MATHEMATICAL THEORY OF
SCATTERING RESONANCES

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Semyon Dyatlov, UC Berkeley and MIT

Maciej Zworski, UC Berkeley
The purpose of this book is to provide a broad introduction to the theory of scattering resonances.

Scattering resonances appear in many branches of mathematics, physics and engineering. They generalize eigenvalues or bound states for systems in which energy can scatter to infinity. A typical state has then a rate of oscillation (just as a bound state does) and a rate of decay. Although the notion is intrinsically dynamical, an elegant mathematical formulation comes from considering meromorphic continuations of Green’s functions or scattering matrices. The poles of these meromorphic continuations capture the physical information by identifying the rate of oscillations with the real part of a pole and the rate of decay with its imaginary part. The resonant state, which is the corresponding wave function, then appears in the residue of the meromorphically continued operator. An example from pure mathematics is given by the zeros of the Riemann zeta function: they are, essentially, the resonances of the Laplacian on the modular surface. The Riemann hypothesis then states that the decay rates for the modular surfaces are all either 0 or $\frac{1}{2}$. A standard example from physics is given by shape resonances created when the interaction region is separated from free space by a potential barrier. The decay rate is then exponentially small in a way depending on the width of the barrier.

In the book we provide an introduction to mathematical techniques used in the study of scattering resonances, concentrating on simplest models but providing references to modern literature and indications of what happens in more general situations. Some chapters (such as Chapter 2 and 3) are meant to be easily accessible and others (such as Chapter 5) somewhat more demanding. The rather substantial set of appendices provides detailed
accounts of most methods needed in the text. A diagram representing the dependencies of various sections is presented at the end of Chapter 1. The choice of topics is necessarily determined by the research interests of the authors and many important aspects of the subject are not covered. We also stayed away from exciting but technical developments such as precise asymptotics for shape resonances, fractal Weyl laws or resonance gaps for chaotic systems.

SD was introduced to scattering resonances by MZ who in turn had the good fortune to be introduced to this field by Richard Melrose. We would like to thank him for his generous guidance and insights and for his foundational results on resonance counting and trace formulas. The viewpoint and many discoveries of Johannes Sjöstrand changed the subject in a fundamental way. MZ was fortunate to maintain a long collaboration with him and would like to thank him for sharing his ideas and expertise over the years. Many other colleagues and collaborators have contributed to our understanding of the subject and special thanks are due to Ivana Alexandrova, Nicolas Burq, Tanya Christiansen, Kiril Datchev, Frédéric Faure, Jeff Galkowski, Colin Guillarmou, Laurent Guillopé, Bernard Helffer, Peter Hintz, Jin Long, Frédéric Naud, Stéphane Nonnenmacher, Galina Perelman, Vesselin Petkov, Antônio Sá Barreto, Plamen Stefanov, Siu-Hung Tang, Jared Wunsch, András Vasy and Georgi Vodev.

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Chapter 2 developed from notes on one dimensional scattering written with Siu-Hung Tang in 2001 [TZ01] – I am grateful to my co-author for his help on that project and for allowing me to use our material.

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Chapter 1

INTRODUCTION

1.1 Resonances in scattering theory
1.2 Semiclassical study of resonances
1.3 Some examples
1.4 Overview

1.1. RESONANCES IN SCATTERING THEORY

Scattering resonances are the replacement of discrete spectral data for problems on non-compact domains. Although this book is intended for mathematical audience and it concentrates on rigorous presentation, a physical motivation plays an essential role in the study of scattering resonances. Even when, as for instance in scattering on the modular surface, the questions have purely mathematical context, the origins lie in physics and it is easiest to relate them in the setting of quantum mechanics.

In quantum mechanics a particle is described by a wave function \( \psi_k \) which is normalized in \( L^2 \), \( \| \psi_k \|_{L^2} = 1 \). The probability of finding the particle in a region \( \Omega \) is the given by the integral of \( |\psi_k(x)|^2 \) over \( \Omega \). A pure state is typically an eigenstate of a Hamiltonian \( P \) and hence the evolved state is given by \( \psi_k(t) := e^{-itP} \psi_k = e^{-itE_k} \psi_k \) where \( P \psi_k = E_k \psi_k \). In particular the probability density does not change when the state is propagated. An example could be given by the Bloch electron in a quantum corral shown in Figures 1.1 and 1.9. However the same figure also shows that the measured states have non-zero “widths” – the peak is not a delta function at \( E_k \) – and hence can be more accurately modeled by resonant states. Scattering or quantum resonances are given as complex numbers \( E_k - i\Gamma_k/2 \) and the
following standard argument of the physics literature explains the meaning of the real and imaginary parts: a time dependent pure resonant state propagates according to \( \psi_k(t) = e^{-itE_k - t\Gamma_k/2} \psi_k(0) \) so that the probability of survival beyond time \( t \) is \( p(t) = |\psi_k(t)|^2/|\psi_k(0)|^2 = e^{-\Gamma_k t} \). This explains why the convention for the imaginary part of a resonance is \( \Gamma_k/2 \). Here we neglected the issue that \( \psi_k(0) \notin L^2 \) which is remedied by taking the probabilities over a bounded interaction region. In the energy representation the wave function is given

\[
\varphi_k(E) := \mathcal{F}^{-1}\psi_k(E) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-it(E_k - E) - t\Gamma_k/2 + i\psi_k(0)} dt
\]

\[
= \frac{1}{\sqrt{2\pi i}} \frac{\psi_k(0)}{E_k - i\Gamma_k/2 - E},
\]

which means that the probability density of the resonance with energy \( E \) is proportional to the square of the absolute value of the right hand side.
If $U(t)$ is a propagator and $f,g$ are states then the correlation is given by $\rho_{f,g}(t) = \langle U(t)f, g \rangle$. For instance the evolution could come from a flow $\varphi_t : M \to M$ on a compact manifold, $U(t)f(x) = f(\varphi_{-t}(x))$. The power spectrum of correlations is given by $\hat{\rho}_{f,g}(\lambda) := \int_0^\infty \rho_{f,g}(t) e^{i\lambda t} \, dt$. Resonances are the poles of $\hat{\rho}_{f,g}(\lambda)$ and are independent of $f$ and $g$. The figure shows a schematic correspondence between the power spectrum for different states and these poles: the real part corresponds to the location of a peak in the power spectrum and the imaginary to its width; the $x$ axis is the real part (frequency $\lambda$), $y$ axis the imaginary (rate of decay), $z$ axis $|\hat{\rho}_{f,g}(\lambda)|$. Unlike the power spectrum which depends on $f$ and $g$, the poles depend only on the system.

Consequently the probability density is

\begin{equation}
\frac{1}{2\pi} \frac{\Gamma_k}{(E - E_k)^2 + (\Gamma_k/2)^2} \, dE,
\end{equation}

and this Lorentzian is the famous Breit-Wigner distribution. In practice there are many deviations from this simple formula, especially at high energies and in the presence of overlapping resonances. In Figures 1.1 and 1.9 we see clear Lorentzian peaks and individual resonances can be recovered. In experiment shown in Fig.1.10 the resonances overlap and the peaks in scattering data do not have the simple interpretation using (1.1.1). Weyl laws for counting of resonant states are more complicated and richer as they involve both energy and rates of decay. Even the leading term can be affected by dynamical properties of the system.
A more abstract version of Figures 1.1 and 1.9 is shown in Fig. 1.2. The dependence of a measured quantity \( (dI/dV) \) on energy can be interpreted as power spectrum of correlations with peaks corresponding to poles of the meromorphic continuation of the power spectrum.

A simple mathematical example – studied in detail in Chapter 2 – is given by scattering by compactly supported potentials in dimension one, see Fig. 1.3 for an example of such potential. Scattering resonances are the rates of oscillations and decay of the solution of the wave equation and Fig.1.4 shows such a solution. We see the main wave escape and some trapped waves bounce in the well created by the potential and leak out. Fig. 1.5 shows the values of the solution at one point. Roughly speaking if \( u(t,x) \) is a solution of the wave equation \( (\partial_t^2 - \partial_x^2 + V(x))u = 0 \) with localized initial
data then
\[ u(t, x) \sim \sum_{\text{Im } \lambda_j > -A} e^{-i\lambda_j t} u_j(x) + O_K(e^{-tA}), \quad x \in K \subseteq \mathbb{R}, \]
where \( \lambda_j \) are complex numbers with \( \text{Im } \lambda_j < 0 \). They are independent of the initial data and are precisely the scattering resonances – see Theorem 2.7 for the precise statement.

Harmonic inversion methods, the first being the celebrated Prony algorithm [Pr95], can then be used to extract scattering resonances (see for instance [WMS88]). The resulting complex numbers, that is the resonances for the potential in Fig. 1.3 are shown in Fig. 1.6.

1.2. SEMICLASSICAL STUDY OF RESONANCES

For some very special systems resonances can be computed explicitly. One famous example is the Eckart barrier: \( -\partial_x^2 + \cosh^{-2} x \). It falls into the
1. SEMICLASSICAL STUDY OF RESONANCES

The general class of Pöschel–Teller potentials which can also be used to compute resonances of hyperbolic spaces or hyperbolic cylinders – see for instance [GZ95a] or [Bo16]. Another example is given by the sphere in which case scattering resonances are zeros of Hankel functions which can be described asymptotically – see [St06] and Fig. 1.7.

![Figure 1.7](image)

Figure 1.7. Resonances for the sphere in three dimensions. For each spherical momentum $\ell$ they are given by solutions of $H^{(2)}_{\ell+\frac{1}{2}}(\lambda) = 0$ where $H^{(2)}_\nu$ is the Hankel function of order $\nu$. Each zero appears as a resonance of multiplicity $2\ell + 1$.

In general however it is impossible to obtain an explicit description of individual resonances. Hence we need to consider their properties and their distribution in asymptotic regimes. For instance in the case of obstacle scattering that could mean the high energy limit. In the case of the sphere in Fig. 1.7 that corresponds to letting the angular momentum $\ell \to +\infty$. For a general obstacle that means considering resonances as $|\lambda| \to +\infty$ and $|\Im \lambda| \ll |\lambda|$.

The high energy limit is a special case of the semiclassical limit. For instance we can consider resonances of the Dirichlet realization of $-\hbar^2 \Delta + V$ on $\mathbb{R}^n \setminus \mathcal{O}$ in bounded subsets of $\mathbb{C}$ as $\hbar \to 0$. When $V \equiv 0$ that corresponds to the high energy limit for obstacle problems and when $\mathcal{O} = \emptyset$ to Schrödinger operators.

In the case of semiclassical Schrödinger operators, the properties of the classical energy surface $\xi^2 + V(x) = E$ can be used to study resonances close to $E \in \mathbb{R}$. Some aspects of that will be presented in §2.8 in one dimension and in Chapters 6 and 7 in more depth. Figure 1.8 shows some of the principles in dimension one. The last set of resonances shown there and labeled as Regge resonances comes from the singularities at the boundary of the support of the potential $V$ shown there. Roughly speaking these
1.3. SOME EXAMPLES

We present here a few recent examples of scattering resonances appearing in physical systems.

Fig. 1.1 shows resonance peaks in a scanning tunneling microscope (STM) experiment by Crommie et al [CLE93]: the position of the peaks corresponds to the resonant states trapped by the corral. The trapping is much more pronounced in the experiment shown in Fig. 1.9. It shows resonance peaks for a STM experiment where isospectral quantum corrals of CO molecules, instead of less densely packed iron atoms, are constructed – see [M08] and references given there. The resonances are very close to eigenvalues of the Dirichlet Laplacian (rescaled by $h^2/m_{\text{eff}}$ where $m_{\text{eff}}$ is the
Figure 1.9. A scanning tunnelling microscope (STM) experiment from the Manoharan Lab at Stanford University [M’08]. The figure shows design and realization of quantum isospectral resonators each assembled from 90 CO molecules on the Cu(111) surface. A scanning tunneling microscope (STM) spectrum is a plot of $dI/dV$ ($I$ being the current) as a function of bias voltage $V$. Each spectrum shows the series of surface state electron resonances inside a bounded quantum corral – see Figure 1.2 for a visualization of the relation between the peaks and the complex resonances. (A to C) Schematics and STM topographies of the Bilby (A), Hawk (B), and Broken Hawk (C) domains. The seven identical $\pi/6$–$\pi/3$–$\pi/2$ triangles composing each shape are shown in red. Blue dots indicate the positions of wall molecules. White crosses mark locations where $dI/dV$ spectroscopy was performed. STM topographs are 15 nm by 15 nm ($V = 10$ mV, $I = 1$ nA). A single CO molecule used for registration between spectra accompanies each nanostructure. (D to F) Spectral fingerprints ($dI/dV$ spectra) acquired throughout Bilby (D), Hawk (E), and Broken Hawk (F). (G) The normalized averages of the Bilby and Hawk spectra match closely, consistent with isospectrality, whereas the average Broken Hawk spectrum clearly differs. Inset: Spectral correlation plot (dashed line denotes perfect match) quantifying Bilby-Hawk isospectrality and its departure in Broken Hawk.
1.3. SOME EXAMPLES

Figure 1.10. The experimental set-up used by the Stöckmann group in Marburg and the Kuhl group in Nice [B\textsuperscript{13}],[P\textsuperscript{12}]. A three disc symmetry reduced system is implemented in a microwave scattering experiment. The hard walls correspond to the Dirichlet boundary condition, that is to odd solutions (by reflection) of the full problem. The absorbing barrier, which produces negligible reflection at the considered range of frequencies, models escape to infinity.

effective mass of the Bloch electron). Mathematical results explaining existence of resonances created by a barrier (here formed by a corral of CO molecules) are presented in §7.3.

Figure 1.10 shows an experimental set-up for microwave cavities used to study scattering resonances for chaotic systems. Density of resonance was investigated in this setting in [P\textsuperscript{12}] and that is related to semiclassical upper bounds in §§3.4,4.3,7.2. In [B\textsuperscript{13}] dependence of resonance free strips on dynamical quantities was confirmed experimentally and Chapter 6 contains related mathematical results and references.

Figure 1.11 shows a MEMS (the acronym for the microelectromechanical systems) resonator. The numerical calculations in that case are based on the complex scaling technique, presented in §4.5, adapted to the finite element methods, and known as the method of perfectly matched layers [Be94].

Figure 1.12 shows the profile of gravitational waves recently detected by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and originating from a binary black hole merger. Resonances for such waves are known by the name of quasi-normal modes in physics literature and
Figure 1.11. On top: a schematic representation of a MEMS (microelectromechanical systems) device: a disk attached by a stem to an absorbing plate. Below, a computed resonant mode for this device: the top is colored by the amount of displacement in the radial direction, and the bottom is colored by the amount of displacement in the vertical direction. The computation was done using the complex scaling/PML (perfectly matched layer) methods by Bindel–Govindjee [BG05]. As explained there, for the resonance state shown in the figure, the coupling between the radial and vertical displacements gives this state a large imaginary part – in other words a significant loss in oscillations.

are the characteristic frequencies of the waves emitted during the ringdown phase of the merger, when the resulting single black hole settles down to its stationary state – see for instance [KS99],[Dy12],[DZ13] and §§5.7,6.3.

1.4. OVERVIEW

To make the presentation more accessible we restrict ourselves to the simplest setting in which the theory is physically and mathematically relevant:
1.4. OVERVIEW

Figure 1.12. Left: an aerial view of the LIGO laboratory in Livingston, Louisiana, US. Right: the gravitational wave signal observed on September 14, 2015 simultaneously by LIGO Livingston (blue) and LIGO Hanford (red); see [A*16],[V*14]. The ringdown oscillations of the kind shown in Fig. 1.5 and connected to quasi-normal modes (which is what scattering resonances are called in general relativity) have not been observed yet. What is shown here are oscillations due to a much stronger effect of black hole merger.

compactly supported perturbations in odd dimensions. Many results, especially the ones based on complex scaling, are valid in all dimensions and for suitable non-compact perturbations but for the clarity of presentation we only provide pointers to the literature. The hope is that once the ideas are grasped in the technically less challenging setting the references will become accessible. In the case of scattering on asymptotically hyperbolic spaces (Chapter 5) we present a general theory as there are few advantages in restricting our attention to the hyperbolic space alone.

We now present brief descriptions of the content of the chapters.

Chapter 2: We cover basic theory of resonances in dimension one. Many fundamental such as outgoing solutions, meromorphic continuation of the resolvent, the relation of resonances to the scattering matrix, trace formulas, and resonance expansions of waves, appear later in more complicated settings.

Chapter 3: Here the theory of scattering by compactly supported potentials in odd dimensions is presented in detail. This chapter can be used as the introduction to the study of more general settings (for instance, in the theory of zero resonances) and to the open problems in scattering by compactly supported potentials.

Chapter 4: This chapter is devoted to black box scattering which allows a unified treatment of many different operators ranging from Laplacians on surfaces with constant curvature cusp ends to obstacle scattering in the Euclidean space.
Chapter 5: One of the recent advances in geometric scattering is Vasy’s approach to meromorphic continuation of resolvents for (even) asymptotically hyperbolic spaces. The method was motivated by the study of scattering for black holes and that connection is also explained.

Chapter 6: Resonance free regions have been investigated in mathematical scattering theory since the seminal work of Lax–Phillips and Vainberg (see §4.6). Semiclassical scattering with its connection to classical/quantum correspondence is the natural setting for investigating resonance free regions.

Chapter 7: This last chapter is concerned with resonances generated by strong trapping phenomena such as the presence of barriers or singularities of \( E \mapsto \int_{V(x) \geq E} dx \). We conclude with expansions of waves for strong trapping.

Appendices: We present notational conventions and references to basis techniques. Proofs of various results which are crucial in the text (such as Fredholm theory or propagation of singularities in the semiclassical setting) are presented in detail.

Acknowledgments for Figures:

- Fig. 1.1 comes from [CLE93] and is reprinted with permission from The American Association for the Advancement of Science. The authors are grateful to Mike Crommie for his allowing us to use this figure.

- Fig. 1.9 comes from [M’08] and is reprinted with permission from The American Association for the Advancement of Science. The authors are grateful to Hari Manoharan for suggesting we use this figure to illustrate quantum resonances in a complex STM setting.

- Fig. 1.10 was provided to us by Ulrich Kuhl and represents an experimental set-up at his lab http://lpmc.unice.fr/Kuhl-Ulrich.html. We are grateful to him and Hans-Jürgen Stöckmann for giving us access to their microwave experiments [B’13],[P’12].

- Fig. 1.11 is an original figure produced by David Bindel. The authors are grateful to him for providing the figure and the caption explaining the connection to [BG05]. We are also grateful to Sanjay Govindjee for discussions related to [BG05].

- Fig. 1.12 comes from [V*14] and made use of figures obtained from the LIGO Open Science Center https://losc.ligo.org, a service of LIGO Laboratory, the LIGO Scientific Collaboration and the Virgo Collaboration. LIGO is funded by the U.S. National Science Foundation. Virgo is funded by the French Centre National de Recherche Scientifique (CNRS), the Italian Istituto Nazionale della Fisica Nucleare (INFN) and the Dutch Nikhef, with contributions by Polish and Hungarian institutes.
Dependence graph of sections
Part 1

POTENTIAL SCATTERING
In the simplest setting of one dimensional scattering by compactly supported potentials we can already observe many general phenomena. In particular, various notions can be explained in a very intuitive setting. Technically, there are also many advantages: we are dealing with ordinary differential equations, the methods of complex analysis apply particularly well and trace class properties hold nicely.
2. OUTGOING AND INCOMING SOLUTIONS

We consider the following class of operators:

\[ P_V = D_x^2 + V(x), \quad D_x := \frac{1}{i} \partial_x, \quad V \in L^\infty_{\text{comp}}(\mathbb{R}). \]

The stationary Schrödinger equation then is

\[ (P_V - z)u = f, \quad z \in \mathbb{C}, \quad f \in L^2(\mathbb{R}), \]

while the dynamical equation is given by

\[ (i\partial_t - P_V)v = F, \quad v|_{t=0} = v_0, \quad v_0 \in L^2(\mathbb{R}), \quad F \in L^1_{\text{loc}}(\mathbb{R}_t; L^2(\mathbb{R}_x)). \]

As we will see below it is sometimes important to consider initial data in different spaces than \( L^2 \).

A solution to the stationary equation (2.1.1) produces a solution to (2.1.2) corresponding to the evolution of the state \( u \):

\[ v(t,x) := e^{-izt}u(x), \quad v_0(x) = u(x), \quad F(x,t) = -e^{-izt}f(x). \]

Outside the support of \( V(x) \) and \( f \), say for \( |x| \geq R \), the solutions of (2.1.1) are given by

\[ u(x) = a_\pm e^{i\sqrt{z}|x|} + b_\pm e^{-i\sqrt{z}|x|}, \quad \pm x \geq R. \]

To consider the dependence on \( z \) we have to choose a branch of \( \sqrt{z} \). We consider \( \sqrt{z} \) defined on \( \mathbb{C} \setminus [0, \infty) \) with Im \( \sqrt{z} > 0 \) everywhere, so that

\[ \pm \lim_{\epsilon \to 0^+} \sqrt{z} \pm i\epsilon =: \pm \sqrt{z} \pm i0 > 0, \quad z \in (0, \infty). \]

When considering \( z \in (0, \infty) \) we write \( \sqrt{z} = \sqrt{z} + i0 \).

**Outgoing and incoming solutions.** A solution to (2.1.1) with \( z > 0 \) is called **outgoing** if

\[ u(x) = a_- e^{-i\sqrt{z}x}, \quad x < -R, \quad u(x) = a_+ e^{i\sqrt{z}x}, \quad x > R. \]

This corresponds to \( v \) given by (2.1.3) moving away from the support of \( V(x) \) – see Figure 2.1. We also note that using our convention

\[ z \notin [0, \infty) \implies u(x) \in L^2(\mathbb{R}). \]

Similarly, the solution to (2.1.1) is called **incoming** if

\[ u(x) = b_- e^{i\sqrt{z}x}, x < -R, \quad u(x) = b_+ e^{-i\sqrt{z}x}, x > R. \]

Although the physical motivation illustrated in Figure 2.1 disappears when \( z \notin (0, \infty) \) we will still use the notions of outgoing and incoming solutions as defined above, paying attention to our convention for \( \sqrt{z} \).
2.1. OUTGOING AND INCOMING SOLUTIONS

In Section 2.2 we will address the problem of constructing outgoing (or incoming solutions) to (2.1.1). That will lead to a natural definition of scattering resonances.

The above definition of outgoing and incoming solutions is given in terms of the Schrödinger equation. We can also consider the wave equation:

\[(2.1.5) \quad (\partial^2_t - PV)v = F, \quad v|_{t=0} = v_0, \quad \partial_t v|_{t=0} = v_1.\]

The stationary equation, formally obtained by taking the Fourier transform in \(t\), is given by

\[(2.1.6) \quad (PV - \lambda^2)u = f, \quad \lambda \in \mathbb{C}.\]

In this case the convention regarding the sign of \(\lambda\) in the definition of outgoing and incoming solutions is somewhat arbitrary. We choose a convention consistent with the choice of \(\sqrt{z}\) above:

\[(2.1.7) \quad \lambda^2 = z, \quad \lambda = \sqrt{z}.\]

In particular,

\[\lambda > 0 \implies \sqrt{(\pm \lambda + i0)^2} = \pm \lambda.\]

The outgoing solution to (2.1.6) with a compactly supported \(f\) is supposed to satisfy

\[(2.1.8) \quad u(x) = a_- e^{-i\lambda x}, \quad x < -R, \quad u(x) = a_+ e^{i\lambda x}, \quad x > R.\]

We now have

\[\text{Im } \lambda > 0 \implies u(x) \in L^2(\mathbb{R}).\]
The solutions to (2.1.6) with \( f = 0 \) and \( \text{Im} \lambda > 0 \) are the eigenfunctions of \( PV \) corresponding to eigenvalues \( \lambda^2 \). Note that the equation (2.1.6) is the same when \( \lambda \) is replaced by \(-\lambda\), but the conditions (2.1.8) change under this operation.

We will use the wave equation motivated \( \lambda \) convention in this chapter, except in §2.8 which is motivated by quantum mechanics\(^1\).

To motivate the study of outgoing solutions to (2.1.6), and the importance of the poles of the meromorphic continuation of the outgoing resolvent, constructed in Section 2.2, we now briefly explain an application to the long-time asymptotics of the wave equation. The presented ideas lead to resonance expansions of waves, studied in detail in Section 2.3.

Consider the initial-value problem for the wave equation
\[
(2.1.9) \quad (-\partial_t^2 - PV)v = F, \quad v|_{t=0} = 0, \quad \partial_tv|_{t=0} = 0.
\]
We assume that
\[
V \in C^\infty_c((-R,R);\mathbb{R}), \quad F \in C^\infty_c((0,\infty)_t \times (-R,R)_x),
\]
for some \( R > 0 \). We take the Fourier transform in time
\[
(2.1.10) \quad u(\lambda, x) := \hat{v}(\lambda, x) := \int_0^\infty e^{it\lambda} v(t, x) \, dt.
\]
The integral (2.1.10) converges for \( \text{Im} \lambda > 0 \), thanks to standard energy estimates for the wave equation – see for instance [Ev98, §7.2.4]. Taking the Fourier transform of (2.1.9) in \( t \), we see that for \( \text{Im} \lambda > 0 \), \( u(\lambda) \) solves the equation (2.1.6):
\[
(2.1.11) \quad (\lambda^2 - PV)u(\lambda) = \hat{F}(\lambda).
\]

On the other hand the d’Alembert’s formula (see [Ev98, §2.4]) shows
\[
(2.1.12) \quad v(t, x) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(t-s, y) \, dy \, ds.
\]
Since we assumed that \( V \) and \( F \) are supported in \( \{|x| \leq R\} \), we obtain
\[
(2.1.13) \quad v(t, x) = v_\pm(x \mp t), \quad \pm x \geq R, \; t \geq 0,
\]
for some functions \( v_\pm \) with
\[
\text{supp } v_+ \subset (-\infty, R), \quad \text{supp } v_- \subset (-R, \infty).
\]

\(^1\)Different conventions for scattering resonances are due to their emergence in different fields. We will discuss those issues as they come along.
2.1. MEROMORPHIC CONTINUATION

Figure 2.2. An illustration of the outgoing property for the wave equation. The values \( u(t, x), u(t', x') \) are obtained by integrating \( -\frac{1}{2}(Vv + F) \) over the shaded triangles. For \( x, x' > R \) and \( t - x = t' - x' \), these triangles have the same intersection with \( \{|x| \leq R\} \supset \text{supp}(Vv + F) \), therefore \( u(t, x) = u(t', x') \).

It follows that for \( \text{Im} \lambda > 0 \), \( u(\lambda) \) is outgoing in the sense of (2.1.8): if \( \pm x \geq R \) then

\[
    u(\lambda, x) = \int_0^\infty v_\pm(x \mp t)e^{i\lambda t}dt = \int_{-R}^\infty v_\pm(\mp s)e^{i\lambda s}e^{\pm i\lambda x}
\]

\[
    = a_\pm(\lambda)e^{\pm i\lambda x}, \quad a_\pm(\lambda) := \int_{-R}^\infty v_\pm(\mp s)e^{i\lambda s}.
\]

Here we used the support properties and \( v_\pm \) to guarantee convergence when \( \text{Im} \lambda > 0 \).

One technique for obtaining asymptotics of \( v(t, x) \) as \( t \to \infty \) is to deform the contour in the Fourier inversion formula

\[
    v(t, x) = \frac{1}{2\pi} \int_{\text{Im} \lambda = c} e^{-it\lambda}u(\lambda)(x) \, d\lambda, \quad c > 0.
\]

For that we need to continue \( u(\lambda) \) meromorphically into the lower half-plane and this is done by requiring that \( u(\lambda) \) solve the equation (2.1.11) with the outgoing conditions (2.1.8), where \( \tilde{F}(\lambda) \) is entire in \( \lambda \) since \( F \) is compactly supported. For \( \text{Im} \lambda > 0 \), we have \( u(\lambda) \in L^2(\mathbb{R}) \), therefore \( u(\lambda) \) for general \( \lambda \) can be viewed as the image of \( \tilde{F}(\lambda) \) under the meromorphic continuation of the resolvent \( (\lambda^2 - PV)^{-1} : L^2 \to L^2, \text{Im} \lambda > 0 \), through the continuous spectrum \( \{|\text{Im} \lambda| = 0\} \) of \( PV \) to the entire complex plane. The existence of this meromorphic continuation, as an operator \( L^2_{\text{comp}} \to L^2_{\text{loc}} \), is established in Section 2.2. After proving additional estimates on \( u(\lambda) \) in the lower half-plane, we can deform the contour in (2.1.14) to the line \( \{|\text{Im} \lambda| = -\nu\} \) with \( \nu > 0 \). The integral along the new contour will be \( O(e^{-\nu t}) \), owing to the
$e^{-it\lambda}$ factor, and we accumulate residues from the poles of $u(\lambda)$. These poles, called resonances, will be the central objects of study in this book.

2.2. MEROMORPHIC CONTINUATION

In this section we solve (2.1.6) for $\lambda \in \mathbb{C}$, with $u$ outgoing, that is satisfying (2.1.8). For that we first consider the case of $V = 0$. In that case $u(x)$ is given by an explicit formula:

$$u(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y) dy.$$  

For $\text{Im} \lambda > 0$ this gives the integral kernel of the free resolvent:

$$R_0(\lambda) := (D_x^2 - \lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im} \lambda > 0,$$

(2.2.1)

$$R_0(\lambda, x, y) = \frac{i}{2\lambda} e^{i\lambda|x-y|},$$

where we use the same notation $R_0(\lambda)$ for the operator and its integral kernel. We should stress that for $\text{Im} \lambda < 0$,

$$\frac{D_x^2 - \lambda^2}{\lambda} = R_0(-\lambda), \quad \frac{D_x^2 - \lambda^2}{\lambda} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im} \lambda < 0.$$  

This means that the spectrum of $D_x^2$ is given by $[0, \infty)$ and it is absolutely continuous as can be shown using the Fourier transform – see §B.1.

From the expression (2.2.1) we see that for fixed $x$ and $y$, $R_0(\lambda, x, y)$ is a meromorphic function of $\lambda \in \mathbb{C}$ defining an operator $C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ which is not bounded on $L^2$ for $\text{Im} \lambda \leq 0$.

Using the notion of a meromorphic family of operators (see Appendix C.3) we summarize these facts as follows.

THEOREM 2.1 (Meromorphic continuation of the free resolvent).

The operator $R_0(\lambda)$ defined by (2.2.1) for $\text{Im} \lambda > 0$ extends to a meromorphic family of operators for $\lambda \in \mathbb{C}$:

$$R_0(\lambda) := L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}).$$

We have

$$\|R_0(\lambda)\|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, R^+)} \leq \frac{1}{|\lambda| \text{Im} \lambda}, \quad \text{Im} \lambda > 0,$$

(2.2.2)

and for $\rho \in C^\infty_c(\mathbb{R})$, supp $\rho \subset [-L, L]$, $\lambda \in \mathbb{C}$,

$$\|\rho R_0(\lambda)\rho\|_{L^2(\mathbb{R}^n) \to H^j(\mathbb{R}^n)} \leq C_{L,j} e^{2L(\text{Im} \lambda)} |\lambda|^{-1} |\lambda|^j, \quad 0 \leq j \leq 2,$$

(2.2.3)

where $x_- := \max(0, -x)$.  

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REMARK. The estimate (2.2.2) and the fact that \(-\Delta R_0(\lambda) = I - \lambda^2 R_0(\lambda)\) immediately imply that

\[
\|R_0(\lambda)\|_{L^2 \to H^k} \leq \frac{(\lambda)^k}{|\lambda| \text{Im } \lambda}, \quad \text{Im } \lambda > 0, \quad 0 \leq k \leq 2.
\]

Proof. 1. The estimate (2.2.2) follows directly from spectral theory, since the spectrum of \(D^2_x\) is equal to \(R^+\):

\[
d(\lambda^2, R^+) = \begin{cases} 2 \text{Im } \lambda \text{Re } \lambda & (\text{Re } \lambda)^2 \geq (\text{Im } \lambda)^2, \\ |\lambda|^2 & (\text{Re } \lambda)^2 \leq (\text{Im } \lambda)^2, \end{cases}
\]

and hence \(d(\lambda^2, \mathbb{R}_+) \geq |\lambda| \text{Im } \lambda\).

2. The estimate (2.2.3) for \(j = 0\) follows from (2.2.1) by Schur’s criterion (A.5.3) since

\[
\int_{-\infty}^{\infty} \rho(x) \rho(y) |R_0(\lambda)(x, y)| \, dx \leq C |\lambda|^{-1} \int_{-\infty}^{\infty} \rho(x) \rho(y) e^{-\text{Im } \lambda |x-y|} \, dx
\]

is bounded by \(C_L |\lambda|^{-1} e^{2L(\text{Im } \lambda)}\) and same is true if integration is performed in the \(y\) variable instead.

3. To obtain the estimate for \(j = 2\) we use elliptic regularity estimate (see for instance [Zw12, Theorem 7.1]): if \(U\) and \(W\) are intervals and \(U \Subset W\) then

\[
\|u\|_{H^2(U)} \leq C(\|u\|_{L^2(W)} + \|D^2_x u\|_{L^2(W)}).
\]

Hence, if \(\tilde{\rho} \in C_c^\infty(\mathbb{R})\) satisfies \(\tilde{\rho} = 1\) near \(\text{supp } \rho\) then

\[
\|\rho u\|_{H^2(\mathbb{R})} \leq C(\|\tilde{\rho} u\|_{L^2(\mathbb{R})} + \|\tilde{\rho} D^2_x u\|_{L^2(\mathbb{R})}).
\]

4. We now apply (2.2.5) to \(u = R_0(\lambda) \rho f\), \(f \in L^2\) so that

\[
\|\rho R_0(\lambda) \rho f\|_{H^2} \leq C(\|\tilde{\rho} R_0(\lambda) \rho f\|_{L^2} + \|\tilde{\rho} D^2_x R_0(\lambda) \rho f\|_{L^2}).
\]

Since \(\tilde{\rho} D^2_x R_0(\lambda) \rho f = \rho f + \tilde{\rho} \lambda^2 R_0(\lambda) \rho f\) is bounded by \(C_L (\lambda) e^{2L(\text{Im } \lambda)} \|f\|_{L^2}\) in \(L^2\) the estimate (2.2.3) with \(j = 2\) follows. Finally, the estimate for \(j = 1\) is obtained by interpolating between the cases \(j = 0\) and \(j = 2\). \(\square\)

For \(V \neq 0\) we have a result which shows that the resolvent of \(R_V = D^2_x + V(x)\) also has a meromorphic continuation.

THEOREM 2.2 (Meromorphic continuation of the resolvent in one dimension). Suppose that \(V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{C})\). Then the resolvent

\[
R_V(\lambda) := (D^2_x + V - \lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \text{Im } \lambda > 0,
\]

is a meromorphic family of operators with a finite number of poles.

The family \(R_V(\lambda)\) extends to a meromorphic family of operators for \(\lambda \in \mathbb{C}\):

\[
R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to H^2_{\text{loc}}(\mathbb{R}).
\]
Proof. 1. We first construct \( R_V(\lambda) \) for \( \text{Im} \lambda \gg 1 \). For that we write
\[
(P_V - \lambda^2)R_0(\lambda) = (D_x^2 - \lambda^2 + V)R_0(\lambda) = I + VR_0(\lambda).
\]
For \( \text{Im} \lambda \gg 1 \), \( \|VR_0(\lambda)\|_{L^2 \to L^2} \leq \|V\|_{\infty}(\text{Im} \lambda)^{-2} \leq 1/2 \), and hence \( I + VR_0(\lambda) \) is invertible using the Neumann series:
\[
(I + VR_0(\lambda))^{-1} = \sum_{j=0}^{\infty} (-1)^j (VR_0(\lambda))^j.
\]
This shows that
\[
R_V(\lambda) := (P_V - \lambda^2)^{-1} = R_0(\lambda)(I + VR_0(\lambda))^{-1}.
\]
If \( \rho \in C_c^\infty((-L, L)) \), \( L > 0 \), then for \( \text{Im} \lambda > 0 \), \( \rho R_0(\lambda) : L^2 \to H^2_0((-L, L)) \) (see (2.2.4)). Hence \( \rho R_0(\lambda) \) is a compact operator on \( L^2 \) by Theorem B.3. By taking \( L \) large enough we can choose \( \rho \) which is equal to 1 on \( \text{supp} V \). In particular, \( \rho V = V \). Hence for \( \text{Im} \lambda > 0 \) the operator \( VR_0(\lambda) = V\rho R_0(\lambda) \) is also compact and we can apply Theorem C.5 (or rather the remark after the theorem since \( VR_0(\lambda) \) has a pole at 0) to see that \( R_V(\lambda) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is a meromorphic family of operators in \( \text{Im} \lambda > 0 \).

2. To obtain a continuation to the entire \( \mathbb{C} \), we define the following meromorphic family of operators:
\[
K(\lambda) := VR_0(\lambda) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{comp}}(\mathbb{R}).
\]
(Strictly speaking the notion of meromorphic families operators is only defined on Banach spaces: what we mean here is that \( \hat{\rho}K(\lambda)\hat{\rho} : L^2 \to L^2 \) is a meromorphic family for any \( \hat{\rho} \in C_c^\infty \).)

The only pole of \( K(\lambda) \) is at \( \lambda = 0 \). With the same \( \rho \in C_c^\infty(\mathbb{R}) \) as in Step 1, \( (1 - \rho)K(\lambda) = 0 \), and hence, by inspection,
\[
(I + K(\lambda)(1 - \rho))^{-1} = I - K(\lambda)(1 - \rho).
\]
As in step 1, we see that for \( \text{Im} \lambda \gg 1 \), \( I + K(\lambda)\rho \) is invertible by a Neumann series argument. We conclude that for \( \text{Im} \lambda \gg 1 \),
\[
(I + K(\lambda))^{-1} = \left( (I + K(\lambda)(1 - \rho))(I + K(\lambda)\rho) \right)^{-1}
= (I + K(\lambda)\rho)^{-1}(I - K(\lambda)(1 - \rho))
\]
3. By (2.2.7), for \( \text{Im} \lambda \gg 1 \),
\[
R_V(\lambda) = R_0(\lambda)(I + K(\lambda)\rho)^{-1}(I - K(\lambda)(1 - \rho)).
\]
For \( \lambda \in \mathbb{C} \setminus 0 \), (2.2.3) shows that \( \rho R_0(\lambda)\rho : L^2(\mathbb{R}) \to H^2_0((-L, L)) \), and hence by Theorem B.3 this operator is compact. Since \( V = V\rho \), we conclude that \( K(\lambda)\rho \) is compact on \( L^2(\mathbb{R}) \), and hence \( I + K(\lambda)\rho \) is a meromorphic
family of Fredholm operators (see §C.3). Theorem C.5 gives a meromorphic continuation of
\[(2.2.10) \quad (I + K(\lambda)\rho)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),\]
to \(\mathbb{C}\).

4. From (2.2.3) we also conclude that for \(\text{Im} \lambda \geq 0\), \(\|K(\lambda)\rho\|_{L^2 \to L^2} \leq C/|\lambda|\). The Neumann series argument and (2.2.9) show that \(R_V(\lambda)\) has only finitely many poles for \(\text{Im} \lambda > 0\). (See Theorem 2.8 for more on that.)

5. We now observe that
\[I - K(\lambda)(1 - \rho) : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{comp}}(\mathbb{R}),\]
and
\[(I + K(\lambda)\rho)^{-1} : L^2_{\text{comp}}(\mathbb{R}) \to L^2_{\text{comp}}(\mathbb{R}).\]

The last property can be checked for \(\text{Im} \lambda \gg 1\) using the Neumann series argument: if \(\chi\rho = \rho, \tilde{\chi}\chi = \chi\) then
\[(1 - \tilde{\chi})(I + K(\lambda)\rho)^{-1}\chi = 0, \quad \text{Im} \lambda \gg 1,\]
and this remains true for all \(\lambda\) by analytic continuation.

Combining these facts with the expression for \(R_V\) given in (2.2.9) we obtain the meromorphy of \(R_V(\lambda)\) for \(\lambda \in \mathbb{C}\) as a family of operators \(L^2_{\text{comp}} \to L^2_{\text{loc}}\). \(\square\)

**DEFINITION.** We call the poles of \(R_V(\lambda)\) *scattering resonances* or simply *resonances*. The multiplicity of a resonance at \(\lambda\) is defined as follows:

\[(2.2.11) \quad m_R(\lambda) := \text{rank} \oint_{\lambda} R_V(\zeta)d\zeta,\]

where the integral is over a small circle containing no other poles of \(R_V\).

When \(\lambda\) is not a resonance we put \(m_R(\lambda) = 0\) which is of course consistent with the above definition.

**REMARKS.**

1. When \(V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R})\) then the operator \(P_V\) is self-adjoint and the existence of \(R_V(\lambda), \text{Im} \lambda > 0\), as a meromorphic operator on \(L^2\) follows from the spectral theorem. The poles occur at \(i\sqrt{-E_j}\) where \(E_j\) are the negative eigenvalues of \(P_V\) – see Figure 1.8. These statements also follow from Theorem 2.1.

2. We also have the following basic fact valid for real valued potentials

\[(2.2.12) \quad V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \implies m_R(\lambda) = 0, \quad \lambda \in \mathbb{R} \setminus \{0\}.\]
This implies that for $\lambda \notin \mathbb{R} \setminus \{0\}$, there exists a limit
\[
\lim_{\varepsilon \to 0^+} R_V(\lambda + i\varepsilon) : L^2_{\text{comp}} \to L^2_{\text{loc}}.
\]
This fact is known as the limiting absorption principle. It follows that the spectrum of $P_V$ is given by the continuous spectrum $[0, \infty)$ and a finite number of negative eigenvalues. We will prove this after the proof of Theorem 2.4 below.

3. Reality of $V$ or, equivalently, self-adjointness of $P_V$ imply the following symmetry of resonances:
\[
(2.2.13) \quad V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \implies m_R(\lambda) = m_R(-\bar{\lambda}), \quad \lambda \in \mathbb{R} \setminus \{0\}.
\]
In fact, we will check the following identity of Schwartz kernels:
\[
R_V(-\bar{\lambda}, y, x) = R_V(\lambda, x, y).
\]
Since both sides are meromorphic in $\lambda$ we only need to check that
\[
(2.2.14) \quad R_V(-\bar{\lambda})^* = R_V(\lambda) \quad \text{for } \text{Im} \lambda > 0,
\]
when both sides are bounded operators on $L^2$ (note that $\text{Im}(-\bar{\lambda}) > 0$ if $\text{Im} \lambda > 0$). Using the correspondence between $\lambda$ and $z$ in (2.1.7) that follows from \((P_V - z)^{-1})^* = (P_V - \bar{z})^{-1}\). \qed

4. It is also useful to express the operator $(I + VR_0(\lambda)\rho)^{-1}$ through $R_V(\lambda)$, as follows:
\[
(2.2.15) \quad (I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho.
\]
This follows immediately from the identity
\[
I - (I + VR_0(\lambda)\rho)^{-1} = VR_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}
\]
and the following formula:
\[
(2.2.16) \quad R_V(\lambda)\rho = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1},
\]
which in turn is true for $\text{Im} \lambda \gg 1$ by considering the Neumann series in (2.2.7) and for general $\lambda$ by analytic continuation.

When $\lambda$ is not a pole of $R_V$, the operator $R_V(\lambda)$ gives solutions to the Helmholtz equation satisfying the outgoing condition:

**THEOREM 2.3 (R_V(\lambda) at regular points).** Assume that $\lambda \in \mathbb{C}$ is not a pole of $R_V$. Then for each $f \in L^2_{\text{comp}}(\mathbb{R})$, $u = R_V(\lambda)f$ is the unique outgoing solution, in the sense of (2.1.8), to the equation $(P_V - \lambda^2)u = f$.

**Proof.** 1. The identity $(P_V - \lambda^2)R_V(\lambda)f = f$ holds for $\text{Im} \lambda > 0$ by the definition of $R_V(\lambda)$ and extends to all $\lambda$ by analytic continuation. Same is true for the outgoing condition, which can be written as
\[
(\partial_x \mp i\lambda)(R_V(\lambda)f)(\pm R) = 0,
\]
where $R > 0$ is such that $\text{supp } V \cup \text{supp } f \subset (-R, R)$. Indeed, for $\text{Im } \lambda > 0$ this condition holds as $R_V(\lambda)f \in L^2$ must be a linear combination of $e^{i\lambda|x|}$ and $e^{-i\lambda|x|}$ for $x > R$, and $e^{-i\lambda|x|}$ is exponentially growing as $|x| \to \infty$.

2. It remains to show that $R_V(\lambda)f$ is the unique outgoing solution to $(P_V - \lambda^2)u = f$ and for that it suffices to prove the identity

\[(2.2.18) \quad u = R_V(\lambda)(P_V - \lambda^2)u \]

for all outgoing functions $u \in H^2_{\text{loc}}$. We note that the outgoing condition guarantees that $(P_V - \lambda^2)u$ is compactly supported.

3. The equation $(2.2.18)$ is true for $\text{Im } \lambda > 0$ by the definition of $R_V(\lambda)$: $u = R_V(\lambda)f \in H^2$ in this case. To handle $\lambda$ in the closed lower half-place, we argue by analytic continuation. For that we decompose any outgoing $u \in H^2_{\text{loc}}$ as

\[u(x) = u_0(x) + \chi_+(x)a_+e^{i\lambda x} + \chi_-(x)a_-e^{-i\lambda x},\]

where $u_0 \in H^2_{\text{comp}}$, and where $\chi_\pm \in C^\infty(\mathbb{R})$ are equal to 1 near $\pm 0$ and to 0 near $\mp \infty$. Then for $\text{Im } \lambda > 0$ each term is in $H^2$ and hence

\[u_0 = R_V(\lambda)(P_V - \lambda^2)u_0, \quad \chi_\pm e^{\pm i\lambda \cdot} = R_V(\lambda)(P_V - \lambda^2)(\chi_\pm e^{\pm i\lambda \cdot}).\]

Since $(P_V - \lambda^2)(\chi_\pm e^{\pm i\lambda \cdot}) \in L^2_{\text{comp}}$, the equations have to be valid for all $\lambda$ at which $R_V$ is holomorphic. Hence $(2.2.18)$ holds for any outgoing $u$.

We next study in detail the singular part of $R_V(\lambda)$, starting from the following statement away from $\lambda = 0$. We use the notation in which tensor product is identified with an operator:

\[(2.2.19) \quad (u \otimes v)(f)(x) := u(x) \int_{\mathbb{R}} v(y)f(y)dy.\]

**THEOREM 2.4 (Singular part of $R_V(\lambda)$ in one dimension).** Suppose $m_R(\lambda_0) > 0$, $\lambda_0 \neq 0$.

1) There exist linearly independent $u_j \in H^2_{\text{loc}}(\mathbb{R})$, $j = 1, \ldots, m_R(\lambda_0)$, such that $u_1$ is outgoing (see $(2.1.8)$) and

\[(2.2.20) \quad (P_V - \lambda_0^2)u_1 = 0, \quad (P_V - \lambda_0^2)u_j = u_{j-1},\]

$1 < j \leq m_R(\lambda_0)$.

2) The Laurent expansion of $R_V(\lambda)$ near $\lambda_0$ is given by

\[(2.2.21) \quad R_V(\lambda) = -\sum_{k=1}^{m_R(\lambda_0)} \frac{(P_V - \lambda_0^2)^{k-1}\Pi_{\lambda_0}}{(\lambda^2 - \lambda_0^2)^k} + A(\lambda, \lambda_0),\]

where $\lambda \mapsto A(\lambda, \lambda_0)$ is holomorphic near $\lambda_0$,

\[\Pi_{\lambda_0} = -\frac{1}{2\pi i} \int_{\lambda_0} R_V(\lambda)2\lambda d\lambda,\]
and
\[(2.2.22) \quad (P_V - \lambda_0^2)^{m_{R}(\lambda_0)} \Pi_{\lambda_0} = 0, \quad \text{Ran} \, \Pi_{\lambda_0} = \text{span} \{ u_1, \ldots, u_{m_{R}(\lambda_0)} \}. \]

3) Suppose that \( V \in L^\infty(\mathbb{R}; \mathbb{R}) \). If \( m_{R}(\lambda_0) = 1 \) then
\[(2.2.23) \quad \Pi_{\lambda_0} = -iu_1 \otimes u_1, \quad (P_V - \lambda_0^2)u_1 = 0, \quad u_1(x) = c_\pm e^{\pm i\lambda_0 x}, \quad \pm x \gg 1. \]

Moreover, \( u_1 \) is normalized as follows: for \( R \) large enough,
\[(2.2.24) \quad -2i\lambda_0 \int_{-R}^{R} u_1(x)^2 \, dx + c_+^2 e^{2i\lambda_0 R} + c_-^2 e^{-2i\lambda_0 R} = 1. \]

**Remark.** In Section 2.7 we will find an interpretation of \( \Pi_{\lambda_0} \), \( \lambda_0 \neq 0 \), as a projection. That will explain our sign convention. It will also give an alternative proof of the normalization of \( u_1 \) in (2.2.24). Generically resonances have multiplicity 1, and \( m_{R}(0) = 0 \).

**Proof.**
1. From the general result about meromorphic continuation in §C.3 we know that for some finite rank operators \( A_k, 1 \leq k \leq K \),
\[ R_V(\lambda) = \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \lambda_0^2)^k} + A(\lambda, \lambda_0), \quad \lambda_0 \neq 0, \]
where \( A(\bullet, \lambda_0) \) is holomorphic near \( \lambda_0 \), and
\[ A_1 = -\Pi_{\lambda_0} := -\frac{1}{2\pi i} \oint_{\lambda_0} R_V(\lambda)2\lambda d\lambda. \]

The residue theorem gives
\[(2.2.25) \quad \frac{1}{2\pi i} \oint_{\lambda_0} R_V(\zeta) d\zeta = \sum_{k=1}^{K} (-1)^{k-1}(2k-2)! \frac{(2\lambda_0)^{-2k+1}}{(k-1)!} A_k. \]

2. We now consider the equation \((P_V - \lambda^2)R_V(\lambda) = I \) near \( \lambda = \lambda_0 \): modulo terms holomorphic near \( \lambda_0 \) we have
\[ (P_V - \lambda^2)R_V(\lambda) = \sum_{k=1}^{K} \left( \frac{(P_V - \lambda_0^2)A_k}{(\lambda^2 - \lambda_0^2)^k} - \frac{A_k}{(\lambda^2 - \lambda_0^2)^{k-1}} \right) \]
\[ = \sum_{k=1}^{K} \frac{(P_V - \lambda_0^2)A_k - A_{k+1}}{(\lambda^2 - \lambda_0^2)^k}, \]
where we use the convention that \( A_k = 0 \) for \( k > K \).

It follows that \( A_{k+1} = (P_V - \lambda_0^2)A_k \) which shows that (2.2.21) holds and \((P_V - \lambda_0^2)^K \Pi_{\lambda_0} = 0 \).
We now need to show the existence of \( u_j \)'s satisfying (2.2.20) and (2.2.22).

The operator \((P_V - \lambda_0^2)\) commutes with \(\Pi_{\lambda_0}\) and \((P_V - \lambda_0^2)^k\Pi_{\lambda_0} = 0\). Hence

\[ P_V - \lambda_0^2 : \text{Ran} \Pi_{\lambda_0} \to \text{Ran} \Pi_{\lambda_0}, \]

is nilpotent and we can put it into a Jordan normal form. That means that there exists a basis of \(\text{Ran} \Pi_{\lambda_0} \subset H^2_{\text{comp}}(\mathbb{R})\) of the form

\[ u_{\ell,j}, \quad 1 \leq \ell \leq L, \quad 1 \leq j \leq k_\ell, \quad \sum_{\ell=1}^{L} k_\ell = K, \]

\[ (P_V - \lambda_0^2)u_{\ell,j} = u_{\ell,j-1}, \quad 1 \leq j \leq k_\ell, \quad u_{\ell,0} := 0. \]

From (2.2.9) we see that each \( u_{\ell,j} \) is a linear combination of functions in \( R_0(\lambda_0)(L^2_{\text{comp}}), \ldots, \partial^K R_0(\lambda_0)(L^2_{\text{comp}}) \). Then by (2.2.1), for \( \pm x \gg 1 \), \( u_{\ell,j}(x) \) is the product of \( e^{\pm i\lambda_0 x} \) with a polynomial in \( x \). Since \((P_V - \lambda_0^2)u_{\ell,1} = 0\), we see that \( u_{\ell,1} \in R_0(\lambda)(L^2_{\text{comp}}) \), that is \( u_{\ell,1} \) is outgoing. But then it is unique up to a multiplicative constant. This shows that \( L = 1 \) and that \( u_j := u_{1,j} \) satisfy (2.2.20). We also see that \( K = \dim \text{Ran} \Pi_{\lambda_0} \).

Returning to (2.2.25) we see from Step 3 that

\[ m_R(\lambda_0) := \text{rank} \oint_{\lambda_0} R_V(\zeta) d\zeta \]

is a nilpotent operator \( N_{\lambda_0} : \text{Ran} \Pi_{\lambda_0} \to \text{Ran} \Pi_{\lambda_0} \). Hence

\[ m_R(\lambda_0) := \text{rank} \oint_{\lambda_0} R_V(\zeta) d\zeta = \text{rank} (I + N_{\lambda_0})\Pi_{\lambda_0} = \text{rank} \Pi_{\lambda_0} = K. \]

This gives (2.2.22).

5. It remains to consider the case of real potentials and of resonances with multiplicity 1.

We first note that the construction in Step 1 of the proof of Theorem 2.2 shows that for \( V \) real the Schwartz kernel of \( R_V(ik) \), \( k \gg 1 \) is real. Since \( R_V(ik) \) is also self-adjoint it follows that \( R_V(ik, x, y) = R_V(ik, y, x) \). By analytic continuation this is true at any value of \( \lambda \). If, near \( \lambda_0 \),

\[ R_V(\lambda) = -(\lambda - \lambda_0)^{-1}\Pi_{\lambda_0} + A(\lambda, \lambda_0), \]

we conclude that the Schwartz kernel of the rank one operator \( \Pi_{\lambda_0} \) is symmetric in \( x \) and \( y \) and hence, \( \Pi_{\lambda_0} = iu_1 \otimes u_1 \) for an outgoing solution of \((P_V - \lambda_0^2)u_1 = 0\).
To prove the normalization condition (2.2.24), fix \( R > 0 \) and \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi(x) = 1 \) for \( x \geq R \), \( \chi(x) = 0 \) for \( x \leq 0 \) and such that \( u_1(x) = c_\pm e^{\pm i\lambda_0 x} \) on \( \text{supp} \chi_\pm \). Put
\[ \tilde{u}_1 = u_1 - c_+ \chi_+ e^{i\lambda_0 x} - c_- \chi_- e^{-i\lambda_0 x} \in H^2_{\text{comp}}(\mathbb{R}), \]
and define for \( \lambda \in \mathbb{C} \) the following outgoing function
\[ u_\lambda := \tilde{u}_1 + c_+ \chi_+ e^{i\lambda x} + c_- \chi_- e^{-i\lambda x}, \quad u_{\lambda_0} = u_1. \]
Then, given that \( (P_V - \lambda^2)u_1 = 0 \), we find for \( \lambda \) near \( \lambda_0 \),
\[ (P_V - \lambda^2)u_\lambda = (P_V - \lambda^2)\tilde{u}_1 - c_+ [\partial^2_x, \chi_+] e^{i\lambda_0 x} - c_- [\partial^2_x, \chi_-] e^{-i\lambda_0 x} = (\lambda - \lambda_0)(-2\lambda_0 \tilde{u}_1 - ic_+ [\partial^2_x, \chi_+] x e^{i\lambda_0 x} + ic_- [\partial^2_x, \chi_-] x e^{-i\lambda_0 x}) + O((\lambda - \lambda_0)^2)_{L^2_{\text{comp}}}. \]
Using the identity (2.2.18) for the outgoing function \( u = u_\lambda \) (\( \lambda \) in a punctured neighborhood of \( \lambda_0 \)), the expansion (2.2.26) and the fact that \( \chi_\pm \) and \( \tilde{u}_1 \) are supported inside \((-R, R)\), we see that
\[ u_\lambda = R_V(\lambda)(P_V - \lambda^2)u_\lambda = (\lambda - \lambda_0)^{-1} \int_{-R}^R u_1(x)(P_V - \lambda^2)u_\lambda(x)dx + O(|\lambda - \lambda_0|). \]
Inserting \( \lambda = \lambda_0 \) gives,
\[ 1 = \int_{-R}^R u_1(-2i\lambda_0 \tilde{u}_1 + c_+ [\partial^2_x, \chi_+] x e^{i\lambda_0 x} + c_- [\partial^2_x, \chi_-] x e^{-i\lambda_0 x}) dx. \]
From the definition of \( \tilde{u}_1 \) and the fact that \( u_1 = c_\pm e^{\pm i\lambda_0 x} \) on \( \text{supp} \chi_\pm \), we then get
\[ 1 = -2i\lambda_0 \int_{-R}^R u_1^2 dx + c_+^2 \int_{-R}^R (2i\lambda_0 \chi_+ + e^{-i\lambda_0 x} [\partial^2_x, \chi_+] e^{i\lambda_0 x} x) e^{2i\lambda_0 x} dx \]
\[ + c_-^2 \int_{-R}^R (2i\lambda_0 \chi_- - e^{i\lambda_0 x} [\partial^2_x, \chi_-] e^{-i\lambda_0 x} x) e^{-2i\lambda_0 x} dx. \]
Now, integration by parts, together with the fact that \( \chi_\pm = 1 \) near \( \pm R \) and \( \chi_\pm = 0 \) near \( \mp R \), shows that
\[ \int_{-R}^R (2i\lambda_0 \chi_+ + e^{-i\lambda_0 x} [\partial^2_x, \chi_+] e^{i\lambda_0 x} x) e^{2i\lambda_0 x} dx = e^{2i\lambda_0 R}, \]
\[ \int_{-R}^R (2i\lambda_0 \chi_- - e^{i\lambda_0 x} [\partial^2_x, \chi_-] e^{-i\lambda_0 x} x) e^{-2i\lambda_0 x} dx = e^{-2i\lambda_0 R}, \]
which finishes the proof of (2.2.24). \( \square \)

We can now provide
2.2. MEROMORPHIC CONTINUATION

Proof of (2.2.12). We need to show that there are no outgoing solutions to $(P_V - \lambda^2)u = 0$ for $\lambda$ real and non-zero (at $\lambda = 0$ the example of $V = 0$ shows that a pole is possible and the outgoing solution is given by $u = 1$). Since $V$ is real $\bar{u}$ is also a solution. Using the notation of (2.1.8) we calculate the Wronskians:

$$W(u, \bar{u}) := \begin{vmatrix} u & \bar{u} \\ u_x & \bar{u}_x \end{vmatrix} = \begin{cases} 2i\lambda |a_-|^2, & x < -R, \\ -2i\lambda |a_+|^2, & x > R. \end{cases}$$

Since the Wronskian for the equation $-\partial_x^2 + V - \lambda^2$ is constant, this is impossible for $\lambda \neq 0$ and $u \neq 0$. □

For $\lambda_0 = 0$ we restrict our attention to real $V$’s, that is to self-adjoint operators $P_V$. We need detailed information about the zero resonance only for resonance expansions and trace formulæ. In both cases we will assume self-adjointness of $P_V$ so that we can use the spectral theorem.

THEOREM 2.5 (Singular part of $R_V(\lambda)$ at 0 in one dimension). Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R})$, $V \neq 0$, and that 0 is a resonance. Then $m_R(0) = 1$ and

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda} + A(\lambda),$$

where $\lambda \mapsto A(\lambda)$ is holomorphic near 0, and

(2.2.27) $\Pi_0 = -iu_1 \otimes u_1$, $P_Vu_1 = 0$; $u_1(x) = c_\pm$, $\pm x \gg 1$,

where $c_\pm \in \mathbb{R} \setminus 0$ and $c_+^2 + c_-^2 = 1$.

Proof. 1. Since $P_V$ is self-adjoint, for $\text{Im} \lambda > 0$, $|\lambda| \ll 1$ (so that we avoid possible eigenvalues which are the poles in $\text{Im} \lambda > 0$), the spectral theorem gives

$$\|R_V(\lambda)\|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, \mathbb{R}^+)} \leq \frac{1}{|\lambda| \text{Im} \lambda}.$$  

This shows that

$$R_V(\lambda) = \frac{A_2}{\lambda^2} + \frac{A_1}{\lambda} + A(\lambda),$$

where $A_j$ are finite rank operators, $L^2_{\text{comp}} \to H^2_{\text{loc}}$. By applying $P_V - \lambda^2$ to $R_V(\lambda)$ we conclude that $P_VA_j = 0$.

2. For $\psi \in C_c^\infty(\mathbb{R})$ and $\rho \in C_c^\infty(\mathbb{R}; [0, 1])$ and $\text{Im} \lambda > 0$, $|\lambda| \ll 1$,

$$\|\rho(A_2 + \lambda A_1 + \lambda^2 A(\lambda))\psi\|_{L^2} \leq \|\rho(A_2 + \lambda A_1 + \lambda^2 A(\lambda))\psi\|_{L^2} \leq |\lambda|^2 \|\lambda^2 R_V(\lambda)\psi\| \leq \frac{|\lambda|^2}{d(\lambda^2, \mathbb{R}^+)} \|\psi\|_{L^2}.$$  

Hence, letting $\lambda = it$, $t \to 0+$, we conclude that

$$\|\rho A_2 \psi\|_{L^2} \leq \|\psi\|_{L^2}.$$  

2. MEROMORPHIC CONTINUATION

Since \( \rho \in C_c^\infty([0, 1]) \) is arbitrary we conclude that \( A_2 \) is bounded on \( L^2(\mathbb{R}) \). The range of \( A_2 \) consists of solutions to \( P_Vu = 0 \), namely \( u = a + bx \), for \( x \gg 1 \), so we conclude that \( a = b = 0 \), which implies that \( A_2 = 0 \).

3. We now show that \( \Pi_0 := A_1 \) has rank 1. Indeed, when \( \lambda \) is not a pole of \( R_V \), by Theorem 2.3 \( R_V(\lambda)(L^2_{\text{comp}}) \) consists of outgoing functions. Taking the meromorphic expansion of the outgoing condition (2.2.17) at \( \lambda = 0 \), we see that the range of \( \Pi_0 \) consists of outgoing functions. Since \( P_V \Pi_0 = 0 \), \( \text{Ran} \Pi_0 \) consists of outgoing solutions to the equation \( P_Vu = 0 \). Since the space of such solutions is at most one-dimensional, we see that \( \Pi_0 \) has rank one.

4. Arguing as in the proof of part 3 of Theorem 2.4, we see that \( \Pi_0 = iu_1 \otimes u_1 \), where \( u_1 \in H^2_\text{loc} \) solves \( P_Vu_1 = 0 \) and is outgoing, that is \( u_1(x) = c_\pm, \pm x \gg 1 \), for some \( c_\pm \in \mathbb{C} \). We have \( c_\pm \neq 0 \) since otherwise \( u_1 \), a solution of an ordinary differential equation, would be identically zero. We also get the condition \( c_+^2 + c_-^2 = 1 \), since the proof of (2.2.24) applies for the zero resonance. Finally, by (2.2.14), we see that \( \Pi_0 \) is antisymmetric: for \( \psi, \phi \in L^2(\mathbb{R}) \), \( \langle \Pi_0 \psi, \phi \rangle_{L^2} = -\langle \psi, \Pi_0 \phi \rangle \). Hence \( \bar{u}_1 = \pm u_1 \), so either \( u_1 \) or \( iu_1 \) is real-valued. The second option is impossible since then \( c_\pm \) are purely imaginary and cannot satisfy \( c_+^2 + c_-^2 = 1 \); we conclude that \( u_1 \), and thus \( c_\pm \) are real.

\[ \square \]

REMARK. We can construct a real-valued potential \( V \in C_c^\infty(\mathbb{R}) \) such that \( R_V \) has a resonance at \( \lambda = 0 \) and the constants \( c_\pm \) in (2.2.27) are any given numbers in \( \mathbb{R} \setminus 0 \) satisfying \( c_+^2 + c_-^2 = 1 \). Indeed, for \( c_+c_- > 0 \) consider a function \( u \in C^\infty(\mathbb{R}; \mathbb{R}) \) which is nonvanishing everywhere and \( u(x) = c_\pm \) for \( \pm x \gg 1 \). Put \( V = u''/u \), then \( P_Vu = 0 \) and by Theorem 2.3, zero is a resonance of \( R_V \). Moreover, the function \( u_1 \) from Theorem 2.5 is a multiple of \( u \). For \( c_+c_- < 0 \) we repeat the same argument, taking \( u \in C^\infty(\mathbb{R}; \mathbb{R}) \) which is nonvanishing except at \( x = 0, u(x) = c_\pm \) for \( \pm x \gg 1 \), and \( u(x) = c_+x \) for \( |x| < 1 \).

EXAMPLE. We present a natural family of potentials which have resonances of multiplicity 2 for some values in the family. This is illustrated in Figure 2.3.

Consider a potential \( V \in C_c^1(\mathbb{R}; \mathbb{R}) \), \( \text{supp} V \subset [-a, a] \) with the property that \( V(x) < -c < 0 \) for, say, \( x \in (-b, b) \), \( 0 < b < a \). We then consider a family of potentials \( \tau V, \tau \geq 1 \), that is we vary the coupling constant in the Schrödinger operator

\[ P_\tau := D_x^2 + \tau V(x) . \]
By applying min-max methods directly (see Theorem B.11) or by using semi-classical Weyl law (with $h^2 = 1/\sqrt{\tau}$ – see for instance [Zw12, Theorem 6.8]) we see that the number of negative eigenvalues of $P_\tau$ grows (proportionally to $\sqrt{\tau}$) as $\tau$ increases.

The construction of $R_{\tau V}(\lambda)$ also shows that for any $R$, resonances in $D(0, R)$ are continuous as functions of $\tau$. This means that eigenvalues, that is resonances on $i\mathbb{R}_+$, are obtained, as $\tau$ increases, from a continuous family of resonances passing through zero.

In view of the symmetry of resonances with respect to the real axis given in (2.2.13), the simplicity of the resonance at $\lambda = 0$ given in Theorem 2.5 and the absence of resonances on $\mathbb{R} \setminus \{0\}$ (see (2.2.12)) it means that two resonances meet on $i\mathbb{R}_-$ before splitting. One of them will move through 0 to become an eigenvalue. This provides a simple example of a resonance, $\lambda_0 \in i\mathbb{R}_-$ for which $m_R(\lambda_0) = 2$.

The multiplicity of a resonance can also be described using Fredholm determinants – see §B.5. For that we define

$$(2.2.28) \quad D(\lambda) := \det(I + VR_0(\lambda)\rho).$$
where \( \rho \in L^\infty_{\text{comp}} \) and \( \rho V = V \). This is allowed as \( VR_0(\lambda) \rho \) is a (meromorphic) family of operators of trace class.

We note that \( D(\lambda) \) is a meromorphic function of \( \lambda \) with a single pole at \( \lambda = 0 \). The multiplicity of a zero of \( D(\lambda) \) is defined in the usual way and we have,

\[
m_D(\lambda) := \frac{1}{2\pi i} \oint \frac{D'(\zeta)}{D(\zeta)} d\zeta,
\]

where the integral is over a positively oriented circle which includes \( \lambda \) and no other pole or zero of \( D(\lambda) \).

**THEOREM 2.6 (Multiplicity of a resonance in one dimension).**

The multiplicities defined by (2.2.11) and (2.2.29) are related as follows

\[
m_D(\lambda) = m_R(\lambda), \quad \lambda \in \mathbb{C} \setminus \{0\}.
\]

For \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \),

\[
m_D(0) = m_R(0) - 1.
\]

**Proof.** The proof is based on the Gohberg–Sigal theory of residues for meromorphic families of operators reviewed in Section C.4.

1. We start with the case of a pole at zero, assuming that \( V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R}) \), \( V \neq 0 \). From (2.2.15),

\[
(I + VR_0(\lambda) \rho)^{-1} = I - VR_V(\lambda) \rho
\]

and Theorem 2.5 we see that \((I + VR_0(\lambda) \rho)^{-1}\) has a simple pole of rank one at 0 if and only if \( R_V(\lambda) \) has a pole at 0. On the other hand \( I + VR_0(\lambda) \rho \) has a simple pole of rank one at 0. Hence Theorem C.7 shows that

\[
I + VR_0(\lambda) \rho = U_1(\lambda)(Q_0 + \lambda^{-1}Q_{-1} + \lambda Q_1)U_2(\lambda),
\]

where

\[
rank Q_{-1} = 1, \quad rank Q_1 = m_R(0), \quad Q_j Q_k = \delta_{jk} Q_j,
\]

and \( U_j(\lambda) \) are invertible and holomorphic. The conclusion then follows from Theorem C.8.

2. Now let \( V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{C}) \), \( V \neq 0 \) and assume that \( m_R(\lambda_0) = 1, \lambda_0 \neq 0 \). From (2.2.32) we see that the pole of the left hand side is simple with a rank one residue. Theorem C.7 shows that near there exist holomorphic invertible operators \( U_j(\lambda) \), such that near \( \lambda_0 \)

\[
(I + VR_0(\lambda) \rho)^{-1} = U_1(\lambda)(P_0 + (\lambda - \lambda_0)^{-1}P_1)U_2(\lambda),
\]

where \( P_i P_j = \delta_{ij} P_j \), \( rank P_1 = 1 \) and \( rank(I - P_0) < \infty \). (There are no polynomial terms as they produce poles of \( I + VR_0(\lambda) \rho \) which is holomorphic near \( \lambda_0 \neq 0 \).) Theorem C.8 then shows that \( m_D(\lambda_0) = 1 \).
2.3. EXPANSIONS OF SCATTERED WAVES

A motivation for the study of resonances is the fact that they describe oscillations and decay of waves for problems on non-compact domains. In this sense they replace eigenvalues and Fourier series expansions. Except for Theorem 2.8 we assume in this section that $V$ is real valued. That is because we need to use methods of spectral theory of self-adjoint operators.

To explain the expansions consider first $P_V = D_x^2 + V$ on $[a, b]$ with Dirichlet (or Neumann) boundary condition. Then the problem

\[
\begin{aligned}
(P_V - \lambda^2)u &= 0 \quad \text{on } (a, b) \\
u(a) &= u(b) = 0
\end{aligned}
\]

has a set of distinct solutions

\[
(i \sqrt{-E_k}, v_k), \quad (\lambda_j, u_j),
\]

$E_N < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty,$

\[
\int_a^b |u_j|^2 dx = \int_a^b |v_k|^2 dx = 1.
\]

We then consider the wave equation

\[
\begin{aligned}
(D_t^2 - P_V)w &= 0 \quad \text{on } \mathbb{R} \times (a, b) \\
w(0, x) &= w_0(x), \quad \partial_t w(0, x) = w_1(x) \quad \text{on } [a, b] \\
w(t, a) &= w(t, b) = 0 \quad \text{on } \mathbb{R}.
\end{aligned}
\]

It can be solved using the eigenfunction expansion (Fourier series in the case when $V \equiv 0$):

\[
w(t, x) = \sum_{k=1}^{N} \cosh(t \sqrt{-E_k}) a_k v_k(x) + \sum_{k=1}^{N} \frac{\sinh(t \sqrt{-E_k})}{\sqrt{-E_k}} b_k v_k(x)
\]

\[
+ \sum_{j=0}^{\infty} \cos(t \lambda_j) c_j u_j(x) + \sum_{j=0}^{\infty} \frac{\sin(t \lambda_j)}{\lambda_j} d_j u_j(x)
\]

(2.3.1)
where
\[ a_k = \int_a^b w_0(x) \overline{v_k(x)} \, dx, \quad b_k = \int_a^b w_1(x) \overline{v_k(x)}(x) \]
\[ c_j = \int_a^b w_0(x) \overline{u_j(x)} \, dx, \quad d_j = \int_a^b w_1(x) \overline{u_j(x)} \, dx. \]

We now give the analogue of (2.3.1) when \([a, b]\) is replaced by \(\mathbb{R}\):

**THEOREM 2.7** (Resonance expansions of scattering waves in one dimension). Let \(V \in L^\infty(\mathbb{R}; \mathbb{R})\) and suppose that \(w(t, x)\) is the solution of

\[
\begin{cases}
(D^2_t - P_V)w(t, x) = 0, \\
w(0, x) = w_0(x) \in H^1_{\text{comp}}(\mathbb{R}), \\
\partial_t w(0, x) = w_1(x) \in L^2_{\text{comp}}(\mathbb{R}).
\end{cases}
\]

Then, for any \(A > 0,
\]

\[
w(t, x) = \sum_{\text{Im} \lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j, \ell}(x) + E_A(t),
\]

where the sum is finite,

\[
\sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} f_{j, \ell}(x) = \text{Res}_{\mu=\lambda_j} \left( (iR_V(\mu)w_1 + \mu R_V(\mu)w_0) e^{-i\mu t} \right),
\]

\[(P_V - \lambda_j)^{m_R(\lambda_j)-\ell} f_{j, \ell} = 0,
\]

and for any \(K > 0,\) such that \(\text{supp } w_j \subset (-K, K),\) there exist constants \(C_{K,A}\) and \(T_{K,A}\)

\[
\|E_A(t)\|_{H^2([K, K])} \leq C_{K,A} e^{-tA} (\|w_0\|_{H^1} + \|w_1\|_{L^2}), \quad t \geq T_{K,A}.
\]

**REMARKS.**

1. For numerical illustrations of this theorem see Figures 1.4 and 1.5.

2. It may at first seem strange that only exponential appear as contributions of negative eigenvalues (compared to sinh and cosh in (2.3.1)): the exponentially decaying terms are absorbed into the error term \(E_A(t)\) when \(A\) is small and are “masked” by the resonance expansion when \(A\) is large.

3. We notice that the error term \(E_A(t)\) is more regular for large times. That corresponds to propagation of singularities: when time is large all singularities leave a compact set. When \(V \in C^\infty_c(\mathbb{R})\) then an examination of the
proof shows that we have the same bound with the right hand side replaced by \( \|E_A(t)\|_{H^k([-K,K])} \) for any \( k \).

Before proving Theorem 2.7 we need the existence of a resonance free region and an estimate for the resolvent:

**Theorem 2.8 (Resonance free regions in one dimension).** Suppose that \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{C}) \). Then for any \( \rho \in C^\infty\text{c}(\mathbb{R}) \) and any \( \delta < 1/|\text{chsupp}V| \) (where chsupp \( V \) is the convex hull of the support of \( V \)) there exist constants \( A, C, T \) such that

\[
\|\rho R_V(\lambda)\rho\|_{L^2 \to H^j} \leq C|\lambda|^{-j-1} e^{T (\text{Im } \lambda)}, \quad j = 0, 1, 2,
\]

for

\[
\text{Im } \lambda \geq -A - \delta \log(1 + |\lambda|), \quad |\lambda| > C_0.
\]

In particular there are only finitely many resonances in the region

\[
\{ \lambda : \text{Im } \lambda \geq -A - \delta \log(1 + |\lambda|) \}.
\]

for any \( A > 0 \).

**Proof.** 1. First we modify the estimate (2.2.3) for the free resolvent

\[
\|\rho_1 R_0(\lambda)\rho_1\|_{L^2 \to L^2} \leq C|\lambda|^{-1} e^{(b-a)|\text{Im } \lambda|}, \quad 0 \leq j \leq 2,
\]

where \( \rho_1 \in L^\infty \) and supp \( \rho_1 \subset [a, b] \).

We then recall (2.2.9):

\[
\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho_1(1 + VR_0(\lambda)\rho_1)^{-1}(1 - VR_0(\lambda)(1 - \rho_1))\rho
\]

where we assumed that \( \rho = 1 \) on supp \( V \), and \( \rho_1 \in L^\infty_{\text{con}}(\mathbb{R}) \) is any function satisfying \( \rho_1 V = V \). In particular we can take \( \rho_1 = 1_{\text{chsupp}V} \). We see now
that (2.3.5) holds for $\lambda$ at which we can invert $I + VR_0(\lambda)\rho_1$ by Neumann series, and that follows from
\[
\|VR_0(\lambda)\rho_1\|_{L^2} \leq \frac{1}{2}.
\]
2. To establish this estimate we put $[a, b] := {\text{chsupp}}V$ and use (2.3.6) to see that for $\text{Im} \lambda > -A - \delta \log(1 + |\lambda|),$
\[
\|VR_0(\lambda)\rho_1\|_{L^2} \leq C\|V\|_{L^\infty} e^{(A + \delta \log(1 + |\lambda|))(b-a)}/|\lambda|
\leq C\|V\|_{L^\infty} \frac{|\lambda|^{-1 + \delta(b-a)}}{|\lambda|} \leq C' \|V\|_{L^\infty} |\lambda|^{-1 + \delta(b-a)} \leq \frac{1}{2},
\]
a provided that $\delta < 1/(b - a)$ and $|\lambda| \geq R.$
3. Returning to (2.3.7) we use the bound (2.2.3) for $\rho R_0(\lambda)\rho_1$ and $\rho R_0(\lambda)(1 - \rho_1)$ terms to obtain (2.3.5). \[\square\]

The idea for obtaining the expansion (2.3.3) is to deform the contour in the representation of the wave propagator based on the spectral theorem.

