-tempered. The semiclassical wavefront set $\mathrm{WF}_h(u) \subset \overline{T}^*M$ is defined as follows: a point $(x_0, \xi_0) \in \overline{T}^*M$ does ***not*** *lie in $\mathrm{WF}_h(u)$ if and only if there exists a neighbourhood U of (x_0, ξ_0) in $\overline{T}^*M$ such that for every properly supported $A \in \Psi^k_h(M)$ with $\mathrm{WF}_h(A) \subset U$,*

$$Au = \mathcal{O}(h^\infty)_{C^\infty}.$$

2. Let $B = B(h) : C^\infty_{\mathrm{c}}(M_2) \to \mathcal{D}'(M_1)$ be an h-tempered family of operators. Then $\mathrm{WF}'_h(B) \subset \overline{T}^*(M_1 \times M_2)$ is defined as follows:

$$\mathrm{WF}'_h(B) = \{(x, \xi, y, \eta) : (x, \xi, y, -\eta) \in \mathrm{WF}_h(\mathcal{K}_B)\} \tag{E.2.11}$$

where $\mathcal{K}_B(x, y) \in \mathcal{D}'(M_1 \times M_2)$ is the Schwartz kernel of B.

Note that $\mathrm{WF}_h(u)$ and $\mathrm{WF}'_h(B)$ are closed sets. Using a pseudodifferential partition of unity, Proposition E.30, we see that

$$\mathrm{WF}_h(u) = \emptyset \iff u = \mathcal{O}(h^\infty)_{C^\infty}, \tag{E.2.12}$$

$$\mathrm{WF}'_h(B) = \emptyset \iff B = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}. \tag{E.2.13}$$

The switch of sign of η in (E.2.11) is motivated by the identity

$$\mathrm{WF}'_h(B^*) = \{(y, \eta, x, \xi) : (x, \xi, y, \eta) \in \mathrm{WF}'_h(B)\}.$$

This follows from the fact that $\mathcal{K}_{B^*}(y, x) = \overline{\mathcal{K}_B(x, y)}$ (if the densities used to define the adjoint and the Schwartz kernels are the same) and the following formula for the wavefront set of the complex conjugate:

$$\mathrm{WF}_h(\bar{u}) = \{(x, -\xi) : (x, \xi) \in \mathrm{WF}_h(u)\}. \tag{E.2.14}$$

To see (E.2.14), we use Definition E.36 and the following identity for the quantization formula (E.1.18):

$$\overline{\mathrm{Op}_h(a)u} = \mathrm{Op}_h(a')\overline{u}, \quad a'(x, \xi) = \overline{a(x, -\xi)}. \tag{E.2.15}$$

A fundamental example of wavefront set calculation is given by

PROPOSITION E.37 (Wavefront set of an oscillatory integral). *Assume that*

$$\varphi(x, \theta) \in C^\infty(U; \mathbb{R}), \quad U \subset M_x \times \mathbb{R}^m_\theta$$

is a smooth function and

$$a(x, \theta; h) \in C^\infty_{\mathrm{c}}(U)$$

is supported inside an h-independent compact set $K_a \subset U$ and has all derivatives bounded uniformly in h. Then the family of smooth functions

$$u(x;h) := \int_{\mathbb{R}^m} e^{\frac{i}{h}\varphi(x,\theta)} a(x,\theta;h)\, d\theta, \quad x \in M, \tag{E.2.16}$$

is h-tempered and satisfies

$$\mathrm{WF}_h(u) \subset \{(x, \partial_x\varphi(x,\theta)) : (x,\theta) \in K_a,\ \partial_\theta\varphi(x,\theta) = 0\}.$$

Proof. By differentiation under the integral sign, we see immediately that u is h-tempered, in fact each of its $C^\infty(M)$ seminorms is bounded polynomially in h. By a partition of unity applied to a, we reduce to the case when $M = \mathbb{R}^n$. Then it suffices to prove that for each $b \in \overline{S}^k_{1,0}(T^*\mathbb{R}^n)$ such that

$$\operatorname{supp} b \cap \{(x, \partial_x\varphi(x,\theta)) : (x,\theta) \in K_a,\ \partial_\theta\varphi(x,\theta) = 0\} = \emptyset, \tag{E.2.17}$$

we have $\mathrm{Op}_h(b)u = \mathcal{O}(h^\infty)_{C^\infty}$ where Op_h is defined by (E.1.18). We write

$$\mathrm{Op}_h(b)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n+m}} e^{i\Phi/h} b(x,\xi;h) a(y,\theta;h)\, d\theta dy d\xi,$$
$$\Phi = \langle x - y, \xi\rangle + \varphi(y,\theta)$$

where to define the integral we can first integrate in θ, y variables and then in ξ. The critical points of the phase Φ in the (θ, y, ξ) variables are given by

$$y = x, \quad \partial_\theta\varphi(x,\theta) = 0, \quad \xi = \partial_x\varphi(x,\theta).$$

By (E.2.17), all critical points lie outside of the support of the amplitude $b(x,\xi;h)a(y,\theta;h)$. If $b(x,\xi;h)$ is compactly supported in ξ, then the method of nonstationary phase (E.1.15) gives $\mathrm{Op}_h(b)u = \mathcal{O}(h^\infty)_{C^\infty}$. For the case of general b, we first integrate in θ, y and then in ξ. For ξ large enough, the integral is $\mathcal{O}(h^\infty\langle\xi\rangle^{-\infty})$ by repeated integration by parts in y, using the inequality $|\partial_y\Phi|^{-1} \leq C\langle\xi\rangle^{-1}$ valid when $|\xi| \gg 1$. $\square$

We next study the behaviour of wavefront sets under pseudodifferential operators:

PROPOSITION E.38. *Let $u = u(h) \in \mathcal{D}'(M)$ be h-tempered and $A \in \Psi^\ell_h(M)$ properly supported. Then*

$$\mathrm{WF}_h(Au) \subset \mathrm{WF}_h(A) \cap \mathrm{WF}_h(u), \tag{E.2.18}$$

$$\mathrm{WF}_h(u) \subset \mathrm{WF}_h(Au) \cup (\overline{T}^*M \setminus \mathrm{ell}_h(A)). \tag{E.2.19}$$

Proof. 1. To see (E.2.18), take first $(x_0,\xi_0) \notin \mathrm{WF}_h(A)$. Define the open set $U := \overline{T}^*M \setminus \mathrm{WF}_h(A)$. Then for each properly supported $B \in \Psi^k_h(M)$ such that $\mathrm{WF}_h(B) \subset U$, by (E.2.5) and (E.2.3) we have $BA = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ and thus by (E.2.10), $B(Au) = \mathcal{O}(h^\infty)_{C^\infty}$; this implies that $(x_0,\xi_0) \notin \mathrm{WF}_h(Au)$.

Now, assume instead that $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$. Then there exists an open neighbourhood V of (x_0, ξ_0) in $\overline{T}^*M$ such that for each properly supported $B \in \Psi_h^k(M)$ with $\mathrm{WF}_h(B) \subset V$, we have $Bu = \mathcal{O}(h^\infty)_{C^\infty}$. Since $\mathrm{WF}_h(BA) \subset V$ by (E.2.5), we have $BAu = \mathcal{O}(h^\infty)_{C^\infty}$ as well, which implies that $(x_0, \xi_0) \notin \mathrm{WF}_h(Au)$. This finishes the proof of (E.2.18).

2. To prove (E.2.19), we use an elliptic parametrix. Let

$$(x_0, \xi_0) \in U := \mathrm{ell}_h(A) \setminus \mathrm{WF}_h(Au).$$

Let $B \in \Psi_h^k(M)$ be properly supported and $\mathrm{WF}_h(B) \subset U$. By Proposition E.32, there exists properly supported $Q \in \Psi_h^{k-\ell}(M)$ such that

$$B = QA + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

Moreover, $\mathrm{WF}_h(Q) \subset \mathrm{WF}_h(B) \subset U$. Therefore by (E.2.18) we have

$$\mathrm{WF}_h(Bu) = \mathrm{WF}_h(QAu) \subset \mathrm{WF}_h(Q) \cap \mathrm{WF}_h(Au) = \emptyset$$

which by (E.2.12) gives $Bu = \mathcal{O}(h^\infty)_{C^\infty}$. Thus $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ as required. □

We give the analog of Proposition E.38 for wavefront sets of operators, where we restrict to the class Ψ_h^{comp} of compactly microlocalized operators (see Definition E.28):

PROPOSITION E.39. *Let $B(h) : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$ be an h-tempered family of operators and $A_1 \in \Psi_h^{\mathrm{comp}}(M_1)$, $A_2 \in \Psi_h^{\mathrm{comp}}(M_2)$. Then*

$$\mathrm{WF}_h'(A_1 B A_2) \subset \mathrm{WF}_h'(B) \cap (\mathrm{WF}_h(A_1) \times \mathrm{WF}_h(A_2)), \tag{E.2.20}$$

$$\mathrm{WF}_h'(B) \cap (\mathrm{ell}_h(A_1) \times \mathrm{ell}_h(A_2)) \subset \mathrm{WF}_h'(A_1 B A_2). \tag{E.2.21}$$

Proof. The Schwartz kernel of $A_1 B A_2$ is given by

$$\mathcal{K}_{A_1 B A_2}(x, y) = (A_1 \otimes A_2^T)\mathcal{K}_B(x, y)$$

where $A_1 \otimes A_2^T : \mathcal{D}'(M_1 \times M_2) \to \mathcal{D}'(M_1 \times M_2)$ is the tensor product of A_1 and the operator

$$A_2^T : \mathcal{D}'(M_2) \to \mathcal{D}'(M_2), \quad \overline{A_2^* v} = A_2^T \overline{v}.$$

By (E.2.15), we see that $A_2^T \in \Psi_h^{\mathrm{comp}}(M_2)$ and $\mathrm{WF}_h(A_2^T), \mathrm{ell}_h(A_2^T)$ are obtained from $\mathrm{WF}_h(A_2), \mathrm{ell}_h(A_2)$ by the map $(y, \eta) \mapsto (y, -\eta)$.

Since $\mathrm{WF}_h(A_1), \mathrm{WF}_h(A_2)$ do not intersect the fiber infinity and the quantization procedure (E.1.18) satisfies

$$\mathrm{Op}_h(a \otimes b) = \mathrm{Op}_h(a) \otimes \mathrm{Op}_h(b), \quad a \in C_c^\infty(T^*\mathbb{R}^n),\ b \in C_c^\infty(T^*\mathbb{R}^{n'}),$$

we have $A_1 \otimes A_2^T \in \Psi_h^{\mathrm{comp}}(M_1 \times M_2)$,

$$\mathrm{WF}_h(A_1 \otimes A_2^T) = \{(x, \xi, y, -\eta) : (x, \xi) \in \mathrm{WF}_h(A_1),\ (y, \eta) \in \mathrm{WF}_h(A_2)\}$$

and a similar statement is true for the elliptic set. It remains to apply Proposition E.38 to the Schwartz kernel $\mathcal{K}_B$ and the operator $A_1 \otimes A_2^T$. $\square$

As an application of Propositions E.38 and E.39, we show

PROPOSITION E.40 (Composition formulæ for wavefront sets).

1. Let $B(h) : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$ and $u(h) \in \mathcal{D}'(M_2)$ be h-tempered, and $Q \in \Psi_h^{\mathrm{comp}}(M_2)$. Then

$$\begin{aligned} \mathrm{WF}_h(BQu) \cap T^*M_1 \ \subset \ \{(x,\xi) :& \\ \exists (y,\eta) \in \mathrm{WF}_h(u) \cap \mathrm{WF}_h(Q) :& \ (x,\xi,y,\eta) \in \mathrm{WF}'_h(B)\}. \end{aligned} \tag{E.2.22}$$

2. Let $B_1(h) : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$ and $B_2(h) : C_c^\infty(M_3) \to \mathcal{D}'(M_2)$ be h-tempered, and $Q \in \Psi_h^{\mathrm{comp}}(M_2)$. Then

$$\begin{aligned} \mathrm{WF}'_h(B_1QB_2) \cap T^*(M_1 \times M_3) \ \subset \ \{(x,\xi,z,\zeta) :& \\ \exists (y,\eta) \in \mathrm{WF}_h(Q) : \ (x,\xi,y,\eta) \in \mathrm{WF}'_h(B_1),& \\ (y,\eta,z,\zeta) \in \mathrm{WF}'_h(B_2)\}.& \end{aligned} \tag{E.2.23}$$

Proof. We give a proof of part 2; part 1 is proved in a similar way. Assume that $(x_0,\xi_0,z_0,\zeta_0) \in T^*(M_1 \times M_3)$ does not lie in the right-hand side of (E.2.23). This implies that $V_1 \cap V_2 \cap \mathrm{WF}_h(Q) = \emptyset$, where

$$V_1 = \{(y,\eta) \in T^*M_2 : (x_0,\xi_0,y,\eta) \in \mathrm{WF}'_h(B_1)\},$$
$$V_2 = \{(y,\eta) \in T^*M_2 : (y,\eta,z_0,\zeta_0) \in \mathrm{WF}'_h(B_2)\}$$

are closed subsets of T^*M_2. That is, $\mathrm{WF}_h(Q) \subset W_1 \cup W_2$ where $W_j = T^*M_2 \setminus V_j$. By a microlocal partition of unity, Proposition E.30, we write

$$Q = Q_1 + Q_2, \quad Q_j \in \Psi_h^{\mathrm{comp}}(M_2), \quad \mathrm{WF}_h(Q_j) \cap V_j = \emptyset.$$

Then there exist $A_1 \in \Psi_h^{\mathrm{comp}}(M_1)$, $A_2 \in \Psi_h^{\mathrm{comp}}(M_3)$ such that

$$(x_0,\xi_0) \in \mathrm{ell}_h(A_1), \quad (\mathrm{WF}_h(A_1) \times \mathrm{WF}_h(Q_1)) \cap \mathrm{WF}'_h(B_1) = \emptyset;$$
$$(z_0,\zeta_0) \in \mathrm{ell}_h(A_2), \quad (\mathrm{WF}_h(Q_2) \times \mathrm{WF}_h(A_2)) \cap \mathrm{WF}'_h(B_2) = \emptyset.$$

By part 1 of Proposition E.39, we have $\mathrm{WF}'_h(A_1B_1Q_1) = \emptyset$, therefore by (E.2.13), $A_1B_1Q_1 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. Similarly $Q_2B_2A_2 = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$. Then

$$A_1(B_1QB_2)A_2 = (A_1B_1Q_1)B_2A_2 + A_1B_1(Q_2B_2A_2) = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}.$$

By part 2 of Proposition E.39, we have $(x_0,\xi_0,z_0,\zeta_0) \notin \mathrm{WF}'_h(B_1QB_2)$ as required. $\square$

REMARK. Without a compactly microlocalized operator Q the products Bu and B_1B_2 are only well-defined under a certain wavefront set condition, and the rules for computing their wavefront sets are more complicated.

For the h-independent case, where WF_h is related to the nonsemiclassical wavefront set (see Exercise E.16), see [**HöI**, Theorems 8.2.13 and 8.2.14].

E.3. SEMICLASSICAL DEFECT MEASURES

In this section, we briefly review the concept of semiclassical defect measures. We refer the reader to [**Zw12**, Chapter 5] for a comprehensive introduction. We use the class Ψ_h^{comp} of compactly supported and compactly microlocalized semiclassical pseudodifferential operators, introduced in Definition E.28.

DEFINITION E.41 (**Semiclassical measures**). *Let M be a manifold with a fixed volume form and consider sequences*

$$h_j \to 0, \quad u_j \in \mathcal{D}'(M).$$

Let μ be a nonnegative Radon measure on T^*M. We say that u_j converges to μ in the sense of semiclassical measures, if for all $A = A(h) \in \Psi_h^{\mathrm{comp}}(M)$,

$$\langle A(h_j)u_j, u_j\rangle_{L^2(M)} \to \int_{T^*M} \sigma_h(A)\, d\mu. \tag{E.3.1}$$

THEOREM E.42 (**Existence of semiclassical measures**). *Assume that $h_j \to 0$, $u_j \in \mathcal{D}'(M)$ are sequences such that for some N,*

$$\|\chi u_j\|_{H^{-N}_{h_j}} \leq C_\chi \quad \text{for all } \chi \in C_{\mathrm{c}}^\infty(M) \tag{E.3.2}$$

with the constant C_χ depending on χ but not on j. Then there exists a subsequence $\{j_k\}$ such that u_{j_k} converges to some measure μ on T^*M.

REMARK. One cannot a priori guarantee that (E.3.1) holds for all $A \in \Psi_h^0(M)$, even if M is compact and $\|u_j\|_{L^2(M)} \leq C$. In fact, having $\|u_j\|_{L^2(M)} = 1$ does not imply that μ is a probability measure – see Exercise E.22.

Proof. 1. Fix a quantization procedure $\mathrm{Op}_h = \mathrm{Op}_h^M$, see Proposition E.15. To prove (E.3.1) it is enough to show that for all $a \in C_{\mathrm{c}}^\infty(T^*M)$ we have

$$I_j(a) \to \int_{T^*M} a\, d\mu \quad \text{where} \quad I_j(a) := \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle. \tag{E.3.3}$$

Indeed, let $A \in \Psi_h^{\mathrm{comp}}(M)$ and put $a := \sigma_h(A)$. Then $a \in C_{\mathrm{c}}^\infty(T^*M)$ and $A = \mathrm{Op}_h(a) + \mathcal{O}(h)_{\Psi_h^{\mathrm{comp}}(M)}$. Since operators in $\Psi_h^{\mathrm{comp}}(M)$ are bounded $H^{-N}_{h,\mathrm{loc}}(M) \to H^N_{h,\mathrm{comp}}(M)$ uniformly in h, we see that $\langle A(h_j)u_j, u_j\rangle = I_j(a) + \mathcal{O}(h_j)$, implying (E.3.1).

2. For each compact set $K \subset T^*M$ there exists a constant C_K such that we have the following bound for all $a \in C_{\mathrm{c}}^\infty(T^*M)$ with $\operatorname{supp} a \subset K$:

$$\limsup_{j\to\infty} |I_j(a)| \leq C_K \sup |a|. \tag{E.3.4}$$

Indeed, there exists $\chi \in C_c^\infty(M)$ depending only on K such that $\operatorname{Op}_h(a) = \chi \operatorname{Op}_h(a)\chi$. Denote $v_j := \operatorname{Op}_{h_j}(\langle\xi\rangle^{-N})\chi u_j \in L^2_{\mathrm{comp}}(M)$. Then

$$\begin{aligned}&\operatorname{Op}_h(a) = \operatorname{Op}_h(\langle\xi\rangle^{-N})^* \operatorname{Op}_h(\langle\xi\rangle^{2N}a) \operatorname{Op}_h(\langle\xi\rangle^{-N}) + \mathcal{O}(h)_{H_h^{-N}\to H_h^N},\\ &\|v_j\|_{L^2} \le C\|\chi u_j\|_{H_{h_j}^{-N}} \le C.\end{aligned}$$

Therefore

$$\begin{aligned}|I_j(a)| &= |\langle \operatorname{Op}_{h_j}(\langle\xi\rangle^{2N}a)v_j, v_j\rangle| + \mathcal{O}(h_j)\|\chi u_j\|^2_{H_{h_j}^{-N}}\\ &\le C\sup|\langle\xi\rangle^{2N}a| + Ch_j^{1/2}\end{aligned} \tag{E.3.5}$$

where the last inequality follows from Proposition E.24. This implies (E.3.4).

3. We now follow the proof of [**Zw12**, Theorem 5.2] which we briefly review here. Let $\{a_\ell\} \subset C_c^\infty(T^*M)$ be a countable set which is dense in the space $C_c(T^*M)$ of compactly supported continuous functions with the sup-norm. For each ℓ, the sequence $I_j(a_\ell)$ is bounded by (E.3.4). Using a diagonal argument we extract a subsequence $\{j_k\}$ such that $I_{j_k}(a_\ell)$ converges as $k \to \infty$ for each ℓ. By (E.3.4) we see that for each $a \in C_c^\infty(T^*M)$ and ℓ

$$\limsup_{k,k'\to\infty} |I_{j_k}(a) - I_{j_{k'}}(a)| \le \limsup_{k,k'\to\infty} |I_{j_k}(a_\ell) - I_{j_{k'}}(a_\ell)| + C\sup|a - a_\ell|,$$

where C does not depend on ℓ, as long as $\operatorname{supp} a_\ell$ is contained in a fixed compact set. Therefore $I_{j_k}(a)$ is a Cauchy sequence. Denote

$$I(a) := \lim_{k\to\infty} I_{j_k}(a) \in \mathbb{C}, \quad a \in C_c^\infty(T^*M).$$

4. The map I is a linear functional on $C_c^\infty(T^*M)$ and (E.3.4) implies that

$$|I(a)| \le C_K \sup|a| \quad \text{when} \quad \operatorname{supp} a \subset K.$$

We have $I(a) \in \mathbb{R}$ if a is real-valued as follows from the fact that $\operatorname{Op}_h(a)^* = \operatorname{Op}_h(a) + \mathcal{O}(h)_{H_h^{-N}\to H_h^N}$. Moreover, $I(a) \ge 0$ if $a \ge 0$ as follows from the sharp Gårding inequality, Proposition E.23. Thus $I(a)$ extends to a nonnegative continuous linear functional on $C_c(T^*M)$, and by a Riesz representation theorem there exists a Radon measure μ on T^*M such that

$$I(a) = \int_{T^*M} a\, d\mu \quad \text{for all} \quad a \in C_c^\infty(T^*M).$$

We thus obtain (E.3.3) for the subsequence $\{j_k\}$. □

We now study semiclassical measures associated to solutions of pseudodifferential equations

$$P(h_j)u_j = f_j \tag{E.3.6}$$

where $P = P(h) \in \Psi_h^k(M)$ is properly supported. The first result, similar to [**Zw12**, Theorem 5.3], gives a support property under the assumption that $f_j = o(1)$:

THEOREM E.43 (**Support of semiclassical measures**). *Assume that* (E.3.1), (E.3.2), (E.3.6) *hold and there exists* N *such that*

$$\|\chi f_j\|_{H_{h_j}^{-N}} = o(1) \quad \text{as } j \to \infty \quad \text{for all} \quad \chi \in C_c^\infty(M). \tag{E.3.7}$$

Then the support of the measure μ *is contained in* $\{\sigma_h(P) = 0\}$, *that is*

$$\mu(\{\sigma_h(P) \neq 0\}) = 0. \tag{E.3.8}$$

Proof. We have for each $A \in \Psi_h^{\mathrm{comp}}(M)$, by (E.3.2) and (E.3.7),

$$\langle APu_j, u_j \rangle = \langle Af_j, u_j \rangle \to 0 \quad \text{as } j \to \infty.$$

On the other hand, $AP \in \Psi_h^{\mathrm{comp}}(M)$, so by (E.1.43) we have

$$\langle APu_j, u_j \rangle \to \int_{T^*M} \sigma_h(A)\sigma_h(P)\, d\mu.$$

It follows that for each $a \in C_c^\infty(T^*M)$

$$\int_{T^*M} \sigma_h(P) a \, d\mu = 0,$$

which immediately implies (E.3.8). $\square$

The next statement, which is a generalization of [**Zw12**, Theorem 5.4], in particular shows that when $P^* = P$ and $f_j = o(h_j)$, the measure μ is invariant under the Hamiltonian flow of $\sigma_h(P)$:

THEOREM E.44 (**Semiclassical measures and Hamiltonian flow**). *Assume that* (E.3.1), (E.3.2), (E.3.6) *hold,* $p := \sigma_h(P)$ *is real-valued, and for some* N

$$\|\chi f_j\|_{H_{h_j}^{-N}} = \mathcal{O}(h_j) \quad \text{as } j \to \infty \quad \text{for all} \quad \chi \in C_c^\infty(M). \tag{E.3.9}$$

Then for every compact set $K \subset T^*M$ *there exists a constant* C_K *such that for all* $a \in C_c^\infty(T^*M)$ *with* $\operatorname{supp} a \subset K$

$$\Big| \int_{T^*M} H_p a \, d\mu \Big| \le C_K \sup |a|. \tag{E.3.10}$$

Under the stronger assumption that $\|\chi f_j\|_{H_{h_j}^{-N}} = o(h_j)$, *we have*

$$\int_{T^*M} (H_p a + 2ba)\, d\mu = 0 \quad \text{for all} \quad a \in C_c^\infty(T^*M), \tag{E.3.11}$$

where $b := \sigma_h(h^{-1} \operatorname{Im} P)$ *and* $\operatorname{Im} P := \frac{1}{2i}(P - P^*) \in h\Psi_h^{k-1}(M)$.

Proof. 1. It is enough to handle the case of real-valued a. Take $A \in \Psi_h^{\mathrm{comp}}(M)$ such that $\sigma_h(A) = a$ and $A^* = A$. We compute

$$\begin{aligned}\frac{\operatorname{Im}\langle f_j, Au_j\rangle}{h_j} &= \frac{\langle APu_j, u_j\rangle - \langle P^* A u_j, u_j\rangle}{2ih_j} \\ &= \frac{\langle [A,P]u_j, u_j\rangle}{2ih_j} + \frac{\langle (\operatorname{Im} P)Au_j, u_j\rangle}{h_j}.\end{aligned} \tag{E.3.12}$$

We have $[A, P], (\operatorname{Im} P)A \in h\Psi_h^{\mathrm{comp}}(M)$ and

$$\sigma_h(h^{-1}[A,P]) = iH_p a, \quad \sigma_h(h^{-1}(\operatorname{Im} P)A) = ba,$$

therefore by (E.3.1), the right-hand side of (E.3.12) converges as $j \to \infty$ to

$$\int_{T^*M} \Big(\frac{H_p a}{2} + ba\Big)\, d\mu.$$

2. Similarly to (E.3.4), if $\operatorname{supp} a \subset K$ then by (E.3.2) and (E.3.9)

$$\limsup_{j\to\infty} \big|h_j^{-1} \operatorname{Im}\langle f_j, Au_j\rangle\big| \le C_K \sup |a|$$

and (E.3.10) follows. If instead $\|\chi f_j\|_{H_{h_j}^{-N}} = o(h_j)$, then the left-hand side of (E.3.12) converges to zero and (E.3.11) follows. $\square$

E.4. PROPAGATION ESTIMATES

In this section, we consider general equations of the form

$$\mathbf{P}u = f, \quad u \in H^s_{\mathrm{loc}}(M). \tag{E.4.1}$$

Here $\mathbf{P} \in \Psi_h^k(M)$ is a properly supported semiclassical pseudodifferential operator on a manifold M. We use the boldface notation for $\mathbf{P}$ because the corresponding principal symbol can be complex valued:

$$\mathbf{p} := \sigma_h(\mathbf{P}) = p - iq, \quad p, q \in S^k(T^*M; \mathbb{R}). \tag{E.4.2}$$

Our goal is to prove estimates of the form

$$\|Au\|_{H_h^s} \le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}} \tag{E.4.3}$$

where $A, B, B_1 \in \Psi_h^0(M)$ are compactly supported and satisfy certain dynamical conditions, $\chi \in C_c^\infty(M)$ is some function depending on the supports of $A, B, B_1, \mathbf{P}$, and N can be any number. More precisely we will prove

- propagation of singularities, Theorem E.47;
- high regularity radial estimate, Theorem E.52; and
- low regularity radial estimate, Theorem E.54.

The estimate (E.4.3) should be compared with the elliptic estimate (E.2.8), which is stronger in the sense that the term $\|Bu\|$ is absent and a weaker norm $\|B_1 f\|_{H_h^{s-k}}$ is used. However, (E.2.8) only holds when $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(\mathbf{P})$, while the propagation estimates of this section are valid on the characteristic set $\{\langle\xi\rangle^{-k}\sigma_h(\mathbf{P}) = 0\}$ as well.

E.4.1. Approximation by smooth functions. In the non-elliptic case, the appearance of the norm $\|B_1 f\|_{H_h^{s-k+1}}$ on the right-hand side of (E.4.3) shows that a stronger regularity assumption on f is needed to obtain a propagation estimate. Equation (E.4.1) with $u \in H^s_{\mathrm{loc}}$ implies only $f \in H^{s-k}_{\mathrm{loc}}$. In other words $\|B_1 f\|_{H_h^{s-k+1}}$ might be infinite.

The following lemma shows that if $f = \mathbf{P}u \in H^{s-k+1}_{\mathrm{loc}}$ then it is enough to verify (E.4.3) for the case of $u \in C^\infty$:

LEMMA E.45 (**Approximation lemma).** *Let $\mathbf{P} \in \Psi^k_h(M)$ be properly supported. Assume that*

$$u \in H^s_{\mathrm{loc}}(M), \quad \mathbf{P}u \in H^{s-k+1}_{\mathrm{loc}}(M). \tag{E.4.4}$$

Fix $h > 0$. Then there exists a sequence $u_j \in C^\infty(M)$ such that for all $\chi \in C^\infty_{\mathrm{c}}(M)$

$$\|\chi(u_j - u)\|_{H_h^s} \to 0, \quad \|\chi(\mathbf{P}u_j - \mathbf{P}u)\|_{H_h^{s-k+1}} \to 0. \tag{E.4.5}$$

Proof. Let $\mathrm{Op}_h = \mathrm{Op}_h^M$ be a quantization procedure from Proposition E.15. Take $\psi \in C^\infty_{\mathrm{c}}(\mathbb{R})$ which is equal to 1 near the origin and define for u satisfying (E.4.4),

$$u_j := \mathrm{Op}_h(a_j)u, \quad a_j(x,\xi) := \psi(|\xi|/j).$$

We have $a_j \in S^{-N}(T^*M)$ for all N, thus $u_j \in C^\infty(M)$.

Since $\mathrm{Op}_h(1) = I$ we compute

$$u_j - u = A_j u, \quad \mathbf{P}u_j - \mathbf{P}u = A_j\mathbf{P}u + B_j u, \tag{E.4.6}$$

where

$$A_j := \mathrm{Op}_h(a_j - 1), \quad B_j := [P, A_j].$$

Fix $\chi \in C^\infty_{\mathrm{c}}(M)$ and take arbitrary $m \in \mathbb{R}$. By Proposition E.22, for each $\ell \geq 0$ the norms $\|\chi A_j\|_{H_h^{m+\ell} \to H_h^m}$ and $\|\chi B_j\|_{H_h^{m+\ell} \to H_h^{m-k+1}}$ are bounded by some $S^\ell_{1,0}(T^*M)$-seminorm of $a_j - 1$. A direct calculation shows that the symbols $a_j - 1$ are bounded uniformly in j in the class $S^0_{1,0}(T^*M)$ and converge to 0 in the class $S^1_{1,0}(T^*M)$. Therefore, as $j \to \infty$ we have

$$\|\chi A_j\|_{H_h^m \to H_h^m} \leq C, \quad \|\chi A_j\|_{H_h^{m+1} \to H_h^m} \to 0.$$

In particular, $\|\chi A_j v\|_{H_h^m} \to 0$ for all $v \in H_{\mathrm{loc}}^{m+1}(M)$. Since $H_{\mathrm{loc}}^{m+1}(M)$ is dense in $H_{\mathrm{loc}}^m(M)$, we see that

$$\|\chi A_j v\|_{H_h^m} \to 0 \quad \text{for all} \quad v \in H_{\mathrm{loc}}^m(M). \tag{E.4.7}$$

Similarly we have

$$\|\chi B_j v\|_{H_h^{m-k+1}} \to 0 \quad \text{for all} \quad v \in H_{\mathrm{loc}}^m(M). \tag{E.4.8}$$

Applying (E.4.7),(E.4.8) with $v := u \in H_{\mathrm{loc}}^s(M)$ and $v := \mathbf{P}u \in H_{\mathrm{loc}}^{s-k+1}(M)$ and recalling (E.4.6), we get (E.4.5). □

As an immediate corollary of Lemma E.45, we obtain

LEMMA E.46. *Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported and assume that $u \in H_{\mathrm{loc}}^s(M), v \in H_{\mathrm{comp}}^{-s+k-1}(M)$ satisfy*

$$\mathbf{P}u \in H_{\mathrm{loc}}^{s-k+1}(M), \quad \mathbf{P}^*v \in H_{\mathrm{comp}}^{-s}(M). \tag{E.4.9}$$

Then

$$\langle \mathbf{P}u, v\rangle_{L^2} = \langle u, \mathbf{P}^*v\rangle_{L^2}. \tag{E.4.10}$$

Proof. Let $\chi \in C_c^\infty(M)$ be equal to 1 near $\operatorname{supp} v$ and near $\operatorname{supp}(\mathbf{P}^*v)$. By Lemma E.45, there exist a sequence $u_j \in C^\infty(M)$ such that

$$\|\chi(u_j - u)\|_{H_h^s} \to 0, \quad \|\chi(\mathbf{P}u_j - \mathbf{P}u)\|_{H_h^{s-k+1}} \to 0.$$

Since u_j is smooth and v is compactly supported, we have

$$\langle \mathbf{P}u_j, v\rangle_{L^2} = \langle u_j, \mathbf{P}^*v\rangle_{L^2},$$

and (E.4.10) follows by taking the limit $j \to \infty$. □

E.4.2. Propagation of singularities. The most standard situation when the bound (E.4.3) holds is when the wavefront set of A (see Definition E.27) is controlled by the elliptic set of B (see Definition E.31) via the Hamiltonian flow of $p = \operatorname{Re}\sigma_h(\mathbf{P})$, provided that $q = -\operatorname{Im}\sigma_h(\mathbf{P})$ has the correct sign:

THEOREM E.47 (Propagation of singularities). *Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported, $p, q \in S^k(T^*M)$ be defined in (E.4.2), and*

$$\varphi_t := \exp(t\langle\xi\rangle^{1-k}H_p) : \overline{T}^*M \to \overline{T}^*M \tag{E.4.11}$$

be the flow of the vector field defined in Proposition E.5. Let $A, B, B_1 \in \Psi_h^0(M)$ be compactly supported and the following ***sign condition*** *hold:*

$$\langle\xi\rangle^{-k}q \geq 0 \quad \text{on } \operatorname{WF}_h(B_1). \tag{E.4.12}$$

Assume finally the following ***control condition****: for each $(x,\xi) \in \operatorname{WF}_h(A)$, there exists $T \geq 0$ such that (see Figure E.1)*

$$\varphi_{-T}(x,\xi) \in \operatorname{ell}_h(B); \quad \varphi_t(x,\xi) \in \operatorname{ell}_h(B_1) \text{ for all } t \in [-T, 0]. \tag{E.4.13}$$

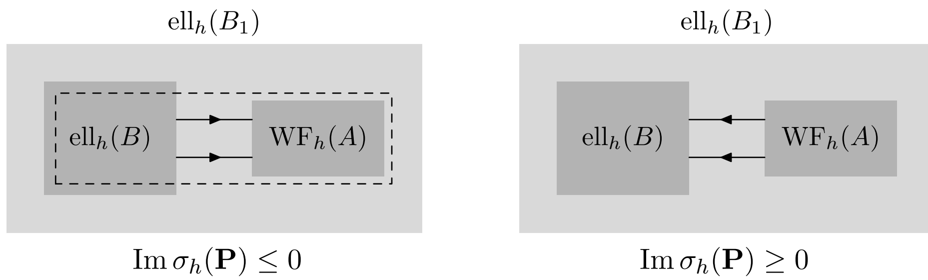


Figure E.1. Propagation of singularities (Theorem E.47), with the flow lines of $\langle \xi \rangle^{1-k} H_p$. The dashed rectangle on the left is the wavefront set of the operator B_2 used in the last step of the proof.

Then there exists $\chi \in C_c^\infty(M)$ such that for all s, N and all $u \in H_{\mathrm{loc}}^s(M)$ such that $f := \mathbf{P}u \in H_{\mathrm{loc}}^{s-k+1}(M)$,

$$\|Au\|_{H_h^s} \le C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + Ch^N \|\chi u\|_{H_h^{-N}}. \tag{E.4.14}$$

REMARKS. 1. The operators A, B, B_1 do not need to be of order 0. For instance, we could instead take $B \in \Psi_h^\ell(M)$ in which case the first term on the right-hand side of (E.4.14) is replaced by $C\|Bu\|_{H_h^{s-\ell}}$. This can be deduced from the original (E.4.14) by introducing the operator $B' := \mathrm{Op}_h(\langle \xi \rangle^{-\ell})B$ where Op_h is defined in Proposition E.15.

2. The normalization (E.4.11) of the Hamiltonian flow is convenient since it extends to the fiber-radial compactification $\overline{T}^*M$ and thus lets us handle singularities at fiber infinity as well. When M is noncompact, the flow φ_t might not be defined for all t; it is then implied in (E.4.13) that $\varphi_t(x, \xi)$ exists for $t \in [-T, 0]$.

3. Applying Theorem E.47 to the operator $-\mathbf{P}$, we can reverse the direction of propagation in (E.4.13) provided that we also reverse the sign condition (E.4.12). (See Figure E.1 and Exercise E.27.) In particular, if $q = 0$, then propagation of singularities applies in both directions. To make the presentation shorter, we state the results for one direction of propagation, but use them in both directions.

4. Propagation of singularities holds under weaker regularity assumptions: if u is merely a distribution and we know that $Bu \in H_{\mathrm{comp}}^s$, $B_1 f \in H_{\mathrm{comp}}^{s-k+1}$, then $Au \in H_{\mathrm{comp}}^s$ and (E.4.14) holds. See Exercise E.31.

5. The power h^{-1} and the norm $\|B_1 f\|_{H_h^{s-k+1}}$ in (E.4.14) are sharp, see Exercises E.27 and E.29.

The first step of the proof of Theorem E.47 is

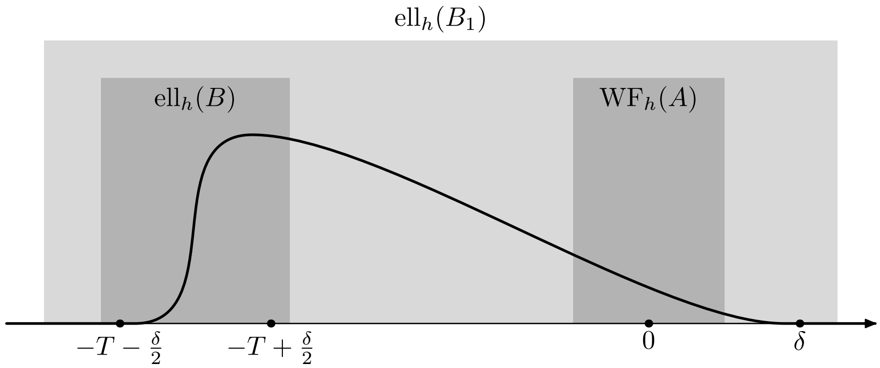


Figure E.2. The escape function ψ_0 along one flow line and the restrictions of $\mathrm{WF}_h(A), \mathrm{ell}_h(B), \mathrm{ell}_h(B_1)$ to this line

LEMMA E.48 (Escape function construction). *Assume that the control condition (E.4.13) holds for all $(x,\xi) \in \mathrm{WF}_h(A)$ and fix $\beta \geq 0$. Then there exists $g \in C_c^\infty(\overline{T}^*M)$ such that* $\operatorname{supp} g \subset \mathrm{ell}_h(B_1)$ *and*

- $g \geq 0$ *everywhere;*
- $g > 0$ *on* $\mathrm{WF}_h(A)$*;*
- $\langle\xi\rangle^{1-k} H_p g \leq -\beta g$ *in a neighbourhood of* $\overline{T}^*M \setminus \mathrm{ell}_h(B)$.

Proof. 1. We first consider the case when $\mathrm{WF}_h(A) = \{(x_0,\xi_0)\}$ consists of a single point. If $(x_0,\xi_0) \in \partial\overline{T}^*M$, then we embed $\overline{T}^*M$ in a manifold without boundary and extend the vector field $\langle\xi\rangle^{1-k}H_p$ there. Using (E.4.13), take $T \geq 0$ such that

$$\varphi_{-T}(x_0,\xi_0) \in \mathrm{ell}_h(B); \quad \varphi_t(x_0,\xi_0) \in \mathrm{ell}_h(B_1) \text{ for all } t \in [-T,0].$$

If $\langle\xi\rangle^{1-k}H_p(x_0,\xi_0) = 0$, then $\varphi_t(x_0,\xi_0) = (x_0,\xi_0)$ for all t. This implies that $(x_0,\xi_0) \in \mathrm{ell}_h(B) \cap \mathrm{ell}_h(B_1)$ and it suffices to take any g such that $g \geq 0$ everywhere, $g(x_0,\xi_0) > 0$, and $\operatorname{supp} g \subset \mathrm{ell}_h(B) \cap \mathrm{ell}_h(B_1)$.

We now assume that $\langle\xi\rangle^{1-k}H_p(x_0,\xi_0) \neq 0$. Taking a smaller value of T if necessary we can guarantee that the trajectory $\{\varphi_t(x_0,\xi_0) \mid t \in [-T,0]\}$ has no self-intersections. Then there exist $\delta > 0$ and a hypersurface Σ passing through (x_0,ξ_0) such that the map

$$\Phi : (-T-\delta,\delta) \times \Sigma \to \overline{T}^*M, \quad \Phi(t,x,\xi) = \varphi_t(x,\xi),$$

is a diffeomorphism onto its image and

$$\Phi\big((-T-\delta,-T+\delta) \times \Sigma\big) \subset \mathrm{ell}_h(B), \quad \Phi\big((-T-\delta,\delta) \times \Sigma\big) \subset \mathrm{ell}_h(B_1).$$

Take a function $\psi \in C_c^\infty((-T-\delta,\delta))$ with the following properties:

- $\psi \geq 0$ everywhere;

- $\psi(0) > 0$;
- $\psi' \leq -\beta\psi$ outside of $(-T-\delta/2, -T+\delta/2)$.

A function ψ_0 with these properties for $\beta = 0$ is easy to construct explicitly, see Figure E.2; for general β, it suffices to put $\psi(t) := e^{-\beta t}\psi_0(t)$. Take also nonnegative $\chi \in C_c^\infty(\Sigma)$ such that $\chi(x_0, \xi_0) > 0$. Then the function

$$g := (\psi \otimes \chi) \circ \Phi^{-1},$$

extended by zero outside of the image of Φ, has the required properties.

2. We now consider the general case. For each $(x_0, \xi_0) \in \mathrm{WF}_h(A)$, let $g_{(x_0,\xi_0)} \in C_c^\infty(\overline{T}^* M)$ be the function constructed in part 1 of the proof. Then $g_{(x_0,\xi_0)} > 0$ on some open neighbourhood $U_{(x_0,\xi_0)}$ of (x_0, ξ_0). Using compactness of $\mathrm{WF}_h(A)$, we cover it with finitely many sets $U_{(x_1,\xi_1)}, \dots, U_{(x_m,\xi_m)}$. The sum of the corresponding functions

$$g = g_{(x_1,\xi_1)} + \cdots + g_{(x_m,\xi_m)}$$

then has the required properties. $\square$

Armed with Lemma E.48, we now prove Theorem E.47 by means of a positive commutator argument (more pedantically, a *negative* commutator argument but we will use the standard nomenclature). The argument below is technically complicated, so we suggest considering the following special case at first reading:

$$s = 0, \quad k = 1, \quad \mathbf{P}^* = \mathbf{P} \in \Psi_h^1(M),$$

where steps 3 and 5 of the proof are not needed, we have

$$G = \mathrm{Op}_h(g), \quad Y = I, \quad \sigma_h(Z) = gH_p g,$$

but we get an additional term $Ch^{1/2}\|B_1 u\|_{H_h^{-1/2}}$ on the right-hand side.

Proof of Theorem E.47. 1. Fix a volume form on M. We use the notation

$$\operatorname{Re} R := \frac{R + R^*}{2}, \quad \operatorname{Im} R := \frac{R - R^*}{2i} \tag{E.4.15}$$

for operators $R : C_c^\infty(M) \to \mathcal{D}'(M)$.

Fix a constant $\beta > 0$, to be chosen later, and let g be the escape function constructed in Lemma E.48. By Proposition E.4, g is a symbol in the class $S^0(T^*M)$. We also fix a metric on M and define $\langle\xi\rangle := \sqrt{1 + |\xi|^2}$.

Let $\mathrm{Op}_h = \mathrm{Op}_h^M$ be a quantization procedure defined in (E.1.38). Take the compactly supported operator

$$G := \mathrm{Op}_h(\langle\xi\rangle^{s+\frac{1-k}{2}} g) \in \Psi_h^{s+\frac{1-k}{2}}(M), \quad \mathrm{WF}_h(G) \subset \mathrm{ell}_h(B_1), \tag{E.4.16}$$

and the properly supported operator

$$Y := \mathrm{Op}_h(\langle\xi\rangle^{\frac{k-1}{2}}) \in \Psi_h^{\frac{k-1}{2}}(M).$$

Using Lemma E.45 we reduce to the case $u \in C^\infty(M)$. Denoting $f = \mathbf{P}u$, we write

$$\operatorname{Im}\langle f, G^*Gu\rangle = \operatorname{Im}\langle(\operatorname{Re}\mathbf{P})u, G^*Gu\rangle + \operatorname{Re}\langle(\operatorname{Im}\mathbf{P})u, G^*Gu\rangle. \tag{E.4.17}$$

We will proceed by bounding from above the terms on the right-hand side.

2. We first write (using that $u \in C^\infty(M)$)

$$\begin{aligned}\operatorname{Im}\langle(\operatorname{Re}\mathbf{P})u, G^*Gu\rangle &= \frac{\langle(\operatorname{Re}\mathbf{P})u, G^*Gu\rangle - \langle G^*Gu, (\operatorname{Re}\mathbf{P})u\rangle}{2i}\\ &= \frac{\langle G^*G(\operatorname{Re}\mathbf{P})u, u\rangle - \langle(\operatorname{Re}\mathbf{P})G^*Gu, u\rangle}{2i}\\ &= h\langle Zu, u\rangle,\end{aligned}$$

where the symmetric compactly supported operator Z is a commutator:

$$Z = \frac{i}{2h}[\operatorname{Re}\mathbf{P}, G^*G] \in \Psi_h^{2s}(M), \quad \mathrm{WF}_h(Z) \subset \mathrm{ell}_h(B_1). \tag{E.4.18}$$

The semiclassical principal symbol of Z is

$$\begin{aligned}\sigma_h(Z) &= \frac{1}{2}\{p, \langle\xi\rangle^{2s+1-k}g^2\}\\ &= \langle\xi\rangle^{2s}\Big(g\langle\xi\rangle^{1-k}H_p g + \Big(s + \frac{1-k}{2}\Big)\frac{\langle\xi\rangle^{1-k}H_p\langle\xi\rangle}{\langle\xi\rangle}g^2\Big).\end{aligned} \tag{E.4.19}$$

Choose a constant $C_1 \geq 0$ independent of the choice of G such that

$$\Big(s + \frac{1-k}{2}\Big)\frac{\langle\xi\rangle^{1-k}H_p\langle\xi\rangle}{\langle\xi\rangle} \leq C_1 \quad \text{on } \mathrm{ell}_h(B_1) \supset \operatorname{supp} g. \tag{E.4.20}$$

By Lemma E.48, we have in a neighbourhood of $\overline{T}^*M \setminus \mathrm{ell}_h(B)$,

$$\langle\xi\rangle^{-2s}\sigma_h\big(Z + (\beta - C_1)(YG)^*(YG)\big) \leq 0.$$

By the microlocal Gårding inequality, Proposition E.34, applied to the operator $-\big(Z + (\beta - C_1)(YG)^*(YG)\big) \in \Psi_h^{2s}(M)$, we get the bound

$$\begin{aligned}h\langle Zu, u\rangle \leq{}& (C_1 - \beta)h\|YGu\|_{L^2}^2 + Ch\|Bu\|_{H_h^s}^2\\ &+ Ch^2\|B_1 u\|_{H_h^{s-1/2}}^2 + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}^2.\end{aligned} \tag{E.4.21}$$

3. We next compute (using again that $u \in C^\infty(M)$)

$$\begin{aligned}\operatorname{Re}\langle(\operatorname{Im}\mathbf{P})u, G^*Gu\rangle &= \operatorname{Re}\langle G(\operatorname{Im}\mathbf{P})u, Gu\rangle\\ &= \langle(\operatorname{Im}\mathbf{P})Gu, Gu\rangle + \operatorname{Re}\langle[G, \operatorname{Im}\mathbf{P}]u, Gu\rangle.\end{aligned}$$

We have $\sigma_h(\operatorname{Im}\mathbf{P}) = -q$. The sign condition (E.4.12) implies that

$$\langle\xi\rangle^{-k}\sigma_h(\operatorname{Im}\mathbf{P}) \leq 0 \quad \text{on } \operatorname{ell}_h(B_1) \supset \operatorname{WF}_h(G).$$

Using a pseudodifferential partition of unity (Proposition E.30), we write $\operatorname{Im}\mathbf{P}$ as a sum of two operators, one of which has nonpositive principal symbol and the wavefront set of the other one does not intersect $\operatorname{WF}_h(G)$. Applying sharp Gårding inequality, Proposition E.23, to the first of these operators, we get

$$\begin{aligned}\langle(\operatorname{Im}\mathbf{P})Gu, Gu\rangle &\leq C_2' h\|Gu\|^2_{H_h^{(k-1)/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}\\ &\leq C_2 h\|YGu\|^2_{L^2} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}\end{aligned} \tag{E.4.22}$$

for some constants C_2', C_2 which are independent of the choice of G; for the last inequality, we used the elliptic estimate, Theorem E.33. Next,

$$\operatorname{Re}\langle[G, \operatorname{Im}\mathbf{P}]u, Gu\rangle = \langle\operatorname{Re}(G^*[G, \operatorname{Im}\mathbf{P}])u, u\rangle.$$

The operator $G^*[G, \operatorname{Im}\mathbf{P}]$ lies in $h\Psi_h^{2s}(M)$ and

$$\sigma_h(h^{-1}G^*[G, \operatorname{Im}\mathbf{P}]) = ig\{g, q\}$$

is purely imaginary; therefore we have $\operatorname{Re}(G^*[G, \operatorname{Im}\mathbf{P}]) \in h^2\Psi_h^{2s-1}(M)$. Since $\operatorname{WF}_h(G^*[G, \operatorname{Im}\mathbf{P}]) \subset \operatorname{ell}_h(B_1)$, we get by Theorem E.33

$$\langle\operatorname{Re}(G^*[G, \operatorname{Im}\mathbf{P}])u, u\rangle \leq Ch^2\|B_1 u\|^2_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}. \tag{E.4.23}$$

Adding together (E.4.22) and (E.4.23), we get

$$\begin{aligned}\operatorname{Re}\langle(\operatorname{Im}\mathbf{P})u, G^*Gu\rangle &\leq C_2 h\|YGu\|^2_{L^2}\\ &\quad + Ch^2\|B_1 u\|^2_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.\end{aligned} \tag{E.4.24}$$

4. Adding (E.4.21) and (E.4.24) and using (E.4.17), we arrive to

$$\begin{aligned}\operatorname{Im}\langle f, G^*Gu\rangle &\leq (C_1 + C_2 - \beta)h\|YGu\|^2_{L^2} + Ch\|Bu\|^2_{H_h^s}\\ &\quad + Ch^2\|B_1 u\|^2_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.\end{aligned}$$

We now put $\beta := C_1 + C_2 + 1$ and use the following corollary of the Cauchy–Schwarz inequality and Theorem E.33:

$$\begin{aligned}|\langle f, G^*Gu\rangle| = |\langle Gf, Gu\rangle| &\leq C\|Gf\|_{H_h^{(1-k)/2}}\|Gu\|_{H_h^{(k-1)/2}}\\ &\leq C\|B_1 f\|_{H_h^{s-k+1}}\|YGu\|_{L^2} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}\end{aligned} \tag{E.4.25}$$

to get

$$\begin{aligned}\|YGu\|^2_{L^2} &\leq C\|Bu\|^2_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}}\|YGu\|_{L^2}\\ &\quad + Ch\|B_1 u\|^2_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|^2_{H_h^{-N}}.\end{aligned} \tag{E.4.26}$$

This implies the following estimate:

$$\begin{aligned}\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} \\ + Ch^{1/2}\|B_1 u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}\end{aligned} \tag{E.4.27}$$

where we have used Theorem E.33 and the fact that $\mathrm{WF}_h(A) \subset \mathrm{ell}_h(G)$ to bound $\|Au\|_{H_h^s}$ in terms of $\|YGu\|_{L^2}$.

5. We finally remove the term $Ch^{1/2}\|B_1 u\|_{H_h^{s-1/2}}$ in (E.4.27). For that, we use induction to prove the following estimate for each $\ell \in \mathbb{N}$:

$$\begin{aligned}\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} \\ + Ch^{\ell/2}\|B_1 u\|_{H_h^{s-\ell/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned} \tag{E.4.28}$$

For $\ell = 1$, this is exactly (E.4.27). Now, assuming that (E.4.28) is true for some ℓ, we prove it for $\ell + 1$. Take compactly supported $B_2 \in \Psi_h^0(M)$ such that the control condition (E.4.13) holds for (A, B, B_1) replaced by (A, B, B_2) and by (B_2, B, B_1); in particular, $\mathrm{WF}_h(B_2) \subset \mathrm{ell}_h(B_1)$. To construct B_2, it suffices to make it microlocalized in a small neighbourhood of the union of segments $\{\varphi_t(x, \xi) \mid t \in [-T, 0]\}$ with $(x, \xi) \in \mathrm{WF}_h(A)$ and T given by (E.4.13) – see Figure E.1.

Applying (E.4.28) to (A, B, B_2) and (E.4.27) to (B_2, B, B_1), we get

$$\begin{aligned}\|Au\|_{H_h^s} &\leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_2 f\|_{H_h^{s-k+1}} \\ &\quad + Ch^{\ell/2}\|B_2 u\|_{H_h^{s-\ell/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}, \\ \|B_2 u\|_{H_h^{s-\ell/2}} &\leq C\|Bu\|_{H_h^{s-\ell/2}} + Ch^{-1}\|B_1 f\|_{H_h^{s-\ell/2-k+1}} \\ &\quad + Ch^{1/2}\|B_1 u\|_{H_h^{s-\ell/2-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned}$$

Combining these estimates (and using Theorem E.33 to bound $\|B_2 f\|$ via $\|B_1 f\|$) we obtain (E.4.28) for $\ell + 1$. The fact that (E.4.28) holds for all ℓ immediately implies (E.4.14), finishing the proof. ☐

We now state a basic positive commutator estimate for the case when $\mathbf{P}$ has real principal symbol, which assumes the existence of an escape function which satisfies a sign condition (E.4.29). This estimate will be used in the proofs of radial source/sink estimates in §E.4.3 below.

LEMMA E.49. *Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported, $p := \mathrm{Re}\,\sigma_h(\mathbf{P})$, $A, B, B_1 \in \Psi_h^0(M)$ be compactly supported, and $g \in C_c^\infty(\overline{T}^*M)$ satisfy*

(1) $\langle\xi\rangle^{-k}\,\mathrm{Im}\,\sigma_h(\mathbf{P}) = 0$ *in a neighborhood of* $\mathrm{supp}\, g$;

(2) $g \geq 0$ *everywhere,* $\mathrm{WF}_h(A) \subset \{g > 0\}$, *and* $\mathrm{supp}\, g \subset \mathrm{ell}_h(B_1)$;

(3) *we have in a neighbourhood of* $\overline{T}^*M \setminus \mathrm{ell}_h(B)$, *for some constant* $\delta > 0$ *and a fixed Riemannian metric on* M *in the definition of* $\langle\xi\rangle$,

$$\langle\xi\rangle^{1-k}\Big(H_p g + \sigma_h(h^{-1}\operatorname{Im}\mathbf{P})g + \Big(s + \frac{1-k}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}g\Big) \le -\delta g. \tag{E.4.29}$$

Then there exists $\chi \in C_c^\infty(M)$ *such that for all* N *and* $u \in H^s_{\mathrm{loc}}(M)$, $f := \mathbf{P}u \in H^{s-k+1}_{\mathrm{loc}}(M)$,

$$\begin{aligned}\|Au\|_{H_h^s} \le{}& C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}}\\ &+ Ch^{1/2}\|B_1 u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned} \tag{E.4.30}$$

REMARK. The expression $\langle\xi\rangle^{1-k}\sigma_h(h^{-1}\operatorname{Im}\mathbf{P})$ does not make sense globally unless $\sigma_h(\operatorname{Im}\mathbf{P}) = \operatorname{Im}\sigma_h(\mathbf{P})$ is identically zero. However, we can define this expression as a smooth real valued function on a neighborhood of $\operatorname{supp} g$ in $\overline{T}^*M$, since $\operatorname{supp}(\langle\xi\rangle^{-k}\sigma_h(\operatorname{Im}\mathbf{P})) \cap \operatorname{supp} g = \emptyset$. Namely we put

$$\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) := \sigma_h(X(h^{-1}\operatorname{Im}\mathbf{P})) \quad \text{near} \quad \operatorname{supp} g \tag{E.4.31}$$

where $X \in \Psi_h^0(M)$ is properly supported and

$$\mathrm{WF}_h(X) \cap \operatorname{supp}(\langle\xi\rangle^{-k}\sigma_h(\operatorname{Im}\mathbf{P})) = \emptyset,$$
$$X = I + \mathcal{O}(h^\infty) \text{ microlocally near } \operatorname{supp} g.$$

Here $X\operatorname{Im}\mathbf{P} \in h\Psi_h^{k-1}(M)$ and two different choices of X produce the same function in a neighbourhood of $\operatorname{supp} g$.

Proof. Using Lemma E.45 we reduce to the case $u \in C^\infty(M)$. As in the proof of Theorem E.47 we put

$$G := \mathrm{Op}_h(\langle\xi\rangle^{s+\frac{1-k}{2}}g), \quad Y := \mathrm{Op}_h(\langle\xi\rangle^{\frac{k-1}{2}}).$$

Following steps 1–2 in the proof of Theorem E.47, we obtain

$$\operatorname{Im}\langle f, G^*Gu\rangle = h\operatorname{Re}\langle Z'u, u\rangle,$$

where

$$Z' = Z + h^{-1}G^*G(\operatorname{Im}\mathbf{P}) \in \Psi_h^{2s}(M), \quad \mathrm{WF}_h(Z') \subset \mathrm{ell}_h(B_1),$$

$Z \in \Psi_h^{2s}(M)$ is defined in (E.4.18), and $h^{-1}G^*G(\operatorname{Im}\mathbf{P}) \in \Psi_h^{2s}(M)$ since $\sigma_h(G^*G(\operatorname{Im}\mathbf{P})) = 0$ by assumption (1). Using (E.4.19), we calculate

$$\begin{aligned}\langle\xi\rangle^{-2s}\sigma_h(Z') ={}& g\langle\xi\rangle^{1-k}H_p g\\ &+ \Big(\langle\xi\rangle^{1-k}\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) + \Big(s + \frac{1-k}{2}\Big)\frac{\langle\xi\rangle^{1-k}H_p\langle\xi\rangle}{\langle\xi\rangle}\Big)g^2.\end{aligned}$$

By (E.4.29), we see that

$$\langle\xi\rangle^{-2s}\sigma_h\big(Z' + \delta(YG)^*(YG)\big) \le 0 \quad \text{in a neighbourhood of } \overline{T}^*M \setminus \mathrm{ell}_h(B).$$

By Proposition E.34 applied to $-\big(Z' + \delta(YG)^*(YG)\big)$, we have

$$\begin{aligned} h\,\mathrm{Re}\langle Z'u, u\rangle \leq & -\delta h\|YGu\|_{L^2}^2 + Ch\|Bu\|_{H_h^s}^2 \\ & + Ch^2\|B_1u\|_{H_h^{s-1/2}}^2 + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}^2. \end{aligned}$$

Arguing similarly to step 4 of the proof of Theorem E.47, we obtain (E.4.30). □

E.4.3. Radial source/sink estimates. We now show that the control condition (E.4.13) can in some situations be relaxed. In these cases, the positivity in the positive commutator argument comes not from the Hamiltonian derivative of the escape function H_pg, but from the other terms in (E.4.29).