Proof of Theorem 2.7. 1. Let us first consider (2.3.2) with $w_0 \equiv 0$ and $w_1 \in H^2(\mathbb{R}), \text{supp} w_1 \subset (-K, K).$

By the spectral theorem, the solution of (2.3.2) can be written as
\[
U(t) := \int_0^\infty \frac{\sin t\lambda}{\lambda} dE_\lambda + \sum_{k=1}^K \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k,
\]
where $\mu_k^2 < 0 (\text{Im} \mu_k > 0)$ are the negative eigenvalues of $P_V$ with $v_k$ the corresponding real valued normalized eigenfunctions (we use the notation (2.2.19)) and $dE_\lambda$ is the spectral measure on $(0, \infty):$

\[
P_V = \int_0^\infty \lambda^2 dE_\lambda + \sum_{k=1}^K \mu_k^2 v_k \otimes v_k, \quad I = \int_0^\infty \lambda^2 dE_\lambda + \sum_{k=1}^K v_k \otimes v_k.
\]

Since for $\mu$ near $\mu_k,$ $R_V(\mu) = (\mu_k^2 - \mu^2)^{-1}(v_k \otimes v_k) + Q_k(\mu),$ where $\mu \mapsto Q_k(\mu)$ is holomorphic near $\mu_k,$ and $\text{Res}_{\mu = \mu_k} (\mu_k^2 - \mu^2)^{-1} = -(2\mu_k)^{-1}$ we have
\[
\sum_{k=1}^K \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k = \sum_{k=1}^K \pm \text{Res}_{\mu = \pm \mu_k} iR_V(\pm \mu)e^{-i\mu t}.
\]
2. Using Stone’s Formula recalled in Theorem B.8 we write the spectral measure $dE_\lambda$ in (2.3.9) in terms of $R_V(\lambda):$
\[
dE_\lambda = \frac{1}{\pi i} (R_V(\lambda) - R_V(-\lambda)) \lambda d\lambda,
\]
where we noted the change of variables (2.1.7): $z = \lambda^2, \pm \lambda = \sqrt{z \pm i0}.$
Hence

\[ w(t) - \sum_{k=1}^{K} \frac{\sin t\mu_k}{\mu_k} v_k \otimes v_k = \frac{1}{\pi i} \int_0^\infty \sin t\lambda (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ = \frac{1}{\pi i} \int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\Sigma_{\epsilon}} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ + \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\sigma_{\epsilon}} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ = \frac{1}{2\pi} \int_{\Sigma_0} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \]

\[ + \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\sigma_{\epsilon}} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda , \]

where \( \Sigma_{\epsilon} \) is the union of \( \mathbb{R}\setminus(-\epsilon,\epsilon) \) with the semicircle \((0, \pi) \ni s \mapsto e^{-is}\), oriented counterclockwise, and \( \sigma_{\epsilon} \) is the same semicircle oriented clockwise. The parameter \( \epsilon_0 \) is chosen so that there are no poles of \( R_V \) in \( D(0, \epsilon_0) \setminus \{0\} \).

To justify convergence of the integral over \( \Sigma_{\epsilon_0} \) we use the spectral theorem (see (2.3.11)) which shows that

\[ (R_V(\lambda) - R_V(-\lambda))(D_2^2 + V) = \lambda^2 (R_V(\lambda) - R_V(-\lambda)) . \]

From that we conclude that for \( \rho \in C_c^\infty \) equal to 1 on \( \text{supp} \, w_1 \),

\[ \rho(R_V(\lambda) - R_V(-\lambda)) \rho w_1 = \rho(R_V(\lambda) - R_V(-\lambda)) w_1 \]

\[ = \rho(R_V(\lambda) - R_V(-\lambda))(1 + \lambda^2)^{-1}(1 + D_2^2 + V) w_1 . \]

Since \( \rho(R_V(\lambda) - R_V(-\lambda)) \rho = O(1) : L^2 \to L^2 \) this shows that the integral converges in \( L^2_{\text{loc}} \).

3. The integral over \( \sigma_{\epsilon} \) converges to 0 as \( \epsilon \to 0+ \) in \( L^2_{\text{loc}} \) unless \( R_V \) has a resonance at 0. In that case we use Theorem 2.5 to see that

\[ \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\sigma_{\epsilon}} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda = \frac{1}{2\pi} \lim_{\epsilon \to 0+} \int_{\sigma_{\epsilon}} e^{-it\lambda} \frac{2\Pi_0 w_1}{\lambda} d\lambda \]

\[ = \frac{1}{\pi} \int_0^\pi \Pi_0 w_1(-ids) \]

\[ = -i\Pi_0 w_1 . \]

4. Now let \( \rho \in C_c^\infty(\mathbb{R}) \) satisfy \( \rho \equiv 1 \) on \([-K, K]\) (recall that we assumed that \( \text{supp} \, w_1 \subset (-K, K) \)). Choose \( R \) large enough so that all the resonances
with \( \text{Im} \lambda > -A - \delta \log(1 + |\text{Re} \lambda|) \) are contained in \(|\lambda| \leq R\). We deform the contour of integration in the integral over \( \sum_{\epsilon} \) using the following contours:

\[
\Gamma := \{ \lambda - it : \lambda \in \mathbb{R} \},
\]

\[
\Gamma_R := \Gamma \cap \{|\text{Re} \lambda| \leq R\},
\]

\[
\gamma^\pm_R = \{ \pm R - it : 0 \leq t \leq A + \epsilon + \delta \log(1 + R) \}, \quad \gamma_R := \gamma^+_R \cup \gamma^-_R,
\]

\[
\gamma^\infty = (-\infty, -R) \cup (R, \infty).
\]

Here we choose \( \epsilon \) so that there are no resonances on \( \Gamma \). We also put

\[
\Omega_A := \{ \lambda : \text{Im} \lambda \geq -A - \epsilon - \delta \log(1 + |\text{Re} \lambda|) \} \setminus \{0\}
\]

and define

\[
\Pi_A(t) := i \sum_{\lambda_j \in \Omega_A} \text{Res}_{\lambda = \lambda_j} (\rho R_V(\lambda) e^{-i\lambda t}).
\]

In this notation (2.3.10) and the residue theorem show that \( U(t) \) defined in (2.3.8) is given by

\[
(2.3.13) \quad \rho U(t) = -i \rho_0(t) + \Pi_A(t) + E_{\gamma_R}(t) + E_{\gamma_R}(t),
\]

where (with natural orientations)

\[
(2.3.14) \quad E_{\gamma}(t) := \frac{1}{2\pi} \int_{\gamma} e^{-i\lambda t} (\rho R_V(\lambda) - R_V(-\lambda)) \rho w_1 d\lambda.
\]

We note that the contributions from the poles of \( R_V(\lambda) \) at \( \lambda = -\mu_k \) cancel the contributions from \( \sin t\mu_k \) – see (2.3.10).

5. Since \( \rho \equiv 1 \) on \( \text{supp} \ w_1 \)

\[
(2.3.15) \quad \| E_{\gamma_R}(t) w_1 \|_{H^1}, \quad \| E_{\gamma^\infty}(t) w_1 \|_{H^1} \to 0, \quad R \to \infty,
\]

In fact, using (2.3.5) and (2.3.12) we obtain

\[
\| E_{\gamma^\infty}(t) w_1 \|_{H^1} \leq C \int_R^\infty (1 + |\lambda|^{-1}) \| w_1 \|_{H^2} \leq \frac{C}{R} \| w_1 \|_{H^2},
\]

and

\[
\| E_{\gamma_R}(t) w_1 \|_{H^1} \leq \frac{C}{1 + R^2} \| w_1 \|_{H^2}. \]

Hence (2.3.15) holds for \( w_1 \in H^2, \text{supp} \ w_1 \subset (-K, K) \).

6. We now return to (2.3.13) and see that

\[
(2.3.16) \quad \rho U(t) w_1 = -i \Pi_0 w_1 + \Pi_A(t) w_1 + E_{\Gamma}(t) w_1,
\]

for \( w_1 \in H^2, \text{supp} \ w_1 \subset (-K, K) \),

where \( E_{\Gamma} \) is defined using (2.3.10) and \( \Gamma \) defined in Step 4.

We now show that for \( t \gg 1 \),

\[
(2.3.17) \quad \| E_{\Gamma}(t) w_1 \|_{H^2} \leq C e^{-tA} \| w_1 \|_{L^2}.
\]
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For that we use (2.3.5) with \( j = 2 \) and \(|\lambda| > R\), and the assumption that there are no poles of \( R_V(\lambda) \) near \( \Gamma \) in a compact set. Thus we obtain:

\[
\| E_\Gamma(t) w_1 \|_{H^2} \leq C e^{-At} \int_{\mathbb{R}} e^{-t \delta \log(1 + |\lambda|)} e^{\delta T \log(1 + |\lambda|)} (1 + |\lambda|)^{-\delta(t-T)+1} \| w_1 \|_{L^2} d\lambda \leq C' e^{-At} \| w_1 \|_{L^2}, \quad t > T + 3/\delta.
\]

Since \( C_c^\infty((-K, K)) \subset H^2 \) is dense in \( L^2([-K, K]) \) the decomposition (2.3.16) and the estimate (2.3.16) are valid for \( w_1 \in L^2 \), supp \( w_1 \subset [-K, K] \) proving theorem for \( w_0 = 0 \).

The case of arbitrary \( w_0 \in H^1_{\text{comp}} \) and \( w_1 \equiv 0 \) follows by replacing \( \sin t\lambda/\lambda \) by \( \cos t\lambda \) in the formula for \( w(t, x) \). \( \square \)

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Outside of the support of \( V \), a solution of

\[(2.4.1) \quad (P_V - \lambda^2)u = 0\]

can be written as a sum of an outgoing and incoming terms

\[
u(x) = u_{\text{in}}(x) + u_{\text{out}}(x), \quad |x| \geq R.
\]

Following the conventions described in the beginning of this chapter,

\[
u_{\text{in}}(x) = b_{\text{sgn}}(x) e^{-i\lambda|x|}, \quad u_{\text{out}}(x) = a_{\text{sgn}}(x) e^{i\lambda|x|}, \quad |x| \geq R.
\]

In scattering we compare the incoming waves with the outgoing ones and mathematically that is captured by the scattering matrix which is defined as follows

\[(2.4.2) \quad S : \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \mapsto \begin{pmatrix} a_+ \\ a_- \end{pmatrix}.
\]

To describe \( S = S(\lambda) \) at frequency \( \lambda \) we need to find solutions to (2.4.1) of the following form:

\[(2.4.3) \quad u^\pm(x) = e^{\pm i\lambda x} + v^\pm(x, \lambda)\]

where \( v^\pm(x, \lambda) \) is outgoing. The functions \( v^\pm \) are easily found using the outgoing resolvent \( R_V(\lambda) \):

\[(2.4.4) \quad v^\pm(x, \lambda) = -R_V(\lambda) \left( V e^{\pm i\lambda x} \right).
\]

This is well defined away from the poles of \( R_V(\lambda) \). In particular, In the self-adjoint case that means that \( u_\pm \) exist for \( \lambda \in \mathbb{R} \setminus 0 \).
REMARK. The strange $\pm$ notation (which is different than the $\pm$ notation of (2.4.2)) is motivated by the higher dimensional setting in which $\pm$ is replace by $\omega \in S^{n-1}$. When $n = 1$, $S^0 = \{+, -\}$.

If we write
\[(2.4.5)\]
\[v_{\text{sgn}(x)}^\pm(\lambda) := e^{-i\lambda|x|}v^\pm(x, \lambda), \quad |x| > R,\]
then (2.4.2) shows that
\[(2.4.6)\]
\[S(\lambda) : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 + v^+_\lambda(\lambda) \\ v^-_\lambda(\lambda) \end{pmatrix},

S(\lambda) : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} v^+_\lambda(\lambda) \\ 1 + v^-_\lambda(\lambda) \end{pmatrix},\]
which means that
\[(2.4.7)\]
\[S(\lambda) = I + A(\lambda), \quad A(\lambda) = \begin{pmatrix} v^+_\lambda(\lambda) & v^-_\lambda(\lambda) \\ v^+_\lambda(\lambda) & v^-_\lambda(\lambda) \end{pmatrix}.\]

**THEOREM 2.9** (Scattering matrix in terms of the resolvent). 1) The coefficients of $A(\lambda)$ are meromorphic functions of $\lambda$ given by the following formula:
\[(2.4.8)\]
\[v^{\omega}_\theta(\lambda) = \frac{1}{2i\lambda} \int_{\mathbb{R}} e^{i\lambda(\omega - \theta)x} V(x)(1 - e^{-i\lambda\omega x} R_V(\lambda)(e^{i\lambda\omega} V)(x)) dx,\]
where $\theta, \omega \in \{+, -\}$.

2) If we put $E_\rho(\lambda) : L^2(\mathbb{R}) \rightarrow \mathbb{C}^2$,
\[(2.4.9)\]
\[E_\rho(\lambda) u := \begin{pmatrix} \int_{\mathbb{R}} e^{-i\lambda x} u(x) \rho(x) dx \\ \int_{\mathbb{R}} e^{i\lambda x} u(x) \rho(x) dx \end{pmatrix},\]
where $\rho \in L^\infty_{\text{comp}}$, $\rho V = V$, then
\[(2.4.10)\]
\[S(\lambda) = I + \frac{1}{2i\lambda} E_\rho(\lambda)(I + VR_0(\lambda)\rho)^{-1}VE_\rho(\bar{\lambda})^*.\]

**Proof.** 1. Since $R_V(\lambda) = R_0(\lambda)(I - VR_V(\lambda))$, we have
\[v^{\omega}_\theta(\lambda) = -e^{-i\lambda\theta y} R_0(\lambda)(I - VR_V(\lambda))(V e^{i\lambda V})(y), \quad \theta y > R,\]
where $\text{supp} \ V \subset [-R, R]$.

Using the explicit formula for $R_0(\lambda)$ we then notice that for $f$ with $\text{supp} \ f \subset [-R, R],$
\[R_0(\lambda) f(y) = -\frac{1}{2i\lambda} e^{i\theta y} \int_{\mathbb{R}} e^{-i\lambda x} f(x) dx, \quad \theta y > R.\]
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Combining the two expressions we obtain (2.4.8).

2. Now we use
\[ R(V(\lambda))V = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}V \] (see (2.2.9)) and \((I + VR_0(\lambda)\rho)^{-1}V = \rho(I + VR_0(\lambda)\rho)^{-1}V\), so that

\[ (2.4.11) \quad v_{\theta}(\lambda) = -e^{-i\lambda y_0}R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}(Ve^{i\omega \lambda \bullet}), \quad \theta y > R, \]

and (2.4.10) follows.

\[ \square \]

INTERPRETATION. 1. We have \(v_{+}^+(\lambda) = v_{-}^-(\lambda)\). This can be seen by comparing the values of the Wronskian:

\[ W(u^+, u^-) := \begin{vmatrix} u^+ & u^- \\ \partial_x u^+ & \partial_x u^- \end{vmatrix} = \begin{cases} -2i\lambda(1 + v_-^-), & x < -R, \\ -2i\lambda(1 + v_+^+), & x > R. \end{cases} \]

Since \(W\) is constant, it follows that for \(\lambda \neq 0\), \(v_{+}^+(\lambda) = v_{-}^-(\lambda)\).

2. The coefficients \(v_{\theta}(\lambda)\) have important physical interpretations:

\[ t(\lambda) = 1 + v_{+}^+(\lambda) \] is the transmission coefficient,

\[ r_+(\lambda) = v_{+}^+(\lambda) \] is the right reflection coefficient,

\[ r_-(\lambda) = v_{-}^+(\lambda) \] is the left reflection coefficient.

This interpretation follows from comparing (2.4.2) and (2.4.6).

3. Changing \(\lambda\) to \(-\lambda\) in the definition of \(S(\lambda)\) shows that when \(S(\lambda)\) and \(S(-\lambda)\) both exist then

\[ (2.4.13) \quad S(-\lambda) = JS(\lambda)^{-1}J, \quad J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

4. When \(V\) is real and \(\lambda \in \mathbb{R} \setminus \{0\}\) and \(u\) solves (2.4.1) we can also take Wronskians of \(u\) and \(\bar{u}\):

\[ W(u, \bar{u}) = \begin{cases} i\lambda(|a_-|^2 - |b_+|^2), & x < -R, \\ i\lambda(|b_-|^2 - |a_+|^2), & x > R, \end{cases} \]

which means \(S\) given by (2.4.2) is unitary. Hence we obtain unitarity of the scattering matrix: \(S(\lambda)^* = S(\lambda)^{-1}\). A meromorphic continuation of this equality gives

\[ (2.4.14) \quad V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \implies S(\bar{\lambda})^* = S(\lambda)^{-1}, \quad \lambda \in \mathbb{C}. \]

This implies that \(v_{\theta}(\lambda)\) are holomorphic for \(\lambda \in \mathbb{R}\).

REMARK. As already remarked above we should think of \(\pm\) as the element of the “sphere”, \(S^0\), in one dimensional space. As we will see the same formula is valid in dimension \(n\) with \(\theta, \omega \in S^{n-1}\). The scattering “matrix” is then given as the sum of the idenity and an operator defined by an integral kernel (2.4.8) in \(S^{n-1} \times S^{n-1}\). Of course the interpretation of reflected and transmited waves is then less clear.
The representation given in Theorem 2.9 gives us important estimates for the coefficients of the scattering matrix in the physical half plane, $\text{Im} \lambda \geq 0$:

**THEOREM 2.10** (Estimates on the scattering matrix). For

$$\text{Im} \lambda \geq 0, \quad |\lambda| \geq C_0,$$

we have

$$\|e^{\mp i\lambda x}R_V(\lambda)V e^{\pm i\lambda \bullet} \|_{L^2 \to L^\infty} \leq \frac{C_1}{|\lambda|}. \quad (2.4.15)$$

Consequently, (2.4.8) implies that for $\text{Im} \lambda \geq 0, \quad |\lambda| \geq C_0$,

$$v_+^+(\lambda) = \frac{1}{2i\lambda} \left( \hat{V}(0) + O(1/|\lambda|) \right),$$

$$v_-^-(\lambda) = \frac{1}{2i\lambda} \left( \hat{V}(0) + O(1/|\lambda|) \right). \quad (2.4.16)$$

**REMARK.** The estimate (2.4.15) implies that $\|e^{\mp i\lambda x}R_V(\lambda)(V e^{\pm i\lambda \bullet}) \|_{L^\infty} \leq C_1/|\lambda|$ for $\text{Im} \lambda \geq 0, \quad |\lambda| \geq C_0$: we simply apply the operator to $1_{\text{supp} V}$. That particular estimate will be used to obtain (2.4.16).

**Proof.**

1. Let

$$R_0^\omega(\lambda) := e^{-i\lambda\omega x}R_0(\lambda)e^{i\lambda\omega \bullet}.$$  

Its Schwartz kernel given by

$$R_0^\omega(\lambda, x, y) := e^{-i\lambda\omega x}R_0(\lambda, x, y)e^{i\lambda\omega y} = \frac{i}{2\lambda} e^{i\lambda(|x-y| - \omega(x-y))}. \quad (2.4.17)$$

As $|x-y| - \omega(x-y) \geq 0$, (2.4.17) shows that for $\text{Im} \lambda \geq 0$ we have

$$\|VR_0^\omega(\lambda) \rho\|_{L^2 \to L^2} \leq C/|\lambda|.$$  

Hence the Neumann series for $(I + VR_0^\omega(\lambda) \rho)^{-1}$ converges for $\text{Im} \lambda \geq 0, \quad |\lambda| > C_0$. Similarly,

$$R_0^\omega(\lambda) \rho = O(1/|\lambda|) : L^2 \to L^\infty, \quad \text{Im} \lambda \geq 0.$$  

2. Recalling (2.2.9),

$$R_V(\lambda)V = R_0(\lambda)(I + VR_0(\lambda) \rho)^{-1}V$$

we see

$$e^{-i\omega x}R_V(\lambda)V e^{i\omega \bullet} = e^{-i\omega x}R_0(\lambda)e^{i\omega \bullet} \left( e^{-i\omega \bullet}(I + VR_0(\lambda) \rho)^{-1}e^{i\omega \bullet} \right) V$$

$$= R_0^\omega(\lambda)(I + VR_0(\lambda) \rho)^{-1}V,$$

where for $\text{Im} \lambda > 0$ and $|\lambda| > C_0$ the convergence is guaranteed by estimates in Step 1. The same estimates then imply (2.4.15). The asymptotic formulas (2.4.16) then follow from the expression for $v_\omega^\omega$ in (2.4.8).  \[\square\]
2.4. SCATTERING MATRIX IN DIMENSION ONE

REMARKS. 1. We should stress that unlike many results in this chapter the statements about the scattering matrix for \( \lambda \) real remain valid for real-valued potentials satisfying very weak decay conditions – see \([Me85]\) for one account of that and for references.

2. The scattering matrix can also be described in the following way:

\[
S(\lambda) = \begin{pmatrix}
i\lambda & \tilde{Y}(\lambda) \\
\bar{X}(\lambda) & X(\lambda) \\
\bar{Y}(-\lambda) & \tilde{X}(\lambda) \\
i\lambda & \bar{X}(\lambda)
\end{pmatrix},
\]

where \( X \) and \( Y \) are naturally defined distributions, compactly supported in the case when \( V \) is compactly supported – see Fig. 2.4. We do not use this representation here but it can be very helpful in the study of resonances (which are then the zeros of \( \tilde{X} \)) and also of inverse problems – see Melin \([Me85]\) and \([TZ01],[Zw87],[Zw01]\).

The determinant of the scattering matrix is related to the determinant defined by (2.2.28):

**THEOREM 2.11 (A determinant identity).** For \( V, \rho \in L^\infty_{\text{comp}} \) satifying \( \rho V = V \), let

\[
D(\lambda) := \det(I + VR_0(\lambda)\rho).
\]

Then

\[
\frac{D(-\lambda)}{D(\lambda)} = \det S(\lambda),
\]

where \( S(\lambda) \) is the scattering matrix.

**Proof.** 1. In the notation of (2.4.9) we write

\[
\rho(R_0(\lambda) - R_0(-\lambda))\rho = \frac{i}{2\lambda} E_\rho(\bar{\lambda})^* E_\rho(\lambda),
\]

that is

\[
E_\rho(\bar{\lambda})^* E_\rho(\lambda) = \rho(x)e^{i\lambda x} \otimes \rho(y)e^{-i\lambda y} + \rho(x)e^{-i\lambda x} \otimes \rho(y)e^{i\lambda y}.
\]

2. We now write

\[
(I + VR_0(-\lambda)\rho) = (I + VR_0(\lambda)\rho)(I - (I + VR_0(\lambda)\rho)^{-1}V(R_0(\lambda) - R_0(-\lambda))\rho) = (I + VR_0(\lambda)\rho)(I - (I + VR_0(\lambda)\rho)^{-1}(iVE_\rho(\tilde{\lambda})^* E_\rho(\lambda)/2\lambda)) = (I + VR_0(\lambda)\rho)(I + T(\lambda)),
\]
Figure 2.5. The distributions $X$ and $Y$ appearing in (2.4.18) are defined as follows: suppose that $\text{supp } V \subset [a, b]$ and solve $(\partial_x^2 - \partial_y^2 + V(x))A_-(x, y) = 0$ with $A_-(x, y) = \delta(x - y)$, $x < a$. Then for $x > b$, $A_-(x, y)$ solves the free wave equation and hence $\partial_y A_-(x, y) = X(x - y) + Y(x + y)$ where $X, Y \in \mathcal{S}'(\mathbb{R})$ and $\text{supp } X \subset [-2(b-a), 0]$, $\text{supp } Y \subset [2a, 2b]$. The original proof of Theorem 2.14 in [Zw87] proceeded by showing that $\text{chsupp } X = [-2(b-a), 0]$ and then applying a theorem of Titchmarsh [HöII, Theorem 16.1.9] on the counting of zeros of Fourier transforms.

where we defined

\[(2.4.22) \quad T(\lambda) := \frac{1}{2i\lambda} (I + VR_0(\lambda)\rho)^{-1}VE_\rho(\lambda)^*E_\rho(\lambda).\]

We note that $T(\lambda) : L^2 \rightarrow L^2$ is a finite rank operator.

3. Hence to prove (2.4.19) we need to show that

\[(2.4.23) \quad \det_{L^2} S(\lambda) = \det_{L^2} (I + T(\lambda)),\]
Putting
\[ A := (2i\lambda)^{-1}(I + VR_0(\lambda)\rho)^{-1}VE(\lambda)^*, \quad B = E(\lambda) \]
we have \( T(\lambda) = AB \). On the hand (2.4.10) shows that \( S(\lambda) = I + BA \). Hence (2.4.23) follows from (B.5.13): \( \det(I + AB) = \det(I + BA) \). 

The multiplicity of a pole of \( S(\lambda) \) and \( S(\lambda)^{-1} \) is defined using the determinant of the scattering matrix. The poles of the scattering matrix are sometimes called scattering poles. Theorem 2.11 combined with Theorem 2.6 gives

**THEOREM 2.12** (Multiplicities of scattering poles in one dimension). The multiplicity of a scattering pole defined by

\[ (2.4.24) \quad m_S(\lambda) = \frac{1}{2\pi i} \text{tr} \oint_{\lambda} S(\zeta)^{-1} \partial_\zeta S(\zeta) d\zeta, \]

where the integral is over a positively oriented circle which includes \( \lambda \) and no other pole or zero of \( \det S(\lambda) \), is related to the multiplicity of a scattering resonance (2.2.11) as follows:

\[ (2.4.25) \quad m_S(\lambda) = m_R(\lambda) - m_R(-\lambda). \]

The scattering matrix is always holomorphic and unitary at zero, and thus does not ‘see’ the resonance at zero directly. However, we have the following

**THEOREM 2.13** (Scattering matrix at zero). For \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \) we have

1) If \( 0 \) is not a pole of \( R_V \), then \( S(0) = -J \), where \( J \) is defined in (2.4.13).

2) If \( 0 \) is a pole of \( R_V \) and \( c_\pm \in \mathbb{R}\setminus\{0\}, c_+^2 + c_-^2 = 1 \), are defined in Theorem 2.5, then

\[ S(0) = \begin{pmatrix} 2c_-c_+ & c_+^2 - c_-^2 \\ c_-^2 - c_+^2 & 2c_-c_+ \end{pmatrix}. \]

Proof. 1. Suppose first that \( 0 \) is not a resonance. Then \( v^\pm(x, 0) \), as defined by (2.4.4), is outgoing by Theorem 2.3. However, the function \( 1 \) is also outgoing at \( \lambda = 0 \), which means that \( u^\pm(x, 0) \) are outgoing solutions to the equation \( P_V u = 0 \). By another application of Theorem 2.3, we see that \( u^\pm = 0 \). Then \( v^\pm = -1 \), and it remains to use (2.4.5).

2. Suppose now that \( 0 \) is a resonance. By Theorem 2.5,

\[ R_V(\lambda) = A(\lambda) + \frac{i}{\lambda}u_1 \otimes u_1, \]
where \( A(\lambda) \) is holomorphic near 0, \( P_V u_1 = 0 \), and \( u_1(x) = c_\pm \) for \( \pm x \gg 1 \). We have
\[
\int u_1(x)V(x) \, dx = \int u_1'(x) \, dx = 0
\]
and thus, by (2.4.4), \( v^\pm(x, \lambda) \) is holomorphic at \( \lambda = 0 \). Therefore, \( v_+(\lambda), v_-(\lambda) \) are holomorphic at \( \lambda = 0 \), and for \( \lambda \) near 0, we have
\[
(2.4.26) \quad u^\pm(x, \lambda) = \begin{cases} 
  e^{\pm i\lambda x} + v_+(\lambda) e^{i\lambda x}, & x \gg 1, \\
  e^{\pm i\lambda x} + v_-(\lambda) e^{-i\lambda x}, & -x \gg 1.
\end{cases}
\]
Since \( P_V u^\pm(x, 0) = 0 \), we see that \( u^\pm(x, 0) \) are multiples of \( u_1(x) \). Therefore,
\[
(2.4.27) \quad c_-(1 + v_+(0)) = c_+(1 + v_-(0)).
\]
Next, differentiating (2.4.26) in \( \lambda \), we find
\[
\partial_\lambda u^\pm(x, 0) = \begin{cases} 
  (\pm 1 + v_+(0)) ix + \partial_\lambda v_+(0), & x \gg 1, \\
  (\pm 1 - v_-(0)) ix + \partial_\lambda v_-(0), & -x \gg 1.
\end{cases}
\]
However, since \( (P_V - \lambda^2) u^\pm(x, \lambda) = 0 \), we have \( P_V \partial_\lambda u^\pm(x, 0) = 0 \). Therefore, the Wronskians \( W(u_1, \partial_\lambda u^\pm) \) are constant. We compute
\[
W(\partial_\lambda u^\pm, u_1) = \begin{cases} 
  ic_+(\pm 1 + v_+(0)), & x \gg 1; \\
  ic_-(\pm 1 - v_-(0)), & -x \gg 1.
\end{cases}
\]
Therefore,
\[
c_+(1 + v_+(0)) = c_-(1 - v_+(0)), \\
c_+(1 - v_+(0)) = c_-(1 - v_-(0)).
\]
Combining these equations with (2.4.27), we obtain the formula for \( S(0) \).

Theorem 2.13 implies the following characterization of the zero resonance in terms of the scattering matrix:
\[
(2.4.28) \quad \det S(0) = (-1)^{m_R(0)+1}.
\]

### 2.5. ASYMPTOTICS FOR THE COUNTING FUNCTION

In this section we will prove a Weyl law for the number of scattering resonances of a compactly supported, bounded, complex valued potential. In higher dimensions only weaker results are known and for the existence of resonances we need to assume that the potential is real valued: as we will see in Chapter 3 a complex valued compactly supported potential in three dimensions may have no resonances at all.
2.5. ASYMPTOTICS FOR THE COUNTING FUNCTION

THEOREM 2.14 (Asymptotics for the number of resonances). Suppose that \( V \in L_\text{comp}^\infty(\mathbb{R}; \mathbb{C}) \). Then

\[
\sum \{ m_R(\lambda) : |\lambda| \leq r \} = \frac{2|\text{chsupp}V|}{\pi} r(1 + o(1)),
\]

as \( r \to \infty \). Here \( \text{chsupp} \) is the convex hull of the support.

In addition, for any \( \epsilon > 0 \),

\[
\sum \{ m_R(\lambda) : |\lambda| \leq r, |\text{Im} \lambda| \geq \epsilon |\text{Re} \lambda| \} = o(r),
\]

as \( r \to \infty \).

MOTIVATION. To obtain the asymptotic formula we use the theory of entire functions of finite type – see §D.2. One standard application of that theory is a proof of Titchmarsh’s theorem stating that if \( g \in L^1(\mathbb{R}) \) then the number of zeros of \( \hat{g} \) in \( D(0, r) \) is equal to \( \pi^{-1}|\text{chsupp} g| r(1 + o(r)) \), as \( r \to \infty \). This follows Theorem D.2, (D.2.9), and from the Paley-Wiener Theorem which shows that the following bound is optimal

\[
|\hat{g}(\lambda)| \leq C e^{a(\text{Im} \lambda) - b(\text{Im} \lambda)^+}, \quad \text{chsupp} g = [a, b].
\]

We will apply these methods to the determinant \( D(\lambda) = \det(I + VR_0(\lambda)\rho) \).

Using the formula \( (2.4.19) \) the growth of \( D(\lambda) \) will be related to the growth of the reflection coefficients \( v_\pm^+(\lambda) \) for \( \text{Im} \lambda > 0 \). Formula \( (2.4.8) \) shows that the reflection coefficients can be considered as nonlinear Fourier transforms of \( V \): the linearizations of \( v_\pm^+(\lambda) \) at \( V = 0 \) are given by \( \hat{V}(\mp 2\lambda)/2i\lambda \). Hence the optimal growth of \( v_\pm^+ \), and consequently of \( D(\lambda) \) can be related to the support of \( V \).

Before proving the theorem we need some estimates for the determinant \( D(\lambda) = \det(I + VR_0(\lambda)\rho) \). These estimates will also be useful in the section on trace formulas.

THEOREM 2.15 (Determinant estimates). There exist constants \( C_j, j = 0, 1, 2, 3 \) such that the determinant \( D(\lambda) \) defined by \( (2.2.28) \) satisfies

\[
\lim_{t \to +\infty} D(e^{i\theta} t) = 1, \quad 0 < \theta < \pi,
\]

\[
|D(\lambda)| \leq C_1(1 + 1/|\lambda|), \quad \text{Im} \lambda \geq 0,
\]

\[
|D(\lambda)| \geq C_2, \quad \text{Im} \lambda > 0, \quad |\lambda| \geq C_0,
\]

\[
|\lambda D(\lambda)| \leq C_3(1 + |\lambda|) \exp(2|\text{chsupp} V/(\text{Im} \lambda)^-|), \quad \lambda \in \mathbb{C},
\]

where \( \text{chsupp} V \) is the convex hull of the support of \( V \).
In addition, if $-\mu_{K}^{2} < -\mu_{K-1}^{2} < \cdots -\mu_{1}^{2} < 0$, $\mu_{j} > 0$, are the eigenvalues of $P_{V}$ then the scattering matrix satisfies

$$
\left| \prod_{k=1}^{K} \frac{\lambda - i\mu_{k}}{\lambda + i\mu_{k}} \det S(\lambda) \right| \leq e^{2|\text{chsupp } V| \text{Im } \lambda}, \quad \text{Im } \lambda \geq 0.
$$

**Remark.** The estimate (2.5.4) will not be needed in this section and is a by-product of the proof of the estimates on $D(\lambda)$. It will be useful in §2.6.

We start with the following lemma concerning trace class norms of the free cut-off resolvent:

**Lemma 2.16.** Suppose that $\rho \in L^{\infty}(\mathbb{R})$ and $\text{supp } \rho \subset [-L, L]$. Then

$$
\|\rho R_{0}(\lambda) \rho\|_{L^{1}} \leq \frac{C \exp(2L(\text{Im } \lambda)_-) - 1}{|\text{Im } \lambda|}, \quad \text{Im } \lambda \neq 0,
$$

$$
\|\rho R_{0}(\lambda) \rho\|_{L^{1}} \leq C + \frac{C}{|\lambda|}, \quad \lambda \in \mathbb{R}.
$$

**Proof.** 1. We start with the case of $\text{Im } \lambda > 0$. In that case, as operators on $L^{2}$,

$$
R_{0}(\lambda) = (D_{x}^{2} - \lambda^{2})^{-1} = (D_{x} - \lambda)^{-1}(D_{x} + \lambda)^{-1}.
$$

The explicit formulae for the Schwartz kernels are given by

$$
(D_{x} \pm \lambda)^{-1}(x, y) = \pm ie^{\pm i\lambda(x-y)}H(\pm(x-y)), \quad \text{Im } \lambda > 0,
$$

where $H(t) = 0$ for $t < 0$ and $H(t) = 1$ for $t \geq 0$.

From this we see that

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} |\rho(x)(D_{x} \pm \lambda)^{-1}(x, y)|^{2} dx dy \leq \frac{2L\|\rho\|_{L^{\infty}}^{2}}{|\text{Im } \lambda|}.
$$

Hence

$$
\|\rho R_{0}(\lambda) \rho\|_{L^{1}}^{2} \leq \|\rho(D_{x} + \lambda)^{-1}\|_{L^{2}} \|\rho(D_{x} - \lambda)^{-1}\|_{L^{2}} \leq \frac{C}{|\text{Im } \lambda|^{2}}.
$$

2. To prove the estimate for $\text{Im } \lambda \leq 0$ we use (2.4.20), (2.4.21) and the fact that

$$
\|u \otimes v\|_{L^{1}} = \|u\|_{L^{2}} \|v\|_{L^{2}}.
$$

This gives,

$$
\|\rho R_{0}(\lambda) \rho\|_{L^{1}} \leq \|\rho R_{0}(-\lambda) \rho\|_{L^{1}} + \frac{1}{|\lambda|} \|\rho e^{i\lambda \cdot \bullet}\|_{L^{2}} \|\rho e^{-i\lambda \cdot \bullet}\|_{L^{2}} \leq \frac{C}{|\text{Im } \lambda|} + \frac{C_{e^{-2L|\text{Im } \lambda|}}}{|\lambda|}.
$$

$$
\leq \frac{C_{e^{2L(\text{Im } \lambda)_-}}}{|\text{Im } \lambda|}.
$$
3. To establish the second inequality in \((2.5.5)\) we first use Theorem 2.1 and Proposition B.20 to see that
\[
\|\rho R_0(\lambda)\rho\|_{L^1} \leq \|\rho R_0(\lambda)\rho\|_{L^2 \to L^2} \leq C(|\lambda| + 1/|\lambda|).
\]
To see the improvement we recall \((B.4.3)\):
\[
\|\rho R_0(\lambda)\rho\|_{L^1} = \max_{\{e_k\},\{f_\ell\}} \sum_{\ell,k} \langle \rho R_0(\lambda)\rho e_k, f_\ell \rangle,
\]
where the maximum is taken over all pairs of orthonormal bases of \(L^2(\mathbb{R})\). Hence the second inequality in \((2.5.5)\) follows from
\[
|h(\lambda)| \leq C_0 + C_0/|\lambda|, \quad h(\lambda) := \sum_{k,\ell} \langle \rho R_0(\lambda)\rho e_k, f_\ell \rangle,
\]
as long as \(C_0\) is independent of the choice of the bases.

4. Consider
\[
h_1(\lambda) := \frac{\lambda h(\lambda)}{\lambda + 2i}.
\]
Then \(h_1(\lambda)\) is holomorphic in the strip \(|\text{Im} \lambda| < 2\). The estimate \((2.5.7)\) shows that
\[
|h_1(\lambda)| \leq C_1 + C_1|\lambda|, \quad |\text{Im} \lambda| \leq 1,
\]
and the first estimate in \((2.5.5)\) (established in Steps 1 and 2 of the proof) shows that
\[
|h_1(\lambda)| \leq C_2, \quad |\text{Im} \lambda| = 1.
\]
The inequality \((2.5.8)\) then follows from the three line theorem (see §D.1).

5. Finally we show the lower bounds on \(D(\lambda)\) for \(\text{Im} \lambda \geq 0\) and \(|\lambda|\) large. Since \(VR_0(\lambda)\rho = \mathcal{O}(1/|\lambda|)_{L^2 \to L^2}\) (Theorem 2.1), \((I + VR_0(\lambda)\rho)^{-1}\) exists and is uniformly bounded on \(L^2\) in that range of \(\lambda\)'s. Hence
\[
D(\lambda)^{-1} = \det((I + VR_0(\lambda)\rho)^{-1})
\]
\[
= \det(I - VR_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1})
\]
\[
\leq \exp \left( \|V\|_\infty \|\rho R_0(\lambda)\rho\|_{L^1} \|I + VR_0(\lambda)\rho\|^{-1}_{L^2} \right)
\]
\[
= \mathcal{O}(1),
\]
which gives the lower bound. \(\square\)

**Proof of Theorem 2.15.** 1. To study \(D(e^{i\theta} t)\) we use \((B.5.15)\) with \(A = 0\) and \(B = VR_0(e^{i\theta} t)\rho\):
\[
|D(e^{i\theta} t) - 1| \leq \|V\|_\infty \|\rho R_0(e^{i\theta} t)\rho\|_{L^1} e^{2\|V\|_\infty \|\rho R_0(e^{i\theta} t)\rho\|_{L^1}}
\]
The first estimate in \((2.5.5)\) shows that the right hand side goes to 0 as \(t \to +\infty\) for \(0 < \theta < \pi\).
2. We now write

\[ V R_0(\lambda)\rho = A(\lambda) + B(\lambda), \quad A(\lambda) := \frac{i}{2\lambda} V \otimes \rho \]

where, \( B(\lambda) \) is holomorphic and, by applying (2.5.6) and the estimates (2.5.5),

\[ \|B(\lambda)\|_{L^1} \leq C, \quad \text{Im} \lambda \geq 0. \]

From (B.5.14) we see

\[ |D(\lambda)| = |\det(I + A(\lambda) + B(\lambda))| \leq \det(I + |A(\lambda)|) \det(I + |B(\lambda)|) \]

\[ \leq e^{\|B(\lambda)\|_{L^1}} \det(I + |A(\lambda)|) \leq C \det(I + |A(\lambda)|). \]

Here (using the notation (2.2.19))

\[ |A(\lambda)| := (A(\lambda)^*A(\lambda))^\frac{1}{2} = \frac{1}{2|\lambda|} \frac{\|V\|^2_2 \|\rho\|^2_2}{\bar{\rho} \otimes \rho} \]

and hence (see Exercise B.2)

\[ \det(I + |A(\lambda)|) = 1 + \frac{\|V\|^2_2 \|\rho\|^2_2}{2|\lambda|}. \]

Returning to (2.5.9) we obtain

(2.5.10) \[ D(\lambda) = \mathcal{O}(1) + \mathcal{O}(1/|\lambda|), \quad \text{Im} \lambda \geq 0. \]

3. For estimates in \( \text{Im} \lambda \leq 0 \) we use Theorem 2.11:

(2.5.11) \[ D(\lambda) = \det S(-\lambda)D(-\lambda). \]

Hence we need to estimate \( \det S(-\lambda) \) for \( \text{Im} \lambda \leq 0. \)

Using (2.4.7) and Theorem 2.10 we see that for \( \text{Im} \lambda \leq 0, \ |\lambda| \geq C_0, \)

(2.5.12) \[ \det S(-\lambda) = 1 - \nu_+(-\lambda)e_+(-\lambda) + \mathcal{O}(1/|\lambda|). \]

Now, (2.4.8) and (2.4.15) show that for \( \text{Im} \lambda \leq 0, \ |\lambda| \geq C_0, \)

\[ |\nu_+(-\lambda)e_+(-\lambda)| \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\text{Im} \lambda(x-y)} |V(x)||V(y)| dxdy \]

\[ \leq C'e^{-2\text{Im} \lambda \text{chsupp} V}, \]

which shows that

(2.5.13) \[ |\det S(-\lambda)| \leq Ce^{2\text{chsupp} V \|\text{Im} \lambda\|}, \quad \text{Im} \lambda \leq 0, \ |\lambda| \geq C_0. \]

From (2.5.10) and (2.5.11) we conclude that

\[ |\lambda D(\lambda)| \leq C(1 + |\lambda|)e^{2\text{chsupp} V \|\text{Im} \lambda\|}, \quad |\lambda| \geq C_0, \]

Since \( \lambda D(\lambda) \) is holomorphic it follows the estimate is valid everywhere. That completes the proof of (2.5.3).
4. To establish (2.5.4) we see that (2.5.13) and unitarity of $S(\lambda)$ for $\lambda \in \mathbb{R}$ shows that

$$g(\lambda) := e^{2i\lambda|\text{chsupp } V|} \prod_{k=1}^{K} \frac{\lambda - i\mu_k}{\lambda + i\mu_k} \det S(\lambda),$$

satisfies

$$|g(\lambda)| = \begin{cases} 1 & \lambda \in \mathbb{R}, \\ O(1) & \text{Im } \lambda \geq 0, \ |\lambda| \geq C_0. \end{cases}$$

Using Theorem 2.12 we see that product over $\mu_k$'s removed the possible singularities of $\det S(\lambda)$. This means that $g(\lambda)$ is holomorphic for $\text{Im } \lambda \geq 0$. But then the Phragmén–Lindelöf theorem (see for instance [Ti39, §5.61] for the particular case needed here) shows that $|g(\lambda)| \leq 1$ for $\text{Im } \lambda \geq 0$ which is (2.5.4).

Proof of Theorem 2.14. 1. Using Theorem 2.6 we will prove the theorem by obtaining an asymptotic formula for the number of zeros of the entire function

$$f(\lambda) := \lambda D(\lambda),$$

where $D(\lambda)$ is defined in (2.2.28). The factor $\lambda$ removes the pole at $\lambda = 0$ – see the second estimate in (2.5.3).

2. By rescaling and translation we can assume that

(2.5.14) $\text{chsupp } V = [-1, 1].$

In view of Theorem D.2 and (D.2.9) it suffices to show that

(2.5.15) $\int_{\mathbb{R}} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty; \quad |f(\lambda)| \leq (1 + |\lambda|)^N e^{4(|\text{Im } \lambda|)}$

and

(2.5.16) $\limsup_{|\lambda| \to \infty} \frac{\log |f(\lambda)|}{|\lambda|} = 4.$

The conditions in (2.5.15) follow immediately from (2.5.3) and we are left to establish (2.5.16), that is to calculate the type of $f$.

3. From (2.5.3), (2.5.11) and (2.5.12) we see that for $\text{Im } \lambda \leq 0, \ |\lambda| \geq C_0$,

$$|\nu_+(-\lambda)v_+(-\lambda)|/C - C \leq |D(\lambda)| \leq C|\nu_+(-\lambda)v_+(-\lambda)| + C.$$

Also, using (2.4.8), (2.4.15) and (2.5.14)

$$|\nu_+(-\lambda)| = O(1) \int_{-1}^{1} e^{2|\text{Im } \lambda x|} dx = O(e^{2|\text{Im } \lambda|}).$$

Hence, the type of $\lambda D(\lambda)$ will be exactly 4 if we show that we cannot have

(2.5.17) $|\nu^{-\omega}(-\lambda)| \leq Ce^{2(1-\delta)|\lambda|}, \quad \text{Im } \lambda \leq 0, \ |\lambda| \geq C$,

with $\delta > 0$ for $\omega = +$ or for $\omega = -$. 
4. Let us choose $\beta > 0$ such that $R_V(\lambda)$ is holomorphic for $\text{Im} \lambda \geq \beta$; that is possible as there are only finitely many poles on $R_V(\lambda)$ in $\text{Im} \lambda \geq 0$. Motivated by Theorem 2.10 we define

$$f_\omega(x, \lambda) := e^{-i\omega \lambda x} R_V(\lambda) (V e^{i\omega \lambda \bullet})(x),$$

which is holomorphic for $\text{Im} \lambda \geq \beta$. Theorem 2.10 (see also the remark following the statement) shows that

$$|f_-(x, \lambda)| \leq C/|\lambda|, \quad \text{Im} \lambda > \beta.$$  

Since $v_+(-\lambda + i\beta)$ is holomorphic for $\text{Im} \lambda \leq 0$, (2.5.17) with $\omega = +$ implies that

$$|v_+(-\lambda + i\beta)| \leq C e^{2(1-\delta)|\lambda|}, \quad \text{Im} \lambda \leq 0, \quad \delta > 0,$$

and we need to find a contradiction to this statement.

We start with (2.4.8): for $\text{Im} \lambda \leq 0$,

$$v_+(-\lambda + i\beta) = -\frac{1}{2(\beta + i\lambda)} \int_{\mathbb{R}} e^{2i\lambda x} V(x) e^{2\beta x} (1 - f_-(x, -\lambda + i\beta)) dx.$$  

We then introduce

$$V_\epsilon^\beta(x) := \mathbf{1}_{[-1,1]}(x) V(x) e^{2\beta x},$$

$$g_\epsilon^\beta(x, \lambda) = \mathbf{1}_{[-1,1]}(x) f_-(x, -\lambda + i\beta).$$

5. Take $\epsilon < \delta$, and define

$$I_\epsilon(2\lambda) := \int_{\mathbb{R}} e^{2i\lambda x} V_\epsilon^\beta(x) (1 - g_\epsilon^\beta(x, \lambda)) dx.$$  

which holomorphic in $\text{Im} \lambda \leq 0$. In the same range of $\lambda$’s

$$\int_{\mathbb{R}} e^{2i\lambda x} \mathbf{1}_{[-1,1]}(x) V(x) e^{2\beta x} (1 - f_-(x, -\lambda - i\beta)) dx =$$

$$O(1) \int_{-1}^{1-\epsilon} e^{-2\text{Im} \lambda x} (1 + O(|\lambda|^{-1})) dx = O(e^{2(1-\epsilon)|\lambda|}).$$

Hence, the fact that $\epsilon < \delta$ and the assumption (2.5.19) show that

$$|I_\epsilon(2\lambda)| \leq C e^{2(1-\epsilon)|\lambda|}.$$

The Paley-Wiener theorem [H61, Theorem 7.3.1] then shows that

$$\hat{I}_\epsilon(x) = 0, \quad x > 1 - \epsilon,$$

that is

$$V_\epsilon^\beta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2i\lambda(y-x)} V_\epsilon^\beta(y) g_\epsilon^\beta(y, \lambda) d\lambda dy.$$
for $1 - \epsilon \leq x \leq 1$. Plancherel’s theorem and the Cauchy Schwartz inequality then imply that

\[
\|V^\beta_\epsilon\|_{L^2} = \left\| \int_{\mathbb{R}} e^{2i\lambda y} V^\beta_\epsilon(y) h^\beta_\epsilon(y, \lambda) dy \right\|_{L^2(d\lambda)} \\
\leq \left\| V^\beta_\epsilon \right\|_{L^2} \left\| h^\beta_\epsilon(\bullet, \lambda) \right\|_{L^2(dy)}_{L^2(d\lambda)} \\
= \left\| V^\beta_\epsilon \right\|_{L^2} \left\| h^\beta_\epsilon \right\|_{L^2(dyd\lambda)}
\]

(2.5.21)

We note that $g^\beta_\epsilon \in L^2(d\lambda)$ because of the $O(1/|\lambda|)$ decay of $f_-$ (2.5.18).

Because of the factor $1_{[1-\epsilon,1]}$ in the definition of $g^\beta_\epsilon$ in (2.5.20), we have

\[
(2\pi)^{-1} \| g^\beta_\epsilon \|_{L^2(dyd\lambda)} \to 0, \ \epsilon \to 0^+.
\]

It follows from (2.5.21) that for $\epsilon$ small enough $\|V^\beta_\epsilon\|_{L^2} = 0$. Recalling (2.5.20) this means that

$V(x) = 0$ for $1 - \epsilon < x < 1$,

contradicting the assumption that $\text{chsupp} \ V = [-1,1]$.

6. The same argument applies under the assumption that (2.5.17) holds for $\omega = -$ and it shows that

$V(x) = 0$ for $-1 < x < -1 + \epsilon$,

leading again to contradiction. Hence (2.5.16) holds and Theorem D.2 gives the asymptotics of resonances.

2.6. TRACE FORMULAS

We will now prove three closely related trace formulas. The first one, the Birman–Kreĭn formula, relates the scattering matrix to the trace of $f(P_V) - f(P)$. The second is a version of the Breit–Wigner approximation (1.1.1) for the effect of resonances on the spectrum. The last formula is a Poisson formula which relates the trace of the wave group to a sum of resonances $\sum e^{-i\lambda_j|t|}$ and is a special case of Melrose’s trace formula described in §3.10. The Birman–Kreĭn formula is also valid in higher dimensions and for more general perturbations. However, Breit–Wigner formulas in higher dimensions are harder to formulate rigorously – see §3.13 for comments on that.

From the technical point of view the trace formulas are consequences of the determinant identity presented in Theorem 2.11.
THEOREM 2.17 (Birman-Kreĭn formula in one dimension). Suppose that \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \). Then for \( f \in \mathcal{S}(\mathbb{R}) \) the operator \( f(P_V) - f(P) \) is of trace class and

\[
\text{tr} (f(P_V) - f(P_0)) = \frac{1}{2\pi i} \int_0^\infty f(\lambda^2) \text{tr} \left( (S(\lambda)^{-1} S(\lambda)) \right) d\lambda + \sum_{k=1}^K f(E_k) + \frac{1}{2} (m_R(0) - 1) f(0),
\]

where \( S(\lambda) \) is the scattering matrix and \( E_K < \cdots < E_1 < 0 \) are the (negative) eigenvalues of \( P_V \).

INTERPRETATION. As in the beginning of Section 2.3 we can compare this result to a result involving eigenvalues. Let us denote the Dirichlet realization of \( P_V \) on \([a,b]\) by \( P^D_V \). The spectrum of \( P^D_V \) is discrete,

\[
E_N < E_{N-1} < \cdots < E_1 < 0 < \lambda_0^2 < \lambda_1^2 < \cdots \to \infty.
\]

For \( f \in \mathcal{S}(\mathbb{R}) \), we have

\[
\text{tr} f(P^D_V) = \sum_{j=0}^\infty f(\lambda_j^2) + \sum_{k=1}^N f(E_k)
\]

which can be written as

\[
\text{tr} f(P^D_V) = \int_0^1 f(\lambda^2) \frac{dN(\lambda)}{d\lambda} d\lambda + \sum_{k=1}^N f(E_k)
\]

where

\[
N(\lambda) = \# \{ \lambda_j^2 : \lambda_j^2 \leq \lambda^2 \}
\]

is the counting function for the positive eigenvalues of \( P^D_V \).

Hence we have the following correspondence between confined (discrete spectrum) and open (continuous spectrum/scattering) problems:

\[
N(\lambda) \leftrightarrow \sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda).
\]

Since \( S(\lambda) \) is unitary for \( \lambda \in \mathbb{R} \) the right hand side is real. Since only the derivatives appear in the formula we can choose the branch of log arbitrarily.

**Proof of Theorem 2.17.** For simplicity we assume that there are no negative eigenvalues as their contribution is easy to analyse.
2.6. TRACE FORMULAS

1. Since we assume that \( V \in L^\infty(\mathbb{R}; \mathbb{R}) \), \( P_V \) is self-adjoint and we can apply Stone’s formula as we did in the proof of Theorem 2.7. That gives

\[
f(P_V) = \frac{1}{2\pi i} \int_0^\infty f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda d\lambda \\
= \frac{1}{4\pi i} \int_\mathbb{R} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda d\lambda ,
\]

where we used the fact that the integrand is even in \( \lambda \). The integral on the left should be understood as an operator \( L^2\text{comp} \rightarrow L^2_{\text{loc}} \)

2. We write

\[
R_V(\lambda) - R_0(\lambda) = -R_V(\lambda)V R_0(\lambda)
\]

(2.6.4)

\[
= -R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1}V R_0(\lambda) .
\]

We note that this operator has a simple pole at \( \lambda = 0 \). The residue is given by \( (u \otimes u + 1 \otimes 1/2)/i \), where \( u \) is given in (2.2.27) and \( u = 0 \) if \( m_R(0) = 0 \).

We define

\[
B(\lambda) := -2\lambda R_V(\lambda) V R_0(\lambda)
\]

(2.6.5)

\[
= -2\lambda R_0(\lambda)(I + V R_0(\lambda)\rho)^{-1}V R_0(\lambda) : L^2\text{comp} \rightarrow L^2_{\text{loc}},
\]

which is a meromorphic family of operators, holomorphic in \( \text{Im} \lambda \geq 0 \). (In view of (2.6.4), the simple pole at \( \lambda = 0 \) is cancelled by the \( \lambda \) factor and we assumed there were no negative eigenvalues.) With this notation we have

\[
f(P_V) - f(P_0) = \frac{1}{4\pi i} \sum_\pm \int_\mathbb{R} f(\lambda^2)B(\pm \lambda)d\lambda .
\]

(2.6.6)

The spectral theorem shows (again we assumed that there are no eigenvalues; easy modifications are needed otherwise) that

\[
\|R_V(\lambda)\|_{L^2 \rightarrow L^2} = \frac{1}{d(\lambda^2, \mathbb{R}_+)} \leq \frac{1}{|\lambda| \text{Im} \lambda}, \quad \text{Im} \lambda > 0 .
\]

(2.6.7)

Applying this with \( V = 0 \) gives, for \( \text{Im} \lambda > 0 \),

\[
\|\lambda R_0(\lambda)\|_{L^2 \rightarrow H^2} \leq C |\lambda| \left( \|D_z^2 R_0(\lambda)\|_{L^2 \rightarrow L^2} + \|R_0(\lambda)\|_{L^2 \rightarrow L^2} \right)
\]

\[
\leq C' |\lambda| (1 + |\lambda|^2) \|R_0(\lambda)\|_{L^2 \rightarrow L^2}
\]

\[
\leq (1 + |\lambda|^2) / \text{Im} \lambda.
\]

This estimate and the fact that \( V \in L^\infty_{\text{comp}} \) show that \( V R_0(\lambda) \) is of trace class for \( \text{Im} \lambda > 0 \) (see Theorem B.3) and that

\[
\|\lambda VR_0(\lambda)\|_{L^1} \leq C \frac{1 + |\lambda|^2}{\text{Im} \lambda}, \quad \text{Im} \lambda > 0 .
\]
Combining this with (2.6.5) and (2.6.7)

\[(2.6.8)\]
\[\|B(\lambda)\|_{L^1} \leq \frac{1 + |\lambda|^2}{|\text{Im } \lambda|^2|\lambda|} \leq \frac{1 + |\lambda|^2}{|\text{Im } \lambda|^2}, \quad \text{Im } \lambda > 0.\]

Let \( g \in \mathcal{S}(\mathbb{C}), \text{ supp } g \subset \{|\text{Im } \lambda| \leq 1\}, \) be an almost analytic extention of \( f(\lambda^2) \) (see B.2):

\[(2.6.9)\]
\[g(\lambda) = f(\lambda^2), \quad \lambda \in \mathbb{R}, \quad \partial_\lambda g(\lambda) = O(|\text{Im } \lambda|)\).

The Cauchy-Green formula (D.1.1) applied to the right hand side of (2.6.6) shows that

\[f(P_V) - f(P_0) = \frac{1}{2\pi i} \left(t_+(f) - t_-(f)\right),\]

\[(2.6.10)\]
\[t_{\pm}(f) := \int_{\text{Im } \lambda > 0} \partial_\lambda g(\lambda)B(\pm \lambda)dm(\lambda).\]

Using (2.6.8) and (2.6.9) we conclude that for any \( N > 0, \) and in particular for \( N \geq 4, \)

\[\|t_{\pm}(f)\|_{L^1} \leq C_N \int_{0 < \pm |\text{Im } \lambda| < 1} |\text{Im } \lambda|^N (1 + |\lambda|)^{-2N+2}|\text{Im } \lambda|^{-3}dm(\lambda) < \infty.\]

This proves the claim that

\[(2.6.11)\]
\[f(P_V) - f(P_0) \in L^1.\]

3. To calculate the trace of \( f(P_V) - f(P_0) \) we use Theorem 2.11. Taking logarithmic derivatives of both sides of (2.4.19) we obtain

\[(2.6.12)\]
\[\text{tr } F(-\lambda) + \text{tr } F(\lambda) = \text{tr } (\partial_\lambda S(\lambda))S(\lambda)^{-1},\]

\[F(\lambda) := -\partial_\lambda (VR_0(\lambda)\rho)(I + VR_0(\lambda)\rho)^{-1} \cdot \]

We note that \( F(\lambda), \lambda \in \mathbb{C}, \) is a meromorphic family of operators in \( L^1(L^2), \)

with no poles in \( \text{Im } \lambda > 0 \) (we assumed that there are no negative eigenvalues). From (2.2.33) (see §C.4) we see that

\[(2.6.13)\]
\[\text{tr } F(\lambda) = -\text{tr}(\lambda^{-1}Q_1 - \lambda^{-1}Q_{-1}) + \varphi(\lambda) = \frac{1 - m_R(0)}{\lambda} + \varphi(\lambda),\]

where \( \varphi(\lambda) \) is holomorphic in \( \text{Im } \lambda \geq 0. \) To estimate \( \varphi(\lambda) \) we recall (see the proof of Theorem 2.8) that

\[\|VR_0(\lambda)\rho\|_{L^2 \rightarrow H^2} \leq C|\lambda|e^{C(\text{Im } \lambda)}, \quad |\lambda| \geq 1.\]

Cauchy estimates (C.3.1) (or an explicit calculation) show that for \( \text{Im } \lambda \geq 0, \)

\[|\lambda| > 1,
\[\|\partial_\lambda VR_0(\lambda)\rho\|_{L^1} \leq C\|\partial_\lambda \rho VR_0(\lambda)\rho\|_{L^2 \rightarrow H^2} \leq C'|\lambda|.\]
From the definition of $F(\lambda)$ and from the invertibility of $I + VR_0(\lambda)\rho$ for $|\lambda| \gg 1$, $\text{Im } \lambda \geq 0$ it now follows that

$$|\text{tr } F(\lambda)| \leq C|\lambda|, \quad |\lambda| \geq C_0, \quad \text{Im } \lambda \geq 0.$$ 

Since $\varphi(\lambda)$ is holomorphic in $\text{Im } \lambda \geq 0$ we obtain.

$$|\varphi(\lambda)| \leq C(1 + |\lambda|), \quad \text{Im } \lambda \geq 0.$$ 

4. We claim that for $\text{Im } \lambda > 0$

(2.6.14) $\text{tr } F(\lambda) = \text{tr } B(\lambda),$ 

where $B(\lambda)$ was defined by (2.6.5).

To see (2.6.14) we use the fact that $R_0(\lambda)$ is bounded on $L^2$ for $\text{Im } \lambda > 0$ and hence

$$\partial_{\lambda}(VR_0(\lambda)\rho) = 2\lambda VR_0(\lambda)^2 \rho.$$ 

Using this, the cyclicity of the trace (Theorem B.4.8 applied twice) and $\rho V = V$, we obtain, always for $\text{Im } \lambda > 0$,

$$\text{tr } F(\lambda) = -2\lambda \text{tr } R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda) = \text{tr } B(\lambda),$$

which is (2.6.14).

5. We now use (2.6.14) and (D.1.1) in (3.9.15):

$$\text{tr } (f(P_V) - f(P_0)) = \frac{1}{2\pi} \sum_{\pm} \int_{\pm \text{Im } \lambda > 0} \partial_{\lambda} g(\lambda) \text{tr } F(\pm \lambda) dm(\lambda)$$

(2.6.15) 

$$= \frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon)} g(\lambda) \text{tr } F(\lambda) d\lambda$$

$$+ \frac{1}{2\pi} \sum_{\pm} \int_{\Omega_{\pm}(\epsilon)} \partial_{\lambda} g(\lambda) \text{tr } F(\lambda) d\mathcal{L}(\lambda),$$

where where

$$\Omega_{\pm}(\epsilon) = D(0, \epsilon) \cap \mathbb{C}_{\pm}, \quad \mathbb{C}_{\pm} := \{ \pm \text{Im } \lambda > 0 \},$$

$$\gamma_+(\epsilon) = \partial(\mathbb{C}_+ \setminus \Omega_+(\epsilon)), \quad \gamma_-(\epsilon) = \partial(\mathbb{C}_+ \cup \Omega_-(\epsilon)),$$

and the boundaries are positively oriented (as boundaries of the indicated sets).

Estimates (2.6.8) (applied using (2.6.14)) and (2.6.9) show that the last term on the right hand side of (3.9.24) is $O(\epsilon^\infty)$. Using (2.6.12) we then see that

$$\frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon)} g(\lambda) \text{tr } F(\lambda) d\lambda = \frac{1}{4\pi i} \int_{\mathbb{R}} f(\lambda^2) \text{tr } \partial_{\lambda} S(\lambda) S(\lambda)^{-1} d\lambda$$

$$+ \frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon) \setminus \mathbb{R}} g(\lambda) \text{tr } F(\lambda) d\lambda + O(\epsilon).$$
(The error term $O(\epsilon)$ comes from passing to the limit from $\gamma_\pm(\epsilon) \cap \mathbb{R}$ to $\mathbb{R}$.)

The structure of $F(\lambda)$ near 0 given in (2.6.13) shows that

$$\frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_\pm(\epsilon) \cap \mathbb{R}} g(\lambda) \tr F(\pm \lambda) d\lambda = (m_R(0) - 1) \frac{f(0)}{4\pi i} \int_{\partial D(0, \epsilon)} \frac{d\lambda}{\lambda} + O(\epsilon)$$

$$= \frac{1}{2} (m_R(0) - 1) f(0) + O(\epsilon).$$

Letting $\epsilon \to 0$ and noting that $\tr S(\lambda)^{-1} \partial S(\lambda)$ is even (see (2.6.12)) we obtain (2.6.1). □

REMARK. An examination of the proof of (2.6.11) shows that $T_V : f \mapsto \tr f(P_V) - f(P_0)$ defines a tempered distribution, $T_V \in \mathcal{S}'(\mathbb{R})$. This will be important in higher dimensions where the properties of $\det S(\lambda)$ are less clear.

The density appearing in (2.6.1) can be expressed in terms of resonances as follows:

THEOREM 2.18 (Breit–Wigner approximation). Suppose that $V \in L_\infty^\comp(\mathbb{R}; \mathbb{R})$. Then

$$(2.6.16) \quad \frac{1}{2\pi i} \tr \partial_\lambda S(\lambda) S(\lambda)^{-1} = -\frac{1}{\pi} |\text{chsupp} V| - \frac{1}{\pi} \sum_j \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2},$$

where the sum is over all non-zero resonances of $P_V$ and it converges for $\lambda \in \mathbb{R}$.

INTERPRETATION. 1. For $\text{Im} \lambda_j < 0$, the density

$$dm_{\lambda_j}(\lambda) = -\frac{1}{\pi} \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2} d\lambda$$

defines a probability measure on $\mathbb{R}$ and is known in this context as the Breit–Wigner Lorentzian. Formally, if $\text{Im} \lambda_j = 0$ then the Lorentzian becomes $\delta(\lambda - \lambda_j)$ which is consistent with the discussion following Theorem 2.17. The Birman–Kreın formula can then be written as

$$\tr (f(P_V) - f(P_0)) = \sum_j \int_0^\infty f(\lambda^2) dm_{\lambda_j}(\lambda) - \frac{|\text{chsupp} V|}{\pi} \int_0^\infty f(\lambda^2) d\lambda$$

$$+ \sum_{k=1}^K f(E_k) + \frac{1}{2} (m_R(0) - 1) f(0).$$

This means that the spectral shift function (see §3.13 for discussion and references) is expressed in terms of resonances. This provides a rigorous formulation of the Breit–Wigner approximation (1.1.1) – see Fig. 2.6.
2.6. TRACE FORMULAS

Figure 2.6. Breit–Wigner approximation in a simple example: The resonances for the potential on top are computed using a code `squarepot.m` by David Bindel [BZ]. The right hand side of (2.6.16) is computed using 40 of those resonances. The ones shown correspond to the peaks in $\text{tr} S(\lambda)^{-1} \partial_\lambda S(\lambda)/2\pi i$. The scattering matrix was computed using the transfer matrix – see Exercise 2.6. The plot was truncated to show the fine feature agreement with (2.6.16).

2. In the proof of we will see that

$$\frac{1}{\pi} \sum_j \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2} = \frac{1}{\pi} \text{chsupp } V + \mathcal{O}(\lambda^{-2}), \quad \lambda \to +\infty.$$  

When $V \in C_c^\infty(\mathbb{R}; \mathbb{R})$ the asymptotics of $\sigma(\lambda) = \text{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1}/2\pi i$ presented in Exercise 2.4 (see Theorem 3.67 for the general version) show that $\mathcal{O}(\lambda^{-2})$ can be replaced by a full asymptotic expansion.
Proof of Theorem 2.18. 1. Theorem 2.12 shows that the zeros of det \( S(\lambda) \) for \( \text{Im}\ \lambda \geq 0 \) are given by \(-\lambda_j\) where \( \lambda_j \) are resonances of \( P_V \). They are then the zeros of

\[
g(\lambda) := e^{2i|\text{chsupp } V|} \prod_{k=1}^{K} \frac{\lambda - i\mu_k}{\lambda + i\mu_k} \det S(\lambda),
\]

where we used the notation of (2.5.4). From that bound we see that \( g(\lambda) \) is holomorphic for \( \text{Im}\ \lambda \geq 0 \) and that \( |g(\lambda)| \leq 1 \) there. Carleman’s estimate (D.1.10) then gives

\[
(2.6.17) \sum_j \left| \frac{\text{Im} \lambda_j}{|\lambda_j|^2} \right| < \infty.
\]

2. Hadamard’s factorization theorem (D.2.7) applied to \( \lambda D(\lambda) \) and the symmetries of \( \det S(\lambda) \) (\( \det S(\lambda)^{-1} = \det S(-\lambda) = \overline{\det S(\lambda)} \) – see (2.4.13) and (2.4.14)) show that

\[
(2.6.18) \det S(\lambda) = e^{ia\lambda} P(\overline{\lambda}), \quad P(\lambda) := \prod_j \left( 1 - \frac{\lambda}{\lambda_j} \right) e^{\frac{\lambda}{\lambda_j}}, \quad a \in \mathbb{R},
\]

where the product is over all non-zero resonances. For \( \lambda \in \mathbb{R} \), (2.6.17) and (2.6.18) give

\[
\frac{1}{2\pi i} \text{tr} \partial_\lambda S(\lambda)S(\lambda)^{-1} = \frac{a}{2\pi} - \frac{1}{2\pi i} \sum_j \left( \frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_j} + \frac{1}{\lambda - \lambda_j} \right)
\]

\[
= \frac{a}{2\pi} - \frac{1}{\pi} \sum_j \left( \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2} - \frac{\text{Im} \lambda_j}{|\lambda_j|^2} \right)
\]

\[
= -B - \frac{1}{\pi} \sum_j \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2}, \quad B := -\frac{a}{2\pi} + \frac{1}{\pi} \sum_j \frac{\text{Im} \lambda_j}{|\lambda_j|^2}.
\]

3. To obtain (2.6.16) it remains to show that \( B = |\text{chsupp } V|/\pi \). We start by estimating \( \text{tr} \partial_\lambda S(\lambda)S(\lambda)^{-1} \) as \( \lambda \to \pm \infty \). For that we use Theorem (2.4.23), \( \det S(\lambda) = \det(I + T(\lambda)) \) where \( T(\lambda) \) is defined by (2.4.22). Then,

\[
| \text{tr} \partial_\lambda S(\lambda)S(\lambda)^{-1} | = | \text{tr} \partial_\lambda T(\lambda)(I + T(\lambda))^{-1} |
\]

\[
\leq \| \partial_\lambda T(\lambda) \|_{L^1} \|(I + T(\lambda))^{-1} \|_{L^2 \to L^2}. \]

Since \( VR_0(\lambda) = O(|\lambda|^{-1})_{L^2 \to L^2} \), \( (I + VR_0(\lambda)\rho)^{-1} = O(1)_{L^2 \to L^2} \), \( \lambda \to \pm \infty \). The definition (2.4.22) gives \( T(\lambda) = O(|\lambda|^{-1})_{L^2 \to L^2} \) so that

\[
(I + T(\lambda))^{-1} = O(1)_{L^2 \to L^2}, \quad \lambda \to \pm \infty.
\]
2.6. TRACE FORMULAS

To estimate the trace class norm of $\partial_{\lambda} T(\lambda)$ we write, in the notation of (2.4.22), and for $\lambda$ real large,

$$\|\partial_{\lambda} T(\lambda)\|_{L^1} = \|\partial_{\lambda} (I + V R_0(\lambda) \rho)^{-1} V \partial_{\lambda} \rho (\lambda)^{-1} E_{\rho}(\lambda)^{*} E_{\rho}(\lambda)\|_{L^1} \leq C \|\partial_{\lambda} (V R_0(\lambda) \rho)\|_{L^2 \rightarrow L^2} \|\partial_{\lambda} \rho (\lambda)^{-1} E_{\rho}(\lambda)^{*} E_{\rho}(\lambda)\|_{L^1} \leq C \lambda^{-2}.$$  

Here we used the bound $\|\partial_{\lambda} V R_0(\lambda) \rho\|_{L^2 \rightarrow L^2} \leq C/|\lambda|$ (obtained from the explicit formula for $R_0(\lambda)$) and the fact that $E_{\rho}(\lambda)^{*} E_{\rho}(\lambda)$ is a finite rank operator (2.4.21) which can be differentiated with respect to $\lambda$ keeping the boundedness in $\lambda$.

4. Returning to the calculation in Step 2 we see that

$$(2.6.19) \quad \frac{1}{\pi} \sum_{j} \frac{|\text{Im} \lambda_j|}{|\lambda - \lambda_j|^2} = B + O(\lambda^{-2}), \quad \lambda \to \pm \infty.$$

For $\delta > 0$ and $\Lambda > |\lambda|$ we split the left hand side into three terms

$$\frac{1}{\pi} \sum_{j} \frac{|\text{Im} \lambda_j|}{|\lambda - \lambda_j|^2} = S_{|\lambda| \leq (1-\delta)\Lambda}(\lambda) + S_{(1-\delta,1+\delta)\Lambda}(\lambda) + S_{(1+\delta)\Lambda,\infty}(\lambda),$$

$$S_I := \frac{1}{\pi} \sum_{|\lambda_j| \in I} \frac{|\text{Im} \lambda_j|}{|\lambda - \lambda_j|^2}, \quad I \subset [0, \infty).$$

We then see that for $|\lambda| \leq \Lambda$ (2.6.17) gives

$$S_{(1+\delta)\Lambda,\infty}(\lambda) \leq \sum_{|\lambda_j| \geq (1+\delta)\Lambda} \frac{|\text{Im} \lambda_j|}{|\lambda_j - \Lambda|^2} \leq \delta^{-2} \sum_{|\lambda_j| \geq (1+\delta)\Lambda} \frac{|\text{Im} \lambda_j|}{|\lambda_j|^2} = o(1),$$

uniformly as $\Lambda \to \infty$. Also, the asymptotics for the counting function of resonances (2.5.1) gives

$$\int_{-\Lambda}^{\Lambda} S_{(1-\delta,1+\delta)\Lambda}(\lambda) d\lambda = O(\delta \Lambda) + o(\Lambda).$$

Hence integrating (2.6.19) from $-\Lambda$ to $\Lambda$ we obtain

$$\frac{1}{\pi} \sum_{|\lambda_j| \leq (1-\delta)\Lambda} \sum_{\pm} \tan^{-1} \left( \frac{\Lambda \pm \text{Re} \lambda_j}{|\text{Im} \lambda_j|} \right) + O(\delta \Lambda) + o(\delta(\Lambda)) = 2B \Lambda + O(1).$$

In view of (2.5.2) we only need to sum over $|\text{Im} \lambda_j| < \delta^2 |\lambda_j|$ at an expense of adding another $o_\delta(\Lambda)$ error term.

Since for $x > 0$, $\tan^{-1}(x)/\pi = 1/2 + O((x)^{-1})$, $|\lambda_j| \leq (1 - \delta)\Lambda$ and for $|\text{Im} \lambda_j| < \delta^2 |\text{Re} \lambda_j|$,

$$\sum_{\pm} \tan^{-1} \left( \frac{\Lambda \pm \text{Re} \lambda_j}{|\text{Im} \lambda_j|} \right) = 1 + O \left( \frac{|\text{Im} \lambda_j|}{\Lambda - |\text{Re} \lambda_j|} \right) = 1 + O(\delta).$$
Combined with (2.5.1) this gives
\[
\frac{2}{\pi} \left| \text{chsupp} \, V \right| \Lambda + \mathcal{O}(\delta \Lambda) + o_\delta(\Lambda) = 2B\Lambda + \mathcal{O}(1).
\]
Since \( \delta \) is arbitrary this shows that \( B = \frac{\left| \text{chsupp} \, V \right|}{\pi} \) as claimed. \( \square \)

**REMARK.** If we use the representation (2.4.18) of the scattering matrix, the transmission coefficient is given by \( t(\lambda) = i\lambda/\tilde{X}(\lambda) \). Exercise (2.10.1) then shows that \( \det S(\lambda) = \tilde{X}(-\lambda)/\tilde{X}(\lambda) \). Our mathematical version of the Breit–Wigner approximation (2.6.16) then follows from results of Titchmarsh \[ Ti26, Theorems IV and VI \] and from \[ Zw87 \] where \( \text{chsupp} \, X = [-2|\text{chsupp} \, V|, 0] \) is established. Here we used heavier complex analysis to establish (2.5.1) from which we obtained (2.6.16) directly.

We now define the following distribution on \( \mathbb{R} \): with the sum over all resonances and \( \varphi \in C^\infty_c((-L, L)) \),

\[
(2.6.20) \quad \text{p.v. } \sum_{\lambda_j \neq 0} e^{-i|t|\lambda_j} (\varphi) := \lim_{R \to \infty} \sum_{|\lambda_j| \leq R} \int_{-R}^R \varphi(t) e^{-i\lambda_j t} dt.
\]

To check that (2.6.20) defines a distribution we integrate by parts twice using \( \partial_t(e^{-i\lambda t}) = (i/\lambda)e^{-i\lambda t} \):

\[
\text{p.v. } \sum_{\lambda_j \neq 0} e^{-i|t|\lambda_j} (\varphi) = \lim_{R \to \infty} \sum_{0 < |\lambda_j| \leq R} \int_0^L (\varphi(t) + \varphi(-t)) e^{-i\lambda_j t} dt
\]

\[
= \lim_{R \to \infty} \sum_{0 < |\lambda_j| \leq R} \left( \frac{2i}{\lambda_j} \varphi(0) + \mathcal{O}\left( \frac{\sup |\varphi^{(2)}|}{|\lambda_j|^2} \right) \right)
\]

Thanks to (2.5.1) \( \sum_{\lambda_j \neq 0} |\lambda_j|^2 < \infty \) while the symmetry of resonances \( \lambda_j \to -\bar{\lambda}_j \) gives

\[
\sum_{0 < |\lambda_j| \leq R} \frac{2i}{\lambda_j} = \sum_{0 < |\lambda_j| \leq R} \frac{2 \text{Im} \lambda_j}{|\lambda_j|^2}.
\]

The estimate (2.6.17) shows that the series on the right converges and hence (2.6.20) is well defined.