More specifically, our estimates will be associated to *radial sources/sinks*, defined as follows:

DEFINITION E.50 (**Radial source/sink**). *Let $\kappa : T^*M \setminus 0 \to \partial\overline{T}^*M$ be the projection map, see (E.1.11). Take $p \in S^k(T^*M;\mathbb{R})$ and consider the flow (see Proposition E.5)*

$$\varphi_t := \exp(t\langle\xi\rangle^{1-k}H_p) : \overline{T}^*M \to \overline{T}^*M. \tag{E.4.32}$$

We say that a nonempty compact φ_t-invariant set

$$L \ \subset\ \{\langle\xi\rangle^{-k}p = 0\} \cap \partial\overline{T}^*M$$

*is a **radial source** for p, if there exists a neighbourhood $U \subset \overline{T}^*M$ of L such that uniformly in $(x,\xi) \in U \cap T^*M$,*

$$\kappa(\varphi_t(x,\xi)) \to L, \quad t \to -\infty; \tag{E.4.33}$$

$$|\varphi_t(x,\xi)| \geq C^{-1}e^{\theta|t|}|\xi|, \quad t \leq 0, \tag{E.4.34}$$

for some $C, \theta > 0$. Here $|\cdot|$ denotes a norm on the fibers of T^*M.

*A **radial sink** for p is by definition a radial source for $-p$. (See Figure E.4 below.)*

REMARKS. 1. The convergence in (E.4.33) is understood as follows: if d is a distance function on $\overline{T}^*M$ then

$$\sup_{(x,\xi)\in U} d\big(\kappa(\varphi_t(x,\xi)), L\big) \to 0 \quad \text{as} \quad t \to -\infty.$$

2. Note that (E.4.33) and (E.4.34) together imply

$$\varphi_t(x,\xi) \to L \quad \text{in } \overline{T}^*M \quad \text{as} \quad t \to -\infty. \tag{E.4.35}$$

3. In order for $p \in S^k(T^*M;\mathbb{R})$ to have a radial source or sink we need $k > 0$. Indeed, assume instead that $k \leq 0$. Let L be a radial source and U be the neighborhood of L from Definition E.50. Denote $\widetilde{p} := \langle\xi\rangle^{-k}p$.

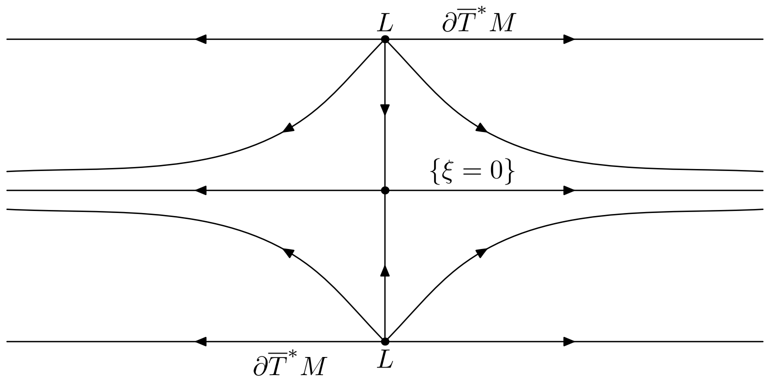


Figure E.3. The phase space picture of the Hamiltonian flow of $p = x\xi$ on $\overline{T}^*\mathbb{R}$. The horizontal direction is x and the vertical direction is a compactification of ξ; the top and bottom lines correspond to $\{\xi = \pm\infty\} \subset \partial\overline{T}^*M$. The set L, consisting of two points, is a radial source.

Take any $(x, \xi) \in U \cap T^*M$, then $p(\varphi_t(x, \xi)) = p(x, \xi)$. On the other hand $\widetilde{p}(\varphi_t(x, \xi)) \to 0$ as $t \to -\infty$ since $\widetilde{p}|_L = 0$. By (E.4.34) we get $p(x, \xi) = 0$. That is, p vanishes on U. In particular, $H_p = 0$ on U which contradicts (E.4.34).

4. We say that $(x, \xi) \in \partial\overline{T}^*M$ is a *radial point* if it is a fixed point for φ_t, that is $\langle\xi\rangle^{1-k}H_p(x, \xi) = 0$. (If p is homogeneous, this means that on the ray $\kappa^{-1}(x, \xi)$ the vector field H_p is radial, i.e. it is parallel to $\xi \cdot \partial_\xi$. This explains the term 'radial point'.) The radial sources/sinks arising in the study of asymptotically hyperbolic manifolds in Chapter 5 do consist of radial points. However, the radial estimates presented here also apply to settings where L contains non-radial points, such as Kerr–de Sitter metrics [**Va13**] and Anosov flows [**DZ16**].

EXAMPLE. Consider the following operator $\mathbf{P} \in \Psi_h^1(\mathbb{R})$:

$$\mathbf{P} := x(hD_x) + i\gamma h = -ihx\partial_x + i\gamma h, \quad \gamma \in \mathbb{C}. \tag{E.4.36}$$

Its principal symbol $p := \sigma_h(\mathbf{P})$ is given by

$$p(x, \xi) = x\xi, \quad H_p = x\partial_x - \xi\partial_\xi.$$

Then the following set is a radial source for p (consisting of radial points):

$$L := \partial\overline{T}^*\mathbb{R} \cap \{x = 0\}.$$

See Figure E.3 for a phase space picture of the flow $\varphi_t = e^{tH_p}$. We encourage the reader to look at Exercises E.34 and E.37 which explain the definitions and statements below for this example.

To make the subprincipal condition in our estimates invariant, we use the following definition. Recall that by Proposition E.4, for $a \in S^{k-1}(T^*M)$ the function $\langle\xi\rangle^{1-k}a$ extends smoothly to $\overline{T}^*M$.

PROPOSITION E.51 (Eventual positivity/negativity). *Assume that $p \in S^k(T^*M;\mathbb{R})$, φ_t is defined by (E.4.32), $L \subset \overline{T}^*M$ is a compact φ_t-invariant set, and $a \in S^{k-1}(T^*M;\mathbb{R})$. The following are equivalent:*

(1) *there exists $T > 0$ such that*

$$\int_0^T (\langle\xi\rangle^{1-k}a) \circ \varphi_t \, dt > 0 \quad \text{on } L; \tag{E.4.37}$$

(2) *there exists $b \in S^0(T^*M;\mathbb{R})$ such that*

$$\langle\xi\rangle^{1-k}(a + H_p b) > 0 \quad \text{on } L. \tag{E.4.38}$$

If the above conditions hold, we say that a is ***eventually positive*** *on L with respect to p. We say a is* ***eventually negative*** *if $-a$ is eventually positive.*

REMARKS. 1. It follows from (E.4.38) that eventual positivity of a does not depend on the choice of the Riemannian metric in the definition of $\langle\xi\rangle$. Moreover, since L is φ_t-invariant, the direction of propagation in (E.4.37) does not matter.

2. In the case when $\langle\xi\rangle^{1-k}H_p$ vanishes on L (in particular, in the application in Chapter 5), eventual positivity of a is simply equivalent to $\langle\xi\rangle^{1-k}a$ being positive on L.

Proof. 1. Assume that there exists $T > 0$ such that (E.4.37) holds. Put

$$b := \frac{1}{T}\int_0^T (T-t)(\langle\xi\rangle^{1-k}a) \circ \varphi_t \, dt \;\in\; S^0(T^*M),$$

then integration by parts shows that

$$\begin{aligned}\langle\xi\rangle^{1-k}H_p b &= \frac{1}{T}\int_0^T (T-t)\partial_t\big((\langle\xi\rangle^{1-k}a)\circ\varphi_t\big)\, dt \\ &= -\langle\xi\rangle^{1-k}a + \frac{1}{T}\int_0^T (\langle\xi\rangle^{1-k}a)\circ\varphi_t \, dt,\end{aligned}$$

therefore

$$\langle\xi\rangle^{1-k}(a + H_p b) = \frac{1}{T}\int_0^T (\langle\xi\rangle^{1-k}a)\circ\varphi_t \, dt > 0 \quad \text{on } L$$

and (E.4.38) follows.

2. Now, assume that (E.4.38) holds. Fix $\varepsilon > 0$ such that

$$a_1 := \langle\xi\rangle^{1-k}(a + H_p b) \geq \varepsilon \quad \text{on } L,$$

and a constant C such that $|b| \leq C$ on L. Since L is φ_t-invariant, we have

$$\begin{aligned}\int_0^T (\langle\xi\rangle^{1-k} a) \circ \varphi_t \, dt &= \int_0^T a_1 \circ \varphi_t \, dt - \int_0^T (\langle\xi\rangle^{1-k} H_p b) \circ \varphi_t \, dt \\ &\geq \varepsilon T - b \circ \varphi_T + b \\ &\geq \varepsilon T - 2C\end{aligned}$$

on L. For T large enough, this implies (E.4.37). □

We are now ready to state the first radial estimate, which gives a priori bounds near a radial source provided we are in sufficiently high Sobolev regularity. (See Figure E.4.)

THEOREM E.52 (High regularity radial estimate). *Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported, $k > 0$, $p := \operatorname{Re} \sigma_h(\mathbf{P}) \in S^k(T^*M)$,*

$$L \ \subset \ \{\langle\xi\rangle^{-k} p = 0\} \cap \partial\overline{T}^* M$$

*be a radial source for p (see Definition E.50) and $\langle\xi\rangle^{-k} \operatorname{Im} \sigma_h(\mathbf{P}) = 0$ near L. Let $s \in \mathbb{R}$ satisfy the following **threshold condition**: the symbol*

$$\sigma_h(h^{-1} \operatorname{Im} \mathbf{P}) + \Big(s + \frac{1-k}{2}\Big) \frac{H_p \langle\xi\rangle}{\langle\xi\rangle} \tag{E.4.39}$$

is eventually negative on L with respect to p as defined in Proposition E.51.

Fix compactly supported $B_1 \in \Psi_h^0(M)$ such that $L \subset \operatorname{ell}_h(B_1)$. Then there exists compactly supported $A \in \Psi_h^0(M)$ such that $L \subset \operatorname{ell}_h(A)$ and $\chi \in C_c^\infty(M)$ such that for all N and $u \in H_{\mathrm{loc}}^s(M)$, $f := \mathbf{P}u \in H_{\mathrm{loc}}^{s-k+1}(M)$

$$\|Au\|_{H_h^s} \leq C h^{-1} \|B_1 f\|_{H_h^{s-k+1}} + C h^N \|\chi u\|_{H_h^{-N}}. \tag{E.4.40}$$

REMARKS. 1. The symbol $\sigma_h(h^{-1} \operatorname{Im} \mathbf{P}) \in S^{k-1}(T^*M; \mathbb{R})$ is defined near L using (E.4.31).

2. The condition that (E.4.39) is eventually negative is independent of the choice of the density in the definition of $\operatorname{Im} \mathbf{P}$ and the metric in the definition of $\langle\xi\rangle$. Moreover, this condition is satisfied for $s > 0$ large enough. See Exercises E.32 and E.33 for details.

3. Changing $\mathbf{P}$ to $-\mathbf{P}$, we obtain an estimate for the case when L is a radial sink. Similarly Theorem E.54 below can be applied to radial sources.

4. Combining Theorem E.47 and Theorem E.52, we get bounds of the form (E.4.40) for each A such that every backwards trajectory of φ_t starting at $\mathrm{WF}_h(A)$ converges to L, as long as the sign condition (E.4.12) holds.

5. A stronger statement is available which gives a priori regularity of u assuming that u lies in a sufficiently high Sobolev class – see Exercise E.35.

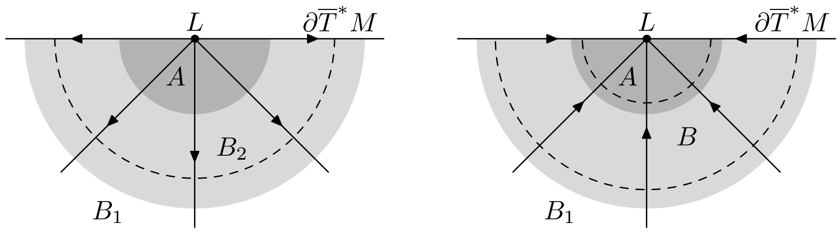


Figure E.4. An illustration of Theorem E.52 for radial sources (left) and Theorem E.54 for radial sinks (right). The horizontal line on the top denotes $\partial\overline{T}^*M$. The dashed half-disk on the left denotes the wavefront set of the operator B_2 used in the proof of Theorem E.52. The dashed half-annulus on the right denotes the wavefront set of B, reflecting the fact that $\mathrm{WF}_h(A)\cap\mathrm{WF}_h(B)\neq\emptyset$ for the specific operators A, B constructed in the proof of Theorem E.54.

Similarly to Theorem E.47, the proof of Theorem E.52 relies on an escape function construction:

LEMMA E.53. *Assume that $L\subset\partial\overline{T}^*M$ is a radial source for p and let $U\subset\overline{T}^*M$ be an open neighbourhood of L. Then there exists a function $\chi\in C_{\mathrm{c}}^\infty(U)$ such that*

- $\chi\geq 0$ *everywhere;*
- $\chi>0$ *on L;*
- $\langle\xi\rangle^{1-k}H_p\chi\leq 0$ *everywhere.*

Proof. By (E.4.35), we may shrink U so that $\varphi_t(x,\xi)\to L$ as $t\to-\infty$ uniformly in $(x,\xi)\in U$. Take $\psi\in C_{\mathrm{c}}^\infty(U;[0,1])$ such that $\psi=1$ near L. Then for $T>0$ large enough, we have

$$t\geq T,\ (x,\xi)\in\operatorname{supp}\psi\quad\Longrightarrow\quad\psi\big(\varphi_{-t}(x,\xi)\big)=1. \tag{E.4.41}$$

Put

$$\chi:=\int_T^{2T}\psi\circ\varphi_t\,dt.$$

Then $\chi\geq 0$ everywhere and, since L is φ_t-invariant, $\chi>0$ on L. We also see from (E.4.41) that $\operatorname{supp}\chi\subset U$. It remains to show that

$$\langle\xi\rangle^{1-k}H_p\chi=\psi\circ\varphi_{2T}-\psi\circ\varphi_T\leq 0.$$

Indeed, suppose that $\psi(\varphi_{2T}(x,\xi))>\psi(\varphi_T(x,\xi))$ for some (x,ξ). Since $0\leq\psi\leq 1$ everywhere, this implies that $\psi(\varphi_{2T}(x,\xi))>0$ and $\psi(\varphi_T(x,\xi))<1$. By (E.4.41) applied to $\varphi_{2T}(x,\xi)$, we arrive to a contradiction. $\square$

We now give the proof of the high regularity radial estimate, using the positive commutator estimate from Lemma E.49:

Proof of Theorem E.52. 1. Using Proposition E.51 and the fact that (E.4.39) is eventually negative on L, choose $b \in C^\infty(\overline{T}^* M; \mathbb{R})$ such that

$$\langle\xi\rangle^{1-k}\Big(\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) + \Big(s + \frac{1-k}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} + H_p b\Big) < 0 \quad \text{on } L. \tag{E.4.42}$$

Let $B_2 \in \Psi_h^0(M)$ be compactly supported and satisfy $L \subset \operatorname{ell}_h(B_2)$; we will fix it in the second step of the proof. Take a neighbourhood U of L such that for some constant $\delta > 0$,

$$U \subset \operatorname{ell}_h(B_2), \tag{E.4.43}$$

$$\langle\xi\rangle^{-k}\operatorname{Im}\sigma_h(\mathbf{P}) = 0 \quad \text{on } U, \tag{E.4.44}$$

$$\langle\xi\rangle^{1-k}\Big(\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) + \Big(s + \frac{1-k}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} + H_p b\Big) \le -\delta \quad \text{on } U. \tag{E.4.45}$$

Let $\chi \in C_c^\infty(U)$ be the function constructed in Lemma E.53 and put

$$g := e^b \chi \in C_c^\infty(U).$$

Put $B := 0$ and take $A \in \Psi_h^0(M)$ which is elliptic on L and satisfies $\operatorname{WF}_h(A) \subset \{\chi > 0\}$. Then the assumptions of Lemma E.49 are satisfied, with B_1 replaced by B_2. In particular, (E.4.29) follows from (E.4.45) together with the inequality $\langle\xi\rangle^{1-k} H_p \chi \le 0$. The estimate (E.4.30) then gives

$$\begin{aligned}\|Au\|_{H_h^s} \le{}& Ch^{-1}\|B_2 f\|_{H_h^{s-k+1}} \\ &+ Ch^{1/2}\|B_2 u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned} \tag{E.4.46}$$

2. It remains to remove the term $Ch^{1/2}\|B_2 u\|_{H_h^{s-1/2}}$ from (E.4.46). Let V be a neighbourhood of L such that

$$V \subset \operatorname{ell}_h(B_1), \quad \langle\xi\rangle^{-k}\operatorname{Im}\sigma_h(\mathbf{P}) = 0 \quad \text{on } V.$$

We choose B_2 elliptic on L and with the following property: for each $(x,\xi) \in \operatorname{WF}_h(B_2)$, the trajectory $\varphi_t(x,\xi)$ converges to L as $t \to -\infty$ and lies inside V for all $t \le 0$. The existence of such operator is guaranteed by (E.4.35).

By propagation of singularities (Theorem E.47), we have

$$\|B_2 u\|_{H_h^{s-1/2}} \le C\|Au\|_{H_h^{s-1/2}} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1/2}} + Ch^N\|\chi u\|_{H_h^{-N}}.$$

Combined with (E.4.46), this gives (using the elliptic estimate, Theorem E.33, to bound $\|B_2 f\|_{H_h^{s-k+1}}$ in terms of $\|B_1 f\|_{H_h^{s-k+1}}$)

$$\|Au\|_{H_h^s} \le Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + Ch^{1/2}\|Au\|_{H_h^{s-1/2}} + Ch^N\|\chi u\|_{H_h^{-N}}.$$

By the interpolation inequality, Proposition E.21, with $\alpha := (2Ch^{1/2})^{-1}$, $r := s - 1/2$, $s_1 := -N$, $s_2 := s$, and putting the resulting term $\frac{1}{2}\|Au\|_{H_h^s}$ on the left-hand side, we get

$$\|Au\|_{H_h^s} \leq Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + Ch^{N+s}\|Au\|_{H_h^{-N}} + Ch^N\|\chi u\|_{H_h^{-N}}.$$

Since N can be chosen arbitrarily large, this implies (E.4.40). □

The second radial estimate bounds the solution near a radial sink, provided that we control it in a punctured neighbourhood of the sink and work in sufficiently low Sobolev regularity. (See Figure E.4.)

THEOREM E.54 (Low regularity radial estimate). *Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported, $k > 0$, $p := \operatorname{Re}\sigma_h(\mathbf{P}) \in S^k(T^*M)$,*

$$L \subset \{\langle\xi\rangle^{-k}p = 0\} \cap \partial\overline{T}^*M$$

*be a radial sink for p (see Definition E.50), and $\langle\xi\rangle^{-k}\operatorname{Im}\sigma_h(\mathbf{P}) = 0$ near L. Let $s \in \mathbb{R}$ satisfy the following **threshold condition**: the symbol*

$$\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) + \Big(s + \frac{1-k}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} \tag{E.4.47}$$

is eventually negative on L with respect to p as defined in Proposition E.51.

Fix compactly supported $B_1 \in \Psi_h^0(M)$ such that $L \subset \operatorname{ell}_h(B_1)$. Then there exist compactly supported $A, B \in \Psi_h^0(M)$ such that $L \subset \operatorname{ell}_h(A)$, $\operatorname{WF}_h(B) \subset \operatorname{ell}_h(B_1) \setminus L$ and there exists $\chi \in C_c^\infty(M)$ such that for all N and $u \in H_{\mathrm{loc}}^s(M)$, $f := \mathbf{P}u \in H_{\mathrm{loc}}^{s-k+1}(M)$, we have

$$\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} + Ch^N\|\chi u\|_{H_h^{-N}}. \tag{E.4.48}$$

REMARKS. 1. Similarly to Theorem E.52, the expression (E.4.47) makes sense near L and its eventual negativity is invariant. Moreover, the threshold condition is satisfied for large negative s – see Exercise E.33.

2. A stronger statement is available: if u is merely a distribution and $Bu \in H^s_{\mathrm{comp}}(M)$, $B_1 f \in H^{s-k+1}_{\mathrm{comp}}(M)$, then $Au \in H^s_{\mathrm{comp}}$. See Exercise E.36.

Proof. As in the proof of Theorem E.52, choose $b \in C^\infty(\overline{T}^*M;\mathbb{R})$ and a neighbourhood U of L such that for some constant $\delta > 0$,

$$U \subset \operatorname{ell}_h(B_1), \tag{E.4.49}$$

$$\langle\xi\rangle^{-k}\operatorname{Im}\sigma_h(\mathbf{P}) = 0 \quad \text{on } U, \tag{E.4.50}$$

$$\langle\xi\rangle^{1-k}\Big(\sigma_h(h^{-1}\operatorname{Im}\mathbf{P}) + \Big(s + \frac{1-k}{2}\Big)\frac{H_p\langle\xi\rangle}{\langle\xi\rangle} + H_p b\Big) \leq -\delta \quad \text{on } U. \tag{E.4.51}$$

Let $\chi_1, \chi_2 \in C_c^\infty(U; [0,1])$ and $\psi \in C_c^\infty(U \setminus L; [0,1])$ satisfy

$$\chi_2 = 1 \quad \text{near } L, \quad \chi_1 = 1 \quad \text{near } \operatorname{supp}\chi_2,$$
$$\psi = 1 \quad \text{near } \operatorname{supp}\chi_1 \cap \operatorname{supp}(1-\chi_2).$$

Using a quantization procedure $\mathrm{Op}_h = \mathrm{Op}_h^M$ from Proposition E.15, put

$$g := e^b\chi_1, \quad A := \mathrm{Op}_h(\chi_2), \quad B := \mathrm{Op}_h(\psi), \quad B_2 := A + B.$$

Then the assumptions of Lemma E.49 are satisfied, with B_1 replaced by B_2. In particular, the condition (E.4.29) follows from (E.4.51) and the fact that $\operatorname{supp}(H_p\chi_1) \subset \{\psi = 1\}$. The estimate (E.4.30) (using Theorem E.33 to bound $\|B_2 f\|_{H_h^{s-k+1}}$ in terms of $\|B_1 f\|_{H_h^{s-k+1}}$) then gives

$$\begin{aligned}\|Au\|_{H_h^s} \leq{}& C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} \\ &+ Ch^{1/2}\|B_2 u\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned} \tag{E.4.52}$$

Recalling that $B_2 = A + B$, we see that

$$\begin{aligned}\|Au\|_{H_h^s} \leq{}& C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}} \\ &+ Ch^{1/2}\|Au\|_{H_h^{s-1/2}} + \mathcal{O}(h^\infty)\|\chi u\|_{H_h^{-N}}.\end{aligned} \tag{E.4.53}$$

It remains to use Proposition E.21 as in the proof of Theorem E.52. $\square$

E.5. HYPERBOLIC ESTIMATES

In this last section we provide a self-contained account of hyperbolic estimates for second order differential operators. They are used in §5.5.

Before presenting the general case, we discuss these estimates in a simple one-dimensional example, explaining their relation to propagation of singularities. Thus, consider the differential operator

$$P = (hD_t)^2 - 1 = -h^2\partial_t^2 - 1 \in \Psi_h^2(\mathbb{R}). \tag{E.5.1}$$

Take cutoff functions $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ satisfying (see Figure E.5)

$$\chi_1 = 1 \quad \text{near } [1,\infty), \quad \chi_2 = 1 \quad \text{near } (-\infty, 1]. \tag{E.5.2}$$

Then the elliptic estimate (Theorem E.33) and propagation of singularities (Theorem E.47) together imply for every N (see Exercise E.38)

$$\begin{aligned}\|(1-\chi_1)u\|_{H_h^1([0,\infty))} \leq{}& Ch^{-1}\|\chi_2 Pu\|_{L^2(\mathbb{R})} + C\|\chi_1\chi_2 u\|_{H_h^1(\mathbb{R})} \\ &+ Ch^N\|u\|_{L^2([-1,2])}.\end{aligned} \tag{E.5.3}$$

Here we assume for simplicity that $u \in C^\infty(\mathbb{R})$. However, by direct ODE analysis we can obtain an estimate without the $h^N\|u\|_{L^2([-1,2])}$ remainder:

$$\|(1-\chi_1)u\|_{H_h^1([0,\infty))} \leq Ch^{-1}\|\chi_2 Pu\|_{L^2([0,\infty))} + C\|\chi_1\chi_2 u\|_{H_h^1(\mathbb{R})}. \tag{E.5.4}$$

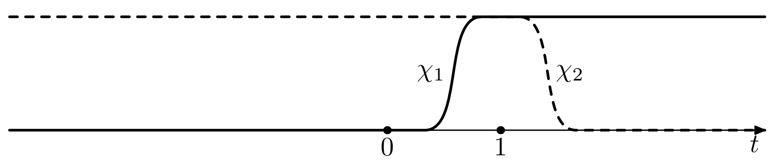


Figure E.5. The cutoffs χ_1, χ_2 from (E.5.2).

It is easy to see (E.5.4) in the special case $Pu = 0$, since then $u = c_+e^{it/h} + c_-e^{-it/h}$ for some $c_\pm \in \mathbb{C}$ and the term $\|\chi_1\chi_2 u\|_{H_h^1(\mathbb{R})}$ controls $|c_\pm|$. See Exercise E.39 for the general case.

The estimate (E.5.4) is stronger than (E.5.3) but it holds under more restrictive assumptions. Indeed, if we instead consider $P = (hD_t)^2 + 1$ then (E.5.3) still holds (we can even remove the term $C\|\chi_1\chi_2 u\|_{H_h^1(\mathbb{R})}$ since P is semiclassically elliptic everywhere), but (E.5.4) fails even when $Pu = 0$ (taking for instance $u = e^{-t/h}$).

Coming back to (E.5.1), we also have the estimate for all $v \in C^\infty(\mathbb{R})$

$$\|(1 - \chi_1)v\|_{H_h^1(\mathbb{R})} \leq Ch^{-1}\|\chi_2 Pv\|_{L^2(\mathbb{R})} \quad \text{if} \quad \operatorname{supp} v \subset [0, \infty). \tag{E.5.5}$$

The estimates (E.5.4)–(E.5.5) can be interpreted in terms of uniqueness for the initial value problem for the operator P: (E.5.4) means that if $Pu = 0$ on $[0, 1]$, then $u|_{[0,1]}$ is controlled by the Cauchy data $(u(1), hu'(1))$. Similarly (E.5.5) means that if $Pv = 0$ on $[0, 1]$ and the Cauchy data $(v(0), hv'(0))$ is equal to 0, then $v = 0$ on $[0, 1]$.

The main results of this section, Theorems E.56 and E.57, prove estimates of the form (E.5.4) and (E.5.5) when P is a hyperbolic operator. These are related to well-posedness of the initial value problem for the wave equation (see Theorem E.61 below).

E.5.1. Statements of the estimates. Throughout this section we assume that $\overline{M}$ is a compact manifold with interior M and boundary ∂M, fix a boundary defining function $t : \overline{M} \to [0, \infty)$ (see Definition 5.1) and a product structure (see (5.1.2))

$$(t, y) : t^{-1}\big([0, 1)\big) \to [0, 1)_t \times (\partial M)_y. \tag{E.5.6}$$

We use this product structure to identify $\{t < 1\} = t^{-1}([0, 1)) \subset \overline{M}$ with $[0, 1) \times \partial M$. (The theorems below apply also to the case when $\overline{M}$ is noncompact but has a compact boundary. The example (E.5.1) corresponds to $\overline{M} = [0, \infty)$.) Using Definition E.25, we consider the Sobolev spaces

$$\bar{H}^s(M), \quad \dot{H}^s(\overline{M}),$$

and the semiclassical norms $\|\bullet\|_{\bar{H}_h^s(M)}$, $\|\bullet\|_{\dot{H}_h^s(\overline{M})}$ on these spaces.

We assume that

$$P \in \mathrm{Diff}_h^2(M)$$

is a second order semiclassical differential operator (see §E.1.1) with coefficients which are smooth up to the boundary ∂M. Then P is bounded uniformly in h on the following spaces:

$$P : \bar{H}_h^s(M) \to \bar{H}_h^{s-2}(M), \quad P : \dot{H}_h^s(\overline{M}) \to \dot{H}_h^{s-2}(\overline{M}).$$

The semiclassical principal symbol of P is a second order polynomial in ξ,

$$p := \sigma_h(P) \in \mathrm{Poly}^2(T^*M).$$

We also consider the *nonsemiclassical principal symbol* p_0, the leading part of p which is a homogeneous second order polynomial in ξ. On $\{t < 1\}$, p and p_0 are functions of (t, y, τ, η) where $\tau \in \mathbb{R}$, $\eta \in T_y^*(\partial M)$ are the momentum variables corresponding to $t \in [0, 1)$, $y \in \partial M$ and

$$p_0(t, y, \tau, \eta) := \tfrac{1}{2}\langle [\partial^2_{(\tau,\eta)} p(t, y, 0, 0)](\tau, \eta), (\tau, \eta)\rangle.$$

DEFINITION E.55 (Hyperbolic operators). *We say that:*

• *P is **hyperbolic** with respect to t on $\{t < 1\}$, if p is real-valued and for each $(t, y, \eta) \in [0, 1) \times T^*(\partial M)$, $\eta \neq 0$, the equation*

$$p_0(t, y, \tau, \eta) = 0, \quad \tau \in \mathbb{R}, \tag{E.5.7}$$

has two distinct real solutions $\tau_\pm$.

• *P is **semiclassically hyperbolic** with respect to t on $\{t < 1\}$, if it is hyperbolic and for each $(t, y, \eta) \in [0, 1) \times T^*(\partial M)$, the equation*

$$p(t, y, \tau, \eta) = 0, \quad \tau \in \mathbb{R}, \tag{E.5.8}$$

has two distinct real solutions $\tau_\pm$.

EXAMPLE. Consider the following operator on $\overline{M} = [0, 1)_t \times \mathbb{S}^1_y$:

$$P = (hD_t)^2 - (hD_y)^2 + V(t, y), \quad V \in C^\infty(\overline{M}; \mathbb{R});$$
$$p(t, y, \tau, \eta) = \tau^2 - \eta^2 + V(t, y), \quad p_0(t, y, \tau, \eta) = \tau^2 - \eta^2.$$

Then P is hyperbolic for any V, but semiclassically hyperbolic only if $V < 0$ on $\overline{M}$. As another example, the operator $(hD_t)^2 - 1$ on the same manifold M is not semiclassically hyperbolic (since it is not hyperbolic), even though (E.5.8) has two real solutions for each (t, y, η).

We now state the main estimates of this section. We start with the case when P is hyperbolic and the constants in the estimates depend on h:

THEOREM E.56 (**Hyperbolic estimate I).** *Assume that P is hyperbolic with respect to t on $\{t < 1\}$. Take $\chi_1, \chi_2 \in C^\infty(\overline{M})$ satisfying*

$$\chi_1 = 1 \quad \text{near } \{t \geq 1\}, \qquad \chi_2 = 1 \quad \text{near } \{t \leq 1\}. \tag{E.5.9}$$

Then for each $u \in \bar{H}^s(M)$ such that $Pu \in \bar{H}^{s-1}(M)$, we have

$$\|(1-\chi_1)u\|_{\bar{H}^s(M)} \leq C\|\chi_2 Pu\|_{\bar{H}^{s-1}(M)} + C\|\chi_1\chi_2 u\|_{\bar{H}^s(M)}, \tag{E.5.10}$$

and for each $v \in \dot{H}^s(\overline{M})$ such that $Pv \in \dot{H}^{s-1}(\overline{M})$, we have

$$\|(1-\chi_1)v\|_{\dot{H}^s(\overline{M})} \leq C\|\chi_2 Pv\|_{\dot{H}^{s-1}(\overline{M})}. \tag{E.5.11}$$

In both cases the constant C may depend on h.

REMARKS. 1. The estimates (E.5.10), (E.5.11) are analogous to the well-posedness of the Cauchy problem for hyperbolic equations. Indeed, the norm $\|\chi_1\chi_2 u\|_{\bar{H}^s(M)}$ on the right-hand side of (E.5.10) controls the behaviour of u near $\{t = 1\}$, and the left-hand side of (E.5.10) controls the norm of u in the region $\{t < 1 - \varepsilon\}$ provided $\operatorname{supp} \chi_1 \subset \{t > 1 - \varepsilon\}$. For (E.5.11), the requirement that $v \in \dot{H}^s(\overline{M})$ corresponds to the vanishing of $v, \partial_t v$ on $\{t = 0\}$. The somewhat peculiar formulation of the estimates (E.5.10), (E.5.11) is useful in proving the Fredholm property of the modified Laplacian in §5.6, see §5.5.2.

2. A stronger version is available, without assuming a priori regularity of u, v – see Exercise E.41.

In general the constants in (E.5.10), (E.5.11) may grow exponentially fast in h, as can be seen by considering the operator $P = -h^2 D_t^2 + 1$ on $\overline{M} = [0, \infty)$. However, when P is semiclassically hyperbolic, we can control these constants uniformly as $h \to 0$:

THEOREM E.57 (**Hyperbolic estimate II**). *Assume that P is semiclassically hyperbolic with respect to t on $\{t < 1\}$. Take cutoff functions χ_1, χ_2 satisfying (E.5.9). Then for each u, v as in Theorem E.56,*

$$\|(1-\chi_1)u\|_{\bar{H}_h^s(M)} \leq Ch^{-1}\|\chi_2 Pu\|_{\bar{H}_h^{s-1}(M)} + C\|\chi_1\chi_2 u\|_{\bar{H}_h^s(M)}, \tag{E.5.12}$$

$$\|(1-\chi_1)v\|_{\dot{H}_h^s(\overline{M})} \leq Ch^{-1}\|\chi_2 Pv\|_{\dot{H}_h^{s-1}(\overline{M})}. \tag{E.5.13}$$

In both cases the constant C is independent of h.

E.5.2. Energy estimate and well-posedness. We start the proofs of Theorems E.56 and E.57 with an energy estimate, proved by a semiclassical version of the factorization method presented in [**HöIII**, §23.2]. In this section we work on the manifold with boundary $[t_0, t_1]_t \times \partial M_y$ where $t_0 < t_1$. (We could in fact replace ∂M with any compact manifold but we keep the notation to be consistent with §E.5.1.) We consider a differential operator $P \in \mathrm{Diff}_h^2([t_0, t_1] \times \partial M)$ with coefficients smooth up to the boundary and say that P is (semiclassically) hyperbolic if it is (semiclassically) hyperbolic with respect to t on the entire $[t_0, t_1] \times \partial M$, see Definition E.55.

Denote $H_y^s := H^s(\partial M)$. For $j \in \mathbb{N}_0$ and $s \in \mathbb{R}$ we use the spaces

$$\begin{gathered} C_t^j H_y^s = C^j([t_0, t_1]; H^s(\partial M)) \subset \mathcal{D}'((t_0, t_1) \times \partial M), \\ \|u\|_{C^j([t_0,t_1];H^s(\partial M))} = \max_{0 \le \ell \le j} \sup_{t \in [t_0,t_1]} \|D_t^\ell u(t)\|_{H^s(\partial M)}. \end{gathered} \tag{E.5.14}$$

For $k \in \mathbb{R}$, denote by $C_t^\infty \Psi_h^k(\partial M)$ the class of operators in $\Psi_h^k(\partial M)$ depending smoothly on $t \in [t_0, t_1]$. Then $D_t : C_t^{j+1} H_y^s \to C_t^j H_y^s$ and any operator in $C_t^\infty \Psi_h^k(\partial M)$ maps $C_t^j H_y^s \to C_t^j H_y^{s-k}$. Later we will also use the spaces

$$L_t^q H_y^s := L^q((t_0, t_1); H^s(\partial M)). \tag{E.5.15}$$

For $u \in C_t^1 H_y^s$ define the quantity

$$E_{s,u}(t) = \|u(t)\|_{H_h^s(\partial M)} + h\|D_t u(t)\|_{H_h^{s-1}(\partial M)}, \quad t \in [t_0, t_1]. \tag{E.5.16}$$

LEMMA E.58 (Energy estimate). *Let $s \in \mathbb{R}$. Then:*

1. Assume that P is semiclassically hyperbolic. Then, in the notation of (E.5.16), for all $t \in [t_0, t_1]$ and $u \in C^2([t_0, t_1]; H^{s+1}(\partial M))$

$$E_{s,u}(t) \le C E_{s,u}(t_0) + C h^{-1} \int_{t_0}^t \|Pu(r)\|_{H_h^{s-1}(\partial M)}\, dr \tag{E.5.17}$$

with the constant C independent of h, t, u.

2. Assume that P is hyperbolic. Then the estimate (E.5.17) holds, but with C depending on h.

REMARKS. 1. Note that we make a stronger regularity assumption on u than requiring the right-hand side of (E.5.17) to be finite. This is the reason for the complications in the proof of well-posedness of the Cauchy problem (Theorem E.61) below.

2. It is not hard to show that Lemma E.58 implies Theorems E.56 and E.57 in the special case $s = 1$, $u \in C^\infty(\overline{M})$, $v \in C_c^\infty(\overline{M})$. However we need the general case which requires the additional arguments in §E.5.3.

Proof. We assume that P is semiclassically hyperbolic, indicating in Step 4 below what changes should be made for the second statement of the lemma.

1. The coefficient of $-h^2 \partial_t^2 = (hD_t)^2$ in P is h-independent and nonvanishing. Multiplying P by a nonvanishing function, we may assume that this coefficient is equal to 1. Since P is semiclassically hyperbolic, using the quadratic formula we factorize

$$p(t, y, \tau, \eta) = \big(\tau - a_1(t, y, \eta)\big)\big(\tau - a_2(t, y, \eta)\big) \tag{E.5.18}$$

where the functions $a_j(t,y,\eta)$ are real-valued symbols in $S^1(T^*\partial M)$ smooth in $t\in[t_0,t_1]$ and satisfying for some $c>0$

$$a_2(t,y,\eta)-a_1(t,y,\eta)\geq c\langle\eta\rangle. \tag{E.5.19}$$

(The lower bound $\sim\langle\eta\rangle$ follows from standard hyperbolicity which is part of the semiclassical definition.)

Using a quantization procedure $\mathrm{Op}_h=\mathrm{Op}_h^{\partial M}$ on ∂M (see Proposition E.15), consider the operators

$$A_j(t)=\mathrm{Op}_h(a_j(t,\bullet))\in C_t^\infty\Psi_h^1(\partial M).$$

We have the approximate factorization

$$\begin{aligned}P&=\big(hD_t-A_1(t)\big)\big(hD_t-A_2(t)\big)\\&\quad+hC_t^\infty\Psi_h^0(\partial M)\,hD_t+hC_t^\infty\Psi_h^1(\partial M)\end{aligned}\tag{E.5.20}$$

and same is true when $A_1(t),A_2(t)$ switch places.

2. We first prove the following estimate for the operators $hD_t-A_j(t)$, valid for any $s\in\mathbb{R}$, $j=1,2$, $v\in C_t^1H_y^{s+1}$, and $t\in[t_0,t_1]$:

$$\begin{aligned}\|v(t)\|_{H_h^s(\partial M)}\leq{}&C\|v(t_0)\|_{H_h^s(\partial M)}\\&+Ch^{-1}\int_{t_0}^t\|(hD_r-A_j(r))v(r)\|_{H_h^s(\partial M)}\,dr.\end{aligned}\tag{E.5.21}$$

To show (E.5.21), take an invertible elliptic operator $Y_s\in\Psi_h^s(\partial M)$ (see Exercise E.11) and put

$$F(t):=\|Y_sv(t)\|_{L^2(\partial M)},\quad C^{-1}\|v(t)\|_{H_h^s(\partial M)}\leq F(t)\leq C\|v(t)\|_{H_h^s(\partial M)}.$$

Since a_j is real-valued, we have $\mathrm{Im}(Y_s^*Y_sA_j(t))\in hC_t^\infty\Psi_h^{2s}(\partial M)$. Therefore

$$\begin{aligned}h\partial_t\big(F(t)^2\big)&=-2\,\mathrm{Im}\langle Y_shD_tv(t),Y_sv(t)\rangle_{L^2(\partial M)}\\&\leq-2\,\mathrm{Im}\langle Y_s(hD_t-A_j(t))v(t),Y_sv(t)\rangle_{L^2(\partial M)}+Ch\|v(t)\|^2_{H_h^s(\partial M)}\\&\leq CF(t)\,\|(hD_t-A_j(t))v(t)\|_{H_h^s(\partial M)}+ChF(t)^2.\end{aligned}$$

The function $F_\varepsilon(t):=\sqrt{F(t)^2+\varepsilon}$ is in $C^1([t_0,t_1])$ for all $\varepsilon>0$ and

$$\partial_tF_\varepsilon(t)\leq Ch^{-1}\|(hD_t-A_j(t))v(t)\|_{H_h^s(\partial M)}+CF(t).$$

Integrating and letting $\varepsilon\to0$, we obtain for all $t\in[t_0,t_1]$

$$F(t)\leq F(t_0)+Ch^{-1}\int_{t_0}^t\|(hD_r-A_j(r))v(r)\|_{H_h^s(\partial M)}\,dr+C\int_{t_0}^tF(r)\,dr.$$

Grönwall's inequality (A.5.1) (applied with $B(t)\equiv C$) gives (E.5.21).

3. Assume that $u \in C_t^2 H_y^{s+1}$. Applying (E.5.21) (with s replaced by $s-1$) to the operator $hD_t - A_{3-j}(t)$ and $v(t) := (hD_t - A_j(t))u(t)$, $j = 1, 2$, and using (E.5.20), we obtain for all $t \in [t_0, t_1]$,

$$\begin{aligned}\|(hD_t - A_j(t))u(t)\|_{H_h^{s-1}(\partial M)} &\le CE_{s,u}(t_0)\\ &+ Ch^{-1}\int_{t_0}^t \|Pu(r)\|_{H_h^{s-1}(\partial M)}\,dr + C\int_{t_0}^t E_{s,u}(r)\,dr.\end{aligned} \tag{E.5.22}$$

By (E.5.19), the operator $A_1(t) - A_2(t)$ is elliptic in the class $\Psi_h^1(\partial M)$ and hence, by the elliptic estimate (Theorem E.33), we have for all $t \in [t_0, t_1]$

$$\begin{aligned}\|u(t)\|_{H_h^s(\partial M)} \le\, &C\|(A_1(t) - A_2(t))u(t)\|_{H_h^{s-1}(\partial M)}\\ &+ Ch\|u(t)\|_{H_h^{s-1}(\partial M)}.\end{aligned} \tag{E.5.23}$$

The first term on the right-hand side is estimated by (E.5.22) and the second one, by $h\|u(t_0)\|_{H_h^{s-1}(\partial M)}$ and the integral of $h\|D_t u(t)\|_{H_h^{s-1}(M)}$; thus

$$\begin{aligned}\|u(t)\|_{H_h^s(\partial M)} \le\, &CE_{s,u}(t_0) + Ch^{-1}\int_{t_0}^t \|Pu(r)\|_{H_h^{s-1}(\partial M)}\,dr\\ &+ C\int_{t_0}^t E_{s,u}(r)\,dr, \quad t \in [t_0, t_1].\end{aligned}$$

We also have

$$h\|D_t u(t)\|_{H_h^{s-1}(\partial M)} \le \|(hD_t - A_1(t))u(t)\|_{H_h^{s-1}(\partial M)} + C\|u(t)\|_{H_h^s(\partial M)}$$

where the first term on the right-hand side is estimated by (E.5.22). Combining the last two estimates, we obtain

$$\begin{aligned}E_{s,u}(t) \le\, &CE_{s,u}(t_0) + Ch^{-1}\int_{t_0}^t \|Pu(r)\|_{H_h^{s-1}(\partial M)}\,dr\\ &+ C\int_{t_0}^t E_{s,u}(r)\,dr, \quad t \in [t_0, t_1].\end{aligned} \tag{E.5.24}$$

The estimate (E.5.17) now follows by Grönwall's inequality (A.5.1).

4. We finally make the weaker assumption that P is hyperbolic and explain how to obtain (E.5.17) with the constant depending on h.

As before, we may assume that the coefficient of $(hD_t)^2$ in P is equal to 1. The discriminant of the quadratic equation $p(t, y, \tau, \eta) = 0$ in τ is asymptotic as $|\eta| \to \infty$ to the discriminant of the equation $p_0(t, y, \tau, \eta) = 0$. Since P is hyperbolic, there exists a constant $C_0 > 0$ such that the operator $P - C_0$ is semiclassically hyperbolic. Thus for all $t \in [t_0, t_1]$

$$E_{s,u}(t) \le CE_{s,u}(t_0) + Ch^{-1}\int_{t_0}^t \|(P - C_0)u(r)\|_{H_h^{s-1}(\partial M)}\,dr.$$

This implies

$$E_{s,u}(t) \le CE_{s,u}(t_0) + Ch^{-1}\int_{t_0}^{t} \|Pu(r)\|_{H_h^{s-1}(\partial M)}\, dr + Ch^{-1}\int_{t_0}^{t} E_{s,u}(r)\, dr.$$

By Grönwall's inequality (A.5.1) now applied with $B(t) = C/h$ we obtain (E.5.17) with the constant $Ce^{C/h}$. □

The rest of this section is devoted to the proof of well-posedness of the Cauchy problem for P – see Theorem E.61 below. Although that is not part of Theorems E.56 and E.57 it is convenient to have it in place. We again adapt the presentation in [**HöIII**, §23.2].

Henceforth we assume that P is hyperbolic. As in the proof of Lemma E.58 we may additionally assume that the coefficient of $(hD_t)^2$ in P is identically equal to 1. We then write

$$P = (hD_t)^2 + (hD_t)P_1 + P_0 \tag{E.5.25}$$

where $P_0 \in \mathrm{Diff}_h^2(\partial M)$, $P_1 \in \mathrm{Diff}_h^1(\partial M)$ depend smoothly on $t \in [t_0, t_1]$.

The following statement will be useful to establish enough regularity in t so that Lemma E.58 can be applied, at the cost of reducing regularity in y. This is an intermediate step in the proof of well-posedness and a stronger statement is eventually valid for solutions to the Cauchy problem (E.5.28). In particular if the right-hand side f and the initial data φ_j lie in C^∞ then we will see that u lies in C^∞ as well – see Theorem E.61.

LEMMA E.59 (**Regularity of weak solutions**). *Assume that $u \in L_t^2 H_y^s$ satisfies $Pu = f$ in the sense of distributions on $(t_0, t_1) \times \partial M$, and $f \in C^\infty([t_0, t_1] \times \partial M)$. Then we have $u \in C_t^j H_y^{s-2-j}$ for all $j \in \mathbb{N}_0$.*

Proof. 1. We need to prove the statement for every fixed h, and for simplicity we may put $h := 1$. It suffices to prove that for all $j \in \mathbb{N}_0$

$$D_t^j u \in L_t^2 H_y^{s-1-j} \tag{E.5.26}$$

where D_t^j is defined in the sense of distributions on $(t_0, t_1) \times \partial M$. Indeed, (E.5.26) implies that $u \in C_t^{j-1} H_y^{s-1-j}$ by Sobolev embedding in one dimension (writing $D_t^{j-1}u$ as the sum of an antiderivative of $D_t^j u$ and a distribution constant in t similarly to step 2 below).

2. For $j = 0$, (E.5.26) is immediate. Next, recalling the definition (E.5.25) of P_0, P_1 and using that $f - P_0 u \in L_t^2 H_y^{s-2}$, define

$$w_1 \in C_t^0 H_y^{s-2}, \quad w_1(t, y) := \int_{t_0}^{t} (f(r, y) - P_0 u(r, y))\, dr.$$

Put

$$w_2 := D_t u + P_1 u - i w_1 \in \mathcal{D}'((t_0, t_1) \times \partial M). \tag{E.5.27}$$

From the definition of w_1 and the equation $Pu = f$ we see that $D_t w_2 = 0$ in the sense of distributions. It follows (see [**HöI**, Theorem 3.1.4$'$]) that w_2 is independent of t, namely $w_2(t, y) = w_3(y)$ where

$$w_3(y) = \int_{t_0}^{t_1} \chi(t) w_2(t, y)\, dt \ \in \ \mathcal{D}'(\partial M)$$

for any fixed $\chi \in C_c^\infty((t_0, t_1))$ satisfying $\int_{\mathbb{R}} \chi(r) dr = 1$. Recalling (E.5.27) we see that

$$w_3(y) = -\int_{t_0}^{t_1} (D_t \chi(t)) u(t, y)\, dt + \int_{t_0}^{t_1} \chi(t)(P_1 u(t, y) - i w_1(t, y))\, dt.$$

Since $u \in L_t^2 H_y^s$ and $P_1 u - i w_1 \in L_t^2 H_y^{s-2}$ we have $w_3 \in H_y^{s-2}$. Writing $D_t u = i w_1 + w_2 - P_1 u$ we see that $D_t u \in L_t^2 H_y^{s-2}$, giving (E.5.26) for $j = 1$.

3. For $j \geq 2$, we argue by induction. Assume that (E.5.26) holds for $0, 1, \dots, j-1$. We write the equation $D_t^{j-2} P u = D_t^{j-2} f$ as

$$D_t^j u = D_t^{j-2} f - D_t^{j-1} P_1 u - D_t^{j-2} P_0 u.$$

From the inductive hypothesis we see that $D_t^j u \in L_t^2 H_y^{s-1-j}$, giving (E.5.26) for j. $\square$

The next statement uses the energy estimate (E.5.17) and the Hahn–Banach theorem to show existence of solutions to the Cauchy problem

$$Pu = f, \quad u|_{t=t_0} = \varphi_0, \quad h D_t u|_{t=t_0} = \varphi_1 \tag{E.5.28}$$

which have weak regularity. Here the equation $Pu = f$ is understood in the sense of distributions on $(t_0, t_1) \times \partial M$.

LEMMA E.60 (**Existence of weak solutions**). *Assume that $s \in \mathbb{R}$, $f \in C^\infty([t_0, t_1] \times \partial M)$, $\varphi_0, \varphi_1 \in C^\infty(\partial M)$. Then there exists a solution u to (E.5.28) in the class*

$$u \in \bigcap_{j \geq 0} C_t^j H_y^{s-j}. \tag{E.5.29}$$

REMARK. In principle the solution u could depend on the choice of s (we have not established uniqueness yet), thus we do not claim that $u \in C^\infty$.

Proof. 1. As before we put $h := 1$. We fix some smooth density on ∂M, which naturally induces a density on $(t_0, t_1) \times \partial M$. Define the following anti-linear form on $C_c^\infty([t_0, t_1) \times \partial M)$ (where P_1 is given by (E.5.25)):

$$\Phi(v) := \langle f, v \rangle_{L^2((t_0,t_1)\times\partial M)} - i\langle \varphi_0, D_t v(t_0) + P_1^* v(t_0) \rangle_{L^2(\partial M)} - i\langle \varphi_1, v(t_0) \rangle_{L^2(\partial M)}.$$

Take arbitrary $s \in \mathbb{R}$. The operator P^* is hyperbolic on $[t_0, t_1] \times \partial M$ with respect to the function $-t$. Thus the energy estimate (E.5.17) (with s replaced by $1 - s$) gives

$$\|v\|_{C_t^0 H_y^{1-s}} + \|v\|_{C_t^1 H_y^{-s}} \leq C\|P^*v\|_{L_t^1 H_y^{-s}} \quad \text{for all } v \in C_c^\infty([t_0, t_1) \times \partial M).$$

Therefore (using that $L_t^2 H_y^{-s} \subset L_t^1 H_y^{-s}$)

$$|\Phi(v)| \leq C\|P^*v\|_{L_t^2 H_y^{-s}} \quad \text{for all } v \in C_c^\infty([t_0, t_1) \times \partial M). \tag{E.5.30}$$

Define the anti-linear form on the space $\{P^*v \mid v \in C_c^\infty([t_0, t_1) \times \partial M)\}$

$$\widetilde{\Phi}(g) := \Phi(v), \quad v \in C_c^\infty([t_0, t_1) \times \partial M), \quad g = P^*v$$

where $\widetilde{\Phi}(g)$ does not depend on the choice of v by (E.5.30).

By (E.5.30) and the Hahn–Banach Theorem, $\widetilde{\Phi}$ extends to a bounded anti-linear form on $L_t^2 H_y^{-s}$. Hence there exists $u \in L_t^2 H_y^s$ such that

$$\Phi(v) = \langle u, P^*v\rangle_{L^2((t_0,t_1)\times\partial M)} \quad \text{for all } v \in C_c^\infty([t_0, t_1) \times \partial M). \tag{E.5.31}$$

2. Taking $v \in C_c^\infty((t_0, t_1) \times \partial M)$ in (E.5.31) and recalling the definition of Φ, we see that $Pu = f$ in $\mathcal{D}'((t_0, t_1) \times \partial M)$. By Lemma E.59 we get $u \in C_t^j H_y^{s-2-j}$ for all $j \in \mathbb{N}_0$. Replacing s by $s + 2$ in the argument in part 1, we obtain a solution to $Pu = f$ in the class (E.5.29).

It remains to show that u satisfies the initial conditions $u(t_0) = \varphi_0$, $D_t u(t_0) = \varphi_1$. Using (E.5.25) with the equation $Pu = f$ and integrating by parts in t, we obtain for all $v \in C_c^\infty([t_0, t_1) \times \partial M)$

$$\begin{aligned} &\langle f, v\rangle_{L^2((t_0,t_1)\times\partial M)} = \langle u, P^*v\rangle_{L^2((t_0,t_1)\times\partial M)} \\ &\quad + i\langle u(t_0), D_t v(t_0) + P_1^* v(t_0)\rangle_{L^2(\partial M)} + i\langle D_t u(t_0), v(t_0)\rangle_{L^2(\partial M)}. \end{aligned}$$

Then (E.5.31) gives

$$\langle u(t_0) - \varphi_0, D_t v(t_0) + P_1^* v(t_0)\rangle_{L^2(\partial M)} + \langle D_t u(t_0) - \varphi_1, v(t_0)\rangle_{L^2(\partial M)} = 0.$$

Since this is true for all $v \in C_c^\infty([t_0, t_1) \times \partial M)$ we obtain the required initial conditions. $\square$

We are now ready to prove

THEOREM E.61 (Well-posedness of the Cauchy problem). *Suppose that P is hyperbolic with respect to t on $[t_0, t_1] \times \partial M$ in the sense of Definition E.55. Fix $h > 0$ and $s \in \mathbb{R}$. Then for any $\varphi_j \in H^{s-j}(\partial M)$, $j = 0, 1$ and $f \in L^1((t_0, t_1); H^{s-1}(\partial M))$ the Cauchy problem (E.5.28) has a unique solution u in the class*

$$u \in C^0([t_0, t_1]; H^s(\partial M)) \cap C^1([t_0, t_1]; H^{s-1}(\partial M)) \tag{E.5.32}$$

and the estimate (E.5.17) holds. Moreover, if P is semiclassically hyperbolic then (E.5.17) holds with constants independent of h.

REMARK. Combining Theorem E.61 and Lemma E.59 we see that if $f \in C^\infty([t_0, t_1] \times \partial M)$ and $\varphi_j \in C^\infty(\partial M)$ then $u \in C^\infty([t_0, t_1] \times \partial M)$.

Proof. 1. We first show existence. Using the notation (E.5.15), choose $f^\ell \in C^\infty([t_0, t_1] \times \partial M)$ and $\varphi_0^\ell, \varphi_1^\ell \in C^\infty(\partial M)$ such that as $\ell \to \infty$,

$$f^\ell \to f \quad \text{in } L^1_t H^{s-1}_y, \quad \varphi_j^\ell \to \varphi_j \quad \text{in } H^{s-j}_y.$$

By Lemma E.60 there exist solutions $u^\ell \in C^2_t H^{s+1}_y$ of the Cauchy problems $Pu^\ell = f^\ell$, $(hD_t)^j u^\ell(t_0) = \varphi_j^\ell$. Applying the energy estimate, Lemma E.58, to the differences $u^\ell - u^m$, we get

$$\begin{aligned} &\|u^\ell - u^m\|_{C^0_t H^s_y} + \|u^\ell - u^m\|_{C^1_t H^{s-1}_y} \\ &\leq C \sum_{j=0}^{1} \|\varphi_j^\ell - \varphi_j^m\|_{H^{s-j}_y} + C\|f^\ell - f^m\|_{L^1_t H^{s-1}_y}. \end{aligned}$$

It follows that u^ℓ is a Cauchy sequence in $C^0_t H^s_y$ and in $C^1_t H^{s-1}_y$. Therefore there exists $u \in C^0_t H^s_y \cap C^1_t H^{s-1}_y$ such that

$$\|u^\ell - u\|_{C^0_t H^s_y} + \|u^\ell - u\|_{C^1_t H^{s-1}_y} \to 0 \quad \text{as } \ell \to \infty.$$

Then u is a solution to the Cauchy problem (E.5.28).

2. Writing the energy estimate (E.5.17) for each u^ℓ and passing to the limit, we obtain the energy bound uniform in $t \in [t_0, t_1]$

$$\begin{aligned} &\|u(t)\|_{H^s_h(\partial M)} + h\|D_t u(t)\|_{H^{s-1}_h(\partial M)} \\ &\leq C\Big(\sum_{j=0}^{1} \|\varphi_j\|_{H^{s-j}_h(\partial M)} + h^{-1}\int_{t_0}^{t} \|f(r)\|_{H^{s-1}_h(\partial M)}\, dr\Big). \end{aligned} \tag{E.5.33}$$

Moreover, if P is semiclassically hyperbolic, then the constants in (E.5.33) are independent of h.

3. It remains to establish uniqueness. To this end, assume that $u \in C^0_t H^s_y \cap C^1_t H^{s-1}_y$ satisfies $Pu = 0$ and $u(t_0) = D_t u(t_0) = 0$. We have $u \in L^2_t H^s_y$, thus by Lemma E.59 we get $u \in C^2_t H^{s-4}_y$. Applying the energy estimate (E.5.17) with s replaced by $s - 5$ we obtain $E_{s-5,u} \equiv 0$ on $[t_0, t_1]$ and thus $u \equiv 0$. $\square$

Theorem E.61 immediately implies the following uniqueness statement for the Cauchy problem in distributions, where we replace vanishing of the Cauchy data by a support property:

PROPOSITION E.62. *Under the conditions of Theorem E.61, assume*

$$u \in \mathcal{D}'((t_0, t_1) \times \partial M), \quad Pu = 0, \quad \operatorname{supp} u \subset (t_0', t_1') \times \partial M$$

where $t_0 \leq t_0' \leq t_1' \leq t_1$ and either $t_0 < t_0'$ or $t_1' < t_1$. Then $u = 0$.