As a consequence of Theorems 2.17 and 2.18 we have the following Poisson formula for resonances:

**THEOREM 2.19 (Poisson formula for resonances).** Suppose that \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \). Then for \( \varphi \in C^\infty_c(\mathbb{R}) \) the operator

\[
\int_{\mathbb{R}} \varphi(t) \left( \cos t \sqrt{P} V - \cos t \sqrt{P_0} \right) dt
\]
is of trace class and, using definition (2.6.20),
\[
2 \tr \left( \cos t \sqrt{P_V} - \cos t \sqrt{P_0} \right)
\]
(2.6.21)
\[
= \text{p.v.} \sum_{\lambda \in \mathbb{C}} m_R(\lambda) e^{-i\lambda |t|} - 2|\text{chsupp } V| \delta_0(t) - 1,
\]
in the sense of distributions on \( \mathbb{R} \).

**Interpretation.**

1. The expansion (2.3.1) leads directly to a trace formula for, say, the Dirichlet realization of \( P_V \) on \([a,b]\). We denote that Dirichlet realization by \( P_V^D \). Assuming for simplicity that there are no non-positive eigenvalues, we have
\[
2 \tr \cos t \sqrt{P_V^D} = \sum_{\lambda^2 \in \text{Spec}(P_V^D)} e^{-i\lambda t}.
\]
Hence the expansion (2.6.21) is an exact analogue of this well-known consequence of the spectral theorem. What is remarkable is the fact that unlike the resonance wave expansions given in Theorem 2.7 the trace formula (2.6.21) is exact.

2. The Poisson formula (2.6.21) remains valid in higher dimensions as (3.10.2) but with a less precise statement at \( t = 0 \).

**Proof of Theorem 2.19.**

1. In the distributional sense,
\[
(2 \cos t \sqrt{P_V})(\varphi) = f(P_V), \quad f(z) := \hat{\varphi}(\sqrt{z}) + \hat{\varphi}(-\sqrt{z}),
\]
where \( f \in C^\infty(\mathbb{R}) \cap \mathcal{S}((0, \infty)) \) (and hence Theorem 2.17 is applicable as we can replace \( f \) by \( \chi f \in \mathcal{S}(\mathbb{R}) \) where \( \chi \in C^\infty(\mathbb{R}) \) vanishes for sufficiently large negative values). Compared to the definition (2.6.20) we see that (2.6.21) is equivalent to
\[
\tr (f(P_V) - f(P_0)) = \lim_{R \to \infty} \sum_{|\lambda_j| \leq R} \int_{-R}^{R} \varphi(t)e^{-i\lambda_j |t|} dt - 2|\text{chsupp } V| \varphi(0) - \hat{\varphi}(0).
\]
Writing \( \sigma(\lambda) := \log \det S(\lambda)/2\pi i \), Theorem 3.51 shows that (since \( \sigma'(\lambda) \) is even)
\[
\tr (f(P_V) - f(P_0)) = \frac{1}{2} \int_{-\infty}^{\infty} f(\lambda^2) \sigma'(\lambda) d\lambda + \sum_{k=1}^{K} f(E_k) + \frac{1}{2}(m_R(0) - 1).
\]
In Step 3 of the proof of Theorem 2.18 we showed that \( \sigma'(\lambda) = \mathcal{O}(\langle \lambda \rangle^{-2}) \) and hence the integral converges.
Hence the proof of the theorem is reduced to showing that
\[ \int_{\mathbb{R}} \hat{\varphi}(\lambda)\sigma'(\lambda)d\lambda + m_{R}(0)\hat{\varphi}(0) + \sum_{\text{Im}\lambda_j > 0} (\hat{\varphi}(\lambda_j) + \hat{\varphi}(-\lambda_j)) \]
(2.6.23)
\[= \lim_{R \to \infty} \sum_{|\lambda_j| \leq R} \int_{\mathbb{R}} \varphi(t)e^{-i\lambda_j|t|}dt - 2|\text{chsupp } V|\varphi(0). \]

We first note that for \(\text{Im}\lambda_j < 0, t \mapsto e^{-i|t|\lambda_j} \) is a tempered distribution. The symmetry \(\lambda_j \mapsto -\bar{\lambda}_j\) then shows that
\[ \mathcal{F}_{t \mapsto \lambda} \left( \sum_{|\lambda_j| \leq R, \text{Im}\lambda_j < 0} e^{-i\lambda_j|t|} \right) = \sum_{|\lambda_j| \leq R, \text{Im}\lambda_j < 0} \frac{2|\text{Im}\lambda_j|}{|\lambda - \lambda_j|^2}. \]

Parseval’s formula \([\text{HöI}, (7.1.8)]\) and (2.6.17) then show that
\[ \lim_{R \to \infty} \sum_{|\lambda_j| \leq R, \text{Im}\lambda_j < 0} \int_{\mathbb{R}} \varphi(t)e^{-i\lambda_j|t|}dt = \frac{1}{\pi} \int \hat{\varphi}(\lambda) \sum_{\text{Im}\lambda_j < 0} \frac{|\text{Im}\lambda_j|}{|\lambda - \lambda_j|^2}d\lambda. \]
(2.6.24)

Inserting (2.6.16) (Theorem 2.18) into (3.10.16) and using \(\int_{\mathbb{R}} \hat{\varphi}(\lambda)d\lambda = 2\pi\varphi(0)\) shows that it remains to prove (recall that resonances in \(\text{Im}\lambda > 0\) lie on the imaginary axis and are square roots of finitely many eigenvalues of \(P_V\))
\[ \frac{1}{\pi} \int_{\mathbb{R}} \hat{\varphi}(\lambda) \sum_{\text{Im}\lambda_j > 0} \frac{|\text{Im}\lambda_j|}{|\lambda - \lambda_j|^2}d\lambda + \sum_{\text{Im}\lambda_j > 0} \int_{\mathbb{R}} \varphi(t)e^{-i\lambda_j|t|}dt \]
(2.6.25)
\[= \sum_{\text{Im}\lambda_j > 0} (\hat{\varphi}(\lambda_j) + \hat{\varphi}(-\lambda_j)) = 2 \sum_{\text{Im}\lambda_j > 0} \int_{\mathbb{R}} \varphi(t)\cos(\lambda_j t)dt. \]

But the same argument which led to (2.6.24) shows that the first term on the left hand side is equal to \(\sum_{\text{Im}\lambda_j > 0} \int_{\mathbb{R}} \varphi(t)e^{i\lambda_j|t|}dt\). Since \(e^{i\lambda_j|t|} + e^{-i\lambda_j|t|} = 2\cos(\lambda_j t)\) this proves (2.6.25), concluding the proof of (2.6.21). \(\square\)

### 2.7. COMPLEX SCALING IN ONE DIMENSION

In this section we present the simplest case of the method of complex scaling which produces a natural family of non-self-adjoint operators whose discrete spectrum consists of resonances.

The idea is to consider \(D_x^2\) as a restriction of the complex second derivative \(D_z^2\) to the real axis thought of as a contour in \(\mathbb{C}\). This contour is then deformed away from the support of \(V\) so that \(P\) can be restricted to it. This provides ellipticity at infinity at the price of losing self-adjointness.
Figure 2.7. Curve $\Gamma$ used in complex scaling. The curve is given by $x \mapsto x + ig(x)$ for a $C^\infty$ function $g$ satisfying $g(x) = 0$ for $-R \leq x \leq R$ and $g(x) = x \tan \theta$ for $|x|$ sufficiently large, where $\theta$ is a given constant.

An account of this method in higher dimensions will be provided in §4.5. Again, in one dimension we can provide a low-tech self-contained presentation.

Let $\Gamma \subset \mathbb{C}$ be a $C^1$ simple curve. We define differentiation and integration of functions mapping $\Gamma$ to $\mathbb{C}$ as follows. Let $\gamma(t)$ be a parametrization $\mathbb{R} \rightarrow \Gamma \subset \mathbb{C}$, and let $f \in C^1(\Gamma)$ in the sense that $f \circ \gamma \in C^1(\mathbb{R})$. We define

$$\partial_{\Gamma}^z f(z_0) = \gamma'(t_0)^{-1} \partial_t (f \circ \gamma)(t_0), \quad \gamma(t_0) = z_0,$$

where the inverse and the multiplication are in the sense of complex numbers. We further define

$$D_{\Gamma}^z = \frac{1}{i} \partial_{\Gamma}^z.$$

By the chain rule, $\partial_{\Gamma}^z f(z_0)$ is independent of parametrization. In fact, if $\alpha$ is another parametrization, $\alpha(s_0) = z_0$, then

$$\gamma'(t_0) = c(s_0) \alpha'(s_0), \quad c(s_0) = (\partial_s (\gamma^{-1} \circ \alpha)(s_0))^{-1} \in \mathbb{R}.$$

Then

$$\alpha'(s_0)^{-1} \partial_s (f \circ \alpha)(s_0) = \alpha'(s_0)^{-1} \partial_s (f \circ \gamma \circ (\gamma^{-1} \circ \alpha))(s_0)$$

$$= \alpha'(s_0)^{-1} \partial_s (\gamma^{-1} \circ \alpha)(s_0) \partial_t (f \circ \gamma)(t_0)$$

$$= \alpha'(s_0)^{-1} c(s_0)^{-1} \partial_t (f \circ \gamma)(t_0)$$

$$= \gamma'(t_0)^{-1} \partial_t (f \circ \gamma)(t_0).$$

If $f$ extends to a $C^1$ function in a neighbourhood of $\Gamma$ and

$$\gamma(t) = \gamma_1(t) + i \gamma_2(t), \quad \gamma_j : \mathbb{R} \rightarrow \mathbb{R},$$
Figure 2.8. Curve $\Gamma$ used in some PML (perfectly matched layer) computations. A typical curve is given by a function $\mathbb{R} \ni x \mapsto x + ig(x)$ where $g(x) = -|x + R|^\alpha$ for $x < -R$, $g(x) = 0$ for $-R \leq x \leq R$, and $g(x) = (x - L)^\alpha$ for $x > R$, where $\alpha > 1$.

then

$$\frac{\partial^\Gamma f(z_0)}{\partial z} = \gamma'(t_0)^{-1}(\partial_x f(z_0)\gamma'_1(t_0) + \partial_y f(z_0)\gamma'_2(t_0))$$

In particular, if $f$ is holomorphic in a neighborhood of $\Gamma$, then the Cauchy-Riemann equation, $\partial_y f = i\partial_x f$, shows that

$$\frac{\partial^\Gamma f}{\partial z} = \partial_x f = \partial_z f,$$

so in this case $\partial^\Gamma$ coincides with the holomorphic differential operator.

To integrate along the curve we can use both the complex contour measure and the arclength measure, denoted

$$dz = \gamma'(t)dt, \quad |dz| = |\gamma'(t)|dt,$$

respectively. We assume that $\gamma \in C^2(\mathbb{R}; \mathbb{C})$. The spaces $L^2(\Gamma)$ and $H^j(\Gamma)$, $j = 1, 2$, are defined using the measure $|dz|$.

Given a potential $V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{C})$ we further assume that

(2.7.1) $\Gamma \cap \mathbb{R} \supset [-L, L]$; $\supp V \subset (-L, L)$. The potential $V$ is then a well defined function on $\Gamma$, so that putting

(2.7.2) $P_{V, \Gamma} := (D^\Gamma_z)^2 + V(z)$,

makes sense.

We make the following assumption on the behaviour of $\Gamma$ at infinity:

(2.7.3) $\exists \theta \in (0, \pi), \ a_\pm \in \mathbb{C}, \ K \in \mathbb{C}$,

$$\Gamma \setminus K = \bigcup_{\pm} \left(a_\pm \pm e^{i\theta}(0, \infty)\right) \setminus K.$$
2.7. COMPLEX SCALING IN ONE DIMENSION

We orient \( \Gamma \) so that \( \text{Im} \, \gamma(t) \to +\infty \) as \( t \to +\infty \): this can be done in view of (2.7.3). We also define
\[
(2.7.4) \quad \Lambda_\Gamma := \{ \lambda \in \mathbb{C} \setminus \mathbb{R}_- : -\theta < \arg \lambda < \pi - \theta \},
\]
where \( \arg : \mathbb{C} \setminus \mathbb{R}_- \to (-\pi, \pi) \). An example of \( \Gamma \) is shown in Fig. 2.7. We remark that one can also consider more general behaviour at infinity such as shown in Fig. 2.8 where \( \Gamma = \{ x + ig(x) : x \in \mathbb{R} \} \) for a specific \( g \).

Let us first consider the case of \( V \equiv 0 \).

**THEOREM 2.20** (Complex scaling for the free Laplacian). For \( \lambda \in \mathbb{C} \setminus 0 \) and \( f \in C^1_0(\Gamma) \), define
\[
(2.7.5) \quad R_{0,\Gamma}(\lambda)f(z) := \frac{i}{2\lambda} \int_{\Gamma} e^{i\lambda \varphi(z,w)} f(w) dw,
\]
\[
\varphi(\gamma(t), \gamma(s)) := \pm(\gamma(t) - \gamma(s)), \quad \pm(t-s) \geq 0.
\]
For \( \lambda \in \Lambda_\Gamma \), \( R_{0,\Gamma}(\lambda) \) extends to an operator \( L^2(\Gamma) \to H^2(\Gamma) \) which is a two sided inverse of \( (D^\Gamma_2)^2 - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma) \).

**Proof.** 1. For \( f \in C^2_0(\Gamma) \) we check by direct calculation that
\[
R_{0,\Gamma}(\lambda)((D^\Gamma_2)^2 - \lambda^2)f(z) = f(z), \quad ((D^\Gamma_2)^2 - \lambda^2)R_{0,\Gamma}(\lambda)f(z) = f(z).
\]
Since \( C^2_0(\Gamma) \) is dense in \( L^2(\Gamma) \) and in \( H^2(\Gamma) \), the result will follow once we show that \( R_{0,\Gamma}(\lambda) \) is bounded on \( L^2 \) for \( \lambda \in \Lambda_\Gamma \).

2. To bound \( R_{0,\Gamma}(\lambda) \) on \( L^2(\Gamma) \) we can, by reparametrization, assume that \( \gamma(t) = b_\pm + e^{i\theta} t \) for \( \pm t \geq C_0 \). Then for \( 0 < \theta + \arg \lambda < \pi \)
\[
\text{Re}(i\lambda \varphi(\gamma(t), \gamma(s))) = -\sin(\theta + \arg \lambda)|\lambda||t-s| + O(1) \leq -\epsilon|\lambda||t-s| + O(1),
\]
for some \( \epsilon > 0 \). This implies that
\[
\sup_{t \in \mathbb{R}} \int_{\mathbb{R}} |e^{i\lambda \varphi(\gamma(t), \gamma(s))}| d|\gamma'(s)| ds < \infty
\]
so that boundedness of \( R_{0,\Gamma}(\lambda) \) on \( L^2(\Gamma) \) follows from Schur’s estimate (A.5.3). \( \square \)

We will now use the inverse of the free operator \( (D^\Gamma_2)^2 - \lambda^2 \) to show that \( P_{V,\Gamma} - \lambda^2, \lambda \in \Lambda_\Gamma \), is a Fredholm operator and to identify the values of \( \lambda \) for which it is not invertible with scattering resonances.

**THEOREM 2.21** (Complex scaling in dimension one). Suppose that \( \Gamma \) satisfies (2.7.3) and \( P_{V,\Gamma} \) is defined by (2.7.2) with \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{C}) \).

1) For \( \lambda \in \Lambda_\Gamma \),
\[
(2.7.6) \quad P_{V,\Gamma} - \lambda^2 : H^2(\Gamma) \to L^2(\Gamma),
\]
is a Fredholm operator and the spectrum of $P_{\Gamma,V}$ in $\Lambda_{\Gamma}$ is discrete.

2) We have

$$m_R(\lambda) = \text{tr}_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{\Gamma,V})^{-1} 2\zeta d\zeta, \quad \lambda \in \Lambda_{\Gamma},$$

where the integral is over any sufficiently small positively oriented curve enclosing $\lambda$ (the value is constant when the curves are sufficiently small). In particular, the eigenvalues of $P_{\Gamma,V}$ in $\Lambda_{\Gamma}$ are independent of $\Gamma$ and coincide with scattering resonances.

3) With the same notation,

$$m_D(\lambda) = \text{tr}_{L^2(\Gamma)} \frac{1}{2\pi i} \oint (\zeta^2 - P_{\Gamma,V})^{-1} 2\zeta d\zeta, \quad \lambda \in \Lambda_{\Gamma},$$

where $m_D(\lambda)$ was defined in (2.2.29).

REMARKS.

1. The operator

$$\Pi_{\lambda,\Gamma} = \frac{1}{2\pi i} \oint \frac{1}{\lambda} (\zeta^2 - P_{\Gamma,V})^{-1} 2\zeta d\zeta : L^2(\Gamma) \to L^2(\Gamma),$$

is a projection – see Theorem C.6. Since the operator $P_{\Gamma,V}$ is not normal the projection is not orthogonal. The trace in (2.7.6) then gives the rank of this projection.

As in the proof of Theorem 2.5 we then see that near an eigenvalue, $\lambda \in \Lambda_{\Gamma},$

$$(P_{\Gamma,V} - \zeta^2)^{-1} = \sum_{j=1}^{m} \frac{(P_{\Gamma,V} - \lambda^2)^{j-1} \Pi}{(\lambda^2 - \zeta^2)^j} + A(\zeta,\lambda),$$

where, for $\zeta$ near $\lambda$, $\zeta \mapsto A(\zeta,\lambda)$ is a holomorphic family of bounded operators on $L^2(\Gamma)$. The fact that the order of the pole is equal to the rank of the projection is a consequence of the fact that we are dealing with ordinary differential equations which can have at most one $L^2$ solution – see step 2 of the proof below.

2. Part 3 of the theorem provides the proof of Theorem 2.6 in the case when $m_R(\lambda) > 1$, for scattering poles with $\arg \lambda > -\pi$. When $V$ is real valued that gives the result for all $\lambda$ as there are no poles on the real axis. (We can also use the symmetry (2.2.13).) For complex valued potentials we need to change the contour to $\overline{\Gamma}$ to obtain the same result for poles near $(-\infty, 0)$.

Proof. 1. For $\lambda \in \Lambda_{\Gamma}$, Theorem 2.20 shows

$$\rho R_{0,\Gamma}(\lambda) : L^2(\Gamma) \to H^2([-L,L]) \hookrightarrow L^2(\Gamma),$$
and hence $VR_{0,\Gamma}(\lambda) = V\rho R_{0,\Gamma}(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ is a compact operator. It follows that
\[(P_{V,\Gamma} - \lambda^2)R_{0,\Gamma}(\lambda) = (I + VR_{0,\Gamma}(\lambda))\]
which implies the Fredholm property of $P_{V,\Gamma} - \lambda^2$, $\lambda \in \Lambda_\Gamma$. Theorem C.5 shows that $(I + VR_{0,\Gamma}(\lambda))^{-1}$ is a meromorphic family of operators which means that the spectrum of $P_{V,\Gamma}$ in $\Lambda_\Gamma$ is discrete.

2. Suppose $\lambda \in \Lambda_\Gamma$ is a resonance of multiplicity $m = m_R(\lambda)$. According to Theorem 2.5 this equivalent to the existence of $u_m : \mathbb{R} \to \mathbb{C}$ satisfying $(P_V - \lambda^2)^m u_m(x) = 0$ and
\[u_1(x) := (P - \lambda^2)^{m-1} u_m(x) = \begin{cases} Ae^{i\lambda x} & x \geq L \\ Be^{-i\lambda x} & x \leq -L \end{cases}\]
This means that $u_m$ satisfies
\[u_m(x) = \begin{cases} P(x)e^{i\lambda x}, & x \geq L, \\ Q(x)e^{-i\lambda x}, & x \leq -L, \end{cases}\]
with polynomials $P$ and $Q$ of degree $m - 1$. We now define $\tilde{u} : \Gamma \to \mathbb{C}$ as follows. We write $\Gamma$ as a disjoint union of connected components,
\[\Gamma = \Gamma_+ \cup [-L, L] \cup \Gamma_-,
\]
so that $\text{Im } z \to \pm \infty$ on $\Gamma_\pm$. We then put
\[\tilde{u}_m(z) = \begin{cases} P(z)e^{i\lambda z}, & z \in \Gamma_+, \\ u_m(z), & z \in [-L, L], \\ Q(z)e^{-i\lambda z}, & z \in \Gamma_- \end{cases}\]
The function $\tilde{u}_m$ clearly satisfies
\[(P_{V,\Gamma} - \lambda^2)^m \tilde{u}_m = 0, \quad (P_{V,\Gamma} - \lambda^2)^{m-1} \tilde{u}_m \neq 0.\]
Since
\[\text{Re}(i\lambda z|_{\Gamma_\pm}) = -\sin(\theta + \arg \lambda)|\lambda||z| + 0(1) < -\epsilon|\lambda||z| + 0(1), \quad \epsilon > 0.\]
It follows that
\[\|\tilde{u}_m\|_{L^2(\Gamma)}^2 \leq \int_{-L}^{L} |u_m(x)|^2 \, dx + \sum_{\pm} C \int_{\Gamma_\pm} |z|^{m-1} e^{-\epsilon|\lambda||z|} \, d|z| < \infty,\]
and the same estimate is valid for $\tilde{u}_j := (P_{V,\Gamma} - \lambda^2)^{m-j} \tilde{u}_m$, $1 \leq j \leq m$. Because $u_1 \in L^2(\Gamma)$ solving $(P_{V,\Gamma} - \lambda^2)u_1 = 0$ is unique up to a multiplicative constant (there is always a solution of the ordinary equation which is not in $L^2$), this shows that for every resonance of multiplicity $m$ we obtain an eigenvalue of $P_{V,\Gamma}$ of algebraic multiplicity $m$ and geometric multiplicity 1.

3. This argument can be reversed. Let $\lambda \in \Lambda_\Gamma$ be an eigenvalue of $P_{V,\Gamma}$ of multiplicity $m$ (and, as explained above, geometric multiplicity 1). Since $e^{-\pi iz}|_{\Gamma_\pm} \notin L^2(\Gamma)$,
this forces the corresponding $\tilde{u}_m$ to be of the form (2.7.10) and by reversing
the construction in step 2 we see that $\lambda$ is a resonance of multiplicity $m$. This proves (2.7.7).

4. To prove (2.7.8) choose $\rho \in C^\infty_c(\mathbb{R})$ with support in $[-L,L]$ and equal to
1 on the support of $V$. In view of (2.7.1) $\rho$ defines a function on $\Gamma$ and the
multiplication operator on $L^2(\Gamma)$: $u(z) \mapsto \rho(z)u(z)$. The definition of $R_{0,\Gamma}$
shows that

$$\rho R_{0,\Gamma}(\lambda)V = \rho R_0(\lambda)V,$$

where the operator on the right is well defined as an operator on $L^2(\Gamma)$. Consequently,

$$(I + \rho R_{0,\Gamma}(\lambda)V)^{-1} = (I + \rho R_0(\lambda)V)^{-1}$$

is a meromorphic family of operators on $L^2(\Gamma)$ – see (2.2.10). Arguing as in
step 3 of the proof of Theorem 2.2 we obtain the following analogue of (2.2.9):

$$(2.7.11) \quad R_{V,\Gamma}(\lambda) = R_{0,\Gamma}(\lambda)(I + VR_{0,\Gamma}(\lambda)\rho)^{-1}(I - VR_{0,\Gamma}(\lambda)(1 - \rho))$$

Since

$$R_{0,\Gamma}(\lambda) = (P_{0,\Gamma} - \lambda^2)^{-1} : L^2(\Gamma) \to H^2(\Gamma),$$

$$I - VR_{0,\Gamma}(\lambda)(1 - \rho) = (I + VR_{0,\Gamma}(\lambda)(1 - \rho))^{-1} : L^2(\Gamma) \to L^2(\Gamma),$$

are holomorphic families of invertible operators, we apply Theorem C.8 to obtain (2.7.7). □

**INTERPRETATION.** The method of complex scaling identifies scattering resonances in conic regions $\Lambda_{\Gamma}$ with eigenvalues of non self-adjoint operators $P_{V,\Gamma}$. We gain the advantage of being able to use methods of spectral theory, albeit in the murkier non-normal setting. The multiplicity of a resonance is now the trace of a projection. The resonant states, that is the outgoing solutions to $(P - \lambda^2)u$, are restrictions to $\mathbb{R}$ of functions which continue holomorphically to functions with $L^2$ restrictions to $\Gamma$. In one dimension this is explicit as seen in (2.7.10). Since have dealt only with compactly supported potentials our contours $\Gamma$ had to coincide with $\mathbb{R}$ near the support of $V$. The method generalizes to the case of potentials which are analytic and decaying to 0 in conic neighbourhoods of $\pm(L,\infty)$. As we will see later on it also generalized to higher dimensions though of course the treatment is not as explicit there.

As an application of the method we present the following result about perturbation of resonances.

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2. COMPLEX SCALING IN ONE DIMENSION

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THEOREM 2.22 (Continuity of resonances under perturbations). Suppose that $V_0 \in L^\infty_{\text{comp}}([-L,L])$ and that $\Omega \subset \mathbb{C}$ is a fixed bounded open set with a $C^1$ boundary $\partial \Omega$ such that there are no resonances of $V_0$ on $\partial \Omega$.

Denoting by $m_V(\lambda)$ the multiplicity of $\lambda$ as a resonance of $V$ there exists $\epsilon$ such that for $V \in L^\infty_{\text{comp}}([-L,L])$ with $\|V-V_0\|_\infty < \epsilon$, we have

$$
\sum_{\lambda \in \Omega} m_{V_0}(\lambda) = \sum_{\lambda \in \Omega} m_V(\lambda).
$$

REMARK. The dependence of $\epsilon$ on $V_0$ and $\Omega$ can be quite dramatic. The simplest example is given by taking a family $V_s = s V$, $s \in [0,1]$, where $V \neq 0$ is a fixed potential in $L^\infty_{\text{comp}}$. From Theorem 2.14 we know that for $s \neq 0$ there are infinitely many resonances while $s = 0$ there is only one resonance at 0.

Proof of Theorem 2.22. 1. Let us first assume that $0 \notin \Omega$. By writing $\Omega$ as a disjoint union of sets with piecewise $C^1$ boundaries and no resonances on those boundaries, we can assume that $\Omega \subset \Lambda_{\Gamma}$ or $\Omega \subset \bar{\Lambda}_{\Gamma}$ for some $\Gamma$ ($\bar{\Gamma} := \{\bar{\lambda} : \lambda \in \Gamma\}$) and we will consider the first of these cases, the other being analogous. (As discussed in Remark 2 after Theorem 2.21 we need the second case for resonances near $(-\infty,0)$ in the case $V$ is not real valued.)

We take contours such that $\Gamma \cap \mathbb{R} \subset [-L-1,L+1]$. Hence the resonances of $V$ in $\Omega$ are the same as eigenvalues of $P_{V_0,\Gamma}$ there. From (2.7.7) we see that

$$
\sum_{\lambda \in \Omega} m_{V_0}(\lambda) = \frac{1}{2\pi i} \text{tr} \int_{\partial\Omega} (\zeta - P_{V_0,\Gamma})^{-1} d\zeta,
$$

2. Since there are no resonances on $\partial \Omega$, there exists $M$ such that

$$
\| (\zeta - P_{V,\Gamma})^{-1} \|_{L^2(\Gamma) \to H^2(\Gamma)} \leq M, \quad \zeta \in \partial \Omega.
$$

For $V \in L^\infty([-L,L])$,

$$
\zeta - P_{V,\Gamma} = (\zeta - P_{V_0,\Gamma})(I - (\zeta - P_{V_0,\Gamma})^{-1}(V - V_0))
$$

Hence for $\|V-V_0\|_\infty < \epsilon$, with epsilon sufficiently small, the last factor can be inverted by Neumann series. It follows that

$$
\| (\zeta - P_{V,\Gamma})^{-1} \|_{L^2(\Gamma) \to H^2(\Gamma)} \leq 2M, \quad \zeta \in \partial \Omega.
$$

With $\rho \in C^\infty_c((-L-1,L+1))$, $\rho = 1$ on $[-L,L]$, we obtain a bound on the trace class norm:

$$
\| \rho(\zeta - P_{V_0,\Gamma})^{-1} \|_{\mathcal{L}_1} \leq \| \rho(\zeta - P_{V_0,\Gamma})^{-1} \|_{L^2(\Gamma) \to H^2(\Gamma)} \leq C(\sup_{\Omega} |\zeta|M) \leq C'M.
$$
We can now estimate
\[
\frac{1}{2\pi} \left| \text{tr} \int_{\partial \Omega} \left( (\zeta - P_{V_0})^{-1} - (\zeta - P_{V})^{-1} \right) d\zeta \right|
\]
\[
= \frac{1}{2\pi} \left| \text{tr} \int_{\partial \Omega} \left( (\zeta - P_{V_0})^{-1}(V - V_0)\rho(\zeta - P_{V})^{-1} \right) d\zeta \right|
\]
\[
\leq CM^2 \|V - V_0\|_{L^\infty} < 1,
\]
if \(\|V - V_0\|_{L^\infty} < \epsilon\) with \(\epsilon\) small enough. Since the left hand side has to take integral values this means it has to be equal to 0. Returning to (2.7.13) that means that we can replace \(V_0\) by \(V\) on the right hand side and that proves (2.7.12).

3. When \(0 \in \Omega\) we need a different argument as the complex scaling method does not work there. However, we can assume that \(\Omega = D(0, r)\) as we can apply the previous argument to \(\Omega \setminus D(0, r)\). We can further assume that \(V_{s_0}\) has a resonance at zero and that \(D(0, r)\) does not contain any other resonances. Theorem 2.6 then shows that (we use the obvious notation for multiplicities depending on the potential)
\[
m_{V_0}(0) - 1 = \left( I + V_0\rho R_0(\zeta)\rho \right)^{-1} V_0 \partial_\zeta (\rho R_0(\zeta)\rho) d\zeta,
\]
where \(\rho \in C^\infty_c(\mathbb{R})\) is equal to 1 on \([-L, L]\). Also,
\[
\| (I + V_0\rho R_0(\zeta)\rho)^{-1} \|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < M, \quad |\zeta| = r.
\]
Since
\[
I + V\rho R_0(\lambda)\rho = (I + V_0\rho R_0(\zeta)\rho) (I + (I + V_0\rho R_0(\lambda)\rho)^{-1}(V - V_0) ),
\]
we that if \(\|V - V_0\|_{L^\infty} < \epsilon\) with \(\epsilon\) small enough then
\[
\| (I + V\rho R_0(\zeta)\rho)^{-1} \|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} < 2M, \quad |\zeta| = r.
\]
We conclude that \(V\) has no resonances on \(|\zeta| = r\) and that
\[
(2.7.14) \sum_{|\lambda| < r} m_V(\lambda) = 1 + \frac{1}{2\pi i} \text{tr} \int_{|\zeta| = r} (I + V\rho R_0(\zeta)\rho)^{-1} V \partial_\zeta (\rho R_0(\zeta)\rho) d\zeta,
\]
noting that right hand side, as an integral of a logarithmic derivative of a meromorphic function, \(D(\lambda)\), takes only integral values. We have
\[
\| \partial_\zeta (\rho R_0(\zeta)\rho) \|_{L_1} \leq C, \quad |\zeta| = r,
\]
see Lemma 2.16 and (C.3.1). Hence we can argue as in step 2 and see that the right hand side in (2.7.14) is equal to 1.

**REMARK.** The reader have probably noticed that the proof in step 3 can be used near any resonance. But as we proved (2.2.30) using complex scaling (see Remark 2 after Theorem 2.21) it is natural to use complex scaling to
give a direct proof of the continuity of resonances. One can also see that the
argument in the above proof is valid for more general families of operators.

The second application is closely related and shows generic simplicity
of scattering resonances. Although we have spent some time and effort
discussing resonances of higher multiplicity\footnote{Higher multiplicities can indeed occur as explained in the example in §2.2. For a more
general construction see [Sj87, §4].} it is important to remember
that higher multiplicities are very special.

**THEOREM 2.23** (Generic simplicity of resonances in dimension
one). For any $L$, there exists $V \subset L^\infty([-L,L],\mathbb{R})$ which is an intersection
of open dense sets in $L^\infty([-L,L],\mathbb{R})$ and such that
$$\forall V \in V, \lambda \in \mathbb{C}, \quad m_R(\lambda) \leq 1.$$ 

**REMARKS.** 1. An intersection of open dense set is sometimes called a
residual set and the property that holds on a residual set is called
generic. Complements of residual sets are called meagre. All this is related to the
classical Baire category theorem which states that a complete metric space
cannot be meagre.

2. As can be seen from the proof, we can replace $L^\infty([-L,L];\mathbb{R})$
with $C^\infty((-L,L];\mathbb{R})$ or other spaces of functions.

We start the proof with the following lemma:

**LEMMA 2.24** (An application of Rouché’s Theorem). Suppose that
$\epsilon \mapsto f(z)$ a family of functions holomorphic in a neighbourhood of $D(0,r_0)$
and that
$$f_\epsilon(z) = z^m - \epsilon + O(\epsilon^2) + O(\epsilon|z|), \quad |z| \leq r_0.$$ 
Then for $\epsilon$ sufficiently small $f_\epsilon(z)$ has exactly $m$ simple zeros in the disc
$D(0,r_0)$,
$$z_k = \left(1 + O(\epsilon^m) \right) + \frac{1}{m} e^{2\pi ik/m}, \quad k = 0, \ldots, m - 1.$$ 

**Proof.** 1. Since
$$|z^m - f_\epsilon(z)| = |\epsilon + O(\epsilon^2 + \epsilon r_0)| \leq C\epsilon < |z|^m, \quad |z| = r_0, \quad \epsilon \ll 1$$
Rouché’s Theorem shows that the number of zeros of $f_\epsilon$ and $z^m$ agree (with
multiplicities) in $|z| < r_0$, that is $f_\epsilon$ has exactly $m$ zeros there.

2. Consider the discs $D_k := D(\epsilon^{1/m} e^{2\pi imk/m}, \rho \epsilon^{1/m}), 0 \leq k \leq m - 1$. We note that
$$\rho < \pi/m \implies D_k \cap D_\ell = \emptyset, \quad k \neq \ell.$$ 
For $z \in \partial D_k$,
$$|z^m - \epsilon - f_\epsilon(z)| \leq C_0 \epsilon^{1+\frac{1}{m}}.$$
On the other hand, if \( \rho > 2C_0\epsilon^{\frac{1}{m}} \) then for \( z \in \partial D_k \),

\[
|z^m - \epsilon| = \epsilon \rho (1 + \mathcal{O}(\rho^2)) > C_0\epsilon^{1+\frac{1}{m}} \geq |z^m - \epsilon - f_\epsilon(z)|,
\]

If \( \epsilon \) is small enough we can choose \( \rho \) so that \( C_0\epsilon^{\frac{1}{m}} < \rho < \pi/m \), and as \( D_k \)'s are then disjoint, Rouché's theorem shows that there is exactly one zero of \( f_\epsilon \) in each \( D_k \). This shows that all \( m \) zeros are simple. \( \square \)

**Proof of Theorem 2.23.** 1. Since for real valued potentials, there are no resonances on \( \mathbb{R}\setminus\{0\} \), it is enough to show that for a residual set of potentials resonances are simple in \( \mathbb{C}\setminus\mathbb{R}_- \).

We introduce the following ordering in \( \mathbb{C}\setminus\mathbb{R}_- \):

\[
z \preceq w \iff |z| < |w| \text{ or } |z| = |w|, \text{ arg}(z) \leq \text{ arg}(w),
\]

where \( \text{arg} : \mathbb{C}\setminus\mathbb{R}_- \to (-\pi, \pi) \). For any \( V \) we order resonances as follows:

\[
\lambda_1 \preceq \lambda_2 \preceq \ldots \lambda_n \preceq \ldots
\]

We then define

\[
\mathcal{V}_n := \{ V \in L^\infty([-L, L]; \mathbb{R}) : m_R(\lambda_j) = 1, j = 1, \ldots, n \}.
\]

2. Theorem 2.22 shows that the set \( \mathcal{V}_n \) is open on \( L^\infty([-L, L]; \mathbb{R}) \): we take \( \Omega \) to be the union of \( n \) disjoint discs, each containing one resonance \( \lambda_j \). Theorem 2.22 shows that for potentials in a neighbourhood of \( V \in \mathcal{V}_n \) the multiplicity does not change in each disc, which means that resonances remain simple.

3. We now show that \( \mathcal{V}_n \) is dense in \( L^\infty([-L, L]; \mathbb{R}) \). That means that for \( V \notin \mathcal{V}_n \), any neighbourhood of \( V \) in \( L^\infty \) contains an element of \( \mathcal{V}_n \). We can choose \( k \geq 0 \) so that ordered resonances of \( V \) satisfy

\[
\lambda_1 = \cdots = \lambda_{m_1} < \lambda_{m_1+1} = \cdots = \lambda_{m_1+m_2} < \cdots < \lambda_{n+k},
\]

\[
|\lambda_{n+k}| < |\lambda_{n+k+1}|, \quad m_1 + m_2 + \cdots m_p = n + k.
\]

Consider

\[
(2.7.15) \quad \lambda := \lambda_{m_1+\cdots+m_{j-1}+1},
\]

and assume that \( m = m_j > 1 \). That means that \( \lambda \neq 0 \) and, using symmetry (2.2.13), we can assume that \( \text{Re} \lambda \geq 0 \). This means that \( \lambda^2 \) is an eigenvalue of \( P_{V,\Gamma} \) for a contour \( \Gamma \) with \( \pi/2 < \theta < \pi \) (see (2.7.3)). Hence there exists \( w_m \in L^2(\Gamma) \) satisfying

\[
(2.7.16) \quad (P_{V,\Gamma} - \lambda^2)^m w_m = 0, \quad w_1 := (P_{V,\Gamma} - \lambda^2)^{m-1} w_m \neq 0,
\]
2.7. COMPLEX SCALING IN ONE DIMENSION

see (2.7.9) and (2.7.10). It is unique up to a multiplicative constant. Following §C.1 we construct the following Grushin problem for $P_{V,Γ}$:

$$
P_{V,Γ}(ζ) := \begin{pmatrix} P_{V,Γ} - ζ^2 & R_- \\ R_+ & 0 \end{pmatrix} : H^2(Γ) ⊕ C → L^2(Γ) ⊕ C,
$$

$$
R_- : C → L^2(Γ), \quad R_- u_- := u_- w_m, \quad ∥w_m∥_{L^2(Γ)} = 1,
$$

$$
R_+ : H^2(Γ) → C, \quad R_+ u := \int_{Γ} uw_1 dz, \quad ∥w_1∥_{L^2(Γ)} = 1,
$$

where the normalization of $w_m$ and $w_1$ was chosen for later convenience. We note that

$$(2.7.17) \quad \ker_{H^2(Γ)}(P_{V,Γ} - λ^2) = Cw_1
$$

and as the index is 0 (changing $λ$ to $ζ$ close to $λ$ produces an invertible operator – see Theorem C.4), the dimension of the coker $(P_{V,Γ} - λ^2)$ is also 1. The function $w_m$ is not in the image $P_{V,Γ} - λ^2$ as otherwise $m$ in (2.7.16) would not be minimal, that is, the multiplicity of $λ$ would be greater than $m$. Hence

$$(2.7.18) \quad (P_{V,Γ} - λ^2)H^2(Γ) + Cw_m = L^2(Γ).
$$

Hence $P_{V,Γ}(λ)$ is invertible and by Lemma C.2 so is $P_{V,Γ}(ζ)$ for $ζ$ close to $λ$. We also check that

$$
P_{V,Γ}(ζ)^{-1} = \begin{pmatrix} E(ζ) & E_{-+}(ζ) \\ E_-(-ζ) & E_{++}(ζ) \end{pmatrix}, \quad E_{-+}(ζ) = (θ^2 - λ^2)^m,
$$

$$
E_-(λ)v = \int_{Γ} v w_m dz, \quad E_+(λ)v_+ = v_+ w_1.
$$

4. Now take $W ∈ L^∞([-L, L], R)$ and consider $P_{V+εW}(ζ)$. If $ε > 0$ is small enough, Lemma C.2 shows that this operator is invertible for $ζ$ close to $λ$ and the corresponding $E_{−+}^ε$ has an expansion given by (C.1.7):

$$
E_{−+}^ε(ζ) = (θ^2 - λ^2)^m - εE_-(λ)WE_+(λ) + O(ε^2)
$$

$$
= (θ^2 - λ^2)^m - ε \int_{-L}^{L} W(x)w_1(x)w_m(x)dx + O(ε|λ - ζ| + ε^2).
$$

Since $w_1$ and $w_m$ solve differential equations (of orders 2 and $2m$ respectively, $w_1 w_m|_{-L,L} ≠ 0$). Hence $∫ W w_1 w_m ≠ 0$ for some choice of $W ∈ L^∞([-L, L]; R)$. We can then apply Lemma 2.24 to see that for $ε > 0$ small enough all the zeros of $E_{−+}^ε(ζ)$ near $λ$ as simple.

5. We can apply this argument to each $λ$ of the form (2.7.15). If $w_{mj}^m$ and $w_1^{ mj}$ denote the corresponding $w_m$ and $w_1$ the condition we need to obtain
simplicity of the eigenvalues $P_{V+\epsilon W;\Gamma}$ near $\lambda^2_j$ is
\[
\int_{-L}^{L} W(x)w_{m_j}^{m_j}(x)\bar{w}_{m_j}^{m_j}(x)dx \neq 0, \quad j = 1, \ldots, p.
\]
But such $W$ exists as $w_{m_j}^{m_j}$ do not vanish identically in $[-L,L]$. We conclude that there exist arbitrarily small $L^\infty([-L,L];\mathbb{R})$ perturbation of $V$ such that the first $n$ resonances of $V + \epsilon W$ are simple. (Note that by continuity the resonances, the perturbation still satisfy $|\lambda_n + k| < |\lambda_n + k + 1|$ if $\epsilon$ is small enough.) We now take
\[
V = \bigcap_{n=1}^{\infty} V_n,
\]
concluding the proof. \qed

2.8. SEMICLASSICAL STUDY OF RESONANCES IN DIMENSION ONE

In this section we will present some results concerning resonances in the semiclassical limit. This means we consider Schrödinger operators with a small parameter $h$:

\[
P = P(h) := (hD_x)^2 + V, \quad V \in L^\infty_{\text{comp}}(\mathbb{R};\mathbb{R}), \quad 0 < h < 1.
\]

REMARK. Since this is the first appearance of $h$ some comments are in place. Although motivated by the Planck constant $\hbar = 1.054571726(47) \times 10^{-34} J \cdot s$, our $h$ in the Schrödinger equation plays the role of $\hbar/\sqrt{2m}$. Its effective size depends in addition on the units of length used and may vary from problem to problem. The semiclassical approximation $h \to 0$ can be applied in situations where that effective size is small. In many problems the semiclassical approximation, although mathematically valid only for $h$ very small, produces good results for $h = 1$.

CONVENTION. Because the motivation in this section comes from quantum mechanics rather than wave propagation we use a different convention for the spectral parameter. We now consider $z \in \mathbb{C} \setminus (-\infty,0]$ with the convention that
\[
\pm \text{Im } z > 0 \implies \pm \text{Im } \sqrt{z} > 0.
\]
From Theorem 2.2 we know that, as an operator $L^2_{\text{comp}} \to L^2_{\text{loc}}$,
\[
((hD_x)^2 + V - z)^{-1} = h^{-2}(D_x^2 + h^{-2}V - \lambda^2)^{-1}, \quad \lambda = \sqrt{z}/h,
\]
2.8. SEMICLASSICAL STUDY OF RESONANCES IN DIMENSION ONE

continues meromorphically from \( \text{Im} \, z > 0, \text{Re} \, z > 0 \) to \( \text{Im} \, z \leq 0, \text{Re} \, z > 0 \). The poles of that meromorphic continuation are denoted by \( \text{Res}(P(h)) \) and the multiplicity is defined as in \( \S \S 2.2 \) and 2.7:

\[
(2.8.1) \quad m(z) := \text{rank} \oint_{z} R(\zeta, h) d\zeta = \frac{1}{2\pi} \text{tr} \oint_{z} (\zeta - (hD_x^\Gamma)^2 - V)^{-1} d\zeta.
\]

2.8.1. Truncated harmonic oscillator. As an example of resonances generated by a well in an island we consider the potential given in Fig. 2.10. The method applies to more general potentials having a unique positive non-degenerate minimum and step singularities at the boundary of the support. For simplicity we present the case of the truncated harmonic oscillator and of the ground state.

Thus we consider the potential \( V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \) defined by

\[
V(x) = \begin{cases} x^2 + 1 & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}
\]

and the operator

\[
(2.8.2) \quad P(h) = (hD_x)^2 + V.
\]

We want to describe the resonance for \( P(h) \) which is close to the lowest eigenvalue of \( (hD_x)^2 + x^2 + 1 \),

\[
(2.8.3) \quad \tilde{z} = 1 + h, \quad \text{with an eigenfunction (ground state)} \quad \tilde{u}(x) = e^{-x^2/2h}.
\]

As \( \tilde{u} \) is even, we can study \( P(h) \) on a half-line with the Neumann boundary condition at \( x = 0 \).

THEOREM 2.25. For \( h \) small enough, there exists a resonance

\[
(2.8.4) \quad z = 1 + h - i4\pi^{-1}h^{\frac{1}{2}}e^{-1/h}(1 + O(h)),
\]

with an even resonant state.

REMARKS. 1. The correction to the real part is of order \( e^{-1/2h} \) and the approximation to the imaginary part is very accurate for relatively large values of \( h \) – see Fig. 2.9

2. A finer analysis shows that for any even \( j \) bounded by a fixed \( h \)-independent \( J \), there exists an even resonant state with

\[
z_j = 1 + h(2j + 1) - i \, h^{\frac{1}{2} - j} c_j e^{-1/h}(1 + O(h)),
\]

\[
c_j := 2^{1+2j} \left( \int_{0}^{\infty} P_j(y)^2 e^{-y^2} \, dy \right)^{-1}.
\]
Here $P_j$ are Hermite polynomials defined recursively by $P_0 = 1$, $P_{j+1} = (2x - \partial_x)P_j$. A similar formula can be given for $j$ odd, and these are the only resonances in

$$[1 - C_1h, 1 + C_1h] - i[0, h\log h]/C_2,$$

where $C_1$ depends on $J$.

**INTERPRETATION.** We see that the eigenvalues $E_j(h) = 1 + (2j + 1)h$ of the (shifted) harmonic oscillator become resonances when the potential is truncated and a finite barrier is created. The resonances are exponentially close to the eigenvalues of the (shifted) harmonic oscillator and in particular we have a perturbative formula for the resonance width (the physical name for the imaginary part):

$$- \Im z_1(h) = 4\pi^{-1/2} e^{-1/2h} (1 + O(h)),$$

corresponding to tunneling through the barrier. The width is exponentially small in terms of the semiclassical parameter $h$. These type of resonances are called *shape* resonances as their properties are affected by the shape of the barrier.
2.8. SEMICLASSICAL STUDY OF RESONANCES IN DIMENSION ONE

Figure 2.10. The potential $V$ and the (vertically rescaled) functions $u_0$ and $\tilde{u}_2$ for $h = 0.01$. TODO the functions.

The proof of Theorem 2.25 requires some preparation. To start we define $u_z \in C^\infty([0,1]), z \in \mathbb{C}$, to be the unique solution of the problem
\begin{equation}
(h^2 D_x^2 + x^2 + 1 - z)u_z = 0, \quad u_z(0) = 1, \quad u_z'(0) = 0,
\end{equation}
and then put
\begin{equation}
X(z) = (X_1(z), X_2(z)) := (u_z(1), hu_z'(1)) \in \mathbb{C}^2.
\end{equation}
Then $z$ is a resonance if and only if
\begin{equation}
X_2(z) - i\sqrt{z}X_1(z) = 0.
\end{equation}
We will apply a contraction mapping principle to the equation (2.8.6) to obtain a resonance exponentially close to $\tilde{z} = 1 + h$. For that, we will need to estimate the first and second derivatives of $X$ in $z$.

To compute the first derivative $\partial_z X$ at $z = \tilde{z}$ we introduce another, growing, solution to $(P(h) - \tilde{z})u = 0$:
\begin{equation}
((hD_x)^2 + x^2 + 1 - \tilde{z})\tilde{v} = 0, \quad \tilde{v}(0) = 0, \quad \tilde{v}'(0) = 1.
\end{equation}
Since the Wronskian, $W(\tilde{u}, \tilde{v}) := \tilde{u}\tilde{v}' - \tilde{u}'\tilde{v} \equiv 1$, we obtain an expression for $\tilde{v}$ in terms of $\tilde{u}$:
\begin{equation}
\tilde{v}(x) = \tilde{u}(x) \int_0^x \frac{1}{\tilde{u}(y)^2} dy = e^{-x^2/2h} \int_0^x e^{y^2/h} dy.
\end{equation}
It is easy to see that
\begin{equation}
|\tilde{v}(x)| + |h\tilde{v}'(x)| \leq C e^{x^2/2h},
\end{equation}
and that\footnote{The dominant contribution to the integral over $0 < y < 1$ comes from $y = 1$ and hence we can make a change of variables $y^2/h = t$ and integrate by parts over $1/2h < t < 1/h$.}
\begin{equation}
\tilde{v}(1) = \frac{1}{2} he^{1/2h}(1 + O(h)), \quad \tilde{v}'(1) = \frac{1}{2} e^{1/2h}(1 + O(h)).
\end{equation}
We define
\[ Y = (Y_1, Y_2) := (\\tilde{v}(1), h\tilde{v}'(1)), \]
and we see from (2.8.8) that
\[ Y = \frac{1}{2}he^{1/2h}(1 + O(h))(1, 1) \]

With this notation we can state the following lemma.

**Lemma 2.26.** With \( Y \) defined by (2.8.9) we have
\[ \partial z X(\\tilde{z}) = -h^{-\frac{3}{2}}\pi^{\frac{1}{2}}Y + O(e^{-1/2h}). \]

**Proof.** 1. For \( u_z \) defined by (2.8.5) the derivative with respect to the parameter \( z \) satisfies the following non-homogeneous equation:
\[ (P(h) - z)\partial_z u_z = u_z; \quad \partial_z u_z(0) = 0, \partial_z u_z'(0) = 0. \]

2. The derivative \( \partial z X(\\tilde{z}) \) can be written as a linear combination of \( X(\\tilde{z}), Y \) with Wronskians at \( x = 1 \) as coefficients:
\[ \partial z X(\\tilde{z}) = W(\partial_z u_z, \tilde{v})(1) \cdot X(\\tilde{z}) - W(\partial_z u_z, \tilde{u})(1) \cdot Y. \]

To compute the Wronskians, we use the following identity, true for all functions \( w_1, w_2 \) and each \( z \in \mathbb{C} \):
\[ h^2 \partial_x W(w_1, w_2) = w_2 \cdot (P(h) - z)w_1 - w_1 \cdot (P(h) - z)w_2. \]

Since \( (P(h) - z)\tilde{u} = 0, \) (2.8.11) and (2.8.13) show that
\[ W(\partial_z u_z, \tilde{u})(1) = h^{-2} \int_0^1 \tilde{u}(x)^2 dx. \]

Similarly
\[ W(\partial_z u_z, \tilde{v})(1) = h^{-2} \int_0^1 \tilde{v}(x)\\tilde{v}(x) dx. \]

Since \( \partial_z u_z(0) = \partial_z u_z'(0) = 0 \), the bound (2.8.7) gives
\[ |W(\partial_z u_z, \tilde{v})(1)| \leq Ch^{-2}. \]

Inserting the estimate \( |X(\\tilde{z})| \leq Ce^{-1/2h} \) in (2.8.12) shows that
\[ \partial X(\\tilde{z}) = -\left(h^{-2} \int_0^1 \tilde{u}(x)^2 dx\right)Y(\\tilde{z}) + O(e^{-1/(2h)}). \]

Calculating
\[ \int_0^1 \tilde{u}_j(x)^2 dx = h^{\frac{1}{2}} \int_0^{1/h} e^{-y^2} dy = h^{\frac{1}{2}}\pi^{\frac{1}{2}}(1 + O(e^{-1/h})), \]
completes the proof. \( \square \)
2.8. SEMICLASSICAL STUDY OF RESONANCES IN DIMENSION ONE

To bound the second derivative $\partial^2_x X$ we need to estimate how fast solutions to the initial value problem for the equation $(P(h) - z)u = 0$ can grow:

**Lemma 2.27.** Let $C_0$ be a fixed constant. Assume that $z$ satisfies

\begin{equation}
1 \leq \text{Re} \, z \leq 1 + C_0 h, \quad |\text{Im} \, z| \leq C_0 h^2.
\end{equation}

Then there exists a constant $C_1$, depending on $C_0$, such that for each $u \in C^\infty([0,1])$, $f = (P(h) - z)u$, and each $x \in [0,1]$, \( e^{-x^2/2h} |u(x)| + |hu'(x) - xu(x)| \)

\begin{equation}
\leq C_1 \left(h^{1/2} |u(0)| + h |u'(0)| + h^{-1} \|e^{-y^2/2h} f(y)\|_{L^2(0,x)}\right).
\end{equation}

**Proof.** 1. Put \( v(x) := e^{-x^2/2h} u(x), \quad g(x) := e^{-x^2/2h} f(x) \), so that

\[-h^2 v'' - 2x hv' + (1 - h - z)v = g,\]

and (2.8.15) becomes

\begin{equation}
1/2 |v(x)| + h |v'(x)| \leq C \left(h^{1/2} |v(0)| + h |v'(0)| + h^{-1} \|g\|_{L^2(0,x)}\right).
\end{equation}

2. We now put $h + \text{Re} \, z - 1 = \nu h$, where $\nu \geq 1$ is bounded uniformly in $h$ and independent of $x$. We have the following estimate valid for $x \geq 0$:

\[
\frac{1}{2} \partial_x (h^2 |v'|^2 + h \nu |v|^2) = \text{Re}(\overline{v} (h^2 v'' + \nu v))
\]

\[= -2xh |v'|^2 + \text{Im} \, z \text{Im}(v \overline{v'}) - \text{Re}(g \overline{v'}) \]

\[\leq C(h^2 |v'|^2 + h^2 |v|^2 + h^{-2} |g|^2) \]

\[\leq C(h^2 |v'|^2 + h^2 |v|^2 + h^{-2} |g|^2).
\]

Since $\partial_x F \leq CF + G$ implies $F(x) \leq e^{C x} F(0) + \int_0^x e^{C(x-y)} G(y) dy$ (Gronwall’s inequality) we arrive to (2.8.16) for $x \in [0,1]$.

With Lemma 2.27 the bound on the second derivative is easy:

**Lemma 2.28.** Assume that $z$ satisfies (2.8.14). Then for some constant $C$ we have

\[|\partial^2_x X(z)| \leq Ch^{-2}e^{1/2h}.
\]

**Proof.** This follows directly by applying Lemma 2.27 to (2.8.5), (2.8.11), and the equation

\[(P(h) - z)\partial^2_x u_z = 2\partial_x u_z, \quad \partial^2_x u_z(0) = 0, \quad \partial^2_x u_z'(0) = 0\]
We can now give the proof of Theorem 2.25. The basic idea is to solve equation (2.8.6) by Newton’s method. From Lemma 2.26 we know that the first derivative at $\tilde{z}$ is exponentially large and Lemma 2.28 provides an estimate for the second derivative in a neighbourhood of $\tilde{z}$. Hence we expect that there is a zero exponentially close to $\tilde{z}$ and the proof below shows that this is in fact the case.

**Proof of Theorem 2.25.**

1. Put
   \[ \Theta(z) := X_2(z) - i\sqrt{z}X_1(z), \]
   so that by (2.8.6), $z$ is a resonance if and only if $\Theta(z) = 0$. Using the explicit formula for the ground state (2.8.3) we see that
   \[ \Theta(\tilde{z}) = -(1 + i)e^{-1/2h}(1 + O(h)). \]
   This, Lemma 2.26 and (2.8.10) give
   \[ \frac{\partial_z \Theta(\tilde{z})}{\Theta(\tilde{z})} = (1 + i)^{-1}e^{1/2h}(1 + O(h))h^{-\frac{1}{2}}(Y_2(\tilde{z}) - iY_1(\tilde{z}) + O(h)|Y(z)|) \]
   \[ = 2^{-1}i\pi^{\frac{1}{2}}h^{-\frac{1}{2}}e^{1/h}(1 + O(h)). \]

2. The equation $\Theta(z) = 0$ is equivalent to $z = \Psi(z)$, where
   \[ \Psi(z) := z - \frac{\Theta(z)}{d_z \Theta(\tilde{z})}. \]
   Take large $N$ and define the disc
   \[ \Omega := \{|z - \tilde{z}| \leq h^N\}. \]
   Then by Lemma 2.28,
   \[ |\partial_z \Theta(z) - \partial_z \Theta(\tilde{z})| \ll |\partial_z \Theta(\tilde{z})|, \quad z \in \Omega, \]
   and thus
   \[ |\partial_z \Psi(z)| \leq 1/2, \quad z \in \Omega. \]
   Then $\Psi : \Omega \to \Omega$ and $|\Psi(z) - \Psi(z')| \leq |z - z'|/2$ for $z, z' \in \Omega$. By contraction mapping principle, the equation $z = \Psi(z)$ has a unique solution $z$ in $\Omega$; this $z$ is then the unique resonance in $\Omega$.

3. To see the asymptotic expansion for $z$, we use that it is the limit of the sequence $z^{(k)}$ defined by
   \[ z^{(0)} = \tilde{z}, \quad z^{(k+1)} = \Psi(z^{(k)}). \]
It is then enough to prove that $z = z^{(1)} + \mathcal{O}(h^{3/2}e^{-1/h})$. Since $|z^{(k+1)} - z^{(k)}| \leq |z^{(k)} - z^{(k-1)}|/2$,

$$|z - z^{(1)}| = \left| \sum_{k=1}^{\infty} (z^{(k+1)} - z^{(k)}) \right| \leq 2|z^{(2)} - z^{(1)}|,$$

and it suffices to show that

$$z^{(2)} = z^{(1)} + \mathcal{O}(h^{3/2}e^{-1/h}).$$

But that is the same as showing that $\Theta(z^{(1)}) / \partial_z \Theta(z^{(1)}) = \mathcal{O}(h^{3/2}e^{-1/h})$. The definition of $z^{(1)}$, Lemma 2.28 and the expressions for $\Theta(\tilde{z})$ and $\partial_z \Theta(\tilde{z})$ is step 1, show that

$$\Theta(z^{(1)}) = \frac{\Theta(\tilde{z}) + \partial_z \Theta(\tilde{z})(z^{(1)} - \tilde{z}) + \mathcal{O}(h^{-2}e^{1/2h})(z^{(1)} - \tilde{z})^2}{\partial_z \Theta(\tilde{z}) + \mathcal{O}(h^{-2}e^{1/2h})(z^{(1)} - \tilde{z})}$$

$$= \frac{\mathcal{O}(h^{-2}e^{1/2h}(h^{3/2}e^{-1/h})^2)}{2^{-1}(1 - i)h^{-1/2}u_x^{1/2h}(1 + \mathcal{O}(h))}$$

$$= \mathcal{O}(h^{-1/2}e^{-2/h}),$$

which is an even stronger estimate. This completes the proof of Theorem 2.25.

### 2.8.2. A general bound on resonance width.

The next result is an easy one dimensional version of a theorem due to Burq (see §6.5 below). It states that for any compactly supported potential the modulus of the imaginary part of the resonance is bounded from below by $\exp(-C/h)$. Theorem 2.25 shows that this bound is optimal. Except for the simple characterization of outgoing solutions, we avoid the use of one dimensional methods to prepare the reader for the proof of the higher dimensional case in §6.5.

To estimate the imaginary we use the following basic fact. Suppose that $V \in L^\infty(\mathbb{R}; \mathbb{R})$ and that $u \in H^2([-R, R])$ solves

$$(hD)_x^2 + V(x) - z)u(x) = 0, \quad x \in [-R, R], \quad z \in \mathbb{C}.$$ 

Then

$$\text{Im} \ z \int_{-R}^{R} |u(x)|^2 \, dx = -h^2 \text{Im} \ u_x \bigg|_{-R}^{R}.$$
Proof of (2.8.17). Since \( u \in C^1([-R, R]) \), the following integration by parts argument is justified:

\[
0 = \int_{-R}^{R} \left( (hD_x)^2 + V - z \right) u \bar{u} - u((hD_x)^2 + V - z) \bar{u} \ dx
= \int_{-R}^{R} (hD_x)^2 u \bar{u} - u(hD_x)^2 \bar{u} \ dx - (z - \bar{z}) \int_{-R}^{R} |u|^2 \ dx
= -h^2 u \bar{u} \bigg|_{-R}^{R} + h^2 u \bar{u} \bigg|_{-R}^{R} - (z - \bar{z}) \int_{-R}^{R} |u|^2 \ dx.
\]

The formula (2.8.17) follows from dividing this by 2i.

Before we use (2.8.17) to estimate resonance width we need the following simple lemma:

**Lemma 2.29.** Suppose that \( M > 0 \). Then for \( u \in H^2_{\text{comp}}(\mathbb{R}) \) we have

\[
\| e^{-Mx/h} (hD_x)^2 e^{Mx/h} u \|_{L^2} \geq M^2 \| u \|_{L^2}.
\]

**Proof.** We define the semiclassical Fourier transform

\[
\mathcal{F}_h u(\xi) := \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} u(x)e^{-i\xi x/h} \ dx,
\]

and recall its basic properties:

\[
\mathcal{F}_h (hD_x u)(\xi) = \xi \mathcal{F}_h u(\xi), \quad \| u \|_{L^2(\mathbb{R})} = \| \mathcal{F}_h u \|_{L^2(\mathbb{R})},
\]

see for instance [Zw12, §3.3]. We then have

\[
\| e^{-Mx/h} (hD_x)^2 e^{Mx/h} u \|_{L^2} = \| (hD_x - iM)^2 u \|_{L^2}
= \| (\xi - iM)^2 \mathcal{F}_h u \|_{L^2}
\geq M^2 \| \mathcal{F}_h u \|_{L^2} = M^2 \| u \|_{L^2},
\]

where we used the fact that

\[
| (\xi - iM)^2 | = | \xi - iM |^2 = | \xi |^2 + M^2 \geq M^2.
\]

We are now ready to prove

**Theorem 2.30** (Lower bounds on resonance width in dimension one). Suppose that \( P(h) := -h^2 \Delta + V, V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R}) \) and that \( E > 0 \). Then there exists \( c = c(V, E) \) such that for \( 0 < h < h_0 \),

\[
\text{Re} \ z \in [E/2, E], \ z \in \text{Res}(P(h)) \implies | \text{Im} \ z | > e^{-c/h}.
\]

**Proof.** 1. Suppose \( z \) is a resonance with \( E/2 \leq \text{Re} \ z \leq E \). In view of Theorem 2.2 this means that there exists \( u \), a resonant state, satisfying

\[
(hD_x)^2 + V - z)u = 0, \ u(x) = A_{\pm} e^{\pm i \sqrt{2}z x/h}, \ \pm x \gg 1, \ A_{\pm} \neq 0.
\]
In view of (2.8.17) the lower bound (2.8.19) will follow from showing that for some $R$,
\begin{equation}
\int_{-R}^{R} |u(x)|^2 \, dx \leq C e^{c/h} |\text{Im} \, u_x \, \bar{u}|_{-R}^R.
\end{equation}

2. We can assume that $\text{Im} \, z > -h$ as otherwise there is nothing to prove. Note that as $\text{Re} \, z \geq E/2$ this implies that $\text{Im} \, \sqrt{z} > -h/C$. Hence for $R$ sufficiently large,
\begin{equation}
|\text{Im} \, u_x \, \bar{u}|_{-R}^R = \text{Re} \, z (|A_+|^2 + |A_-|^2) e^{-2R \text{Im} \, \sqrt{z}/h}
\geq \frac{1}{C} \int_{R}^{R+1} |A_+ e^{i\sqrt{z}/h}|^2 \, dx + \frac{1}{C} \int_{-R}^{-R-1} |A_- e^{i\sqrt{z}/h}|^2 \, dx
= \frac{1}{C} \int_{R \leq |x| \leq R+1} |u(x)|^2 \, dx,
\end{equation}
where the constant $C$ depends on $R$ and $E$.

3. From (2.8.21) we see that (2.8.20) follows from the estimate
\begin{equation}
\int_{-R}^{R} |u(x)|^2 \, dx \leq C e^{c/h} \int_{R \leq |x| \leq R+1} |u(x)|^2 \, dx.
\end{equation}
(This is easy by using ODE methods but we proceed in a more complicated way indicating some aspects of higher dimensional methods such as (6.5.27).)

To establish (2.8.22) we use Lemma 2.29: for $M^2 > \|V\|_{L^\infty} + 1$, (2.8.18) shows that for $v \in H^2_{\text{comp}}$,
\begin{equation}
\|e^{-Mx/h}(hD_x)^2 + V) e^{Mx/h} v\|_{L^2} \geq \|e^{-Mx/h}(hD_x)^2 e^{Mx/h} v\|_{L^2} - \|V v\|_{L^2}
\geq (M^2 - \|V\|_{L^\infty}) \|v\|_{L^2} \geq \|v\|_{L^2}.
\end{equation}

We apply this to $v = e^{Mx/h} \chi u$ where $\chi \in C^\infty([-R+1, R-1); [0,1])$ is equal to 1 on $[-R+\epsilon, R-\epsilon]$:
\begin{equation}
\int_{-R}^{R} |u(x)|^2 \, dx \leq e^{2MR/h} \|e^{-Mx/h} \chi u\|_{L^2(\mathbb{R})}^2
\leq e^{2MR/h} \|e^{-Mx/h} (hD_x)^2 + V) \chi u\|_{L^2}^2
\leq 2e^{4MR/h} \|\chi' hD_x u\|_{L^2}^2
\leq C e^{4MR/h} \int_{R \leq |x| \leq R+1} |u(x)|^2 \, dx,
\end{equation}
where the last inequality follows from the fact that $\text{supp} \, \chi' \subset \{R \leq |x| \leq R+1\}$ and $u(x) = A_\pm e^{i\sqrt{z}/h} |x|$ there. Combining this estimate with (2.8.23) gives (2.8.22) completing the proof. □
2.9. NOTES

For more information about the structure of the resolvent at $\lambda = 0$ see Jensen–Nenciu [JN01] and references given there.

Theorem 2.14 was proved in some special cases in Regge [Re58] and in general (for $V \in L^1_{\text{comp}}(\mathbb{R}; \mathbb{R})$) in [Zw87]. Different proofs were given by Froese [Fr97] and Simon [Si00]. Here we followed [Fr97], where complex valued potentials were allowed. That paper also treats certain non-compactly supported potentials.

The importance of the Carleman estimate (2.6.17), (D.1.10) in one dimensional scattering seems to go back to Selberg [Se53] in scattering on finite volume hyperbolic surfaces. The reason why scattering on finite volume surfaces is effectively one dimensional will be explained in Example 3 in §4.1 (see also Example 3 in §4.2).

For recent advances in the study of resonances for potentials in dimension one see Korotyaev [Ko04], [Ko05], [Ko14] and references given there.

The presentation of complex scaling in Section 2.7 owes a lot to unpublished notes of Kiril Datchev.

For more general one dimensional “well-in-an-island” potentials the resonances were described by Helffer–Sjöstrand [HS86, §11] – see also Servat [Se04] and Dalla Venezia–Martinez [DM17] for more recent accounts and references.

2.10. EXERCISES

Section 2.1

1. Show that in the notation of (2.1.13) we have

$$v_+(z) = -\frac{1}{2} \int_0^\infty \int_{z+\tau}^R (Vv + F)(\tau, y) dy d\tau,$$

$$v_-(z) = -\frac{1}{2} \int_0^\infty \int_{-R}^{z-\tau} (Vv + F)(\tau, y) dy d\tau.$$

Section 2.2

2. Find an approximation for resonances for a step potential,

$$V(x) = \begin{cases} 
0 & |x| > L, \\
V & |x| \leq L,
\end{cases}$$

**Hint:** Use the characterization of a resonant state

$$(D_x^2 + V(x) - \lambda^2)u = 0, \quad u(x) = a_{\pm} e^{i\lambda|x|}, \quad |x| \geq L.$$
2.10. EXERCISES

Section 2.4

3. Use (2.4.13) to show that in the notation of (2.4.12),
\[
\text{(2.10.1) } \det S(\lambda) = \frac{t(\lambda)}{t(-\lambda)}.
\]

4. Suppose that \( V \in \mathcal{C}_c^{\infty}(\mathbb{R}; \mathbb{R}) \). Use Theorem 2.11 to show that
\[
\sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda) = \sum_{j=1}^{J} a_j \lambda^{-j} + \mathcal{O}(\lambda^{-J-1}), \quad \lambda \to \infty,
\]
where
\[
a_1 = -\frac{1}{2\pi} \int V(x) dx, \quad a_2 = \frac{1}{8\pi} \int V(x)^2 dx.
\]
**Hint:** For operators with \( \|A\| < 1 \) and of trace class (see §B.4 for a review of the trace class) we have \( \log \det(I - A) = \text{tr} \log(I - A) = -\sum_{k=1}^{\infty} k^{-1} \text{tr} A^k \) and this can be applied with \( A := -VR_0(\lambda)\rho \) for \( \lambda \gg 1 \). To evaluate the traces split integrals involving \( |x - y| \) to integrals over \( x > y \) and \( y < x \) and integrate by parts. This result is a special case of Theorem 3.67.

Section 2.6

5. Check carefully that the proof of Theorem 2.17 applies when there are negative eigenvalues.

6. Suppose that \( x_0 < x_1 < \cdots < x_N \) and \( V_j \in \mathbb{R}, \ j = 1, \cdots, J \). Define
\[
\text{(2.10.2) } V(x) := \begin{cases} 
0 & x \leq 0 \\
V_j & x_{j-1} < x \leq x_j, \ 0 < j \leq N \\
0 & x > x_N.
\end{cases}
\]
Find an expression for \( S(\lambda) \) using transfer matrices (which should also be computed):
\[
M_{\text{step}}(k_-, k_+) : \begin{pmatrix} a_- \\ b_- \end{pmatrix} \to \begin{pmatrix} a_+ \\ b_+ \end{pmatrix},
\]
\[
u(x) = a_\pm e^{ik_\pm x} + b_\pm e^{-ik_\pm x}, \quad \pm x \geq 0, \quad u \in C^1(\mathbb{R}),
\]
\[
M_{\text{free}}(K) = \begin{pmatrix} e^{iK} & 0 \\ 0 & e^{-iK} \end{pmatrix}.
\]
This method was used to compute the scattering phase for the data in Fig. 2.6.
**Hint:** Compute the transfer matrix for the scattering problem with the potential \( V \):
\[
M_{V}(\lambda) : \begin{pmatrix} a_- \\ b_- \end{pmatrix} \to \begin{pmatrix} a_+ \\ b_+ \end{pmatrix},
\]
$u(x) = u(x) = a_{\pm}e^{ik_{\pm}x} + b_{\pm}e^{-ik_{\pm}x}, \; \pm x \gg 1, \quad (D_x^2 + V(x) - \lambda^2)u = 0$
as a product of $M_{\text{step}}$ and $M_{\text{free}}$ for appropriate choices of the parameters. The scattering matrix sends incoming data to outgoing data:

$$S(\lambda) : \begin{pmatrix} a_- \\ b_+ \end{pmatrix} \rightarrow \begin{pmatrix} a_+ \\ b_- \end{pmatrix}$$

(see (2.4.2) where the notation is different!) and $M_V$ and $S$ can be related.
Chapter 3

SCATTERING RESONANCES IN ODD DIMENSIONS

3.1 Free resolvent in odd dimensions
3.2 Meromorphic continuation
3.3 Resolvent at zero energy
3.4 Upper bounds on the number of resonances
3.5 Complex valued potentials with no resonances
3.6 Outgoing solutions and Rellich’s Theorem
3.7 The scattering matrix
3.8 More on distorted plane waves
3.9 The Birman–Kreǐn trace formula
3.10 The Melrose trace formula
3.11 Scattering asymptotics
3.12 Existence of resonances for real potentials
3.13 Notes

In this section we will consider the simplest higher dimensional situation: scattering by compactly supported potentials in odd dimensions. Some of the results presented in Chapter 2 are valid in this case with proofs requiring only small modifications. Other results, such as asymptotics, or even sharp lower bounds, for the number of scattering poles, are not known.

The main advantage of odd dimensions greater than one is the strong Huyghens principle for the wave equation: if $\Box u = 0$ and the support of
initial data lies in $|x| < R$ then support of $u(t, \bullet)$ lies in $|t| - R < |x| < |t| + R$. The weak Huyghens principle valid in all dimensions says only that the support of $u(t, \bullet)$ lies in $|x| < |t| + R$.

One consequence of the strong Huyghens principle is the analytic continuation of $(-\Delta - \lambda^2)^{-1}$ from $\text{Im} \lambda > 0$ to $\mathbb{C}$.

3. FREE RESOLVENT IN ODD DIMENSIONS

The outgoing resolvent of the free Laplacian is defined just as in the case of dimension one:

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \text{Im} \lambda > 0.$$  

Its existence follows from using the Fourier transform which provides an explicit diagonalization of $-\Delta$:

$$R_0(\lambda) \varphi(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} \frac{1}{|\xi|^2 - \lambda^2} \hat{\varphi}(\xi) d\xi, \quad \text{Im} \lambda > 0,$$

This formula is of course valid in all dimension but the operator $R_0(\lambda)$ has much nicer properties when $n$ is odd.

### 3.1. Relation to the wave equation

We will starts our presentation with the properties of the wave equation. Thus we consider its unique forward fundamental solution:

$$\Box E_+ := (\partial_t^2 - \Delta) E_+ = \delta_0(x) \delta_0(t), \quad \text{supp} \ E_+ \subset \{ t \geq 0 \}.$$  

For $n$ odd we have a particularly nice expression for the distribution $E_+$. Its action on $\varphi \in C^\infty_c(\mathbb{R}_t \times \mathbb{R}^n_x)$ is given by

$$\langle E_+, \varphi \rangle = \int_0^\infty \langle E_+(t), \varphi(t, \bullet) \rangle dt,$$

and we have a distributional convergence $E_+(t) \to \delta_0$ as $t \to 0$ – see [Ev98, §2.4.1] or [HöI, Section 6.2].

The crucial fact seen from this expression is the support property of $E_+$: for odd $n \geq 3$

$$\text{supp} \ E_+ = \{(x, t) : |x| = |t|, t \geq 0\}.$$
This implies the strong Huyghens principle:
\[
\Box u = f, \quad \text{supp } f \subset [-R, R]_t \times B_{\mathbb{R}^n}_R(0, R), \quad u|_{t=-R} = 0 \quad \Rightarrow \quad u(t, x) = 0, \quad \text{for } |x| < t - 2R.
\]

The weak Huyghens principle valid in all dimensions says that \(u(t, x) = 0\) for \(|x| > t + 2R\).

The distribution \(E_+(t)\) appearing in (3.1.4) is used to solve the initial value problem:
\[
\Box u = 0, \quad u(0, x) = \varphi_0(x), \quad \partial_t u(0, x) = \varphi_1(x),
\]
\[
\begin{align*}
 u(t, x) &= E_+(t) * \varphi_1(x) + \partial_t E_+(t) * \varphi_0(x), \quad \varphi_j \in C^\infty(\mathbb{R}^n), \quad t \geq 0.
\end{align*}
\]
Here \(u * v\) denotes the convolution of a compactly supported distribution \(u\) with a smooth function \(v\). Putting \(E(t) = -E_+(-t), \quad t < 0, \quad E(0) = \delta_0\), gives
\[
E(t) \in C^\infty(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^n)).
\]

The solution of (3.1.6) can also be given using the spectral decomposition of \(-\Delta\) and the functional calculus – this corresponds to the Fourier transform decomposition:
\[
\begin{align*}
 u(t, x) &= \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} \varphi_1(x) + \cos(t\sqrt{-\Delta}) \varphi_0(x), \\
 f(\sqrt{-\Delta}) \varphi(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(|\xi|) e^{i(x, \xi)} \hat{\varphi}(\xi) d\xi,
\end{align*}
\]
where \(f(\rho) = \sin t\rho/\rho\) or \(f(\rho) = \cos t\rho\).

If we write \(U(t) := \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}\), then a comparison with (3.1.6) gives the Schwartz kernel of \(U(t)\):
\[
U(t, x, y) = E(t, x - y),
\]
see [HöI, Section 6.1] for the details on the pull back (by \((x, y) \mapsto x - y\) here) of distributions.

The strong Huyghens principle (3.1.5) implies that
\[
(U(t)v)(x) = 0, \quad t > \text{sup}\{|x - y|: y \in \text{supp } v\}.
\]

For future reference we note that the spectral representation immediately gives
\[
\partial_t^k U(t) : H^s(\mathbb{R}^n) \rightarrow H^{s-k+1}(\mathbb{R}^n), \quad k \in \mathbb{N}, \quad s \in \mathbb{R}.
\]
3. FREE RESOLVENT IN ODD DIMENSIONS

The free resolvent $R_0(\lambda)$ given by (3.1.2) can be written using $U(t)$:

\[(3.1.11)\quad R_0(\lambda) = \int_0^\infty e^{i\lambda t} U(t) dt.\]

In fact, for $\text{Im}\lambda > 0$,

\[\frac{1}{|\xi|^2 - \lambda^2} = \int_0^\infty \frac{\sin t|\xi|}{|\xi|} e^{it\lambda} dt, \quad \text{Im}\lambda > 0,\]

where the integral converges since $\sup_{\lambda \in \mathbb{R}} |\sin t\lambda/\lambda| = |t|$. The formula (3.1.11) then follows from (3.1.2) and (3.1.7).

This representation gives us the following important result:

**THEOREM 3.1 (Free resolvent in odd dimensions).** Suppose that $n \geq 3$ is odd. Then the resolvent defined by

\[R_0(\lambda) = (\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\]

for $\text{Im}\lambda > 0$, continues analytically to an entire family of operators

\[R_0(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2_{\text{loc}}(\mathbb{R}^n).\]

For any $\rho \in C_c^\infty(\mathbb{R}^n)$ we have the following estimates:

\[(3.1.12)\quad \rho R_0(\lambda) \rho = O((1 + |\lambda|)^{-1} e^{L(\text{Im}\lambda)} ) : L^2(\mathbb{R}^n) \to H^j(\mathbb{R}^n),\]

\[j = 0, 1, 2, \text{ where } L > \text{diam supp } \rho := \sup\{|x - y| : x, y \in \text{supp } \rho\}.\]

**Proof.** 1. For the statement about holomorphy it suffices show that for any $\rho \in C_c^\infty(\mathbb{R}^n)$,

\[\rho R_0(\lambda) \rho : L^2 \to L^2\]

continues from $\text{Im}\lambda > 0$ to an entire family of bounded operators.

2. If $L > \text{diam supp } \rho$ then (3.1.9) gives $\rho U(t) \rho = 0$ for $t \geq L$. Then (3.1.11) shows that, for $\text{Im}\lambda > 0$ at first,

\[(3.1.13)\quad \rho R_0(\lambda) \rho = \int_0^L e^{i\lambda t} \rho U(t) \rho dt.\]

The right hand side is now defined and, as an operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, holomorphic for $\lambda \in \mathbb{C}$.

3. Since $U(t) = \sin t\sqrt{-\Delta}/\sqrt{-\Delta}$, and $\sup_{\lambda \in \mathbb{R}} |\sin t\lambda/\lambda| = |t|$, we have

\[
\|U(t)\|_{L^2 \to H^1} \leq C\|U(t)\|_{L^2 \to L^2} + C\|\sqrt{-\Delta}U(t)\|_{L^2 \to L^2}
= O(|t| + 1).
\]

This and (3.1.13) give the bound (3.1.12) for $j = 1$. For $j = 0$ we write

\[
\lambda \rho R_0(\lambda) \rho = \int_0^L D_t(e^{i\lambda t}) \rho U(t) \rho dt = - \int_0^L e^{i\lambda t} \rho D_t U(t) \rho dt.
\]
3.1. FREE RESOLVENT IN ODD DIMENSIONS

We have
\[ D_t U(t) = -i \cos t \sqrt{-\Delta} = \mathcal{O}_{L^2 \to L^2}(1), \]
and the bound (3.1.12) for \( j = 0 \) follows.

4. Finally, we consider (3.1.12) for \( j = 2 \). Suppose that \( \rho_1 \in C_c^\infty(\mathbb{R}^n) \) satisfies
\[ \rho_1 = 1 \text{ near } \text{supp } \rho, \quad \text{diam}(\text{supp } \rho_1) < L. \]
Since \((-\Delta - \lambda^2) R_0(\lambda) = I\), we have
\[
\| \rho R_0(\lambda) \rho \|_{L^2 \to H^2} \leq C \| \Delta \rho R_0(\lambda) \rho \|_{L^2 \to L^2} + C \| \rho R_0(\lambda) \rho \|_{L^2 \to L^2} \\
\leq C \| \rho \Delta R_0(\lambda) \rho \|_{L^2 \to L^2} + C \| \Delta, \rho \|_{L^2 \to L^2} \rho R_0(\lambda) \rho \|_{L^2 \to L^2} \\
+ C \| \rho R_0(\lambda) \rho \|_{L^2 \to L^2} \\
\leq C |\lambda|^2 \| \rho R_0(\lambda) \rho \|_{L^2 \to L^2} + C \| \rho_1 R_0(\lambda) \rho_1 \|_{L^2 \to H^1} \\
+ C \| \rho R_0(\lambda) \rho \|_{L^2 \to L^2} + C,
\]
for some constants \( C \) (which may change from line to line). Hence (3.1.12) for \( j = 2 \) follows from the estimates for \( j = 0, 1 \). \( \square \)

The wave equation representation and the formula for \( E(t) \) (and hence \( U(t) \) in view of (3.1.8)) given in (3.1.4) can be used to derive an explicit formula for the Schwartz kernel of \( R_0(\lambda) \), \( R_0(\lambda, x, y) \). Instead we take a direct approach based on the Fourier transform representation (3.1.2).

3.1.2. An explicit formula for \( R_0(\lambda) \) in odd dimensions. We start with the following

**Lemma 3.2** (Oscillatory integrals over \( S^{n-1} \)). Suppose that \( n \geq 3 \) is odd and \( d\omega \) denotes the standard measure on \( S^{n-1} \) (induced from the Lebesgue measure on \( \mathbb{R}^n \), \( S^{n-1} := \{ x : |x| = 1, x \in \mathbb{R}^n \} \)).

Then for \( \zeta \in \mathbb{R} \), and \( x \in \mathbb{R}^n \),
\[
(3.1.14) \quad \int_{S^{n-1}} e^{i\zeta(\omega, x)} d\omega = 2\pi^{n-1/2} \left( e^{i|\zeta||x|} F_n(\zeta|x|) + e^{-i|\zeta||x|} F_n(-\zeta|x|) \right),
\]
where \( F_n(k) \) is given by
\[
(3.1.15) \quad F_n(k) := e^{-2ik} \left( -\partial_k \right)^{n-3/2} \left( e^{2ik/ik^{n-1/2}} \right),
\]

**Remark.** The integral in (3.1.14) can be expressed using the Bessel function \( J_{n-2} \) but we take a direct approach.
Proof. 1. It is clear that the right hand side is a function of $\zeta$ and $|x|$. Hence we can assume that $x = |x|e_1$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Using the parametrization of $S^{n-1}$ by $B_{\mathbb{R}^n-1}(0, 1)$:

$$\omega = (\pm \sqrt{1 - r^2}, r\theta), \quad \theta \in S^{n-2}, \quad 0 \leq r \leq 1,$$

and putting $c_n := \text{vol}(S^{n-2})$ and $k := \zeta|x|$, we have

$$\int_{S^{n-1}} e^{ik\langle \omega, e_1 \rangle} d\omega = c_n \int_0^1 \left( e^{ik\sqrt{1 - r^2}} + e^{-ik\sqrt{1 - r^2}} \right) \frac{r^{n-2}}{\sqrt{1 - r^2}} dr$$

$$= 2c_n \int_0^1 \cos(ky)(1 - y^2)^{\frac{n-3}{2}} dy$$

$$= 2c_n (1 + \partial_k^2)^{\frac{n-3}{2}} \int_0^1 \cos(ky) dy =$$

$$= c_n (1 + \partial_k^2)^{\frac{n-3}{2}} \left( \frac{e^{ik} - e^{-ik}}{ik} \right).$$

2. Putting $D_k := (1/i)\partial_k$, we factorize

$$(1 + \partial_k^2)^{\frac{n-3}{2}} = (1 + D_k)^{\frac{n-3}{2}} (1 - D_k)^{\frac{n-3}{2}}$$

$$= (e^{-ik} D_k e^{ik})^{\frac{n-3}{2}} (-e^{ik} D_k e^{-ik})^{\frac{n-3}{2}}$$

$$= e^{-ik} D_k^{\frac{n-3}{2}} e^{2ik} (-D_k)^{\frac{n-3}{2}} e^{-ik}$$

$$= e^{-ik} (-\partial_k)^{\frac{n-3}{2}} e^{2ik} (-\partial_k)^{\frac{n-3}{2}} e^{-ik}.$$ 

Hence

$$(1 + \partial_k^2)^{\frac{n-3}{2}} \left( \frac{e^{ik}}{ik} \right) = e^{-ik} (-\partial_k)^{\frac{n-3}{2}} e^{2ik} (-\partial_k)^{\frac{n-3}{2}} (1/ik)$$

$$= (\frac{n-2}{2})! e^{-ik} (-\partial_k)^{\frac{n-3}{2}} \left( e^{2ik} / ik^{n-1} \right),$$

with the action on $-e^{-ik}/ik$ obtained by taking complex conjugates. We now recall that $c_n = 2\pi^{(n-1)/2} / (\frac{n-3}{2})!$ which gives (3.1.14).

From Lemma 3.2 we obtain a formula for the the Schwartz kernel of $R_0(\lambda)$:

**THEOREM 3.3 (Schwartz kernel of the resolvent in odd dimensions).** Suppose that $n \geq 3$ is odd. Then the Schwartz kernel of the resolvent $R_0(\lambda)$ defined in Theorem 3.1 is given

$$R_0(\lambda, x, y) = \frac{e^{i\lambda|x-y|}}{|x-y|^{n-2}} P_n(\lambda|x-y|),$$

$$P_n(k) := i2^{-n+1}n^{-\frac{n-1}{2}}k^{n-2}F_n(k),$$

$$F_n(k) := i2^{-n+1}n^{-\frac{n-1}{2}}k^{n-2}F_n(k),$$

where $F_n(k)$ is the normalized $n$-th Legendre polynomial.
where \( F_n \) is given in (3.1.14); \( P_n \) is a polynomial of degree \((n - 3)/2\) with

\[
P_n(0) = \frac{(n - 3)!}{\pi^{n/2} 2^{n-1}(\frac{n-3}{2})!},
\]

(3.1.17)

\[
\left( z^{-\frac{n-3}{2}} P_n(1/z) \right) \big|_{z=0} = \frac{1}{4\pi (2\pi i)^{\frac{n-3}{2}}}.
\]

**Remark.** Formulæ (3.1.17) give the extremal coefficients of the polynomial \( P_n(z) \): \( P_n(0) \) determines the leading asymptotic of \( R_0(\lambda, x, y) \) as \(|x - y| \to 0\) and, \((z^{-\frac{n-3}{2}} P_n(1/z))_{z=0} \), the leading asymptotic as \(|x - y| \to \infty\).

In dimension \( n = 3 \), (3.1.16) takes the simple form

\[
R_0(\lambda, x, y) = e^{i\lambda|x-y|} \frac{1}{4\pi|x-y|},
\]

Proof. 1. We prove the formula for \( \text{Im} \lambda > 0 \) and continue both sides analytically in \( \lambda \). We start by rewriting (3.1.2) using polar coordinates \( \xi = \zeta \omega, \omega \in S^{n-1}, \zeta \in \mathbb{R} \) and Lemma 3.2:

\[
R_0(\lambda, x, y) = \frac{1}{(2\pi)^n} \int_0^\infty \int_{S^{n-1}} e^{i\zeta(\omega, x-y)} \frac{d\omega}{\zeta^2 - \lambda^2} d\zeta^n \frac{1}{\zeta^2 - \lambda^2} d\zeta
\]

\[
= \frac{1}{2(2\pi)^n} \int_{S^{n-1}} e^{i\zeta(\omega, x-y)} \frac{d\omega}{\zeta^2 - \lambda^2} d\zeta
\]

\[
= \frac{1}{2^n\pi^{\frac{n+1}{2}}} \int_\mathbb{R} e^{i|z-y|} F_n(\zeta|x-y|) \left( \frac{\zeta^{n-1}}{\zeta^2 - \lambda^2} \right) d\zeta + \frac{1}{2^n\pi^{\frac{n+1}{2}}} \int_\mathbb{R} e^{-i|z-y|} F_n(-\zeta|x-y|) \left( \frac{\zeta^{n-1}}{\zeta^2 - \lambda^2} \right) d\zeta.
\]

We note that switching to the integral over \((0, \infty)\) to the integral over \( \mathbb{R} \) was justified as for odd \( \zeta^{n-1} = (-\zeta)^{n-1} \).

2. For \(|x - y| \neq 0\), the functions \( F_n(\pm \zeta|x-y|)\zeta^{n-1} \) are holomorphic in \( \zeta \) and hence we can deform the contours in the the two integrals on the right hand side: to \( \text{Im} \zeta = N \to \infty \) and \( \text{Im} \zeta = -N \to \infty \), respectively. For the first integral we obtain a contribution from the pole at \( \zeta = \lambda \) and for the second from the pole at \( \zeta = -\lambda \). The residue theorem then gives

\[
R_0(\lambda, x, y) = 2^{-n+1} \pi^{-\frac{n-1}{2}} e^{i\lambda|x-y|} \lambda^{n-2} F_n(\lambda|x-y|),
\]

(3.1.18)

and this gives (3.1.16).
To compute the highest and lowest coefficients of $P_n$ we use (3.1.15) which gives

$$P_n(0) = \pi^{\frac{n-1}{2}} 2^{-n+1} \left(\frac{n-1}{2}\right) \left(\frac{n-1}{2} + 1\right) \cdots \left(\frac{n-1}{2} + \frac{n-5}{2}\right) = \pi^{\frac{n-1}{2}} 2^{-n+1} \frac{(n-3)!}{\left(\frac{n-3}{2}\right)!}.$$ 

Similarly we obtain the formula for the highest coefficient of $P_n(z)$. □

We finish with an explicit version of Stone’s formula (B.1.13) for the free Laplacian in odd dimensions:

**Theorem 3.4 (Stone’s formula for the free Laplacian).** Suppose $n$ is odd and $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, $\text{Im} \lambda > 0$. Then the analytic continuation of the Schwartz kernel $R_0(\lambda, x, y)$ satisfies

$$(3.1.19) \quad R_0(\lambda, x, y) - R_0(-\lambda, x, y) = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{S^{n-1}} e^{i\lambda \langle \omega, x - y \rangle} d\omega, \quad \lambda \in \mathbb{C},$$

where $d\omega$ denotes the standard measure on $S^{n-1}$.

**Remark.** We refer to (3.1.19) as Stone’s formula as it is special case of a formula valid for all self-adjoint operators – see Theorem B.8. The right hand side is related to the spectral measure of $-\Delta$ obtained using the Fourier transform and the left hand side is the difference of boundary values of resolvents at the real axis: for $\lambda > 0$,

$$R_0(\lambda) = \lim_{\epsilon \to 0^+} (-\Delta - \lambda^2 - i\epsilon)^{-1}, \quad R_0(-\lambda) = \lim_{\epsilon \to 0^+} (-\Delta - \lambda^2 + i\epsilon)^{-1},$$

where the limits are taken in the sense of operators $C^\infty_c(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ (with finer formulations possible).

**Proof.** We could use contour deformation starting with (3.1.2) as in §3.1.4 below but we can use (3.1.18) and Lemma 3.2.

Indeed, from (3.1.18) we see that

$$R_0(\lambda, x, y) - R_0(-\lambda, x, y) = i 2^{-n+1} \pi^{\frac{n-1}{2}} \lambda^{n-2} (e^{ik F_n(k)} + e^{-ik F_n(-k)})|_{k = \lambda|x-y|},$$

which combined with (3.1.14) gives (3.1.19). □
3.1.3. Asymptotic behaviour of $R_0(\lambda)f$. The next theorem gives asymptotics of $R_0(\lambda)f$, $f \in \mathcal{E}'(\mathbb{R}^n)$ as $|x| \to \infty$ for $\lambda \neq 0$.

This result does not depend on the parity of the dimension.

**THEOREM 3.5 (Outgoing asymptotics).** Suppose that $n > 1$ and $f \in \mathcal{E}'(\mathbb{R}^n)$ is a compactly supported distribution (or $f \in \mathcal{S}(\mathbb{R}^n)$).

Then for $\lambda \in \mathbb{R} \setminus 0$,

$$R_0(\lambda)f(|x|\theta) = e^{i\lambda|x||x|^{-\frac{n-1}{2}}} h(|x|, \theta),$$

(3.1.20)

$$h(x, \theta) = \sum_{j=0}^{\infty} |x|^{-j}h_j(\theta), \quad h_0(\theta) = \frac{1}{4\pi} \left( \frac{\lambda}{2\pi i} \right)^{\frac{1}{2}(n-3)} \hat{f}(\lambda \theta),$$

as $|x| \to \infty$.

**Proof.** 1. The proof is based on the formula (3.1.16) for the Schwartz kernel of $R_0(\lambda)$ and the following expansions valid as $|x| \to \infty$:

(3.1.21)

$$|x - y| = |x|(1 - 2<x|x|, y)/|x| + |y|^2/|x|^2)^{\frac{1}{2}}$$

$$= |x| - <x|x|, y> + (|y|^2/2 - <x|x|, y)^2/4)/|x| + \cdots$$

$$= |x| - <x|x|, y> + \sum_{k=1}^{K-1} a_k(y, x/|x|)|x|^{-k} + \mathcal{O}(|y|^{K+1}|x|^{-K}).$$

The last bound is valid for $y \in \mathbb{R}^n$, $|x| > 1$ and $|a_k(y, \omega)| \leq C_k|y|^{k+1}$.

Similarly,

(3.1.22)

$$|x - y|^{-p} = |x|^{-p}(1 + 2<x|x|, y)/|x| + |y|^2/|x|^2)^{-p}$$

$$= |x|^{-p}(1 + (1 + 2p<x|x|, y)/|x| + \cdots$$

$$= |x|^{-1}(1 + \sum_{k=1}^{K-1} b_k(y, x/|x|)|x|^{-k} + \mathcal{O}(|y|^K|x|^{-K}),$$

where $|b_k(y, \omega)| \leq C_k|y|^k$.

2. We now use (3.1.16) and (3.1.17) (see the remark following the theorem) to write

$$R_0(\lambda, x, y) = \frac{1}{4\pi} \frac{\lambda^\frac{n-3}{2}}{2\pi i} \frac{e^{i\lambda|x-y|}}{|x-y|^{\frac{n+1}{2}}} (1 + \cdots + a_n|x-y|^{-\frac{n+3}{2}}).$$

Pairing this with $f(y) \in \mathcal{E}'(\mathbb{R}^n)$ (or integrating against $f(y) \in \mathcal{S}(\mathbb{R}^n)$) and using expansions (3.1.21) and (3.1.22) gives (3.1.20). □
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3.1.4. Continuation of the resolvent using contour deformation.

We will now consider another way of continuing the resolvent kernel $R_0(\lambda, x, y)$. To streamline the notation we will write

$$R_0(\lambda, x, y) = R_0(\lambda, x - y),$$

and we think of $R_0(\lambda, x)$ as a distribution in the $x$ variables. Then for $\varphi \in C_c^\infty(\mathbb{R}^n)$ (for any $n$, odd or even)

$$R_0(\lambda)(\varphi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\hat{\varphi}(\xi)}{|\xi|^2 - \lambda^2} d\xi, \quad \text{Im} \lambda > 0,$$

where the left hand side is understood as the distributional pairing of $R_0(\lambda)(x)$ and $\varphi(x)$.

**REMARK.** The spectral representation (3.1.23) immediately implies bounds on the free resolvent for $\text{Im} \lambda > 0$:

$$\|R_0(\lambda)\|_{L^2 \to H^k} \simeq \sup_{t > 0} \frac{(1 + t)^{k/2}}{|t - \lambda^2|} \leq C' \frac{\langle \lambda \rangle^k}{|\lambda| \text{Im} \lambda}, \quad k = 0, 1, 2.$$  (3.1.24)

This estimate is independent of the dimension.

We can re-write the integral in (3.1.23) using polar coordinates in $\xi = \rho \theta$, $\rho \in (0, \infty)$, $\theta \in S^{n-1}$ so that

$$R_0(\lambda)(\varphi) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} \int_0^\infty \int_{S^{n-1}} \frac{\tilde{\varphi}(\rho \theta)}{\rho^2 - \lambda^2} \rho^{n-1} d\rho d\theta, \quad \text{Im} \lambda > 0,$$

where $d\theta$ is the element of integration on $S^{n-1}$.

We can rewrite the integral in $\rho$ as a contour integral over the following contours:

$$\gamma_n = \begin{cases} \mathbb{R}, & \text{oriented from } -\infty \text{ to } +\infty, \quad \text{for } n \text{ odd} \\ \mathbb{R}_- + \mathbb{R}_+, \quad \mathbb{R}_\pm \text{ oriented from } 0 \text{ to } \pm \infty, & \text{for } n \text{ even} \end{cases}$$

Figure 3.1. Contour deformation used to define $R_0(\lambda)$ for $\text{Im} \lambda \geq 0$. 

$$\gamma_n(\lambda), \text{ n odd}$$

$$\gamma_n(\lambda), \text{ n even}$$
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\[ R_0(\lambda)(\varphi) = \frac{1}{2} \frac{1}{(2\pi)^n} \int_{S^n} \int_{S^n} \frac{\hat{\varphi}(\rho\theta)}{\rho^2 - \lambda^2 \rho^{n-1}} d\rho d\theta, \quad \text{Im} \lambda > 0. \tag{3.1.25} \]

We can now deform the contours in a \( \lambda \)-dependent way, as shown in Fig. 3.1 for \( \lambda \in \mathbb{R} \setminus \{0\} \). For \( \text{Im} \lambda < 0 \) the further deformation leads to contours shown in Fig. 3.2 (\( n \) odd) and Fig. 3.3 (\( n \) even):

\[ \gamma_n(\lambda) = \gamma_n + \delta_n(\lambda) + \delta_n(-\lambda), \]

where, for some \( r < |\text{Im} \zeta| \),

\[ \delta_n(\zeta) = \begin{cases} \partial D(\zeta, r) & \text{for Im} \zeta < 0 \text{ and } n \text{ odd}, \\ -\partial D(\zeta, r) & \text{Im} \zeta > 0 \text{ and } n \text{ odd}, \end{cases} \]

where the boundary of a disc is positively oriented.

For \( n \) odd the contour integrals over \( \delta_n(\pm \lambda) \) can be absorbed into \( \gamma_n \) as \( \lambda \) crosses the real axis again and This shows that we can continue \( R_0(\lambda) \varphi, \varphi \in C_c^\infty \) to a holomorphic function in \( \mathbb{C} \setminus \{0\} \). This gives the holomorphic continuation of the distributional kernel, \( R_0(\lambda, x, y) \).

For \( n \) even the contour integrals over \( \delta_n(\pm \lambda) \) cannot be absorbed into \( \gamma_n \) as \( \lambda \) crosses the real axis again due to the wrong orientation: that means that \( R_0(\lambda, x, y) \) continues to the logarithmic cover of \( \mathbb{C} \setminus \{0\} \) when \( n \) is even. The integrals over \( \delta_n(\pm \lambda) \) can be evaluated by the residue theorem and that shows that for \( n \) even

\[ R_0(\lambda e^{i\pi})(x) = R_0(\lambda)(x) + \frac{\ell}{2\pi} (-1)^{n-2}(i+1) \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{S^{n-1}} e^{i\lambda(x, \omega)} d\omega. \]
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Figure 3.3. Contour deformations used to define $R_0(\lambda)$ for $\lambda > 0$ for $n$ even. Now $R_0(e^{-i\pi/2}\lambda) = R_0(\lambda)$ can be expressed using an integral over the circular contour which doubles rather than gets absorbed.

The resolvent is defined on the logarithmic plane.

3.1.5. Additional estimates. We conclude this section with two low energy estimates which will be useful in §3.9. In particular, they can be omitted till that section is reached.

**Lemma 3.6.** Suppose that $\rho \in C_c^\infty(\mathbb{R}^n)$. Then for any $C_0 > 0$ and $k$ there exist $C_1$ such that

\[ \| \lambda \rho R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) \rho \|_{L^2 \to H^2} \leq C_1, \]

for $0 \leq |\lambda| \leq C_0 \leq \frac{1}{2} \text{Im} \lambda_0 \leq 2C_0$, $\text{Im} \lambda \geq 0$.

**Proof.**

1. It is convenient to prove a stronger estimate for the same range of $\lambda$ and $\lambda_0$:

\[ \| e^{-\langle x \rangle} \lambda R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) e^{-\langle x \rangle} \|_{L^2 \to H^2} \leq C. \]

2. We first prove

\[ \| e^{-\langle x \rangle} \lambda R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) e^{-\langle x \rangle} \|_{L^2 \to L^2} \leq C, \]

for any $k \geq 0$. In fact, using $R_0(\lambda_0) R_0(\lambda) = (\lambda^2 - \lambda_0^2)^{-1} (R_0(\lambda) - R_0(\lambda_0))$

\[ R_0(\lambda) R_0(\lambda_0)^k R_0(\lambda) = a_k (\lambda^2 - \lambda_0^2)^{-k} R_0(\lambda)^2 + b_k (\lambda^2 - \lambda_0^2)^{-k} R_0(\lambda) \]

\[ + \sum_{\ell=0}^{k-1} c_{k\ell} (\lambda^2 - \lambda_0^2)^{-2-\ell} R_0(\lambda_0)^{k+1-\ell}. \]

Since $|\lambda^2 - \lambda_0|^2 > (\text{Im} \lambda_0)^2 - |\lambda|^2 > 3C_0^2 > 0$ and since $\|R_0(\lambda)\|_{L^2 \to L^2} \leq C|\lambda_0|^{-1} (\text{Im} \lambda_0)^{-1} \leq CC_0^2$ (see (3.1.24)) we only need to show that

\[ \| e^{-\langle x \rangle} R_0(\lambda) e^{-\langle x \rangle} \|_{L^2 \to L^2} \leq C, \]
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\[ \| \epsilon^{-x} \lambda R_0(\lambda)^2 e^{-x} \|_{L^2 \to L^2} \leq C. \]

The first estimate follows the representation of the Schwartz kernel of \( R_0(\lambda) \) given in Theorem 3.3 and the Schur criterion (A.5.3). For the second estimate we note that

\[ \lambda R_0(\lambda)^2 = \frac{1}{2} \partial_\lambda R_0(\lambda) \]

which gives an expression for the Schwartz kernel of \( \lambda R_0(\lambda)^2 \). Then the Schur criterion gives the second estimate.

3. We start with the following observation: suppose that \( e^{-x} u \in H^2(\mathbb{R}^n) \).

Then with \( \| \| := \| \cdot \|_{L^2} \)

\( (3.1.29) \quad \| \Delta(e^{-x} u) \| \leq C \| e^{-x} \Delta u \| + C \| e^{-x} u \|. \)

In fact,

\[ \Delta(e^{-x} u) = e^{-x} (\Delta u - \langle x \rangle ^{-1} x \cdot \nabla u - (n + (n - 1)|x|^2) \langle x \rangle ^{-3} u) \]
\[ = e^{-x} \Delta u - \langle x \rangle ^{-1} x \cdot \nabla (e^{-x} u) - O(\langle x \rangle ^{-1}) e^{-x} u. \]

Thus,

\[ \| \Delta(e^{-x} u) \|^2 \leq 2 \| e^{-x} \Delta u \|^2 + 2 \| \nabla (e^{-x} u) \|^2 + C \| e^{-x} u \|^2 \]
\[ \leq 2 \| e^{-x} \Delta u \|^2 - 2 \langle (\Delta e^{-x} u), e^{-x} u \rangle + C \| e^{-x} u \|^2 \]
\[ \leq 2 \| e^{-x} \Delta u \|^2 + \frac{1}{2} \| \Delta(e^{-x} u) \|^2 + (C + 2) \| e^{-x} u \|^2. \]

Since we assumed \( e^{-x} u \in H^2 \), the integration by parts was justified. Also, we can move the \( \Delta(e^{-x} u) \) term to the left hand side, proving (3.1.29).

4. The estimate (3.1.29) shows that

\[ \| e^{-x} \lambda R_0(\lambda) R_0(\lambda)^k R_0(\lambda) e^{-x} \|_{L^2 \to H^{2k}} \leq \]
\[ C \| e^{-x} \lambda R_0(\lambda) \Delta^k R_0(\lambda)^k R_0(\lambda) e^{-x} \|_{L^2 \to L^2} \]
\[ + C \| e^{-x} \lambda R_0(\lambda) R_0(\lambda)^k R_0(\lambda) e^{-x} \|_{L^2 \to L^2}. \]

But this follows from (3.1.28) by iterating the identity \( -\Delta R_0(\lambda) = I + \lambda^2 \) \( k \) times. \( \square \)

The second lemma provides a basic weighted estimates on the resolvent up to the real axis near 0 energy. It can be refined in many ways – see [JK79],[Je80a],[Mu82] for classical estimates and [Va18] for recent developments.

**LEMMA 3.7.** Suppose that \( s \geq 0, s \notin \mathbb{N}. \) Then, for \( C_0, C_1 > 0 \) there exists \( C_1 \) such that for

\[ |\lambda| \leq C_0 \text{Im} \lambda \leq C_1, \]

\( (3.1.30) \quad \| \langle x \rangle ^{-s} R_0(\lambda) \langle x \rangle ^{-s} \|_{L^2 \to L^2} \leq C_3 + C_3 |\lambda|^{s-2}. \)
Proof. From Theorem 3.3 we know that for $\text{Im} \lambda \geq 0$,

$$|R_0(\lambda, x, y)| \leq \sum_{k=0}^{\frac{n-3}{2}} c_k |\lambda|^k |x - y|^{2 + k - n} e^{-\text{Im} \lambda |x - y|}.$$  

Using $\langle x \rangle \langle y \rangle \geq \langle x - y \rangle$ we have for $s \geq 0$,

$$\langle x \rangle^{-s} |x - y|^{2 + k - n} \langle y \rangle^{-s} \leq |x - y|^{2 + k - n} \langle x - y \rangle^{-s}. $$

Since for $s \geq 0$, $s \notin \mathbb{N}$,

$$\int_\mathbb{R} \langle x \rangle^{-s} |R_0(\lambda, x, y)| \langle y \rangle^{-s} dx \leq \sum_{k=0}^{\frac{n-3}{2}} c_k |\lambda|^k \int_0^\infty (1 + r)^{-s} r^{1+k} e^{-\text{Im} \lambda r} dr$$

$$\leq \sum_{k=0}^{\frac{n-3}{2}} c_k |\lambda|^k (C + \int_1^\infty r^{1+k-s} e^{-\text{Im} \lambda r} dr)$$

$$\leq \sum_{k=0}^{\frac{n-3}{2}} c_k |\lambda|^k (C + C|\lambda|^{-(2 + k - s)})$$

$$\leq C + C|\lambda|^{s-2}.$$  

the Schur criterion (A.5.3) gives (3.1.30).  

3.2. MEROMORPHIC CONTINUATION

In this chapter we define scattering resonances for compactly supported potentials in odd dimension and present some of their basic properties.

3.2.1. Continuation of the resolvent. Once we have established the properties of the free resolvent in odd dimensions the properties of

$$R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1}, \quad \text{Im} \lambda \gg 0, $$

$$V \in L^\infty(\mathbb{R}^n, \mathbb{C}), \quad n \geq 3, \quad \text{odd}, \quad \Delta = \sum_{j=1}^n \partial_{x_j}^2,$$

follow exactly as in one dimension. The situation is even simpler as we do not have a resonance at zero for $R_0(\lambda)$.

In particular the proof of the following theorem is exactly the same as in the one dimensional case:

**THEOREM 3.8** (Meromorphic continuation of the resolvent). Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$ and that $n \geq 3$ is odd. Then the

$$R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1} : L^2 \rightarrow L^2, \quad \text{Im} \lambda > 0,$$
is a meromorphic family of operators with finitely many poles. It extends to
a meromorphic family of operators for:
\[ R_V(\lambda) := L^2_{\text{comp}} \longrightarrow L^2_{\text{loc}}, \quad \lambda \in \mathbb{C}. \]

Just as in the proof of Theorem 2.2 we have a formula for the meromor-
phically continued resolvent. Let \( \rho \in L^\infty_{\text{comp}}(\mathbb{R}^n) \) satisfy \( \rho \equiv 1 \) on \( \text{supp} V \).

Then
\[
(3.2.1) \quad R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1 - \rho)).
\]
We also have
\[
(3.2.2) \quad \rho R_V(\lambda) \rho = (I + \rho R_0(\lambda)V)^{-1}\rho R_0(\lambda)\rho.
\]

**REMARK.** For future reference we make the following observation about
the Schwartz kernel of \( R_V(\lambda) \): if \( V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C}) \) then
\[
(3.2.3) \quad R_V(\lambda, x, y) = R_V(\lambda, y, x).
\]
In fact, since \( R_0(\lambda, x, y) = R_0(\lambda, y, x) \) this follows from (3.2.2) as the two
expressions for \( \rho R_V(\lambda) \rho \) are transposes of each other and \( \rho \) can be chosen
to be equal to 1 on arbitrarily large sets.

Scattering resonances are the poles of \( R_V(\lambda) \) and their multiplicities,
\( m_R(\lambda) \) are defined by
\[
(3.2.4) \quad m_V(\lambda) = m_R(\lambda) := \dim \text{span} \{A_1(L^2_{\text{comp}}), \cdots A_J(L^2_{\text{comp}})\},
\]
where
\[
R_V(\zeta) = \sum_{j=1}^{J} \frac{A_j}{(\zeta - \lambda)^j} + A(\zeta, \lambda),
\]
with \( \zeta \mapsto A(\zeta, \lambda) \) holomorphic near \( \lambda \).

As we will see in Theorem 3.23 this definition coincides with the defini-
tion (2.2.11) away from 0:
\[
(3.2.5) \quad m_V(\lambda) = m_R(\lambda) := \dim \text{span} A_1(L^2_{\text{comp}}) = \text{rank} \oint_{\lambda} R_V(\zeta)2\zeta d\zeta.
\]

We use notation \( m_V \) to emphasize the dependence on the potential \( V \).
When there is no ambiguity \( m_R \) is used to distinguish this multiplicity from
the multiplicity \( m_S(\lambda) \) defined in \( \S 3.7 \) using the scattering matrix.

The structure of the singular part of the resolvent at a pole is described
by following the proof of 1) in Theorem 2.4:
THEOREM 3.9 (Singular part of \( R_V(\lambda) \) II). Suppose \( m_R(\mu) > 0 \), \( \mu \neq 0 \). Then for some integer \( K(\mu) \leq m_R(\mu) \),

\[
R_V(\lambda) = -\sum_{k=1}^{K(\mu)} \frac{(P - \mu^2)^{k-1}}{(\lambda^2 - \mu^2)^k} \Pi_\mu + A(\lambda, \mu),
\]

where \( \lambda \mapsto A(\lambda, \mu) \) is holomorphic near \( \mu \), and

\[
(3.2.7) \quad \Pi_\mu = -\frac{1}{2\pi i} \oint_{\mu} R_V(\lambda) 2\lambda d\lambda, \quad (P_V - \mu^2)^{K(\mu)} \Pi_\mu = 0.
\]

REMARKS. 1. The expansion (3.2.6) takes a particularly simple form when \( m_R(\mu) = 1 \):

\[
(3.2.8) \quad R_V(\lambda) = \frac{u \otimes u}{\lambda - \mu} + A(\lambda, \mu), \quad u \in H^2_{loc}(\mathbb{R}^n), f \in L^2_{comp}(\mathbb{R}^n),
\]

where for \( f \in L^2_{comp}(\mathbb{R}^n) \), \( (u \otimes u) f(x) := u(x) \int_{\mathbb{R}^n} u(y) f(y) dy \). In fact, since \( K(\mu) = 1 \) and \( \Pi_\mu \) is of rank one the Schwartz kernel of the residue is given by \( u(x) v(y) \) for some \( u, v \in H^2_{loc}(\mathbb{R}^n) \). But the Schwartz kernel of the resolvent satisfies (3.2.3) (see also Exercise 3.4) which means that we can choose \( v(x) = u(x) \).

2. A generalization of Theorem 3.23 to a large class of compactly supported perturbations of \(-\Delta\) is given in §4.2. A discussion of outgoing solutions is also provided there.

3.2.2. Resonance expansions of scattered waves. The proofs of Theorem 2.8 on resonance free regions and of Theorem 2.7 apply without any modifications to the case of higher odd dimensions. Thus we obtain

THEOREM 3.10 (Resonance free regions II). Suppose that

\[
V \in L^\infty_{comp}(\mathbb{R}^n; \mathbb{C}), \quad n \geq 3, \quad \text{odd}.
\]

Then for any \( \rho \in C^\infty_c(\mathbb{R}^3) \) there exist constants \( A, C, T \) depending on \( \rho \) such that

\[
(3.2.9) \quad \|\rho R_V(\lambda)\rho\|_{L^2 \to H^j} \leq C|\lambda|^{j-1} e^{T(\text{Im} \lambda)}, \quad j = 0, 1, 2,
\]

for

\[
\text{Im} \lambda \geq -A - \delta \log(1 + |\lambda|), \quad |\lambda| > C_0, \quad \delta < 1/\text{diam} (\text{supp} V).
\]

In particular there are only finitely many resonances in the region

\[
\{ \lambda : \text{Im} \lambda \geq -A - \delta \log(1 + |\lambda|) \}.
\]

for any \( A > 0 \).
3.2. MEROMORPHIC CONTINUATION

THEOREM 3.11 (Resonance expansions of scattering waves II). Let \( V \in L^\infty(\mathbb{R}^n; \mathbb{R}) \) for \( n \geq 1 \) odd, and suppose that \( w(t, x) \) is the solution of

\[
\begin{align*}
(D_t^2 - P_V)w(t, x) &= 0, \\
w(0, x) &= w_0(x) \in H^1_{\text{comp}}(\mathbb{R}), \\
\partial_t w(0, x) &= w_1(x) \in L^2_{\text{comp}}(\mathbb{R}).
\end{align*}
\]

Let and \( \{ \lambda_j \} \) be the set of resonances of \( P_V \) (including \( \{ i \sqrt{-E_k} \}_{k=1}^N \), where \( E_N < \cdots \leq E_2 \leq E_1 \leq 0 \) are the eigenvalues of \( P_V \)).

Then, for any \( A > 0 \),

\[
w(t, x) = \sum_{m_R(\lambda_j) - 1}^{m_R(\lambda_j)} \sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} w_{j,\ell}(x) + E_A(t),
\]

where the sum is finite and

\[
\sum_{\ell=0}^{m_R(\lambda_j)-1} \lambda_j^\ell e^{-i\lambda_j t} w_{j,\ell}(x) = \text{Res}_{\lambda = \lambda_j} \left( (iR_V(\lambda)w_1 + \lambda R_V(\lambda)w_0) e^{-i\lambda t} \right),
\]

\[
(P_V - \lambda_j)^{k+1} w_{j,k} = 0,
\]

and for any \( K > 0 \), such that \( \text{supp } w_j \subset [-K, K] \), there exist constants \( C_{K,A} \) and \( T_{K,A} \)

\[
\| E_A(t) \|_{H^2([-K,K])} \leq C_{K,A} e^{-tA} \left( \| w_0 \|_{H^1} + \| w_1 \|_{L^2} \right), \quad t \geq T_{K,A}.
\]

3.2.3. Finite rank perturbations and multiplicities. To handle agreement of multiplicities it is convenient to work with simple poles. This section as well as Theorem 3.45 below deal with these thorny issues and can be safely skipped at first reading.

The theory presented above applies without changes to \( V \) of the form

\[
V = V_0 + V_1, \quad V_0 \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{C}),
\]

\[
V_1 = \sum_{j=1}^J f_j \otimes g_j, \quad f_j, g_j \in L^\infty_{\text{comp}}(\mathbb{R}^n, \mathbb{C}),
\]

that is to potentials replaced by a sum of a potential and a finite rank perturbation. The advantage of this approach is the ease with which the multiplicities can be split. That will allow a simple treatment of the relation between multiplicities of the scattering matrix and the resolvent.
We have already seen that the structure of the singular part of the
resolvent can be quite complicated and both statements and proofs are easier
when the resonances are simple, that is when multiplicity is equal to 1.

We start with a simple lemma which is a more precise version of Theorem
C.3 in a special case. From (C.2.2) we see that for a compact operator on a
Hilbert space \( \dim \ker(I + K) = \dim \coker(I + K) = \dim \ker(I + K^*) \).

**Lemma 3.12** (Grushin problem for Fredholm operators). Suppose
that \( K : H \to H \) is a compact operator on a Hilbert space \( H \). Let \( v_1, \cdots, v_m \)
be an orthonormal basis of \( \ker(I + K) \) and \( w_1, \cdots, w_m \) be an orthonormal basis
of \( \ker(I + K^*) \). Define

\[
R_+ : \mathbb{C}^m \to H, \quad R_+ u := \sum_{j=1}^{m} u_j w_j,
\]

\[
R_- : \mathbb{C}^m \to H, \quad R_- u := \sum_{j=1}^{m} u_j w_j,
\]

Then

\[
\begin{bmatrix}
I + K & R_-
\end{bmatrix}^{-1} \begin{bmatrix}
R_+ & E_+
\end{bmatrix} = \begin{bmatrix}
E & E_+
\end{bmatrix} : H \oplus \mathbb{C}^m \to H \oplus \mathbb{C}^m,
\]

where

\[
E_+ : \mathbb{C}^m \to H, \quad E_+ z = \sum_{j=1}^{m} z_j v_j, \quad z \in \mathbb{C}^m,
\]

\[
E_- : H \to \mathbb{C}^m, \quad E_- u = \langle u, v_j \rangle_H w_j.
\]

The next lemma deals with simplicity of zeros of determinants:

**Lemma 3.13** (Simplicity of zeros). Suppose that \( M(\lambda) \) is a holomor-
phic family of \( m \times m \) matrices and

\[
\det M(\lambda) = \lambda^p g(\lambda), \quad g(0) \neq 0, \quad M(0) = 0.
\]

There exists a matrix \( A \) such that for any holomorphic family of \( m \times m \)
matrices, \( \lambda \mapsto M(\lambda, \epsilon) \) satisfying

\[
M(\lambda, \epsilon) = M(\lambda) + \epsilon A + f(\lambda, \epsilon), \quad \| f(\lambda, \epsilon) \| = \mathcal{O}(\epsilon^2 + |\lambda| \epsilon),
\]

there exists \( \epsilon_0 \) and \( r_0 \) such that for \( 0 < \epsilon < \epsilon_0 \), \( \det M(\lambda, \epsilon) \) has exactly \( p \)
simple zeros in \( D(0, r_0) \).

**Proof.** 1. We note that for \( 0 < |\lambda| \leq r_0 \) with \( r_0 \) small enough, \( \det M(\lambda) \neq 0 \)
and hence \( M(\lambda) \) is invertible. It follows that for \( |\lambda| = r_0 \)

\[
\| M(\lambda)^{-1} (M(\lambda, \epsilon) - M(\lambda)) \| \leq C r_0 \epsilon < 1,
\]

if \( \epsilon < \epsilon_0 \) for some \( \epsilon_0 \) chosen small enough depending on \( r_0 \) and \( A \). The matrix
valued version of Rouché’s Theorem C.9 now shows that the numbers the
number $\rho$ of $\det M(\lambda, \epsilon)$ in $|\lambda| < r_0$ is equal to $p$, the multiplicity of the only zero of $\det M(\lambda)$ there.

2. We can apply Lemma C.10 to $M(\lambda)$, so that

$$M(\lambda) = E(\lambda)M_0(\lambda)F(\lambda),$$

where we identified $\text{Im} P_j$. Let $0 < \theta \ll 0$ and put $D_k = \text{diag}(1, e^{i\theta}, \ldots, e^{i(k-1)\theta})$. We then put

$$A_0 = D_{m_1}P_1 + D_{m_2}P_2 + \cdots + D_{m_r}P_r,$$

where we identified $\text{Im} P_j$ with $\mathbb{C}^{m_j}$ and $\mathbb{C}^m$ with $\mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_r}$. If $\theta$ is small enough, the zeros of $\det(M_0(\lambda) - \epsilon A_0)$ are simple and are given by

$$\lambda_{k_j, q, \ell} = e^{1/k_j}e^{2\pi i/k_j q},$$

$$\ell = 0, \ldots, k_j, \quad q = 0, \ldots, m_j - 1, \quad j = 1, \ldots, r.$$ One can check that there exists $c_0 > 0$ such that for $(k, q, \ell) \neq (k', q', \ell')$,

$$(3.2.14) \quad |\lambda_{k, q, \ell} - \lambda_{k', q', \ell'}| > c_0 \max(e^{\frac{1}{k_j}}, e^{\frac{1}{k_j'}}),$$

provided $\epsilon$ and $\theta$ are smaller than some fixed constant (depending on $k_j$'s and $m_j$'s).

4. Putting $A = E(0)^{-1}A_0F(0)^{-1}$ we have

$$E(\lambda)^{-1}M(\lambda, \epsilon)F(\lambda)^{-1} = M_0(\lambda) - \epsilon A_0 + e(\lambda, \epsilon),$$

$$||e(\lambda, \epsilon)|| = O(\epsilon^2 + |\lambda|\epsilon)$$

and we are looking for zeros of the determinant of the right hand side. Consider $U = D(\lambda_{k_j, q, \ell}, e^{1/k_j}\rho)$. In view of (3.2.14) different $U$'s are disjoint of $\rho < c_0$. Then for $\lambda \in \partial U$, and $\rho$ small enough (independently of $\epsilon$),

$$||(M_0(\lambda) - \epsilon A_0)^{-1}|| = \left( \min_{j=1, \ldots, r} \min_{0 \leq q \leq m_j - 1} |\lambda_{k_j} - e^{i\theta q}| \right)^{-1} \leq \frac{k\epsilon^{-1}}{\rho - O(\rho^2)}.$$

Hence, for $\lambda \in \partial U$,

$$||(M_0(\lambda) - \epsilon A_0)^{-1}e(\lambda, \epsilon)|| \leq \frac{C(|\lambda| + \epsilon)}{\rho - O(\rho^2)} \leq C\epsilon\frac{1}{k_j}\rho^{-1} < 1,$$

if $C\epsilon\frac{1}{k_j} < \rho$. An application of Theorem C.9 shows that the determinant of $M(\lambda, \epsilon)$ has exactly one zero in $U$. In view of (3.2.14) there exists $p$ disjoint
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discs with such properties for the corresponding \(\lambda_{k,j,q}r_0\)'s. It follows that all the zeros of this determinant in \(D(0,r_0)\) are simple. □

We can now prove

**THEOREM 3.14** (Multiplicity splitting). Suppose that \(V \in L_{\text{comp}}^\infty(\mathbb{R}^n; \mathbb{C})\) and that for some \(\lambda_0 \in \mathbb{C}\), \(m_V(\lambda_0) > 1\). Then there exists a finite rank perturbation

\[
W = \sum_{i,j=1}^m f_i \otimes g_j, \quad f_i, g_j \in C_c^\infty(\mathbb{R}^n; \mathbb{C}),
\]

and constants \(\epsilon_0\) and \(r_0\) such that for \(0 < \epsilon < \epsilon_0\),

\[
\sum_{|\lambda - \lambda_0| < r_0} m_V + \epsilon W(\lambda) = m_V(\lambda_0),
\]

\[
m_{V + \epsilon W}(\lambda) \leq 1, \quad |\lambda - \lambda_0| < r_0.
\]

**Proof.** 1. Since

\[
R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1 - \rho)) = (I + VR_0(\lambda)\rho)^{-1} = I - VR_V(\lambda)\rho,
\]

we see that simplicity of a pole of \(R_V\) is equivalent to the simplicity of a pole of \((I + VR_0(\lambda)\rho)^{-1}\) and we will consider that operator.

2. We apply Lemma 3.12 with \(K = VR_0(\lambda_0)\rho\). From (3.1.12) we have

\[
\|VR_0(\lambda)\rho - VR_0(\lambda_0)\rho\|_{L^2 \to L^2} \leq \|VR_0(\lambda_0)\rho\|_{L^2 \to L^2} + \epsilon \|VR_0(\lambda)\rho\|_{L^2 \to L^2} \leq C_1(\lambda - \lambda_0^2) + \epsilon e^{C_0|\lambda|},
\]

where \(C_1\) depends on \(V\) and \(W\). Hence, for \(\lambda\) sufficiently close to \(\lambda_0\) and \(\epsilon\) small enough we can use the same \(R_\pm\) as for \(VR_0(\lambda_0)\rho\) to obtain a well posed Grushin problem:

\[
\begin{bmatrix}
I + (V + \epsilon W)R_0(\lambda)\rho & R_- \\
R_+ & 0
\end{bmatrix}^{-1} = \begin{bmatrix}
E^-(\lambda) & E_+^e(\lambda) \\
E^-_e(\lambda) & E^e_+(\lambda)
\end{bmatrix},
\]

3. The expansion (C.1.7) gives

\[
E^e_{-+}(\lambda) = E_{-+}(\lambda) - \epsilon A + \mathcal{O}(|\lambda| + \epsilon^2),
\]

where

\[
A = E_-WE_+ = B^TC, \quad B_{ij} := \int_{\mathbb{R}^n} f_i(x)\overline{w_j(x)}, \quad C_{ij} := \int_{\mathbb{R}^n} g_i(x)v_j(x)dx.
\]

Since the sets \(\{w_j\}_{j=1}^m\) and \(\{v_j\}_{j=1}^m\) are linearly independent we can find \(f_j\)'s and \(g_j\)'s so that \(B\) and \(C\) are arbitrary matrices. This means that we can
choose $A$ is in Lemma 3.13. Since the poles of $I + (V + \epsilon W)R_0(\lambda)\rho$ are the zeros of $\det E_{\pm}(\lambda)$ the conclusion follows.

As the first application we record the following fact:

**Theorem 3.15 (Multiplicity as a Trace).** Let $m_R(\lambda)$ be defined by (3.2.4) and suppose $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfies $\rho V = V$. Then

$$m_R(\lambda) = \frac{1}{2\pi i} \text{tr} \oint_\lambda (I + VR_0(\zeta)\rho)^{-1} \partial_\zeta (VR_0(\zeta)\rho) d\zeta,$$

where the integral is over a positively oriented circle containing $\lambda$ and no other possible pole of $R_V$.

**Proof.**

1. As explained in the beginning of this section we can replace $V$ by more general operators of the form $V = V_0 + V_1$ where $V_0$ is a potential $V_1$ is a finite rank perturbation,

$$V_0 \in L^\infty(\mathbb{R}^n; \mathbb{C}), \quad V_1 = \sum_{j=1}^J f_j \otimes g_j, \quad f_j, g_j \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C}).$$

If the support of $V_0$ and of $f_j, g_j$’s is contained in a fixed compact set, say $B(0, R)$ then the map $V \mapsto H_V(\lambda)$ is continuous. Hence, by Rouché’s Theorem, if $D(\lambda_0, r_0) \subset \mathbb{C}$ is such that there are no zeros of $D_V(\lambda)$ or $H_V(\lambda)$ on $\partial D(\lambda_0, r_0)$ then for any fixed $W$ with the same properties as $V$,

$$\sum_{\lambda \in D(\lambda_0, r_0)} m_H(\lambda) = \sum_{\lambda \in D(\lambda_0, r_0)} m_{H_{V+\epsilon W}}(\lambda), \quad 0 \leq \epsilon < \epsilon_0.$$

2. Theorem 3.14 shows that for any $V$ there exists $W$ such that the poles of $R_{V+\epsilon W}(\lambda)$, $0 < \epsilon < \epsilon_0$ near $\lambda_0$ are all simple.

In view of (3.2.16) the same holds for the poles of $(I + VR_0(\lambda)\rho)^{-1}$. Then (3.2.18) and the first statement in (3.2.15) show that it is sufficient to prove (3.2.17) in the case of simple poles of $R_V(\lambda)$ and $(I + VR_0(\lambda)\rho)^{-1}$.

3. Suppose that $\lambda$ is such a pole. Then (C.4.6) (the first part of Theorem C.8) gives

$$\frac{1}{2\pi i} \text{tr} \oint_\lambda (I + VR_0(\zeta)\rho)^{-1} \partial_\zeta (VR_0(\zeta)\rho) d\zeta = 1 = m_R(\lambda),$$

proving the claim. □
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Consider the operator $P_V, V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, where $n > 1$ is odd. Denote by $H_0$ the eigenspace of $P_V$ at 0:

$$H_0 := \{ v \in H^2(\mathbb{R}^n) : P_V v = 0 \},$$

and let

$$\Pi_0 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

be the orthogonal projector onto $H_0$.

We describe the structure of the resolvent $R_V(\lambda)$ at $\lambda = 0$, starting with the following

**Lemma 3.16.** We have

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda),$$

as operators $L^2_{\text{comp}} \to L^2_{\text{loc}}$, where $\lambda \mapsto A(\lambda)$ is holomorphic near 0 and $A_1 : L^2_{\text{comp}} \to L^2_{\text{loc}}$ is a symmetric operator such that $P_V A_1 = 0$.

**Remark.** In the Lemma, the symmetry of $A_1$ means that $\langle A_1 \varphi, \psi \rangle = \langle \varphi, A_1 \psi \rangle$, for $\varphi, \psi \in L^2_{\text{comp}}(\mathbb{R}^n)$.

**Proof.** 1. The upper bound on the resolvent in the upper half-plane,

$$\| R_V(it) \|_{L^2 \to L^2} \leq \frac{1}{t^2}, \quad t \in (0, \varepsilon),$$

and the fact that $\lambda \mapsto R_V(\lambda)$ is a meromorphic family of operators imply that we have the following decomposition:

$$R_V(\lambda) = -\frac{A_2}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda),$$

where $A_j : L^2_{\text{comp}} \to L^2_{\text{loc}}$ are finite rank operators and $A(\lambda)$ is holomorphic near $\lambda = 0$. Since $R_V(it)$ is self-adjoint for $t > 0$, we see that for $\varphi, \psi \in L^2_{\text{comp}}(\mathbb{R}^n)$,

$$\langle A_2 \varphi, \psi \rangle = \lim_{t \to 0^+} \langle t^2 R_V(it) \varphi, \psi \rangle = \lim_{t \to 0^+} \langle \varphi, t^2 R_V(it) \psi \rangle = \langle \varphi, A_2 \psi \rangle,$$

and

$$\langle A_1 \varphi, \psi \rangle = \lim_{t \to 0^+} \langle (t R_V(it) + t^{-1} A_2) \varphi, \psi \rangle$$

$$= \lim_{t \to 0^+} \langle \varphi, (t R_V(it) + t^{-1} A_2) \psi \rangle = \langle \varphi, A_1 \psi \rangle.$$

Hence $A_j$ are symmetric on $L^2_{\text{comp}}$. Using $(P_V - \lambda^2) R_V(\lambda) \psi = \psi, \psi \in L^2_{\text{comp}}(\mathbb{R}^n)$ we obtain $P_V A_2 = 0$.

2. We now observe that (3.3.2) shows that $A_2$ is bounded $L^2 \to L^2$ and that (3.3.3) is valid for all $\varphi, \psi \in L^2$, that is, $A_2$ is selfadjoint. Since $P_V A_2 = 0$, the
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range of $A_2$ is contained in $H_0$. To show that $A_2 = \Pi_0$, it remains to verify that for each $v \in H_0$, we have $A_2v = v$, and this follows by substituting $\varphi = v$ in (3.3.3):

$$\langle v, \psi \rangle = \langle t^2 R_V(itv, \psi) \rangle \rightarrow \langle A_2v, \psi \rangle \quad \text{as } t \rightarrow 0^+.$$  

In dimensions 5 or greater, we have $A_1 = 0$:

**THEOREM 3.17.** Assume that $n \geq 5$. Then

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + A(\lambda),$$

where $A(\lambda) : L^2_{\text{comp}} \rightarrow L^2_{\text{loc}}$ is holomorphic at $\lambda = 0$.

**Proof.** 1. Since $R_0(\lambda)$ is injective on $L^2_{\text{comp}}$ ($(-\Delta - \lambda^2)$ is its left inverse), it follows from (3.2.1) that, for $\lambda$ near 0,

$$R_V(\lambda) = R_0(\lambda)(-\tilde{A}_2/\lambda^2 + \tilde{A}_1/\lambda + \tilde{A}(\lambda)),$$

where $\tilde{A}_j, \tilde{A}(\lambda) : L^2_{\text{comp}} \rightarrow L^2_{\text{comp}}$ and $\tilde{A}(\lambda)$ is holomorphic near 0. For $n \geq 5$, in the notation of Theorem 3.3, and for $\varphi \in L^2_{\text{comp}}(\mathbb{R}^n)$,

$$R_0(0)\varphi(x) = P_n(0) \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}} dy \in L^2(\mathbb{R}^n),$$

$$\partial_\lambda R_0(0)\varphi(x) = (iP_n(0) + P'_{n}(0)) \int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-3}} dy \in L^2(\mathbb{R}^n),$$

and the asymptotics are given by $c/|x|^{n-2}$ and $c'/|x|^{n-3}$, respectively. For $n \geq 7$ this means that $R_0(0)\varphi, \partial_\lambda R_0(0)\varphi \in L^2(\mathbb{R}^n)$. This is also true for $n = 5$ as using (3.1.16) we check that $iP_5(0) + P'_5(0) = 0$. Hence

$$R_0(0)\varphi, \partial_\lambda R_0(0)\varphi \in L^2, \quad \text{for } \varphi \in L^2_{\text{comp}}, n \geq 5.$$

From this we conclude that

$$A_1(L^2_{\text{comp}}) = \left( R_0(0)\tilde{A}_1 - \partial_\lambda R_0(0)\tilde{A}_2 \right)(L^2_{\text{comp}}) \subset R_0(0)(L^2_{\text{comp}}) + \partial_\lambda R_0(0)(L^2_{\text{comp}}) \subset L^2.$$

Since $P_VA_1 = 0$, it follows that $A_1 : L^2_{\text{comp}} \rightarrow H_0$.

2. We now take $\psi \in L^2_{\text{comp}}, v \in H_0$, and consider, for $t > 0$,

$$0 = t \langle R_V(it)v, \psi \rangle - t^{-1} \langle v, \psi \rangle = \langle v, tR_V(it)\psi - t^{-1}v, \psi \rangle$$

$$= i(v, A_1\psi) + t\langle v, A(it)\psi \rangle + t^{-1} \langle v, \Pi_0\psi \rangle - t^{-1} \langle v, \psi \rangle$$

$$= i(v, A_1\psi) + t\langle v, A(it)\psi \rangle + t^{-1} \langle \Pi_0v, \psi \rangle - t^{-1} \langle v, \psi \rangle$$

$$\rightarrow i(v, A_1\psi), \quad t \rightarrow 0^+,$$

as $\Pi_0v = v$. Since $A_1\psi \in H_0$ we can take $v = A_1\psi$ to conclude that $A_1 \equiv 0$.  

\[\square\]
3. RESOLVENT AT ZERO ENERGY

We now concentrate on the interesting case \( n = 3 \). We start by analysing the asymptotic behaviour of the elements of \( H_0 \):

**Lemma 3.18.** Assume that \( n = 3 \) and \( v \in H_0 \). Then:

1. \( v = R_0(0)f \), where \( f = -Vv = -\Delta v \in L^2_{\text{comp}}(\mathbb{R}^3) \) and \( \int_{\mathbb{R}^3} f = 0 \).
2. Uniformly in \( \theta \in S^2 \) and locally uniformly in \( y \in \mathbb{R}^3 \),

\[
v(y + r\theta) = \frac{1}{4\pi r} \sum_{j=1}^{3} b_j \theta_j + \frac{3}{8\pi r^3} \sum_{j,k=1}^{3} (B_{jk} - 2b_j y_k) \theta_j \theta_k - \frac{1}{8\pi r^3} \sum_{j=1}^{3} (B_{jj} - 2b_j y_j) + \mathcal{O}\left(\frac{1}{r^4}\right), \quad r \to +\infty,
\]

where

\[
b_j(v) = \int_{\mathbb{R}^3} x_j f(x) \, dx, \quad B_{jk}(v) = \int_{\mathbb{R}^3} x_j x_k f(x) \, dx.
\]

3. For \( y \in \mathbb{R}^3 \) and \( r > 0 \),

\[
I_v(r, y) := \int_{S^2} v(y + r\theta) \, d\theta = \mathcal{O}(r^{-4}), \quad r \to +\infty
\]

locally uniformly in \( y \).

**Proof.** 1. We have for all \( \lambda \in \mathbb{C} \), \( (-\Delta - \lambda^2)R_V(\lambda) : L^2_{\text{comp}} \to L^2_{\text{comp}} \) and, using (3.2.1),

\[
R_V(\lambda) = R_0(\lambda)(-\Delta - \lambda^2)R_V(\lambda).
\]

From (3.3.1), we see that \( \Pi_0 = R_0(0)(-\Delta)\Pi_0 \), and, since \( P_V\Pi_0 = 0 \), that

\[
(-\Delta)\Pi_0 = -V\Pi_0 : L^2_{\text{comp}} \to L^2_{\text{comp}}.
\]

By Lemma 3.16, \( v \in H_0 \) is in the image of \( \Pi_0 \). Therefore, \( v = R_0(0)f \), where \( f = (-\Delta)v = -Vv \in L^2_{\text{comp}} \).

2. Since \( R_0(0)(x, y) = \frac{1}{4\pi|x-y|} \), we write

\[
v(y + r\theta) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(x)}{|x-y-r\theta|} \, dx = \frac{1}{4\pi r} \int_{\mathbb{R}^3} \frac{f(x)}{|\theta - r^{-1}(x-y)|} \, dx
\]

\[
= \frac{1}{4\pi r} \int_{\mathbb{R}^3} f(x)(1 - 2r^{-1}\langle\theta, x-y\rangle + r^{-2}|x-y|^2)^{-1/2} \, dx.
\]

We now use the Taylor expansion \((1 + s)^{-1/2} = 1 - \frac{1}{2}s + \frac{3}{8}s^2 + \mathcal{O}(s^3)\). In particular, we get

\[
v(r\theta) = \frac{1}{4\pi r} \int_{\mathbb{R}^3} f + \mathcal{O}\left(\frac{1}{r^2}\right);
\]

\[\text{This detailed analysis will not be needed in the rest of the book but it contains ideas behind the important study of more general potentials – see references in §3.13.}\]
since \( v \in L^2 \), this implies that \( \int_{\mathbb{R}^3} f = 0 \). Expanding \((1 - 2r^{-1}\langle \theta, x - y \rangle + r^{-2}|x - y|^2)^{-1/2} \) up to an \( \mathcal{O}(r^{-3}) \) remainder, we get (3.3.5).

3. Using the formulas
\[
\int_{S^2} \theta_j d\theta = 0, \quad \int_{S^2} \theta_j \theta_k d\theta = \frac{4\pi}{3}\delta_{jk},
\]
we see that the spherical integrals of the terms on the right-hand side of (3.3.5) are zero, except for the \( \mathcal{O}(r^{-4}) \) remainder. \( \square \)

We now want to understand the asymptotics, as \( t \to 0^+ \), of the function \( R_V(it)v \in L^2 \) for \( v \in H_0 \). We first consider \( R_0(it)v \):

**Lemma 3.19.** Assume that \( v \in H_0 \). Then we have the following asymptotic expansion in \( L^2_{\text{loc}} \) as \( t \to 0^+ \):

\[
R_0(it)v = K_v + tJ_v + \mathcal{O}(t^{3/2}),
\]
where, using the notation of (3.3.7),
\[
K_v(y) = \frac{1}{4\pi} \int_0^\infty rI_v(r, y) \, dr, \quad J_v(y) = -\frac{1}{4\pi} \int_0^\infty r^2 I_v(r, y) \, dr.
\]
Moreover, \( J_v(y) \) is a polynomial of order 1 in \( y \) and
\[
\partial_y^j J_v(y) = -\frac{b_j(v)}{12\pi}, \quad 1 \leq j \leq 3,
\]
where \( b_j(v) \) are defined in (3.3.6).

**Proof.** We write
\[
R_0(it)v(y) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-t|x-y|}}{|x-y|} v(x) \, dx = \frac{1}{4\pi} \int_0^\infty re^{-tr} I_v(r, y) \, dr.
\]
We now use the expansion \( e^{-s} = 1 - s + \mathcal{O}(s^{3/2}) \), valid uniformly in \( s \in [0, \infty) \), with \( s = tr \). By part 3 of Lemma 3.18, we have locally uniformly in \( y \), \( I_v(r, y) = \mathcal{O}(r^{-4}) \) as \( r \to +\infty \). We moreover have \( I_v(r, y) = \mathcal{O}(1) \) uniformly in \( r, y \), since \( v \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) \). Then the integral
\[
\int_0^\infty r^{1+\alpha} I_v(r, y) \, dr
\]
converges absolutely for \( \alpha = 0, 1, 3/2 \), which gives (3.3.10).

To show (3.3.11), we use the divergence theorem for the vector field \( v(x)e_j \) to obtain
\[
\partial_y^j \int_0^R r^2 I_v(r, y) \, dr = \partial_y^j \int_{B(y, R)} v(x) \, dx = \int_{B(y, R)} \partial_j v(x) \, dx
\]
\[
= R^2 \int_{S^2} \theta_j v(y + R\theta) \, d\theta.
\]
By (3.3.5), this converges as \( R \to +\infty \) to \( b_j/3 \), locally uniformly in \( y \), proving (3.3.11). \( \square \)

The operator \( v \mapsto K_v \) can be characterized in terms of the principal value integrals:

**LEMMA 3.20.** Let \( v \in H_0 \) and \( \varphi \in L^2_{\text{comp}} \) and put \( u := R_0(0)\varphi \in L^2_{\text{loc}}. \) Then the limit

\[
\langle v, u \rangle_0 := \lim_{R \to +\infty} \int_{B(y,R)} v(x) \overline{u(x)} \, dx
\]

exists, is independent of \( y \), and

\[
\langle v, u \rangle_0 = \langle K_v, \varphi \rangle.
\]

**Proof.** For \( y, y' \in \mathbb{R}^3 \) and large \( R \), we proceed as in (3.3.12) to obtain

\[
\partial_j \int_{B(y',R)} \frac{v(x)}{|x-y|} \, dx = R^2 \int_{\mathbb{R}^2} \theta_j \frac{v(y' + R\theta)}{|y' - y + R\theta|} \, d\theta.
\]

Since \( v(x) = \mathcal{O}(|x|^{-2}) \) by (3.3.5), we see that (3.3.15) is \( \mathcal{O}(R^{-1}) \), locally uniformly in \( y, y' \), implying that

\[
\int_{B(y,R)} \frac{v(x)}{|x-y|} \, dx - \int_{B(y',R)} \frac{v(x)}{|x-y|} \, dx = \mathcal{O}(R^{-1}).
\]

Then for each fixed \( y' \),

\[
\langle K_v, \varphi \rangle = \frac{1}{4\pi} \lim_{R \to +\infty} \int_{\mathbb{R}^3} \overline{\varphi(y)} \int_{B(y,R)} \frac{v(x)}{|x-y|} \, dxdy
\]

\[
= \frac{1}{4\pi} \lim_{R \to +\infty} \int_{\mathbb{R}^3} \overline{\varphi(y)} \int_{B(y',R)} \frac{v(x)}{|x-y|} \, dxdy
\]

\[
= \lim_{R \to +\infty} \int_{B(y',R)} \overline{u(x)} v(x) \, dx,
\]

yielding (3.3.13). \( \square \)

To characterize \( A_1 \), we consider the following space of resonant states at zero:

\[
\widetilde{H}_0 := \{ v \in H^2_{\text{loc}}(\mathbb{R}^3) : P_v v = 0, \ v = R_0(0)(-\Delta v) \}.
\]

By part 1 of Lemma 3.18, we see that \( H_0 \subset \widetilde{H}_0 \). Moreover, by (3.3.9) (which applies to any function of the form \( R_0(0)f \) with \( f \in L^2_{\text{comp}} \)) and since \( |x|^{-2} \) lies in \( L^2 \) near infinity, we see that

\[
H_0 = \left\{ v \in \widetilde{H}_0 : \int_{\mathbb{R}^3} \Delta v(x) \, dx = 0 \right\}.
\]

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Since $\Delta v = Vv$ it follows immediately that $H_0$ has codimension at most one in $\tilde{H}_0$:

\[(3.3.17)\] 
\[\tilde{m}_R(0) := \dim(\tilde{H}_0/H_0) \leq 1.\]

**LEMMA 3.21.** 1. The image of $A_1$ lies inside $\tilde{H}_0$.

2. If $v \in \tilde{H}_0$, then

\[(3.3.18)\] 
\[v - A_1(V) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Delta v(x) \, dx \in H_0.\]

**Proof.**

1. It follows from (3.3.8) that

\[iA_1 = R_0(0)(-\Delta)A_1 + \partial_\lambda R_0(0)\Delta \Pi_0.\]

Since

\[\partial_\lambda R_0(0)(x,y) = \frac{i}{4\pi} \text{ and } \int_{\mathbb{R}^3} \Delta v = 0, \quad v \in H_0,\]

we have

\[\partial_\lambda R_0(0)\Delta \Pi_0 = 0.\]

Therefore, $A_1 = R_0(0)(-\Delta)A_1$. Together with $P_VA_1 = 0$ this shows that the image of $A_1$ lies in $\tilde{H}_0$.

2. From by the resolvent identity,

\[R_{V}(\lambda) = R_0(\lambda) - R_0(\lambda)VR_0(\lambda),\]

we see that for $\rho \in C_c^\infty(\mathbb{R}^3)$ equal to one near supp $V$, $R_0(\lambda)V = R_0(\lambda)\rho(P_V - \lambda^2)R_0(\lambda)V : L^2 \to L^2_{l_0c}$. We now apply (3.3.19) to $-v$, recalling from Lemma 3.18 that $v = -R_0(0)Vv$:

\[-R_0(\lambda)Vv = -\left(\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda)\right)\rho(P_V - \lambda^2)R_0(\lambda)Vv\]

\[= \Pi_0\rho v + \frac{i}{2} \Pi_0\rho P_V\partial_\lambda^2 R_0(0)Vv - iA_1\rho P_V\partial_\lambda R_0(0)Vv + \lambda g(\lambda),\]

where $g(\lambda) \in L^2_{l_0c}(\mathbb{R}^3)$ is holomorphic. (The singular terms have to cancel out as the left hand side is holomorphic in $\lambda$). Putting $\lambda = 0$ we obtain

\[v = \Pi_0(\rho v + \frac{1}{2} \rho P_V\partial_\lambda^2 R_0(0)Vv) - iA_1\rho P_V\partial_\lambda R_0(0)Vv.\]

3. It follows that for $g = -i\rho P_V\partial_\lambda R_0(0)Vv$, we have $v - A_1g \in H_0$. We now calculate

\[g(x) = \frac{V(x)}{4\pi} \int_{\mathbb{R}^3} V(y)v(y)dy = \frac{V(x)}{4\pi} \int_{\mathbb{R}^3} \Delta v(y)dy\]

completing the proof. \[\square\]

We finally split $A_1$ into two parts, one produced by functions from $H_0$ and one orthogonal to it in a certain sense:
LEMMA 3.22. Define the operator

\[(3.3.20)\quad T : L^2_{\text{comp}}(\mathbb{R}^3) \to L^2_{\text{loc}}(\mathbb{R}^3), \quad Tf(x) = \frac{1}{12\pi} \int_{\mathbb{R}^3} \langle x, y \rangle f(y) \, dy.\]

In the notation of (3.3.17) we have:

1. If \( \bar{m}_R(0) = 0 \), then \( A_1 = \Pi_0 VTV_0 \).
2. If \( \bar{m}_R(0) = 1 \), then

\[(3.3.21)\quad A_1 = \Pi_0 VTV_0 + 4\pi(u_0 \otimes \bar{u}_0),\]

where \( u_0 \) is the unique element of \( \tilde{H}_0 \) satisfying

\[(3.3.22)\quad u_0(x) = \frac{1}{4\pi|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \quad \langle u_0, v \rangle_0 = 0 \text{ for } v \in H_0,\]

where \( \langle \cdot, \cdot \rangle_0 \) is defined by (3.3.13).

REMARK. Since Lemma 3.18 (part 1) shows that \( \Pi_0(V) = 0 \) we could replace the operator \( T \) in (3.3.21) by the operator

\[(3.3.23)\quad G_3 f(x) := -\frac{1}{24\pi} \int_{\mathbb{R}^3} |x-y|^2 f(y) \, dy.\]

Proof. 1. Suppose that \( v \in H_0 \) and \( \psi \in L^2_{\text{comp}} \) and take \( \rho \in C_\infty_c(\mathbb{R}^3) \) such that \( \rho V = V \) and \( \rho \psi = \psi \). We write for \( t > 0 \),

\[(3.3.24)\quad \langle v, \psi \rangle = t^2 \langle R_V(it)v, \psi \rangle = t^2 \langle v, R_V(it)\psi \rangle = \rho(-\Delta + t^2)R_V(it)\psi,\]

where in the last equality we used that

\[(-\Delta + t^2)R_V(it)\psi = (I - VR_V(it))\psi = \rho(I - VR_V(it))\psi = \rho(-\Delta + t^2)R_V(it)\psi.\]

2. We next write the following expansion in \( L^2_{\text{comp}} \) as \( t \to +0 \) (note that \( (-\Delta)\Pi_0 = -\Pi_0 V \) and \( (-\Delta)A_1 = -VA_1 \)):

\[(3.3.25)\quad \rho(-\Delta + t^2)R_V(it)\psi = \frac{(-\Delta)\Pi_0\psi}{t^2} + \frac{(-\Delta)A_1\psi}{t} + \mathcal{O}(1).\]

3. Combining (3.3.10) (Lemma 3.19) with (3.3.24) and (3.3.25) gives

\[\langle v, \psi \rangle = \langle K_v, (-\Delta)\Pi_0\psi \rangle + t\langle J_v, (-\Delta)\Pi_0\psi \rangle + t\langle K_v, (-\Delta)A_1\psi \rangle + \mathcal{O}(t^{3/2}).\]

The terms next to the first power of \( t \) give in the limit \( t \to 0^+ \),

\[(3.3.26)\quad \langle J_v, (-\Delta)\Pi_0\psi \rangle + \langle K_v, (-\Delta)A_1\psi \rangle = 0.\]
4. We evaluate the first term in (3.3.26). By part 1 of Lemma 3.18, we have
\[ \int_{\mathbb{R}^3} (-\Delta) \Pi_0 \psi = 0. \]
By part 3 of the same lemma, we then have
\[ \langle J_v, (-\Delta) \Pi_0 \psi \rangle = -\frac{1}{12\pi} \sum_{j=1}^3 b_j(v) \int_{\mathbb{R}^3} x_j (-\Delta \Pi_0 \psi)(x) \, dx. \]

Since \( b_j(v) \) is equal to the integral of \(-x_j Vv\), we have
\[ \langle J_v, (-\Delta) \Pi_0 \psi \rangle = -\frac{1}{12\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \langle x, y \rangle V(x)V(y)v(x)\Pi_0 \psi(y) \, dx \, dy = -\langle v, \Pi_0 VTV \Pi_0 \psi \rangle. \]

5. Returning to (3.3.26), by Lemma 3.20
\[ \langle K_v, (-\Delta) A_1 \psi \rangle = \langle v, A_1 \psi \rangle_0, \]
which inserted into (3.3.26) gives
\[ \langle v, (A_1 - \Pi_0 VTV) \Pi_0 \psi \rangle_0 = 0. \]
Thus the image of \( \tilde{A}_1 := A_1 - \Pi_0 VTV \Pi_0 \) lies inside \( \tilde{H}_0 \) and is orthogonal to \( H_0 \) with respect to the generalization \( \langle \cdot, \cdot \rangle_0 \) of the \( L^2 \) inner product on \( H_0 \). In the case \( \tilde{m}_R(0) = 0 \), we then have \( \tilde{A}_1 = 0 \).

6. We now consider the case \( \tilde{m}_R(0) = 1 \). Let \( u_0 \in \tilde{H}_0 \) be defined by (3.3.22): from (3.3.16) we see that such \( u_0 \) exists and is unique. The discussion in Step 5 shows that the image of \( \tilde{A}_1 \) is contained in the span of \( u_0 \). Since \( \tilde{A}_1 \) is symmetric, we have for some \( c \in \mathbb{C} \),
\[ \tilde{A}_1 = c(u_0 \otimes \bar{u}_0). \]
To find \( c \), we apply (3.3.18) to \( u_0 \). Since \( V \) is \( L^2 \) orthogonal to the space \( H_0 \) by part 1 of Lemma 3.3.5, we have \( \Pi_0(V) = 0 \). Moreover, using the expansion in (3.3.22) and the fact that \( u_0 = R_0(0)(-\Delta u_0) \) we obtain (see (3.3.9))
\[ \int_{\mathbb{R}^3} (-\Delta u_0)(x) \, dx = \int_{\mathbb{R}^3} (-V(x)u_0(x)) \, dx = 1. \]

From (3.3.18) we obtain
\[ u_0 + \frac{\tilde{A}_1(V)}{4\pi} \in H_0. \]
However, \( \tilde{A}_1(V) = -cu_0 \) and thus \( c = 4\pi \), finishing the proof. \( \square \)

We summarize the findings of this section in
THEOREM 3.23 \((\mathbf{R}_V(\lambda) \text{ near } 0 \text{ for } n \geq 3 \text{ odd})\). 1) Suppose that \(V \in L^\infty_{\text{comp}}(\mathbb{R}; \mathbb{R})\) and that \(m_R(0) > 0\). Then

\[(3.3.28) \quad \mathbf{R}_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{iA_1}{\lambda} + A(\lambda),\]

where \(\lambda \mapsto A(\lambda)\) is holomorphic near 0, \(\Pi_0\) is the orthogonal projection onto the space of \(L^2\) solutions to \(P_Vu = 0\) and \(A_1\) is described by Lemma 3.22.