Proof. Without loss of generality we assume that $\operatorname{supp} u \subset (t_0, t_1'] \times \partial M$ where $t_0 < t_1' < t_1$. As in the proof of Lemma E.60, fix a density on ∂M.

Take arbitrary $f \in C_c^\infty((t_0, t_1) \times \partial M)$. Since P^* is hyperbolic, by Theorem E.61 there exists

$$v \in C^\infty([t_0, t_1] \times \partial M), \quad P^* v = f, \quad v|_{t=t_0} = 0, \quad D_t v|_{t=t_0} = 0.$$

We have $\operatorname{supp} f \subset [\tilde t, t_1) \times \partial M$ for some $\tilde t \in (t_0, t_1)$. By the uniqueness for the Cauchy problem for P^* on $[t_0, \tilde t\,] \times \partial M$, we get $\operatorname{supp} v \subset [\tilde t, t_1) \times \partial M$ as well. Thus $\operatorname{supp} u \cap \operatorname{supp} v$ is a compact subset of $(t_0, t_1) \times \partial M$, which gives

$$0 = \langle Pu, v \rangle = \langle u, P^* v \rangle = \langle u, f \rangle.$$

Since f was chosen arbitrary, we see that $u = 0$. $\square$

E.5.3. Proofs of hyperbolic estimates. Theorem E.61 uses the spaces $C_t^j H_y^{s-j}$, while Theorems E.56 and E.57 are stated in the spaces $H^s_{t,y}$. To pass between these spaces, we introduce the family $H^{s,r}$ of Sobolev spaces indexed by two parameters $s, r \in \mathbb{R}$. (See [**HöIII**, Appendix B] for a more detailed introduction in the nonsemiclassical setting.) We first define these on $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_y$: $H^{s,r}(\mathbb{R}^{n+1}) \subset \mathscr{S}'(\mathbb{R}^{n+1})$ has the semiclassical norm

$$\begin{aligned} \|u\|_{H_h^{s,r}(\mathbb{R}^{n+1})} &= \|\langle hD_y \rangle^r \langle (hD_t, hD_y) \rangle^s u\|_{L^2(\mathbb{R}^{n+1})} \\ &= \|\langle h\eta \rangle^r \langle (h\tau, h\eta) \rangle^s \hat u(\tau, \eta)\|_{L^2(\mathbb{R}^{n+1})}. \end{aligned} \tag{E.5.34}$$

Here $\langle h\eta \rangle = \sqrt{1 + |h\eta|^2}$, $\langle (h\tau, h\eta) \rangle = \sqrt{1 + (h\tau)^2 + |h\eta|^2}$, and we use the unitary Fourier transform. The spaces $H_h^{s,r}$ belong to the class of generalized Sobolev spaces in the sense of [**Zw12**, §8.3].

If Op_h is the standard quantization on $\mathbb{R}^{n+1}$ defined in (E.1.18) and a symbol $a(t, y, \tau, \eta)$ satisfies the derivative bounds

$$|\partial^\alpha_{(t,y,\tau,\eta)} a(t, y, \tau, \eta)| \le C_\alpha \langle \eta \rangle^{r'} \langle (\tau, \eta) \rangle^{s'}$$

then by [**Zw12**, Theorem 8.10] we have

$$\|\operatorname{Op}_h(a)\|_{H^{s,r}(\mathbb{R}^{n+1}) \to H^{s-s', r-r'}(\mathbb{R}^{n+1})} \le C. \tag{E.5.35}$$

Similarly to §E.1.8 we define Sobolev spaces of compactly supported distributions $H^{s,r}_{h,\mathrm{comp}}(\mathbb{R} \times \partial M)$ modeled locally on $H_h^{s,r}(\mathbb{R}^{n+1})$. An equivalent norm on $H^{s,r}_{h,\mathrm{comp}}$ (for elements supported in a fixed compact set) is given by $\|Y_r u\|_{H_h^s(\mathbb{R} \times \partial M)}$ where $Y_r \in \Psi_h^r(\partial M)$ is an invertible elliptic operator independent of t (see Exercise E.11).

As in §E.5.2 we will work on the manifold with boundary $[t_0, t_1] \times \partial M$ where $t_0 < t_1$. Define the space

$$\widetilde H^{s,r} = \widetilde H^{s,r}([t_0, t_1) \times \partial M) \subset \mathcal{D}'((-\infty, t_1) \times \partial M)$$

consisting of distributions in $H^{s,r}$ which have supported ($\dot{H}$) behavior on $\{t_0\} \times \partial M$ and extendable ($\bar{H}$) behavior on $\{t_1\} \times \partial M$. More precisely,

$$u \in \widetilde{H}^{s,r} \iff \begin{cases} \exists\, U \in H^{s,r}_{\mathrm{comp}}(\mathbb{R} \times \partial M),\ \operatorname{supp} U \subset [t_0, t_1+1] \times \partial M, \\ \text{such that } u = U|_{(-\infty,t_1)\times\partial M}. \end{cases}$$

The semiclassical norm on $\widetilde{H}^{s,r}$ is defined as

$$\|u\|_{\widetilde{H}^{s,r}_h} := \inf \|U\|_{H^{s,r}_h},$$

with the infimum over all possible extensions U. Similarly we define the space $\widetilde{H}^s_h$, replacing $H^{s,r}_{h,\mathrm{comp}}$ by the usual Sobolev space $H^s_{h,\mathrm{comp}}$.

Using (E.5.35) we have the following properties, with constants in the norm equivalence and the norm bounds uniform in h (see the beginning of §E.5.2 for notation):

- $\widetilde{H}^{s,0}_h = \widetilde{H}^s_h$ and $\widetilde{H}^{0,r}_h = L^2((t_0,t_1); H^r_h(\partial M))$;
- $A \in C^\infty_t \Psi^k_h(\partial M) \implies A : \widetilde{H}^{s,r}_h \to \widetilde{H}^{s,r-k}_h$;
- $hD_t : \widetilde{H}^{s,r}_h \to \widetilde{H}^{s-1,r}_h$.

The local model for $\widetilde{H}^{s,r}_h([t_0,t_1) \times \partial M)$ is

$$\begin{aligned} &\widetilde{H}^{s,r}_h([t_0,t_1)\times\mathbb{R}^n) \\ &= \{U|_{(-\infty,t_1)\times\mathbb{R}^n} : U \in H^{s,r}_h(\mathbb{R}^{n+1}),\ \operatorname{supp} U \subset [t_0,\infty)\times\mathbb{R}^n\}. \end{aligned}$$

(Here the restriction $\operatorname{supp} U \subset [t_0, t_1+1]$ is removed by multiplying by a cutoff function in t. The only reason we made this restriction in the first place is to avoid defining a global norm on $H^{s,r}_h(\mathbb{R} \times \partial M)$.)

Following [**HöIII**, Theorem B.2.4], we next consider the Fourier multiplier on $\mathscr{S}'(\mathbb{R}^{n+1})$

$$\Lambda_{s,r} = \big(ihD_t + \langle hD_y\rangle\big)^s \langle hD_y\rangle^r, \quad s, r \in \mathbb{R}.$$

Here $(i\tau + \langle\eta\rangle)^s$ is well defined as $\operatorname{Re}(i\tau + \langle\eta\rangle) \geq 1$ with the branch chosen positive at $\tau = 0$.

It follows from (E.5.34) that $\Lambda_{s,r} : H^{s',r'}_h(\mathbb{R}^{n+1}) \to H^{s'-s,r'-r}_h(\mathbb{R}^{n+1})$ is a unitary operator for all s', r'.

For any $t_0 \in \mathbb{R}$ the operator $\Lambda_{s,r}$ preserves the space of tempered distributions supported in $[t_0,\infty) \times \mathbb{R}^n$. Indeed, $\Lambda_{s,r}$ is the convolution operator with the inverse Fourier transform of $(ih\tau + \langle h\eta\rangle)^s \langle h\eta\rangle^r$, which is supported in $[0,\infty) \times \mathbb{R}^n$. The latter can be seen either by a direct computation [**HöI**, Example 7.1.17] or by the Paley–Wiener–Schwartz Theorem [**HöI**, Theorem 7.3.1], since for each η, the function $\tau \mapsto (ih\tau + \langle h\eta\rangle)^s$ is holomorphic and polynomially bounded in $\operatorname{Im} \tau \leq 0$.

Since $\widetilde{H}_h^{s,r}([t_0,t_1)\times\mathbb{R}^n)$ is the quotient of the space of distributions in $H_h^{s,r}(\mathbb{R}^{n+1})$ supported in $[t_0,\infty)\times\mathbb{R}^n$ by those supported in $[t_1,\infty)\times\mathbb{R}^n$, we see that $\Lambda_{s,r}$ induces a unitary operator

$$\widetilde{\Lambda}_{s,r}:\widetilde{H}_h^{s',r'}([t_0,t_1)\times\mathbb{R}^n)\to\widetilde{H}_h^{s'-s,r'-r}([t_0,t_1)\times\mathbb{R}^n). \tag{E.5.36}$$

We collect further properties of the spaces $\widetilde{H}_h^{s,r}$ used below in

LEMMA E.63. *We have*

$$s+r\le s'+r',\ s\le s' \implies \|u\|_{\widetilde{H}_h^{s,r}}\le C\|u\|_{\widetilde{H}_h^{s',r'}}, \tag{E.5.37}$$

$$\|u\|_{\widetilde{H}_h^{s,r}}\le C\big(\|u\|_{\widetilde{H}_h^{s-1,r+1}}+\|hD_tu\|_{\widetilde{H}_h^{s-1,r}}\big). \tag{E.5.38}$$

Moreover, for each $f\in\widetilde{H}^{s-1,r}$ there exist $f_j\in\widetilde{H}^{s,r-j}$, $j=0,1$, such that

$$f=f_1+hD_tf_0,\quad \|f_j\|_{\widetilde{H}_h^{s,r-j}}\le C\|f\|_{\widetilde{H}_h^{s-1,r}}. \tag{E.5.39}$$

Proof. 1. The inequality (E.5.37) follows from the inequality $\|u\|_{H_h^{s,r}(\mathbb{R}^{n+1})}\le \|u\|_{H_h^{s',r'}(\mathbb{R}^{n+1})}$ which is immediate from (E.5.34).

2. Applying a partition of unity to u we see that to show (E.5.38) it suffices to prove that

$$\|u\|_{\widetilde{H}_h^{s,r}([t_0,t_1)\times\mathbb{R}^n)}\le\|u\|_{\widetilde{H}_h^{s-1,r+1}([t_0,t_1)\times\mathbb{R}^n)}+\|hD_tu\|_{\widetilde{H}_h^{s-1,r}([t_0,t_1)\times\mathbb{R}^n)}.$$

By (E.5.36) this is equivalent to

$$\|\widetilde{\Lambda}_{s,r}u\|_{L^2}\le\|\widetilde{\Lambda}_{s-1,r+1}u\|_{L^2}+\|\widetilde{\Lambda}_{s-1,r}hD_tu\|_{L^2},$$

and this follows from the identity

$$\widetilde{\Lambda}_{s,r}u=\widetilde{\Lambda}_{s-1,r+1}u+i\widetilde{\Lambda}_{s-1,r}hD_tu.$$

3. Applying a partition of unity to f, we reduce (E.5.39) to the case $f\in\widetilde{H}^{s-1,r}([t_0,t_1)\times\mathbb{R}^n)$. Then we put

$$f=f_1+hD_tf_0,\quad f_0:=i\widetilde{\Lambda}_{-1,0}f,\quad f_1:=\widetilde{\Lambda}_{-1,1}f$$

and use (E.5.36). $\square$

We now show well-posedness for the operator P in the spaces $\widetilde{H}^{s,r}$. Here the support condition at $t=t_0$ forces the Cauchy data to be equal to 0.

THEOREM E.64. *Assume that P is hyperbolic with respect to t on $[t_0,t_1]\times\partial M$ in the sense of Definition E.55. Fix $h>0$ and $s,r\in\mathbb{R}$. Denote $\widetilde{H}^{s,r}=\widetilde{H}^{s,r}([t_0,t_1)\times\partial M)$. Then for any $f\in\widetilde{H}^{s-1,r}$ there exists unique*

$$u\in\widetilde{H}^{s+1,r-1},\quad Pu=f\ \text{in}\ \mathcal{D}'((-\infty,t_1)\times\partial M). \tag{E.5.40}$$

Moreover we have the estimate

$$\|u\|_{\widetilde{H}_h^{s+1,r-1}([t_0,t_1)\times\partial M)} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}([t_0,t_1)\times\partial M)}. \tag{E.5.41}$$

If P is semiclassically hyperbolic, then the constant in (E.5.41) *is independent of h.*

Proof. 1. We extend P to a hyperbolic operator on $[t_0-\varepsilon,t_1]\times\partial M$ for some $\varepsilon>0$. Then uniqueness of solutions to (E.5.40) follows immediately from Proposition E.62 and the fact that $u\in\widetilde{H}^{s+1,r-1}$ implies $\operatorname{supp} u\subset[t_0,t_1)\times\partial M$. Similarly to (E.5.25) we assume that the coefficient of $(hD_t)^2$ in P is equal to 1 and write

$$P=(hD_t)^2+(hD_t)P_1+P_0,\quad P_j\in C_t^\infty\operatorname{Diff}_h^{2-j}(\partial M). \tag{E.5.42}$$

In the estimates below, the constants are independent of h if P is assumed to be semiclassically hyperbolic.

2. We first consider the case $s=1$, establishing weaker regularity of u. We have

$$f\in\widetilde{H}^{0,r}=L^2((t_0,t_1);H^r(\partial M))\subset L^1((t_0-\varepsilon,t_1);H^r(\partial M)),$$

where we extend f by 0 in $(t_0-\epsilon,t_0)\times\partial M$. By Theorem E.61 there exists

$$\begin{gathered}u\in C^0([t_0-\varepsilon,t_1];H^{r+1}(\partial M))\cap C^1([t_0-\varepsilon,t_1];H^r(\partial M)),\\ Pu=f,\quad u|_{t=t_0-\varepsilon}=D_tu|_{t=t_0-\varepsilon}=0,\\ \sup_t\|u(t)\|_{H_h^{r+1}(\partial M)}\le Ch^{-1}\|f\|_{\widetilde{H}_h^{0,r}}.\end{gathered}$$

By the uniqueness for the Cauchy problem for P on $[t_0-\varepsilon,t_0]\times\partial M$, u defines a distribution in $\mathcal{D}'((-\infty,t_1)\times\partial M)$ supported in $[t_0,t_1)\times\partial M$. Since $C^0\subset L^2$ we get

$$\|u\|_{\widetilde{H}_h^{0,r+1}}\le Ch^{-1}\|f\|_{\widetilde{H}_h^{0,r}}. \tag{E.5.43}$$

3. We now consider the case $s\ge 1$. By (E.5.37)

$$\|f\|_{\widetilde{H}_h^{0,s+r-1}}\le C\|f\|_{\widetilde{H}_h^{s-1,r}}. \tag{E.5.44}$$

Let u be the solution to $Pu=f$ constructed in Step 2. Then by (E.5.43) and (E.5.44)

$$\|u\|_{\widetilde{H}_h^{0,s+r}}\le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}}. \tag{E.5.45}$$

To improve the regularity of u we claim the following bound valid for all $m\le s$:

$$\|u\|_{\widetilde{H}_h^{m+1,s+r-m-1}}\le C\|u\|_{\widetilde{H}_h^{m,s+r-m}}+C\|f\|_{\widetilde{H}_h^{s-1,r-1}}. \tag{E.5.46}$$

To show (E.5.46) we use (E.5.42) to write $(hD_t)^2u = f - (hD_t)P_1u - P_0u$ and (E.5.37) to estimate

$$\|(hD_t)^2u\|_{\widetilde{H}_h^{m-1,s+r-m-1}} \le C\|u\|_{\widetilde{H}_h^{m,s+r-m}} + C\|f\|_{\widetilde{H}_h^{s-1,r-1}},$$
$$\|hD_tu\|_{\widetilde{H}_h^{m-1,s+r-m}} \le C\|u\|_{\widetilde{H}_h^{m,s+r-m}}.$$

By (E.5.38) these give

$$\|hD_tu\|_{\widetilde{H}_h^{m,s+r-m-1}} \le C\|u\|_{\widetilde{H}_h^{m,s+r-m}} + C\|f\|_{\widetilde{H}_h^{s-1,r-1}}.$$

Applying (E.5.38) again we obtain (E.5.46).

Now, arguing by induction on m with (E.5.45) as the base and (E.5.46) as the step, we get

$$\|u\|_{\widetilde{H}_h^{m+1,s+r-m-1}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}} \quad \text{for} \quad m = -1, 0, 1, \dots, \lfloor s \rfloor.$$

Applying this with $m := \lfloor s \rfloor$ and using (E.5.37) we get

$$\|u\|_{\widetilde{H}_h^{s,r}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

Finally, by another application of (E.5.46) we get the desired bound (E.5.41):

$$\|u\|_{\widetilde{H}_h^{s+1,r-1}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

4. It remains to consider the case $s < 1$. We handle it by decreasing induction on s. Namely, we assume that the statement of the theorem holds for $s+1$ (and any r) and show it for s. Take $f \in \widetilde{H}^{s-1,r}$ and write using (E.5.39)

$$f = f_1 + hD_tf_0, \quad \|f_j\|_{\widetilde{H}_h^{s,r-j}} \le C\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

By the inductive hypothesis there exist $w_j \in \widetilde{H}_h^{s+2,r-j-1}$ satisfying

$$Pw_j = f_j, \quad \|w_j\|_{\widetilde{H}_h^{s+2,r-j-1}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

Put $u_1 := w_1 + hD_tw_0$, then by (E.5.37)

$$\|u_1\|_{\widetilde{H}_h^{s+1,r-1}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}},$$
$$Pu_1 - f = g := [P, hD_t]w_0,$$
$$\|g\|_{\widetilde{H}_h^{s,r-1}} \le Ch\|w_0\|_{\widetilde{H}_h^{s+1,r}} \le C\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

Here we used that $[P, hD_t] \in (hD_t)hC_t^\infty \operatorname{Diff}_h^1 + hC_t^\infty \operatorname{Diff}_h^2$ by (E.5.42).

Using the inductive hypothesis again, we find u_2 such that

$$Pu_2 = -g, \quad \|u_2\|_{\widetilde{H}_h^{s+2,r-2}} \le Ch^{-1}\|f\|_{\widetilde{H}_h^{s-1,r}}.$$

It remains to put $u := u_1 + u_2$ and recall the uniqueness proved in step 1. $\square$

We can finally give

Proofs of Theorems E.56 and E.57. We give a single proof of both theorems, noting that the estimates below are uniform in h if we assume that P is semiclassically hyperbolic. We fix cutoffs χ_1, χ_2 satisfying (E.5.9), put $t_0 := 0$, and take $t_1 \in (0,1)$ such that $\operatorname{supp}(1-\chi_1) \subset \{t < t_1\}$.

1. First assume that $u \in \bar H^s(M)$ and $Pu \in \bar H^{s-1}(M)$. Fix

$$\chi \in C^\infty(\overline{M}), \quad \operatorname{supp}(1-\chi_1) \cap \operatorname{supp}(1-\chi) = \emptyset, \quad \operatorname{supp}\chi \subset \{t < t_1\}.$$

Put $w := \chi u$, then $\operatorname{supp} w \subset \{t < t_1\}$. Thus w lies in the space $\widetilde H^s((0,t_1] \times \partial M)$ defined similarly to $\widetilde H^s([t_0,t_1) \times \partial M)$ but with the roles of t_0, t_1 reversed: w is extendable on $t = 0$ and supported on $t = t_1$. Similarly we have

$$f := Pw = \chi P u + [P,\chi] u \in \widetilde H^{s-1}((0,t_1] \times \partial M).$$

Applying Theorem E.64 (with t replaced by $-t$ and $r := 0$) to w, and using that $\widetilde H^{s+1,-1} \subset \widetilde H^s$ by (E.5.37), we get the bound

$$\|w\|_{\widetilde H^s_h((0,t_1]\times\partial M)} \leq C h^{-1} \|f\|_{\widetilde H^{s-1}_h((0,t_1]\times\partial M)}.$$

Combining this with the estimates (where in the second estimate below we use that $\chi_1\chi_2 = 1$ near the support of $[P,\chi] \in h\,\mathrm{Diff}^1_h(\overline{M})$)

$$\begin{aligned}\|(1-\chi_1)u\|_{\bar H^s_h(M)} &\leq C\|w\|_{\widetilde H^s_h((0,t_1]\times\partial M)},\\ \|f\|_{\widetilde H^{s-1}_h((0,t_1]\times\partial M)} &\leq C\|\chi_2 P u\|_{\bar H^{s-1}_h(M)} + Ch\|\chi_1\chi_2 u\|_{\bar H^s_h(M)}\end{aligned}$$

we get (E.5.10), (E.5.12).

2. Now assume that $v \in \dot H^s(\overline{M})$ and $Pv \in \dot H^{s-1}(\overline{M})$. Let w be the restriction of v to $[0,t_1) \times \partial M$. Then by Theorem E.64 we get

$$\|w\|_{\widetilde H^s_h([0,t_1)\times\partial M)} \leq C h^{-1} \|Pv\|_{\widetilde H^{s-1}_h([0,t_1)\times\partial M)}$$

Combining this with the estimates

$$\begin{aligned}\|(1-\chi_1)v\|_{\dot H^s_h(\overline{M})} &\leq C\|w\|_{\widetilde H^s_h([0,t_1)\times\partial M)},\\ \|Pv\|_{\widetilde H^{s-1}_h([0,t_1)\times\partial M)} &\leq C\|\chi_2 P v\|_{\dot H^{s-1}_h(\overline{M})}\end{aligned}$$

we get (E.5.11), (E.5.13). □

E.6. NOTES

For a fixed h, say $h = 1$, the theory presented here is the standard theory of pseudodifferential operators known as *microlocal analysis*. Good introductions include Alinhac–Gérard [**AG07**] and Grigis–Sjöstrand [**GS94**]. A major treatise is Hörmander [**HöI**]–[**HöIV**].

The semiclassical theory with various applications is presented in several texts: Robert [**Ro05**], Helffer [**He88**] (potential wells, Witten complex), Martinez [Ma02a] (FBI transform), Dimassi–Sjöstrand [DS99] (fine spectral asymptotics), Zworski [Zw12] (broad introduction, semiclassical defect measures), Guillemin–Sternberg [GS13] (functorial approach, Maslov indices), Combescure–Robert [CR12] (coherent states). The reader can consult these works for history of the subject and further references. Here we only mention that in the semiclassical setting, the defect measures of §E.3 were introduced by Gérard [Gé91] and Lions–Paul [LP93].

Although most of the material reviewed in this appendix can be found in the abovementioned texts, Chapter 5 requires an extension of the theory in which propagation estimates are provided uniformly in the standard microlocal *and* semiclassical senses. This includes our definition of the semiclassical wave front as subset of $T^*M \cup \partial\overline{T}^*M$ coming from [Dy12] and [DZ16] and radial propagation estimates, also from [DZ16]. The latter were originally presented in the context of scattering theory by Melrose [Me94] and were first used for scattering on asymptotically hyperbolic manifolds by Vasy [Va13]. See Dyatlov–Guillarmou [DG14] for more general propagation estimates.

The presentation of semiclassical hyperbolic estimates in §E.5 is a self-contained adaptation of Hörmander's treatment in [HöIII, §23.2 and Appendix B.2].

E.7. EXERCISES

Section E.1

1. Let M be a manifold and $A : C^\infty(M) \to C^\infty(M)$ a linear operator. For $f \in C^\infty(M)$, let $\operatorname{ad}_f(A) = [f, A]$ be the commutator of A with the multiplication operator by f. Let $k \in \mathbb{N}_0$.

(a) Show that $A = A(h) \in \mathrm{Diff}_h^k(M)$ if and only if $\operatorname{ad}_f A \in h\,\mathrm{Diff}_h^{k-1}(M)$, $h\partial_h A - kA \in \mathrm{Diff}_h^{k-1}(M)$ for each $f \in C^\infty(M)$, with $\mathrm{Diff}_h^{-1}(M) := 0$.

(b) Let $A \in \mathrm{Diff}_h^k(M)$. Show that $\sigma_h(A) = a_0 + a_1 + \cdots + a_k$, where a_ℓ is a homogeneous polynomial of degree ℓ on the fibers of T^*M satisfying

$$(\operatorname{ad}_f^\ell A)g(x) = \ell!(ih)^\ell a_\ell(x, df(x))g(x) + \mathcal{O}(h^{\ell+1})$$

for all h-independent $f, g \in C^\infty(M)$.

2. Let M be a compact manifold, $a \in C^\infty(T^*M)$, and fix a Riemannian metric g on M. Show that $a \in S^0_{1,0}(T^*M)$ if and only if

$$\sup_{\tau>1} \|a_\tau\|_{C^N(\{1\le|\xi|_g\le 2\})} < \infty$$

for each N, where $a_\tau(x, \xi) := a(x, \tau\xi)$.

3. Suppose $a \in S^k_{1,0}(T^*M)$ for some $k > 0$ and $|a| \le 1$ everywhere. Show that $a \in S^\varepsilon_{1,0}(T^*M)$ for all $\varepsilon > 0$.

4. Construct a symbol $a \in \overline{S}^0_{1,0}(T^*\mathbb{R})$ and a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $a(\varphi(x), \varphi'(x)^{-1}\xi)$ does not lie in $\overline{S}^0_{1,0}(T^*\mathbb{R})$.

5. This exercise shows that for each fixed h, the class of operators $\Psi^k_h(M)$ does not depend on h, and explains how to change semiclassical parameter in quantization. Let $h, \tau \in (0, 1]$, $a \in S^k(T^*M)$ be h-independent, and Op^M_h be a quantization procedure defined in Proposition E.15. Show that

$$\operatorname{Op}^M_{\tau h}(a) = \operatorname{Op}^M_h(a_\tau) \quad \text{where} \quad a_\tau(x, \xi) := a(x, \tau\xi). \tag{E.7.1}$$

6. This exercise outlines an alternative proof of Proposition E.19.

(a) Use (E.1.50) to reduce to case $k = s = 0$.

(b) Fix constants $N \in \mathbb{N}_0$, $C_0 > \sup|a|$. Use induction on N to construct $b_N \in \overline{S}^0_{1,0}(T^*\mathbb{R}^n)$, $r_N \in \overline{S}^{-N}_{1,0}(T^*\mathbb{R}^n)$ such that

$$C_0^2 = \operatorname{Op}_h(a)^* \operatorname{Op}_h(a) + \operatorname{Op}_h(b_N)^* \operatorname{Op}_h(b_N) + h^N \operatorname{Op}_h(r_N).$$

(c) Show that for $r \in \overline{S}^{-N}_{1,0}(T^*\mathbb{R}^n)$ and $N \ge n+1$, the norm $\|\operatorname{Op}_h(r)\|_{L^2 \to L^2}$ is bounded uniformly in h. (Hint: use integration by parts in ξ to show that the Schwartz kernel $\mathcal{K}_{\operatorname{Op}_h(r)}(x, y)$ is $\mathcal{O}(h^{-n}\langle(x-y)/h\rangle^{-R})$ for each $R \in \mathbb{N}_0$. Then apply Schur's inequality (A.5.3).)

(d) Argue similarly to the proof of Proposition E.24 to show

$$\|\operatorname{Op}_h(a)u\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathscr{S}(\mathbb{R}^n),$$

where C is independent of u and h, and finish the proof.

7. Let $\overline{M}$ be a compact manifold with boundary $\partial M \ne \emptyset$ and interior M. Consider the map

$$T_s : \dot{H}^s(\overline{M}) \to \bar{H}^s(M), \quad u \mapsto u|_M. \tag{E.7.2}$$

(a) Assume that $s \in (-\frac{1}{2}, \frac{1}{2})$. Show that T_s is an isomorphism. (Hint: show that multiplication operators by $\mathbf{1}_M$ and $\mathbf{1}_{M_{\mathrm{ext}} \setminus M}$ extend to bounded operators on $H^s(M_{\mathrm{ext}})$, see e.g. [TaI, Proposition 4.5.3]. From here deduce the bound

$$\|u\|_{H^s} + \|v\|_{H^s} \le C\|u + v\|_{H^s}, \quad u \in \dot{H}^s(\overline{M}),\ v \in \dot{H}^s(M_{\mathrm{ext}} \setminus M).$$

This shows that T_s is injective and has closed range. To show surjectivity, use that $\dot{H}^s$ is dual to $\bar{H}^{-s}$.)

(b) Assume that $s < -\frac{1}{2}$. Show that T_s is not injective. (Hint: take a delta function.)

(c) Assume that $s > \frac{1}{2}$. Show that T_s is not surjective.

Section E.2

8. Let (M,g) be a Riemannian manifold and $u \in \mathcal{D}'(M)$. Use Theorem E.33 to show that if $\Delta_g u$ is smooth, then so is u. (Hint: let $P = -h^2\Delta_g$ and use the fact that $A \in \Psi_h^0(M)$ is smoothing when $\mathrm{WF}_h(A) \cap \partial\overline{T}^*M = \emptyset$.)

Is the result still true if (M,g) is a Lorentzian manifold (see §5.7) and Δ_g is replaced by the d'Alembert–Beltrami operator?

9. Use Theorem E.33 to show that for each $\chi_0 \in C_c^\infty(\mathbb{R})$, there exists $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\|\chi_0 u\|_{L^2} \le C\|\chi(hD_x + i)u\|_{L^2} + \mathcal{O}(h^\infty)\|\chi u\|_{L^2} \tag{E.7.3}$$

for all $u \in C^\infty(\mathbb{R})$. Give a direct proof of (E.7.3) and show that the $\mathcal{O}(h^\infty)$ term there cannot be removed.