2) For \(n \geq 5\), \((3.3.28)\) holds with \(A_1 = 0\).

3.4. UPPER BOUNDS ON THE NUMBER OF RESONANCES

As in the case of dimension one we will estimate the number of resonances using a suitable determinant. We start with

LEMMA 3.24 (Trace class properties). For \(V, \rho \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C}), n \geq 3, \text{ odd,}\)

\[(3.4.2) \quad (VR_0(\lambda)\rho)^p, \quad p \geq \frac{n+1}{2},\]

is an entire family of trace class operators.

Proof. 1. We first estimate the characteristic values of \(\rho_1 R_0(\lambda)\rho_1\) where \(\rho_1 \in C^\infty(\mathbb{R}^n)\). If \(\text{supp} \rho_1 \subset B(0, R)\) we can consider

\[(3.4.1) \quad \rho_1 R_0(\lambda)\rho_1 : L^2(\mathbb{T}_R^n) \rightarrow L^2(\mathbb{T}_R^n), \quad \mathbb{T}_R := \mathbb{R}^n/R\mathbb{Z}^n.\]

2. Then, using \((B.3.7)\) and then \((B.3.9)\), we have

\[(3.4.4) \quad s_j(\rho_1 R_0(\lambda)\rho_1) \leq s_j((-\nabla_{\mathbb{T}_R^n})^2 + 1)^{-\ell/2}(\nabla_{\mathbb{T}_R^n} + 1)^{\ell/2} \rho_1 R_0(\lambda)\rho_1\]

\[\leq C j^{-2\ell/n}\|\rho_1 R_0(\lambda)\rho_1\|_{L^2 \rightarrow H^{2\ell}}.\]

Theorem 3.1 gives

\[(3.4.3) \quad s_j(\rho_1 R_0(\lambda)\rho_1) \leq C \min(|\lambda|^{-1}, j^{-1/n}, |\lambda| j^{-2/n}) \exp(C(\text{Im} \lambda)_-).\]

3. By taking \(\rho_1 = 1\) on \(\text{supp} \rho \cup \text{supp} V\) we can use \((B.3.7)\) again to see that \((3.4.3)\) holds for \(s_j(V R_0(\lambda)\rho)\). Using \((B.3.6)\) we see that

\[s_j((VR_0(\lambda)\rho)^p) \leq C_1 |\lambda|^{p} j^{-2p/n} \exp(C_1(\text{Im} \lambda)_-).\]

when \(p \geq (n + 1)/2\)

\[\sum_j s_j((VR_0(\lambda)\rho)^p) < \infty\]

which means the operator is of trace class. \(\square\)
3.4. UPPER BOUNDS ON THE NUMBER OF RESONANCES

Since for \( n \geq 2 \), \( VR_0(\lambda) \) is no longer of trace class we cannot use the determinant defined by (2.2.28).

**DEFINITION 3.25.** Suppose that \( n \geq 3 \) is odd. Using Lemma 3.24 the following definition is justified: for \( \rho \in C^\infty_c \) equal to 1 near the support of \( V \),

\[
H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}).
\]

We sometimes write \( H = H_V \) to emphasize the dependence on the potential.

**THEOREM 3.26 (Multiplicity of a resonance II).** Let the functions \( H \) be given by (3.4.4) and let \( m_H(\lambda) \) be the multiplicity of \( \lambda \) as a zero of \( H(\lambda) \).

Then, in the notation of (3.2.4),

\[
m_R(\lambda) \leq m_H(\lambda), \quad \lambda \in \mathbb{C}.
\]

**Proof.**

1. Arguing as in the proof of Theorem 3.15 it is enough to prove (3.4.5) when \( m_R(\lambda) \leq 1 \).

2. As \( n \) is odd,

\[
I - (VR_0(\lambda)\rho)^{n+1} = (I - VR_0(\lambda)\rho + \cdots - (VR_0(\lambda)\rho)^n)(I + VR_0(\lambda)\rho).
\]

Hence if \( \lambda \) is a simple pole of \( (I + VR_0(\lambda)\rho)^{-1} \) then the operator \( I - (VR_0(\lambda)\rho)^{n+1} \) has a non-empty kernel. That implies that \( H(\lambda) = 0 \), that is, \( m_H(\lambda) \geq 1 = m_R(\lambda) \) completing the proof of (3.4.5). \( \square \)

**DISCUSSION.** To obtain a determinant for which the zeros would agree with resonances with multiplicities we could use regularized determinants – see [Si79b] – and put

\[
D(\lambda) := \det(I + VR_0(\lambda)\rho), \quad p \geq \frac{n+1}{2}.
\]

However one can show that, except when \( n = 3 \), \( D(\lambda) \) grows too fast as \( \text{Im} \lambda \to -\infty \). This makes estimates on the number of zeros unwieldy.

The determinant \( H(\lambda) \) is introduced to remedy the growth problem but we pay by introducing additional zeros. For bounds on the growth of the number of resonances, which is all we are able to do precisely, that of course does not matter. The choice of \( n+1 \) as the power of \( VR_0(\lambda)\rho \) was arbitrary as in view of Lemma 3.24 we could have taken any \( p \geq (n+1)/2 \). It turns out convenient in the proof of Theorem 3.28.

The main result of this section is the following upper bound
THEOREM 3.27 (Upper bounds on the number of resonances I).
Suppose that \( n \geq 3 \) is odd and that \( V \in L_\infty^{\text{comp}}(\mathbb{R}^n; \mathbb{C}) \). Let \( m_R(\lambda) \) be the multiplicity of a resonance at \( \lambda \) as defined in (2.2.11).

Then

\[
\sum_{|\lambda| \leq r} m_R(\lambda) \leq C_V r^n.
\]

INTERPRETATION. In the case of \(-\Delta + V\) on a bounded domain, for instance on \( \mathbb{T}^n \), the spectrum is discrete and for \( V \in L_\infty(\mathbb{T}^n; \mathbb{R}) \) we have the asymptotic Weyl law for the number of eigenvalues:

\[
|\{ \lambda : \lambda^2 \in \text{Spec}(-\Delta_{\mathbb{T}^n} + V), \ |\lambda| \leq r \}| = c_n \text{vol}(\mathbb{T}^n) r^n (1 + O(1/r)),
\]

where the eigenvalues are included according to their multiplicities.

In the case of \(-\Delta + V\) on \( \mathbb{R}^n \) the discrete spectrum is replaced by the discrete set of resonances. Hence the bound (3.4.6) is an analogue of the Weyl law. Except in dimension one (see Theorem 2.14) the issue of asymptotics or even optimal lower bounds remains unclear at the time of writing (see Section 3.13 for references).

Jensen’s formula, see (D.1.5) in §D.2, and (3.4.5) show that Theorem 3.27 is an immediate consequence of an estimate on \( H(\lambda) \):

THEOREM 3.28 (Determinant bounds I). If \( V \in L_\infty^{\text{comp}}(\mathbb{R}^n; \mathbb{C}) \) and \( \rho \in C_\infty^c(\mathbb{R}^n) \) is equal to one on \( \text{supp} V \), then for some constant \( A \),

\[
H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}),
\]

satisfies

\[
|H(\lambda)| \leq A \exp(A|\lambda|^n).
\]

In particular, we have

\[
\sum_{|\lambda| \leq r} m_H(\lambda) \leq C_V r^n.
\]

Proof. 1. We use the Weyl inequality (B.5.5) to see that

\[
|H(\lambda)| \leq \prod_{k=1}^{\infty} (1 + s_k((VR_0(\lambda)\rho)^{n+1})).
\]

We then use (B.5.5) to see that

\[
s_k((VR_0(\lambda)\rho)^{n+1}) \leq \|V\|_{\infty}^{n+1} (s_{k/(n+1)}(\rho R_0(\lambda)\rho))^{n+1}.
\]

\[
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\]

3. UPPER BOUNDS ON THE NUMBER OF RESONANCES
Hence we need to estimate \( s_j(\rho R_0(\lambda)\rho) \) for \( \rho \in C_c^\infty(\mathbb{R}^n) \).

2. We start with easier estimates in the physical half-plane \( \text{Im}\lambda \geq 0 \). We apply (3.4.3) to obtain
\[
s_j(\rho R_0(\lambda)\rho) \leq \frac{C_j}{n^{1/n}},
\]
which inserted in (3.4.9) gives
\[
s_k((VR_0(\lambda)\rho)^{n+1}) \leq \frac{C_1}{k^{(n+1)/n}}.
\]
Using this in (3.4.8) we then get
\[
H(\lambda) \leq \exp \left( \sum_{k=1}^{\infty} s_k((VR_0(\lambda)\rho)^{n+1}) \right)
\]
\[
\leq \exp \left( C_1 \sum_{k=1}^{\infty} k^{-(n+1)/n} \right)
\]
\[
\leq C_2,
\]
that is, \( H(\lambda) \) is uniformly bounded for \( \text{Im}\lambda \geq 0 \).

3. To obtain estimates for \( \text{Im}\lambda < 0 \) we use (3.1.19) to write
\[
\rho(R_0(\lambda) - R_0(-\lambda))\rho = a_n \lambda^{-2} E_{\rho}(\bar{\lambda})^* E_{\rho}(\lambda),
\]
(3.4.10)
\[
E_{\rho}(\lambda)u(\omega) := \int_{\mathbb{R}^n} e^{i\lambda\langle \omega, x \rangle} \rho(x)u(x)dx,
\]
\[
E_{\rho}(\lambda) : L^2(\mathbb{R}^n) \rightarrow L^2(S^{n-1}).
\]
Hence for \( \text{Im}\lambda < 0 \) (B.3.5) gives
(3.4.11)
\[
s_j(\rho R_0(\lambda)\rho) \leq a_n |\lambda|^{-2}\|E_{\rho}(\lambda)\|s_{[j/2]}(E_{\rho}(\lambda)) + s_{[j/2]}(\rho R_0(-\lambda)\rho)
\]
\[
\leq C \exp(C|\lambda|) s_{[j/2]}(E_{\rho}(\lambda)) + C j^{-1/n}.
\]

4. To estimate \( s_j(E_{\rho}(\lambda)) \) we use the Laplacian on the sphere, \(-\Delta_{S^{n-1}}\), and (B.3.6):
(3.4.12)
\[
s_j(E_{\rho}(\lambda)) \leq s_j((-\Delta_{S^{n-1}} + 1)^{-\frac{\ell}{2}}\|(-\Delta_{S^{n-1}} + 1)^{\frac{\ell}{2}} E_{\rho}(\lambda)\|
\]
\[
\leq C_j \frac{j^{-2\ell/(n-1)}}{\|(-\Delta_{S^{n-1}} + 1)^{\frac{\ell}{2}} E_{\rho}(\lambda)\|}
\]
\[
\leq C_j \frac{j^{-2\ell/(n-1)}}{\exp(C_1|\lambda|)(2\ell)!}.
\]
Here we used the fact that for \( \rho \) with support in \( B(0, R) \),
\[
\|(-\Delta_{S^{n-1}} + 1)^{\frac{\ell}{2}} E_{\rho}(\lambda)\| \leq C_\rho \sup_{\omega \in S^{n-1}, |x| \leq R} \left| (-\Delta_{S^{n-1}} + 1)^{\frac{\ell}{2}} e^{i\lambda\langle x, \omega \rangle} \right|,
\]
and we estimated \( \sup \) using, essentially, the Cauchy estimates.
We now optimize the estimate (3.4.12) in $\ell$: by Stirling’s formula

$$C_4 \ell^{-2\ell/(n-1)}(2\ell)! \leq (j/(C_4 \ell^{n-1}))^{-2\ell/(n-1)} \exp(-j \frac{1}{n-1} / C_4),$$

if $\ell = (j/C_4 \ell^{n-1})^{\frac{1}{2}}$.

This gives

(3.4.13) \[ s_j(E_\rho(\lambda)) \leq C_2 \exp \left( C_2|\lambda| - j \frac{1}{n-1} / C_4 \right). \]

5. Going back to (3.4.9) and (3.4.11) we obtain

$$s_k((V R_0(\lambda)\rho)^{n+1}) \leq C_3 \exp \left( C_3|\lambda| - k \frac{1}{n-1} / C_3 \right) + C_3 k^{-\frac{n+1}{n}}.$$

In particular,

(3.4.14) \[ s_k((V R_0(\lambda)\rho)^{n+1}) \leq \begin{cases} C_4 \exp(C_4|\lambda|), & k \leq C_4|\lambda|^{n-1} \\ C_4 k^{-\frac{n+1}{n}}, & k \geq C_4|\lambda|^{n-1}. \end{cases} \]

Returning to (3.4.8) we use (3.4.14) as follows

$$|H(\lambda)| \leq \prod_{k \leq C_4|\lambda|^{n-1}} \exp(C_4|\lambda|) \left( \exp \sum_{k \geq C_4|\lambda|^{n-1}} C_4 k^{-(n+1)/n} \right)$$

\[ \leq \exp(C_5|\lambda|^n), \]

which completes the proof. \( \square \)

**REMARK.** The exponent $n$ in (3.4.6) is optimal as shown by the case of radial potentials. Let $V(x) = v(|x|)(R-|x|)^0_0$, where $v$ is a $C^2$ even function, and $v(R) > 0$. Then, see [Zw89a],

(3.4.15) \[ \sum \{ m_R(\lambda) : |\lambda| \leq r \} = C_R r^q(1 + o(1)). \]

The constant $C_R$ and its appearance in (3.4.6) is explained and discussed in [St06].

### 3.5. COMPLEX VALUED POTENTIALS WITH NO RESONANCES

As we have seen in Theorem 2.14 one dimensional complex valued compactly supported non-zero potentials always have infinitely many resonances with a counting functions satisfying a nice asymptotic formula. The situation is dramatically different in higher dimensions where complex valued potentials may have no resonances at all.
3.5. COMPLEX VALUED POTENTIALS WITH NO RESONANCES

**Theorem 3.29** (Complex valued potentials with no resonances). Let \((r, \theta, x')\) be cylindrical coordinates in \(\mathbb{R}^{k+2}\), where \(k \geq 1\) is odd:

\[
x = (x_1, x_2, x') \quad x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x' \in \mathbb{R}^k.
\]

Suppose that \(V \in L^\infty_{\text{comp}}(\mathbb{R}^3; \mathbb{C})\) is of the following form:

\[
V(x) = e^{i \theta m} W(r, x'), \quad W \in L^\infty_{\text{comp}}([0, \infty) \times \mathbb{R}^k).
\]

Then, if \(m \neq 0\), the resolvent \(R_V(\lambda)\) is entire in \(\mathbb{C}\), that is the operator 

\[-\Delta + V\]

has no resonances.

**Remark.** We can easily place conditions on \(W\) so that

\(V \in C^\infty_c(\mathbb{R}^{2+k}; \mathbb{C}).\)

Before starting the proof we need two simple lemmas

**Lemma 3.30** (Fourier decomposition of the resolvent). Let \(\Pi_\ell\) be the projection onto the \(\ell\)’th Fourier mode:

\[
(3.5.1) \quad \Pi_\ell u(r, \theta, x') := e^{i \theta \ell} \frac{1}{2\pi} \int_0^{2\pi} u(r, \varphi, x') e^{-i \varphi} d\varphi.
\]

Then for \(\rho \in C^\infty_c(\mathbb{R}^{2+k}), \rho = \rho(r, x')\), we have

\[
\| \Pi_\ell \rho R_0(\lambda) \rho \Pi_\ell \|_{L^2 \to L^2} \leq \frac{C(\lambda) e^{C(\Re \lambda)}}{\langle \ell \rangle}, \quad \ell \in \mathbb{Z}.
\]

**Proof.** 1. Because we chose \(\rho\) to be independent of \(\theta\), \(\Pi_\ell\) commutes with \(\rho R_0(\lambda) \rho\). Put

\[
u := \rho R_0(\lambda) \rho \Pi_\ell f, \quad f \in L^2.
\]

Then (3.1.12) gives

\[
l_\ell^u \leq C(\lambda) e^{C(\Re \lambda)} \| f \|_{L^2}.
\]

2. On the other hand

\[
l_\ell^u \geq l_\ell^u \geq \langle -\Delta u, u \rangle
\]

\[
= \int_{\mathbb{R}^{n-2}} \int_0^\infty \int_0^{2\pi} (D_r^2 - (i/r)D_r - \Delta_{x'} + \ell^2/r^2) u \bar{u} d\theta dr dx'
\]

\[
= \int_{\mathbb{R}^{n-2}} \int_0^\infty \int_0^{2\pi} (|D_r u|^2 + |D_{x'} u|^2 + \ell^2/r^2) u \bar{u} d\theta dr dx'
\]

\[
\geq \langle (\ell^2/r^2) u, u \rangle_{L^2} \geq \ell^2 \| u \|_{L^2}^2/C,
\]

where the last inequality followed from the fact that \(r\) is bounded on the support of \(u\) by \(\max_{x \in \text{supp} \rho} |x|\) so that \(\ell^2/r^2 \geq \ell^2/C\). Combining this with (3.5.3) proves (3.5.2). \(\square\)
The next lemma is an elementary statement about sequences:

**LEMMA 3.31 (Two sided sequences).** Let \( \{a_j\}_{j=-\infty}^{\infty} \) be a sequence satisfying \( a_j \to 0 \), \( j \to \pm\infty \). Suppose that for some \( m \in \mathbb{Z} \setminus \{0\} \) and \( J \in \mathbb{N} \) we have the following property: for each \( j \) there exists \( C_j \geq 0 \) such that

\[
|a_{j+m}| \leq C_j |a_j|, \quad \text{and} \quad C_j \leq 1 \text{ for } |j| \geq J,
\]

for some \( J \).

Then \( a_j \equiv 0, \ j \in \mathbb{Z} \).

**Proof.** Fix \( j \in \mathbb{Z} \) and use (3.5.4) to obtain

\[
|a_j| \leq C_{j-m} |a_{j-m}| \leq \cdots \leq \prod_{k=1}^{p} C_{j-km} |a_{j-mp}|
\]

\[
\leq K |a_{j-mp}| \to 0, \quad p \to \infty, \quad K := \prod_{|\ell| < J} C_\ell \geq \prod_{|j-mk| < J} C_{j-km}.
\]

This shows that \( a_j = 0 \) as claimed. \qed

**Proof of Theorem 3.29.** 1. In view of (3.2.16) if \( m_R(\lambda) > 0 \) for some \( \lambda \) then \( (I + VR_0 \rho)^{-1} \) has a pole for any \( \rho \in C_{\infty}^\infty(\mathbb{R}^{2+k}) \) such that \( \rho = 1 \) on \( \text{supp} \ V \). In particular we can take \( \rho = \rho(r, x') \).

Hence there exists \( u \in L^2 \) such that

\[
u = -V R_0(\lambda) \rho u = -V \rho R_0(\lambda) \rho \, u.
\]

2. We now use the structure of \( V, V(r, \theta, x') = e^{im\theta} W(r, x') \), to calculate

\[
\Pi_{j+m} u = \Pi_{j+m} \left( e^{im\theta} W \rho R_0(\lambda) \rho u \right)
\]

\[
= e^{im\theta} \Pi_j W \rho R_0(\lambda) \rho \Pi_j u.
\]

Lemma 3.30 now shows that

\[
\|\Pi_{j+m} u\|_{L^2} \leq \frac{C(\lambda) e^{C|\lambda|}}{\langle j \rangle} \|\Pi_j u\|_{L^2}.
\]

If we put

\[
a_j := \|\Pi_j u\|_{L^2}, \quad C_j := \frac{C(\lambda) e^{C|\lambda|}}{\langle j \rangle},
\]

then the assumptions of Lemma 3.31 are satisfied. Thus \( \Pi_j u = 0 \) for all \( j \) which means that \( u = 0 \) and there is no resonance at \( \lambda \). \qed
3.6. OUTGOING SOLUTIONS AND RELLICH’S THEOREM

In Section 2.4 the scattering matrix mapped incoming to outgoing components of solutions to
\[(P_V - \lambda^2)w = 0.\]
The intuition behind the notion of incoming and outgoing components of a solution to (3.6.1) was presented in §2.1.

A conceptually similar procedure is used in the case of scattering in higher dimensions with asymptotic formulae such as (3.1.20) replacing explicit representations involving \(\exp(i\lambda|x|)\). The starting point is the same as in (2.4.3): we consider solutions to (3.6.1) of the form
\[(3.6.2) w(x, \lambda, \omega) = e^{-i\lambda\langle x, \omega \rangle} + u(x, \lambda, \omega),\]
where \(u\) is outgoing in the sense defined below. It is obtained using the resolvent \(R_V(\lambda)\), except at the possible poles:
\[(3.6.3) u(x, \lambda, \omega) := -R_V(\lambda)(Ve^{-i\lambda\langle \bullet, \omega \rangle}).\]

First we need to define outgoing solutions and show that \(R_V(\lambda)\) is well defined for \(\lambda \in \mathbb{R} \setminus \{0\}\).

**Definition 3.32.** A solution \(u\) to \((P_V - \lambda^2)u = f, \lambda \in \mathbb{R} \setminus \{0\}, f \in L^2_{\text{comp}}(\mathbb{R}^n)\) is called outgoing if there exists \(g \in L^2_{\text{comp}}(\mathbb{R}^n)\) such that
\[(3.6.4) u = R_0(\lambda)g,\]
where \(R_0(\lambda)\) is the resolvent given by (3.1.12).

A solution \(u\) is called incoming if \(u = R_0(-\lambda)g, \lambda \in \mathbb{R} \setminus \{0\}, \) for some \(g \in L^2_{\text{comp}}(\mathbb{R}^n)\).

**Interpretation.** The asymptotic expansion in Theorem (3.5) shows that the outgoing (+) and incoming (−) solutions satisfy
\[u(x) = e^{\pm i\lambda|x|} a \left( \frac{x}{|x|} \right) + O \left( \frac{1}{|x|^\frac{n+1}{2}} \right), \quad |x| \to \infty.\]
Hence \(u\) can be interpreted as a spherical wave with \(a(x/|x|)\) giving the intensity at different directions \(x/|x|\). The different signs \(\lambda\) show that \(R_0(\pm)g\) is holomorphic in \(\pm \text{Im} \lambda \in \mathbb{R} \setminus \{0\}\). The wave equation interpretation discussed in §2.1 shows that the corresponding time dependent solutions are supported in \(\pm t > C\), that is are outgoing/incoming. See also the self-contained discussion after Theorem 4.9 and Exercise 4.3.
3. OUTGOING SOLUTIONS AND RELLICH’S THEOREM

In particular we see that \( u \) given by (3.6.3) is outgoing provided that \( \lambda \) is not a pole of \( R_V(\lambda) \):

\[
u(x, \lambda, \omega) = -R_V(\lambda)(Ve^{-i\lambda \langle \bullet, \omega \rangle}) = R_0(\lambda)f, \]
\[
f = -(I + VR_0(\lambda)\rho)^{-1}(Ve^{-i\lambda \langle \bullet, \omega \rangle}) \in L^2_{\text{comp}}(\mathbb{R}^n). \]

When \( V \) is real valued and \( \lambda \in \mathbb{R} \setminus \{0\} \) then Rellich’s important result (Theorem 3.33) states that there are no outgoing solutions to (3.6.1). In other words, \( R_V(\lambda) \) has no non-zero real poles:

**Theorem 3.33 (Rellich’s uniqueness theorem I).** Suppose that \( V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}) \) is real valued. Then for \( \lambda \in \mathbb{R} \setminus \{0\} \) there are no outgoing solutions to

\[
(P_V - \lambda^2)u = 0.
\]

Equivalently, \( R_V(\lambda) \) has no poles for \( \lambda \in \mathbb{R} \setminus \{0\} \).

**Proof.** 1. We first show that having an outgoing solution for \( \lambda > 0 \) is equivalent to \( R_V \) having a pole at \( \lambda \).

One implication follows directly from (3.2.1): if \( R_V \) has a pole at \( \lambda \) then \( I + VR_0(\lambda)\rho \) is not invertible which by the Fredholm property (see §C.2) means that it has a non-empty kernel. If \( g = -VR_0(\lambda)\rho g \), then \( g = \rho g \in L^2_{\text{comp}} \) and \( u := R_0(\lambda)g \) solves

\[
(P_V - \lambda^2)u = (-\Delta - \lambda^2)R_0(\lambda)g + VR_0(\lambda)g = g + VR_0(\lambda)g = 0.
\]

Hence (3.6.4) holds.

Conversely, suppose that \( u = R_0(\lambda)g, g \in L^2_{\text{comp}} \) solves \((P_V - \lambda^2)u = 0\).

By the same argument we see \((I + VR_0(\lambda)\rho)g = 0\) and by Theorem 3.26, \( R_V \) has a pole at \( \lambda \).

2. The proof now proceeds by contradiction. So suppose that \( R_V \) has a pole at \( \lambda > 0 \). From (3.2.1) we see (as in Step 1) that there exists \( g \in L^2 \) such that \( g = -VR_0(\lambda)g \). Defining \( w = -R_0(\lambda)g \) we obtain \( Vw = g \), and hence,

\[
(P_V - \lambda^2)w = 0, \quad w = -R_0(\lambda)Vw.
\]

Theorem 3.5 shows that

\[
w = R_0(\lambda)(Vw)(x) = e^{i\lambda |x|} \left( h \left( \frac{x}{|x|} \right) + O \left( \frac{1}{|x|} \right) \right),
\]

where

\[
h(\theta) = c_n \lambda^{n-3} \sqrt{Vw}(\lambda \theta).
\]

In particular,

\[
(\partial_r - i\lambda)w = O(r^{-\frac{n+1}{2}}), \quad r := |x|.
\]
3. Since $\lambda$ is real we have
\[
0 = \int_{B(0,R)} (w(P_V - \lambda^2)\bar{w} - (P_V - \lambda^2)w\bar{w})dx
\]
\[= \int_{B(0,R)} (\bar{w}\Delta w - w\Delta \bar{w})dx = \int_{\partial B(0,R)} (\partial_r w\bar{w} - w\partial_r \bar{w})dS \tag{3.6.7}
\]
Using (3.6.5) and (3.6.6) we obtain
\[
0 = 2i\lambda \int_{\partial B(0,R)} |w|^2dS + \mathcal{O}(R^{-n}) \int_{\partial B(0,R)} dS
\]
which gives
\[
\int_{\partial B(0,R)} |w|^2dS = \mathcal{O}(R^{-1}).
\]
Taking $R \to \infty$ this implies, in the notation of (3.6.5), that
\[
0 = \int_{S^{n-1}} |h(\theta)|^2d\theta = |c_n|^2|\lambda|^{n-3} \int_{S^{n-1}} |\hat{V}w(\lambda\theta)|^2d\theta.
\]
4. We conclude that
\[
\hat{V}w(\xi) = 0, \quad \langle \xi, \xi \rangle = \lambda^2, \quad \xi \in \mathbb{R}^n.
\]
If we put
\[
\Sigma := \{ \xi \in \mathbb{C}^n : \langle \xi, \xi \rangle = \lambda^2 \},
\]
then $\Sigma$ is a connected complex hypersurface in $\mathbb{C}^n$ and the entire function $\hat{V}w(\xi)$ vanishes on $\Sigma \cap \mathbb{R}^n$. It follows that $\hat{V}w(\xi) = 0$ on $\Sigma$. From that we see that
\[
\frac{\hat{V}w(\xi)}{\langle \xi, \xi \rangle - \lambda^2}
\]
is an entire function of $\xi \in \mathbb{C}^n$.
Since
\[
(\langle \xi, \xi \rangle - \lambda^2)\hat{w}(\xi) = \hat{V}w(\xi),
\]
Paley-Wiener theorem as applied in [HöI, Theorem 7.3.2] shows that $w \in \mathcal{E}'$.

To complete the proof we need the following lemma which is a simple version of a Carleman estimate (see [Zw12, §7.2] and references given there):

**Lemma 3.34.** For every $R > 0$ there exists $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that for $h > 0$ and $u \in H^2(\mathbb{R}^n)$ with $\text{supp} \ u \subset B(0, R)$ we have
\[
\|h^2 e^{\varphi/h} \Delta e^{-\varphi/h} u\|_{L^2} \geq ch^{\frac{1}{2}} \|u\|_{L^2}. \tag{3.6.8}
\]

**Proof.** Let us first assume that $u \in C_c^\infty(\mathbb{R}^n)$. Put
\[
P_\varphi := -h^2 e^{\varphi/h} \Delta e^{-\varphi/h}.
\]
3. OUTGOING SOLUTIONS AND RELLICH’S THEOREM

Then
\[
\|P_\varphi u\|_{L^2}^2 = \langle P_\varphi u, P_\varphi u \rangle = \langle P_\varphi^* P_\varphi u, u \rangle
\]
\[
= \langle P_\varphi^* u, u \rangle + \langle [P_\varphi^*, P_\varphi] u, u \rangle
\]
\[
= \|P_\varphi^* u\|_{L^2}^2 + \langle [P_\varphi^*, P_\varphi] u, u \rangle
\]
\[
\geq \langle [P_\varphi^*, P_\varphi] u, u \rangle. \tag{3.6.9}
\]

2. From (3.6.9) we see that it suffices to construct \( \varphi \) such that for all \( u \in C_c^\infty(B(0, R)) \)
\[
\langle [P_\varphi^*, P_\varphi] u, u \rangle \geq c^2 h \|u\|_{L^2}.
\]
A calculation shows that
\[
P_\varphi u = -h^2 \Delta u + 2h(\partial \varphi, \partial u) - |\partial \varphi|^2 u + h(\Delta \varphi)u,
\]
\[
P_\varphi^* u = -h^2 \Delta u - 2h(\partial \varphi, \partial u) - |\partial \varphi|^2 u + h(\Delta \varphi)u.
\]
Using the identity
\[
[P_\varphi^*, P_\varphi] = \frac{1}{2} [P_\varphi + P_\varphi^*, P_\varphi - P_\varphi^*]
\]
we compute
\[
[P_\varphi^*, P_\varphi] u = -8h^3 \sum_{j,k=1}^n \partial_{x_j x_k}^2 \varphi \cdot \partial_{x_j x_k}^2 u + 4h(\partial \varphi, \partial |\partial \varphi|^2) u
\]
\[
- 8h^3 (\partial(\Delta \varphi), \partial u) - 2h^3 (\Delta^2 \varphi) u. \tag{3.6.10}
\]
We choose \( \varphi(x) := |x|^2/2 + Mx_1 \) for some large constant \( M \). Then the contribution from the second line of (3.6.10) is zero and we compute
\[
[P_\varphi^*, P_\varphi] u = 8h(-h^2 \Delta u + |x + Me_1|^2 u).
\]
If \( M \geq R + 1 \) then for \( u \in C_c^\infty(B(0, R)) \),
\[
\langle [P_\varphi^*, P_\varphi] u, u \rangle \geq 8h(\|hD_x u\|_{L^2}^2 + \|u\|_{L^2}^2) \geq 8h \|u\|_{L^2}^2.
\]

3. The last inequality proved (3.6.8) for \( u \in C_c^\infty(B(0, R)) \). Since both sides are finite for \( u \in H^2 \), \( \text{supp} \ u \subset B(0, R) \) an approximation argument completes the proof. \( \square \)

5. To complete the proof of Theorem 3.33 we now apply Lemma 3.34 to \( u = e^{\varphi/h} w \) where \( w \) comes from Step 4 of the proof: \( w \in H^2 \), \( \text{supp} \ w \subset B(0, R) \). We have
\[
0 = h^2 \|e^{\varphi/h}(P_V - \lambda^2)w\|_{L^2} = \|e^{\varphi/h}(-h^2 \Delta + h^2 V - h^2 \lambda^2)e^{-\varphi/h}u\|_{L^2}
\]
\[
\geq \|e^{\varphi/h}(-h^2 \Delta)e^{-\varphi/h}u\|_{L^2} - Ch^2 \|u\|_{L^2}
\]
\[
\geq ch^{\frac{3}{2}} \|u\|_{L^2} - Ch^2 \|u\|_{L^2} \geq (c/2)h^{\frac{3}{2}} \|u\|_{L^2},
\]

\[\|u\|_{H^2} \leq (c/2)h^{\frac{3}{2}} \|u\|_{L^2}.
\]
if \( h \) is small enough. But this means that \( u \equiv 0 \) which implies that \( w \equiv 0 \). Hence we have no outgoing solutions to the homogeous equation when \( \lambda \in \mathbb{R} \setminus \{0\} \). □

Rellich’s uniqueness theorem holds in a stronger form which will be useful later in this section and also when we consider more general perturbations:

THEOREM 3.35 (Rellich’s uniqueness theorem II). Suppose \( P \) is a self-adjoint operator with domain \( H^2(\mathbb{R}^n) \) such that for \( \chi \in C_\infty^\infty(B(0,2R)) \), \( \chi = 1 \) in \( B(0,R) \) we have \( P(1 - \chi) = -\Delta(1 - \chi) \). Suppose that \( \lambda > 0 \) and \( u \in H^2_{\text{loc}} \) satisfies

\[
(P - \lambda^2)u = 0, \quad \lim_{R \to \infty} \int_{\partial B(0,R)} |(\partial_r - i\lambda)u|^2 dS = 0.
\]

Then

\[
(3.6.12) \quad u(x) = 0 \text{ for } |x| > R.
\]

REMARKS. 1. The second condition in (3.6.11) is implied by a stronger condition that

\[
(\partial_r - i\lambda)u = o(r^{-\frac{n-1}{2}}),
\]

which is called the Sommerfeld radiation condition. As we saw in (3.6.6) (Step 2 of the proof of Theorem 3.33) it typically arises in an even stronger form

\[
(\partial_r - i\lambda)u = O(r^{-\frac{n+1}{2}}).
\]

2. The specific structure of \( P \) is unimportant and this is our first encounter with more general operators than \(-\Delta + V\). What matters is the fact that \( P \) coincides with \(-\Delta\) outside a compact set and that it is self-adjoint. The assumptions about the domain of \( P \) can also be relaxed as we will see in the chapter on black box scattering.

Proof. 1. With \( \chi \) as in the statement of theorem we have

\[
(-\Delta - \lambda^2)(1 - \chi)u = [\Delta, \chi]u =: f \in C_\infty^\infty(\mathbb{R}^n).
\]

We claim that \((1 - \chi)u = R_0(\lambda)f\). To see that put

\[
w := (1 - \chi)u - R_0(\lambda)f, \quad (-\Delta - \lambda^2)w = 0.
\]

Then from (3.6.11) and the asymptotics of \( R_0(\lambda)f \) in (3.1.20) we see that

\[
(3.6.13) \quad \int_{\partial B(0,R)} |G|^2 dS = o(1), \quad G := \frac{1}{2\pi}(\partial_r - i\lambda)w.
\]
We now use Green’s formula (see Step 3 of the proof of Theorem 3.33 for a similar argument):

\[
0 = \frac{1}{2\pi} \int_{B(0,R)} (w(-\Delta - \lambda^2)\dot{w} - \dot{\lambda}(\Delta - \lambda^2)w) \, dx
\]

\[
= \frac{1}{2\pi} \int_{B(0,R)} (\dot{w} \Delta w - w \Delta \dot{w}) \, dx = \frac{1}{2\pi} \int_{\partial B(0,R)} (\partial_r \dot{w} - w \partial_r \dot{w}) \, dS
\]

\[
= \int_{\partial B(0,R)} (|w|^2 + 2\Re G\dot{w}) \, dS \geq \frac{1}{2} \int_{\partial B(0,R)} |w|^2 \, dS - 2\int_{\partial B(0,R)} |G|^2 \, dS.
\]

This and (3.6.13) (which was derived from the assumption (3.6.11)) imply that

\[
\varphi(R) := \int_{\partial B(0,R)} |w|^2 \, dS \to 0, \quad R \to \infty.
\]

It follows that \(\frac{1}{R} \int_{B(0,R)} |w(x)|^2 \, dx = \frac{1}{R} \int_0^R \varphi(r) \, dr \to 0, \quad R \to \infty.\) (Note that this implies that \(w \in \mathcal{S}'(\mathbb{R}^n)\) and hence \(\dot{w}\) makes sense as a distribution.) From \((-\Delta - \lambda^2)w = 0\) we have \(\sup \dot{w} \subset \{ |\xi|^2 = \lambda^2 \}.\) To see that \(w = 0\) we apply the following result to \(u = \dot{w}.\) It is a special case of [H61, Theorem 7.1.27]:

**Lemma 3.36.** Suppose that \(u \in \mathcal{E}'(\mathbb{R}^n)\) satisfies

\[
\text{supp } u \subset \partial B(0,1) \quad \text{and} \quad \lim_{R \to \infty} \frac{1}{R} \int_{B(0,R)} |\hat{u}(\xi)|^2 \, d\xi = 0.
\]

Then \(u \equiv 0.\)

**Proof.** 1. Since \(u\) is compactly supported we see that \(\hat{u} \in C^\infty(\mathbb{R}^n).\) Suppose that \(\chi \in C^\infty_c(B(0,1))\) and \(\int \chi(x) \, dx = 1.\) Put \(\chi_\varepsilon(x) := \varepsilon^{-n} \chi(x/\varepsilon)\) and define \(C^\infty_c(\mathbb{R}^n) \ni u_\varepsilon := u * \chi_\varepsilon \to u \in \mathcal{D}'(\mathbb{R}^n).\)

Since \(\hat{u}_\varepsilon(\xi) = \hat{\chi}_\varepsilon \hat{u},\) Plancherel’s formula gives

\[
(2\pi)^n \|u_\varepsilon\|^2 = \int_{\mathbb{R}^n} |\hat{u}_\varepsilon(\xi)|^2 |\hat{\chi}_\varepsilon(\xi)|^2 \, d\xi
\]

\[
\leq \left( \int_{|\xi| \leq 1} |\hat{u}_\varepsilon(\xi)|^2 |\hat{\chi}_\varepsilon(\xi)|^2 \, d\xi \right) \sup_{|\xi| \leq 1} |\hat{\chi}_\varepsilon(\xi)|^2 d\xi
\]

\[
\leq \frac{1}{\varepsilon} \sup_{|\eta| \leq 1} |\hat{\chi}(\eta)|^2 \varepsilon \int_{|\xi| \leq 1/\varepsilon} |\hat{u}(\xi)|^2 \, d\xi
\]

\[
+ \frac{1}{\varepsilon} \sum_{j=1}^\infty 2^j \sup_{|\eta| \leq 2^j} |\hat{\chi}(\eta)|^2 2^{-j} \varepsilon \int_{|\xi| \leq 2^j/\varepsilon} |\hat{u}(\xi)|^2 \, d\xi
\]

\[
\leq \frac{1}{\varepsilon} C_\chi \sup_{R > 1/\varepsilon} \frac{1}{R} \int_{B(0,R)} |\hat{u}(\xi)|^2 \, d\xi,
\]
3.6. OUTGOING SOLUTIONS AND RELLICH’S THEOREM

where

\[ C_\chi = \sup_{|\eta| \leq 1} |\hat{\chi}(\eta)|^2 + \sum_{j=1}^{\infty} \sup_{2^{j-1} \leq |\eta| \leq 2^j} 2^j |\hat{\chi}(\eta)|^2. \]

The sum converges as \( \hat{\chi} \in \mathcal{S} \) and thus \( \hat{\chi}(\eta) = \mathcal{O}(\langle \eta \rangle^{-\infty}) \).

Consequently, the hypothesis of the lemma gives

\[ (3.6.14) \quad \|u_\epsilon\|_2^2 \leq \frac{C}{\epsilon} \sup_{R>1/\epsilon} \mathcal{R} \int_{B(0,R)} |\hat{u}(\xi)|^2 d\xi =: K(\epsilon)/\epsilon, \quad \lim_{\epsilon \to 0} K(\epsilon) = 0. \]

2. Now suppose that \( \psi \in C^\infty_c(\mathbb{R}^n) \). Because of the support condition on \( u \) we know that \( \text{supp} u_\epsilon \subset \text{supp} u + \text{supp} \chi_\epsilon \subset \partial B(0,1) + B(0,\epsilon) =: A_\epsilon \). Using (3.6.14) we see that

\[ |u(\psi)|^2 = \lim_{\epsilon \to 0} |u_\epsilon(\psi)|^2 \leq \lim_{\epsilon \to 0} \|u_\epsilon\|^2 \int_{A_\epsilon} |\psi(x)|^2 dx \]

\[ \leq \lim_{\epsilon \to 0} K(\epsilon) \times \lim_{\epsilon \to 0} \epsilon^{-1} \int_{A_\epsilon} |\psi(x)|^2 dx = \lim_{\epsilon \to 0} K(\epsilon) \int_{\partial B(0,1)} |\psi(x)|^2 dS = 0. \]

Since \( \psi \) was an arbitrary smooth function it follows \( u \equiv 0 \). \qed

2. Step 1 of the proof of Theorem 3.35 and Lemma 3.35 show that

\( (1-\chi)u = R_0(\lambda)f, \quad f = [\Delta, \chi]u \in C^\infty_c(\mathbb{R}^n). \)

For \( \chi_1 \in C^\infty_c(\mathbb{R}^n) \) equal to 1 on the support of \( \chi \), the expansion (3.1.20) and Green’s formula give (see (3.6.7))

\[ (3.6.15) \quad \frac{1}{i} \langle [-\Delta, \chi_1]R_0(\lambda)f, R_0(\lambda)f \rangle = |c_n|^2 \lambda^{n-2} \int_{\mathbb{S}^{n-1}} |\hat{f}(\lambda \theta)|^2 d\theta. \]

3. On the other hand,

\[ \langle [-\Delta, \chi_1]R_0(\lambda)f, R_0(\lambda)f \rangle = \langle [-\Delta, \chi_1]u, u \rangle \]

\[ = \langle [-\Delta - \lambda^2, \chi_1]u, u \rangle \]

\[ = \langle [P - \lambda^2, \chi_1]u, u \rangle \]

\[ = \langle \chi_1 u, (P - \lambda^2)u \rangle - \langle (P - \lambda^2)u, \chi_1 u \rangle \]

\[ = 0. \]

Returning to (3.6.15) we see that \( \hat{f}(\lambda \theta) \equiv 0, \theta \in \mathbb{S}^{n-1} \).

4. We now argue as in Step 4 of the proof of Theorem 3.33: \( \hat{f}(\xi)/(\langle \xi, \xi \rangle - \lambda^2) \) is entire and hence \( (1-\chi)u \) is compactly supported. \qed
INTERPRETATION. In the formula (3.6.15) the left hand side depends on \( \chi_1 \) while the right hand side does not. Another way to of stating this formula is

\[
\frac{1}{i} \langle [-\Delta, \chi_1] R_0(\lambda) f, R_0(\lambda) f \rangle = \text{Im} \langle R_0(\lambda) f, f \rangle,
\]

which follows from (3.6.16) applied with \(-\Delta\) in place of \(P\) and with \(\chi_1 = 1\) on \(\text{supp} f\). The Stone’s formula (3.1.19) the right hand side can be expressed using the spectral pojection of \(-\Delta\). From this (3.6.15) follows directly without using the expansion (3.1.20).

The expression on the left of (3.6.17) is called the quantum flux of \(R_0(\lambda)f\) (or of more general solutions). The fact that \(R_0(\lambda)f\) is outgoing is reflected in the fact that the flux is positive; it is negative for an incoming solution.

Using Rellich’s uniqueness theorem the condition of being outgoing can be formulated in the following equivalent ways. For the proof see the hints in Exercise 3.5.

**THEOREM 3.37 (Outgoing solutions).** Suppose that \(V \in L^\infty_\text{comp}(\mathbb{R}^n; \mathbb{R})\), \(f \in \mathcal{E}'(\mathbb{R}^n)\) is a compactly supported distribution and that \(u\) solves

\[
(P_V - \lambda^2)u = f, \quad \lambda \in \mathbb{R} \setminus \{0\}.
\]

Then the following conditions are equivalent:

i) \(u(x) = e^{i\lambda |x| a(x/|x|)}|x|^{-(n-1)/2} + \mathcal{O}(|x|^{-(n+1)/2})\), as \(|x| \to \infty\), where the expansion can be differentiated,

ii) \((\partial/\partial r - i\lambda)u = o(r^{-(n-1)/2})\), as \(r \to \infty\), \(r = |x|\),

iii) \(u = R_V(\lambda)f\),

iv) \(u = R_0(\lambda)g\), for some \(g \in \mathcal{E}'(\mathbb{R}^n)\).

When \(V \in C^\infty_c(\mathbb{R}^n; \mathbb{R})\) and \(f \in C^\infty_c(\mathbb{R}^n)\) then \(g \in C^\infty_c(\mathbb{R}^n)\).

As in dimension one we want to decompose the solution (3.6.2) into incoming and outgoing terms. The scattering matrix will then relate these two terms.

**THEOREM 3.38 (Decomposition of free plane waves).** For \(\lambda \in \mathbb{R} \setminus \{0\}\), we have, in the sense of distributions in \(x/|x| \in S^{n-1}\)

\[
e^{-i\lambda(x,\omega)} \sim \frac{1}{(\lambda|x|)^{\frac{n-1}{2}}} \left( c_n^+ e^{-i\lambda |x|} \delta_\omega(x/|x|) + c_n^- e^{i\lambda |x|} \delta_- \omega(x/|x|) \right),
\]

as \(|x| \to \infty\), where

\[
c_n^\pm = (2\pi)^{\frac{n-1}{2}} e^{\pm \frac{\pi}{4} (n-1)i}.
\]
More precisely for $\varphi \in C^\infty (\mathbb{S}^{n-1})$,

$$(\lambda r)^{n-1 \over 2} \int_{\mathbb{S}^{n-1}} e^{-i \lambda r (\omega, \theta)} \varphi (\theta) d\theta = c_+^n e^{-i \lambda r} \varphi (\omega) + c_-^n e^{i \lambda r} \varphi (-\omega) + \mathcal{O}(1/r),$$

as $r \to \infty$, with a full expansion in powers of $r$.

**INTERPRETATION.** We consider

$$\lambda - n - 1 \over 2 c_+ \delta_+ (\theta)$$

as leading coefficients of the incoming (+) and outgoing (−) components of $\exp(-i \lambda (x, \omega))$, even though that is valid only in the sense of distributions. This is an analogue of the decomposition of $\exp(\pm i \lambda x)$, $x \in \mathbb{R}$,

$$e^{\pm i \lambda x} = e^{-i \lambda |x|} (\pm x)_0^0 + e^{i \lambda |x|} (\pm x)_0^0, \quad x \neq 0.$$

**Proof.** To prove the results we use the method of stationary phase – see for instance [Zw12, §3.5] or [HöI, §7.7].

1. We can assume that $\omega = (1, 0, \cdots, 0)$. Then the function $\langle \theta, \omega \rangle = \theta_1$ has two critical points on $\mathbb{S}^{n-1}$, corresponding to $\theta_1 = \pm 1$. Hence we can assume that $\varphi$ is supported near the two poles $\theta_1 = \pm \lambda$ – the other contributions are $\mathcal{O}(1/r\lambda)$ as the phase is non-stationary.

2. Near the two poles we can use coordinates $t \in \mathbb{R}^{n-1}$, $\theta = (\pm \sqrt{1 - |t|^2}, t) \in \mathbb{S}^{n-1}$. Then, for $\varphi$ supported near $\theta_1 = \pm 1$ ($t = 0$) we have

$$\int_{\mathbb{S}^{n-1}} e^{-i \lambda r (\omega, \theta)} \varphi (\theta) d\theta = \int_{B_{\mathbb{S}^{n-1}}(0, 1)} e^{i \lambda r \sqrt{1 - |t|^2}} \varphi (\pm \sqrt{1 - |t|^2}, t) J(t) dt,$$

where $J(t) = 1 + \mathcal{O}(t^2)$.

3. The Hessian of the phase at $t = 0$ is given by $\pm i r \lambda I_{\mathbb{S}^{n-1}}$ and hence the method of stationary phase gives

$$\int_{B_{\mathbb{S}^{n-1}}(0, 1)} e^{i \lambda r \sqrt{1 - |t|^2}} \varphi (\pm \sqrt{1 - |t|^2}, t) J(t) dt 
= \left( 2 \pi \over r \lambda \right)^{n-1 \over 2} e^{\pm i (n-1) \over 2} \left( \varphi (\pm 1, 0) + \mathcal{O} \left( 1 \over r \lambda \right) \right),$$

with a full asymptotic expansion in powers of $(r \lambda)^{-1}$.

4. A general $\varphi$ can be written as a sum of functions which are supported near $\theta_1 = \pm 1$, and in the non-stationary region. That gives the result. □

**REMARK.** The proof gives a more precise result which we formulate as follows: for $\varphi \in C^\infty (\mathbb{S}^{n-1})$,

$$(3.6.20) \int_{\mathbb{S}^{n-1}} e^{-i \lambda r (\omega, \theta)} \varphi (\theta) d\theta = e^{-i \lambda r} a^+ (\lambda r, \omega) (\varphi) + e^{i \lambda r} a^- (\lambda r, \omega) (\varphi),$$
where

\[ a^\pm(\rho,\omega,\theta) \in S_{\text{phg}}^{-\frac{n-1}{2}}((0,\infty)\rho; C^\infty(S_\omega^{n-1}, \mathcal{D}'(S_\theta^{n-1})) \]

which means that for every \( k \geq 1, \)

\[
|a^\pm(\rho, \bullet)(\varphi)| \leq C\rho^{-\frac{n-1}{2}}\|\varphi\|_{C^{n+1}}, \quad \rho > 0,
\]

\[
a^\pm(\rho,\omega)(\varphi) = \rho^{-\frac{n-1}{2}} \sum_{j=0}^{k-1} \rho^{-j} a^\pm_j(\omega)(\varphi) + \rho^{-\frac{n-1}{2}-k} A^\pm_k(\rho,\omega)(\varphi),
\]

\[
a_0^\pm(\omega) = c^\pm_0 \delta_\omega, \quad |a^\pm_j(\omega)(\varphi)| \leq C_j\|\varphi\|_{C^{2j}},
\]

\[
|A_k(\rho,\omega)(\varphi)| \leq C_{pk\ell}\|\varphi\|_{C^{2k+n}}, \quad \text{uniformly in } \rho > 0.
\]

This means in particular that \( a^\pm_j(\omega) \) is a family of distribution of order \( 2j \). The expansion (3.6.21) is a distributional formulation of the stationary phase estimate [HöI, (7.7.13)]. It will be useful in the proof of Theorem 3.51.

The next result shows that the incoming and outgoing scattering patterns are naturally paired. Later it will allow us to establish the unitarity of the scattering matrix.

**THEOREM 3.39 (Boundary pairing).** Let \( P \) be a self-adjoint operator with domain \( H^2(\mathbb{R}^n) \), and such that for \( \chi \in C^\infty_c(B(0,2R); \mathbb{R}) \), \( \chi = 1 \) in \( B(0,R) \) we have \( P(1-\chi) = -\Delta(1-\chi) \).

Suppose that \( u_\ell \in H^2_{\text{loc}}(\mathbb{R}^n), \ell = 1,2 \) satisfy

\[
(P - \lambda^2)u_\ell = F_\ell \in \mathcal{S}(\mathbb{R}^n), \quad \lambda \in \mathbb{R} \setminus \{0\},
\]

\[
u_\ell(r\theta) = r^{-\frac{n+1}{2}} \left( e^{i\lambda r} f_\ell(\theta) + e^{-i\lambda r} g_\ell(\theta) \right) + \mathcal{O}(r^{-\frac{n+1}{2}}), \quad \theta \in S^{n-1},
\]

with \( f_\ell, g_\ell \in C^\infty(S^{n-1}) \), and the expansion is also valid for derivatives with respect to \( \partial_r \).

Then

\[
2i\lambda \int_{S^{n-1}} \left( g_1 \tilde{g}_2 - f_1 \tilde{f}_2 \right) d\omega = \int_{\mathbb{R}^n} \left( F_1 \tilde{u}_2 - u_1 \tilde{F}_2 \right) dx.
\]

**Proof.** 1. We note that the integral on the right hand side is well defined as \( F_\ell \in \mathcal{S} \) and \( u_\ell \in L^\infty \) (in view of the expansions). If \( \chi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \) is as in
3.7. THE SCATTERING MATRIX

In this section we will define and describe the scattering matrix for $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, $n \geq 3$, odd. Except for the behaviour near $\lambda = 0$ and the fact that we use the properties of the resolvent, the parity of the dimension is not very important here.
To define the scattering matrix we go back to (3.6.2)
\begin{align}
    e(\lambda, \omega, x) &= e^{-i\lambda(x, \omega)} + u(x, \lambda, \omega),
    \quad (P_V - \lambda^2) w = 0,
    \\
    u(x, \lambda, \omega) &= -R_V(\lambda)(V e^{-i\lambda(\bullet, \omega)}). 
\end{align}
(3.7.1)

Theorem 3.33 shows that \( w \) is defined for \( \lambda \in \mathbb{R} \setminus \{0\} \). We see from ii) in Theorem 3.37 that \( u \) is an outgoing spherical wave. The scattering matrix will be defined as the operator relating the leading incoming and outgoing terms, normalized so that it is the identity when \( V = 0 \).

Using Theorems 3.37 and 3.38 we write the leading terms in \( w \) of (3.7.1) as follows:
\begin{align}
    e(\lambda, \omega, r\theta) &\sim c_n^+ (\lambda r)^{-\frac{n-1}{2}} \left( e^{-i\lambda r} \delta_\omega(\theta) + e^{i\lambda r} i^{1-n} \left( \delta_{-\omega}(\theta) + b(\lambda, \theta, \omega) \right) \right),
\end{align}
(3.7.2)

Here \( b(\lambda, \theta, \omega) \) gives the leading part of the asymptotics of \( u(r\theta, \lambda, \omega) \) as \( r \to \infty \):
\begin{align}
    u(r\theta, \lambda, \omega) &= c_n^- (\lambda r)^{-\frac{n-1}{2}} e^{i\lambda r} b(\lambda, \theta, \omega) + \mathcal{O}(r^{-\frac{n+1}{2}}),
\end{align}
(3.7.3)

and the constants are
\begin{align}
    c_n^\pm &= e^{\pm \frac{\pi}{4} (n-1)i(2\pi)^{\frac{n}{4}}}. 
\end{align}
(3.7.4)

**Definition 3.40.** The **absolute scattering matrix** maps the incoming terms to the outgoing terms in (3.7.2):
\begin{align}
    S_{\text{abs}}(\lambda) : \delta_\omega(\theta) &\mapsto i^{1-n} \left( \delta_{-\omega}(\theta) + b(\lambda, \theta, \omega) \right).
\end{align}
(3.7.5)

The relative scattering matrix is defined as
\begin{align}
    S(\lambda) : \delta_\omega(\theta) &\mapsto \delta_\omega(\theta) + b(\lambda, \theta, -\omega),
\end{align}
(3.7.6)
where \( b \) is given in (3.7.2).

The action on delta functions determines the integral kernel defining \( S(\lambda) \) as an operator on \( C^\infty(S^{n-1}) \):
\begin{align}
    f(\theta) &= \int_{S^{n-1}} \delta_\omega(\theta) f(\omega) d\omega 
    \mapsto \int_{S^{n-1}} \left( \delta_{-\omega}(\theta) + b(\lambda, \theta, -\omega) \right) f(\omega) d\omega,
\end{align}

that
\begin{align}
    S(\lambda) f(\theta) &= f(\theta) + \int_{S^{n-1}} b(\lambda, \theta, -\omega) f(\omega) d\omega.
\end{align}
See Theorem 3.42 for an equivalent definition of scattering matrices.

We observe that for \( V = 0 \) we have
\begin{align}
    S_{\text{abs},0}(\lambda) f(\theta) &= i^{1-n} f(-\theta).
\end{align}
3.7. THE SCATTERING MATRIX

The scattering matrix was obtained by normalizing $S_{\text{abs}}$ by this free absolute scattering matrix:

$$S(\lambda) = S_{\text{abs}}(\lambda)S_{\text{abs},0}(\lambda)^{-1} = i^{n-1}S_{\text{abs}}(\lambda)J, \quad Jf(\theta) := f(-\theta).$$

**INTERPRETATION.** The function $b(\lambda, \theta, \omega)$ is called the *scattering amplitude* (up to a normalizing factor). It measures the intensity of the spherical scattered wave in the direction $\theta$, following an interaction of the incident plane wave in the direction of $-\omega$ (coming from the point $\omega$ at infinity) – see Fig. 3.4. We will see below that

$$(3.7.7) \quad b(\lambda, \theta, \omega) = b(\lambda, \omega, \theta) \quad \text{for } V \in L^\infty(\mathbb{R}^n; \mathbb{R}).$$

This is a higher dimensional version of the symmetry of the left and right transmission coefficients – see (2.4.12) and (2.4.13).

We have the following description of $S(\lambda)$:

**THEOREM 3.41 (Description of the scattering matrix I).** *The scattering matrix given by (3.7.6) defines an operator*

$$S(\lambda) = I + A(\lambda) : L^2(S^{n-1}) \rightarrow L^2(S^{n-1}),$$

*where $A(\lambda) : \mathcal{D}'(S^{n-1}) \rightarrow C^\infty(S^{n-1})$, is given by*

$$(3.7.8) \quad A(\lambda) = a_n \lambda^{n-2}E_\rho(\lambda)(I + VR_0(\lambda)\rho)^{-1}VE_\rho(\bar{\lambda})^*, \quad E_\rho : L^2(\mathbb{R}^n) \rightarrow L^2(S^{n-1}), \quad E_\rho(\lambda)(\omega, x) := \rho(x)e^{-i\lambda(x,\omega)},$$

*where $\rho \in C^\infty_c(\mathbb{R}^n)$ is equal to one on $\text{supp} V$ and $a_n = (2\pi)^{-n+1}/2i$.\"
In particular the Schwartz kernel \((\theta, \omega) \mapsto A(\lambda, \theta, \omega)\) is real analytic and can be written as

\[
a_n \lambda^{n-2} \int_{\mathbb{R}^n} e^{i\lambda(\omega-\theta,x)}V(x)(1 - e^{-i\lambda(\omega,x)})R_V(\lambda)(e^{i\lambda(\omega,\cdot)}V(\cdot))(x)dx.
\]

**Proof.**

1. The definition of \(u\) in (3.7.1) gives

\[
u = R_0(\lambda)f, \quad f(x) := -(I + VR_0(\lambda)\rho)^{-1}(Ve^{-i\lambda(\cdot,\omega)})(x).
\]

From Theorem 3.5 and (3.7.3) we obtain

\[
b(\lambda, \theta, \omega) = \frac{1}{4\pi} \frac{\lambda^{n-2}}{(2\pi)^{n+1} i^{n-1}} \hat{f}(\lambda \theta) = \frac{1}{2i(2\pi)^{n-1}} \int_{\mathbb{R}^n} e^{-i\lambda(x,\theta)}(I + VR_0(\lambda)\rho)^{-1}(Ve^{-i\lambda(\cdot,\omega)})(x)dx,
\]

which gives (3.7.8).

2. To see (3.7.9) we recall that

\[
(I + VR_0(\lambda)\rho)^{-1}V = (I - VR_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1})V = (I - VR_V(\lambda))V,
\]

from which the formula follows immediately.

The next result gives an intrinsic definition of the scattering matrix as the operator mapping incoming to outgoing data at infinity:

**Theorem 3.42 (Solution with a prescribed incoming part).** Suppose that \(V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}), \lambda \in \mathbb{R} \setminus \{0\}\). Then for any \(g \in C^\infty(S^{n-1})\) there exist unique \(f \in C^\infty(S^{n-1})\) and \(v \in H^2_{\text{loc}}(\mathbb{R}^n)\) such that

\[
(P_V - \lambda^2)v = 0,
\]

\[
v(r\theta) = r^{-\frac{n-1}{2}} \left( e^{i\lambda r}f(\theta) + e^{-i\lambda r}g(\theta) \right) + O(r^{-\frac{n+1}{2}}).
\]

Moreover, the scattering matrices defined by (3.7.5) and (3.7.6) relate \(g\) and \(f\) as follows:

\[
S_{\text{abs}}(\lambda) : g(\theta) \mapsto f(\theta),
\]

\[
S(\lambda)J : g(-\theta) \mapsto e^{-n-1}f(\theta).
\]

**Proof.**

1. For \(g \in C^\infty(S^{n-1})\) consider

\[
u_0(x) := b_n \lambda \frac{n-1}{2} \int_{\mathbb{S}^{n-1}} g(\omega)e^{-i\lambda(x,\omega)}d\omega,
\]

\[
b_n = 1/c_n^+ = (2\pi)^{-\frac{n-1}{2}} e^{-\frac{\pi}{2}(n-1)}.
\]

Then \((-\Delta - \lambda^2)u_0 = 0\) and Theorem 3.38 gives asymptotics of \(u_0\).
2. We then put
\[ v(x) = u_0(x) - R_V(\lambda)(Vu_0)(x) = b_n \lambda^{\frac{n-1}{2}} \int_{S^{n-1}} g(\omega)w(x, \lambda, \omega) d\omega, \]
where \( w \) is given by (3.7.1). This gives the desired \( v \) and (3.7.6) (that is, the fact that \( S(\lambda) = i^{n-1} S_{\text{abs}}(\lambda)J, Jg(-\theta) = g(\theta) \)) shows that (3.7.11) holds.

3. The uniqueness of \( v \) follows from Theorem 3.35.

\[ \square \]

**REMARK.** For any given \( g \in C^\infty(S^{n-1}) \) we can find \( v_0 \in C^\infty(\mathbb{R}^n) \) such that, for \( \lambda \in \mathbb{R} \setminus \{0\} \),
\[ (-\Delta - \lambda^2)v \in \mathcal{S}(\mathbb{R}^n), \]
\[ v_0(r\theta) = r^{-\frac{n-1}{2}} e^{-i\lambda r} F(r, \theta), \quad F(r, \theta) \sim \sum_{j=0}^\infty F_j(\theta) r^{-j}, \quad F_0 = g, \]
\[ F_{j+1} = \frac{1}{2i(j+1)\lambda} \left( -\Delta_{S^{n-1}} + \frac{(n-1)(n-3)}{4} - j(j+1) \right) F_j. \]

In particular, the terms in the asymptotic expansions of \( F \) are determined by the leading term, \( g \).

To obtain (3.7.12) we write
\[ r^{\frac{n-1}{2}} \partial_r r^{-\frac{n-1}{2}} = \partial_r - \frac{n-1}{2r}, \]
so that, in polar coordinates,
\[ -r^{\frac{n-1}{2}} \Delta r^{-\frac{n-1}{2}} = -\partial_r^2 + \frac{(n-1)(n-3)}{4r^2} - \frac{1}{r^2} \Delta_{S^{n-1}}, \]
and hence
\[ -r^{\frac{n-1}{2}} e^{i\lambda r} \Delta(r^{-\frac{n-1}{2}} e^{-i\lambda r} F_j(\theta) r^{-j}) = -r^{-j-2} ij \lambda F_j(\theta) + r^{-j-2} \left( -\Delta_\theta + \frac{(n-1)(n-3)}{4} - j(j+1) \right) F_j(\theta). \]

Hence, we see that \( v_0^j(r, \theta) := r^{-\frac{n-1}{2}} e^{i\lambda r} \sum_{j=0}^j F_j(\theta) r^{-j}, \) satisfies \((-\Delta - \lambda^2)v_0^j = \mathcal{O}(r^{-j-1}), r \to \infty.\) Constructing \( v_0 \) using Borel’s argument (see for instance \( [Zw12, \text{ Theorem 4.15}] \)) gives
\[ (3.7.13) \quad (-\Delta - \lambda^2)v_0 \in \mathcal{S}(\mathbb{R}^n). \]

Conversely, if \( v_0 \) satisfies (3.7.13), and has an expansion in (3.7.12), then the terms are determined by the leading term, \( F_0, \) as in (3.7.12). The solution \( v \) in (3.7.10) is obtained by putting
\[ v = v_0 - R_V(\lambda)(P_V - \lambda^2)v_0. \]

We can now state basic properties of the scattering matrix:
THEOREM 3.43 (Properties of the scattering matrix). For $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$, odd, the scattering matrix, $S(\lambda)$, is meromorphic in $\mathbb{C}$ with poles of finite rank, and it satisfies

\begin{equation}
S(\lambda)^{-1} = JS(-\lambda)J, \quad Jf(\theta) := f(-\theta), \quad \lambda \in \mathbb{C}.
\end{equation}

There are only finitely many poles in the closed upper half plane and for $\Im \lambda > 0$, $\lambda^2 \in \text{Spec}(P_V)$.

When $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$ then

\begin{equation}
S(\lambda)^{-1} = S(\bar{\lambda})^*, \quad \lambda \in \mathbb{C}.
\end{equation}

In particular, $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ and holomorphic on $\mathbb{R}$.

REMARK. Since $S(\lambda) = I + A(\lambda)$ where $A(\lambda)$ is of trace class (see Theorem 3.41) the determinant, $\det S(\lambda)$, is well defined – see §B.5. From (3.7.15), (B.5.12) and Proposition B.27 we see that

\begin{equation}
(\det S(\lambda))^{-1} = \det S(\lambda) \quad \text{for } V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}).
\end{equation}

We also have,

\begin{equation}
(\det S(\lambda))^{-1} = \det S(-\lambda), \quad \text{for } V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C}).
\end{equation}

In fact, from (3.7.14), (B.5.12) and Proposition B.27 (or the fact that $\text{tr}(JA(-\lambda)J)^k = \text{tr} A(-\lambda)^k$) we have

\begin{equation}
\det S(\lambda)^{-1} = \det(I + JA(-\lambda)J) = \det(I + A(-\lambda)) = \det S(-\lambda).
\end{equation}

Proof of Theorem 3.43. 1. The meromorphy of $\lambda \mapsto S(\lambda)$ follows from the meromorphy of $\lambda \mapsto R_V(\lambda)$ and the representation (3.7.8).

2. Theorem 3.42, and (3.7.11) in particular, show that $S_{\text{abs}}(\lambda)^{-1} = S_{\text{abs}}(-\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and hence for all $\lambda$, in the sense of meromorphic families of operators. Since $S(\lambda) = i^{n-1}S_{\text{abs}}(\lambda)J$, we have

\begin{align*}
S(\lambda) &= i^{n-1}S_{\text{abs}}(\lambda)J = i^{n-1}S_{\text{abs}}(\lambda)^{-1}J \\
&= i^{n-1}(i^{1-n}S(\lambda)J)^{-1}J = i^{2(n-1)}JS(\lambda)^{-1}J \\
&= JS(\lambda)^{-1}J,
\end{align*}

since $n$ is odd.

3. The unitarity of $S_{\text{abs}}(\lambda)$ for real valued $V$ follows from Theorems 3.39 and 3.42. In the notation of (3.7.10), we apply (3.6.22) with $u_1 = u_2 = u$, $g_1 = g_2 = g$ and $f_1 = f_2 = S_{\text{abs}}(\lambda)g$ to obtain

\begin{equation}
\|S(\lambda)g\|^2_{L^2(\mathbb{S}^{n-1})} = \|g\|^2_{L^2(\mathbb{S}^{n-1})}.
\end{equation}
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It follows that \( S(\lambda) \) is unitary for \( \lambda \in \mathbb{R} \setminus \{0\} \). Since Theorem 3.41 shows that \( \lambda \mapsto S(\lambda) \) is a meromorphic family of operators, 0 must be a removable singularity and \( S(\lambda) \) is unitary for \( \lambda \in \mathbb{R} \).

4. The equation (3.7.15) holds for \( \lambda \in \mathbb{R} \) and, as both sides are meromorphic it extends to all of \( \mathbb{C} \).

\[ \Box \]

Proof of (3.7.7). We can now prove the symmetry of the scattering amplitudes. For \( \lambda = ik, \ k \gg 1, \ \re V(ik) = (P_V + k^2)^{-1} \) is self-adjoint (we assume here \( V \) is real valued). This and (3.2.3) show that \( \re V(ik,x,y) \) is real valued. The expression for the scattering matrix kernel (3.7.8) shows that \( i^{-n}b(ik,\theta,\omega) \) is real – the \( i \) factors cancel.

On the other hand,

\[ S_{\text{abs}}(ik)^* = S_{\text{abs}}(-ik)^{-1} = S_{\text{abs}}(ik), \ k \in \mathbb{R}, \]

and hence \( S_{\text{abs}}(ik) \) is self-adjoint. From (3.7.5) we see that the Schwartz kernel is real and hence \( b(ik,\theta,\omega) = b(ik,\omega,\theta) \). By analytic continuation this remains valid for all \( \lambda \).

We now present a different formula for the scattering matrix. The potential \( V \) does not appear explicitly and hence it can be used for more general (compactly supported) perturbations. In fact, it holds for any black box perturbation which we will consider later in this book.

**THEOREM 3.44 (Description of the scattering matrix II).** Let \( P_V, \rho, \) and \( E_\rho(\lambda) \) be as in Theorem 3.41. Choose \( \chi_i \in C^\infty_c(\mathbb{R}^n), \ i = 1, 2, 3, \)

\[ \chi_i|_{\supp V} = 1, \ \chi_{i+1}|_{\supp \chi_i} = 1, \ \chi_3 = \rho. \]

Then the scattering matrix is given by

(3.7.18) \[ S(\lambda) = I + a_n\lambda^{-2}E_\rho(\lambda)[\Delta,\chi_1]R_V(\lambda)[\Delta,\chi_2]E_\rho(\bar{\lambda})^*, \]

where \( a_n := (2\pi)^{-n+1}/2i. \)

**Proof.** 1. For \( h_1, h_2 \in C^\infty(\mathbb{S}^{n-1}) \) let us put

\[ u_1 = ((1 - \chi_2)E(\bar{\lambda})^* - R_V(\lambda)[\Delta,\chi_2]E_\rho(\bar{\lambda})^*)h_1, \]

\[ u_2 = (1 - \chi_1)E(\bar{\lambda})^*h_2, \]

where

\[ E(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1}), \ E(\lambda)u(\omega) := \int_{\mathbb{R}^n} u(x)e^{-i\lambda(\omega)}dx, \]

and \( E(\bar{\lambda})^* : L^2(\mathbb{S}^{n-1}) \to L^2_{\text{loc}}(\mathbb{R}^n). \)

We check that, thanks to the support properties of \( \chi_j \)’s and \( \rho, \)

\[ F_1 := (P_V - \lambda^2)u_1 = 0 \]
and
\[ F_2 := (P_V - \lambda^2)u_2 = [\Delta, \chi_1]E_\rho(\bar{\lambda})^*h_2. \]

2. We first assume that \( \lambda \in \mathbb{R} \setminus \{0\} \). We see that \( u_1 \) satisfies the assumptions of Theorem 3.39 with
\[ g_1(\theta) = c_n^2 \lambda^{-\frac{n-1}{2}}h_1(-\theta), \quad c_n = e^{-\frac{i}{4}\pi(n-1)(2\pi)^{\frac{1}{2}(n-1)}}, \]
\[ f_1(\theta) = S_{abs}(\lambda)g_1(\theta) = c_n i^{1-n}\lambda^{-\frac{n-1}{2}}S(\lambda)h_1(\theta). \]

In fact, since \( R_V(\lambda) \) is the outgoing resolvent, the only incoming contribution comes from the free term \((1 - \chi_1)E_\rho(\bar{\lambda})^*h_1\) and this follows from Theorems 3.38 (formula for \( g_1 \)) and 3.42 (formula for \( f_1 \)). Note the change of sign in the exponent in \( E(\lambda)^* \).

Theorem 3.38 then shows that for \( u_2 \) we have asymptotic expansion with
\[ g_2(\theta) = c_n^2 \lambda^{-\frac{n-1}{2}}h_2(-\theta), \quad f_2(\theta) = c_n i^{1-n}\lambda^{-\frac{n-1}{2}}h_2(\theta). \]

3. We define
\[
(3.7.19) \quad G(\lambda) := E_\rho(\lambda)[\Delta, \chi_1]R_V(\lambda)[\Delta, \chi_2]E_\rho(\bar{\lambda})^*,
\]
which is an operator \( L^2(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1}) \).

Using the fact that \((1 - \chi_2)[\Delta, \chi_1] = 0\), we have
\[ u_1 F_2 = -R_V(\lambda)[\Delta, \chi_2]E_\rho(\bar{\lambda})^*h_1 \times [\Delta, \chi_1]E_\rho(\lambda)^*h_2, \]
and since \( F_1 = 0 \) and \([\Delta, \chi_1]^* = -[\Delta, \chi_1]\),
\[
(3.7.20) \quad \int_{\mathbb{R}^n} (F_1 \bar{u}_2 - u_1 \bar{F}_2)dx = \langle R_V(\lambda)[\Delta, \chi_2]E_\rho(\bar{\lambda})^*h_1, [\Delta, \chi_1]E_\rho(\lambda)^*h_2 \rangle_{L^2(\mathbb{R}^3)} = -\langle E_\rho(\lambda)[\Delta, \chi_1]R_V(\lambda)[\Delta, \chi_2]E_\rho(\bar{\lambda})^*h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})} = -\langle G(\lambda)h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})}.
\]

4. On the other hand the pairing formula (3.6.22) and the expressions for \( f_\ell \) and \( g_\ell \) from Step 2, give
\[
(3.7.21) \quad \int_{\mathbb{R}^n} (F_1 \bar{u}_2 - u_1 \bar{F}_2)dx = 2i\lambda \left( \langle g_1, g_2 \rangle_{L^2(\mathbb{S}^{n-1})} - \langle f_1, f_2 \rangle_{L^2(\mathbb{S}^{n-1})} \right) = 2i\lambda^{-n+2}(2\pi)^{n-1}\langle (I - S(\lambda))h_1, h_2 \rangle_{L^2(\mathbb{S}^{n-1})}.
\]
Comparing (3.7.21) with (3.7.20) we see that
\[ A(\lambda)h_2 = (S(\lambda) - I)h_2 = a_n \lambda^{n-2}G(\lambda), \]
which, recalling the definition of \( G(\lambda) \) (3.7.19), proves (3.7.18) for \( \lambda \in \mathbb{R} \setminus \{0\} \). Analytic continuation shows that the formula is valid, as an equality of two meromorphic families of operators, for \( \lambda \in \mathbb{C} \).
REMARK. It is interesting to note that the representation (3.7.18) does not depend on the cut-off functions, and that we can reverse the condition $\chi_2 \equiv 1$ on the support of $\chi_1$ to $\chi_1 \equiv 1$ on the support of $\chi_2$. Both facts follow directly from the properties of the scattering matrix but here we propose a direct argument based on considering quantum flux.

Suppose that $\chi_2$ is equal to one on the supports of functions $\chi_1$, $\tilde{\chi}_1$, which are equal to 1 near $\text{supp} V$. We claim that

$$E_{\rho}(\lambda)[\Delta, \chi_2]R_V(\lambda)[\Delta, \chi_1 - \tilde{\chi}_1]E_{\rho}(\tilde{\lambda})^* \equiv 0.$$ 

This will follow from showing that

$$(\Delta - \lambda^2)v_j = 0, \ j = 1, 2 \implies \langle R_V(\lambda)[\Delta, \chi_1 - \tilde{\chi}_1]v_1, [\Delta, \chi_2]v_2 \rangle_{L^2(\mathbb{R}^n)} = 0.$$ 

This however is clear as the left hand side is equal to

$$\langle R_V(\lambda)(-P_V(\chi_1 - \tilde{\chi}_1) - (\chi_1 - \tilde{\chi}_1)\Delta)v_1, [\Delta, \chi_2]v_2 \rangle_{L^2(\mathbb{R}^n)} = 0,$$

since $(\chi_1 - \tilde{\chi}_1)[\Delta, \chi_2] = 0$. Similarly, if $\chi_1 \equiv 1$ on the support of $\tilde{\chi}_1$, and $\tilde{\chi}_1 \equiv 1$ near supp $V$, then

$$E_{\rho}(\lambda)[\Delta, \chi_2 - \tilde{\chi}_1]R_V(\lambda)[\Delta, \chi_1]E_{\rho}(\tilde{\lambda})^* \equiv 0,$$

which shows that we can switch the conditions on $\chi_1$ and $\chi_2$. Yet another argument of the same type shows that for the free resolvent we have

$$E_{\rho}(\lambda)[\Delta, \chi_2]R_0(\lambda)[\Delta, \chi_1]E_{\rho}(\tilde{\lambda})^* \equiv 0.$$ 

In the study of resonances the following theorem provides a crucial connection between singularities of the scattering matrix and the resolvent.

**THEOREM 3.45 (Multiplicities of scattering poles II).** Suppose that $S(\lambda)$ is the scattering matrix for $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$, odd.