10. This exercise introduces a family of regularizing operators which is used in Exercises E.31, E.35, and E.36 to show stronger versions of propagation of singularities and radial estimates.

We fix $h := 1$ and denote by $\Psi_0^k(M)$ the class of nonsemiclassical pseudodifferential operators with symbols in $S_{1,0}^k(T^*M)$, see the remark at the end of §E.1.7. We use a regularization parameter $\varepsilon \in (0,1]$ and let $\Psi_{0,\varepsilon}^k(M)$ be the class of semiclassical pseudodifferential operators with symbols in $S_{1,0}^k(T^*M)$ where ε takes the role of h. Let Op and Op_ε be the corresponding quantization procedures, defined in (E.1.38). Fix $r > 0$ and define (recalling (E.7.1))

$$X_\varepsilon := \mathrm{Op}(\langle\varepsilon\xi\rangle^{-r}) = \mathrm{Op}_\varepsilon(\langle\xi\rangle^{-r}) \in \Psi_{0,\varepsilon}^{-r}(M).$$

(a) Using Proposition E.32, construct

$$Y_\varepsilon := \mathrm{Op}(q_\varepsilon) = \mathrm{Op}_\varepsilon(q) \in \Psi_{0,\varepsilon}^r(M), \quad q_\varepsilon(x,\xi) := q(x,\varepsilon\xi;\varepsilon)$$

where $q = \langle\xi\rangle^r + \mathcal{O}(\varepsilon)_{S_{1,0}^{r-1}(T^*M)}$, such that

$$X_\varepsilon Y_\varepsilon = I + \mathcal{O}(\varepsilon^\infty)_{\Psi^{-\infty}}, \quad Y_\varepsilon X_\varepsilon = I + \mathcal{O}(\varepsilon^\infty)_{\Psi^{-\infty}}. \tag{E.7.4}$$

(b) Show that $X_\varepsilon \in \Psi_0^0(M)$, $Y_\varepsilon \in \Psi_0^r(M)$ are bounded uniformly in ε.

(c) Let $A = \mathrm{Op}(a) \in \Psi_0^k(M)$. Show that uniformly in ε,

$$X_\varepsilon A Y_\varepsilon = A + i\,\mathrm{Op}\left(\langle\varepsilon\xi\rangle^r\{a, \langle\varepsilon\xi\rangle^{-r}\}\right) + \mathcal{O}(1)_{\Psi_0^{k-2}(M)} \tag{E.7.5}$$

and $\langle\varepsilon\xi\rangle^r\{a,\langle\varepsilon\xi\rangle^{-r}\}$ is bounded uniformly in ε in the class $S_{1,0}^{k-1}(T^*M)$. Show moreover that the wavefront set $\mathrm{WF}(X_\varepsilon A Y_\varepsilon)$ (a subset of $\partial\overline{T}^*M$ obtained by fixing $h := 1$ in Definition E.27) is contained in $\mathrm{WF}(A)$ with estimates uniform in ε. (Hint: write $X_\varepsilon A Y_\varepsilon - A = [X_\varepsilon, A]Y_\varepsilon + \mathcal{O}(\varepsilon^\infty)_{\Psi^{-\infty}}$.

Write N terms of the full nonsemiclassical symbol of $[X_\varepsilon, A]Y_\varepsilon$, with N large enough depending on r, and estimate the remainder using part (b).)

(d) Let $u \in H^{s-r}_{\mathrm{comp}}(M)$. Assume that $\|X_\varepsilon u\|_{H^s}$ is bounded uniformly as $\varepsilon \to 0$. Show that $u \in H^s_{\mathrm{comp}}(M)$. (Hint: use part (c) to reduce to the case $s = 0$ and $M = \mathbb{R}^n$. Then use the Monotone Convergence Theorem.)

11. Let M be a compact manifold and $k \in \mathbb{R}$; fix a quantization procedure (E.1.38). For $\varepsilon > 0$ define

$$A := \mathrm{Op}_h(\langle \varepsilon\xi\rangle^k) \in \Psi^k_h(M).$$

Arguing similarly to part (a) of Exercise E.10, show that if ε is fixed small enough and $h \in (0,1]$ then A is invertible as an operator on $\mathcal{D}'(M)$ and $A^{-1} \in \Psi^{-k}_h(M)$.

12. Show that a family of operators $B : C^\infty_{\mathrm{c}}(M_2) \to \mathcal{D}'(M_1)$ is h-tempered if and only if for each $\chi_1 \in C^\infty_{\mathrm{c}}(M_1), \chi_2 \in C^\infty_{\mathrm{c}}(M_2)$ there exist C, N such that $\|\chi_1 B \chi_2\|_{H^N_h \to H^{-N}_h} \leq Ch^{-N}$.

13. Under the assumptions of Proposition E.37, use stationary phase in the y, ξ variables to show that for each $B \in \Psi^k_h(M)$, Bu has the form (E.2.16) modulo an $\mathcal{O}(h^\infty)_{C^\infty}$ remainder, with the amplitude

$$b(x,\theta;h) = \sigma_h(B)(x, \partial_x\varphi(x,\theta))a(x,\theta) + \mathcal{O}(h)_{C^\infty_{\mathrm{c}}}.$$

14. Assume that the conditions of Proposition E.37 are satified, and φ is nondegenerate in the sense that $\partial_{\theta_1}\varphi, \dots, \partial_{\theta_m}\varphi$ are linearly independent on the set $\{\partial_\theta\varphi = 0\}$. Show that the set

$$\Lambda_\varphi = \{(x, \partial_x\varphi(x,\theta)) \mid \partial_\theta\varphi(x,\theta) = 0\}$$

is an immersed Lagrangian submanifold of T^*M.

15. Let $u = u(h) \in \mathcal{D}'(\mathbb{R}^n)$ be h-tempered. Use Proposition E.38 to show that $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ if and only if there exists a function $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a neighbourhood V of ξ_0 in the radial compactification of $\mathbb{R}^n$ such that the Fourier transform $\widehat{\chi u}$ satisfies

$$\widehat{\chi u}(\xi/h) = \mathcal{O}(h^\infty\langle\xi\rangle^{-\infty}), \quad \xi \in V.$$

16. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define the standard (nonsemiclassical) wavefront set

$$\mathrm{WF}(u) \subset T^*\mathbb{R}^n \setminus 0 \tag{E.7.6}$$

following [HöI, §8.1]: a point $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ does not lie in $\mathrm{WF}(u)$ if and only if there exists a function $\chi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a conic neighbourhood V of ξ_0 in $\mathbb{R}^n$ such that

$$\widehat{\chi u}(\xi) = \mathcal{O}(\langle\xi\rangle^{-\infty}), \quad \xi \in V.$$

(a) Prove that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is independent of h, then

$$\mathrm{WF}_h(u) = \mathrm{WF}(u) \cup \kappa(\mathrm{WF}(u)) \cup (\operatorname{supp} u \times \{0\})$$

where $\kappa : T^*\mathbb{R}^n \setminus 0 \to \partial\overline{T}^*\mathbb{R}^n$ is defined in (E.1.11). In particular, $\mathrm{WF}(u) = \mathrm{WF}_h(u) \cap (T^*\mathbb{R}^n \setminus 0)$. This formula gives a way to define $\mathrm{WF}(u) \subset T^*M \setminus 0$ for a distribution u on a manifold M.

(b) Show that

$$\operatorname{sing\,supp} u = \pi(\mathrm{WF}(u))$$

where $\pi(x, \xi) = x$ and $\operatorname{sing\,supp} u$ is the complement of elements of $\mathbb{R}^n$ having open neighbourhoods U such that $u|_U \in C^\infty(U)$.

17. Calculate the semiclassical wavefront sets of the following distributions on $\mathbb{R}^n$:

(a) $e^{-\frac{|x|^2}{2h}}$;

(b) $\chi(x/h)$, where $\chi \in \mathscr{S}(\mathbb{R}^n)$;

(c) $e^{-1/h}\delta_0(x)$.

18. For two manifolds M_1, M_2, show that the natural map

$$T^*M_1 \times T^*M_2 \to T^*(M_1 \times M_2)$$

extends to continuous maps

$$\overline{T}^*M_1 \times T^*M_2 \to \overline{T}^*(M_1 \times M_2), \quad T^*M_1 \times \overline{T}^*M_2 \to \overline{T}^*(M_1 \times M_2),$$

but not to a continuous map $\overline{T}^*M_1 \times \overline{T}^*M_2 \to \overline{T}^*(M_1 \times M_2)$. (This explains why we do not handle the fiber infinity in Proposition E.40.)

19. Let $A \in \Psi_h^k(M)$. Prove that

$$\mathrm{WF}'_h(A) = \{(x, \xi, x, \xi) : (x, \xi) \in \mathrm{WF}_h(A)\},$$

where the left-hand side uses (E.2.11) and the right-hand side, Definition E.27.

20. Let $B(h) : C_c^\infty(M_2) \to \mathcal{D}'(M_1)$ be h-tempered. Show that a point $(x, \xi, y, \eta) \in T^*(M_1 \times M_2)$ does not lie in $\mathrm{WF}'_h(B)$ if and only if there exists neighbourhoods $U(x, \xi) \subset T^*M_1$, $V(y, \eta) \subset T^*M_2$ such that for each h-tempered family of functions $f \in C_c^\infty(M_2)$, we have

$$\mathrm{WF}_h(f) \subset V \implies \mathrm{WF}_h(Bf) \cap U = \emptyset.$$

(Hint: use Propositions E.38 and E.39.)

Section E.3

21. Compute the semiclassical limiting measures for the following families of functions on $\mathbb{R}^n$:

(a) $u(x; h) = (\pi h)^{-n/4} e^{-\frac{|x-x_0|^2}{2h} + \frac{i\langle x, \xi_0\rangle}{h}}$ where $(x_0, \xi_0) \in T^*\mathbb{R}^n$ is fixed;

(b) $u(x;h) = e^{\frac{i\varphi(x)}{h}} b(x)$ where $\varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^n)$;

(c) $u(x;h) = h^{-n/2}\chi(x/h)$ where $\chi \in \mathscr{S}(\mathbb{R}^n)$.

22. Assume that M is a compact manifold and $u(x;h) = e^{\frac{ir(h)\varphi(x)}{h}} b(x)$ where $\varphi \in C^\infty(M;\mathbb{R})$, $b \in C^\infty(M)$, $d\varphi \neq 0$ on $\operatorname{supp} b$, and $r(h) > 0$, $r(h) \to \infty$ as $h \to 0$. Show that u converges to the measure $\mu = 0$. Conclude that when $b \not\equiv 0$ we cannot have (E.3.1) for all $A \in \Psi_h^0(M)$, for any choice of the sequence $h_j \to 0$.

23. Assume that M is a compact manifold and $P = P(h) \in \Psi_h^k(M)$ is elliptic at fiber infinity, that is $\partial \overline{T}^* M \subset \operatorname{ell}_h(P)$. Assume that $h_j \to 0$, $u_j \in L^2(M)$ satisfy as $j \to \infty$

$$\|u_j\|_{L^2} \leq C, \quad \|P(h_j)u_j\|_{H_h^{-k}} \to 0,$$

and u_j converges to some measure μ. Show that the convergence statement (E.3.1) is satisfied for all $A \in \Psi_h^0(M)$. In the special case when $\|u_j\|_{L^2} = 1$, show that μ is a probability measure.

Section E.4

24. Fix $\varepsilon > 0$. Show that Lemma E.46 does not hold under the assumptions

$$u \in H^s_{\mathrm{loc}},\ \mathbf{P}u \in H^{s-k+1+\varepsilon}_{\mathrm{loc}}, \quad v \in H^{-s+k-1-\varepsilon}_{\mathrm{comp}},\ \mathbf{P}^* v \in H^{-s}_{\mathrm{comp}},$$

using the example $\mathbf{P} = x \in \Psi_h^0(\mathbb{R})$, $u = (x + i0)^{-1}$, and $v = \delta_0$.

25. Let M be a compact manifold with a fixed smooth density and X be a divergence free vector field. Using Lemma E.46, prove that the operator

$$\mathbf{P} := \frac{1}{i} X : \mathscr{D}_{\mathbf{P}} \to L^2(M), \quad \mathscr{D}_{\mathbf{P}} = \{u \in L^2(M) \mid \mathbf{P}u \in L^2(M)\}$$

is self-adjoint on $L^2(M)$ where $\mathbf{P}u$ is understood in the sense of distributions.

26. Let $\mathbf{P} \in \Psi_h^k(M)$ be properly supported and $p = \sigma_h(\mathbf{P})$ be real-valued. Consider a segment

$$\gamma := \{e^{t\langle\xi\rangle^{1-k} H_p}(x_0, \xi_0) \mid t \in [0, T]\}, \quad (x_0, \xi_0) \in \overline{T}^* M.$$

Use Theorem E.47 to prove the following statement: for each h-tempered family $u(h) \in C^\infty(M)$ such that $\gamma \cap \mathrm{WF}_h(\mathbf{P}u) = \emptyset$, we have either $\gamma \subset \mathrm{WF}_h(u)$ or $\gamma \cap \mathrm{WF}_h(u) = \emptyset$.

27. Let $q \in C^\infty(\mathbb{R})$ be a nonnegative function and $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R})$ satisfy $\operatorname{supp} \chi_1 \subset (0, \infty)$ and $\chi_2(0) \neq 0$.

(a) Use Theorem E.47 to show that there exists $\chi \in C_c^\infty(\mathbb{R})$ such that

$$\|\chi_1 u\|_{L^2} \leq C\|\chi_2 u\|_{L^2} + Ch^{-1}\|\chi(hD_x - iq)u\|_{L^2} + \mathcal{O}(h^\infty)\|\chi u\|_{L^2} \tag{E.7.7}$$

for all $u \in C^\infty(\mathbb{R})$. Give a direct proof of (E.7.7).

(b) When $q \equiv 0$, show h^{-1} cannot be replaced in (E.7.7) by $h^{-\delta}$ for $\delta < 1$.

(c) Does (E.7.7) still hold when q is allowed to be negative? (Hint: consider $q(x) = x - 1$.)

28. Let χ_1, χ_2 be as in Exercise E.27. Conjugating by $h^{-x} = e^{\log(1/h)x}$, show that (E.7.7) implies the following estimate for all $u \in C^\infty(\mathbb{R})$:

$$\begin{aligned} \|\chi_1 u\|_{L^2} \leq & Ch^{-R}\|\chi_2 u\|_{L^2} + Ch^{-1-R}\|\chi(hD_x + ih\log(1/h))u\|_{L^2} \\ & + \mathcal{O}(h^\infty)\|\chi u\|_{L^2} \end{aligned} \tag{E.7.8}$$

for some choice of $\chi \in C_c^\infty(\mathbb{R})$ and some $R > 0$ depending on χ_1, χ_2. (Note that (E.7.7) does not apply to the operator $hD_x + ih\log(1/h)$ since in this case $q = -h\log(1/h)$ is not $\mathcal{O}(h)$ and has the wrong sign.) The estimate (E.7.8) can be viewed as a toy model for the resolvent bound in logarithmic regions in §6.4.

29. Let $\mathbf{P} := hD_x$ on $M := \mathbb{R}_x \times \mathbb{S}^1_\theta$. Show that $Ch^{-1}\|B_1 f\|_{H_h^{s-k+1}}$ cannot in general be replaced in (E.4.14) by $Ch^{-1}\|B_1 f\|_{H_h^{s-k+\delta}}$ for $\delta < 1$.

30. Show that Theorem E.47 applies under the following weaker regularity assumption: $u \in \mathcal{D}'(M)$, $B_1 u \in H^s_{\text{comp}}(M)$, $B_1 f \in H^{s-k+1}_{\text{comp}}(M)$. (Hint: apply the original version of the theorem to $B_2 u$ for the right choice of $B_2 \in \Psi^0_h(M)$.)

31. This exercise outlines a proof of the following strengthening of Theorem E.47: if $u \in \mathcal{D}'(M)$, $f = \mathbf{P}u$, then

$$Bu \in H^s_{\text{comp}}(M),\ B_1 f \in H^{s-k+1}_{\text{comp}}(M) \quad \Longrightarrow \quad Au \in H^s_{\text{comp}}(M). \tag{E.7.9}$$

The estimate (E.4.14) then follows from Exercise E.30 (arguing similarly to step 5 in the proof of Theorem E.47, showing that $B_2 u \in H^s_{\text{comp}}(M)$). It is enough to show (E.7.9) for any fixed h, for simplicity we put $h := 1$.

(a) Fix $r > 0$ and let $X_\varepsilon = \operatorname{Op}(\langle \varepsilon\xi\rangle^{-r}) \in \Psi^{-r}_{0,\varepsilon}(M)$, $Y_\varepsilon \in \Psi^r_{0,\varepsilon}(M)$, $\varepsilon > 0$, be the families of operators from Exercise E.10. Put

$$\mathbf{P}_\varepsilon := X_\varepsilon \mathbf{P} Y_\varepsilon \in \Psi^k_0(M).$$

Applying Exercise E.30 to $\mathbf{P}_\varepsilon$ and $X_\varepsilon u$ and using Exercise E.10(c), show the following estimate with bounds uniform in ε:

$$\|X_\varepsilon A u\|_{H^s} \leq C\|X_\varepsilon B u\|_{H^s} + C\|X_\varepsilon B_1 f\|_{H^{s-k+1}} + C\|\chi u\|_{H^{-N}}$$

for all $u \in \mathcal{D}'(M)$, $f := \mathbf{P}u$ such that $B_1 u \in H^{s-r}_{\text{comp}}(M)$ and $B_1 f \in H^{s-r-k+1}_{\text{comp}}(M)$.

(b) Complete the proof by noting that $B_1 u, B_1 f \in H^{-N}_{\text{comp}}$ for large enough N, fixing r large enough depending on N, and using Exercise E.10(d).

32. Show that if a is eventually positive on L with respect to p (see Proposition E.51) and $a_1 \in S^0(T^*M;\mathbb{R})$, then $a + H_p a_1$ is eventually positive as well. Use this to show that eventual negativity of (E.4.39) and (E.4.47) is independent of:

(a) the metric in the definition of $\langle\xi\rangle$; in fact $\langle\xi\rangle$ may be replaced by any positive symbol in $S^1(T^*M)$ which is elliptic on L; and

(b) the density on M in the definition of $\operatorname{Im}\mathbf{P}$. (Hint: use the identity $\psi^{-1}\mathbf{P}^*\psi = \mathbf{P}^* + \psi^{-1}[\mathbf{P}^*,\psi]$ valid for positive $\psi \in C^\infty(M)$.)

33. Use (E.4.34) and (E.4.37) to show that $\langle\xi\rangle^{-1}H_p\langle\xi\rangle$ is eventually negative on a radial source and eventually positive on a radial sink. Deduce that

(a) if $s_1 < s_2$ and (E.4.39) is eventually negative for $s = s_1$, then it is eventually negative for $s = s_2$;

(b) if $s_1 < s_2$ and (E.4.47) is eventually negative for $s = s_2$, then it is eventually negative for $s = s_1$;

(c) (E.4.39) is eventually negative for $s > 0$ large enough;

(d) (E.4.47) is eventually negative for $-s > 0$ large enough.

34. Consider the operator $\mathbf{P} := x(hD_x) + i\gamma h$ and its radial source $L = \partial\overline{T}^*\mathbb{R} \cap \{x = 0\}$, see (E.4.36) and Figure E.3.

(a) Show that (E.4.39) is eventually negative on L if and only if $s > \operatorname{Re}\gamma + \frac{1}{2}$.

(b) Let $s > \operatorname{Re}\gamma + \frac{1}{2}$. Use Theorems E.47 and E.52 to show that for each compactly supported $A \in \Psi_h^0(\mathbb{R})$ such that $\operatorname{WF}_h(A) \cap \{\xi = 0\} = \emptyset$, there exists $\chi \in C_c^\infty(\mathbb{R})$ such that for each

$$u \in H^s_{\mathrm{loc}}(\mathbb{R}), \quad f := \mathbf{P}u \in H^s_{\mathrm{loc}}(\mathbb{R}),$$

we have the following estimate for all N:

$$\|Au\|_{H_h^s} \le Ch^{-1}\|\chi f\|_{H_h^s} + Ch^N\|\chi u\|_{H_h^{-N}}. \tag{E.7.10}$$

(c) Assume that $s < \operatorname{Re}\gamma + \frac{1}{2}$ and $\gamma \notin \mathbb{N}_0$. Show that (E.7.10) no longer holds, by taking $u(x) = x_+^\gamma/\Gamma(\gamma+1)$ and using Exercise E.16 and the formulas for the Fourier transform of u, see [HöI, Example 7.1.17].

35. Following the strategy of Exercise E.31, obtain the following strengthening of Theorem E.52: if (E.4.39) is eventually negative for some $s' < s$, $u \in \mathcal{D}'(M)$, $f = \mathbf{P}u$, and $B_1 u \in H^{s'}_{\mathrm{comp}}(M)$, $B_1 f \in H^{s-k+1}_{\mathrm{comp}}(M)$, then $Au \in H^s_{\mathrm{comp}}(M)$ and (E.4.40) holds.

Hint: The main difference from Exercise E.31 is to use (E.7.5) and Exercise E.33 to verify that the threshold condition holds on H^s for $\mathbf{P}_\varepsilon$ uniformly in ε when $r := s - s'$. For that you need to analyse the contribution of the second term on the right-hand side of (E.7.5). Assume for simplicity that $\langle\xi\rangle^{-k}H_p\langle\xi\rangle < 0$ near L and use the identity

$$\langle\varepsilon\xi\rangle^r H_p\langle\varepsilon\xi\rangle^{-r} = -r\frac{\varepsilon^2\langle\xi\rangle^2}{\langle\varepsilon\xi\rangle^2}\cdot\frac{H_p\langle\xi\rangle}{\langle\xi\rangle}.$$

36. Following the strategy of Exercise E.31, obtain the following strengthening of Theorem E.54: if $u \in \mathcal{D}'(M)$, $f = \mathbf{P}u$, and $Bu \in H^s_{\mathrm{comp}}(M)$, $B_1 f \in H^{s-k+1}_{\mathrm{comp}}(M)$, then $Au \in H^s_{\mathrm{comp}}(M)$ and (E.4.48) holds. (As in Exercise E.35, the new component is the verification of the threshold condition for $\mathbf{P}_\varepsilon$. However, now it is reasonable to assume that $\langle\xi\rangle^{-k}H_p\langle\xi\rangle > 0$.)

37. Show that the conclusion of Exercise E.36 is false for operator $-\mathbf{P}$, where $\mathbf{P}$ was studied in Exercise E.34, when $s > \operatorname{Re}\gamma + \frac{1}{2}$ (that is, the threshold condition fails). (Hint: consider $u(x) = x_+^\gamma/\Gamma(\gamma+1)$ and use that it is smooth away from zero.)

Section E.5

38. Prove the estimate (E.5.3). (Hint: first bound $\|(1-\chi_1)u\|_{H^1_h([0,\infty))}$ by $\|\chi_3(1-\chi_1)u\|_{H^1_h(\mathbb{R})}$ where $\chi_3 \in C^\infty_{\mathrm{c}}((-1,2))$, $\chi_3 \equiv 1$ near $[0,1]$. Now, write $\chi_3(1-\chi_1) = A_1 + A_2$ where $A_1 \in \Psi^0_h(\mathbb{R}), A_2 \in \Psi^{\mathrm{comp}}_h(\mathbb{R})$ are compactly supported on $(-1,1)$, $\mathrm{WF}_h(A_1) \subset \mathrm{ell}_h(P)$, and for each $(t,\tau) \in \mathrm{WF}_h(A_2)$ there exists $s \in \mathbb{R}$ such that $e^{sH_p}(t,\tau) \in \{\chi_1\chi_2 = 1\}$ where $p = \sigma_h(P)$.)

39. Prove the estimates (E.5.4) and (E.5.5):

(a) Show that it is enough to prove the estimates

$$\|u\|_{H^1_h([0,1])} \le Ch^{-1}\|Pu\|_{L^2([0,1])} + C\|u\|_{H^1_h([1-\varepsilon,1])} \tag{E.7.11}$$

$$\|v\|_{H^1_h([0,1])} \le Ch^{-1}\|Pv\|_{L^2([0,1])} \tag{E.7.12}$$

where $\|u\|^2_{H^1_h([a,b])} := \|u\|^2_{L^2([a,b])} + \|h\partial_t u\|^2_{L^2([a,b])}$ and $\chi_1\chi_2 = 1$ on $[1-\varepsilon,1]$.

(b) Prove (E.7.11)–(E.7.12), integrating the identities

$$hD_t\big(e^{\pm it/h}(hD_t \mp 1)u(t)\big) = e^{\pm it/h}Pu(t).$$

40. Show that an operator P is hyperbolic (in the sense of Definition E.55) if and only if for each $(x,\xi) \in T^*\overline{M}$ with $x \in \{t < 1\}$ and $\xi \notin \operatorname{span}(dt)$, the equation $p_0(x, \xi + \tau dt) = 0$ has two distinct real solutions $\tau_\pm$. Show a similar statement for semiclassical hyperbolicity. Conclude that (semiclassical) hyperbolicity depends only on P and t, but not on the choice of the product structure (E.5.6).

41. Following the proofs of Theorems E.56–E.57 and using Proposition E.62 obtain the following strengthening of these theorems:

(a) if $u \in \mathcal{D}'(M)$ and $\chi_2 P u \in \bar{H}^{s-1}(M)$, $\chi_1\chi_2 u \in \bar{H}^s(M)$, then $(1-\chi_1)u \in \bar{H}^s(M)$ and (E.5.10), (E.5.12) hold;

(b) if $v \in \mathcal{D}'(M)$ extends by 0 to a distribution on a manifold without boundary containing $\overline{M}$ and $\chi_2 P v \in \dot{H}^{s-1}(\overline{M})$, then $(1-\chi_1)v \in \dot{H}^s(\overline{M})$ and (E.5.11), (E.5.13) hold.

Bibliography

[A*16] B.P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), *Observation of Gravitational Waves from a Binary Black Hole Merger,* Phys. Rev. Lett. **116** (2016), 061102.

[Ag86] S. Agmon, *Spectral theory of Schrödinger operators on Euclidean and non-Euclidean spaces,* Comm. Pure Appl. Math. **39**(1986), Number S, Supplement.

[Ag98] S. Agmon, *A perturbation theory of resonances,* Comm. Pure Appl. Math. **51**(1998), 1255–1309. (Erratum ibid. **52**(1999), 1617–1618.)

[AC71] J. Aguilar and J.M. Combes, *A class of analytic perturbations for one-body Schrödinger Hamiltonians,* Comm. Math. Phys. **22**(1971), 269–279.

[Ah78] L.V. Ahlfors, *C*omplex analysis. An introduction to the theory of analytic functions of one complex variable, Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 1978.

[AT14] I. Alexandrova and H. Tamura, *Resonances in scattering by two magnetic fields at large separation and a complex scaling method,* **256**(2014), 398–448.

[AG07] S. Alinhac and P. Gérard, *Pseudo-differential Operators and the Nash–Moser Theorem,* American Math Society, 2007.

[ABG70] M.F. Atiyah, R. Bott and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients I,* Acta Math. **124**(1970), 109–189.

[BaMB] A. Bachelot and A. Motet-Bachelot, *Les résonances d'un trou noir de Schwarzschild,* Ann. Inst. H. Poincaré Phys. Théor. **59**(1993), 3–68.

[BC71] E. Balslev and J.M. Combes, *Spectral properties of many-body Schrödinger operators wth dilation analytic interactions,* Comm. Math. Phys. **22**(1971), 280–294.

[BGR82] C. Bardos, J.-C. Guillot and J. Ralston, *La relation de Poisson pour l'équation des ondes dans un ouvert non borné. Application à la théorie de la diffusion,* Comm. Partial Differential Equations **7**(1982), 905–958.

[BLR87] C. Bardos, G. Lebeau and J. Rauch, *Scattering frequencies and Gevrey 3 singularities,* Invent. Math. **90**(1987), 77–114.

[B*13] S. Barkhofen, T. Weich, A. Potzuweit, U. Kuhl, H.-J. Stöckmann, and M. Zworski, *Experimental observation of spectral gap in microwave n-disk systems,* Phys. Rev. Lett. **110**(2013), 164102.

[Ba16] D. Baskin, *An explicit description of the radiation field in 3+1-dimensions,* preprint, arXiv:1604.02984.

[BSW16] D. Baskin, E. Spence and J. Wunsch, *Sharp high-frequency estimates for the Helmholtz equation and applications to boundary integral equations,* SIAM J. Math. Anal. **48**(2016), 229–267.

[BVW15] D. Baskin, A. Vasy and J. Wunsch, *Asymptotics of radiation fields in asymptotically Minkowski space,* Amer. J. Math. **137**(2015), 1293–1364.

[BW13] D. Baskin and J. Wunsch, *Resolvent estimates and local decay of waves on conic manifolds,* J. Diff. Geom. **95**(2013), 183–214.

[BZH10] M.C. Barr, M.P. Zaletel, E.J. Heller, *Quantum corral resonance widths: lossy scattering as acoustics,* Nano letters **10**(2010), 3253–3260.

[BW15] M. Bledsoe and R. Weikard, *The inverse resonance problem for left-definite Sturm-Liouville operators,* J. Math. Anal. Appl., **423**(2015), 1753–1773.

[Be94] J. P. Berenger, *A perfectly matched layer for the absorption of electromagnetic waves,* J. Comp. Phys. **114**(1994), 185–200.

[BG05] D. Bindel and S. Govindjee, *Elastic PMLs for resonator anchor loss simulations,* Int. J. Num. Meth. Eng. **64**(2005), 789–818.