If we define

$$(3.7.22) \quad m_S(\lambda) = -\frac{1}{2\pi i} \text{tr} \int S(\zeta)^{-1}\partial_\zeta S(\zeta)d\zeta,$$

where the integral is over a positively oriented circle which includes $\lambda$ and no other pole or zero of $\det S(\lambda)$, then

$$(3.7.23) \quad m_S(\lambda) = m_R(\lambda) - m_R(-\lambda).$$
Proof. 1. The results in this section apply equally well to $V$ replaced by $V = V_0 + V_1$ where

\begin{equation}
V_0 \in L^\infty, \quad V_1 = \sum_{j=1}^m f_j \otimes g_j,
\end{equation}

for some fixed $R$ (we then choose $\rho \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(0, R)$.

2. We first see that if $(I + VR_0(\lambda)\rho)^{-1}$ is holomorphic on $|\xi \pm \lambda| = r$ then for $V'$ close to $V$, in operator norm (and with a fixed bound on the support as in (3.7.24))

\begin{equation}
\sum_{|\lambda - \zeta| < r} m_{SV}(\zeta) = \sum_{|\zeta - \lambda| < r} m_{SV'}(\zeta).
\end{equation}

In fact, we can use (3.7.8) and (3.7.14) to see that that $S_V(\zeta)^{-1} = JS_V(-\zeta)J$ exists and is bounded on $|\zeta - \lambda| = r$. Since

\begin{equation}
(I + VR_0(\lambda)\rho)^{-1} - (I + V'R_0(\lambda)\rho)^{-1} = (I + VR_0(\lambda)\rho)^{-1}(V' - V)\rho R_0(\lambda)\rho(I + V'R_0(\lambda)\rho)^{-1}
\end{equation}

we see that if $\|V' - V\|_{L^2 \to L^2}$ is sufficiently small invertibility of $I + VR_0(\lambda)\rho$ implies invertibility of $I + V'R_0(\lambda)\rho$ and we can estimate the norm of the difference of the inverses. The the formula for $A(\zeta)$ in (3.7.8) then shows that

\begin{equation}
\|S_V(\zeta)^{-1}(S_V(\zeta) - S_V'(\zeta))\| < 1, \quad |\zeta - \lambda| = r.
\end{equation}

This and Theorem C.9 give (3.7.25).

We can always find arbitrary small $r$'s for which $(I + VR_0(\zeta)\rho)^{-1}$ is holomorphic on $|\zeta \pm \lambda| = r$. The formula (3.7.25) implies that the poles and zeros of $\det S_V(\lambda)$ depend continuously on $V$ in compact sets.

3. The continuity statement above and Theorem 3.14 show that we only need to prove (3.7.23) if $m_R(\pm \lambda) \leq 1$ (the proof of that theorem applies to splitting multiplicities of any finite number of resonances). Hence suppose that $R_V(\zeta)$ has a pole of multiplicity 1 at $\lambda$.

Let us consider the case of $\lambda \neq 0$ first. Then by (3.2.8),

\begin{equation}
R_V(\zeta) = \frac{u \otimes u}{\lambda - \zeta} + H(\zeta, \lambda),
\end{equation}

where $\zeta \mapsto H(\zeta, \lambda)$ is holomorphic near $\lambda$, $(P_V - \lambda^2)u = 0$, $u = R_0(\lambda)f$, $f \in L^\infty_{\text{comp}}$, $f \neq 0$. Theorem 3.44 shows that

\begin{equation}
S(\zeta) = S_0(\zeta, \lambda) - a_n\lambda^{n-2} U_1 \otimes U_2 \frac{U_1 \otimes U_2}{\zeta - \lambda},
\end{equation}

3. THE SCATTERING MATRIX
where $\zeta \mapsto S_0(\zeta, \lambda)$ is holomorphic near $\lambda$ and

$$U_j := E_\rho(\lambda)[\Delta, \chi_j]u.$$  

(Note that the change of sign between (3.7.26) and (3.7.27) comes from $[\Delta, \chi_2]^* = -[\Delta, \chi_2]$.)

We now claim that

$$U_1(\theta) = U_2(\theta) = U(\theta) := \hat{V}u(\lambda \theta).$$

In fact, using the equation $P_Vu = \lambda^2 u,$

$$[\Delta, \chi_j]u = \Delta \chi_j u - \chi_j \Delta u = (\Delta + \lambda^2) \chi_j u + Vu.$$  

The contribution of the first term vanishes since

$$F(\Delta + \lambda^2) \chi_j u(\lambda \theta) = (\lambda^2 + |\xi|^2) \chi_j u \neq 0.$$  

Step 4 of the proof of Theorem 3.33 shows that $U \equiv 0$ would imply that $u$ is compactly supported which is impossible, as shown in Step 5 of that proof.

4. This means that a simple pole of $R_V (\zeta)$ at $\lambda$, implies (3.7.27) with $U_1 = U_2 \neq 0$. Theorem C.7 shows that near $\lambda$

$$S(\zeta) = G(\zeta)(Q_{-1}(\zeta - \lambda) + Q_k(\zeta - \lambda)^k + \cdots + Q_0) F(\zeta),$$

where $Q_{-1}$ has rank 1 and $G$ and $F$ are holomorphic and invertible near $\lambda$. Since we assumed that $R_V (\zeta)$ has at most a simple pole at $-\lambda$ we also have, for $\zeta$ near $\lambda$

$$S(-\zeta) = \bar{E}(\zeta)(\bar{Q}_{-1}(\zeta - \lambda) + \bar{Q}_k(\zeta - \lambda)^k + \cdots + \bar{Q}_0) \bar{F}(\zeta),$$

where rank $\bar{Q}_{-1} = m_R(-\lambda)$.

By (3.7.14)

$$S(-\zeta) = JS(\zeta)^{-1} J = J F(\zeta)^{-1} (Q_{-1}(\zeta - \lambda) + Q_k(\zeta - \lambda)^k + \cdots + Q_0) E(\zeta)^{-1} J$$

which means that $k \leq 1$ and $\tilde{Q}_1 = Q_{-1}, \tilde{Q}_1 = Q_1$. We conclude that

$$m_S(\lambda) = \text{rank } Q_{-1} = \text{rank } Q_1 = m_R(\lambda) - m_R(-\lambda),$$

which, by our reduction to the case of simple poles, proves (3.7.23) for all $\lambda \neq 0$.

5. It remains to discuss the case of $\lambda = 0$. We can again assume that $m_R(\lambda) \leq 1$. Then (3.7.18) shows that $S(\lambda)$ is holomorphic near 0 and using $S(\lambda)^{-1} = JS(\lambda)^{-1} J$ so is its inverse. (When $V$ is real valued the scattering matrix is unitary on the real axis and hence holomorphic and invertible near 0.) That means that $m_S(0) = 0$. \qed
We have
\[ S(\lambda) = I - a_3 \lambda^{-1} E_\rho(0)[\Delta, \chi_1] \Pi_0[\Delta, \chi_2] E_\rho(0)^* \]
\[ - a_3 \partial_x E_\rho(\lambda)|_{\lambda=0} [\Delta, \chi_1] \Pi_0[\Delta, \chi_2] E_\rho(0) \]
\[ - a_3 E_\rho(0)[\Delta, \chi_1] \Pi_0[\Delta, \chi_2] \partial_x E_\rho(\lambda)^*|_{\lambda=0} \]
\[ + i \alpha_3 E_\rho(0)[\Delta, \chi_1] A_1[\Delta, \chi_2] E_\rho(0)^* + O(\lambda). \]

We have \( \Pi_0 = \sum_{j=1}^J u_j \otimes \bar{u}_j \) where \( u_j \in H^2(\mathbb{R}^3) \) and \( P_V u_j \). Then, using \( \chi_\ell \Delta u_j = \chi_\ell V u_j = V u_j = \Delta u_j \) and part 1 of Lemma 3.18 we have
\[ E_\rho(0)[\Delta, \chi_\ell] u_j = \int_{\mathbb{R}^3} (\Delta(\chi_\ell u_j)(x) - \Delta u_j(x)) \, dx = 0, \quad \ell = 1, 2. \]

Hence \( E_\rho(0) \Pi_0[\Delta, \chi_\ell] = 0 \), and most terms in (3.7.29) disappear and in view of Lemma 3.22 and (3.3.27),
\[ S(\lambda) = I + (2\pi)^{-1} E_\rho(0)[\Delta, \chi_1](u_0 \otimes \bar{u}_0)[\Delta, \chi_2] E_\rho(0)^*, \]
\[ P_V u_0 = 0, \quad - \int_{\mathbb{R}^3} V(x) u_0(x) \, dx = 1. \]

Now,
\[ E_\rho(0)[\Delta, \chi_\ell] u_0 = \int_{\mathbb{R}^3} (\Delta(\chi_\ell u_0)(x) - V(x) u_0(x)) \, dx = 1, \]
and this gives (3.7.28). The change of sign comes for \([\Delta, \chi_2]^* = -[\Delta, \chi_2]\).
(We also note that once we see that \( S(0) - I \) is of the form \( cl \otimes 1 \) then \( c \) is determined by \( S(0) = S(0)^{-1} \).)

We conclude this section with a useful results relating the determinant of the scattering matrix to the determinant of an operator acting on \( L^2(\mathbb{R}^n) \).

**Theorem 3.46 (Trace identities).** Suppose that
\[ V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C}), \quad n \geq 3, \quad \text{odd}, \]
and that \( \rho \in C^\infty_c(\mathbb{R}^n) \) is equal to 1 on \( \text{supp} \, V \). Let
\[ T(\lambda) := (I + VR_0(\lambda) \rho)^{-1}(V(R_0(\lambda) - R_0(-\lambda))\rho) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \]
Then \( T(\lambda) \) is of trace class and
\[ \det S(\lambda) = \det(I - T(\lambda)). \]
Proof. 1. The operator $T(\lambda)$ is of trace class since

$$
\rho(R_0(\lambda) - R_0(-\lambda)) : L^2(\mathbb{R}^n) \to H^k([-R, R]),
$$

for any $k$, provided that $\rho \in C^\infty(\mathbb{R}^n)$, supp $\rho \subset B(0, R)$.

2. We will prove the formula for $\lambda \in \mathbb{R}$. For that we first write

$$
Z(\lambda) = b_n \lambda^{n-2} E_\rho(\lambda)(I - VR_0(\lambda) \rho) V E_\rho(\lambda)^* = b_n \lambda^{n-2} E_\rho(\lambda)(I + VR_0(\lambda) \rho)^{-1} V E_\rho(\lambda)^*,
$$

(3.7.31)

where $b_n = -a_n = i (2\pi)^{1-n}/2$.

3. To prove (3.7.30) all we need to show is that for all $k \in \mathbb{N}$

$$
(3.7.32) \quad \text{tr} T(\lambda)^k = \text{tr} Z(\lambda)^k.
$$

In fact, (3.7.32) shows for $t \in \mathbb{C}$, $|t| \ll 1$ (so that the log can be defined),

$$
\log \det(I - tT(\lambda)) = \text{tr} \log(I - tT(\lambda)) = \text{tr} \log(I - tZ(\lambda)) = \log \det(I - tZ(\lambda)).
$$

It follows that $\det(I - tZ(\lambda)) = \det(I - tT(\lambda))$ for $|t|$ small enough, and by analytic continuation in $t$, for $t = 1$.

4. To establish (3.7.32) we use (3.1.19) for $\lambda \in \mathbb{R}$:

$$
\rho(R_0(\lambda) - R_0(-\lambda)) \rho = b_n \lambda^{n-2} E_\rho(\lambda)^* E_\rho(\lambda)
$$

in the definition of $T(\lambda)$:

$$
T(\lambda) = b_n \lambda^{n-2} (I + VR_0(\lambda) \rho)^{-1} V E_\rho(\lambda)^* E_\rho(\lambda).
$$

Let $A = b_n \lambda^{n-2} E_\rho(\lambda)$, $B = (I + VR_0(\lambda) \rho)^{-1} V$, $C = E_\rho(\lambda)^*$ so that $T = BCA$ and $Z = ABC$,

see (3.7.31). The operators $A$ and $C$ are of trace class as operators between different spaces:

$$
A : H_1 \to H_2, \quad B : H_1 \to H_1, \quad C : H_2 \to H_1,
$$

and $B$ is a bounded operator. Cyclicity of trace shows that

$$
\text{tr}_{H_1}(ABC)^n = \text{tr}_{H_1} A(BCA)^{n-1} BC
= \text{tr}_{H_2} BCA(BCA)^{n-1}
= \text{tr}_{H_2} (BCA)^n.
$$

This gives (3.7.32). $\square$
3. MORE ON DISTORTED PLANE WAVES

Distorted plane waves were already defined in (3.7.1) were used to define the scattering matrix. Here we will study them further.

We start with an explicit form of Stone’s formula. Its abstract form for $P_V$ given in (B.1.13) follows from general spectral theory but the results of §3.6 give us an analogue of Theorem 3.4 which was stated for $P_0 = -\Delta$.

**THEOREM 3.47** *(Stone’s formula for $P_V$).* Let $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$.

For $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega \in S_n$ define $e(\lambda, \omega, x)$ by (3.6.2):

\begin{equation}
(3.8.1) \quad e(\lambda, \omega, x) := e^{-i\lambda \langle x, \omega \rangle} - R_V(\lambda)(Ve^{-i\lambda \langle \cdot, \omega \rangle})(x).
\end{equation}

Then

\begin{equation}
(3.8.2) \quad e(\lambda, x, \omega) = e(-\lambda, x, \omega),
\end{equation}

and

\begin{equation}
(3.8.3) \quad R_V(\lambda, x, y) - R_V(-\lambda, x, y) = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{S_{n-1}} e(\lambda, \omega, x)e(\lambda, \omega, y)d\omega.
\end{equation}

The Schwartz kernel of spectral measure of $P_V$ corresponding to the continuous spectrum is given by

\begin{equation}
(3.8.4) \quad dE(\lambda, x, y) = \int_{S_{n-1}} e(\lambda, \omega, x)e(\lambda, \omega, y)d\omega \frac{\lambda^{n-1}d\lambda}{(2\pi)^n},
\end{equation}

where $u_k$’s are normalized eigenfunctions of $P_V$ corresponding to eigenvalues $E_k$, $E_K < E_{K-1} \leq \cdots \leq E_1 \leq 0$.

**INTERPRETATION.** The functions defined in (3.8.1) (and earlier in (3.6.2)) are called *distorted plane waves*. That is because $\exp(-i\lambda \langle \omega, x \rangle)$ is a free plane wave in the sense that its inverse Fourier transform in $\lambda$ is equal to $2\pi \delta(t - \langle \omega, x \rangle)$ which is a plane wave in the direction of $\omega$: a wave in the sense of solving the wave equation, $(\partial_t^2 - \Delta)\hat{e} = 0$.

Formula (3.8.3) combined with (B.1.13) provides a description of the continuous spectral measure of $P_V$ in terms of distorted plane waves. As we will see in Theorem 3.49 the scattering matrix intertwines $e(-\lambda, \cdot, x)$ and $e(\lambda, \cdot, x)$.

**REMARK.** The definition (3.8.1) shows that $\lambda \mapsto \lambda^{n-1}e(\lambda, \omega, x)$ extends to a meromorphic family for $\lambda \in \mathbb{C}$:

\begin{equation}
(3.8.5) \quad \lambda^{\frac{n-1}{2}}e(\lambda, \omega, x) \in C^\infty(\mathbb{R}_\lambda \times S_{n-1}^\omega; H^2_{\text{loc}}(\mathbb{R}^n_\omega)),
\end{equation}
This follows from Theorem 3.23.

When \( n = 3 \), then the term \( \Pi_0 \) does not contribute a pole of order two, \( \lambda^{-2} \), in (3.8.1) as \( \Pi_0(V) = 0 \) – see Step 1 of the proof of Lemma 3.18. But, due to a possible resonance at 0, we may have a singularity \( \lambda^{-1} \) canceled by \( \lambda^{\frac{n-1}{2}} = \lambda \).

For \( n \geq 5 \) there is \( \Pi_0(V) \) may not be zero and hence we may have a pole \( \lambda^{-2} \). Since \( \frac{n-1}{2} \geq 2 \) it is cancelled by \( \lambda^{\frac{n-1}{2}} \) and \( \lambda \mapsto \lambda^{\frac{n-1}{2}} e(\lambda, \omega, x) \) is smooth.

**Proof of Theorem 3.47.** 1. To see (3.8.2) we note that self-adjointness of \( PV \) shows that for \( \text{Im} \lambda > 0 \), \( R_V(\lambda)^* = R_V(-\bar{\lambda}) \). Combining this with (3.2.3) we obtain
\[
R_V(\lambda)^* u = R_V(\lambda) u = R_V(-\bar{\lambda}) u,
\]
and this remains valid for \( \lambda \in \mathbb{R} \) and \( u \in L^2_{\text{comp}} \). Hence, for \( \lambda \in \mathbb{R} \),
\[
e(\lambda, \omega, x) = e^{i\lambda \langle x, \omega \rangle} - R_V(\lambda) \left( V e^{i\lambda \langle x, \omega \rangle} \right)
= e^{i\lambda \langle x, \omega \rangle} - R_V(-\lambda) \left( V e^{i\lambda \langle x, \omega \rangle} \right)
= e(-\lambda, \omega, x).
\]

2. We now note that (3.8.3) is equivalent to (3.8.6)
\[
\langle (R_V(\lambda) - R_V(-\lambda)) \varphi, \varphi \rangle = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}^n} e(\lambda, \omega, x) \varphi(x) dx \right]^2 d\omega,
\]
for any \( \varphi \in C_\infty_c(\mathbb{R}^n) \).

For \( \text{Im} \lambda < 0 \), \( R_V(-\lambda)^* = R_V(\bar{\lambda}) \) and hence by analytic continuation,
\[
\langle R_V(-\lambda) \varphi, \varphi \rangle = \langle \varphi, R_V(\bar{\lambda}) \varphi \rangle, \quad \lambda \in \mathbb{C}.
\]

If \( \text{supp} \varphi \subset B_R := B(0, R) \) then, using (3.8.7) and \( (P_V - \lambda^2) R_V(\pm \lambda) \varphi = \varphi \), we obtain
\[
\langle (R_V(\lambda) - R_V(-\lambda)) \varphi, \varphi \rangle = \langle R_V(\lambda) \varphi, \varphi \rangle - \langle \varphi, R_V(\lambda) \varphi \rangle = \langle R_V(\lambda) \varphi, \varphi \rangle_{L^2(B_R)} - \langle \varphi, R_V(\lambda) \varphi \rangle_{L^2(B_R)}
= \langle R_V(\lambda) \varphi, (P_V - \lambda^2) R_V(\lambda) \varphi \rangle_{L^2(B_R)}
- \langle (P_V - \lambda^2) R_V(\lambda) \varphi, R_V(\lambda) \varphi \rangle_{L^2(B_R)}
= \langle \Delta R_V(\lambda) \varphi, R_V(\lambda) \varphi \rangle_{L^2(B_R)}
- \langle R_V(\lambda) \varphi, \Delta R_V(\lambda) \varphi \rangle_{L^2(B_R)}.
\]
3. Since $R_V(\lambda)\varphi(x) \in C^\infty(\mathbb{R}^n \setminus \text{supp} V)$ we can apply Green’s formula which shows that the left hand side of (3.8.6) is equal to

$$2i \text{Im} \int_{\partial B(0,R)} \partial_r R_V(\lambda)\varphi(y)R_V(\lambda)\varphi(y)dS(y),$$

where $dS(y)$ is the standard measure on the sphere $\partial B(0,R)$.

Hence we need to find asymptotics of the resolvent kernels. The answer is given in the following analogue of (3.1.20):

**Lemma 3.48.** Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n;\mathbb{R})$ and $\lambda \in \mathbb{R} \setminus \{0\}$. For $y \in \mathbb{R}^n$ we have

$$R_V(\lambda, r\omega, y) = e^{i\lambda r} \frac{\lambda^{n-3}}{r^{n-2}} c_n e(\lambda, y, \omega) + O(r^{-\frac{n+1}{2}}), \quad r \to \infty,$$

with a full (differentiable) expansion in powers of $1/r$ valid uniformly for $y$ in compact subsets of $\mathbb{R}^n$.

**Proof.** 1. We recall that $R_V(\lambda) = R_0(\lambda) - R_0(\lambda)VR_V(\lambda)$.

Since

$$R_V(\lambda, r\omega, y) = R_V(\delta(\bullet - y))(r\omega), \quad \delta(\bullet - y) \in \mathcal{E}'(\mathbb{R}^n)$$

we can apply Theorem 3.5 to see that

$$\lambda^{-\frac{n-3}{2}} c_n^{-1} e^{-i\lambda r} r^{-\frac{n-1}{2}} R_V(\lambda, r\omega, y) = e^{-i\lambda(y, \omega)}$$

$$- \int_{\mathbb{R}^n} e^{-i\lambda(y', \omega)} V(y') R_V(\lambda, y', y)dy' + O(r^{-1}),$$

with a full asymptotic expansion in powers of $r$.

2. The symmetry of $R_V(\lambda, y, y')$ (see (3.2.3)) shows that the integral on the right hand side is equal to $R_V(\lambda)(Ve^{-i\lambda(\bullet, \omega)}) (y)$ which means that the right hand side is equal to $e(\lambda, y, \omega) + O(1/r)$. □

4. We complete the proof of (3.8.6), and hence of (3.8.3) by inserting (3.8.9) into (3.8.8) and letting $R \to \infty$. More precisely,

$$r \partial_r [R_V(\lambda)\varphi](R\omega)R_V(\lambda)\varphi(R\omega) =$$

$$i |c_n|^2 R^{-n+2} \lambda^{n-2} \left| \int_{\mathbb{R}^n} e(\lambda, \omega, y)\varphi(y)dy \right|^2 + O(R^{-n+1}).$$
Hence the integral in (3.8.8) is equal to

\[ 2i|c_n|^2 \lambda^{n-2} \int_{S^{n-1}} \left| \int_{\mathbb{R}^n} e(\lambda, \omega, y) \varphi(y) dy \right|^2 d\omega + O(R^{-1}). \]

Inserting the value of \( c_n \) from (3.8.9) and letting \( R \to \infty \) gives (3.8.6) and that is equivalent to (3.8.3).

5. To obtain (3.8.4) we apply Stone’s formula given in Theorem B.8. □

The scattering matrix intertwines distorted plane waves:

**THEOREM 3.49.** In the notation of Theorem 3.47 define

\[ (3.8.10) \quad E_V(\lambda) f(\omega) := \int_{\mathbb{R}^n} e(\lambda, \omega, x) f(x) dx, \quad f \in L^2_{\text{comp}}(\mathbb{R}^n) \]

\[ E_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2(S^{n-1}). \]

Then, \( E_V(\lambda) = S(\lambda) J E_V(-\lambda) \)

where \( S(\lambda) \) is the scattering matrix and \( J f(\theta) = f(-\theta) \). In other words,

\[ (3.8.12) \quad S(\lambda) e(-\lambda, -\bullet, x) = e(\lambda, \bullet, x). \]

**INTERPRETATION.** With \( E_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2(S^{n-1}) \). Theorem 3.47 can be stated as follows:

\[ (3.8.13) \quad \begin{align*}
R_V(\lambda) - R_V(-\lambda) &= i \lambda^{n-2} (2\pi)^{-n-1} E_V(\bar{\lambda})^* E_V(\lambda), \\
dE_\lambda &= \lambda^{n-1} (2\pi)^{-n} E_V(\lambda)^* E_V(\lambda), \quad \lambda > 0.
\end{align*} \]

In fact, using (3.8.2) and (3.8.3) we see that

\[ (3.8.14) \quad R_V(\lambda) - R_V(-\lambda) = i \lambda^{n-2} (2\pi)^{-n-1} E_V(-\lambda)^* E_V(-\lambda), \quad \lambda \in \mathbb{R} \setminus \{0\}, \]

and changing \( \lambda \) to \(-\lambda\) gives (3.8.13) for \( \lambda \) real once we remember that \( n \) is odd. We then continue both sides meromorphically noting that \( \lambda \mapsto E_V(\bar{\lambda})^* \) is meromorphic.

We can use (3.8.11) to obtain a formula in which the singularities are “pushed into” the scattering matrix:

\[ R_V(\lambda) - R_V(-\lambda) = \frac{i}{2} \lambda^{n-2} (2\pi)^{-n} E_V(\lambda)^* S(\lambda) J E_V(-\lambda), \quad \lambda \in \mathbb{C}. \]

This means that for \( \text{Im} \lambda < 0 \), except for \( S(\lambda) \), all the factors on the right hand side are holomorphic (except for finitely many poles coming from negative eigenvalues).
Proof of Theorem 3.49. 1. We start by writing the definition of $E_V$ in terms of operators:

\[(3.8.15) \quad E_V(\lambda) = E_0(\lambda)(I - V R_V(\lambda)),\]

where we used definitions (3.8.1), (3.8.10) and (3.2.3). Recalling from (3.7.9) that

\[S(\lambda) = I + A(\lambda), \quad A(\lambda) = a_n E_0(\lambda)V(I - R_V(\lambda)V)E_0(\lambda)^* ,\]

we need to show that

\[(3.8.16) \quad E_V(\lambda) - J E_V(-\lambda) = A(\lambda) J E_V(-\lambda),\]

and it is enough to consider $\lambda \in \mathbb{R}$.

2. For $\lambda \in \mathbb{R}$, we apply Theorem 3.47 (as rephrased in (3.8.14)) to obtain

\begin{align*}
E_V(\lambda) - J E_V(-\lambda) & = - E_0(\lambda)V(R_V(\lambda) - R_V(-\lambda)) \\
& = a_n \lambda^{n-2} E_0(\lambda) V E_V(-\lambda)^* E_V(-\lambda) \\
& = a_n \lambda^{n-2} E_0(\lambda) V(I - R_V(\lambda)V)E_0(-\lambda)^* E_V(-\lambda) \\
& = A(\lambda) J E_V(-\lambda)
\end{align*}

where we also used $R_V(-\lambda)^* = R_V(\lambda)$ and $E_0(-\lambda)^* = E_0(\lambda)^* J$. This gives (3.8.16) completing the proof. \hfill \Box

3.9. THE BIRMAN–KREĬN TRACE FORMULA

The Birman–Kreĭn formula gives an expression for $\text{tr}(f(P_V) - f(P_0))$ in terms of the determinant of the scattering matrix. It was given in one dimension in Theorem 2.17 and we now proceed to the case of potential scattering in all odd dimensions. We consider the case of real potentials so that the Schrödinger operator $P_V$ is self-adjoint.

Theorems 2.17 below is valid without much change in all dimensions and for much less restrictive classes of potentials. That is not the case with the trace formulae of Theorem 2.19 and 3.53 which cannot hold in even dimensions and are delicate for more general perturbations.

We start with\(^2\)

**THEOREM 3.50 (Trace class property of $f(P_V) - f(P_0)$).** Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$. For $f \in \mathcal{S}(\mathbb{R})$

\[(3.9.1) \quad f(P_V) - f(P_0) \in \mathcal{L}_1(L^2(\mathbb{R}^n)),\]

\(^2\)If the reader is interested in the case of dimension 3 only then Theorem 3.50 can be skipped as the proof of Theorem 3.51 for $n = 3$ provides a direct argument for the trace class property.
that is, the operator on the left hand side is of trace class and
\[ T_V : f \mapsto \text{tr}(f(P_V) - f(P_0)), \quad f \in \mathcal{S}(\mathbb{R}) \]
defines an element of \( \mathcal{S}'(\mathbb{R}) \).

In addition if \( 1_B(0,R) \) is the indicator function of \( B(0,R) \), then
\[ (3.9.2) \quad 1_B(0,R) f(P_V) \in L^1(L^2(\mathbb{R}^n)), \]
and
\[ (3.9.3) \quad \text{tr}(f(P_V) - f(P_0)) = \lim_{R \to \infty} \text{tr} 1_B(0,R) (f(P_V) - f(P_0)). \]

Proof. 1. Since \( f \in \mathcal{S} \), we can write \( f(z) = (z + i)^{-N}g(z), \quad g \in \mathcal{S}(\mathbb{R}). \)
We then apply the Helffer-Sjöstrand formula for functions of self-adjoint operators (see §B.2):
\[ (3.9.4) \quad f(P_V) - f(P_0) = \frac{1}{\pi} \int_{\mathbb{C}} \left( (P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)(P_0 + i)^{-N} \right) \bar{\partial}_z \bar{g}(z) dm(z), \]
where \( dm(z) \) is the Lebesgue measure on \( \mathbb{C} \) and
\[ \bar{g} \in \mathcal{S}(\mathbb{C}), \quad \text{supp } g \subset \{ |\text{Im } z| < 1 \}, \]
and is an almost analytic extension of \( g \).

2. We write
\[ (P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)^{-1}(P_0 + i)^{-N} = A + B, \]
where
\[ A := ((P_V - z)^{-1} - (P_0 - z)^{-1})(P_0 + i)^{-N} \]
\[ = -(P_V - z)^{-1}V(P_0 + i)^{-N}(P_0 - z)^{-1}, \]
and
\[ B := (P_V - z)^{-1}((P_V + i)^{-N} - (P_0 + i)^{-N}) \]
\[ = -(P_V - z)^{-1}\sum_{k=1}^{N} (P_V + i)^{-N+k-1}V(P_0 + i)^{-k}, \]
where the formula for \( B \) is easily proved by induction on \( N \).

3. Arguing as in (B.3.10) we see that for \( \rho \in C^\infty_c(\mathbb{R}^n) \), singular values satisfy
\[ (3.9.6) \quad s_j(\rho(P_0 + i)^{-k}) = s_j((P_0 + i)^{-k} \rho) \leq C j^{-2k/n}. \]
We claim that the same estimate is valid with \( P_0 \) replaced by \( P_V \):
\[ (3.9.7) \quad s_j(\rho(P_V + i)^{-k}) = s_j((P_V + i)^{-k} \rho) \leq C j^{-2k/n}. \]
In fact, we prove this by induction. The case of \( k = 0 \) is immediate and assume that (3.9.7) holds for \( k \). Without loss of generality we can take \( \rho \) satisfying \( \rho V = V \), using the same decomposition as in (3.9.5), applying (B.3.5), (3.9.6) and the induction hypothesis, \( (N := k + 1) \)

\[
s_j((P_V + i)^{-k-1}\rho) \leq s_{j/N}((P_0 + i)^{-k-1}\rho)
\]

\[
+ C \sum_{\ell=1}^{k+1} s_{j/N}((P_V + i)^{-k+\ell-1}V(P_0 + i)^{-\ell}\rho)
\]

\[
\leq C j^{-2k/n}
\]

\[
+ C \sum_{\ell=1}^{k+1} s_{j/2N}((P_V + i)^{-k+\ell-1}\rho)s_{j/2N}((P_0 + i)^{-\ell}\rho)
\]

\[
\leq C j^{-2(k+1)/n} + C' \sum_{\ell=1}^{k+1} j^{-2(k+1-\ell)/n} j^{-2\ell/n}
\]

\[
\leq C'\ j^{-2(k+1)/n}.
\]

4. Returning to step 2, we use (3.9.7) to obtain

\[
s_j((P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)^{-1}(P_0 + i)^{-N})
\]

\[
\leq s_{[j/2]}(A) + s_{[j/2]}(B)
\]

\[
\leq \|(P_V - z)^{-1}\|\|(P_0 - z)^{-1}\|s_{[j/2]}(V(P_0 + i)^{-N})
\]

\[
+ \|(P_V - z)^{-1}\|\sum_{k=1}^{N} s_{[j/2N]}((P_V + i)^{-N+k-1}V(P_0 + i)^{-k})
\]

\[
\leq C|\text{Im } z|^{-2j^{2N/n}}.
\]

Expressing the trace class norm (B.4.2) using singular values we see that

(3.9.8) \( (P_V - z)^{-1}(P_V + i)^{-N} - (P_0 - z)^{-1}(P_0 + i)^{-N} = \mathcal{O}(|\text{Im } z|^{-2})_{\mathcal{L}^1} \),

if \( N > n/2 \). Combined with (3.9.4) we obtain (3.9.1).

5. The estimates (3.9.7) also show that for \( \rho \in C_c^\infty(\mathbb{R}^n) \) equal to 1 on \( B(0,R) \) and \( N > n/2 \),

\[
1_{B(0,R)}(P_V + i)^{-N}(P_V - z)^{-1} = 1_{B(0,R)} \rho(P_V + i)^{-N}(P_V - z)^{-1}
\]

\[
= \mathcal{O}(|\text{Im } z|^{-1})_{\mathcal{L}^1},
\]

which gives (3.9.2).

6. To see (3.9.3) we claim that

(3.9.9) \( 1_{\mathbb{R}^n \setminus B(0,R)}(P_0 - z)^{-1}V = \mathcal{O}(R^{-M}|\text{Im } z|^{-M-1})_{\mathcal{L}^1} \),
for sufficiently large $M$. Using (3.9.4) with $N = 0$ and the resolvent identity, this gives a quantitative version of (3.9.3).

To see (3.9.9) we choose $\rho \in C^\infty_c$, $\rho \equiv 1$ on supp $V$ (independent of $R$) and $\psi_{j,R} \in C^\infty_c$, $1 \leq j \leq J$, such that supp $\psi_{j+1,R} \equiv 1$ on supp $\psi_{j,R}$, $j < J$, supp $\psi_{1,R} \equiv 1$ on supp $\rho$ and

$$\partial^\alpha \psi_{j,R} = O_{\alpha,J}(R^{-|\alpha|}), \quad \text{supp} \psi_{j,R} \subseteq B(0,R).$$

In particular supp $\psi_{j,R} \cap \text{supp} \partial \psi_{j+1,R} = \emptyset$. This fact will be crucial in the next calculation.

Since the estimate (3.9.9) is equivalent to the estimate for the adjoint, we can estimate the trace class norm of

$$\rho(P_0 - z)^{-1} 1_{\mathbb{R}^n \setminus B(0,R)} = \rho \psi_{1,R} \cdots \psi_{J,R} (P_0 - z)^{-1} 1_{\mathbb{R}^n \setminus B(0,R)}$$

$$= \rho \psi_{1,R} \cdots \psi_{J-1,R} (P_0 - z)^{-1} [P_0, \psi_{J,R}] (P_0 - z)^{-1} 1_{\mathbb{R}^n \setminus B(0,R)}$$

$$= \rho (P_0 - z)^{-1} [P_0, \psi_{1,R}] (P_0 - z)^{-1} \cdots [P_0, \psi_{J,R}] (P_0 - z)^{-1} 1_{\mathbb{R}^n \setminus B(0,R)}.$$

From estimates on derivatives of $\psi_{j,R}$ we see that

$$[P_0, \psi_{j,R}] = \mathcal{O}(R^{-1})_{H^{s+2}((\mathbb{R}^n)) \to H^{s+1}((\mathbb{R}^n))}.$$

Also $(P - z)^{-1} = \mathcal{O}(|\text{Im } z|^{-1})_{H^s \to H^{s+2}}$ and hence

$$[P_0, \psi_{j,R}](P_0 - z)^{-1} = \mathcal{O}(R^{-1} |\text{Im } z|^{-1})_{H^s((\mathbb{R}^n)) \to H^{s+1}((\mathbb{R}^n))}.$$

We conclude that

$$\rho(P_0 - z)^{-1} 1_{\mathbb{R}^n \setminus B(0,R)} = \mathcal{O}(|\text{Im } z|^{-J-1}R^{-J})_{L^2(\mathbb{R}^n) \to H^{J+2}(B(0,R_0))},$$

for $R_0$ such that supp $\rho \subset B(0,R_0)$. For $J$ large enough $\|1_{B(0,R_0)} A\|_{L^1} \leq C\|A\|_{L^2 \to H^{J+2}}$, and that concludes the proof of (3.9.9). As stated after that estimate gives a strong version of (3.9.3).

7. The proof that $f \mapsto \text{tr}(f(P_V) - f(P_0))$ defines a tempered distribution follows from estimates on $\partial g$ in terms of the a finite number of semi-norms $\sup |\langle \lambda \rangle^m \partial^j g|$. These follow from the construction of $g$ in §B.2. \hfill $\square$

We are now ready for the main result of this section:

**THEOREM 3.51 (The Birman–Kreǐn formula).** Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, where $n \geq 3$ is odd.

Then for $f \in \mathcal{S}(\mathbb{R})$ the operator $f(P_V) - f(P)$ is of trace class and

$$\text{tr}(f(P_V) - f(P_0)) = \frac{1}{2\pi i} \int_0^\infty f(\lambda^2) \text{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda)) \, d\lambda$$

(3.9.10)

$$+ \sum_{k=1}^K f(E_k) + \frac{1}{2} \bar{m}_R(0)f(0),$$
where $S(\lambda)$ is the scattering matrix and $E_K < E_{K-1} \leq \cdots \leq E_1 \leq 0$ are the eigenvalues of $P_V$ and $\tilde{m}_R(0)$ is defined by (3.3.17) (and can be non-zero only when $n = 3$).

**REMARKS.**

1. For the heuristic interpretation of

$$\sigma'(\lambda) := \frac{1}{2\pi i} \text{tr} S(\lambda)^{-1} \partial_\lambda S(\lambda)$$

see the discussion after Theorem 2.17.

2. At this stage the integral on the right hand side of (3.9.10) is meant as distributional pairing of $\sigma' \in \mathcal{S}'$ with $f(\lambda^2)$ – see Theorem 3.50. In §3.11 we will see that $\sigma'$ is polynomially bounded so that the integral converges in the usual sense.

3. In Lemma 3.52 §3.11 we will use $f(s) = e^{-ts}$: that is allowed as $f(P_V) = (\chi f)(P_V)$ for $\chi \in C^\infty(\mathbb{R})$, supp $\chi \subset (\min \text{Spec}(P_V) - 1, \infty)$, $\chi \equiv 1$ on $[\min \text{Spec}(P_V), \infty)$. We then have $\chi f \in \mathcal{S}(\mathbb{R})$.

In dimension three a complication arises from the possibility of the resonance at zero – see Theorem 3.23. On the other hand the trace class properties are easier and hence the proof we presented in dimension one (see §2.6) applies and in fact is somewhat easier as $R_0(\lambda)$ is now holomorphic at zero.

**Proof of Theorem 3.51 for $n = 3$.** 1. As in (3.8.4), the spectral theorem and Stone’s formula show that

$$f(P_V) = \sum_{k=1}^K f(E_k) u_k \otimes \bar{u}_k + \frac{1}{4\pi i} \int_{\mathbb{R}} f(\lambda^2)(R_V(\lambda) - R_V(-\lambda))2\lambda d\lambda,$$

where we used the fact that the integrand is even to change the integration from $(0, \infty)$ to $\mathbb{R}$. The operator $(R_V(\lambda) - R_V(-\lambda))2\lambda$ is smooth in $\lambda \in \mathbb{R}$ as an operator $L^2_{\text{comp}} \to L^2_{\text{loc}}$. For simplicity we will now assume that there are no negative eigenvalues: their contribution in the formula is clear.

2. We write

$$R_V(\lambda) - R_0(\lambda) = -R_V(\lambda)VR_0(\lambda)$$

$$= -R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda).$$

This operator can have a pole at $\lambda = 0$ as described in Lemma 3.16. In the notation of that lemma we define

$$B(\lambda) := 2\lambda(R_V(\lambda) - R_0(\lambda)) + \frac{2\Pi_0}{\lambda}$$

$$= -2\lambda R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda) + \frac{2\Pi_0}{\lambda},$$

$$B(\lambda) : L^2_{\text{comp}}(\mathbb{R}^3) \to L^2_{\text{loc}}(\mathbb{R}^3),$$

(3.9.11)
which is a meromorphic family of operators, holomorphic for \( \text{Im} \lambda \geq 0 \). The possible \( 1/\lambda \) term in \( R_V(\lambda) \) at \( \lambda = 0 \) is cancelled by the \( \lambda \) factor and the spectral pole \(-\Pi_0/\lambda^2\) by the the last term on the right hand side.

With this notation we have (we assumed for simplicity of exposition that there are no negative eigenvalues)

\[
(3.9.12) \quad f(P_V) - f(P_0) = f(0)\Pi_0 + \frac{1}{4\pi i} \sum_\pm \int_{\mathbb{R}} f(\lambda^2)B(\pm \lambda)d\lambda.
\]

We recall from (3.1.24) that for \( \text{Im} \lambda \geq 0 \) \( \|R_0(\lambda)\|_{L^2 \rightarrow H^2} \leq C(\lambda)^2/|\lambda| \text{Im} \lambda \). Arguing as in (3.4.2) (with \( n = 3 \)) this gives

\[
\begin{align*}
s_j(R_0(\lambda)\rho) &= s_j(\rho R_0(\bar{\lambda})) \\
&= s_j((-\Delta_{\tau^k_h} + 1)^{-1}(-\Delta_{\tau^k_h} + 1)\rho R_0(\bar{\lambda})) \\
&\leq s_j((-\Delta_{\tau^k_h} + 1)^{-1}\|R_0(\bar{\lambda})\|_{L^2 \rightarrow H^2} \\
&= O((\lambda)^2/|\lambda| \text{Im} \lambda)^{-2/3}.
\end{align*}
\]

Since there are no poles for \( \lambda \neq 0, \text{Im} \lambda \geq 0 \),

\[
(3.9.13) \quad \|(I + VR_0(\lambda)\rho)^{-1}\|_{L^2 \rightarrow L^2} = O \left( (\lambda)^2/|\lambda|^2 \right), \quad \text{Im} \lambda > 0.
\]

In fact, for \( |\lambda| \gg 1 \) this follows from (3.1.12) and for \( \lambda \) near 0 from (3.3.4). Hence,

\[
\begin{align*}
&\quad \|B(\lambda)\|_{L^1} \leq \frac{2}{|\lambda|} + 2|\lambda|\sum_{j=1}^{\infty} s_j(R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda)) \\
&\leq \frac{2}{|\lambda|} + \frac{C(\lambda)^2}{|\lambda|} \sum_{j=1}^{\infty} s_{[j/2]}(R_0(\lambda)\rho)^2 \\
&\leq \frac{C(\lambda)^6}{|\lambda|^3(\text{Im} \lambda)^2} \sum_{j=1}^{\infty} j^{-4/3} \leq \frac{C(\lambda)^6}{|\lambda|^3(\text{Im} \lambda)^2} \leq \frac{C(\lambda)^6}{(\text{Im} \lambda)^2}.
\end{align*}
\]

3. Let \( g \in \mathcal{S}(\mathbb{C}), \text{supp} g \subset \{|\text{Im} \lambda| \leq 1\} \), be an almost analytic extension of \( f(\lambda^2) \), see §B.2. The Cauchy-Green formula (D.1.1) applied to the right hand side of (3.9.12) shows that (with \( dm(\lambda) \) the Lebesgue measure on \( \mathbb{C} \))

\[
(3.9.15) \quad f(P_V) - f(P_0) = \frac{1}{2\pi} (t_+(f) - t_-(f)),
\]

\[
t_\pm(f) := \int_{\pm \text{Im} \lambda > 0} \partial_\lambda g(\lambda)(B(\pm \lambda)dm(\lambda)).
\]

Since \( \partial_\lambda g(\lambda) = O(|\text{Im} \lambda|^\infty(\lambda)^{-\infty}) \), the estimate (3.9.14) shows that \( t_\pm(f) \in L_1 \). In particular, this implies directly that the \( f(P_V) - f(P_0) \) is of trace class and that \( f \mapsto \text{tr}(f(P_V) - f(P_0)) \) defines a tempered distribution – of course this also follows from Theorem 3.50.
3. To relate the trace of (3.9.15) to the scattering matrix we reformulate Theorem 3.46 as follows:

\[(3.9.16) \quad \det S(\lambda) = \det((I + VR_0(\lambda)\rho)^{-1}(I + VR_0(-\lambda)\rho)).\]

This is valid since, in the notation of Theorem 3.46,

\[I - T(\lambda) = I - (I + VR_0(\lambda)\rho)^{-1}(V(R_0(\lambda) - R_0(-\lambda))\rho) = (I + VR_0(\lambda)\rho)^{-1}(I + VR_0(-\lambda)\rho).\]

Since \((-\lambda^2 - \lambda^2)\partial_\lambda R_0(\lambda) = 2\lambda R_0(\lambda),\) elliptic estimates (see for instance [Zw12, Theorem 7.1]) show that

\[\rho\partial_\lambda R_0(\lambda)\rho = O\left((\lambda)^2 e^{C(\text{Im}\lambda)}\right) : L^2(\mathbb{R}^3) \to H^4(\mathbb{R}^3)\]

and that implies that

\[(3.9.17) \quad \|\partial_\lambda(VR_0(\lambda)\rho)\|_{L^1(L^2(\mathbb{R}^3))} \leq (\lambda)^2 e^{C(\text{Im}\lambda)}, \quad \lambda \in \mathbb{C}.\]

Taking logarithmic derivatives of both sides of (3.9.16) gives

\[(3.9.18) \quad \text{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} = \text{tr} F(-\lambda) + \text{tr} F(\lambda),\]

\[F(\lambda) := -\partial_\lambda(VR_0(\lambda)\rho)(I + VR_0(\lambda)\rho)^{-1}.\]

We note that \(F(\lambda), \lambda \in \mathbb{C},\) is a meromorphic family of operators in \(L_1(L^2),\)

with no poles in \(\text{Im} \lambda > 0.\)

Theorem 3.15 and the Gohberg–Sigal theory, see §C.4, show that, near \(\lambda = 0,\)

\[(3.9.19) \quad I + VR_0(\lambda)\rho = U_1(\lambda)(Q_2\lambda^2 + Q_1\lambda + Q_0)U_2(\lambda),\]

where the operators \(U_j(\lambda)\) are invertible and holomorphic as function of \(\lambda,\)

\[Q_1Q_2 = \delta_{ij}Q_{ij}, \quad \text{rank}(I - Q_0) < \infty,\]

\[\text{rank} Q_2 = m_R(0) - m_R(0) = \text{tr} \Pi_0, \quad \text{rank} Q_1 = \bar{m}_R(0).\]

(We have \((I + VR_0)^{-1} = U_2(\lambda)^{-1}(Q_2\lambda^{-2} + Q_1\lambda^{-1} + Q_0)U_1(\lambda)^{-1};\) the total multiplicity of the pole is \(m_R\) while the rank of the \(\lambda^{-2}\) has to be \(\text{tr} \Pi_0.)\)

Taking a logarithmic derivative of (3.9.19) gives

\[(3.9.20) \quad \text{tr} F(\lambda) = -\text{tr}(2\lambda^{-1}Q_2 + \lambda^{-1}Q_1) + \varphi(\lambda)\]

\[= -\frac{1}{\lambda}(2 \text{tr} \Pi_0 + \bar{m}_R(0)) + \varphi(\lambda),\]

where \(\varphi(\lambda)\) is holomorphic in \(\text{Im} \lambda \geq 0.\) In view of (3.9.13) and (3.9.17), we have

\[(3.9.21) \quad |\varphi(\lambda)| \leq C(\lambda)^2, \quad \text{Im} \lambda \geq 0.\]
4. We claim that for \( \text{Im} \lambda > 0 \)
\[
(3.9.22) \quad \text{tr} F(\lambda) = \text{tr} \left( B(\lambda) - \frac{2\Pi_0}{\lambda} \right),
\]
where \( B(\lambda) \) was defined by (3.9.11). To see this we use the fact that \( R_0(\lambda) \) is bounded on \( L^2 \) for \( \text{Im} \lambda > 0 \) and hence \( \partial_\lambda (VR_0(\lambda)\rho) = 2\lambda VR_0(\lambda)^2\rho \). Using this, the cyclicity of the trace, and \( \rho V = V \), we obtain, always for \( \text{Im} \lambda > 0 \),
\[
\text{tr} F(\lambda) = -2\lambda \text{tr} R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}VR_0(\lambda) = \text{tr} \left( B(\lambda) - \frac{2\Pi_0}{\lambda} \right),
\]
which is (3.9.22).

5. Let
\[
(3.9.23) \quad h(\lambda) := \text{tr} F(\lambda) + 2\text{tr} \Pi_0 = -\frac{\tilde{m}_R(0)}{\lambda} + \varphi(\lambda).
\]
In this notation, (3.9.22) and (D.1.1) used in (3.9.15) give
\[
(3.9.24) \quad \text{tr} (f(P_\nu) - f(P_0)) = \frac{1}{2\pi} \sum_{\pm} \int_{\pm \text{Im} \lambda > 0} \partial_\lambda g(\lambda) h(\pm \lambda) dm(\lambda)
\]
\[
= \frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon)} g(\lambda) h(\pm \lambda) d\lambda + \frac{1}{2\pi} \sum_{\pm} \int_{\Omega_{\pm}(\epsilon)} \partial_\lambda g(\lambda) h(\pm \lambda) dm(\lambda),
\]
where
\[
\Omega_{\pm}(\epsilon) = D(0, \epsilon) \cap \mathbb{C}_{\pm}, \quad \mathbb{C}_{\pm} := \{\pm \text{Im} \lambda > 0\},
\]
\[
\gamma_{\pm}(\epsilon) = \partial (\mathbb{C}_{\pm} \setminus \Omega_{\pm}(\epsilon)), \quad \gamma_{-}(\epsilon) = \partial (\mathbb{C}_{+} \cup \Omega_{-}(\epsilon)),
\]
and the boundaries are positively oriented (as boundaries of the indicated sets).

Estimates (3.9.21) and \( \partial_\lambda g(\lambda) = \mathcal{O}(|\text{Im} \lambda|^\infty(\lambda)^{-\infty}) \) show that the last term on the right hand side of (3.9.24) is \( \mathcal{O}(\epsilon) \) as \( \epsilon \to 0 \). Also, (3.9.18) and (3.9.23) imply that
\[
\frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon) \cap \mathbb{R}} g(\lambda) h(\pm \lambda) d\lambda
\]
\[
= \frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon) \cap \mathbb{R}} f(\lambda^2) (\text{tr} F(\pm \lambda) \pm 2\text{tr} \Pi_0\lambda^{-1}) d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\epsilon}^{\infty} f(\lambda^2) \text{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{0}^{\infty} f(\lambda^2) \text{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} d\lambda + \mathcal{O}(\epsilon),
\]
as the $\lambda^{-1}$ terms cancel. Hence,

$$\frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon)} g(\lambda) h(\pm \lambda) d\lambda = \frac{1}{2\pi i} \int_{0}^{\infty} f(\lambda^2) \text{tr} \partial_\lambda S(\lambda) S(\lambda)^{-1} d\lambda$$

$$+ \frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon) \setminus \mathbb{R}} g(\lambda) h(\pm \lambda) d\lambda + \mathcal{O}(\epsilon).$$

The structure of $h(\lambda)$ near 0 given in (3.9.23) shows that

$$\frac{1}{4\pi i} \sum_{\pm} \int_{\gamma_{\pm}(\epsilon) \setminus \mathbb{R}} g(\lambda) h(\pm \lambda) d\lambda = m_R(0) f(0) + \mathcal{O}(\epsilon).$$

Letting $\epsilon \to 0$ and noting that $\text{tr} S(\lambda)^{-1} \partial S(\lambda)$ is even (see (3.9.18)) we obtain (3.9.10).

For odd dimensions $n \geq 5$ we do not have a possibility of a zero resonance but the arguments are complicated by weaker trace class properties. The following somewhat laborious lemma will be used to establish (3.9.10) for functions supported near 0:

**LEMMA 3.52.** Suppose that $n \geq 5$ and $E_K < E_{K-1} \leq \cdots \leq E_1 < 0$ are the eigenvalues of $P_V$ included according to their multiplicities. Then for sufficiently large $M$ and $N$,

$$\text{tr}((P_V + M)^{-N} e^{-tP_V} - (P_0 + M)^{-N} e^{-tP_0}) = \sum_{k=1}^{K} (E_k + M)^{-N} e^{-tE_k} + o(1), \ t \to +\infty. \quad (3.9.25)$$

**REMARK.** A more precise asymptotics for $t \to +\infty$ are valid once we establish (3.9.10) and restrictions on $N$ or $M$ are needed either – see Exercise 3.11.

**Proof.** 1. For $\epsilon$ small enough define the contour (see Figure 3.5)

$$\gamma_\epsilon = (i\epsilon - e^{-\frac{\epsilon}{2}}[0, \infty)) \cup (i\epsilon + e^{\frac{\epsilon}{2}}[0, \infty))$$

oriented from left to right$^3$. We also define

$$\Omega := \bigcup_{0 \leq \epsilon \leq \epsilon_0} \gamma_\epsilon \cap D(0, 2\epsilon_0), \ \epsilon_0 \ll 1, \quad (3.9.26)$$

$^3$ This choice of orientation comes from the fact that we are integrating $(P_V - z)^{-1} dz$, $z = \lambda^2$ which is then consistent with the residue theorem.
3.9. THE BIRMAN–KREÎN TRACE FORMULA

We choose \( \epsilon_0 \) small enough so that for no poles of \( R_V(\lambda) \) other than 0 belong to \( \Omega \). With this notation, for \( \text{Im} \lambda_0 \gg 1 \),

\[
e^{-tP_V}(P_V - \lambda_0^2)^{-N} - e^{-tP_0}(P_0 - \lambda_0^2)^{-N} = \sum_{E_k < 0} (E_k + i)^{-N} e^{-tE_k} u_k \otimes u_k
\]

\[
+ \frac{1}{\pi i} \int_{\gamma_\epsilon} (\lambda R_V(\lambda)R_V(\lambda_0)^N - \lambda R_0(\lambda)R_0(\lambda_0)^N) e^{-t\lambda^2} d\lambda.
\]

In (3.9.8) (in the proof of Theorem 3.50) we showed that

\[(3.9.27) \quad \|R_V(\lambda)R_V(\lambda_0)^N - R_0(\lambda)R_0(\lambda_0)^N\|_{L_1} \leq C |\text{Im} \lambda|^{-4}, \quad \lambda \in \gamma_\epsilon,
\]

which shows that

\[(3.9.28) \quad \sum_{E_k < 0} (E_k + M)^{-N} e^{-tE_k} + \frac{1}{\pi i} \int_{\gamma_\epsilon} f(\lambda)e^{-t\lambda^2} d\lambda,
\]

where

\[(3.9.29) \quad f(\lambda) := \text{tr} (\lambda R_V(\lambda)R_V(\lambda_0)^N - \lambda R_0(\lambda)R_0(\lambda_0)^N), \quad \lambda_0 := i\sqrt{M}.
\]

Away from a neighbourhood of 0 the estimate (3.9.27) is all that we need and we now concentrate on \( \lambda \in \Omega \), with \( \Omega \) given in (3.9.26).

2. Since \( n \geq 5 \), Theorem 3.17 shows that

\[
\lambda R_V(\lambda)\rho = \lambda R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1} = \lambda R_0(\lambda)\rho(I - VR_V(\lambda)\rho)
\]

\[
= \lambda^{-1} R_0(\lambda)V\Pi_0\rho + \lambda R_0(\lambda)\rho B(\lambda)\rho_1,
\]

where \( \rho_1 \in C_c^\infty(\mathbb{R}^n) \), \( \rho_1\rho = \rho \), and

\[
B(\lambda) := I - V A(\lambda)\rho : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
\]

is a holomorphic family of operators for \( \lambda \in \Omega \) (see (3.9.26)). Using

\[(3.9.30) \quad R_0(\lambda)V\Pi_0 = -R_0(\lambda)((-\Delta - \lambda^2) + \lambda^2)\Pi_0 = -(I + \lambda^2 R_0(\lambda))\Pi_0,
\]
and the adjoint of this identity for $\bar{\lambda}$, we obtain

$$\lambda R_V(\lambda) - \lambda R_0(\lambda) = -\lambda R_V(\lambda)VR_0(\lambda)$$

$$= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}R_0(\lambda)V\Pi_0 VR_0(\lambda)$$

$$= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}(I + \lambda^2 R_0(\lambda))\Pi_0(I + \lambda^2 R_0(\lambda))$$

$$= -\lambda R_0(\lambda)\rho B(\lambda)VR_0(\lambda) - \lambda^{-1}\Pi_0$$

$$- \lambda R_0(\lambda)\Pi_0 - \lambda\Pi_0 R_0(\lambda) - \lambda^3 R_0(\lambda)\Pi_0 R_0(\lambda).$$

The goal of the above identity is to obtain as few factors of $R_0(\lambda)$ and as many powers of $\lambda$ as possible.

3. We rewrite $f$ in (3.9.29) as follows:

$$f(\lambda) = \text{tr} \left( (\lambda R_V(\lambda) - \lambda R_0(\lambda))R_0(\lambda_0)^N + \lambda R_V(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N) \right).$$

This gives the decomposition

$$f(\lambda) = a(\lambda) + b(\lambda) + c(\lambda),$$

$$a(\lambda) := -\lambda^{-1} \text{tr} \Pi_0 R_V(\lambda_0)^N, \quad b(\lambda) = b_1(\lambda) + b_2(\lambda)$$

$$b_1(\lambda) := -\lambda \text{tr} R_0(\lambda)\rho B(\lambda)VR_0(\lambda)R_0(\lambda_0)^N,$n

$$b_2(\lambda) := -\lambda \text{tr} (2R_0(\lambda)\Pi_0 + \lambda^2 R_0(\lambda)\Pi_0 R_0(\lambda))R_0(\lambda_0)^N,$n

$$c(\lambda) := \text{tr} \lambda A(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N),$$

$$A(\lambda) := R_V(\lambda) + \lambda^{-2}\Pi_0,$n

where $A(\lambda)$ is holomorphic in a neighbourhood of 0 by Theorem 3.17.

The contribution of $a(\lambda)$ is straightforward:

$$\frac{1}{\pi i} \int_{\gamma_\epsilon} a(\lambda)e^{-t\lambda^2}d\lambda = -m_R(0)(-\lambda_0^2)^{-N} \frac{1}{\pi i} \int_{\gamma_\epsilon} \lambda^{-1}e^{-t\lambda^2}d\lambda$$

$$= -m_R(0)(-\lambda_0^2)^{-N} \frac{1}{\pi i} \int_{\gamma_\epsilon^1} \lambda^{-1}e^{-t\lambda^2}d\lambda$$

$$= m_R(0)(-\lambda_0^2)^{-N},$$

where we deformed $\gamma_\epsilon$ to $\gamma_\epsilon^1 := ([\mathbb{R} \setminus (-\epsilon, \epsilon)] \cup (\partial D(0, \epsilon) \cap \text{Im} \lambda > 0))$ oriented from left to right: the contributions over the real axis cancel and integration over the clockwise oriented half circle produces $-\pi i$.

4. We now move to the analysis of $b_1(\lambda)$. Since $\rho R_0(\lambda)R_0(\lambda_0)^N, \lambda \in \gamma_\epsilon, \epsilon > 0$, is of trace class, cyclicity of the trace shows that

$$b_1(\lambda) = \lambda \text{tr} B(\lambda)VR_0(\lambda)R_0(\lambda_0)^N R_0(\lambda)\rho.$$
3.9. The Birman–Krein Trace Formula

If we take $N$ sufficiently large, then by Lemma 3.6 we have

$$|b_1(\lambda)| \leq C\|B(\lambda)\|_{L^2 \rightarrow L^2} \|\lambda \rho R_0(\lambda) R_0(\lambda_0)^N R_0(\lambda)\|_{L^1}$$

$$\leq C\|B(\lambda)\|_{L^2 \rightarrow L^2} \|\lambda \rho R_0(\lambda) R_0(\lambda_0)^N R_0(\lambda)\|_{L^2 \rightarrow H^{n+1}} \leq C,$$

for $\lambda \in \Omega$. This shows that we can deform $\gamma_\epsilon$ to $\gamma_0$ and hence

$$\int_{\gamma_\epsilon} b_1(\lambda)e^{-t\lambda^2}d\lambda = \int_{\gamma_0} b_1(\lambda)e^{-t\lambda^2}d\lambda = O(1) \int_{\gamma_0} e^{-t \text{Re} \lambda^2}d|\lambda| = O(t^{-\frac{1}{2}}) = o(1).$$

5. To analyse $b_2$ let $\Pi_0 = \sum_{j=1}^J u_j \otimes u_j$,

$$u_j = -R_0(0)V u_j = O(|x|^{2-n}) \in L^p(\mathbb{R}^n), \quad p > \frac{n}{n-2},$$

in particular for some $p < 2$ when $n \geq 5$. We then use Exercise 3.2 and Young’s inequality (A.5.2) to see that for $\lambda \in \Omega$,

$$\|R_0(\lambda) u_j\|_{L^2} \leq C|\lambda|^{q-1}/q \|u_j\|_{L^p}, \quad \frac{1}{q} + \frac{1}{p} = \frac{3}{2}.$$ But since we can use $p < 2$ this means that we can take $q > 1$ and hence,

$$\|R_0(\lambda)\Pi_0\|_{L^2 \rightarrow L^2} \leq C|\lambda|^{-2-\delta}, \quad \delta > 0.$$ It follows that

$$\lambda R_0(\lambda)\Pi_0 R_0(\lambda_0)^N = (|\lambda|^{-1+\delta}) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n), \quad \lambda \in \Omega,$$

and since the operator of finite rank the same estimate is valid for the trace class norm. The same estimate holds for the term involving $\lambda^3 R_0(\lambda)\Pi_0 R_0(\lambda)$ (or an even better estimate if we use Exercise 3.3). We can now deform the contour to $\gamma_0$ and

$$\int_{\gamma_\epsilon} b_2(\lambda)e^{-t\lambda^2}d\lambda = \int_{\gamma_0} b_2(\lambda)e^{-t\lambda^2}d\lambda = \int_{\gamma_0} O(|\lambda|^{-1+\delta})e^{-t \text{Re} \lambda^2}d|\lambda| = O(t^{-\delta/2}) = o(1).$$

6. We continue with the analysis of the term $c(\lambda)$. For that we use (3.9.30) and the resolvent identity to write

$$\lambda A(\lambda) = T_1(\lambda) + T_2(\lambda),$$

$$T_1(\lambda) := \lambda R_0(\lambda) + \lambda R_0(\lambda)V A(\lambda)V R_0(\lambda) - \lambda R_0(\lambda) V R_0(\lambda),$$

$$T_2(\lambda) := -\lambda R_0(\lambda)\Pi_0 - \lambda \Pi_0 R_0(\lambda) + \lambda^3 R_0(\lambda)\Pi_0 R_0(\lambda).$$

We then write

$$c(\lambda) = c_1(\lambda) + c_2(\lambda), \quad c_j(\lambda) := \text{tr} T_j(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N).$$
7. To analyse $c_1(\lambda)$ we use (3.9.5):

\[(3.9.36)\quad c_1(\lambda) = -\sum_{k=1}^{N} \text{tr} T_1(\lambda) R_V(\lambda_0)^{N+1-k} V R_0(\lambda_0)^k.\]

We claim the following: for $s \geq \frac{3}{2}$,

\[(3.9.37)\quad \|\langle x \rangle^{-s} T_1(\lambda) \langle x \rangle^{-s} \|_{L^2 \to L^2} = \mathcal{O}(1), \quad \lambda \in \Omega,\]

and for any $r \in \mathbb{R}$,

\[(3.9.38)\quad s_j(\langle x \rangle^r R_V(\lambda_0)^k \rho), \quad s_j(\rho R_0(\lambda_0)^k \langle x \rangle^r) \leq C j^{-k/n}.

Assuming (3.9.37) and (3.9.38) for now we can estimate $c_1(\lambda)$ as follows. We first note that (3.9.38) give

$$s_j(\langle x \rangle^{-s} R_V(\lambda_0)^{N+1-k} V R_0(\lambda_0)^k \langle x \rangle^s) \leq C j^{-(N+1)/k}.$$ Combining this with (3.9.37) and (3.9.36) give

$$\|T_1(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N)\|_{L^1(\langle x \rangle^s L^2)} \leq C, \quad \lambda \in \gamma_0, \quad s \geq s_0.$$ Using Lemma B.30 we see that

$$c_1(\lambda) = \text{tr}_{\langle x \rangle^s L^2} T_1(\lambda)(R_V(\lambda_0)^N - R_0(\lambda_0)^N) = \mathcal{O}(1), \quad \lambda \in \Omega.$$ Hence we can deform $\gamma_\epsilon$ to $\gamma_0$ and that gives, as in (3.9.33)

\[(3.9.39)\quad \int_{\gamma_\epsilon} c_1(\lambda)e^{-t\lambda^2} d\lambda = \mathcal{O}(t^{-\frac{1}{2}}).\]

6. The term $c_2$ is treated the same as $b_2$ in Step 5. We now put (3.9.39), (3.9.35), (3.9.33), (3.9.32) and (3.9.31) in (3.9.28) to obtain (3.9.25).

It remains to provide the two missing proofs.

7. Proof of (3.9.37). We use Lemma 3.7 so that (writing $\|Q\| := \|Q\|_{L^2 \to L^2}$), so that on $\gamma_0$,

$$\|\langle x \rangle^{-s} T_1(\lambda) \langle x \rangle^{-s} \| \leq |\lambda|(1 + \|\langle x \rangle^{-s} R_0(\lambda)^{N} \langle x \rangle^{-s} \|) (1 + \|\langle x \rangle^{s} V(\lambda)^{s} \|) \leq C|\lambda|(1 + |\lambda|^{-2+s})^2 \leq C, \quad s \geq \frac{3}{2}.$$ 8. Proof of (3.9.38). For $r = 0$ the estimates were proved in Step 3 of the proof of Theorem 3.50. Since $s_j(\lambda) = s_j(A^* A)^{\frac{1}{2}},$

$$s_j(\langle x \rangle^r R_V(\lambda_0)^k \rho)^2 = s_j(\rho R_V(\lambda_0)^k \langle x \rangle^{2r} R_V(-\lambda_0)^k \rho)^2 \leq s_j(\rho R_V(\lambda_0)^k) \|\langle x \rangle^{2r} R_V(-\lambda_0)^k \langle x \rangle^{-2r} \|_{L^2 \to L^2} \leq C j^{-2k/n} \|\langle x \rangle^{2r} R_V(-\lambda_0)^k \langle x \rangle^{-2r} \|_{L^2 \to L^2},$$
where we used the estimate with \( r = 0 \). To see that the norm on the right hand side is finite we first note that if \( M (\lambda_0 = i\sqrt{M}) \) is large enough\(^4\) then 
\[
\| \langle x \rangle^{2r} V R_0(\lambda_0) \rho(x)^{-2r} \|_{L^2 \to L^2} \leq C_r M^{-1} < \frac{1}{2}.
\]
A Neumann series argument then shows that
\[
\| \langle x \rangle^{2r} \left( I + V R_0(\lambda_0) \rho \right)^{-1} \langle x \rangle^{-2r} \|_{L^2 \to L^2} \leq 2,
\]
and using (3.14.1) and (3.2.1) we obtain
\[
\| \langle x \rangle^{2r} R V(\lambda_0) \langle x \rangle^{-2r} \|_{L^2 \to L^2} \leq C.
\]
Returning to (3.9.40) we obtain the first inequality in (3.9.38). The second inequality follows by taking \( V = 0 \).

We are now ready for the somewhat involved and computational proof of (3.9.10) in higher dimensions.

**Proof of Theorem 3.51 for \( n \geq 5 \).** In the proof we first assume that \( f \in C^\infty_c(\mathbb{R} \setminus \{0\}) \) and prove (3.9.10) in that case. Let \( T_V \in \mathcal{S}'(\mathbb{R}) \) denote 
\[
T_V(f) = \text{tr}(f(P_V) - f(P_0))
\]
This means that
\[
T_V|_{\mathbb{R} \setminus \{0\}}(E) = \sum_{E_k < 0} \delta_{E_k}(E) + E_+^0 \partial_E \left[ \sigma(\sqrt{E}) \right],
\]
where
\[
\sigma(\lambda) := \frac{1}{2\pi i} \log \det(I + S(\lambda)), \quad E_+^0 = \begin{cases} 1 & E > 0 \\ 0 & E \leq 0. \end{cases}
\]
with \( \sigma(\lambda) \) defined up to a constant – see Theorem 3.67 for more on that.

Hence
\[
T_V - \sum_{E_k < 0} \delta_{E_k}(E) + E_+^0 \partial_E \left[ \sigma(\sqrt{E}) \right] = \sum_{j=0}^J c_j \delta_{E_j}^j(E).
\]
We test this against \((E + i)^{-N} e^{-iE}\) (see Remark 3 after Theorem 3.51) and apply Lemma 3.52 to see that \( c_0 = m_R(0) \) and \( c_j \equiv 0 \) for \( j > 0 \).

1. We first observe that on the right hand side (3.9.10) we have
\[
\text{tr} S(\lambda)^{-1} \partial_\lambda S(\lambda) = \text{tr} S(\lambda)^* \partial_\lambda S(\lambda) = \text{tr} S_{\text{abs}}(\lambda)^* \partial_\lambda S_{\text{abs}}(\lambda),
\]
and hence we can work with the absolute scattering matrix given by (3.7.5).

2. We define
\[
\tilde{e}(\lambda, \omega, r\theta) = (2\pi)^{-\frac{n-1}{2}} e^{\frac{n}{2}(n-1)i} \lambda^{\frac{n-1}{2}} e(\lambda, \omega, r\theta),
\]
with a similar definition for \( e_0 \). By (3.8.5) \( \lambda \mapsto e(\lambda) \) is holomorphic on \( \mathbb{R} \).

We then rewrite (3.6.20) as follows:
\[
\tilde{e}_0(\lambda, \omega, r\theta) := e^{i\lambda r} a(r, \theta, \omega) + e^{-i\lambda r} \tilde{a}(r, \omega, \theta),
\]
\(^4\)This is the only place that the requirement that \( M \) is large is used; we are not interested in optimality of estimates here since the conclusions of the lemma will be strengthened once Theorem 3.51 is proved.
where we suppressed the dependence on $\lambda$ in $a$ and $\tilde{a}$. The coefficients are distribution valued symbols

$$
\frac{n-1}{2} a, \frac{n-1}{2} \tilde{a} \in S^0_{\text{phg}}((0, \infty)_r, C^\infty(S_{\omega}^{n-1}, \mathcal{D}'(S_{\theta}^{n-1}))),
$$

where the notation means that we have a full asymptotic expansions in $r$,

$$
a(r, \omega, \theta) \sim r^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} r^{-j} a_j(\theta, \omega), \quad a_j \in C^\infty(S_{\omega}^{n-1}, \mathcal{D}'(S_{\theta}^{n-1})),
$$

with error bounds described in (3.6.21) and

$$
a_0(\omega, \theta) = \delta_{-\omega}(\theta).
$$

The difference,

$$
\tilde{e}(\lambda, \omega, r\theta) - \tilde{e}_0(\lambda, \omega, r\theta) = e^{i\lambda r} B(r, \omega, \theta),
$$

is given by

$$
B \in S^{-(n-1)/2}_{\text{phg}}((R_0, \infty)_r, C^\infty(S_{\omega}^{n-1} \times S_{\theta}^{n-1})),
$$

$$
B(r, \omega, \theta) \sim r^{-\frac{n-1}{2}} \sum_{j=0}^{\infty} r^{-j} B_j(\theta, \omega), \quad B_j \in C^\infty(S_{\omega}^{n-1} \times S_{\theta}^{n-1})),
$$

and

(3.9.43) \quad B_0(\omega, \theta) = b(\lambda, \omega, \theta), \quad B_0(\theta, \omega) = B_0(\omega, \theta),

where $b$ appears in (3.7.5), the definition of $S_{\text{abs}}(\lambda)$ and the symmetry (3.7.7) was proved after Theorem 3.43. All of the expansions above are uniform for $\lambda > \epsilon > 0$ which is sufficient as we assume that $f \in C^\infty_c((0, \infty))$.

The condition that $r > R_0$ comes from the fact that $x \mapsto \tilde{e}(\lambda, x, \omega)$ may not be smooth for $x \in \text{supp } V$, if $V$ is not smooth. We only use the above expressions asymptotically so this restriction is not important.

3. In the above notation the Schwartz kernel of the absolute scattering matrix (3.7.5) is given by

(3.9.44) \quad S_{\text{abs}}(\lambda, \theta, \omega) = i^{1-n}(a_0(\theta, \omega) + B_0(\theta, \omega)).

Recalling (3.9.41) and using the symmetry (3.9.43), we have

(3.9.45) \quad \text{tr } S(\lambda)^{-1} \partial_\lambda S(\lambda) = \int_{S^{n-1}} \int_{S^{n-1}} (\tilde{a}_0(\theta, \omega) + \tilde{B}_0(\theta, \omega)) \partial_\lambda B_0(\theta, \omega) d\theta d\omega,

where the integral is meant in the sense of distributional pairing in $\theta$.

4. Using (3.8.13), we rewrite the spectral measure in (3.8.4) as

(3.9.46) \quad dE_\lambda = \tilde{E}_V(\lambda)^* \tilde{E}_V(\lambda) \frac{d\lambda}{2\pi}, \quad \lambda > 0,
where
\[ \tilde{E}_V(\lambda)f(\theta) := \int_{\mathbb{R}^n} \tilde{e}(\lambda, \theta, x)f(x)dx, \text{ } \tilde{E}_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1}). \]

Putting \( B_r := B(0, r) \) We note that (3.8.5) gives
\[ (3.9.47) \quad 1_l B_r \tilde{E}_V(\lambda)^* \tilde{E}_V(\lambda) 1_l B_r \in L^1(L^2(\mathbb{R}^n)). \]

Applying (3.9.46) we get
\[ f(P_V) - \sum_{k=0}^{K} f(E_k)u_k \otimes u_k = \frac{1}{2\pi} \int_0^\infty f(\lambda^2) \tilde{E}_V(\lambda)^* \tilde{E}_V(\lambda)\{d\lambda, \]
which combined with (3.9.3) gives
\[ \text{tr}(f(P_V) - f(P_0)) - \sum_{k=1}^{K} f(E_k) \]
\[ = \lim_{r \to \infty} \text{tr} 1_{B_r}(f(P_V) - f(P_0)) 1_{B_r} - \sum_{k=1}^{K} f(E_k) \]
\[ = \lim_{r \to \infty} \frac{1}{2\pi} \int_0^\infty f(\lambda^2) \text{tr} 1_{B_r}(\tilde{E}_V(\lambda)^* \tilde{E}_V(\lambda) - \tilde{E}_0(\lambda)^* \tilde{E}_0(\lambda)) 1_{B_r} d\lambda. \]

Using (3.9.47) and (B.4.11) (the example at the end of §B.4)
\[ \text{tr} 1_{B_r} \tilde{E}_V(\lambda) \tilde{E}_V(\lambda)^* 1_{B_r} = \int_{B_r} \int_{\mathbb{S}^{n-1}} |\tilde{e}(\lambda, x, \omega)|^2 d\omega dx. \]

We conclude that
\[ (3.9.48) \quad \text{tr}(f(P_V) - f(P_0)) - \sum_{k=1}^{K} f(E_k) \]
\[ = \lim_{r \to \infty} \frac{1}{2\pi} \frac{1}{2\pi} \int_0^\infty f(\lambda^2) d\lambda \int_{\mathbb{S}^{n-1}} d\omega \int_{B_r} dx(|\tilde{e}|^2 - |\tilde{e}_0|^2). \]

(Of course \(|\tilde{e}_0|^2 = \lambda^{n-1}\) but it is useful to keep it as is.)

5. We now use the Maass–Selberg method for converting the integral in \( x \in B_r \) to an integral over \( \partial B_r \). It is based on the following simple identity:
\[ (P_V - \lambda^2)\partial_\lambda \tilde{e} = 2\lambda \tilde{e}, \]
with an analogue valid for \( P_0 \) and \( \tilde{e}_0 \).
Hence, using the fact that \( \lambda \neq 0 \) is real, we use Green’s formula and put \( D = (1/i)\partial \): 

\[
\int_{B_r} |\tilde{e}|^2 dx = \frac{1}{2\lambda} \int_{B_r} (P_V - \lambda^2) \partial_\lambda \tilde{e} \tilde{e} dx \\
= \frac{1}{2\lambda} \int_{B_r} ((P_V - \lambda^2) \partial_\lambda \tilde{e}\tilde{e} - \partial_\lambda \tilde{e}(P_V - \lambda^2) \tilde{e}) dx \\
= \frac{1}{2\lambda} \int_{B_r} (-\Delta \partial_\lambda \tilde{e}\tilde{e} + \partial_\lambda \partial_\lambda \tilde{e}) dx \\
= \frac{1}{2\lambda} \int_{S^{n-1}} \left( (D_r \lambda \tilde{e}) \partial_\lambda \tilde{e} + D_\lambda \tilde{e} D_r \tilde{e} \right) r^{n-1} d\theta, \ x = r\theta. 
\]

(3.9.49)

6. Before inserting (3.9.49) into (3.9.48) we make three observations:

A) if we write \( \tilde{e} = \tilde{e}_0 + \tilde{w} \) then, applying (3.9.49) in (3.9.48) shows that no terms that are quadratic in \( \tilde{e}_0 \) remain;

B) since \( \tilde{w}(r\theta, \omega) \) is smooth in \( \theta \), products of terms derived from \( \tilde{w} \) and \( \tilde{e}_0 \) can be expressed using the expansions in Step 4, with the integrals of products understood as distributional pairings;

C) all terms with factors of \( e^{\pm 2i\lambda r} \) vanish in the \( r \to \infty \) limit due to the integration against \( f(\lambda^2) \in C^\infty_c(\mathbb{R}_\lambda) \); this means the only distributional pairings come from terms involving \( a \) and \( B \).