[BZ] D. Bindel and M. Zworski, *Theory and computation of resonances in 1d scattering,* `http://www.cs.cornell.edu/%7Ebindel/cims/resonant1d/`

[BK62] M.Sh. Birman and M.G. Kreĭn, *On the theory of wave operators and scattering operators,* Dokl. Akad. Nauk. SSSR **144**(1962), 475–478.

[Bo02] J.-F. Bony, *Minoration du nombre de résonances engendrées par une trajectoire fermée,* Comm. PDE **27**(2002), 1021–1078.

[BBR14] J.-F. Bony, V. Bruneau and G. Raikov, *Counting function of characteristic values and magnetic resonances,* Comm. Partial Differential Equations, **39**(2014), 274–305.

[BBR10] J.-F. Bony, N. Burq, and T. Ramond, *Minoration de la résolvante dans le cas captif,* Comptes Rendus Acad. Sci, Mathematique **348**(23-24)(2010), 1279–1282.

[B*16] J.-F. Bony, S. Fujiie, T. Ramond and M. Zerzeri, *Resonances for homoclinic trapped sets,* Asterisque **405**(2018).

[BH08] J.-F. Bony and D. Häfner, *Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric,* Comm. Math. Phys. **282**(2008), 697–719.

[BS01] J.-F. Bony and J. Sjöstrand, *Trace formula for resonances in small domains,* J. Funct. Anal. **184**(2001), 402–418.

[Bo14] D. Borthwick, *Distribution of resonances for hyperbolic surfaces,* Exp. Math. **23**(2014), 25–45.

[Bo16] D. Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces,* Birkhäuser, 2nd edition, 2016.

[BP02] D. Borthwick and P. Perry, *Scattering poles for asymptotically hyperbolic manifolds,* Trans. Amer. Math. Soc. **354**(2002), 1215–1231.

[BW14] D. Borthwick and T. Weich, *Symmetry reduction of holomorphic iterated function schemes and factorization of Selberg zeta functions,* to appear in J. Spectr. Theory, arXiv:1407.6134.

[BG16] D. Borthwick and C. Guillarmou, *Upper bounds on the number of resonances on geometrically finite hyberbolic manifolds,* Journal of EMS. **18**(2016), 997–1041.

[BW16] D. Borthwick and T. Weich, *Symmetry reduction of holomorphic iterated function schemes and factorization of Selberg zeta functions,* J. Spectr. Theory **6**(2016), 267–329.

[BD18] J. Bourgain and S. Dyatlov, *Spectral gaps without the pressure condition,* Ann. of Math. **187**(2018), 825–867.

[BGS11] J. Bourgain, A. Gamburd and P. Sarnak, *Generalization of Selberg's 3/16 theorem and affine sieve,* Acta Math. **207**(2011), 255–290.

[BP03] V. Bruneau and V. Petkov, *Meromorphic continuation of the spectral shift function,* Duke Math. J. **116**(2003), 389–430.

[Bu98] N. Burq, *Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel,* Acta Math. **180**(1998), 1–29.

[Bu02a] N. Burq, *Lower bounds for shape resonances widths of long range Schrödinger operators,* Amer. J. Math. **124**(2002), 677–735.

[Bu02b] N. Burq, *Semi-classical estimates for the resolvent in nontrapping geometries,* IMRN **2002**(5), 221–241.

[Bu04] N. Burq, *Smoothing effect for Schrödinger boundary value problems,* Duke Math. J. **123**(2004), 403–427.

[BGH10] N. Burq, C. Guillarmou and A. Hassell, *Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics,* Geom. Funct. Anal. **20**(2010), 627–656.

[BZ01] N. Burq and M. Zworski, *Resonance expansions in semi-classical propagation,* Comm. Math. Phys. **232**(2001), 1–12.

[Bu62] V.S. Buslaev, *Trace formulas for the Schrödinger operator in a three-dimensional space,* Dokl. Akad. Nauk SSSR **143**(1962), 1067–1070.

[Bu75] V.S. Buslaev, *The asymptotic behavior of the spectral characteristics of exterior problems for the Schrödinger operator,* Izv. Akad. Nauk SSSR Ser. Mat. **39**(1975), 149–235.

[CW15] H. Cao and J. Wiersig, *Dielectric microcavities: Model systems for wave chaos and non-Hermitian physics,* Rev. Mod. Phys. **87**(2015), 61–111.

[CV02] F. Cardoso and G. Vodev, *Uniform estimates of the resolvent of the Laplace-Beltrami operator on infinite volume Riemannian manifolds II,* Ann. Henri Poincaré 3:4, 673–691, 2002.

[C*18] V. Cardoso, J.L. Costa, K. Destounis, P. Hintz, and A. Jansen, *Quasinormal Modes and Strong Cosmic Censorship,* Phys. Rev. Lett. **120**(2018), 031103.

[Ca36] T. Carleman, *Über die asymptotische Verteilung der Eigenwerte partieller Differentialgleichungen,* Ber. Verk. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **88**(1936), 119–134.

[Ch98] T. Christiansen, *Spectral asymptotics for compactly supported perturbations of the Laplacian on* $\mathbb{R}^n$*,* Comm. Partial Differential Equations, **23**(1998), 933–948.

[Ch99] T. Christiansen, *Some lower bounds on the number of resonances in Euclidean scattering,* Math. Res. Letters **6**(1999), 203–211.

[Ch06] T. Christiansen, *Schrödinger operators with complex-valued potentials and no resonances,* Duke Math J. **133**(2006), 313–323.

[CH05] T. Christiansen and P. Hislop, *The resonance counting function for Schrödinger operators with generic potentials,* Math. Res. Lett. **12**(2005), 821–826.

[Ch17a] T. Christiansen, *Lower bounds for resonance counting functions for obstacle scattering in even dimensions,* Amer. J. Math. **139**(2017), 617–640.

[Ch17b] T. Christiansen, *A sharp lower bound for a resonance-counting function in even dimensions,* Ann. Inst. Fourier **67**(2017), 579–604.

[CdV81a] Y. Colin de Verdière, *Une formule de traces pour l'opérateurs de Schrödinger dans* $\mathbb{R}^3$*,* Ann. Sci. Éc. Norm. Sup. 4éme série **14**(1981), 27–39.

[CdV81b] Y. Colin de Verdière, *Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein,* C. R. Acad. Sci. Paris Sér. I Math. **293**(1981), 361–363.

[CdV83] Y. Colin de Verdière, *Pseudo-laplaciens. II,* Ann. Inst. Fourier **33**(1983), 87–113.

[CR12] M. Combescure and D. Robert, *Coherent states and applications in mathematical physics,* Theoretical and Mathematical Physics, Springer, Dordrecht, 2012.

[Cl18] D.C. Clary, *Spiers Memorial Lecture. Introductory lecture: quantum dynamics of chemical reactions,* Faraday Discuss., **212**(2018), 9–32.

[CLE93] M.F. Crommie, C.P. Lutz and D.M. Eigler, *Confinement of electrons to quantum corrals on a metal surface,* Science **262**(1993), 218–220.

[DM17] M. Dalla Venezia and A. Martinez, *Widths of highly excited shape resonances,* Ann. Henri Poincaré **18**(2017), 1289–1304.

[Da09] K. Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets,* Comm. Math. Phys. **286**(3)(2009), 837–850.

[Da14] K. Datchev, *Quantitative limiting absorption principle in the semiclassical limit,* Geometric and Functional Analysis **24**(2014), 740–747.

[Da16] K. Datchev, *Resonance free regions for nontrapping manifolds with cusps,* Analysis and PDE **9**(2016) 907–953.

[DD13] K. Datchev and S. Dyatlov, *Fractal Weyl laws for asymptotically hyperbolic manifolds,* Geom. Funct. Anal. **23**(2013), 1145–1206.

[DDZ15] K. Datchev, S. Dyatlov and M. Zworski, *Resonances and lower resolvent bounds,* J. Spectr. Theory **5**(2015), 599–615.

[DV12a] K. Datchev and A. Vasy, *Gluing semiclassical resolvent estimates via propagation of singularities,* IMRN **2012**(23), 5409–5443.

[DV12b] K. Datchev and A. Vasy, *Propagation through trapped sets and semiclassical resolvent estimates,* Annales de l'Institut Fourier, **62**(2012), 2379–2384.

[Da95] E.B. Davies, *Spectral theory and differential operators,* Cambridge University Press, 1995.

[De92] J.-M. Delort, *F.B.I. Transformation: Second Microlocalization and Semilinear Caustics,* Lecture Notes in Mathematics **1522**, Springer, 1992.

[DS99] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit,* Cambridge University Press, 1999.

[DZ03] M. Dimassi and M. Zerzeri, *A local trace formula for resonances of perturbed periodic Schrödinger operators,* J. Funct. Anal. **198**(2003), 142–159.

[DV13] T.-C. Dinh and D.-V. Vu, *Asymptotic number of scattering resonances for generic Schrödinger operators,* Comm. Math. Phys. **326**(2014), 185–208.

[DN17] T.-C. Dinh and V.-A. Nguyen, *Distribution of scattering resonances for generic Schrödinger operators,* arXiv:1709.06375.

[Do96] S. Doi, *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds,* Duke Math. J. **82**(1996), 679–706.

[Dr15a] A. Drouot, *A quantitative version of Hawking radiation,* Ann. Inst. Henri Poincaré. **18**(2017), 757–806.

[Dr15b] A. Drouot, *Scattering resonances for highly oscillatory potentials,* Ann. Sci. Éc. Norm. Supér. **51**(2018), 865–925.

[Dr17] A. Drouot, *Stochastic stability of Pollicott–Ruelle resonances,* Comm. Math. Phys. **356**(2017), 357–396.

[DVW14] V. Duchêne, I. Vukićević and M. Weinstein, *Scattering and localization properties of highly oscillatory potentials,* Comm. Pure Appl. Math., **67**(2014), 83–128.

[DGM18] T. Duyckaerts, A. Grigis and A. Martinez, *Resonance widths for general Helmholtz Resonators with straight neck,* Duke Math. Journal, **165**(2018) 2793–2810.

[Dy11a] S. Dyatlov, *Quasi-normal modes and exponential energy decay for the Kerr–de Sitter black hole,* Comm. Math. Phys. **306**(2011), 119–163.

[Dy11b] S. Dyatlov, *Exponential energy decay for Kerr–de Sitter black holes beyond event horizons,* Math. Res. Lett. **18**(2011), 1023–1035.

[Dy12] S. Dyatlov, *Asymptotic distribution of quasi-normal modes for Kerr–de Sitter black holes,* Ann. Inst. Henri Poincaré **13**(2012), 1101–1166.

[Dy15a] S. Dyatlov, *Resonance projectors and asymptotics for r-normally hyperbolic trapped sets,* Journal of AMS **28**(2015), 311–381.

[Dy15b] S. Dyatlov, *Asymptotics of linear waves and resonances with applications to black holes,* Comm. Math. Phys. **335**(2015), 1445–1485.

[Dy16] S. Dyatlov, *Spectral gaps for normally hyperbolic trapping,* Ann. Inst. Fourier **66**(2016), 55–82.

[Dy19] S. Dyatlov, *Improved fractal Weyl bounds for hyperbolic manifolds,* with an appendix by D. Borthwick, S. Dyatlov and T. Weich, J. Eur. Math. Soc. **21**(2019), 1595–1639.

[DG10] S. Dyatlov and S. Ghosh, *Symmetry of bound and antibound states in the semiclassical limit for a general class of potentials,* Proc. AMS **138**(2010), 3203–3210.

[DG14] S. Dyatlov and C. Guillarmou, *Pollicott–Ruelle resonances for open systems,* Ann. Inst. Henri Poincaré (A) **17**(2016), 3089–3146.

[DJ16] S. Dyatlov and L. Jin, *Resonances for open quantum maps and the fractal uncertainty principle,* Comm. Math. Phys. **354**(2017), 269–316.

[DJ18] S. Dyatlov and L. Jin, *Semiclassical measures on hyperbolic surfaces have full support,* Acta Math. **220**(2018), 297–339.

[DZa16] S. Dyatlov and J. Zahl, *Spectral gaps, additive energy, and a fractal uncertainty principle,* Geom. Funct. Anal. **26**(2016), 1011–1094.

[DZ13] S. Dyatlov and M. Zworski, *Trapping of waves and null geodesics for rotating black holes,* Phys. Rev. D **88**(2013), 084037.

[DZ15] S. Dyatlov and M. Zworski, *Stochastic stability of Pollicott–Ruelle resonances,* Nonlinearity, **28**(2015), 3511–3534.

[DZ16] S. Dyatlov and M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis,* Ann. Sci. Ec. Norm. Supér. **49**(2016), 543–577.

[DZ18] S. Dyatlov and M. Zworski, *Fractal Uncertainty for Transfer Operators,* International Mathematics Research Notices, rny026, https://doi.org/10.1093/imrn/rny026.

[EFS15] L. Ermann, K.M. Frahm and D.L. Shepelyansky, *Google matrix analysis of directed networks,* Rev. Mod. Phys. **87**(2015), 1261–1310.

[Ev98] L. C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics **19**, American Math Society, 1998.

[FP72] L.D. Faddeev and B.S. Pavlov, *Scattering theory and automorphic functions,* Seminar of Steklov Mathematical Institute of Leningrad **27**(1972), 161–193.

[FT13] Frédéric Faure and Masato Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows,* Comptes rendus – Mathématique **351**(2013), 385–391.

[Fa77] J.D. Fay, *Fourier coefficients of the resolvent for a Fuchsian group,* J. Reine Angew. Math., **293–294**(1977), 143–203

[FG] C. Fefferman and C.R Graham, *The ambient metric,* Annals of Mathematics Studies **178**. Princeton University Press, Princeton, NJ, 2012.

[FL90] C. Fernández and R. Lavine, *Lower bounds for resonance widths in potential and obstacle scattering,* Comm. Math. Phys. **128**(1990), 263–284.

[Fr97] R. Froese, *Asymptotic distribution of resonances in one dimension,* J. Diff. Eq. **137**(1997), 251–272.

[FHP91] R. Froese, P. Hislop and P. Perry, *The Laplace operator on hyperbolic three manifolds with cusps of non-maximal rank,* Invent. Math. **106**(1991), 295–333.

[FLM11] S. Fujiié, A. Lahmar-Benbernou, and A. Martinez, *Width of shape resonances for non globally analytic potentials,* J. Math. Soc. Japan **63** (2011), 1–78.

[Ga14] J. Galkowski, *Distribution of Resonances in Scattering by Thin Barriers,* arXiv:1404.3709.

[Ga16] J. Galkowski, *Resonances for Thin Barriers on the Circle,* Journal of Physics A. 49 (2016), 125205.

[Ga17] J. Galkowski, *A Quantitative Vainberg Method for Black Box Scattering,* Comm. Math. Phys. **349**(2017), 527–549.

[Ga19] J. Galkowski, *The Quantum Sabine Law for Resonances in Transmission Problems,* arXiv:1511.05091, to appear in Pure and Applied Analysis.

[GS15] J. Galkowski and H. Smith, *Restriction bounds for the free resolvent and resonances in lossy scattering,* International Mathematics Research Notices (2015), 7473–7509.

[Gan15] O. Gannot, *From quasimodes to resonances: exponentially decaying perturbations,* Pacific J. Math **277**(2015), 77–97.

[Gan14] O. Gannot, *A global definition of quasinormal modes for Kerr–AdS Black Holes,* Ann. Inst. Fourier **68**(2018), 1125–1167.

[GrZw] C. Robin Graham and Maciej Zworski, *Scattering matrix in conformal geometry,* Invent. Math. **152**(2003), 89–118.

[GR89] P. Gaspard and S.A. Rice, *Semiclassical quantization of the scattering from a classically chaotic repellor,* J. Chem. Phys. **90**(1989), 224–2254.

[Gé88] C. Gérard, *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes,* Mémoires de la Société Mathématique de France Sér. 2, **31**(1988), 1–146.

[GM89] C. Gérard and A. Martinez, *Semiclassical asymptotics for the spectral function of long range Schrödinger operators,* J. Funct. Anal. **84**(1989), 226–254.

[GMR89] C. Gérard, A. Martinez and D. Robert, *Breit–Wigner formulas for the scattering phase and the total scattering cross-section in the semi-classical limit,* Comm. Math. Phys. **121**(1989), 323–336.

[GS87] C. Gérard and J. Sjöstrand, *Semiclassical resonances generated by a closed trajectory of hyperbolic type,* Comm. Math. Phys. **108**(1987), 391–421.

[GS88] C. Gérard and J. Sjöstrand, *Resonances en limite semiclassique et exposants de Lyapunov,* Comm. Math. Phys. **116**(1988), 193-213.

[Gé91] P. Gérard, *Mesures semi-classiques et ondes de Bloch,* in *Séminaire Équations aux Dérivées Partielles 1990–1991,* exp. XVI. École Polytech., Palaiseau, 1991.

[GK69] I. C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators,* Translations of Mathematical Monographs **18**, American Mathematical Society, 1969.

[GS71] I. Gohberg and E. Sigal, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché,* Math. U.S.S.R. Sbornik **13**(1971), 607–629.

[Go73] W. L. Goodhue, *Scattering theory for hyperbolic systems with coefficients of Gevrey type,* Trans. Amer. Math. Soc. **180**(1973), 337–346.

[G*10] A. Goussev, R. Schubert, H. Waalkens and S. Wiggins, *Quantum theory of reactive scattering in phase space,* Adv. Quant. Chem. **60**(2010), 269–332.

[GL91] C.R. Graham and J.M. Lee, *Einstein metrics with prescribed conformal infinity on the ball,* Adv. Math. **87**(1991), 186–225.

[GM14] A. Grigis and A. Martinez, *Resonance widths for the molecular predissociation,* Anal. PDE **7**(2014), 1027–1055

[GS94] A. Grigis and J. Sjöstrand, *Microlocal Analysis for Differential Operators, An Introduction,* Cambridge University Press, 1994.

[Gu05] C. Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds,* Duke Math. J. **129**(2005), 1–37.

[GM12] C. Guillarmou and R. Mazzeo, *Resolvent of the Laplacian on geometrically finite hyperbolic manifolds,* Inv. Math. **187**(2012), 99–144.

[GS13] V. Guillemin and S. Sternberg, *Semiclassical analysis,* International Press, Boston, 2013.

[Gu84] L. Guillopé, *Asymptotique de la phase de diffusion pour l'opérateur de Schrödinger dans* $\mathbb{R}^n$*,* Séminaire Équations aux dérivées partielles (dit "Goulaouic-Schwartz") (1984–1985), Exp. No. 5. For the thesis containing the proofs see https://www.math.sciences.univ-nantes.fr/~guillope/LG/these_1981.pdf

[GLZ04] L. Guillopé, K.K. Lin and M. Zworski, *The Selberg zeta function for convex co-compact Schottky groups,* Comm. Math. Phys. **245**(2004), 149–176.

[GZ95a] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances on non-compact Riemann surfaces,* J. Func. Anal. **129**(1995), 364–389.

[GZ95b] L. Guillopé and M. Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity,* Asymptotic Anal. **11**(1995), 1–22.

[GZ97] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces,* Ann. of Math. **145**(1997), no. 3, 597–660.

[GZ99] L. Guillopé and M. Zworski, *Wave trace for Riemann surfaces,* Geom. Funct. Anal. **6**(1999), 1156–1168.

[HL94] T. Hargé and G. Lebeau, *Diffraction par un convexe,* Invent. Math. **118**(1994), 161–196.

[Ha89] W.K. Hayman, *Subharmonic Functions,* vol.II, Academic Press, London, 1989.

[He88] B. Helffer, *Semi-classical Analysis for the Schrödinger Operator and Applications,* Lecture Notes in Mathematics **1336**, Springer, 1988.

[He13] B. Helffer, *Spectral theory and its applications,* Cambridge University Press, 2013.

[HS86] B. Helffer and J. Sjöstrand, *Resonances en limite semiclassique,* Bull. Soc. Math. France **114**, no. 24–25, 1986.

[He96] E. Heller, *Quantum proximity resonances,* Phys. Rev. Lett. **77**(1996), 4122–4125.

[HJ18] L. Hillairet and C. Judge, *Hyperbolic triangles without embedded eigenvalues,* Ann. of Math. **187**(2018), 301–377.

[Hi15] P. Hintz, *Resonance expansions for tensor-valued waves on asymptotically Kerr-de Sitter spaces,* arXiv:1502.03183, to appear in J. Spectr. Theory.

[HV14a] P. Hintz and A. Vasy, *Non-trapping estimates near normally hyperbolic trapping,* Math. Res. Lett. **21**(2014), 1277–1304.

[HV14b] P. Hintz and A. Vasy, *Global analysis of quasilinear wave equations on asymptotically Kerr-de Sitter spaces,* arXiv:1404.1348.

[HV15] P. Hintz and A. Vasy, *Semilinear wave equations on asymptotically de Sitter, Kerr-de Sitter and Minkowski spacetimes,* Analysis &PDE **8**(2015), 1807–1890.

[HV16] P. Hintz and A. Vasy, *The global non-linear stability of the Kerr-de Sitter family of black holes,* Acta Mathematica, **220**(2018), 1–206.

[HZ17] P. Hintz and M. Zworski, *Wave decay for star-shaped obstacles in $\mathbb{R}^3$: papers of Morawetz and Ralston revisited,* Math. Proc. R. Ir. Acad. **117A**(2017), 47–62.

[HZ18] P. Hintz and M. Zworski, *Resonances for obstacles in hyperbolic space,* Comm. Math. Phys. **359**(2018), 699–731.

[HP03] M. Hitrik and I. Polterovich, *Resolvent Expansions and Trace Regularizations for Schrödinger Operators,* Contemporary Math. **327**(2003), 161–173.

[HPS77] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Mathematics, Springer, 1977.

[HS89] P.D. Hislop and I.M. Sigal, *Semiclassical theory of shape resonances in quantum mechanics,* Mem. Amer. Math. Soc. **78**(1989).

[HS96] P.D. Hislop and I.M. Sigal, *Introduction to Spectral Theory: with Applications to Schrödinger operators,* Applied Mathematical Sciences **113**, Springer 1996.

[HZ09] J. Holmer and M. Zworski, *Breathing patterns in nonlinear relaxation,* Nonlinearity, **22**(2009), 1259–1301.

[HöI] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis,* Springer Verlag, 1983.

[HöII] L. Hörmander, *The Analysis of Linear Partial Differential Operators II. Differential Operators with Constant Coefficients,* Springer Verlag, 1983.

[HöIII] L. Hörmander, *The Analysis of Linear Partial Differential Operators III. Pseudo-Differential Operators,* Springer Verlag, 1985.

[HöIV] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV. Fourier Integral Operators,* Springer Verlag, 1985.

[Hu86] W. Hunziker, *Distortion analyticity and molecular resonance curves,* Ann. Inst. H. Poincaré Phys. Théor. **45**(1986), 339–358.

[Ik83] M. Ikawa, *On the poles of the scattering matrix for two strictly convex obstacles,* J. Math. Kyoto Univ. **23**(1983), **1**, 127–194.

[Ik88] M. Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies,* Ann. Inst. Fourier, **38**(1988), 113–146.

[In86] A. Intissar, *A polynomial bound on the number of the scattering poles for a potential in even-dimensional spaces,* Comm. Partial Differential Equations **11**(1986), 367–396.

[Iv16] V. Ivrii, *100 years of Weyl law,* Bull. Math. Sci. (2016) `http://link.springer.com/journal/13373`

[J*14] T-C. Jagau, D. Zuev, K. B. Bravaya, E. Epifanovsky, and A.I. Krylov, *A Fresh Look at Resonances and Complex Absorbing Potentials: Density Matrix-Based Approach,* J. Phys. Chem. Lett. **5**(2014), 310–315.

[JN10] D. Jakobson and F. Naud, *Lower bounds for resonances of infinite area Riemann surfaces,* Analysis & PDE **3**(2010), 207–225.

[JN12] D. Jakobson and F. Naud, *On the critical line of convex co-compact hyperbolic surfaces,* Geom. Funct. Anal. **22**(2012), 352–368.

[JN14] D. Jakobson and F. Naud, *Resonances and convex co-compact congruence subgroups of $PSL_2(\mathbb{Z})$,* Israel J. Math. **213**(2016), 443–473

[Je80a] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(RR^m)$, $m \geq 5$,* Duke Math. J. **47**(1980), 57-80.

[Je80b] A. Jensen, *Resonances in an abstract analytic scattering theory,* Ann. Inst. H. Poincaré Sect. A (N.S.) **33**(1980).

[Je81] A. Jensen, *Time delay in potential scattering: Some "geometric" results,* Comm. Math. Phys. **82**(1981), 436–456.

[JK79] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions,* Duke Math. J. **46**(1979), 583–611.

[JN01] A. Jensen and G. Nenciu, *A unified approach to resolvent expansions at thresholds,* Rev. Math. Phys. **13**(2001), 717–754. (Erratum ibid. **16**(2004), 675–677.)

[JN06] A. Jensen and G. Nenciu, *The Fermi Golden Rule and its form at thresholds in odd dimensions,* Comm. Math. Phys. **261**(2006), 693–727.

[Ji14] L. Jin, *Resonance-free region in scattering by a strictly convex obstacle,* Ark. Mat. **52**(2014), 257–289.

[Ji15] L. Jin, *Scattering resonances of convex obstacles for general boundary conditions,* Comm. Math. Phys. **335**(2015), 759–807.

[JZ16] L. Jin and M. Zworski, *A local trace formula for Anosov flows,* with an appendix by F. Naud, Ann. Henri Poincaré (A) **18**(2017), 1–35.

[K*97] J.A. Katine, M.A. Eriksson, A.S. Adourian, R.M. Westervelt, J.D. Edwards, A. Lupu-Sax, E.J. Heller, K.L. Campman and A.C. Gossard, *Point contact conductance of an open resonator,* Phys. Rev. Lett. **79**(1997), 4806–4809.

[Ka80] T. Kato, *Perturbation theory of linear operators,* Springer Verlag, Berlin, Heidelberg, 1980.

[KZ95] F. Klopp and M. Zworski, *Generic simplicity of resonances,* Helv. Phys. Acta **68**(1995), 531–538.

[KV19] F. Klopp and M. Vogel, *Semiclassical resolvent estimates for bounded potentials,* Pure and Applied Analysis, **1**(2019), 1–25.

[KS99] K. Kokkotas and B. Schmidt, *Quasi-Normal Modes of Stars and Black Holes,* Living Rev. Relativity **2**(1999), 2.

[KMBK13] M.J. Körber, M. Michler, A. Bäcker, and R. Ketzmerick, *Hierarchical fractal Weyl laws for chaotic resonance states in open mixed systems,* Phys. Rev. Lett. **111**(2013), 114102.

[Ko04] E. Korotyaev, *Stability for inverse resonance problem,* Int Math Res Notices (2004) 2004 (73), 3927–3936.

[Ko05] E. Korotyaev, *Inverse resonance scattering on the real line,* Inverse Problems **21**(2005), 325–241.

[Ko14] E. Korotyaev, *Lieb-Thirring type inequality for resonances,* Bull. Math. Sci. **7**(2017), 211–217.

[LP68] P. D. Lax and R.S. Phillips, *Scattering theory,* Academic Press, 1968.

[LP76] P. D. Lax and R.S. Phillips, *Scattering theory for automorphic forms,* Annals of Math. Studies **87**, Princeton, 1976.

[LP78] P. D. Lax and R.S. Phillips, *The time delay operator and a related trace formula,* Topics in functional analysis. Advances in Nath. Suppl.Studies **3**(1978), 197–295.

[LP82] P.D. Lax and R.S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces,* J. Funct. Anal. **46**(1982), 280–350.

[Le84] G. Lebeau, *Régularité Gevrey 3 pour la diffraction,* Comm. Partial Differential Equations **9**(1984), 1437–1494.

[LZ16] M. Lee and M. Zworski, *A Fermi golden rule for quantum graphs,* J. Math. Phys. **57**(2016), 092101, 17 pp.

[Le64] B. Ja. Levin, *Distribution of zeros of entire functions,* American Mathematical Society Translations of Mathematical Monographs, Volume 5, 1964.

[Li02] K.K. Lin, *Numerical study of quantum resonances in chaotic scattering,* J. Comp. Phys. **176**(2002), 295-329.

[LP93] P.-L. Lions and T. Paul, *Sur les mesures de Wigner,* Rev. Mat. Iberoamericana **9**(1993), 553–618.

[LSZ03] W. Lu, S. Sridhar, and M. Zworski, *Fractal Weyl laws for chaotic open systems,* Phys. Rev. Let. **91**(2003), 154101.

[WMS88] M. Wei, G. Majda and W. Strauss, *Numerical computation of the scattering frequencies for acoustic wave equations,* J. Comp. Phys. **75**(1988), 345–358.

[Ma88] N. Mandouvalos, *Spectral theory and Eisenstein series for Kleinian groups,* Proc. London Math. Soc. **57**(1988), 209–238.

[Mar88] A.S. Markus, *Introduction to the spectral theory of polynomial operator pencils,* Translations of Mathematical Monographs **71**, American Mathematical Society, Providence, RI, 1988.

[Ma02a] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis,* Springer, 2002.

[Ma02b] A. Martinez, *Resonance free domains for non globally analytic potentials,* Ann. Henri Poincaré **4**(2002), 739–756. (Erratum ibid. **8**(2007), 1425–1431.)

[Maz] R.R. Mazzeo, *Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds,* Amer. J. Math. **113**(1991), 25–45.

[MM87] R.R. Mazzeo and R.B. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature,* J. Funct. Anal. **75**(1987), 260–310.

[Me85] A. Melin, *Operator methods for inverse scattering on the real line,* Comm. Partial Differential Equations **10**(1985), 677–766.

[Me82] R.B. Melrose, *Scattering theory and the trace formula of the wave group,* J. Funct. Anal. **45**(1982), 429–440.