7. Noting that

\[
D_r e^{i\lambda r} = e^{i\lambda r} (D_r + \lambda), \quad D_\lambda e^{i\lambda r} = e^{i\lambda r} (D_\lambda + r),
\]

the observations made in Step 6 and the expansions in Step 4 show that

\[
\text{tr} (f(P_V) - f(P_0)) - \sum_{k=1}^{K} f(E_k) \\
= \lim_{r \to \infty} \int_{\mathbb{R}} f(\lambda^2) \frac{d\lambda}{4\pi\lambda} \int_{S^{n-1}} d\omega \int_{S^{n-1}} d\theta \ r^{n-1} C(r),
\]

(3.9.50)

where

\[
C(r) = C(r, \theta, \omega) \in S_{\text{phg}}^{-n-2}((0, \infty)_r; \mathcal{D}'(S^{n-1}_0 \times S^{n-1}_\omega)),
\]

is given by

\[
C(r) := (D_r + \lambda)(D_\lambda + r)aB + (D_r + \lambda)(D_\lambda + r)B\bar{a} + (D_r + \lambda)(D_\lambda + r)B\bar{B} + (D_\lambda + r)a\bar{D} + (D_\lambda + r)a\bar{B}B + (D_\lambda + r)B\bar{a} + (D_\lambda + r)B\bar{B}. 
\]

We note that differentiation in \( r \) decreases the order in \( r \) and differentiation in \( \lambda \) preserves it. We now group terms according to their order in \( r \): all
terms of size $O(r^{-n})$ will disappear in the limit (remember the $r^{n-1}$ factor in (3.9.50)). Hence,

$$C(r) = 2\lambda r a \bar{B} + 2\lambda r B \bar{a} + 2\lambda r |B|^2 + D_r(ra) \bar{B} + 2\lambda D_\lambda a \bar{B} + D_r(rB) \bar{B} + \lambda D_\lambda B \bar{B} + arD_r \bar{B} + BrD_r a + \lambda D_\lambda B \bar{B} + BrD_r B + O(r^{-n})$$

$$= 2 \text{Re} (rD_r a \bar{B} + rD_r B \bar{B} + rD_r B \bar{a} + \lambda D_\lambda a r \bar{D}_r B) + 2r \lambda \text{Re} (2a \bar{B} + |B|^2) - i \text{Re} (2a \bar{B} + |B|^2) + 2\lambda D_\lambda a \bar{B} + 2\lambda D_\lambda \bar{a} B \bar{B} + O(r^{-n}),$$

where $O(r^{-n})$ is meant in the sense of distributional expansion in Step 4.

8. The coefficient of $r^{-n+2}$ in the expansion of $C(r)$ is given by

$$2 \lambda \text{Re} (2a_0 (\theta, \omega) B_0 (\theta, \omega) + |B_0 (\theta, \omega)|^2).$$

We claim that the integral of this term with respect to $\omega$ (or $\theta$) is equal to 0. In fact, unitarity of the scattering matrix $S_{\text{abs}}(\lambda)$ and symmetry of $b$ imply that

$$\int_{\mathbb{S}^{n-1}} (\delta_\omega (\theta) + b(\theta, \omega)) (\delta_\gamma (\omega) + \bar{b}(\bar{\omega}, \bar{\gamma})) d\omega = \delta_\gamma (\theta),$$

which means that

$$(3.9.51) \quad b(\theta, -\gamma) + \bar{b}(\gamma, -\theta) + \int_{\mathbb{S}^{n-1}} b(\theta, \omega) \bar{b}(\gamma, \bar{\omega}) d\omega \equiv 0.$$ 

On the other hand, putting $\gamma = \theta$ in (3.9.51),

$$\int_{\mathbb{S}^{n-1}} (2 \text{Re} (a_0 (\theta, \omega) \bar{B}_0 (\theta, \omega)) + |B_0 (\theta, \omega)|^2) d\omega$$

$$= 2 \text{Re} b(\theta, -\theta) + \int_{\mathbb{S}^{n-1}} |b(\theta, \omega)|^2 d\omega \equiv 0.$$ 

9. To compute the coefficient of $r^{-n+1}$ we note that $D_\lambda a_0 = 0$ and that one of the terms vanishes due to the unitarity of $S_{\text{abs}}$. Another satisfies

$$2 \text{Re} (rD_r a \bar{B} + rD_r B \bar{B} + rD_r B \bar{a} + \lambda D_\lambda a r \bar{D}_r B)$$

$$= 2 \text{Re} \left( \frac{n-1}{2} i (a_0 B_1 + \bar{a}_0 B_1 + |B_0|^2) \right) r^{-n+1} + O(r^{-n}) = O(r^{-n}).$$

Hence we are left with

$$4 \lambda \text{Re} (a_0 \bar{B}_1 + a_1 \bar{B}_0 + B_1 \bar{B}_0) + 2\lambda \partial_\lambda B_0 (\bar{a}_0 + \bar{B}_0).$$

In view of (3.9.45) and (3.9.50) the second term is exactly what appears in (3.9.10). Hence it remains to show that the first vanishes. For that we use (3.7.12) to see that

$$a_1 = \frac{1}{2 i \lambda} \left( -\Delta_{\mathbb{S}^{n-1}} + \frac{(n-1)(n-3)}{4} \right) a_0, \quad B_1 = \frac{1}{2 i \lambda} \left( -\Delta_{\mathbb{S}^{n-1}} + \frac{(n-1)(n-3)}{4} \right) B_0.$$
This means that (in the sense of distributional pairing),

\[ 4\lambda \text{Re} \int_{S^{n-1}} (a_0 \bar{B}_1 + a_1 \bar{B}_0 + B_0 \bar{B}_1) d\theta \]

\[ = 2 \text{Re} i \int_{S^{n-1}} (-a_0 \Delta_{S^{n-1}} \bar{B}_0 + \Delta_{S^{n-1}} a_0 \bar{B}_0 + \Delta_{S^{n-1}} B_0 \bar{B}_0 - b_n |B_0|^2) d\theta \]

\[ = 2 \text{Re} i \int_{S^{n-1}} (-\Delta_{S^{n-1}} a_0 \bar{B}_0 + \Delta_{S^{n-1}} a_0 \bar{B}_0 - |\nabla B_0|^2) d\theta \]

\[ = -2 \text{Re} i \int_{S^{n-1}} |\nabla B_0|^2 d\theta = 0. \]

And this completes the proof. □

3.10. THE MELROSE TRACE FORMULA

The next theorem is the odd dimensional generalization of Theorem 2.19 and it connects resonances with the trace of the wave group. We first observe that for we can define the distribution

\[ \varphi \mapsto \sum_{\lambda \in \mathbb{C}} m_R(\lambda) \int_{\mathbb{R}} t^n \varphi(t) e^{-i|\lambda|t} dt, \quad \varphi \in C_0^\infty(\mathbb{R}). \]

To see this, suppose that supp \( \varphi \subset [-R, R] \). With \((n + 1)\) integrations by parts based on \( (i/\lambda) \partial_t e^{-i\lambda t} = e^{-i\lambda t} \) we see

\[ \left| \int_0^\infty t^n \varphi(\pm t) e^{-i\lambda t} dt \right| = CR(1 + |\lambda|)^{-(n+1)} e^{R(\text{Im} \lambda)_+} \sup_{0 \leq k \leq n+1} |\varphi^{(k)}|. \]

Let

(3.10.1) \[ N(r) := \sum \{m_R(\lambda) : 0 < |\lambda| \leq r\}, \]

so that by Theorem 3.27 we have \( N(r) \leq C_V r^n \). Since there are only finitely many \( \lambda \)'s with \( \text{Im} \lambda > 0 \),

\[ \left| \sum_{\lambda \in \mathbb{C}} m_R(\lambda) \int_0^\infty t^n (\varphi(t) + \varphi(-t)) e^{-i\lambda t} dt \right| \leq C \sup_{0 \leq k \leq n+1} |\varphi^{(k)}| \sum_{\lambda \neq 0} m_R(\lambda) |\lambda|^{-n-1} \]

\[ = C \sup_{0 \leq k \leq n+1} |\varphi^{(k)}| \left( C + \int_0^\infty r^{-n-1} dN(r) \right) \]

\[ \leq C \sup_{0 \leq k \leq n+1} |\varphi^{(k)}| \left( C + C \int_0^\infty r^{-n-2} dr \right) \leq C \sup_{0 \leq k \leq N} |\varphi^{(k)}|, \]

where the constants depend on \( R \) and \( V \).
3.10. THE MELROSE TRACE FORMULA

We now present a trace formula in which resonances appear in an almost the same way as eigenvalues:

**THEOREM 3.53 (Trace formula for resonances).** Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, where $n \geq 3$ is odd. Then,

$$2^n \text{tr} \left( \cos t \sqrt{P} - \cos t \sqrt{P_0} \right) = t^n \sum_{\lambda \in \mathbb{C}} m_T(\lambda) e^{-i\lambda |t|},$$

(3.10.2)
in the sense of distributions on $\mathbb{R}$ and where

$$m_T(\lambda) := \begin{cases} m_R(\lambda) & \lambda \neq 0 \\ 2m_\mathcal{R}(0) - \tilde{m}_\mathcal{R}(0) & \lambda = 0. \end{cases}$$

(3.10.3)

**REMARKS.**

1. As explained after Theorem 2.19 in the case of $-\Delta + V$ on a bounded domain in $\mathbb{R}^n$ (with, say, Dirichlet boundary conditions) this result is an immediate consequence of the spectral theorem for self-adjoint operators with discrete spectra. It is quite remarkable that the same theorem holds (in odd dimensions, and for compactly supported perturbations) in an exactly the same form for resonances. The formula remains valid for any “black box” (see §4.1) compactly supported perturbations in odd dimensions [SZ94], [Zw97].

2. A power of $t$ in (3.10.2) is needed as there are many possible extensions of the distribution $\sum_{\lambda \in \mathbb{C}} m_R(\lambda) \exp(-i\lambda |t|)$ from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$.

3. The trace formula (3.10.2) is a consequence of the Birman–Kreĭn formula and of the Hadamard factorization the scattering determinant, $\det S(\lambda)$, as a meromorphic function – see Theorem 3.54 below. That was not the original proof – see §3.13 – but in applications it is sometimes easier to use the factorization of $\det S(\lambda)$ directly; see §3.12.

**THEOREM 3.54 (Factorization of the scattering matrix).** Suppose that $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$ where $n \geq 1$ is odd. Then

$$\det S(\lambda) = (-1)^m e^{g(\lambda)} \frac{P(-\lambda)}{P(\lambda)},$$

(3.10.4)

$$P(\lambda) := \prod E_n(\lambda/\mu)^{m_R(\mu)}, \quad E_n(z) := (1 - z)e^{z + z^2 + \cdots + z^n/n},$$

$$g(\lambda) = a_n \lambda^n + a_{n-2} \lambda^{n-2} + \cdots + a_1 \lambda$$

When $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$ then

$$P(-\lambda) = \overline{P(\lambda)}, \quad a_n \in i\mathbb{R}, \quad m = \bar{m}_R(0),$$

(3.10.5)

where $\bar{m}_R(0)$ is defined by (3.3.17).
Proof. 1. Theorem 3.45 already shows that (3.10.4) holds with $g$ given an entire function. From (3.7.17) we see that $g$ has to be odd. Hence all we need to show is that $g$ is a polynomial of degree at most $n$.

2. We will first establish two preliminary bounds

$$|\det S(\lambda)| \leq \begin{cases} C \exp C|\lambda|^n, & \text{Im } \lambda \geq 0, |\lambda| > C, \\ C \exp C|\lambda|^{n^2}, & \lambda \notin \bigcup_{m_H(\mu) > 0} D(\mu, \langle \mu \rangle^{-n-\epsilon}), \end{cases}$$

where $m_H(R)$ was defined before (3.4.5). In view of (3.7.30) this will follow from estimates on the characteristic values of the operator $T(\lambda)$ appearing there, in the spirit of the proof of Theorem 3.27.

3. To apply estimates on characteristic values we use (3.4.10) to write

$$T(\lambda) = c_n (I + VR_0(\lambda)\rho)^{-1} V E_\rho(\bar{\lambda})^* E_\rho(\lambda).$$

From (3.1.12) we see that

$$\| (I + VR_0(\lambda)\rho)^{-1} \| \leq C, \quad \text{Im } \lambda \geq 0, \quad |\lambda| \geq C.$$  

Hence in the same range of $\lambda$’s we have

$$s_j(T(\lambda)) \leq C|\lambda|^{n-2} \|E_\rho(\bar{\lambda})^* s_j(E_\rho(\lambda))\|.$$  

Applying (4.3.39) we obtain (with a different constant $C$)

$$s_j(T(\lambda)) \leq C \exp \left( C|\lambda| - j^{1/n-1}/C \right), \quad \text{Im } \lambda \geq 0, \quad |\lambda| \geq C.$$  

The Weyl inequality can now be applied as in part 5 of the proof of Theorem 3.27 and that gives

$$|\det(I - T(\lambda))| \leq \prod_{j=0}^{\infty} (1 + s_j(T(\lambda)))$$

$$\leq \prod_{j \leq C|\lambda|^{n-1}} (1 + e^{C|\lambda|}) \exp \sum_{j \geq 1} e^{-j^{1/n-1}/C}$$

$$\leq C \exp C|\lambda|^n, \quad \text{Im } \lambda \geq 0, \quad |\lambda| \geq C.$$  

This proves the first part of (3.10.6).

4. We now consider the case of $\lambda$ outside of a union of discs containing resonances. First we note that for any $\epsilon > 0$, there exists a sequence $r_k \to \infty$, such that

$$\forall k, \quad \partial D(0, r_k) \cap \bigcup_{m_H(\mu) > 0} D(\mu, \langle \mu \rangle^{-n-\epsilon}) = \emptyset,$$

which follows from Theorem 3.28 as it implies that

$$\sum_{\mu \in \mathbb{C}} m_H(\mu)\langle \mu \rangle^{-n-\epsilon} < \infty.$$
3.10. THE MELROSE TRACE FORMULA

(This estimate shows that the sum of radii of the discs in (3.10.7) is finite; if \( \partial D(0, r) \) intersected at least one of the discs for all \( r > r_0 \), the sum of radii would have to be infinite.)

To estimate \( \| (I + VR_0(\lambda)\rho)^{-1} \| \) away from resonances we use (B.5.18) with \( p = n + 1 \):

\[
\| (I + VR_0(\lambda)\rho)^{-1} \| \leq \frac{G(\lambda)}{|H(\lambda)|},
\]

where

\[
G(\lambda) := \prod_{j=0}^{\infty} (1 + s_j(\lambda)^{n+1}), \quad H(\lambda) := \det(I - (VR_0(\lambda)\rho)^{n+1}).
\]

Theorem 3.27 shows that \( H(\lambda) \) is an entire function of order \( n \), and its proof shows that

\[
G(\lambda) \leq C \exp(C|\lambda|^n).
\]

The minimum modulus theorem for entire functions of order \( n \) (see (D.2.6)) shows that

\[
|H(\lambda)| \geq \exp(-C|\lambda|^n + \epsilon), \quad \lambda \notin \bigcup_{m_H(\mu) > 0} D(\mu, \langle \mu \rangle - n - \epsilon).
\]

Hence for \( \lambda \)'s in the same set we obtain

\[
\| (I + VR_0(\lambda)\rho)^{-1} \| \leq C \exp(C|\lambda|^n + \epsilon).
\]

Returning to singular values of \( T(\lambda) \) this gives

\[
s_j(T(\lambda)) \leq C \exp\left(C|\lambda|^n + \epsilon - \frac{j^{n-1}}{n-1}/C\right), \quad \lambda \notin \bigcup_{m_H(\mu) > 0} D(\mu, \langle \mu \rangle - n - \epsilon).
\]

The same argument as before proves the second part of (3.10.6).

5. We now recall the estimates on Weierstrass products (see (D.2.5) and (D.2.6) in §D.2):

\[
e^{-C|\lambda|^n + \epsilon} \leq |P(\pm \lambda)| \leq e^{C|\lambda|^n + \epsilon}, \quad \pm \lambda \notin \bigcup_{m_H(\mu) > 0} D(\mu, \langle \mu \rangle - n - \epsilon).
\]

Hence in the same set of \( \lambda \)'s we have

\[
|\exp(g(\lambda))| = |\det S(\lambda)| \frac{|P(\lambda)|}{|P(-\lambda)|} \\
\leq C \exp(C|\lambda|^n + C|\lambda|^n + \epsilon) \\
\leq C \exp C|\lambda|^n.
\]

We can now use this on circles of radius \( r_k \) satisfying (3.10.7). The maximum principle then shows that the above estimate holds everywhere. Hence,

\[
\text{Re} g(\lambda) \leq C|\lambda|^n,
\]
and an application of the Borel-Carathéodory theorem (D.1.7) gives

$$|g(\lambda)| \leq C|\lambda|^{n^2},$$

which implies that $g$ is a polynomial.

6. To see that $g(\lambda)$ is a polynomial of degree $n$ we apply the same strategy as in the proof of (3.10.8) but for $\text{Im} \lambda \geq 0, |\lambda| \geq C$. This way we can use the first estimate in (3.10.6). That gives

$$\text{Re} \, g(\lambda) \leq C_\epsilon |\lambda|^{n+\epsilon}, \quad \text{Im} \lambda \geq 0, \quad |\lambda| \geq C.$$ 

For $n \geq 2$ any polynomial satisfying this bound has to have degree at most $n$.

7. The statement (3.10.5) about $P(\lambda)$ and the polynomial $g$ when $V$ is real valued comes from the unitarity of the scattering matrix. For the factor $(-1)^m$ see (3.7.28) and Exercise 3.9. □

Before proving Theorem 3.53 we need a bound on $\log \det S(\lambda)$ for $\lambda \in \mathbb{R}$. A much more precise result will be presented in Theorem 3.11.41 but the point is that the lemma depends only on the upper bound on the counting function in Theorem 3.27 and the factorization of the scattering matrix in Theorem 3.54. Hence, it can be used in more general situations.

**Lemma 3.55.** Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$ and $n \geq 1$ is odd. Then for any $\epsilon > 0$ there exists $C_\epsilon$ such that

$$|\log \det S(\lambda)| \leq C_\epsilon \langle \lambda \rangle^{n+\epsilon}, \quad \lambda \in \mathbb{R}. \tag{3.10.9}$$

**Proof.** 1. Let

$$s(\lambda) := -i \log \det S(\lambda), \quad \lambda \in \mathbb{R}, \quad s(0) = 0.$$ 

(We do not divide by $2\pi$ in order to simplify the subsequent formulas.) In view of (3.10.5),

$$\det S(\lambda) = (-1)^m e^{g(\lambda)} \frac{P(\bar{\lambda})}{P(\lambda)}.$$ 

Since for $\lambda \in \mathbb{R}$ and $\mu \neq 0$,

$$\partial_\lambda \left( \log \left( 1 - \frac{\lambda}{\mu} \right) - \log \left( 1 - \frac{\lambda}{\bar{\mu}} \right) \right) = -\frac{2i \text{Im} \mu}{|\lambda - \bar{\mu}|^2},$$

we have

$$\partial_\lambda^{n+1} s(\lambda) = -2 \sum_{\mu \neq 0} m_R(\mu) \partial_\lambda^n \frac{\text{Im} \mu}{|\lambda - \mu|^2}, \tag{3.10.10}$$
where the sum converges uniformly for \( \lambda \) in compact sets. That is guaranteed by Theorem 3.27 as, with \( N(r) \) defined by (3.10.1),

\[
\sum_{\mu \neq 0} m_R(\mu) |\mu|^{-n-1} = C + \int_0^\infty r^{-n-1}dN(r) < \infty.
\]

2. Choose \( \zeta \in C_c^\infty((-3, 3); [0, 1]) \) equal to 1 on \([-2, 2]\) and define \( s_1(\lambda) \) by

\[
s'_1(\lambda) := -2 \sum_{\mu \neq 0} m_R(\mu) \zeta \left( \frac{|\mu|}{\lambda} \right) \frac{\text{Im} \frac{\mu}{\lambda}}{|\lambda - \mu|^2}, \quad s_1(0) = 0.
\]

The upper bound on the number of resonances (3.4.6) and the fact that

\[
\int_{\mathbb{R}} \frac{\text{Im} \lambda}{|\lambda - \mu|^2} d\lambda = \pi,
\]

show

\[
|s_1(\lambda)| \leq C(\lambda)^n.
\]

3. We also define

\[
s_2(\lambda) := s(\lambda) - s_1(\lambda),
\]

and use (3.10.10) and (3.10.12) to obtain

\[
\partial^{n+1}_\lambda s_2(\lambda) = -2 \sum_{\mu \neq 0} m_R(\mu) \partial^n_\lambda \left[ \left( 1 - \zeta \left( \frac{|\mu|}{\lambda} \right) \right) \frac{\text{Im} \mu}{|\lambda - \mu|^2} \right]
\]

where

\[
A := -2 \sum_{\mu \neq 0} m_R(\mu) \left( 1 - \zeta \left( \frac{|\mu|}{\lambda} \right) \right) \partial^n_\lambda \frac{\text{Im} \mu}{|\lambda - \mu|^2}
\]

and

\[
B := 2 \sum_{\mu \neq 0} m_R(\mu) \sum_{k=1}^n \binom{n}{k} \partial^k_\lambda \zeta \left( \frac{|\mu|}{\lambda} \right) \partial^{n-k}_\lambda \left( \frac{\text{Im} \mu}{|\lambda - \mu|^2} \right).
\]

The sum in the definition of \( B \) is finite as the support of \( \zeta' \) restricts \( \mu \)'s to \( 2|\lambda| \leq |\mu| \leq 3|\lambda| \). Moreover, \( \partial^k_\lambda \zeta \left( |\mu|/\lambda \right) = O(|\lambda|^{-k}) \) and

\[
\partial^{n-k}_\lambda \left( \frac{i \text{Im} \mu}{|\lambda - \mu|^2} \right) = \partial^{n-k}_\lambda \left( (\lambda - \bar{\mu})^{-1} - (\lambda - \bar{\mu})^{-1} \right)
\]

\[
= O(|\lambda - \mu|^{-n-k-1})
\]

\[
= O(|\mu|^{-n-k-1}), \quad |\mu| > 2|\lambda|.
\]

Hence, using (3.4.6),

\[
|B| \leq C \sum_{2|\lambda| \leq |\mu| \leq 3|\lambda|} m_R(\mu) |\lambda|^{-k} |\mu|^{-n+k-1} = O(|\lambda|^{-1}).
\]
To estimate $A$ we also use (3.4.6):

$$|A| \leq C \sum_{|\mu| \geq 2|\lambda|} m_R(\mu)|\mu|^{-n-1} = C \int_{2|\lambda|}^{\infty} r^{-n-1}dN(r) = \mathcal{O}(|\lambda|^{-1}).$$

We conclude that

$$|\partial^{n+1}_\lambda s_2(\lambda)| = \mathcal{O}(\langle \lambda \rangle^{-1}) \implies s_2(\lambda) = \mathcal{O}(\langle \lambda \rangle^{n+\epsilon}).$$

Combined with (3.10.13) this concludes the proof. □

Proof of Theorem 3.53. 1. Let $\varphi := t^k \psi$, $\psi \in C_0^\infty(\mathbb{R})$. In the distributional sense,

$$(2 \bullet^k \cos \bullet \sqrt{P_V})(\psi) = f(P_V), \quad f(z) := \hat{\varphi}(\sqrt{z}) + \hat{\varphi}(-\sqrt{z}),$$

where $f \in C^\infty(\mathbb{R}) \cap \mathcal{S}(0, \infty)$. It follows that (3.10.2) is equivalent to

$$\text{tr} (f(P_V) - f(P_0)) = \sum_{\lambda \in \mathbb{C}} m_T(\lambda) \int_0^\infty (\varphi(t) + \varphi(-t))e^{-i\lambda t}dt,$$

where $m_T$ is defined in (3.10.3).

Writing $\sigma(\lambda) := \log \det S(\lambda)/2\pi i$, $\sigma(0) = 0$, Theorem 3.51 shows that (since $\sigma'(\lambda)$ is even)

$$\text{tr} (f(P_V) - f(P_0)) = \frac{1}{2} \int_{-\infty}^\infty f(\lambda^2)\sigma'(\lambda)d\lambda + \sum_{k=1}^K f(E_k) + \frac{1}{2} \tilde{m}_R(0),$$

where the integral is understood as a pairing of $f(\lambda^2) \in \mathcal{S}(\mathbb{R}_\lambda)$ and $\sigma' \in \mathcal{S}'(\mathbb{R})$. We claim that

$$\int_{-\infty}^{\infty} f(\lambda^2)\sigma'(\lambda)d\lambda = \lim_{r \to \infty} \int_{-r}^{r} f(\lambda^2)\sigma'(\lambda)d\lambda.$$

In fact, using (3.10.9) we have

$$\int_{\mathbb{R}} f(\lambda^2)\sigma'(\lambda)d\lambda = - \int_{\mathbb{R}} \sigma(\lambda) f'(\lambda^2)2\lambda d\lambda = - \lim_{r \to \infty} \int_{-r}^{r} \sigma(\lambda) f'(\lambda^2)2\lambda d\lambda$$

$$= \lim_{r \to \infty} \left( \int_{-r}^{r} f(\lambda^2)\sigma'(\lambda)d\lambda + 2f(r^2)\sigma(r) \right)$$

$$= \lim_{r \to \infty} \int_{-r}^{r} f(\lambda^2)\sigma'(\lambda)d\lambda.$$
This also shows that we can take the limit along any sequence \( r_j \to \infty \).
Hence the proof of the theorem is reduced to showing that for some \( r_j \to \infty \),
\[
\frac{1}{2\pi i} \lim_{j \to \infty} \int_{-r_j}^{r_j} \hat{\phi}(\lambda) \partial_\lambda (\log \det S(\lambda))d\lambda + (2m_R(0) - \tilde{m}_R(0)) \hat{\phi}(0) + \sum_{\text{Im } \mu > 0} m_R(\mu) (\hat{\phi}(\mu) + \hat{\phi}(-\mu))
\]
\[
= \sum_{\mu \in \mathbb{C}} m_T(\mu) \int_0^\infty (\varphi(t) + \varphi(-t)) e^{-i\mu t} dt.
\]
\[(3.10.16)\]

2. We define \( \varphi_\pm(t) := t^n \psi(t) \in C_k^k(\mathbb{R}) \), so that \( \varphi = \varphi_+ + (-1)^n \varphi_- \).
Integration by parts shows that
\[(3.10.17)\]
\[|\partial^\ell_\lambda \hat{\varphi}_\pm(\lambda)| \leq C (1 + |\lambda|)^{-n-1-\ell} e^{C(\text{Im } \lambda)_\pm}.
\]
It is enough to prove the theorem for \( \varphi \) replaced by \( \varphi_\pm \) and we will present the + case, the − case being analogous.

3. In the notation of (3.10.4),
\[
\partial_z E_n(z) = -(1 - z)^{-1} + 1 + \cdots + z^{n-1} = -z^n (1 - z)^{-1}.
\]
Hence the factorization in (3.10.4) gives
\[
\partial_\lambda (\log \det S(\lambda)) = g'(\lambda) + \sum_{\mu \neq 0} m_R(\mu) \partial_\lambda (\log E_n(-\lambda/\mu) - \log E_n(\lambda/\mu))
\]
\[
= g'(\lambda) + \sum_{\mu \neq 0} m_R(\mu) \left(\frac{\lambda}{\mu}\right)^n ((\lambda + \mu)^{-1} - (\lambda - \mu)^{-1}).
\]
Since \( m_R(\mu) = 0 \) for \( \mu \in \mathbb{R} \setminus \{0\} \) the sum converges uniformly in \( \lambda \in [-r, r] \).
Also, if \( \chi_+(t) := i^n t^n_+ \psi \) then \( \hat{\varphi}_\pm(\lambda) = \partial^n_\lambda \hat{\psi}_+ \) and
\[
\lim_{r \to \infty} \int_{-r}^{r} \hat{\varphi}_+(\lambda) g'(\lambda)d\lambda = \lim_{r \to \infty} \int_{-r}^{r} \partial^n_\lambda \hat{\chi}_+(\lambda) g'(\lambda)d\lambda = 0.
\]
(The polynomial \( g' \) has degree at most \( n-1 \) and \( \partial^n_\lambda \hat{\chi}_+(\lambda) = O((\lambda)^{-1-\ell}) \) for \( 0 \leq \ell \leq n \).) Hence,
\[(3.10.18)\]
\[
\int_{-r}^{r} \hat{\varphi}_+(\lambda) \partial_\lambda (\log \det S(\lambda))d\lambda = \sum_{\mu \neq 0} m_R(\mu) \Phi(\mu, r),
\]
where
\[
\Phi(\mu, r) := \int_{-r}^{r} \hat{\varphi}_+(\lambda) ((\lambda + \mu)^{-1} - (\lambda - \mu)^{-1})d\lambda.
\]
4. We now claim that for any $\epsilon > 0$ there exists a sequence $r_j \to \infty$ such when $m_R(\mu) > 0$

\[
(3.10.19) \quad \frac{1}{2\pi i} \Phi(\mu, r_j) = \begin{cases} 
\hat{\varphi}_+(\mu) + O(|\mu|^{-n-\frac{3}{2}}r_j^{-\frac{1}{2}}) & \text{Im } \mu < 0, \\
-\hat{\varphi}_+(\mu) + O(r_j^{-\epsilon}) & \text{Im } \mu > 0.
\end{cases}
\]

Let

$$
\Gamma_r = \partial D(0, r) \cap \{\text{Im } \lambda \leq 0\},
$$

be oriented counterclockwise. If for $\lambda \in \Gamma_r$ we have $m_R(\pm \lambda) = 0$, then the residue theorem shows that

$$
\Phi(\mu, r) = 2\pi i \text{sgn}(\text{Im } \lambda) \hat{\varphi}_+(\mu)
$$

$$
+ \int_{\Gamma_r} \hat{\varphi}_+(\lambda) \frac{(\lambda/\mu)^n}{(\lambda + \mu)^{-1} - (\lambda - \mu)^{-1}} d\lambda,
$$

and we need to estimate the last term.

We first note that if $|\mu| < r/2$ or $|\mu| > r/2$ then

\[
(3.10.20) \quad \int_{\Gamma_r} \hat{\varphi}_+(\lambda) \frac{(\lambda/\mu)^n}{(\lambda + \mu)^{-1} - (\lambda - \mu)^{-1}} d\lambda = O(|\mu|^{-n-\frac{3}{2}}r^{-\frac{1}{2}}).
\]

In fact, using (3.10.17) we see that for $\lambda \in \Gamma_r$

$$
\varphi_+(\lambda) \frac{(\lambda/\mu)^n}{(\lambda + \mu)^{-1} - (\lambda - \mu)^{-1}} = O(|\mu|^{-n}r^{-n}) \min(|\mu|^{-1}, r^{-1}) O(|\mu|^{-\frac{3}{2}}r^{-n-\frac{1}{2}}).
$$

Since the length of $\Gamma_r$ is $\pi r$, (3.10.20) follows.

5. For $r/2 < \mu < 2r$ we will use the following

**LEMMA 3.56.** Suppose that $h$ is holomorphic in a neighbourhood of $\Gamma_r$ and $\mu \notin \Gamma_r$, $r > 1$. Then

\[
(3.10.21) \quad \int_{\Gamma_r} h(\lambda) (\lambda - \mu)^{-1} d\lambda = O(\langle \log d(\mu, \Gamma_r) \rangle + \log r) \max_{\Gamma_r} (|h| + r|h'|),
\]

where $d(\mu, \Gamma_r) = \min_{\lambda \in \Gamma_r} |\lambda - \mu|$.

**Proof.** We define $\log(\lambda - \mu)$ for $\lambda \in \mathbb{C} \setminus (\mu + i[0, \infty))$ if $|\mu| < r$ or $\text{Im } \mu > 0$ and for $\lambda \in \mathbb{C} \setminus (\mu - i[0, \infty))$ otherwise. In particular, $\log(\lambda - \mu)$ is well and holomorphic on a neighbourhood of $\Gamma_r$. Then,

\[
\int_{\Gamma_r} h(\lambda) (\lambda - \mu)^{-1} d\lambda = \int_{\Gamma_r} h(\lambda) \partial_\lambda \log(\lambda - \mu) d\lambda
\]

\[
= h(r) \log(r - \mu) - h(-r) \log(-r - \mu) - \int_{\Gamma_r} h'(\lambda) \log(\lambda - \mu) d\lambda
\]

\[
= O(|\log |r - \mu|| + |\log |r + \mu|| + 4\pi) \max_{\Gamma_r} (|h|)
\]

\[
+ O(\max_{\lambda \in \Gamma_r} |\log(\lambda - \mu)|) r \max_{r \in \Gamma_r} |h'|.
\]
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Since
\[
\max_{\lambda \in \Gamma_r} |\log(\lambda - \mu)| \leq 2\pi + |\log d(\mu, \Gamma_r)| + \log(2r)
\]
and
\[
|\log |\mu \pm r|| \leq |\log d(\mu, \Gamma_r)| + \log(2r),
\]
(3.10.21) follows. \qed

We now choose a sequence \( r_j \to \infty \) so that
\[
\forall j, \quad \Gamma_{r_j} \cap \bigcup_{\mu \in \Omega} D(\mu, \langle \mu \rangle - n - 1) = \emptyset.
\]
As in the case of (3.10.7) this follows from (3.4.6).

We now apply Lemma (3.56) with
\[
(3.10.22) \quad h(\lambda) := \hat{\varphi}_+ (\lambda/\mu)^n, \quad \max_{\Gamma_r} (|h| + |h'|) = O(|\mu|^{-n-1}).
\]
where to get the estimate we used (3.10.17). We have
\[
m_R(\pm \mu) > 0 \implies d(\mu, \Gamma_{r_j}) > |\mu|^{-n-1},
\]
and (3.10.21) gives, for \( r_j/2 \leq |\mu| \leq 2r_j \),
\[
\int_{\Gamma_{r_j}} \hat{\varphi}_+ (\lambda/\mu)^n (\lambda \pm \mu)^{-1} d\lambda = O(|\mu|^{-n} \log |\mu| r_j^{-1})
\]
\[
= O(|\mu|^{-n} \log |\mu| r_j^{-\frac{1}{2}})
\]
\[
= O(|\mu|^{-n} r_j^{-\frac{1}{2}}).
\]
Combining this with (3.10.20) gives (3.10.19).

6. Returning to (3.10.18) we see that (3.10.19) and (3.10.11) give
\[
\frac{1}{2\pi i} \int_{-r_j}^{r_j} \hat{\varphi}_+ (\lambda) d\lambda (\log \det S(\lambda)) d\lambda =
\sum_{\text{Im} \mu < 0} m_R(\mu) \hat{\varphi}_+ (\mu) - \sum_{\text{Im} \mu > 0} m_R(\mu) \hat{\varphi}_+ (\mu) + O(r_j^{-\frac{1}{2}}).
\]
This proves (3.10.16) which, as explained in Step 1 gives (3.10.2). \qed

**Remark.** The proof of Theorem 3.53 relies only on the upper bound on the counting function for resonances and the factorization of \( \det S(\lambda) \) in Theorem 3.54. We did not use any specific results about the distribution of resonances.
3.11. SCATTERING ASYMPTOTICS

Our next result about the scattering matrix concern asymptotics of the scattering phase, \( \log \det S(\lambda)/2\pi \). As explained after Theorem 2.17 the scattering phase, also known as the scattering winding number, is the analogue of the counting function for eigenvalues of a Schrödinger operator on a bounded domain. It is of intrinsic interest but it will also play an important role in establishing existence of infinitely – see §3.12.

The proof consists of a number of steps, each of independent interest and each useful in other situations. We first describe the structure of the scattering amplitude as \( \lambda \to \infty \). That is done by viewing the resolvent dynamically and relating it to the Schrödinger propagator. We then prove that the scattering phase has an expansion as \( \lambda \to \infty \). Using heat trace asymptotics we then compute leading coefficients of that expansions.

In this section we for the first time in the book use microlocal/semiclassical methods – we will refer to Appendix E for what is needed but will assume familiarity with basic notation for classes of pseudodifferential operators and symbols (see §E.1).

3.11.1. Semiclassical structure of the scattering matrix. The semiclassical parameter \( h \), in the notation of Appendix E is \( h := 1/\lambda \). The semiclassical Hamiltonian is

\[
P := h^2 P_V = -h^2 \Delta + h^2 V, \quad V \in C^\infty_\text{c}(\mathbb{R}^n; \mathbb{R}).
\]

The potential term, \( h^2 V \), is a very weak perturbation of the free Hamiltonian \(-h^2 \Delta\). That is in contrast to the examples considered in §2.8 and to the theory presented in Part 3 of this book.

**THEOREM 3.57** (Scattering amplitude as a pseudodifferential operator). Suppose that \( V \in C^\infty_\text{c}(\mathbb{R}^n, \mathbb{C}) \). Let \( A(\lambda) \) be as in Theorem 3.41. Then

\[
A_h(E) := A(\sqrt{E}/h) : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}), \quad E > 1.
\]

is a family of semiclassical pseudodifferential operators depending smoothly on \( E \).

\[
\partial^k_E A_h(E) \in h\Psi^\text{comp}_h(\mathbb{S}^{n-1}), \quad k = 0, 1, \ldots .
\]

Moreover,

\[
(3.11.1) \quad \sigma_h(h^{-1} A_h(E))(\theta, \xi) = \frac{1}{2\pi} E^{-1/2} \int_{\mathbb{R}} V(-\xi/\sqrt{E} + s\theta) \, ds,
\]

where \( \xi \in \{ \eta \in \mathbb{R}^n : \langle \eta, \theta \rangle = 0 \} = T_\theta \mathbb{S}^{n-1} \subset \mathbb{R}^n \), and \( T^*\mathbb{S}^{n-1} \) is identified with \( T_\theta \mathbb{S}^{n-1} \) using the standard metric on the sphere.
The first step of the proof is yet another formula for the scattering matrix, similar to that in Theorem 3.44:

**THEOREM 3.58 (Description of the scattering matrix III).** Let $P_V$, $\rho$, and $E_\rho(\lambda)$ be as in Theorem 3.41. Take $\chi \in C_\infty_c(\mathbb{R}^n)$ such that $\chi = 1$ near $\text{supp } V$ and $\rho = 1$ near $\text{supp } \chi$. Then

$$A(\lambda) = c_n \lambda^{n-2} E_\rho(\lambda) [\Delta, \chi] R_V(\lambda) V E_\rho(\bar{\lambda})^*, \quad c_n = (2\pi)^{1-n}/2i.$$  

**Proof.** Recall that the operator $A(\lambda)$ is given by

$$A(\lambda)f(\theta) = \int_{\mathbb{R}^{n-1}} b(\lambda, \theta, -\omega) f(\omega) d\omega,$$

with $b(\lambda, \theta, \omega)$ defined in (3.7.3). The function $u$ in that definition is given by

$$u(x, \lambda, -\omega) = -R_V(\lambda)(V e^{i\lambda \langle \cdot, \omega \rangle}) = -R_V(\lambda) V E_\rho(\bar{\lambda})^* \delta_\omega.$$  

We have $(-\Delta - \lambda^2) u = (P_V - \lambda^2) u = 0$ on $\text{supp}(1 - \chi)$ and thus

$$(-\Delta - \lambda^2)(1 - \chi(x)) u(x, \lambda, -\omega) = [\Delta, \chi] u(x, \lambda, -\omega).$$  

The function $(1 - \chi) u$ is outgoing and Theorem 3.37 shows that

$$(1 - \chi(x)) u(x, \lambda, -\omega) = -R_0(\lambda) [\Delta, \chi] R_V(\lambda) V E_\rho(\bar{\lambda})^* \delta_\omega(x).$$  

The formula (3.11.2) follows from Theorem 3.5.  

We now analyse the expression (3.11.2) for $\lambda = \sqrt{E}/h$, where $E > 0$ varies in a fixed compact set, say $E \in [1, 2]$, and $h \to 0$. The operators $[\Delta, \chi]$ and multiplication by $V$ (denoted by $V$) are differential and $E_\rho(\lambda), E_\rho(\lambda)^*$ have an explicit oscillatory integral form.

Therefore the only component of (3.11.2) which needs to be understood further is the resolvent $R_V(\lambda)$. We rewrite in the semiclassical form:

$$R_h(E) := h^{-2} R_V(\sqrt{E}/h), \quad E \in [1, 2].$$  

We start with the following microlocalization statement about the free resolvent away from the diagonal:

**LEMMA 3.59.** For fixed $E > 0$, the free semiclassical resolvent

$$R_{0,h}(E) := h^{-2} R_0(\sqrt{E}/h),$$  

satisfies

$$\text{WF}_h'(R_{0,h}(E)) \cap (T^*\mathbb{R}^n)^2 \cap \{x \neq y\} \subset \{(x, \xi, x + t\xi, \xi) : |\xi|^2 = E, \ t \geq 0\}.$$  

(3.11.3)
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Proof. 1. Let \( \chi \in C^\infty(\mathbb{R}^{2n}) \) satisfy
\[
\chi_\delta(x,y) = \begin{cases} 
0 & |x-y| < \delta, \\
1 & |x-y| > \delta.
\end{cases}
\]
If \( R_{0,h}(E,x,y) \) is the Schwartz kernel of \( R_{0,h}(E) \) it is enough to show that
\[(3.11.4) \quad WF_h(\chi_\delta R_{0,h}) \subset \{(x,x + t\xi, \xi, -\xi) : |\xi|^2 = E, t \geq 0, x \in \mathbb{R}^n \}.
\]
(See §E.2 for a review of wave front sets.)

2. Theorem 3.3 shows that the smooth function \( (\chi_\delta R_{0,h}(E))(x,y) \) can be written as
\[
h^{-\frac{n+1}{2}} e^{\frac{i}{2} \varphi(x,y)} a_\delta(x,y,h),
\]
where \( \varphi(x,y) = \sqrt{E} |x-y| \) and \( \partial_{x,y} a_\delta = O_{\alpha,\delta}(1) \). Its wave front set is then given by the standard formula (see for instance [Zw12, Example (iii), §8.4]):
\[
\{(x,y, \partial_x \varphi(x,y), \partial_y \varphi(x,y), (x,y) \in \text{supp } a_\delta \}.
\]
But that gives (3.11.4) and hence (3.11.3). \( \square \)

We now write \( R_h(E) \) using the semiclassical Schrödinger propagator,
\[
\exp(-itP/h), \quad P = h^2 P_V.
\]
This is motivated by the following formula valid for \( \text{Im } E > 0 \)
\[(3.11.5) \quad R_h(E) = \frac{i}{h} \int_0^\infty e^{itE/h} e^{-itP/h} dt.
\]
The integral (3.11.5) converges as an operator \( L^2 \to L^2 \) since \( e^{-itP/h} \) is unitary and \( e^{itE/h} \) is exponentially decaying for \( \text{Im } E > 0 \) as \( t \to +\infty \). Hence (3.11.5) follows from the spectral theorem.

This is no longer true when \( E \in \mathbb{R} \), however we have a microlocal approximation statement given in the next theorem. It relies strongly on a translation of the estimate (2.3.5) to the semiclassical setting: for all \( k \),
\[(3.11.6) \quad \|\rho R_h(E)\rho\|_{L^2 \to H^k} \leq C_k/h, \quad E \in [1, 2].
\]
We call this a non-trapping resolvent estimate – more general versions for more complicated operators will be studied in Chapter 6, see Theorems 6.10, 6.16 and 6.22.

LEMMA 3.60 (Parametrix for \( R_h(E) \)). Assume that \( B \in \Psi^{\text{comp}}(\mathbb{R}^n), \chi \in C_c^\infty(\mathbb{R}^n) \), and \( V \) are all supported in \( \{|x| < R\} \) and that
\[
WF_h(B) \subset \{1/2 \leq |\xi| \leq 2\}.
\]
Then for \( T \geq 8R \),
\[(3.11.7) \quad \chi R_h(E)B = \frac{i}{h} \int_0^T \chi e^{itE/h} e^{-itP/h} B dt + O(h^\infty)_{D' \to C_c^\infty}.
\]
Proof. 1. Consider an $h$-tempered family $f = f(h) \in \mathcal{D}'(\mathbb{R}^n)$ (see §E.2.3) and define a family $v = v(h)$ by

$$v := \left( R_h(E) B - \frac{i}{h} \int_0^T \chi_1 e^{itE/h} e^{-itP/h} B dt \right) f \in \mathcal{D}'(\mathbb{R}^n).$$

Here $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ is a cutoff such that $\chi_1 = 1$ on $B(0, R + 10T)$. Note that by the nontrapping estimate (3.11.6), $v$ is also $h$-tempered. We calculate $g := (P - E)v$

$$= Bf + \chi_1 \int_0^T \partial_t (e^{itE/h} e^{-itP/h}) Bf dt - \frac{i}{h} \int_0^T e^{itE/h}[P, \chi_1] e^{-itP/h} Bf dt$$

$$= \chi_1 e^{iT E/h} e^{-iT P/h} Bf - \frac{i}{h} \int_0^T e^{itE/h}[P, \chi_1] e^{-itP/h} Bf dt \in C_c^\infty(\mathbb{R}^n).$$

From [Zw12, Theorem 12.5] we have

(3.11.8) $WF_h(e^{-itP/h}Bf) \subset \{(x + 2t\xi, \xi) : |x| \leq R, \ |\xi| \in [1/2, 2]\}.$

Then $WF_h(e^{-itP/h}Bf) \cap \text{supp}(1 - \chi_1) = \emptyset$ for $t \in [0, T]$. Therefore, $[P, \chi_1] e^{-itP/h} Bf = O(h^\infty) C_c^\infty.$

Putting $t = T \geq 8R$ in (3.11.8) we see that $(x, \xi) \in WF_h(g)$ implies that $x = y + 2R \xi \ |y| \leq R$ and hence

$$\langle x, \xi \rangle = T|\xi|^2 + \langle y, \xi \rangle \geq T/4 - 2R \geq 0.$$

In other words,

(3.11.9) $WF_h(g) \subset \{(x, \xi) : |x| \geq R, \ |x, \xi| \geq 0, \ |\xi| \in [1/2, 2]\}.$

2. We next claim that

(3.11.10) $v = R_{0,h}(E)g + O(h^\infty) C_c^\infty.$

Indeed, both $v$ and $R_{0,h}(E)g$ are outgoing, therefore by Theorem 3.37

$$v - R_{0,h}(E)g = R_h(E)(P - E)(v - R_{0,h}(E)g)$$

$$= -R_h(E)(h^2VR_{0,h}(E)g).$$

From (3.11.3) and (3.11.9), we obtain that

(3.11.11) $WF_h(R_{0,h}(E)g) \cap \{|x| < R\} = \emptyset.$

Thus $VR_{0,h}(E)g = O(h^\infty) C_c^\infty$ and (3.11.10) follows from (3.11.6).

3. Finally, by (3.11.11), we have $\chi R_{0,h}(E)g = O(h^\infty) C_c^\infty$, therefore $\chi v = O(h^\infty) C_c^\infty$. Since this is true for any $h$-tempered family $f(h)$, (3.11.7) follows. □

To write an oscillatory integral expression for $R_h(E)$ and thus for $A_h(E)$, we need to write such an expression for $e^{-itP/h}$.
**THEOREM 3.61 (Parametrix for the Schrödinger propagator).**
Suppose that $V \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$ and $B \in \Psi^{\text{comp}}_h(\mathbb{R}^n)$. Then for $|t| \leq T$,
\begin{equation}
(3.11.12) \quad \exp(-itP/h)B = \exp(-ithP)V)B = U_B(t) + O_T(h^\infty)_{\mathcal{Y}\rightarrow C^\infty},
\end{equation}
where for $f \in C_c^\infty(\mathbb{R}^n)$,
\begin{equation}
U_B(t)f(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y,\xi) - \frac{i}{h}|\xi|^2} b(t,x,\xi,h)f(y)dyd\xi,
\end{equation}
\begin{equation}
b(t,x,\xi,h) \sim \sum_{j=0}^\infty h^j b_j(t,x,\xi), \quad b_0(t,x,\xi) = \sigma_h(B)(x-2t\xi,\xi).
\end{equation}

**Proof.** The proof follows the WKB construction – see [Zw12, §10.2] – much simplified due to the fact that the flow is explicit and linear.

1. We are looking for $a(t,x,\xi,h)$ solving the following equation asymptotically in $h$:
\begin{equation}
(hD_t - h^2\Delta + h^2V) \left( e^{\frac{i}{h}(x,\xi)} - \frac{i}{h}|\xi|^2 b(t,x,\xi,h) \right) = 0,
\end{equation}
\begin{equation}
b(0,x,\xi,h) = b(x,\xi), \quad B = b(x,hD).
\end{equation}
We will solve for $a$ as an asymptotic expansion,
\begin{equation}
b(t,x,\xi,h) \sim \sum_{j=0}^\infty h^j b_j(t,x,\xi), \quad b_0(0,x,\xi) = b(x,\xi), \quad b_j(0,x,\xi) = 0.
\end{equation}

Since
e^{-\frac{i}{h}(x,\xi)+\frac{i}{h}|\xi|^2}(hD_t - h^2\Delta)e^{\frac{i}{h}(x,\xi)} - \frac{i}{h}|\xi|^2 = \frac{h}{i}(\partial_t + 2\langle \xi, \partial_x \rangle) - h^2\Delta,
we have the following set of equations for the terms in the expansion ($b_{-1} \equiv 0$):
\begin{equation}
(\partial_t + 2\langle \xi, \partial_x \rangle) b_j(t,x,\xi) = -i(-\Delta x + V(x))b_{j-1}(t,x,\xi).
\end{equation}
These are solved by
\begin{equation}
b_0(t,x,\xi) = b(x-2t\xi,\xi)
\end{equation}
\begin{equation}
b_j(t,x,\xi) = \frac{1}{i} \int_0^t (-\Delta x + V(x-2s\xi))b_{j-1}(s,x-2s\xi,\xi)ds, \quad j \geq 1.
\end{equation}
Since $b$ is compactly supported in $(x,\xi)$ so are $b_j$’s (with the size of the support depending on $t$) which shows that $U_B(t) : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ – see [Zw12, Theorem 4.1].

2. It remains to justify the estimate on the remainder in (3.11.13) and for that we will use Duhamel’s formula and the mapping properties of $\exp(-itP/h)$ – see [Zw12, §10.1]. We have shown so far that
\begin{equation}
(ih\partial_t - P)U_B(t) = r(t) \in O_T(h^\infty)_{\mathcal{Y}\rightarrow \mathcal{Y}}, \quad U_B(0) = B.
\end{equation}
Hence, by Duhamel’s formula,

$$e^{-itP/h}B - U_B(t) = -\frac{i}{h} \int_0^t e^{-i(t-s)P/h}R(s)ds.$$  

(3.11.15)

Using [Zw12, (10.1.10)] with $m(x, \xi) = 1 + |\xi|^2$ we see that $e^{-i(t-s)P/h}$ is $H^k_h \to H^k_h$ for all $k$ which implies that $e^{-i(t-s)P/h}R(s) = O(h^\infty)_{\mathcal{J}^r} \to H^N$ for all $N$. Inserting this into (4.6.15) shows that $e^{-itP/h}B - U_B(t) : O_T(h^\infty)_{\mathcal{J}^r} \to C^\infty$ as claimed.

Proof of Theorem 3.57. 1. To simplify notation we assume that $E = 1$ and drop $E$ from all the formulas. Since $A_h(E) = A_h/\sqrt{E}$ that is justified if $E \in K \subseteq (0, \infty)$.

By (3.11.2), we write

$$A_h := A_h(1) = \frac{1}{2\pi} (2\pi)^{1-n} h^{4-n} \delta_h[\Delta, \chi] R_h V \mathcal{E}_h^*, \quad R_h := R_h(1).$$

where (with $d\omega$ denoting the measure on $S^{n-1}$ induced from $\mathbb{R}^n$)

$$\mathcal{E}_h^* f(x) := \int_{S^{n-1}} e_h^*(x,\omega) f(\omega) d\omega.$$

As in the proof of (3.11.3) we see that

$$\text{WF}'(\mathcal{E}_h^*) \subset \{(x, \omega; \omega, x - \langle x, \omega \rangle \omega) \} \subset T^* \mathbb{R}^n \times T^* S^{n-1},$$

where we identified $T^*_\theta S^{n-1} \subset T^*_\theta \mathbb{R}^n$, $\theta \in S^{n-1} \subset \mathbb{R}^n$.

Take $B \in \Psi^\comp_0(\mathbb{R}^n)$ such that

$$B = 1 \quad \text{microlocally on } \{x \in \text{supp } V, \ |\xi| \in [2/3, 3/2]\};$$

$$\text{WF}_h(B) \subset \{|x| < R, \ |\xi| \in [1/2, 2]\}.$$

(3.11.16)

This and the estimate of the wave front set of $\mathcal{E}_h^*$ imply that

$$V \mathcal{E}_h^* = BV \mathcal{E}_h^* + O(h^\infty)_{\mathcal{D}'(S^{n-1})} \to C^\infty(\mathbb{R}^n).$$

Therefore, by Lemma 3.60, for fixed large $T > 0$,

$$A_h(E) = \frac{\pi h^{3-n}}{(2\pi)^n} \delta_h[\Delta, \chi] \int_0^T e^{it/h} e^{-itP/h} BV \mathcal{E}_h^* dt$$

$$+ O(h^\infty)_{\mathcal{D}'(S^{n-1})} \to C^\infty(\mathbb{R}^{n-1}).$$

(3.11.17)

2. We may replace $e^{-itP/h}B$ in (3.11.17) by $U_B(t)$ from Theorem 3.61. Also, we may replace the integral from 0 to $T$ in (3.11.17) by the integral against a function $\psi(t) \in C^\infty_c(0, T)$ such that $\psi = 1$ on $[\delta, T - \delta]$ for $\delta$ small enough.
(The supports of $[\Delta, \chi]$ and $V$ are disjoint and we can use propagation result [Zw12, Theorem 12.5].) Differentiating the formula (3.11.13) in $x$, we find

$$[\Delta, \chi]U_B(t)f(x) = \frac{2i}{\hbar}(2\pi\hbar)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y,\xi)} e^{-t(\xi^2)} b(t, x, \xi, h) f(y) \, dy \, d\xi,$$

$$\bar{b}(t, x, \xi, h) \sim \sum_{j=0}^{\infty} h^j \bar{b}_j(t, x, \xi), \quad \bar{b}_0(t, x, \xi) = \langle \xi, d\chi(x) \rangle b_0(t, x, \xi).$$

We thus obtain the following integral formula, valid modulo an $O(h^\infty)_{D^\nu \rightarrow C^{\infty}}$ remainder:

$$A_h g(\theta) \equiv \frac{i\hbar}{(2\pi\hbar)^{2n-1}} \int_{\mathbb{R}^{2n+1} \times \mathbb{S}^{n-1}} e^{\frac{i}{\hbar}\Phi} \tilde{a}g(\omega) \, dt \, dx \, dy \, d\xi \, d\omega,$$

$$\Phi(t, x, y, \xi, \omega, \theta) = \langle x - y, \xi \rangle + t(1 - |\xi|^2) + \langle (y, \omega) - (x, \theta) \rangle,$$

$$\tilde{a} \sim \sum_{j=0}^{\infty} h^j \tilde{a}_j, \quad \tilde{a}_0(t, x, y, \xi) = \psi(t)V(y)\langle \xi, d\chi(x) \rangle \sigma_h(B)(x - 2t\xi, \xi).$$

The support of the symbol $\tilde{a}(t, x, y, \xi, h)$ is contained in

$$\{ t \in (0, T), \ y \in \text{supp} \ V, \ x \in \text{supp} \chi, \ |\xi| \in [1/2, 2] \}.$$

Moreover, we note for future reference that for $|\xi|^2 = 1 \in [1, 2]$, it follows from (3.11.16) that

$$a_0(t, x, x - 2t\xi, \xi) = 1_{\mathbb{R}^+}(t) V(x - 2t\xi)\langle \xi, d\chi(x) \rangle.$$

3. We now use the integral formula for $A_h(E)$ to show that it is an $h$-pseudodifferential operator. We refer to §E.1.5 for the definition of a pseudodifferential operator on a manifold ($\mathbb{S}^{n-1}$ in our case).

First we note that if $\theta \neq \omega$ then $|\xi - \theta|^2 + |\xi - \omega|^2 \geq \frac{1}{2} |\theta - \omega|^2 > 0$ and

$$-h^2(\xi - \theta)^2 + |\xi - \omega|^2 - (\Delta x + \Delta y) e^{\frac{i}{\hbar}\Phi} = e^{\frac{i}{\hbar}\Phi}.$$

Hence, repeated integration by parts in $x, y$ in the integral in (3.11.18) show that for $\chi', \chi'' \in C^{\infty}(\mathbb{S}^{n-1})$ and supp $\chi' \cap$ supp $\chi'' = \emptyset$, we have $\chi' A_h \chi'' = O(h^\infty)_{D^\nu \rightarrow C^{\infty}}$.

It then remains to show that for $\chi'$ supported in a small coordinate neighbourhood, $\chi' A_h \chi'$ is a classical pseudodifferential operator. Without loss of generality, we consider the coordinate neighbourhood $\{ \theta_n > 0 \}$ with the coordinate $\theta'$, where $\theta = (\theta', \sqrt{1 - |\theta'|^2})$. For $\chi'$ supported in this neighbourhood, the symbol of $\chi' A_h \chi'$ is given by oscillatory testing:

$$a_{\chi'}(\theta', \eta) := e^{-\frac{i}{\hbar} \langle \theta', \eta \rangle} \chi'(\theta')(A_h \chi' (e^{\frac{i}{\hbar}\Phi}(\theta', \eta)))((\theta'), \theta', \eta \in \mathbb{R}^{n-1}).$$
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We need to calculate \( a(\theta', \eta) \) and show that it is indeed a classical symbol. Then \( \chi' A_h \chi' = a(\theta', hD_{\theta'}, h) \) (note that we are using here the standard quantization (E.1.12) and not the Weyl quantization).

4. Using the oscillatory integral expression for \( A_h \), we write

\[
a' \chi'(\theta', \eta) = (2\pi)^{1-2n}i h^{2-2n} \int_{\mathbb{R}^{4n}} e^{i \Phi(t,x,y,\xi,\omega,\theta)} (\omega'-\theta', \eta) \, dt \, dx \, dy \, d\xi \, d\omega' .
\]

The phase in the integral is

\[
\langle x - y, \xi \rangle + \langle \omega' - \theta', \eta \rangle + t(1 - |\xi|^2) + \langle y, \omega \rangle - \langle x, \theta \rangle .
\]

The critical points in \((y, \xi)\) are given by \( \xi = \omega \), \( y = x - 2t\xi \) and the critical Hessian has determinant 1 and signature 0. Hence, the stationary phase method in the \((y, \xi)\) variables gives

\[
a' \chi'(\theta', \eta) = (2\pi)^{1-n}i h^{2-n} \int_{\mathbb{R}^{2n}} e^{i \Phi^* a} \, dt \, dx \, d\omega' ,
\]

\[
\Phi^*(t, x, \omega', \theta') = \langle \omega' - \theta', \eta \rangle + \langle x', \omega' - \theta' \rangle + x_n \left( \sqrt{1 - |\omega'|^2} - \sqrt{1 - |\theta'|^2} \right) ,
\]

\[
a^*(t, x, \omega', \theta') \sim \sum_{j=0}^{\infty} h^j a^*_j(t, x, \omega', \theta') ,
\]

\[
a^*_0(t, x, \omega', \theta') = \frac{\chi'(\omega') \chi'(\theta') \tilde{a}_0(t, x, x - 2t\omega, \omega)}{\sqrt{1 - |\omega'|^2}} .
\]

We next apply stationary phase in the \((x', \omega')\) variables: the critical points are given by \( \omega' = \theta' \) and \( x' = -\eta \) and the Hessian has determinant 1 and signature 0. Hence,

\[
a' \chi'(\theta', \eta) = ih \int_{\mathbb{R}^2} a^\flat dt dx_n , \quad a^\flat \sim \sum_{j=0}^{\infty} h^j a^\flat_j ,
\]

where \( a^\flat_0(t, x_n, \theta', \eta) = \)

\[
\frac{\chi'(\theta')^2 \tilde{a}_0(t, -\eta + x_n\theta'/\theta_n, x_n, -\eta + x_n\theta'/\theta_n - 2t\theta', x_n - 2t\theta_n, \theta)}{\sqrt{1 - |\theta'|^2}} .
\]

Note that \( a^\flat \) is compactly supported, as follows from the support condition on \( \tilde{a} \). Integrating in \( t, x_n \), we see that \( h^{-1}a' \) is a compactly supported classical symbol and thus \( A_h \in h\Psi^{\text{comp}}(\mathbb{S}^n) \).
5. It remains to calculate the principal symbol of $h^{-1}A_h$. A change of variables $x_n = \theta_n(s + 2t)$ gives

$$
\sigma_h(h^{-1}A_h)(\theta', \eta)
= i \int_0^\infty dt \int \mathcal{R} ds V(-\eta,0 + s\theta) \langle \theta, d\chi(-\eta,0 + (s + 2t)\theta) \rangle
= \frac{1}{2i} \int \mathcal{R} V(-\eta,0 + s\theta) \, ds,
$$

which proves (3.11.1) with $E = 1$.

6. We now consider the derivatives of $A_h(E)$ with respect to $E$. The relation

$$A_h(E) = A_h/\sqrt{E}(1) \quad \text{or the construction of the expansions)}$$

shows that the terms in the expansion depend smoothly on $E$. On the other hand, in the notation of Step 3,

$$\chi'\partial_E^k A_h(E) \chi'' = O\left(\frac{1}{t} \rightarrow C_\infty(h^\infty)\right),$$

and $\chi'\partial_E^k A_h(E) \chi'' = a_{\chi',k}(\theta', hD_{\theta'}, h)$, where

$$\partial_\alpha \partial_\beta a_{\chi',k} = O\left(h^{-N(k,|\alpha|+|\beta|)}\right)$$

(that follows from (3.11.2) and (3.11.6)). Also $a_{\chi'}(\theta', \eta) = \tilde{a}_{\chi'}(\theta', \eta) + O(h^\infty)$, and $\tilde{a}_{\chi'}$ has differentiable expansions in $h$.

The interpolation inequality (A.5.4) (applied with $m = 0$, $\ell = M + k$) now gives

$$\sup_{|\alpha|+|\beta|=M} \|\partial_\alpha \partial_\beta a_{\chi',k} - \partial_E^k \tilde{a}_{\chi'}\|_{L^\infty} \leq C_{M,k} \|a_{\chi'} - \tilde{a}_{\chi'}\|_{L^\infty}^{\frac{1}{M+k+1}} \left(\left(\sup_{|\alpha|+|\beta|=M+1} \|\partial_\alpha \partial_\beta a_{\chi',k+1}\|_{L^\infty} + C_{M,k}'\right)^{\frac{M+k}{M+k+1}} \right)
\leq O(h^\infty)O(h^{-N_M,k+1}) = O(h^\infty).$$

It follows that $a_{\chi',k}$ is a symbol with a full asymptotic expansion. □

An important consequence of Theorem 3.57 is the existence of the expansion for the scattering phase:

**THEOREM 3.62 (Existence of an asymptotic expansion).** Define the derivative of the scattering phase, $\sigma'(\lambda)$, by

$$
(3.11.20) \quad \sigma'(\lambda) := \frac{1}{2\pi i} \text{tr} S(-\lambda)\partial_\lambda S(\lambda).
$$

Then there exists a sequence $b_1, b_2, \ldots$ such that

$$
\sigma'(\lambda) \sim \sum_{j=1}^\infty b_j \lambda^{n-2-j}, \quad \lambda \to +\infty,
$$

where $n = \dim \mathcal{H}$. 

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and

\begin{equation}
    b_1 = -\frac{(n-2) \text{Vol}(S^{n-1})}{8(2\pi)^n} \int_{\mathbb{R}^n} V(x) \, dx.
\end{equation}

**REMARK.** In Theorem 3.67 we will show that only even terms appear in the expansion and will provide a method for computing the coefficients. That will show that the integrated expansion is also valid – see (3.11.41).

**Proof.** 1. Since \( S(\lambda) = I + A(\lambda) \) and \( A_k(E) = A(\sqrt{E}/h) \), \( \partial_\lambda S(\lambda) = h\partial E A(1)/2, \lambda = 1/h \) and

\begin{equation}
    \sigma'(\lambda) = \frac{1}{2\pi i} \text{tr}(S(\lambda)^* \partial_\lambda S(\lambda)) = \frac{h}{4\pi i} \text{tr}((I + A_h(1)^*) \partial E A_h(1))
\end{equation}

Theorem 3.57 shows that \( A_h(1), \partial E A_h(1) \in h\Psi^\text{comp}_h(S^{n-1}) \). Its proof, and the composition formula for pseudodifferential operators (see Proposition E.8), also gives asymptotic expansions of full symbols of operators localized to coordinate patches:

\begin{equation}
    \begin{aligned}
    b_{\chi'}(y, hD_y, h) &= \chi'(I + A_h(1)) \partial E A_h(1) \chi', \\
    b_{\chi'}(y, \eta, h) &\sim \sum_{k=1}^{\infty} h^f b_{\chi',k}(y, \eta), \\
    b_{\chi',k} &\in C^\infty_c(B(0,1)_y \times (B(0,2) \setminus B(0, \frac{1}{2})))_\eta,
    \end{aligned}
\end{equation}

\( y = \theta' \in \mathbb{R}^{n-1}, (\theta', (1 - |\theta'|^2)^{\frac{1}{2}}) \in S^{n-1} \) – see Step 3 of that proof for the notation. Choosing a partition of unity \( \sum_{j=1}^{J} \chi_j^2 = 1, \chi_j \in C^\infty(S^{n-1}), \) (3.11.24) gives

\[
\text{tr}_{L^2(S^{n-1})}(I + A_h(1)) \partial E A_h(1) = \sum_{j=1}^{J} \text{tr}_{L^2(S^{n-1})} b_{\chi_j}(y, hD_y, h) = \frac{1}{(2\pi h)^{n-1}} \int_{T^*\mathbb{R}^{n-1}} b_{\chi_j}(y, \eta, h) \, dy \, d\eta \\
\sim \sum_{k=1}^{\infty} a_k h^{-n-1+k},
\]

\( a_k := \sum_{j=1}^{J} \int_{T^*\mathbb{R}^{n-1}} b_{\chi_j,k}(y, \eta) \, dy \, d\eta. \) Returning to (3.11.23) we obtain (3.11.21).

2. The first coefficient is given by the integral of the principal symbol of \( h^{-1} \partial E A_h(1) \) which we compute using (3.11.1) (note that \( \xi \in \{ \eta : \langle \eta, \theta \rangle = \)
0} = T_0 S^{n-1} \subset \mathbb{R}^n):$

\begin{align*}
    b_1 &= \frac{1}{2(2\pi)^n} \partial_E|_{E=1} \sigma_h(h^{-1}A_h(E)) \\
    &= -\frac{1}{4(2\pi)^n} \partial_E|_{E=1} \left( E^{-1/2} \int_{\mathbb{R}} ds \int_{T^*S^{n-1}} d\xi d\theta V(-\xi/\sqrt{E} + s\theta) \right) \\
    &= -\frac{1}{8(2\pi)^n} \int_{\mathbb{R}} ds \int_{T^*S^{n-1}} d\xi d\theta (\xi \cdot \nabla V(-\xi + s\theta) - V(-\xi + s\theta)).
\end{align*}

We can integrate parts in $\xi$ by assuming, without loss of generality that $\theta = (0, 1) = e_n$ and $\xi = (\xi', 0), \xi' \in \mathbb{R}^{n-1}:$

$$
\int_{\mathbb{R}^{n-1}} (\xi', 1) \cdot \nabla V(-((\xi', 0) + se_n)) d\xi' = \int_{\mathbb{R}^{n-1}} (n - 2)V(\xi + s\theta) d\xi'.
$$

Hence,

\begin{align*}
    b_1 &= -\frac{n - 2}{8(2\pi)^n} \int_{\mathbb{R}} ds \int_{T^*S^{n-1}} d\theta d\xi V(\xi + s\theta) \\
    &= -\frac{n - 2}{8(2\pi)^n} \int_{\mathbb{R}^n \times S^{n-1}} V(x) d\theta dx = -\frac{(n - 2) \text{Vol}(S^{n-1})}{8(2\pi)^n} \int_{\mathbb{R}^n} V(x) dx,
\end{align*}

concluding the proof of (3.11.22). \qed

It would be tempting to compute the coefficients of the expansions in Theorem 3.57 directly using symbolic calculus and the structure of the parametrix for the propagator and we did this for the first term in the expansion (3.11.22) However, it more convenient to compute the terms appearing in the expansion by using the Birman–Kre˘ın in formula (3.9.10) and heat trace asymptotics of Theorem 3.64 below.

3.11.2. Heat trace asymptotics. We start with a representation of the resolvent useful for $\text{Im} \lambda > 0:$

**LEMMA 3.63 (Another expansion for the resolvent).** Suppose $V \in C^\infty_c(\mathbb{R}^n; \mathbb{C}).$ Then for any $M \in \mathbb{N},$

\begin{equation}
    R_V(\lambda) = \sum_{m=0}^{M} X_m R_0(\lambda)^{m+1} + R_V(\lambda) X_{M+1} R_0(\lambda)^{M+1}, \tag{3.11.25}
\end{equation}

where the operators $X_m$ are defined by induction as follows:

\begin{equation}
    X_0 := I, \quad X_{m+1} = -VX_m + [X_m, P_0], \tag{3.11.26}
\end{equation}

where $V$ (as elsewhere) means the multiplication operator $f \mapsto Vf.$ For $m > 0, X_m$ have order $\leq m - 1$ and compactly supported coefficients.
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Proof. 1. For \( M = 0 \) (3.11.25) states that

\[
R_V(\lambda) - R_0(\lambda) = R_V(\lambda)X_1R_0(\lambda) = -R_V(\lambda)VR_0(\lambda),
\]

which is the resolvent identity.

2. Assuming that (3.11.25) holds for \( M \) replaced with \( M - 1 \), the inductive step means proving that

(3.11.27) \( R_V(\lambda)X_MR_0(\lambda)^M - X_MR_0(\lambda)^{M+1} = R_V(\lambda)X_{M+1}R_0(\lambda)^{M+1} \).

To see this we use the definition of \( X_{M+1} \) to write (we suppress the \( \lambda \) dependence)

\[
R_VX_M - R_VX_{M+1}R_0 = R_V(X_M + VX_MR_0 + P_0X_MR_0 - X_MP_0R_0)
= R_V(X_M + (P_V - \lambda^2)X_MR_0 - X_M(P_0 - \lambda^2)R_0)
= R_VX_M + X_MR_0 - R_VM
= X_MR_0.
\]

Multiplying this on the right by \( R_0^M \) gives (3.11.27). \( \Box \)

We can now study heat trace asymptotics.

**THEOREM 3.64.** Suppose that \( V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \). Then,

\[
e^{-tP_V} - e^{-tP_0} \in L^1_1(L^2(\mathbb{R}^n)), \quad t > 0,
\]

and for any \( K \in \mathbb{N} \),

(3.11.28) \( \text{tr} \left( e^{-tP_V} - e^{-tP_0} \right) = \frac{1}{(4\pi t)^{n/2}} \sum_{k=1}^{K} a_k(V)t^k + O(t^{K+1-n/2}), \)

where

(3.11.29) \( a_1(V) = -\int V(x)dx, \quad a_2(V) = \frac{1}{2} \int V(x)^2dx. \)

Proof. 1. Functional calculus for self-adjoint operators shows that

(3.11.30) \( e^{-tP_V} - e^{-tP_0} = \frac{1}{2\pi i} \int_{\Gamma_c} e^{-tz}((P_V - z)^{-1} - (P - z)^{-1})dz, \)

\( \Gamma_c : s \mapsto z(s) := c + i|s|e^{i\text{sgn}(s)\pi/4}, \quad s \in \mathbb{R}, \quad c < E_K. \)

That follows from a deformation of a contour involving \((P_V - z)^{-1}\) and \((P_0 - z)^{-1}\) separately (where \( t \geq 0 \) is needed) – see Fig. 3.6. The integral on the right hand side converges in operator norm on \( L^2 \) as \( \text{Re} z(s) \sim |s|, \)
\( s \to \pm \infty, \) and

\[
\|(P_V - z)^{-1} - (P_0 - z)^{-1}\| = \|(P_V - z)^{-1}V(P_0 - z)^{-1}\|
\leq C|z|^{-2}, \quad z \in \Gamma_c.
\]
Figure 3.6. Contours in the $z$-plane ($z = \lambda^2$) used to express $e^{-tP_V} - e^{-tP_0}$ in terms of the resolvent. The contour in black provides the usual expression of $e^{-tP_V}$ and $e^{-tP_0}$. The red contour $s \mapsto c + |s|e^{i\text{sgn}(s)\pi/4}, s \in \mathbb{R}$, provides an expression for $e^{-tP_V} - e^{-tP_0}$.

Theorem 3.50 or a direct argument based on (3.11.30) also show that $e^{-tP_V} - e^{-tP_0}$ is of trace class. The trace can then be computed by integrating the Schwartz kernel over the diagonal.

2. The formula
\[
\frac{1}{2\pi i} \int_{\Gamma_c} e^{-tz}(P_0 - z)^{-m-1} dz = \frac{t^m}{m!} e^{-tP_0}
\]

and Lemma 3.63 show
\[
(3.11.31) \quad \text{tr}(e^{-tP_V} - e^{-tP_0}) = \sum_{m=1}^{M} \frac{t^m}{m!} \text{tr}(X_m e^{-tP_0}) + \text{tr} e_M(t),
\]

where
\[
(3.11.32) \quad e_M(t) := \frac{1}{2\pi i} \int_{\Gamma_c} e^{-tz}(P_V - z)^{-1} X_{M+1}(P_0 - z)^{-M-1} dz.
\]

3. We first analyze the terms in the sum on the right hand side of (3.11.31). The Schwartz kernel of $e^{-tP_0}$ is given by
\[
K(t, x, y) = \frac{1}{(4\pi t)^{3/2}} e^{-|x-y|^2/4t},
\]

see for instance [Ev98, Chapter 1] or [HöI, Theorem 3.3.3].
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Since for \( m \geq 1 \), \( X_m \) is a differential operator of order \( m - 1 \), with compactly supported coefficients,

\[
\frac{t^m}{m!} \text{tr} (X_m e^{-tP_0}) = \frac{t^m}{m!(4\pi t)^{n/2}} \int_{\mathbb{R}^n} (X_m e^{-|x-y|^2/4t}) |_{x=y} dx
\]

\[= \frac{1}{(4\pi t)^{n/2}} t^{m-\left[\frac{m-1}{2}\right]} \sum_{k=0}^{\left[\frac{m-1}{2}\right]} a_{m,k} t^k = O(t^{\left[\frac{m}{2}\right]+1-\frac{n}{2}}),
\]

which means that the expansion makes formal sense. Grouping the coefficients according to the powers of \( t \) gives the coefficients in the expansion and a calculation based on (3.11.26) gives (3.11.29). In fact,

\[
X_1 = V, \quad X_2 = V^2 - 2\nabla V \cdot \nabla - \Delta V,
\]

\[
X_3 = -2 \sum_{j,k} \partial_{x_j} \partial_{x_k} V \partial_{x_j} \partial_{x_k} + \tilde{X}_3,
\]

where \( \tilde{X}_3 \) is an operator of order 1. Hence,

\[
a_1(V) = (4\pi t)^{n/2} \text{tr}(X_1 e^{-tP_0}) = -\int_{\mathbb{R}^n} (V(x)e^{-|x-y|^2/4t}) |_{x=y} dx = -\int_{\mathbb{R}^3} V(x)dx.
\]

Also,

\[
(4\pi t)^{n/2} \text{tr}(X_2 e^{-tP_0}) = \int_{\mathbb{R}^n} (X_2 e^{-|x-y|^2/4t}) |_{x=y} dx = \int_{\mathbb{R}^n} V(x)^2 dx,
\]

and

\[
(4\pi t)^{n/2} \text{tr}(X_3 e^{-tP_0}) = -2 \int_{\mathbb{R}^n} \left( \sum_{j,k} V_{x_j x_k} \partial_{x_j} \partial_{x_k} e^{-|x-y|^2/4t} \right) |_{x=y} dx + O(t^{-1})
\]

\[= 4 \int_{\mathbb{R}^n} \Delta V dx + O(t^{-1}) = O(t^{-1}).
\]

Using this in (3.11.31) gives \( a_2(V) = \frac{1}{2} \int_{\mathbb{R}^n} V(x)^2 dx \).