[Me84a] R.B. Melrose, *Growth estimates for the poles in potential scattering,* unpublished manuscript, 1984.

[Me84b] R.B. Melrose, *Polynomial bounds on the distribution of poles in scattering by an obstacle,* Journées "Équations aux Dérivées partielles", Saint-Jean de Monts, 1984.

[Me88] R.B. Melrose, *Weyl asymptotics for the phase in obstacle scattering,* Comm. Partial Differential Equations **13**(1988), 1431–1439.

[Me94] R.B. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces,* in *Spectral and scattering theory* (M. Ikawa, ed.), Marcel Dekker, 1994.

[Me95] R.B. Melrose, *Geometric Scattering Theory,* Cambridge University Press, 1995.

[MSV14] R.B. Melrose, A. Sá Barreto and A. Vasy, *Analytic continuation and semiclassical resolvent estimates on asymptotically hyperbolic spaces,* Comm. Partial Differential Equations **39**(2014), 452–511.

[MU] R.B. Melrose and G. Uhlmann, *An introduction to microlocal analysis,* on-line book, http://www-math.mit.edu/~rbm/books/imaast.pdf

[MS99] M. Merkli and I.M. Sigal, *A time-dependent theory of quantum resonances,* Comm. Math. Phys. **201**(1999), 549–576.

[M*08] C.R. Moon, L.S. Mattos, B. K. Foster, G. Zeltzer, W.Ko and H. C. Manoharan, *Quantum Phase Extraction in Isospectral Electronic Nanostructures,* Science **319**(2008), 782–787.

[Mo61] C. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation,* Comm. Pure. Appl. Math. **14**(1961), 561–568.

[MRS77] C. Morawetz, J. Ralston, and W. Strauss, *Decay of solutions of the wave equation outside nontrapping obstacles,* Comm. Pure. Appl. Math. **30**(1977), 447–508.

[Mu82] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations,* J. Funct. Anal. **49**(1982), 10–56.

[NSZ03] S. Nakamura, P. Stefanov, and M. Zworski, *Resonance expansions of propagators in the presence of potential barriers,* J. Funct. Anal. **205**(2003), 180–205.

[Na14] F. Naud, *Density and location of resonances for convex co-compact hyperbolic surfaces,* Invent. Math. **195**(2014), 723–750.

[Na15] F. Naud, *Borne de Weyl fractale et résonances,* Séminaire BOURBAKI, Novembre 2015 68ème année, 2015–2016, no. 1107, http://www.bourbaki.ens.fr/TEXTES/1107.pdf

[Ne04] L. Nedelec, *Multiplicity of resonances in black box scattering,* Canad. Math. Bull. **47**(2004), 407–416.

[N02] R.G. Newton, *Scattering theory of waves and particles,* 2nd Edition, Dover, 2002.

[No11] S. Nonnenmacher, *Spectral problems in open quantum chaos,* Nonlinearity **24**(2011), R123–R167.

[NS08] S. Nonnenmacher and E. Schenck, *Resonance distribution in open quantum chaotic systems,* Phys. Rev. E **78**(2008), 045202.

[NSZ11] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, *From open quantum systems to open quantum maps,* Comm. Math. Phys. **304**(2011), 1–48.

[NSZ14] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, *Fractal Weyl law for open quantum chaotic maps,* Ann. of Math. (2) **179**(2014), 179–251.

[NZ07] S. Nonnenmacher and M. Zworski, *Distribution of resonances for open quantum maps,* Comm. Math. Phys. **269**(2007), 311–365.

[NZ09a] S. Nonnenmacher and M. Zworski, *Quantum decay rates in chaotic scattering,* Acta Math. **203**(2009), 149–233.

[NZ09b] S. Nonnenmacher and M. Zworski, *Semiclassical resolvent estimates in chaotic scattering,* Applied Mathematics Research eXpress 2009; doi: 10.1093/amrx/abp003

[NZ15] S. Nonnenmacher and M. Zworski, *Decay of correlations for normally hyperbolic trapping,* Invent. Math. **200**(2015), 345–438.

[OW16] H. Oh and D. Winter, *Uniform exponential mixing and resonance free regions for convex co-compact congruence subgroups of $SL_2(\mathbb{Z})$,* Journal of AMS, **29**(2016), 1069–1115.

[Pa75] S. J. Patterson, *The Laplacian operator on a Riemann surface I, II, III,* Comp. Math. **31**(1975), 83–107; **32**(1976), 71–112; **33**(1976), 227–259.

[Pa76] S. J. Patterson, *The limit set of a Fuchsian group,* Acta Math. **136**(1976), 241–273.

[Pe89] P. A. Perry, *The Laplace operator on a hyperbolic manifold. II. Eisenstein series and the scattering matrix,* J. Reine Angew. Math. **398**(1989), 67–91.

[PS10] V. Petkov and L. Stoyanov, *Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function,* Anal.&PDE **3**(2010), 427–489.

[PZ99] V. Petkov and M. Zworski, *Breit–Wigner approximation and distribution of resonances,* Comm. Math. Phys. **204**(1999), 329–351, Erratum, Comm. Math. Phys. **214**(2000), 733–735.

[PZ01] V. Petkov and M. Zworski, *Semi-classical estimates on the scattering determinant,* Ann. H. Poincaré, **2**(2001), 675–711.

[PS85] R.S. Phillips and P. Sarnak, *On cusp forms for co-finite subgroups of $PSL(2,\mathbb{R})$,* Invent. Math. **80**(1985), 339–364.

[Po85] G. Popov, *Asymptotics of Green's functions in the shadow,* C. R. Acad. Bulgare Sci. **38**(1985), 1287–1290.

[P*12] A. Potzuweit, T. Weich, S. Barkhofen, U. Kuhl, H.-J. Stöckmann, and M. Zworski, *Weyl asymptotics: from closed to open systems,* Phys. Rev. E. **86**(2012), 066205.

[Pr95] G. R. Prony, *Essai éxperimental et analytique: sur les lois de la dilatabilité,* J. École Polytechnique, Floréal et Plairial, an III (1795), **1**, cahier 22, 24-76.

[Ra69] J. Ralston, *Solutions of the wave equation with localized energy,* Comm. Pure Appl. Math. **22**(1969), 807–823.

[Ra78] J. Ralston, *Addendum to: "The first variation of the scattering matrix"* (J. Differential Equations **21**(1976), no. 2, 378–394) by J. W. Helton and Ralston. J. Differential Equations **28**(1978), no. 1, 155–162.

[RS80] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I, Functional Analysis,* Academic Press, 1980.

[RS79] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. III, Scattering Theory,* Academic Press, 1979.

[Re58] T. Regge, *Analytic properties of the scattering matrix,* Il Nuovo Cimento, **8**(1958), 671–679.

[Re07] W. P. Reinhardt, *Complex Scaling in Atomic and Molecular Physics, In and Out of External Fields,* AMS Proceedings Series: Proceedings of Symposia in Pure Mathematics **76**(2007), 357–377.

[RM95] U.V. Riss and H.D. Meyer, *Reflection-Free Complex Absorbing Potentials,* J. Phys. B **28**(1995), 1475–1493.

[Ro05] D. Robert, *Autour de l'approximation semi-classique,* **128** (2005), Progress in Mathematics **68**, Birkhauser 1987.

[RT15] I. Rodnianski and T. Tao, *Effective limiting absorption principles and applications,* Comm. Math. Phys. **333**(2015), 1–95.

[Ro01] M. Rouleux, *Absence of resonances for semiclassical Schrödinger operators with Gevrey coefficients,* Hokkaido Math. J. **30**(2001), 475–517.

[Sá01] A. Sá Barreto, *Remarks on the distribution of resonances in odd dimensional Euclidean scattering,* Asymptot. Anal. **27**(2001), 161–170.

[SZ96] A. Sá Barreto and M. Zworski, *Existence of resonances in potential scattering,* Comm. Pure Appl. Math. **49**(1996), 1271–1280.

[SZ97] A. Sá Barreto and M. Zworski, *Distribution of resonances for spherical black holes,* Math. Res. Lett. **4**(1997), 103–121.

[Sc07] W. Schlag, *Dispersive estimates for Schrödinger operators: a survey,* in *Mathematical aspects of nonlinear dispersive equations,* 255–285, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[SWW06] R. Schubert, H. Waalkens, S. Wiggins, *Efficient computation of transition state resonances and reaction rates from a quantum normal form,* Phys. Rev. Lett. **96**(2006), 218302.

[Se53] A. Selberg, *Göttingen lecture notes* in *Collected papers, vol. 1* Springer, Berlin, with a foreword by K. Chandrasekharan (1989).

[Se04] E. Servat, *Résonances en dimension un pour l'opérateur de Schrödinger,* Asymptot. Anal. **39**(2004), 187–224.

[Sh16] J. Shapiro, *Semiclassical resolvent bounds in dimension two,* to appear in Proc. AMS.

[Sh19] J. Shapiro, *Semiclassical resolvent bound for compactly supported L^∞ potentials,* to appear in J. Spec. Theory.

[Si72] B. Simon, *Quadratic form techniques and the Balslev–Combes theorem,* Comm. Math. Phys. **27**(1972), 1–9.

[Si73] B. Simon, *Resonances in n–body quantum systems with dilation analytic potentials and the foundations of time-dependent perturbation theory,* Ann. of Math. **97**(1973), 247–274.

[Si79a] B. Simon, *The definition of molecular resonance curves by the method of exterior complex scaling,* Phys. Lett. **71A**(1979), 211–214.

[Si79b] B. Simon, *Trace ideals and their applications,* LMS Lecture Note Series **35**, Cambridge University Press, Cambridge-New York, 1979.

[Si00] B. Simon, *Resonances in one dimension and Fredholm determinants,* J. Funct. Anal. **178**(2000), 396–420.

[Sj82] J. Sjöstrand, *Singularités analytiques microlocales,* Astérisque **95**, 1982.

[Sj87] J. Sjöstrand, *Semiclassical resonances generated by nondegenerate critical points,* in Pseudodifferential operators (Oberwolfach, 1986), 402–429, Lecture Notes in Math. **1256**, Springer, Berlin, 1987.

[Sj90] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems,* Duke Math. J. **60**(1990), 1–57.

[Sj96a] J. Sjöstrand, *A trace formula and review of some estimates for resonances,* in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[Sj96b] J. Sjöstrand, *A trace formula for resonances and application to semi-classical Schrödinger operators,* Séminaire Équations aux dérivées partielles (1996-1997) Volume 1996–1997, 1–13.

[Sj02] J. Sjöstrand, *Lectures on resonances,* version préliminaire, printemps 2002.

[Sj14] J. Sjöstrand, *Weyl law for semi-classical resonances with randomly perturbed potentials,* Mém. Soc. Math. Fr. (N.S.) **136**(2014), vi+144 pp.

[SV97] J. Sjöstrand and G. Vodev, *Asymptotics of the number of Rayleigh resonances. With an appendix by Jean Lannes,* Math. Ann. **309**(1997), 287–306.

[SZ91] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles,* J. Amer. Math. Soc. **4**(1991), 729–769.

[SZ93] J. Sjöstrand and M. Zworski, *Lower bounds on the number of scattering poles,* Comm. Partial Differential Equations, **18**(1993), 847-858.

[SZ94] J. Sjöstrand and M. Zworski, *Lower bounds on the number of scattering poles II,* J. Func. Anal. **123**(1994), 336-367.

[SZ95] J. Sjöstrand and M. Zworski, *The complex scaling method for scattering by strictly convex obstacles,* Ark. Mat. **33**(1995), 135–172.

[SZ99] J. Sjöstrand and M. Zworski, *Asymptotic distribution of resonances for convex obstacles,* Acta Math. **183**(1999), 191–253.

[SZ02] J. Sjöstrand and M. Zworski, *Quantum monodromy and semiclassical trace formulae,* J. Math. Pure Appl. **81** (2002), 1–33.

[SZ07a] J. Sjöstrand and M. Zworski, *Fractal upper bounds on the density of semiclassical resonances,* Duke Math. J. **137**(2007), 381–459.

[SZ07b] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems,* Ann. Inst. Fourier **57**(2007), 2095–2141.

[SM92] T. Seideman and W.H. Miller, *Calculation of the cumulative reaction probability via a discrete variable representation with absorbing boundary conditions,* J. Chem. Phys. **96**(1992), 4412–4422.

[SZ16] H. Smith and M. Zworski, *Heat traces and existence of scattering resonances for bounded potentials,* Ann. Inst. Fourier **66**(2016), 455–475.

[SW98] A. Soffer and M.I. Weinstein, *Time dependent resonance theory,* Geom. Funct. Anal. **8**(1998), 1086–1128.

[St94] P. Stefanov, *Stability of resonances under smooth perturbations of the boundary,* Asymptotic Anal. **9**(1994), 291–296.

[St99] P. Stefanov, *Quasimodes and resonances: sharp lower bounds,* Duke Math. J. **99**(1999), 75–92.

[St00] P. Stefanov, *Resonances near the real axis imply existence of quasimodes,* C. R. Acad. Sci. Paris Sér. I Math., 330(2):105-108, 2000.

[St01] P. Stefanov, *Resonance expansions and Rayleigh waves,* Math. Res. Lett. **8**(2001), 107–124.

[St05] P. Stefanov, *Approximating resonances with the complex absorbing potential method,* Comm. Partial Differential Equations **30**(2005), 1843–1862.

[St06] P. Stefanov, *Sharp upper bounds on the number of the scattering poles,* J. Funct. Anal. **231**(2006), 111–142.

[SV96] P. Stefanov and G. Vodev, *Neumann resonances in linear elasticity for an arbitrary body,* Comm. Math. Phys. **176**(1996), 645–659.

[Su79] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions,* Inst. Hautes Études Sci. Publ. Math. **50**(1979), 171–202.

[Ta15] H. Tamura, *Aharonov–Bohm effect in resonances for scattering by three solenoids at large separation,* preprint 2015; *Aharonov–Bohm effect in resonances for scattering by three solenoids,* Proc. Japan Acad. Ser. A Math. Sci. **91**(2015), 45–49.

[TZ98] S.H. Tang and M. Zworski, *From quasimodes to resonances,* Math. Res. Lett. **5**(1998), 261–272.

[TZ00] S.H. Tang and M. Zworski, *Resonance expansions of scattered waves,* Comm. Pure Appl. Math. **53**(2000), 1305–1334.

[TZ01] S.H. Tang and M. Zworski, *Potential scattering on the real line,* unpublished notes, `https://math.berkeley.edu/~zworski/tz1.pdf`

[TaI] M.E. Taylor, *Partial Differential Equations I – Basic Theory,* Applied Math Sciences series **115**, Springer, Berlin, 2011.

[TaII] M.E. Taylor, *Partial Differential Equations II – Qualitative Studies of Linear Equations,* Applied Math Sciences series **116**, Springer, Berlin, 2011.

[Ti26] E.C. Titchmarsh, *The zeros of certain integral functions,* Proc. London Math. Soc. (2) **25**(1926), 283–302.

[Ti39] E.C. Titchmarsh, *The theory of functions,* 2nd edition, Oxford University Press, 1939.

[Ti86] E.C. Titchmarsh, *The theory of the Riemann zeta-function. Second edition,* Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.

[Uh76] K. Uhlenbeck, *Generic properties of eigenfunctions,* Amer. J. Math. **98**(1976), 1059–1078.

[Va73] B.R. Vaĭnberg, *Exterior elliptic problems that depend polynomially on the spectral parameter and the asymptotic behavior for large values of the time of the solutions of nonstationary problems,* (Russian) Mat. Sb. (N.S.) **92**(134)(1973), 224–241.

[Va89] B.R. Vaĭnberg, *Asymptotic Methods in Equations of Mathematical Physics,* Gordon and Breach, 1989.

[V*14] M. Vallisneri et al. *The LIGO Open Science Center,* Proceedings of the 10th LISA Symposium, University of Florida, Gainesville, May 18-23, 2014; arXiv:1410.4839.

[Va13] A. Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces,* with an appendix by Semyon Dyatlov, Invent. Math. **194**(2013), 381–513.

[Va12] A. Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates,* Inverse problems and applications. Inside Out II, edited by Gunther Uhlmann, Cambridge University Press, MSRI publications **60**(2012).

[Va18] A. Vasy, *Resolvent on Riemannian scattering (asymptotically conic) spaces,* arXiv:1808.06123.

[VZ00] A. Vasy and M. Zworski, *Semiclassical estimates in asymptotically Euclidean scattering,* Comm. Math. Phys. **212** (2000), 205–217.

[Vo92] G. Vodev, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian,* Comm. Math. Phys. **146**(1992), 205–216.

[Vo94a] G. Vodev, *Sharp bounds on the number of scattering poles in even-dimensional spaces,* Duke Math. J. Volume **74**(1994), 1–17.

[Vo94b] G. Vodev, *Sharp bounds on the number of scattering poles in the two dimensional case,* Math. Nachrichten, **170**(1994), 287–297.

[Vo00] G. Vodev, *Exponential bounds of the resolvent for a class of noncompactly supported perturbations of the Laplacian, Math. Res. Lett*, 7:3, 287–298, 2000.

[Vo13] G. Vodev, *Semi-classical resolvent estimates for Schrödinger operators, Asymptot. Anal.* 81:2, 157–170, 2013.

[Vo14] G. Vodev. *Semi-classical resolvent estimates and regions free of resonances, Math. Nach.* **287**(2014), 825–835.

[Vo19] G. Vodev, *Semi-classical resolvent estimates for short-range L^∞ potentials,* Pure and Applied Analysis **1**(2019), 207–214.

[Wa15] C. Warnick, *On quasinormal modes of asymptotically Anti-de Sitter black holes,* Comm. Math. Phys. **333**(2015), 959–1035.

[Wi56] C.H. Wilcox, *A generalization of theorems of Rellich and Atkinson,* Proc. A.M.S **7**(1956), 271–276.

[WZ00] J. Wunsch and M. Zworski, *Distribution of resonances for asymptotically euclidean manifolds,* J. Diff. Geom. **55**(2000), 43–82.

[WZ11] J. Wunsch and M. Zworski, *Resolvent estimates for normally hyperbolic trapped sets,* Ann. Inst. Henri Poincaré (A), **12**(2011), 1349–1385.

[Ya92] D.R. Yafaev, *Mathematical Scattering Theory. General Theory,* Translations of Mathematical Monographs **105**, Amer. Math. Soc., 1992.

[Ya09] D.R. Yafaev, *Mathematical scattering theory : analytic theory,* Mathematical Surveys and Monographs **158**, Amer. Math. Soc., 2009.

[Zw87] M. Zworski, *Distribution of poles for scattering on the real line,* J. Funct. Anal. **73**(1987), 277–296.

[Zw89a] M. Zworski, *Sharp polynomial bounds on the number of scattering poles of radial potentials,* J. of Funct. Anal. **82**(1989), 370–403.

[Zw89b] M. Zworski, *Sharp polynomial bounds on the number of scattering poles,* Duke Math. J. **59**(1989), 311–323.

[Zw90] M. Zworski, *unpublished,* 1990.

[Zw94] M. Zworski, *Counting scattering poles,* in *Spectral and scattering theory,* M. Ikawa ed., Lecture notes in pure and applied mathematics **161**, Marcel Dekker 1994.

[Zw97] M. Zworski, *Poisson formula for resonances,* Séminaire E.D.P. (1996–1997), Exposé no XIII,

[Zw98] M. Zworski, *Poisson formula for resonances in even dimensions,* Asian J. Math. **2**(1998), 609–617.

[Zw99] M. Zworski, *Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces,* Invent. Math. **136**(1999), 353–409.

[Zw01] M. Zworski, *A remark on isopolar potentials,* SIAM Journal on Mathematical Analysis **32**(2001), 1324–1326.

[Zw12] M. Zworski, *Semiclassical Analysis,* Graduate Studies in Mathematics **138**, Amer. Math. Soc., 2012.

[Zw16] M. Zworski, *Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited,* J. Spectr. Theory. **6**(2016), 1087–1114.

[Zw17] M. Zworski, *Mathematical study of scattering resonances,* Bull. Math. Sci. **7**(2017), 1–85.

[Zw18] M. Zworski, *Scattering resonances as viscosity limits,* in "Algebraic and Analytic Microlocal Analysis", M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch, eds. Springer, 2018.

Index

$\bullet^s_\pm$, 488
analytic Fredholm theory, 522
asymptotic expansion, *see* symbol
asymptotically hyperbolic manifold, 316
  canonical coordinates, 318
  even, 320
  even extension, 321

black box Hamiltonian, 229
black box scattering, 225
black hole
  ergoregion, 362
  event horizon, 363
  Kerr–de Sitter, 362
  mode stability, 364
  Schwarzschild–de Sitter, 362, 376
Borel–Carathéodory theorem, 537
boundary defining function, 315
  canonical, 318
Breit–Wigner approximation, 16, 74, 219, 476

C^∞, C^∞_c, 485, 488
$C^j_t H^s_y$, 598
canonical 1-form, 549
Carleman estimates, 144, 419
Cauchy–Green formula, 535
complex scaling, 81, 275
Con_p, 341
control condition, 579
cotangent bundle, 385, 545
  fiber-radially compactified, 548
cutoff chart, 552

$\mathcal{D}'$, 488
D_x, 544
Δ_g, 226
Δ_θ, 279
Diff^k_h, 544
differential operator, 544, 551
domain of operator, 493

$\mathcal{E}'$, 488
ell_h, 567
elliptic, 567
  estimate, 567
  parametrix, 567
energy estimate, 598
escape, 382
escape function, 411, 580, 591
essential support, 564
eventually positive/negative, 589

Fermi Golden Rule, 264, 308
fiber infinity, 548
finite volume surface, 229
Fredholm determinant, 509
free resolvent, 36, 106

$\tilde{g}$, 359
Gårding inequality, 563, 568
$\Gamma^\pm$, $\Gamma^\pm_\bullet$, 382
Gohberg–Sigal theory, 524
Grönwall's inequality, 486
Grushin problem, 516

h, 92, 379, 544
$h^\infty\Psi^{-\infty}$, 554

h-tempered, 569
H_p, 380, 549
H^s_{comp}, H^s_{loc}, H^s, 485, 562
$\bar{H}^s$, $\dot{H}^s$, 564
H^s_h, 560, 561
$H^s_{h,\mathrm{comp}}$, $H^s_{h,\mathrm{loc}}$, 561
$\bar{H}^s_h$, $\dot{H}^s_h$, 564
$H^{s,r}_h$, 605
$\widetilde{H}^s, \widetilde{H}^{s,r}_h$, 606
Hadarmard's factorization theorem, 540
Hamiltonian vector field, 380, 549
hyperbolic cylinder, 317
  associated spacetime, 361
  canonical coordinates, 318
  even extension, 322
hyperbolic operator, 596
  Cauchy problem, 603
  estimate, 596, 597
hyperbolic space, 313
  associated spacetime, 360
  canonical coordinates, 318
  even extension, 321

indicial roots, 323

Jensen's formula, 537

K, $K_\bullet$, 382
κ, 548
Kato's local smoothing, 434

$L_\pm$, 334
L^2, L^2_{loc}, L^2_{comp}, 485
$\mathcal{L}_1(H_1, H_2)$, $\mathcal{L}_1(H)$, 500
$\mathcal{L}(X)$, $\mathcal{L}(X, Y)$, 486
Lax–Phillips theory, 218
Liouville measure, 385
locally finite, 490

m_D, 48
m_R, 39, 119, 238
m_S, 61, 160, 273
$\widetilde{m}_R(0)$, 131, 134
m_V, 88, 119
maximum principle
  rectangle, 536
  semiclassical, 448
Mazzeo–Melrose theory, 371
meromorphic continuation
  modified Laplacian, 350
  resolvent, 37, 118, 233, 353
  scattering matrix, 56, 156, 268
method of nonstationary phase, 550
method of stationary phase, 550
microlocal equivalence, 566
modified Laplacian, 327
  hyperbolic cylinder, 329
  hyperbolic space, 329
modular surface, 269
multiplicity, *see* resonances

normally hyperbolic trapping, 400

$\mathcal{O}(F)_X, \mathcal{O}(F)_{X\to Y}$, 486
$\mathcal{O}(h^\infty)$, 486
obstacle scattering, 229
Op_h, 551, 557
Op^M_h, 557
operator
  closable, 493
  closed, 493
  compactly microlocalized, 566
  compactly supported, 489
  essentially self-adjoint, 494
  Fredholm, 517
  of trace class, 500
  properly supported, 489
  regular, 489
  self-adjoint, 494, 495
  smoothing, 489
  symmetric, 494
  unitary, 494
outgoing resolvent, *see* scattering resolvent
outgoing solution
  asymptotically hyperbolic, 324, 353
  dimension one, 32
  Euclidean scattering, 141, 241, 259

p
  asymptotically hyperbolic, 328
  Schrödinger operator, 380
$P(h)$, 379
$P_h(\omega)$, 327
$P(\lambda)$, 327
$\widetilde{P}(\lambda)$, 366
$P_\psi(\lambda)$, $P_{\psi,h}(\omega)$, 328
P_θ, 279
P_V, 37, 118
$\widetilde{P}(z)$, 396
perfectly matched layer, 83, 309
Poisson bracket, 484
positive commutator method, 413, 582, 585

principal symbol
commutator, 545, 558
differential operator, 545
product, 545, 558
vector field, 545
pseudodifferential operator, 556
product structure, 316
canonical, 318
propagation of singularities, 579
pseudodifferential operator, 544
change of variables, 553
nonsemiclassical, 560
on $\mathbb{R}^n$, 551
on manifolds, 555
partition of unity, 566
Ψ_h^{comp}, 566
Ψ_0^k, 613
$\Psi_{\delta,h}^k$, 560
Ψ_h^k, 555

quantization
on $\mathbb{R}^n$, 551
on manifolds, 557
quasi-normal modes, 366

$R_0(\lambda)$, 36, 106
$R(\lambda)$, 233, 323, 325, 353
$R_\theta(\lambda)$, 280
$R_V(\lambda)$, 37, 119
$R(z, h)$, 93, 380, 388
radial
estimate, 590, 593
point, 588
sets for asymptotically hyperbolic manifolds, 334
source/sink, 587
reference operator, 244
Rellich's uniqueness theorem, 142, 145, 259, 354
Rellich–Kondrachov's theorem, 493
resolvent, *see* scattering resolvent
resonance at zero, 134
obstacle scattering, 261
one dimension, 45
potential scattering, 126
resonance counting
asymptotics
convex obstacles, 310
normally hyperbolic trapping, 402, 406
one dimension, 62
lower bounds
from closed orbits, 427, 476
semiclassical, 461
upper bounds
black box scattering, 249
fractal, 475
potential scattering, 136, 217
semiclassical, 440
resonance expansion
black hole, 368
non-trapping black box, 307
potential scattering, 50, 121
strong trapping
Schrödinger equation, 468
wave equation, 472
resonance free regions
general semiclassical, 101, 423
logarithmic
non-trapping black box, 302
non-trapping semiclassical, 413
potential scattering, 51, 120
semiclassical overview, 381
strips
black hole, 367
hyperbolic trapping, 427
non-trapping, 388
normally hyperbolic trapping, 403
resonances
asymptotically hyperbolic, 352, 353
black box Hamiltonians, 238
black hole, 366
hyperbolic cylinder, 355
hyperbolic space, 355
multiplicity
agreement with complex scaling, 289
agreement with scattering matrix, 61, 160, 273
black box scattering, 238
generic simplicity, 89, 292
potential scattering, 39, 119
one dimension, 39
potential scattering, 119
shape, 18, 95, 103
resonant state
asymptotically hyperbolic, 352, 353
black box Hamiltonians, 241
potential scattering, 42, 120
$\text{Res}(P)$, $\text{Res}(P(h))$, 93, 380
Rouché's Theorem for operators, 526, 532

$S_{1,0}^k$, 546

$\overline{S}^k_{1,0}$,$\overline{S}^k$,$\overline{S}^k_h$, 550
S^k, 546
S^k_δ, 559
S^k_h, 546
$S^{-\infty}$, 547
scattering matrix
  black box scattering, 267
  modular surface, 272
  one dimension, 55
  potential scattering, 152
  surfaces with cusps, 269
  time dependent definition, 218
scattering resolvent, 39, 119
  asymptotically hyperbolic, 353
  black box scattering, 233
  lower bounds, 434
  potential scattering, 37
  semiclassical, 380
  upper bounds
    non-trapping, 51, 120, 302, 393, 398, 413
    normally hyperbolic trapping, 403
    trapping, 423, 443
Schur's inequality, 487
Schwartz kernel, 489
semiclassical measure, 394, 574
semiclassically outgoing, 357, 391, 397
σ_h, 545, 556
$\Sigma_\pm$, $\widehat{\Sigma}_\pm$, 332
singular values, 497
Sobolev space, 560
  manifold with boundary, 563
spectral gap, *see* resonance free regions
spectrum, 494
`splinepot.m`, 19
`squarepot.m`, 13, 75
Stone's formula, 495
symbol, 545
  asymptotic expansion, 546
  polyhomogeneous, 546
  positively homogeneous, 546
symplectic form, 549

$\overline{T}^*M$, 548
tails, incoming/outgoing, 382
trace formula
  Bardos–Guillot–Ralston, 219
  Birman–Kreĭn, 69, 172
  for resonances, 78, 188, 454
  Lax–Phillips, 219
  Melrose, 188, 219
  Selberg, 370
  Sjöstrand, 454
trapped set, 382
  homoclinic, 427, 467
  hyperbolic, 427
  normally hyperbolic, 400, 427

wavefront set, 564
  composition, 573
  distribution, 570
  nonsemiclassical, 614
  operator, 570
  pseudodifferential operator, 565
Weierstrass product, 540
Weyl inequalities, 507
WF, 614
WF_h, 565, 570
WF'_h, 570

Young's inequality, 487

zero resonance, *see* resonance at zero