4. To estimate the trace class norm of the remainder \( e_M(t) \) in (3.11.32) we start with estimates on the integrand. Uniformly for \( \text{Re} \ z \leq E_k - 1 \) we have \( \|(-\Delta - z)^{-1}\|_{L^2} \leq 1/|z| \) and

\[
\|(-\Delta - z)^{-1}\|_{L^2} \simeq \|\Delta(-\Delta - z)^{-1}\|_{L^2} + \|(-\Delta - z)^{-1}\|_{L^2} \leq C.
\]
3. SCATTERING ASYMPTOTICS

For $0 \leq s \leq 2$, we can use Hölder’s inequality with $p = 2/s$ and $q = 2/2 - s$ to obtain

$$
\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (\xi)^{2s}|\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^{2-2s} (\langle \xi \rangle^2 |\hat{u}(\xi)|)^s d\xi
\leq \|\hat{u}\|_{L^2}^{2-2s} \|\langle \xi \rangle^2 \hat{u}\|_{L^2}^s = \|u\|_{L^2}^{2-2s} \|u\|_{H^s}^s.
$$

Commuting $\Delta$ through the resolvent, it follows that

$$
\|(-\Delta - z)^{-1}\|_{H^r \rightarrow H^{s+r}} \leq C|z|^{-1+s/2}, \quad 0 \leq s \leq 2,
$$
on the contour, uniformly with respect to $\text{Re} z < -1$. Consequently as long $M \geq n \implies s := n + 1 \frac{M}{M + 1} \leq 2$,

we can iterate the estimate to obtain

$$
\|(-\Delta - z)^{-M-1}\|_{L^2 \rightarrow H^{n+1}} \leq C_M |z|^{-(1-\frac{n+1}{2(M+1)})(M+1)} = C_M |z|^{-M+\frac{n-1}{2}}.
$$

Since $X_{M+1}$ is a differential operator with coefficients supported in $|x| \leq R$ we obtain

$$
\|X_{M+1}(-\Delta - z)^{-M-1}\|_{L^1} \leq C \|X_{M+1}(-\Delta - z)^{-M-1}\|_{L^2(\mathbb{R}^n) \rightarrow H^{n+1}(B(0,R))}
\leq C' |z|^{-M+\frac{n-1}{2}}.
$$

5. Returning to (3.11.32) we deform the contour of integration to $s \mapsto -1/t + is$, $s \in \mathbb{R}$. Then, using the uniformity of the above estimates for $\text{Re} z < E_K - 1$, and the bound $\|P_V - z\|^{-1}_{L^2 \rightarrow L^2} \leq C/|z|$, we obtain

$$
\|e_M(t)\|_{L^1} \leq C \int_{-1/t-i\infty}^{-1/t+i\infty} e^{t \text{Re} z} \|X_{M+1}(-\Delta - z)^{-M-1}\|_{L^1} \frac{|dz|}{|z|}
\leq C' \int_{-1/t-i\infty}^{-1/t+i\infty} |z|^{-M+\frac{n-1}{2}} \frac{|dz|}{|z|}
\leq C' t^{M+\frac{n-1}{2}} \int_{-1-i\infty}^{-1+i\infty} |w|^{-M+\frac{n-1}{2}} \frac{|dw|}{|w|}
= O(t^{M-\frac{n-1}{2}}),
$$

where the integral converges if $M \geq n$. Combined with (3.11.33) that gives an estimate of the remainder in (3.11.28). □

3.11.3. Asymptotic expansion. To relate the asymptotic expansion of $\sigma'$ in Theorem 3.62 with the heat trace asymptotics in Theorem 3.64 we will use the following elementary fact:
LEMMA 3.65. For $m \in \mathbb{Z}$ and $t > 0$ define

$$u_m(t) := \int_1^\infty \lambda^m e^{-t\lambda^2} d\lambda \in C^\infty((0, \infty)_t).$$

Then

(3.11.34)

$$u_m(t) - \frac{1}{2} \Gamma \left( \frac{m+1}{2} \right) t^{-\frac{m+1}{2}} \in C^\infty([0, \infty)_t), \quad m \geq 0 \ or \ m \in -2\mathbb{N},$$

$$u_m(t) - \frac{(-1)^{k+1}}{2k!} t^k \log t \in C^\infty([0, \infty)_t), \quad m = -2k - 1, \ k \in \mathbb{N},$$

where $C^\infty([0, \infty)_t)$ denotes functions which are smooth up to 0 in $t$, $\mathbb{N} = \{0, 1, \cdots \}$.

Proof. 1. We first consider the case of $m \geq 0$. Then

$$u_m(t) - \int_0^\infty \lambda^m e^{-t\lambda^2} d\lambda \in C^\infty((0, \infty)_t),$$

and

$$\int_0^\infty \lambda^m e^{-t\lambda^2} d\lambda = t^{-\frac{m+1}{2}} \int_0^\infty x^m e^{-x^2} dx = \frac{1}{2} \Gamma \left( \frac{m+1}{2} \right) t^{-\frac{m+1}{2}},$$

proving the claim.

2. When $m = -2k$, $k \in \mathbb{N}$ then,

$$(-\partial_t)^k u_m(t) = u_0(t) = \frac{1}{2} \Gamma \left( \frac{1}{2} \right) t^{-\frac{1}{2}} + w_0(t), \quad w_0 \in C^\infty([0, \infty)_t).$$

Integrating $k$ times from 1 to $t$ ($t \to 0^+$) gives the desired expression up to a smooth additive term.

3. When $m = -2k - 1$, $k \in \mathbb{N}$ then,

$$(-\partial_t)^{k+1} u_m(t) = u_1(t) = \frac{1}{2} t^{-1} + w_1(t), \quad w_1 \in C^\infty([0, \infty)_t).$$

We again integrate $k + 1$ times from 1 to $t$ ($t \to 0^+$) which gives the second case of (3.11.34).

THEOREM 3.66 (Asymptotics of $\sigma'(\lambda)$). Let $\sigma'(\lambda)$ be given by (3.11.20). Then, only even powers appear in the expansion (3.11.21):

(3.11.35) $\sigma'(\lambda) \sim \sum_{k=1}^{\infty} c'_k(V) \lambda^{n-2k-1}$, $c'_k(V) := \frac{2a_k(V)}{\Gamma \left( \frac{n}{2} - k \right) (4\pi)^{\frac{n}{2}}}$,

where $a_k(V)$ are given by (3.11.28). Moreover,

(3.11.36) $\int_0^\infty \left( \sigma'(\lambda) - \sum_{k=1}^{n \frac{1}{2}} c'_k(V) \lambda^{n-2k-1} \right) d\lambda = -K - \frac{1}{2} \tilde{m}_R(0)$,

where $K$ is the number of eigenvalues of $P_V$ and $\tilde{m}_R(0)$ is the multiplicity of the zero resonance (3.3.17).
Proof. 1. To obtain (3.11.35) we apply Theorem 3.51 to the function \( f(\lambda) = e^{-t\lambda^2} \) (see Remark after the statement of Theorem 3.51):

\[
(3.11.37) \quad \int_0^\infty e^{-t\lambda^2} \sigma'(\lambda) d\lambda = \text{tr} (e^{-tP_V} - e^{-tP_0}) - \sum_{k=1}^K e^{-tE_k} - \frac{1}{2} \tilde{m}_R(0).
\]

The right hand side has an expansion

\[
(3.11.38) \quad \frac{1}{(4\pi t)^{n/2}} \sum_{k=1}^\infty a_k t^k + \sum_{k=1}^\infty b_k t^k - (K + \frac{1}{2} \tilde{m}_R(0)), \quad t \to 0+
\]

where the second (smooth) sum comes from the Taylor expansion of the eigenvalue contributions.

2. The asymptotic expansion (3.11.21) means that for \( \lambda > 1 \) and any \( J \),

\[
\sigma'(\lambda) = \sum_{j=1}^J b_j \lambda^{n-j-2} + O(\lambda^{n-J-3}).
\]

In the notation of Lemma 3.65 we then see that for \( J \geq n-2 \),

\[
(3.11.39) \quad \int_0^\infty \sigma'(\lambda) e^{-t\lambda^2} d\lambda - \sum_{j=1}^J b_j u_{n-j-2}(t) \in C^{[\lfloor \frac{n+3}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor]}(0, \infty).
\]

Comparison of (3.11.34) with (3.11.37) shows that \( b_j = 0 \) for \( j = 2k \), and \( b_{2k-1} = c'_k(V) \) where \( c'_k(V) \) are given in (3.11.35).

3. It remains to show that (3.11.36) holds. Applying (3.11.39) with \( J = n-2 \) (which corresponds to \( k = (n-1)/2 \)) shows that

\[
G(t) := \int_0^\infty \left( \sigma'(\lambda) - \sum_{k=1}^{n-1} c'_k(V) \lambda^{n-2k-1} \right) e^{-t\lambda^2} d\lambda
\]

is a continuous function on \([0, \infty)\). The integrand in the definition of \( G(t) \) is uniformly bounded by \( C(\lambda)^{-2} \) as \( t \geq 0 \) and hence \( G(0+) \) is equal to the left hand side of (3.11.36). Comparison with (3.11.38) gives \( G(0+) = K + \frac{1}{2} \tilde{m}_R(0) \), completing the proof. \( \square \)

For completeness we include a result about the asymptotic behaviour of the actual scattering phase:

**THEOREM 3.67 (Asymptotics of the scattering phase).** Suppose that \( V \in C_c^\infty(\mathbb{R}^n, \mathbb{C}) \) where \( n \geq 1 \) is odd. Define the scattering phase

\[
(3.11.40) \quad \sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda),
\]

by demanding that \( \sigma'(\lambda) \) is given by (3.11.20) and that \( \sigma(0) = \tilde{m}_R(0)/2 \) where \( \tilde{m}_R(0) \) is given in (3.3.17).
Then there exists a sequence $c_k(V)$ such that, as $\lambda \to +\infty$,

$$
(3.11.41) \quad \sigma(\lambda) \sim \sum_{k=1}^{n-1} c_k(V)\lambda^{n-2k} - K + \sum_{k=\frac{n+1}{2}}^{\infty} c_k(V)\lambda^{n-2k},
$$

where $K$ is the number of eigenvalues of $P_V$ and

$$
c_k(V) = \frac{a_k(V)}{\Gamma(\frac{n}{2} - k + 1)(4\pi)^{\frac{n}{2}}},
$$

with $a_k(V)$ are given in (3.11.37). In particular,

$$
(3.11.42) \quad c_1(V) = -\frac{1}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} V(x)dx,
$$

$$
(3.11.42) \quad c_2(V) = \frac{1}{2\Gamma(\frac{n}{2} - 1)(4\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} V(x)^2dx.
$$

Proof. 1. We know from (3.7.28) that $\det S(0) = (-1)^m R(0)$ (see also (3.10.4)). Hence our choice of the value of $\sigma(0)$ determines a branch of the logarithm.

2. We know that $\sigma'(\lambda)$ is an even function and hence $\sigma(\lambda) - \frac{1}{2}\bar{m}(0)$ is odd. In view of (3.11.36), we have for $\lambda > 0$,

$$
\sigma(\lambda) = \sum_{k=1}^{n-1} \lambda^{n-2k} - K - \int_{-\lambda}^{\infty} \left( \sigma'(\tau) - \sum_{k=1}^{n-1} c_k'(V)\tau^{n-2k-1} \right) d\tau.
$$

Integrating (3.11.35) and noting that

$$
\frac{2}{(n-2k)\Gamma(\frac{n}{2} - k)} = \frac{1}{\Gamma(\frac{n}{2} - k + 1)}
$$

gives (3.11.41). \qed

3.12. EXISTENCE OF RESONANCES FOR REAL POTENTIALS

Theorem 2.14 implies that any complex valued compactly supported potential in one dimension has infinitely many resonances. In Section 3.5 earlier in this chapter we have shown that there exist complex valued compactly supported potentials in higher dimensions with no resonances. We will now prove existence of infinitely many resonances for arbitrary potentials $V \in C^\infty_0(\mathbb{R}^n; \mathbb{R})$. It is based on the Birman–Krein formula (2.6.1) applied with $f(s) = e^{-ts}$ and heat trace asymptotics as $t \to 0$. 
THEOREM 3.68 (Existence of resonances). Suppose that
\[ V \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), \quad \text{with } n \geq 3, \text{ odd.} \]
Then
\[ \sum_{\lambda \in \mathbb{C}} m_R(\lambda) = \infty, \]
that is, \( V \) has infinitely many scattering resonances.

REMARK. To show that there have to be some resonances we only use the Birman–Krein formula and the factorization of the scattering matrix from §§3.9 and 3.10 respectively. The asymptotic analysis of §3.11 is needed to show that there are infinitely many resonances.

Proof. 1. We first assume that of \( R_V(\lambda) \) has no poles at all. Then Theorem 3.54 implies that
\[
\sigma(\lambda) := \frac{1}{2\pi i} \log \det S(\lambda) = b_n\lambda^n + b_{n-2}\lambda^{n-2} + \cdots + b_1\lambda, \quad b_j \in \mathbb{R}.
\]
(3.12.1)
Using (2.6.1) with \( f(s) := e^{-st} \) we obtain
\[
\text{tr}(e^{-tP_V} - e^{-tP_0}) = \int_0^\infty f(\lambda^2)\sigma'(\lambda)d\lambda + m_0
\]
(3.12.2)
\[= \alpha_n b_n t^{-\frac{n}{2}} + \alpha_{n-2} b_{n-2} t^{-\frac{n-2}{2}} + \cdots + \alpha_1 b_1 t^{-\frac{1}{2}} + m_0,
\]
where \( m_0 = m_R(0) - \frac{1}{2} \widetilde{m}_R(0) \) and \( \alpha_k = k \int_0^\infty x^{k-1}e^{-x^2}dx = \Gamma\left(\frac{k}{2} + 1\right) \neq 0. \)
Comparison with (3.11.41) and (3.11.42) shows that \( b_n = 0 \) and
\[b_{n-2} = -\beta_{n-2} \int_{\mathbb{R}^n} V(x)dx, \quad b_{n-4} = \beta_{n-4} \int_{\mathbb{R}^n} V(x)^2dx \neq 0.
\]
This gives an immediate contradiction when \( n = 3 \) as (3.12.2) contradicts the formula (3.11.28).

2. To obtain a contradiction for \( n > 3 \). We consider the behaviour of \( \sigma(\lambda) \) as \( \lambda \to 0 \). Suppose that \( R_V(\lambda) \) is entire. Theorem 3.17 shows
\[R_V(\lambda)(Ve^{i\lambda(\cdot,\omega)})(x) = B(\lambda, x, \omega),
\]
where \( B \) is holomorphic in \( \lambda \) near 0 and smooth in \( x \in \mathbb{R}^n, \omega \in S^{n-1}. \) The formula (3.7.8) then shows that
\[\|A(\lambda)\|_{L_1} = O(\lambda^{n-2}), \quad \lambda \to 0.\]
Using this we see that near 0,
\[ 2\pi i \sigma(\lambda) = \log \det(I + A(\lambda)) = \text{tr} \log(I + A(\lambda)) = \mathcal{O}(\|A(\lambda)\|_{L^1}) = \mathcal{O}(\lambda^{n-2}). \]
Comparing this with (3.12.1) we see that \( \sigma(\lambda) = b_{n-2} \lambda^{n-2} \). But this contradicts the fact that \( b_{n-4} \neq 0 \).

3. It remains to show that the number of resonances is infinite. Again we proceed by contradiction using Theorems 3.54 and 3.67. Suppose that there exists a finite number of non-zero resonances. Hence suppose that \(-\mu_1^2 \leq \cdots \leq -\mu_{\lambda_1}^2 < 0, \mu_k > 0\), are the negative eigenvalues of \( PV \) and that \( \rho_j, \rho_j < 0, j = 1, \cdots, J_1, \lambda_j \neq -\lambda_j, j = 1 \cdots, J_2 \) are a finite set of resonances with \( \text{Re} \lambda_j > 0 \). Factorization (3.10.4) shows that
\[
\det S(\lambda) = (-1)^m e^{g(\lambda)} \prod_{k=1}^{K'} \frac{\lambda + i\mu_k}{\lambda - i\mu_k} \prod_{j=1}^{J_1} \frac{\lambda + i\rho_j}{\lambda - i\rho_j} \prod_{j=1}^{J_2} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}.
\]
Hence for \( \lambda \in \mathbb{R} \) and with \( b(\lambda) := g'(\lambda)/2\pi i \),
\[
\sigma'(\lambda) - g'(\lambda) = -\frac{1}{\pi} \sum_{k=1}^{K'} \frac{\mu_k}{\lambda^2 + \mu_k^2} - \frac{1}{\pi} \sum_{j=1}^{J_1} \frac{\rho_j}{\lambda^2 + \rho_j^2} - \frac{1}{\pi} \sum_{j=1}^{J_2} \left( \frac{\text{Im} \lambda_j}{|\lambda - \lambda_j|^2} + \frac{\text{Im} \lambda_j}{|\lambda + \lambda_j|^2} \right),
\]
That means that
\[
\int_0^\infty (\sigma'(\lambda) - g'(\lambda)) \, d\lambda = -\frac{1}{2} K' + \frac{1}{2} J_1 + J_2.
\]
On the other hand, (3.11.36) shows that the left hand side is equal to \(-K' - m_0, m_0 := m_R(0) + \frac{1}{2} \tilde{m}_R(0) \geq \frac{1}{2} m_R(0) \geq 0\), and hence,
\[
0 \geq -\frac{1}{2} K' - m_0 = \frac{1}{2} J_1 + J_2 \geq 0,
\]
which means that both sides vanish. Since we showed that some resonances exist this gives a contradiction. \( \square \)

3.13. NOTES

This chapter presented odd dimensional potential scattering from the perspective of the study of resonances. For a direct treatment of obstacle scattering in dimension three see Taylor [TaII, Chapter 9].

Results in §3.1 are classical. The proof of Theorem 3.1 comes from Vodev [Vo92]. The contour deformation argument in §3.1.4 is a baby version of homological conditions for the existence of lacunas for hyperbolic equations
due to Petrovsky – see Atiyah–Bott–Gårding [ABG70] for a detailed presentation. We learned this many years ago from Johannes Sjöstrand – see [Sj02, §2.1]. Exercise 3.3 was suggested by Gilles Carron.

The discussion of the resonance of zero in §3.3 can be used as an introduction, in the spirit of PDE, to classical work of Jensen–Kato [JK79] (where (3.3.21) is described using the operator given in (3.3.23)). For further discussion of the threshold behaviour for non-compactly supported potentials see Jensen–Nenciu [JN01], Rodnianski–Tao [RT15] and references given there.

The proof of Theorem 3.27 is based on ideas of Melrose who proved the bound

$$\sum \{ m_R(\lambda) : |\lambda| \leq r \} \leq C V r^{n+1}.$$  

The optimal bound (3.4.6) was proved in [Zw89b]. Our presentation uses a substantial simplification of the argument due to Vodev [Vo92] – see Chapter 4 for further applications of these methods.

For the early history of Rellich’s uniqueness theorem, Sommerfeld radiation patterns and of outgoing solution see Wilcox [Wi56]. Our definition that $u = R_0(\lambda) g$ for $g \in \mathcal{E}'(\mathbb{R}^n)$ is equivalent to the more classical definition which in dimension three states that for $R_0$ sufficiently large and $|x| > R_0$,

$$(3.13.1) \quad u(x) = \frac{1}{4\pi} \int_{\partial B(0,R_0)} (u(y) \partial_r \left( \frac{e^{i\lambda|x-y|}}{|x-y|} \right) - \frac{e^{i\lambda|x-y|}}{|x-y|} \partial_r u(y)) dS(y),$$

see Exercise 3.6.

The class of examples in Theorem 3.29 was constructed by Christiansen [Ch06].

The interpretation of the (absolute) scattering matrix as mapping the the incoming “boundary data” to incoming “boundary data” was emphasized by Melrose in many geometric settings [Me95]. Here the boundary refers to the boundary at infinity. The more traditional interpretation in which the scattering matrix provides a mapping between distorted plane waves (see Theorem 3.47) is given in (3.8.12). In time dependent scattering theory the scattering operator, $S$, is defined using wave operators:

$$W_\pm u := \lim_{t \to \pm \infty} e^{itP_V} e^{-itP_0} u, \quad u \in L^2(\mathbb{R}^n), \quad S := W_+ W_-.$$  

The scattering operator commutes with $P_0 = -\Delta$ (as formally follows from the definitions of $W_\pm$ which intertwine $P_V$ and $P_0$) and hence we can decompose it using the spectral decomposition of $-\Delta$:

$$S = \int_0^\infty S(\lambda) dE_\lambda^0, \quad f(-\Delta) = \int_0^\infty f(\lambda^2) dE_\lambda^0.$$
and our scattering matrix $S(\lambda)$ is produced. Lax–Phillips [LP68], Reed–Simon [RS79], Hörmander [HöII, §14.4], Newton [N02], Yafaev [Ya92],[Ya09] and Taylor [TaII, Chapter 8] can be consulted for different mathematical perspectives on the subject. Analytic properties of the scattering matrix were discussed early on by Jensen [Je80b].

Since this is books is devoted to the study of scattering resonance we focus on the stationary, energy dependent scattering matrix. The general formula (3.7.18) giving $S(\lambda)$ in terms of the resolvent is valid for general compactly supported perturbations (see Theorem 4.26) and comes from Petkov–Zworski [PZ01].

The trace identity in Theorem 3.46 was proved by Buslaev in [Bu62]. Theorem 3.67 and further references can be found in [Gu84]. The Birman–Kreǐn formula goes back to the classical paper [BK62] and is related to the more general study of spectral shift functions. For that connection and for references see Yafaev [Ya92, Chapter 8] and [Ya09, Chapter 9].

Trace formulas relating the wave group to the resonances were established by Lax–Phillips [LP78] and Bardos–Guillot–Ralston [BGR82] (in the closely related obstacle case) but for $t > 2R$, where supp $V \subset B(0, R)$. The case of $t > 0$ was established by Melrose [Me82] and generalized by Sjöstrand–Zworski [SZ94]. A different proof with a more precise statement at $t = 0$ was given in [Zw97] (see also [GZ97]) and our exposition provides a more detailed account of the argument there. The argument for obtaining the correct power of $t$ in (3.10.2) based on Lemma 3.56 was suggested by Jeff Galkowski.

Lemma 3.55 reverse engineers [Me88, §4]. The terms in the definition (3.10.12) there can be recognized as the Breit–Wigner Lorentzians (1.1.1). Further analysis can lead to a justification of the Breit–Wigner formulas in the presence of many resonances – see [PZ99],[PZ01]. For the Breit-Wigner approximation in the semiclassical limit and for isolated resonances see Gérard–Martinez–Robert [GMR89].

Theorem 3.67 is due to Colin de Verdière [CdV81a] when $n = 3$ and Guillopé [Gu84] for other dimensions, see also Buslaev [Bu75] and Christiansen [Ch98] and references given there. Analysis of heat trace asymptotics follows Hitrik–Polterovich [HP03] where more general potentials were considered – see that paper for references and also [SZ16] for a more direct approach to heat trace expansions.

That a smooth compactly supported potential (or any superexponentially decaying potential) in any odd dimension has infinitely many resonances was proved in [SZ96] but the method there was less direct.
There have been many improvements since. Christiansen and Hislop [CH05] proved that for a generic $L^\infty_{\text{comp}}(\mathbb{R}^n, \mathbb{R})$ (or $C^\infty_{\text{c}}(\mathbb{R}^n, \mathbb{R})$) potential the exponent $n$ in the polynomial bound (3.4.6) is optimal. That relied on the existence of a lower bound given by (3.4.15) and is also true for generic complex valued potentials. That paper can be consulted for intermediate results on lower bounds. Here we mention Christiansen [Ch99] and Sá Barreto [Sá01] who proved that
\[
\limsup_{r \to \infty} \frac{N(r)}{r} > 0,
\]
and their methods inspired our presentation.

More recently Smith–Zworski [SZ16] showed that any $V \in H^{n-3/2}(\mathbb{R}^n) \cap L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$, has infinitely many resonances and any $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R})$ has some resonances (all for $n$ odd).

### 3.14. EXERCISES

**Section 3.1**

We assume here that $n \geq 3$ is odd and write $R_0(\lambda, x, y) = R_0(\lambda, x - y)$ for the Schwartz kernel of the free resolvent.

1. Show that for any $r \in \mathbb{R}$ and $c_0$ there exists a constant $C_1$ such that for $\text{Im } \lambda \geq c_0$ we have
\[
\|\langle x \rangle^r R_0(\lambda) \langle x \rangle^{-r} \|_{L^2 \to L^2} \leq C_1,
\]
that is,
\[
R_0(\lambda) : \langle x \rangle^{-r} L^2(\mathbb{R}^2) \to \langle x \rangle^{-r} L^2(\mathbb{R}^n).
\]

**Hint.** Use Theorem 3.3 and Schur’s criterion (see the proof of the Lemma 3.7) noting that $\langle x \rangle^{-r} \langle y \rangle^{r} \leq \langle x - y \rangle^{r}$. 

2. Show that for any $C_0 > 0$ there exists $C_1$ such that for such that for $|\lambda|/C_0 \leq \text{Im } \lambda \leq C_0$,
\[
\|R_0(\lambda, \bullet)\|_{L^q(\mathbb{R})} \leq C_1 |\lambda|^{-2+n(q-1)/q}, \quad 1 \leq q < \frac{n}{n-2}.
\]

3. Use the definition of $P_n$ in Theorem 3.3 to show that $\lambda \mapsto R_0(\lambda)^2 = \partial_\lambda R_0(\lambda)/2\lambda$ is analytic near 0 as an operator from $L^2_{\text{comp}} \to L^2_{\text{loc}}$. Conclude that for $n \geq 5$, Lemma 3.6 can be improved to
\[
\|\rho R_0(\lambda) R_0(\lambda) \rho\|_{L^2 \to H^k} \leq C_1, \quad \rho \in C^\infty_{\text{c}}(\mathbb{R}^n),
\]
for $0 \leq |\lambda| \leq C_0 \leq \frac{1}{2} \text{Im } \lambda_0 \leq 2C_0$, $\text{Im } \lambda \geq 0$. 


4. Suppose that $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$. Show directly that the Schwartz kernel of $R_V(\lambda)$ satisfies

\begin{equation}
R_V(\lambda, x, y) = R_V(\lambda, y, x),
\end{equation}

in the sense of distributions – that is $R_V(\lambda) = j^* R_V(\lambda) \in \mathcal{D}'(\mathbb{R}_x^n \times \mathbb{R}_y^n)$, $j(x, y) = (y, x)$.

**Hint:** it is enough to prove (3.14.4) for $\text{Im} \lambda \gg 1$ and that follows from having

\[
\int_{\mathbb{R}^n} R_V(\lambda) f_1(x) f_2(x) dx = \int_{\mathbb{R}^n} f_1(x) R_V(\lambda) f_2(x) dx, \quad f_j \in L^2(\mathbb{R}^n), \quad \text{Im} \lambda \gg 1.
\]

But for that we can put $F_j(x) := R_V(\lambda) f_j \in H^2$, and use integration by parts: $\int F_1(P_V - \lambda^2) F_2 = \int (P_V - \lambda^2) F_1 F_2$.

Section 3.6

5. Prove Theorem 3.37.

**Hint:** (i) $\Rightarrow$ (ii) is obvious and (iv) $\Rightarrow$ (i) follows from the expansion (3.1.20). Since for $\lambda \in \mathbb{R} \setminus \{0\}$, $R_V(\lambda) \rho = R_0(\lambda) \rho (I + VR_0(\lambda) \rho)^{-1}$, and hence $(I + VR_0(\lambda) \rho)^{-1} (R_0(\lambda)$ is injective on $L^\infty_{\text{comp}}(\mathbb{R}^n)$) have no not poles (Theorem 3.33), we see that (iii) $\Rightarrow$ (iv). We finally get (ii) $\Rightarrow$ (iii) by applying Theorem 3.35 to $u - R_V(\lambda) f$. For the last statement use elliptic regularity: for $V$ and $f$ smooth, $u$ is smooth.

6. Let $R_0(\lambda, x, y)$ be the Schwartz kernel of $R_0(\lambda)$ and suppose that $u = R_0(\lambda) f$ where $f \in \mathcal{E}'(\mathbb{R}^n)$ and $\text{Im} \lambda > 0$. Suppose that $\mathcal{O}$ is a bounded open set with a smooth boundary, supp $f \subseteq \mathcal{O}$. Show that for $u \notin \mathcal{O}$,

\begin{equation}
u(x) = \int_{\partial \mathcal{O}} (u(y) \partial_\nu R_0(\lambda, x, y) - R_0(\lambda, x, y) \partial_\nu u(y)) dS(y),\end{equation}

where $\partial_\nu = \langle \nu(y), \partial \rangle$, $\nu(y)$ is the outward unit normal vector to $\partial \mathcal{O}$ and $dS(y)$ is the surface measure on $\partial \mathcal{O}$. Deducce (3.13.1).

7. Show that if $(P_V - \lambda^2) u = g$, $g \in \mathcal{E}'(\mathbb{R}^n)$ and (3.14.5) holds then there exists $f \in \mathcal{E}'(\mathbb{R}^n)$ such that $u = R_0(\lambda) f$.

Section 3.7

8. Suppose that $\lambda > 0$. For a given $f \in C^\infty(S^{n-1})$ construct $v_0$ such that

\begin{equation} (-\Delta - \lambda^2) v_0 \in \mathcal{D}'(\mathbb{R}^n), \quad v_0(r \theta) \sim \frac{e^{i \lambda r}}{r^{n-2}} f(\theta). \end{equation}

Show that $v_0 = R_V(\lambda)(P_V - \lambda^2) v_0$. 

Section 3.2
9. Show that for \( n = 3 \), \( \det S(0) = (-1)^m \) where \( m \) is the multiplicity of resonance at zero and that for \( n \geq 5 \), \( \det S(0) = 1 \).

10. Suppose that \( V \in L^\infty(B(0,R);\mathbb{R}) \) and that that \( i\mu_j, j = 1, \ldots, J \), are the poles of the scattering matrix with \( \mu_j > 0 \). Then

\[
(3.14.6) \quad \|S(\lambda)\| \leq e^{2R \Im \lambda} \prod_{j=1}^{J} \frac{\lambda + i\mu_j}{\lambda - i\mu_j}, \quad \Im \lambda > 0.
\]

**Hint:** Consider

\[
S_1(\lambda) := e^{2R i \lambda} \prod_{j=1}^{J} \frac{\lambda - i\mu_j}{\lambda + i\mu_j} S(\lambda),
\]

which is holomorphic in \( \Im \lambda \geq 0 \) and \( \|S_1(\lambda)\| = 1 \) for \( \Im \lambda = 0 \). The bound (3.14.6) is equivalent to \( \|S_1(\lambda)\| \leq 1 \). Representation of the scattering matrix (3.7.8) shows that \( \|S_1(\lambda)\| \leq C(1 + |\lambda|^N) \) for \( \Im \lambda > 0 \). Applying the Phragmén–Lindelöf principle to \( S_1(\lambda) \) in the upper half plane (see for instance [Ti86, §5.61]) gives \( \|S_1(\lambda)\| \leq 1, \Im \lambda \geq 0 \), and that proves (3.14.6).

Section 3.11

11. Prove the following refinement of Lemma 3.52:

\[
\text{tr}(e^{-tP_V} - e^{-tP_0}) = \sum_{k=1}^{K} e^{-tE_k} + \frac{1}{2} \widetilde{m}_R(0) + t^{-\frac{1}{2} - \frac{(n-5)}{2}} \sum_{k=0}^{K} t^{-k} h_k + O(t^{-K - \frac{1}{2}}),
\]

as \( t \to +\infty \). What are the improvements if there is no zero eigenvalue or resonance?

**Hint:** Use Theorem 3.51 and the facts that \( \sigma'(\lambda) \) is polynomially bounded (Theorem 3.62), \( \sigma'(\lambda) = O(\lambda^{n-5}) \) for \( n \geq 5 \) and \( O(1) \) for \( n = 3 \) near 0 (use Theorem 3.58 and the structure of the resolvent at 0 from §3.3).
Part 2

GEOMETRIC SCATTERING
In Chapters 2 and 3 we studied general properties of resonances in scattering by compactly supported potential. More general compactly supported perturbations include metric perturbations and obstacle scattering. They offer many new interesting and relevant physical features such as presence of trapping – see Fig. 4.1.

For general results of the type seen in the case of potential scattering it is convenient to replace a specific perturbation by an abstractly defined \textit{black box} perturbation.

The following table shows the basic differences and analogies in the case when $n$ is odd.

Here $P$ denotes a self-adjoint operator equal to $-\Delta$ outside $B(0, R_0)$ – see Section 4.1 for precise assumptions. The operator $P$ is assumed to act on a Hilbert space $\mathcal{H}$ with an orthogonal decomposition $\mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0))$. The orthogonal projection onto the first component – the \textit{black box} – is denoted $1_{B(0, R_0)}$. 
### Figure 4.1. An example of trapped trajectories in obstacle scattering.

<table>
<thead>
<tr>
<th>$-\Delta + V$</th>
<th>Black Box</th>
</tr>
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<tr>
<td>Meromorphy of the resolvent $R_V(\lambda) : L^2_{\text{comp}} \to L^2_{\text{loc}}$; Theorem 3.8.</td>
<td>If $\mathbb{1}<em>{B(0,R_0)}(P - i)^{-1}$ is compact then $R(\lambda) : \mathcal{H}</em>{\text{comp}} \to \mathcal{H}_{\text{loc}}$ is meromorphic; Theorem 4.4.</td>
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<tr>
<td>Upper bound on the number of resonances, $N(r) \leq C r^n$; Theorem 3.27.</td>
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<tr>
<td>Trace formula for resonances; Theorem 3.53.</td>
<td>Trace formulae hold if for some $k$ $\mathbb{1}_{B(0,R_0)}(P - i)^{-k} \in L_1(\mathcal{H}, \mathcal{H})$; [SZ94], [Zw97].</td>
</tr>
<tr>
<td>Pole free regions; Theorem 3.10</td>
<td>Geometric assumptions about the classical flow are needed; Theorems 4.43, 6.10, 6.16, [Ma02b], [SZ07a, §3].</td>
</tr>
<tr>
<td>Resonance expansions of waves; Theorem 3.11</td>
<td>Delicate when there are no large pole free regions; Theorem 7.20, [TZ00].</td>
</tr>
</tbody>
</table>
4.1. GENERAL ASSUMPTIONS

Let $\mathcal{H}$ be a complex Hilbert space with an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{R_0} \oplus L^2(\mathbb{R}^n \setminus B(0, R_0)).$$

Here $R_0 > 0$ is fixed and

$$B(x, R) = \{ y \in \mathbb{R}^n : |x - y| < R \}.$$

The orthogonal projections on the first and the second summands in (4.1.1) can be thought of as restrictions of elements of $\mathcal{H}$ to $B(0, R_0)$ and $\mathbb{R}^n \setminus B(0, R_0)$:

$$u \mapsto 1_{B(0,R_0)}u =: u|_{B(0,R_0)} \in \mathcal{H}_{R_0},$$

$$u \mapsto 1_{\mathbb{R}^n \setminus B(0,R_0)}u =: u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2(\mathbb{R}^n \setminus B(0,R_0)).$$

If $\chi \in L^\infty(\mathbb{R}^n)$ and $\chi \equiv c_0 \in \mathbb{C}$ (is equal to a constant) on $B(0, R_0)$ then we define

$$\chi u := c_0 \left( u|_{B(0,R_0)} \right) + \left( \chi|_{\mathbb{R}^n \setminus B(0,R_0)} \right) \left( u|_{\mathbb{R}^n \setminus B(0,R_0)} \right),$$

where the restriction of $\chi$ is the restriction of a function in $L^\infty(\mathbb{R}^n)$ to a subset of $\mathbb{R}^n$.

We define a smaller space of compactly supported elements of $\mathcal{H}$ as

$$\mathcal{H}_{\text{comp}} := \{ u \in \mathcal{H} : u|_{\mathbb{R}^n \setminus B(0,R_0)} \in L^2_{\text{comp}}(\mathbb{R}^n \setminus B(0,R_0)) \},$$

and a larger spaces of vectors locally in $\mathcal{H}$:

$$\mathcal{H}_{\text{loc}} := \mathcal{H}_{R_0} \oplus L^2_{\text{loc}}(\mathbb{R}^n \setminus B(0,R_0)).$$

We now assume that $P(h), \ 0 < h \leq 1$, is a family of unbounded self-adjoint operators, $P(h) : \mathcal{H} \rightarrow \mathcal{H}$, with the domain $\mathcal{D} \subset \mathcal{H}$, independent of $h$. We assume that

$$1_{\mathbb{R}^n \setminus B(0,R_0)} \mathcal{D} \subset H^2(\mathbb{R}^n \setminus B(0,R_0)).$$

Outside of the “black box” $B(0, R_0)$ the operator is equal to the semiclassical Laplacian in the following sense:

$$1_{\mathbb{R}^n \setminus B(0,R_0)}(P(h)u) = -h^2 \Delta(u|_{\mathbb{R}^n \setminus B(0,R_0)}), \quad u \in \mathcal{D},$$

where the right hand side defines an element of $L^2(\mathbb{R}^n \setminus B(0,R_0))$ thanks to (4.1.4).

Condition (4.1.4) is complemented by the condition

$$v \in H^2(\mathbb{R}^n), \quad v|_{B(0,R_0+\epsilon)} \equiv 0 \quad \text{for some } \epsilon > 0 \quad \Longrightarrow \quad v \in \mathcal{D}.$$

(Since $v = 1_{\mathbb{R}^n \setminus B(0,R_0)} v$, $v$ defines an element of $\mathcal{H}$ and hence this statement makes sense.)
For vectors \( v \) satisfying the condition in (4.1.6) we have
\[
P(h)v = -\hbar^2 \Delta v. \tag{4.1.7}
\]
In fact, if \( u \in \mathcal{D} \) then \( u|_{\mathbb{R}^n \setminus B(0,R_0)} \in H^2(\mathbb{R}^n \setminus B(0,R_0)) \) and (recalling that \( v \in H^2(\mathbb{R}^n) \) and that it vanishes in \( B(0,R_0 + \epsilon) \))
\[
\langle P(h)v, u \rangle_{\mathcal{H}} = \langle v, P(h)u \rangle_{\mathcal{H}} = \langle v, (P(h)u)|_{\mathbb{R}^n \setminus B(0,R_0)} \rangle_{L^2(\mathbb{R}^n \setminus B(0,R_0))} = \langle v, -\hbar^2 \Delta(v|_{\mathbb{R}^n \setminus B(0,R_0)}) \rangle_{L^2(\mathbb{R}^n \setminus B(0,R_0))} = \langle -\hbar^2 \Delta v, u \rangle_{\mathcal{H}}.
\]

We equip \( \mathcal{D} \) with (\( h \)-dependent) Hilbert space norms given by
\[
\| u \|_{D_h}^2 := \| u \|_{\mathcal{H}}^2 + \| P(h)u \|_{\mathcal{H}}^2, \quad u \in \mathcal{D}, \tag{4.1.8}
\]
using \( \| u \|_{\mathcal{H}} \) when \( h = 1 \). Using the functional calculus of \( P(h) \) (see §B.2) we can define more general spaces \( \mathcal{D}_\alpha \) with norms
\[
\| u \|_{\mathcal{D}_h}^\alpha := \| (P(h) + i)^\alpha u \|_{\mathcal{H}}, \quad u \in \mathcal{D}, \tag{4.1.9}
\]
From (4.1.5) it follows that for \( \varphi \in C_\infty(\mathbb{R}^n \setminus B(0,R_0)) \),
\[
(4.1.10) \quad u \in \mathcal{D}_h^\alpha \implies \varphi u \in H_{\mathcal{H}}^\alpha(\mathbb{R}^n)
\]
The spaces \( \mathcal{D}_{\text{comp}} \) and \( \mathcal{D}_{\text{loc}} \) are defined using (4.1.2) and (4.1.3)
\[
\mathcal{D}_{\text{comp}} := \mathcal{D} \cap \mathcal{H}_{\text{comp}}, \tag{4.1.11}
\]
\[
\mathcal{D}_{\text{loc}} := \{ u \in \mathcal{H}_{\text{loc}} : \chi \in C_\infty(\mathbb{R}^n), \chi|_{B(0,R_0)} \equiv 1 \Rightarrow \chi u \in \mathcal{D} \}.
\]
Since \( \mathcal{D} \) is dense in \( \mathcal{H} \) (\( P \) is a self-adjoint operator – see §B.1.2) we also see that \( \mathcal{D}_{\text{comp}} \) is dense in \( \mathcal{H}_{\text{comp}} \) in the sense that for each \( u \in \mathcal{H}_{\text{comp}} \) there exists \( u_j \in \mathcal{D}_{\text{comp}} \) such that \( u_j \to u \) in \( \mathcal{H} \). The same conclusion holds for \( \mathcal{D}^k_{\text{comp}} \) where \( \mathcal{D}^k \) is the domain of \( P^k \).

Finally we assume
\[
(4.1.12) \quad \mathbb{1}_{\mathbb{R}^n \setminus B(0,R_0)}(P(h) + i)^{-1} \text{ is compact.}
\]

**DEFINITION 4.1 (Black box Hamiltonians).** A family of unbounded selfadjoint operators, \( P(h) \), \( 0 < h < 1 \), on a complex Hilbert \( \mathcal{H} \) satisfying (4.1.1) is called a semiclassical black box Hamiltonian if (4.1.4),(4.1.5),(4.1.6) and (4.1.12) hold. If \( P \) satisfies (4.1.4),(4.1.5),(4.1.6) and (4.1.12) for \( h = 1 \) we call it a black box Hamiltonian.

**REMARK.** A black box formalism is also possible for other operators than \( -\hbar^2 \Delta \) – see Sjöstrand [Sj96a] for the development of that theory and §4.7 for more references.
EXAMPLES. 1. Potential scattering. Let \( V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}) \) with \( \text{supp} V \subset B(0, R_0) \). If \( \mathcal{H} = L^2(\mathbb{R}^n) \), \( \mathcal{D} = H^2(\mathbb{R}^n) \) and \( P(h) := -h^2 \Delta + V(x) \) then all the black box assumptions are satisfied. This is the case of scattering by compactly supported potentials presented in Chapters 2 and 3. Assumptions (4.1.1), (4.1.4), (4.1.5) and (4.1.6) are also satisfied for more singular compactly supported potentials for which \( P(h) = -h^2 \Delta + V(x) \) has self-adjoint extensions. For instance we can take \( V \geq 0, V \in L^2(0, R_0) \). However, (4.1.12) may not hold.

2. Obstacle scattering. Suppose that \( \mathcal{O} \subset \overline{B(0, R_0)} \) is an open set such that \( \partial \mathcal{O} \) is a smooth hypersurface in \( \mathbb{R}^n \). Let \( \mathcal{H} = L^2(\mathbb{R}^n \setminus \mathcal{O}) \), and

\[
\mathcal{D} = H^2(\mathbb{R}^n \setminus \mathcal{O}) \cap H^1_0(\mathbb{R}^n \setminus \mathcal{O}) = \{ u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : u|_{\partial \mathcal{O}} = 0 \}
\]

and \( P = -\Delta \) (the self-adjoint Dirichlet Laplacian on \( \mathbb{R}^n \setminus \mathcal{O} \)). This is the case of Dirichlet obstacle scattering.

We can also take the Neumann Laplacian in which case \( \mathcal{D} = \{ u \in H^2(\mathbb{R}^n \setminus \mathcal{O}) : \partial_n u|_{\partial \mathcal{O}} = 0 \} \), where \( \partial_n \) is the normal derivative with respect to \( \partial \mathcal{O} \).

3. Scattering on finite volume surfaces. Let \( (X, g) \) be a complete Riemannian surface with the following decomposition

\[
X = X_1 \cup X_0, \quad \partial X_0 = \partial X_1, \quad \text{smooth}.
\]

and

\[
(X_1, g|_{X_1}) = (S^1_a \times [a, \infty), dr^2 + e^{-2r}d\theta^2), \quad a > 0, \quad S^1 = \mathbb{R}/2\pi \mathbb{Z}.
\]

We define

\[
(4.1.13) \quad \mathcal{H} = \mathcal{H}_a \oplus L^2([a, \infty), dr), \quad \mathcal{H}_a = L^2(X_0) \oplus \mathcal{H}_a^{0},
\]

where (with \( \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \))

\[
\mathcal{H}_a^{0} = \left\{ \{a_n(r)\}_{n \in \mathbb{Z}^*} : a_n \in L^2([a, \infty)), \sum_{n \in \mathbb{Z}^*} \int_a^\infty |a_n(r)|^2 dr < \infty \right\}.
\]

For

\[
u = (u|_{X_0}, \{a_n(r)\}_{n \in \mathbb{Z}^*}, a_0(r)) \in L^2(X_0) \oplus \mathcal{H}_a^{0} \oplus L^2([a, \infty)) = \mathcal{H},
\]

the projections are defined by

\[
\mathbf{1}_{[0,a)} u = u|_{[0,a)} : (u|_{X_0}, \{a_n(r)\}_{n \in \mathbb{Z}^*}) \in \mathcal{H}_a, \quad \mathbf{1}_{[a,\infty)} = u|_{[a,\infty)} = a_0(r).
\]

The norm on \( \mathcal{H} \) is given by

\[
(4.1.14) \quad \| u \|^2_{\mathcal{H}} := \int_{X_0} |u|_{X_0}|^2 d\text{Vol}_g + \sum_{n \in \mathbb{Z}^*} \int_a^\infty |a_n(r)|^2 dr.
\]
The space $H^1(X)$ is defined by requiring that $|du|^2_0$ is integrable with respect to $d\text{Vol}_g$, where $|\cdot|^2_0$ is defined using the dual metric $\sum_{i,j} g^{ij}(x) \xi_i \xi_j$, $(g^{ij}) = (g_{ij})^{-1}$. We can identify $H^1(X)$ with the subset of $\mathcal{H}$ since for $u \in H^1(X)$, the restriction of $u$ to any circle $\{r_0\} \times \mathbb{S}_a^1$, $r_0 > a$, is in $L^2(\mathbb{S}_a^1)$ and can be expanded into Fourier series:

$$H^1(X) \ni u \mapsto \left( u|_{X_0}, \{e^{-r/2}u_n(r)\}_{n \in \mathbb{Z}^*}, e^{-r/2}u_0(r) \right),$$

(4.1.15)

$$u_n(r) := \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r, \theta)e^{-in\theta}d\theta, \quad r > a.$$

The factor $e^{-r/2}$ comes from the change of the Riemannian volume $d\text{Vol}_g|_{X_1} = e^{-r}drd\theta$ to the volume $drd\theta$ used in (4.1.14).

Fubini’s theorem shows for $u \in L^2(X)$, $u_n(r)$ in (4.1.15) exist for almost every $r > a$ and define a function in $L^2$. Hence the map in (4.1.15) extends to an isomorphism

$$\iota : L^2(X) \simeq \mathcal{H}.$$

For $u \in C_c^{\infty}(X)$,

$$\Delta_g u|_{X_1} = \left( e^{-r}\partial_r e^r\partial_r + e^{-2r}\partial^2_{\theta} \right) u(r, \theta)$$

(4.1.16)

$$= \sum_{n \in \mathbb{Z}} (\partial^2_r - \frac{1}{4} - e^{2r}n^2)(e^{-r/2}u_n(r))e^{in\theta+r/2}.$$

We now define $P$ as the Friedrichs extension of $-\Delta_g - \frac{1}{2}$ on $C_c^{\infty}(X) \subset H^1(X)$ identified using (4.1.15) with a subset of $\mathcal{H}$, obtained from the quadratic form

$$Q_g(u, u) = \sum_{n \in \mathbb{Z}} \int_{a}^{\infty} \left( |\partial_r a_n(r)|^2 + n^2 e^{2r}|a_n(r)|^2 \right) dr$$

(4.1.17)

$$+ \int_{a}^{\infty} |\partial_r a_0(r)|^2 dr + \int_{X_0} (|du|^2_0 - \frac{1}{4}|u|^2_0) d\text{Vol}_g,$$

where $a_n(r) := e^{-r/2} \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r, \theta)$. This gives a self-adjoint operator with the domain $\mathcal{D}$ which is the image of $H^2(X)$ under the map $\iota$ in (4.1.15).

The spaces $\mathcal{H}$ and the operator $P$ satisfy the black box assumptions (4.1.1), (4.1.4), (4.1.5) and (4.1.6) with $\mathbb{R}^n \setminus B(0, R_0)$ replaced by a half-line $[a, \infty)$. To check the condition (4.1.12), that is the fact that

(4.1.18)  \[ 1_{[0,a]} (P + i)^{-1} \] is a compact operator on $\mathcal{H}$,

we first note that on the first component of $\mathcal{H}_a$, the operator is

(4.1.19)  \[ 1_{X_0} (P + i)^{-1} = 1_{X_0} (-\Delta_g - \frac{1}{4} + i)^{-1} \iota^{-1} : \mathcal{H} \rightarrow H^2(X_0), \]
and hence it is compact on $\mathcal{H} \simeq L^2(X)$. Denoting by $1_{\mathcal{H}_a}$ the orthogonal projection onto $\mathcal{H}_a$ in (4.1.13) and using (4.1.16) we see that

\begin{equation}
1_{\mathcal{H}_a} (P + i)^{-1} u = \{b_n(r)\}_{n \in \mathbb{Z}^*}, \quad 1_{\mathcal{H}_a} u = \{a_n(r)\}_{n \in \mathbb{Z}^*},
\end{equation}

\[-\partial_r^2 + e^{2r} n^2 + i) b_n(r) = a_n(r), \quad n \neq 0, \quad r > a.

Since $i^{-1}(P + i)^{-1} u \in H^2(X) \subset H^1(X)$, we see from (4.1.17) that

\begin{equation}
\sum_{n \in \mathbb{Z}^*} \int_a^\infty (|\partial_r b_n(r)|^2 + n^2 e^{2r} |b_n(r)|^2) dr \leq C \|u\|_H^2
\end{equation}

But this and an adaptation of the Rellich–Kondrachev criterion (Theorem B.3, Exercise 4.2) shows that the map

\begin{equation}
 u \mapsto \{b_n(r)\}_{n \in \mathbb{Z}^*} \in \mathcal{H}_a
\end{equation}

is compact. That completes the proof of (4.1.18) and shows that $P$ satisfies the assumptions of Definition 4.1.

The same procedure work for surfaces of the form

\begin{equation}
 X = X_0 \cup X_1 \cup \cdots \cup X_N, \quad \partial X_0 = \bigcup_{j=1}^N \partial X_j,
\end{equation}

\[(X_j, g|_{X_j}) \simeq ([a_j, \infty)_r \times (\mathbb{R}/b_j \mathbb{Z})_\theta, dr^2 + e^{-2r} d\theta^2),
\]

$a_j \in \mathbb{R}$, $b_j > 0$.

Example 3 shows that $\mathcal{H}_R_0$ can be a genuinely abstract Hilbert space unlike the geometrically simple spaces $\mathcal{H}_R_0 = L^2(B(0, R_0))$ and $\mathcal{H}_R_0(B(0, R_0) \setminus \mathcal{O})$ in Examples 1 and 2 respectively. We will use Example 3 to illustrate other interesting facts such as the Fermi Golden Rule.

4.2. MEROMORPHIC CONTINUATION

In this section we will prove that for a black box Hamiltonians, $P(h)$, defined in §4.1, the resolvent

\[(P(h) - \lambda^2)^{-1} : \mathcal{H} \rightarrow \mathcal{D}, \quad \text{Im} \lambda > 0, \quad \lambda^2 \notin \text{Spec}(P(h)),
\]

continues meromorphically as an operator from $\mathcal{H}_{\text{comp}}$ to $\mathcal{D}_{\text{loc}}$. When $n$ is odd the continuation is to $\lambda \in \mathbb{C}$ and when $n$ is even to the logarithmic cover of $\mathbb{C}$:

\begin{equation}
 \Lambda = \exp^{-1}(\mathbb{C} \setminus \{0\}).
\end{equation}

Since it does not require much additional work, we temporarily break our simplifying assumption that $n$ is odd.
To obtain meromorphic continuation the condition (4.1.12) is essential. Our taking \( \lambda^2 \) as the spectral parameter is discussed in \( \S 2.1 \). We recall that \( \text{Im}\lambda > 0 \) corresponds to the spectral parameter \( \lambda^2 \in \mathbb{C} \setminus [0, \infty) \).

Since the statement about meromorphic continuation is purely functional analytic in nature we consider \( P = P(1) \) only. We start with two lemmas.

**LEMMA 4.2 (Compactness with a larger cut-off).** Suppose that \( P \) is a black box Hamiltonian. For \( R > R_0 \) let \( 1_{B(0,R)} \) be the orthogonal projection onto \( \mathcal{H}_{R_0} \oplus L^2(B(0,R) \setminus B(0,R_0)) \). Then for \( \lambda^2 \notin \text{Spec}(P) \), \( \text{Im}\lambda > 0 \), the operators

\[
(4.2.2) \quad 1_{B(0,R)}(P - \lambda^2)^{-1}, \quad (P - \lambda^2)^{-1} 1_{B(0,R)}, \quad R \geq R_0,
\]

are compact \( \mathcal{H} \to \mathcal{H} \).

**Proof.** 1. For \( R = R_0 \) and the first operator in (4.2.2) compactness follows from (4.1.12) and the resolvent identity:

\[
1_{B(0,R_0)}(P - \lambda^2)^{-1} = 1_{B(0,R_0)}(P + i)^{-1} + 1_{B(0,R_0)}(P + i)^{-1}(i + \lambda^2)(P - \lambda^2)^{-1}.
\]

(A compact operator composed with a bounded operator gives a compact operator.)

2. To handle the case of \( R > R_0 \) we note that the inclusion

\[
H^2(B(0,R) \setminus B(0,R_0)) \hookrightarrow L^2(B(0,R) \setminus B(0,R_0)),
\]

is compact. Since \( (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D} \), (4.1.4) shows that

\[
(1_{B(0,R)} - 1_{B(0,R_0)})(P - \lambda^2)^{-1}
\]

is compact.

Hence the first operator in (4.2.2) is compact.

3. For \( \text{Im}\lambda > 0 \) we have \( \text{Im}(\bar{\lambda}) > 0 \). Hence \( 1_{B(0,R)}(P - (-\bar{\lambda})^2)^{-1} \) is compact. By taking the adjoint we see that the second operator in (4.2.2) is compact. \( \Box \)

The next lemma is a version of the free resolvent estimate (3.1.24). For future reference we formulate it in the semiclassical version:

**LEMMA 4.3 (Estimates on the resolvent in the physical half plane).** Suppose that \( P(h) \) is a semiclassical black box Hamiltonian. Then for \( k = 0,1,2 \) and \( \tau > 0 \), we have

\[
\|1_{\mathbb{R}^n \setminus B(0,R_0)}(P(h) - i\tau)^{-1}\|_{\mathcal{H} \to H^k_{\mathbb{R}^n \setminus B(0,R_0)}} \leq C\langle\tau\rangle^{k/2}\tau^{-1},
\]

where \( \|u\|_{H^k_{\mathbb{R}^n}(\Omega)}^2 := \sum_{|\alpha| \leq k} \|(hD)^{\alpha}u\|_{L^2(\Omega)}^2 \).
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**Proof.** 1. Self-adjointness of \( P(h) \) implies that
\[
\|(P(h) - i\tau)^{-1}\|_{\mathcal{H} \to \mathcal{H}} = \frac{1}{\tau}.
\]
This and (4.1.1) give (4.2.3) for \( k = 1 \). Since
\[
P(h)(P(h) - i\tau)^{-1} = I + i\tau(P(h) - i\tau)^{-1}
\]
we see that
\[
\|(P(h) - i\tau)^{-1}\|_{\mathcal{H} \to \mathcal{D}_h} \leq C\langle \tau \rangle \frac{1}{\tau},
\]
where the norm on \( \mathcal{D}_h = \mathcal{D} \) is given by (4.1.8). This and (4.1.4) give (4.2.3) for \( k = 2 \).

2. From the interpolation estimate
\[
\|u\|_{H^1_{h}(\mathbb{R}^n \setminus B(0,R_0))} \leq C\|u\|_{L^2(\mathbb{R}^n \setminus B(0,R_0))}^{\frac{1}{2}} \|u\|_{H^2_{h}(\mathbb{R}^n \setminus B(0,R_0))}^{\frac{1}{2}},
\]
(see for instance \([\text{Ev}98, \S 5.4]\)) we obtain (4.2.3) for \( k = 1 \) from the estimates for \( k = 0, 2 \). \( \square \)

We are now ready for the main result of this section. We state it for \( h = 1 \):

**THEOREM 4.4 (Meromorphic continuation for black box Hamiltonians).** Suppose that \( P \) is a black box Hamiltonian in the sense of Definition 4.1. Then
\[
R(\lambda) := (P - \lambda^2)^{-1} : \mathcal{H} \to \mathcal{D} \text{ is meromorphic for } \text{Im} \lambda > 0.
\]
In particular, the spectrum of \( P \) in \((−\infty, 0)\) is discrete.

Moreover, when \( n \) is odd, the resolvent in (4.2.4) extends to a meromorphic family
\[
R(\lambda) : \mathcal{H}_{\text{comp}} \to \mathcal{D}_{\text{loc}}, \quad \lambda \in \mathbb{C}.
\]
When \( n \) is even (4.2.5) holds with \( \mathbb{C} \) replaced by the logarithmic plane \( \Lambda \) defined in (4.2.1).

**Proof.** 1. We first consider the case of \( \text{Im} \lambda > 0 \). We choose \( \chi_0 \in C^\infty_0(\mathbb{R}^n; [0, 1]) \) with the property that
\[
\chi_0(x) \equiv 1 \text{ for } x \in B(0, R_0 + \epsilon), \quad \epsilon > 0.
\]
We then choose \( \chi_j \in C^\infty_0(\mathbb{R}^n; [0, 1]), \ j = 1, 2, 3 \) so that
\[
\chi_j(x) \equiv 1 \text{ for } x \in \text{supp} \chi_{j-1}, \quad \text{supp} \chi_j \subset B(0, R),
\]
for some fixed \( R > R_0 \).
We also choose \( \lambda_0 \) with \( \text{Im} \lambda_0 > 0, \lambda_0^2 \notin \text{Spec}(P) \) and define
\[
Q(\lambda, \lambda_0) := Q_0(\lambda) + Q_1(\lambda_0),
\]
(4.2.7)
\[
Q_0(\lambda) := (1 - \chi_0)R_0(\lambda)(1 - \chi_1), \quad Q_1(\lambda_0) := \chi_2(P - \lambda_0^2)^{-1}\chi_1.
\]

Here \( R_0(\lambda) = (-\Delta - \lambda^2)^{-1} \) is the free resolvent – see §3.1.

From (4.1.7) we deduce that \( P(1 - \chi_0) = -\Delta(1 - \chi_0) \), and hence
(4.2.8) \( (P - \lambda^2)Q_0(\lambda) = 1 - \chi_1 + K_0(\lambda), \quad K_0(\lambda) := -[\Delta, \chi_0]R_0(\lambda)(1 - \chi_1). \)

Since \( \text{Im} \lambda > 0 \) and \( \text{supp} \chi_0 \subset B(0, R) \), we have
\[
\notag [\Delta, \chi_0]R_0(\lambda)(1 - \chi_1) : L^2 \to H^2(B(0, R) \setminus B(0, R_0)).
\]

We conclude that \( K_0(\lambda) \) is a compact operator on \( \mathcal{H} \).

The corresponding contribution of \( Q_1(\lambda_0) \) is
(4.2.9) \( (P - \lambda^2)Q_1(\lambda_0) = \chi_1 + K_1(\lambda, \lambda_0), \quad K_1(\lambda, \lambda_0) := (\lambda_0^2 - \lambda^2)\chi_2(P - \lambda_0^2)^{-1}\chi_1 + [P, \chi_2](P - \lambda_0^2)^{-1}\chi_1. \)

Since \( \text{supp} \chi_2 \subset B(0, R) \), compactness of operators (4.2.2) shows that
\[ K_1(\lambda, \lambda_0) : \mathcal{H} \to \mathcal{H} \text{ is a compact operator.} \]

We remark that this conclusion is valid for any \( \lambda \in \mathbb{C} \) for \( n \) odd and \( \lambda \in \Lambda \) for \( n \) even.

2. Putting (4.2.7),(4.2.8) and (4.2.9) together gives
(4.2.10) \( (P - \lambda^2)Q(\lambda, \lambda_0) = I + K(\lambda, \lambda_0), \quad K(\lambda, \lambda_0) := K_0(\lambda) + K_1(\lambda, \lambda_0), \)

where \( \text{Im} \lambda > 0, K(\lambda, \lambda_0) \) is a compact operator. By Theorem C.5, \( (I + K(\lambda, \lambda_0))^{-1} : \mathcal{H} \to \mathcal{H} \) will form a meromorphic family of operators in \( \text{Im} \lambda > 0 \) if we show that that inverse exists for a suitably chosen \( \lambda \). That will be achieved after making a suitable choice of \( \lambda_0 \).

We choose
(4.2.11) \( \lambda_0 = e^{i\pi/4}\mu, \quad \mu \gg 1, \)

so that Lemma 4.3 (applied with \( h = 1 \)) gives
\[
\notag \| [P, \chi_2](P - \lambda_0^2)^{-1}\chi_1 \|_{\mathcal{H} \to \mathcal{H}} \leq C\| 1_{\mathbb{R}^n \setminus B(0, R_0)}(P - i\mu^2)^{-1} \|_{\mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0, R_0))} \leq \frac{C}{\mu} \ll 1, \quad \mu \gg 1,
\]
and
\[
\notag \| [\Delta, \chi_0]R_0(\lambda_0)(1 - \chi_1) \|_{\mathcal{H} \to \mathcal{H}} \leq C\| 1_{\mathbb{R}^n \setminus B(0, R_0)}(\not\Delta - i\mu^2)^{-1} \|_{\mathcal{H} \to H^1(\mathbb{R}^n \setminus B(0, R_0))} \leq \frac{C}{\mu} \ll 1.
\]
Definitions of $K_0(\lambda)$, $K_1(\lambda, \lambda_0)$ and $K(\lambda, \lambda_0)$ in (4.2.8), (4.2.9) and (4.2.10) respectively then give

$$\|K(\lambda_0, \lambda_0)\|_{\mathcal{H} \to \mathcal{H}} \leq \|K_0(\lambda_0)\|_{\mathcal{H} \to \mathcal{H}} + \|K(\lambda_0, \lambda_0)\|_{\mathcal{H} \to \mathcal{H}} \ll 1.$$  

This implies that for our choice of $\lambda_0$ and for $\lambda = \lambda_0$, $I + K(\lambda, \lambda_0)$ is invertible on $\mathcal{H}$. Hence, Theorem C.5 implies that

$$(I + K(\lambda, \lambda_0))^{-1} : \mathcal{H} \to \mathcal{H}, \quad \text{Im} \lambda > 0,$$

is a meromorphic family of operators. Hence (4.2.10) shows that for Im $\lambda > 0$ and $\lambda^2 \notin \text{Spec}(P)$,

$$(4.2.12) \quad (P - \lambda^2)^{-1} = Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0))^{-1}.$$  

That provides the meromorphy of the left hand side as an operatora $\mathcal{H} \to \mathcal{H}$ in Im $\lambda > 0$. Thus we have proved (4.2.4).

3. We now consider meromorphic extension to the lower half plane and start with the case of $n$ odd. We will use (4.2.10) with $\lambda_0$ given by (4.2.11).

For $\chi_3$ with the property (4.2.6) and $K(\lambda, \lambda_0)$ given in (4.2.10) we have

$$(1 - \chi_3)K(\lambda, \lambda_0) = 0.$$  

Hence

$$(4.2.13) \quad I + K(\lambda, \lambda_0) = (I + K(\lambda, \lambda_0)(1 - \chi_3))(I + K(\lambda, \lambda_0)\chi_3),$$

$$(I + K(\lambda, \lambda_0)(1 - \chi_3))^{-1} = I - K(\lambda, \lambda_0)(1 - \chi_3).$$

From (4.2.12) we see that for Im $\lambda > 0$ (as meromorphic families of operators)

$$(4.2.14) \quad (P - \lambda^2)^{-1} = Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0)\chi_3)^{-1}(I - K(\lambda, \lambda_0)(1 - \chi_3)).$$

4. To conclude the proof of (4.2.5) we observe the following facts valid for $n$ odd:

$$\mathbb{C} \ni \lambda \mapsto K_0(\lambda)\chi_3, \quad \mathbb{C} \ni \lambda \mapsto K_1(\lambda, \lambda_0)$$

are meromorphic families of compact operators on $\mathcal{H}$.

We note that if $n \geq 3$ then the families are in fact holomorphic – see §3.1. When $n = 1$ the only pole is at $\lambda = 0$, see §2.2.

In Step 2 of the proof we showed that $I + K(\lambda_0, \lambda_0)$ is invertible on $\mathcal{H}$. The first equation in (4.2.13) gives

$$(I + K(\lambda_0, \lambda_0)\chi_3)^{-1} = (I + K(\lambda_0, \lambda_0))^{-1}(I + K(\lambda_0, \lambda_0)(1 - \chi_3)).$$

Consequently Theorem C.5 shows that

$$(4.2.15) \quad \lambda \mapsto (I + K(\lambda, \lambda_0)\chi_3)^{-1} : \mathcal{H} \to \mathcal{H}$$
is a meromorphic family of operators on \( \mathbb{C} \). We then note that
\[ \mathbb{C} \ni \lambda \mapsto Q(\lambda, \lambda_0) \] is a meromorphic family of operators \( \mathcal{H} \to \mathcal{D}_{\text{loc}} \), and
\[ \lambda \mapsto I - K(\lambda, \lambda_0)(1 - \chi_3) : \mathcal{H}_{\text{com}} \to \mathcal{H}_{\text{com}} \subset \mathcal{H} \]

In the last mapping property, the only term which is not bounded on \( \mathcal{H} \) comes from \( K_0(\lambda) \). But we do have
\[ [\Delta_0, \chi_3] R_0(\lambda)(1 - \chi_1) : \mathcal{H}_{\text{com}} \to \mathcal{H}_{\text{com}} \subset \mathcal{H} \]

Using (4.2.15) we conclude that
\[ \lambda \mapsto R(\lambda) := Q(\lambda, \lambda_0)(I + K(\lambda, \lambda_0)\chi_3)^{-1}(I - K(\lambda, \lambda_0)(1 - \chi_3)) \]
is a meromorphic family of operators \( \mathcal{H}_{\text{com}} \to \mathcal{D}_{\text{loc}} \) for \( \lambda \in \mathbb{C} \). In view of (4.2.14) this gives the meromorphic extension of \( (P - \lambda^2)^{-1}, \text{Im} \lambda > 0 \), acting between these spaces.

5. For \( n \) even we use the fact that for the logarithmic plane \( \Lambda \),\[ \Lambda \ni \lambda \mapsto R_0(\lambda) : L^2_{\text{com}}(\mathbb{R}^n) \to H^2_{\text{loc}}(\mathbb{R}^n), \]
is a holomorphic family of operators. Hence all the statement in steps 3 and 4 of the proof hold if \( \mathbb{C} \) is replaced by \( \Lambda \) and lead to the construction of the meromorphic extension \( R(\lambda) : \mathcal{H}_{\text{com}} \to \mathcal{D}_{\text{loc}}, \lambda \in \Lambda \).

As an immediate consequence of Theorem 4.4 we have a general result about spectra of black box Hamiltonians:

**THEOREM 4.5 (Spectrum of black box Hamiltonians).** Let \( P \) satisfy the assumptions of Theorem 4.4 and let \( n \) be odd. Then
\[ \text{Spec}(P) = \text{Spec}_{\text{cont}}(P) \cup \text{Spec}_{\text{pp}}(P), \text{ Spec}_{\text{cont}}(P) = [0, \infty), \]
(4.2.16)
\[ \text{Spec}_{\text{pp}}(P) = \{z_j\}_{j=1}^{N_+}, \quad z_j \leq z_{j+1}, \]
where \( N_\pm \) can take values \( \pm \infty \), the multiplicities are finite, and the set \( \{z_j\}_{j=1}^{N_+} \) is discrete.

**REMARK.** The theorem is also valid for \( n \) even but a more detailed analysis is needed near 0 – see Vodev [Vo94a],[Vo94b].

**EXAMPLES.**
1. **Potential scattering.** For \( P(h) = -h^2\Delta + V, V \in L^\infty_{\text{com}}(\mathbb{R}^n) \) we obtain meromorphic extensions of \( (P(h) - \lambda^2)^{-1} : L^2_{\text{com}}(\mathbb{R}^n) \to H^2_{\text{loc}}(\mathbb{R}^n) \) to \( \lambda \in \mathbb{C} \) for \( n \) odd and to \( \lambda \in \Lambda \) for \( n \) even. The proof of Theorem 4.4 is a generalization of the proofs of Theorems 2.2 and 3.8.

2. **Obstacle scattering.** Denote by \( -\Delta_\partial \) the Dirichlet realization of \( -\Delta \) on \( \mathbb{R}^n \setminus \partial \) – see Example 2 in §4.2. Then Theorem 4.4 shows that
\[ (-\Delta_\partial - \lambda^2)^{-1} : L^2(\mathbb{R}^n \setminus \partial) \to H^2(\mathbb{R}^n \setminus \partial) \cap H^1_0(\mathbb{R}^n \setminus \partial), \]
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continues meromorphically to $\mathbb{C}$ when $n$ is odd, and to $\Lambda$ when $n$ is even.

One can formulate this result in terms of Green’s function of $-\Delta_O$. For $\text{Im } \lambda > 0$ we consider the Schwartz kernel $G(\lambda, x, y)$ defined by

$$(-\Delta_O - \lambda^2)^{-1} f(x) = \int_{\Omega} G(\lambda, x, y)f(y)dy, \quad f \in C^\infty_c(\Omega), \quad \Omega := \mathbb{R}^n \setminus \overline{O},$$

where by local elliptic theory (see for instance [TaI, §5.11])

$$G(\lambda, x, y) \in C(\Omega \times \Omega \setminus \Delta(\Omega)), \quad G(\lambda, x, y) = G(\lambda, y, x),$$

(4.2.17)

Theorem 4.4 shows that for fixed $x \neq y$ the function $\lambda \mapsto G(\lambda, x, y)$ extends to a meromorphic function on $\mathbb{C}$ or $\Lambda$ depending on $n$ being odd or even. There are no poles for $\text{Im } \lambda > 0$. Since for fixed $y$, $u(x) := G(\lambda, x, y) - G(-\lambda, x, y)$ solves the equation $(-\Delta_O - \lambda^2)u = 0$, it follows that $u \in C^\infty(\Omega)$ and that the properties (4.2.17) hold for all $\lambda$.

In the case of obstacle scattering a direct approach to meromorphic continuation can be given using boundary layer potentials – see [TaII, §9.7].

3. Scattering on finite volume surfaces. Let $(X, g)$ be a Riemannian surface with finitely many cusps – see Example 3 in §4.1 and (4.1.22). Theorem 4.4 shows that the resolvent of the Laplacian

$$(-\Delta_g - \frac{1}{4} - \lambda^2)^{-1} : L^2(X) \to H^2(X), \quad \text{Im } \lambda > 0,$$

continues meromorphically to $\mathbb{C}$ as an operator $L^2_{\text{comp}}(X) \to H^2_{\text{loc}}(X)$. (Strictly speaking we should use for $R_0(\lambda)$ the resolvent for, say, Dirichlet realization of $-\partial^2_s$ on $[a - 1, \infty)$ – the straightforward modifications are left to the reader.) Since $-\Delta_g \geq 0$, Theorem 4.5 show that there are only finitely many eigenvalues of $-\Delta_g$ in $[0, 1/4]$ and hence

$$\text{Spec}_{\text{cont}}(-\Delta_g) = [1/4, \infty), \quad \text{Spec}_{\text{pp}}(-\Delta_g) = \{E_j\}_{j=0}^N \cup \{z_j\}_{j=1}^M,$$

$$0 = E_0 < E_1 \leq \cdots \leq E_N \leq \frac{1}{4}, \quad \frac{1}{4} < z_1 \leq z_2 \leq \cdots,$$

and $M$ can take the value $+\infty$ (infinitely many eigenvalues embedded in the continuous spectrum) or 0 (no embedded eigenvalues). A famous example for which $M = +\infty$ is given by the modular surface $X = \text{SL}_2(\mathbb{Z})\backslash \text{SL}_2(\mathbb{R})$ – see [LP76]. As we will see for a generic metric $g$, $M = 0$.

The significance of the poles of the continuation of $(-\Delta_g - \frac{1}{4} - \lambda^2)^{-1} : L^2_{\text{comp}}(X) \to H^2_{\text{loc}}(X)$ will be discussed in §4.4 when we define distorted plane waves and the scattering matrix.
DEFINITION 4.6 (Resonances for black box Hamiltonians). Let $P$ be a black box Hamiltonian ($P = P(h)$ or is independent of $h$). Then $\lambda \in \mathbb{C}$ for $n$ odd and $\lambda \in \Lambda$ (the logarithmic plane (4.2.1)) for $n$ even is a resonance of $P$ if $\lambda$ is a poles of the meromorphic extension of $(P - \lambda^2)^{-1}$ given in Theorem 4.4.

The multiplicity of a pole at $\lambda$ is defined as

$$m_R(\lambda) := \text{rank} \oint_{\lambda} R(\zeta) d\zeta,$$

where the integral is over a circle containing no other pole of $R(\zeta)$ than $\lambda$.

The set of resonances will be denoted by $\text{Res}(P)$.

REMARK. The convention of meromorphically extending $(P - \lambda^2)^{-1}$, that is taking $\lambda^2$ as a spectral parameter, is useful when global extensions are considered and is motivated by the wave equation – see 2.1. Sometimes, especially when considering problems motivated by quantum mechanics and the Schrödinger equation, it is more convenient to use $z = \lambda^2$ as spectral parameter. Away from 0 the two conventions are clearly equivalent. When confusion is unlikely we will sometimes write $\text{Res}(P)$ for the image of the set of resonances under the map $\lambda \mapsto z = \lambda^2$. As we will see in Theorem 4.7

$$\lambda \neq 0 \implies m_R(\lambda) = \text{rank} \oint_{\lambda} R(\zeta) 2\zeta d\zeta,$$

so the multiplicities agree when consider $R$ as a function of $\lambda$ or $\lambda^2$.

We remark that the different conventions are unavoidable in a subject touching different disciplines – see §1.1.

The next result describes the structure of the singular part of the resolvent.

THEOREM 4.7 (Singular part of $R_V(\lambda)$ for black box Hamiltonians). In the notation of the definition above suppose that $m_R(\lambda) > 0$, $\lambda \neq 0$. Then there exists $M_\lambda \leq m_R(\lambda)$ such that

$$R(\zeta) = -\sum_{k=1}^{M_\lambda} \frac{(P - \lambda^2)^{k-1}}{(\zeta^2 - \lambda^2)^k} \Pi_\lambda + A(\zeta, \lambda),$$

where $\zeta \mapsto A(\zeta, \lambda)$ is holomorphic near $\lambda$,

$$\Pi_\lambda := -\frac{1}{2\pi i} \oint_{\lambda} R(\zeta) 2\zeta d\zeta, \quad (P - \lambda^2)^{M_\lambda} \Pi_\lambda = 0,$$

and

$$m_R(\lambda) = \text{rank} \Pi_\lambda := \text{dim} \Pi_\lambda(\mathcal{H}_{\text{comp}}).$$
In addition for any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi \equiv 1$ in a neighbourhood of $B(0, R_0)$, we have

\begin{equation}
(4.2.22) \quad m_R(\lambda) = \text{rank} \oint_{\lambda} R(\zeta) \chi 2\zeta d\zeta,
\end{equation}

where the integral is over a circle containing no other pole of $R(\zeta)$ than $\lambda$.

**Proof.** 1. The first two statements (4.2.19) and (4.2.20) follow from steps 1 and 2 of the proof of Theorem 2.4. These can be read without referring to the material of Chapter 2 so we do not reproduce them here.

2. We now prove (4.2.21). The operator $\Pi_\lambda$ has finite rank and hence, for any $k \in \mathbb{N}$,

\begin{equation}
(4.2.23) \quad \Pi_\lambda(\mathcal{H}_{\text{comp}}) = \Pi_\lambda(\mathcal{D}_k^{\text{comp}}).
\end{equation}

(See the comment after (4.1.11).) If the rank of $\Pi_\lambda$ is $N_\lambda$ then there exist

\begin{align*}
V_\lambda &\subset \mathcal{D}_k^{\text{comp}}, \quad W_\lambda \subset \mathcal{D}_k^{\text{comp}}, \quad \dim V_\lambda = \dim W_\lambda = N_\lambda, \\
W_\lambda = \Pi_\lambda(\mathcal{H}_{\text{comp}}) = \Pi_\lambda(V_\lambda), \quad (P - \lambda^2) : W_\lambda \rightarrow W_\lambda.
\end{align*}

We then put

\begin{equation*}
B_\lambda := (\Pi_\lambda|_{V_\lambda})^{-1}(P - \lambda^2)\Pi_\lambda, \quad B_\lambda : \mathcal{H}_{\text{comp}} \rightarrow V_\lambda,
\end{equation*}

and note that in the notation of (4.2.20)

\begin{equation*}
B_\lambda^{N_\lambda} = 0.
\end{equation*}

The residue theorem and (4.2.19) give

\begin{equation}
(4.2.24) \quad \frac{1}{2\pi i} \oint_{\lambda} R(\zeta) d\zeta = \Pi_\lambda \left( \sum_{k=1}^{M_\lambda} (-1)^{k-1} \frac{(2k-2)!}{(k-1)!} (2\lambda)^{-2k+1} B_\lambda^{k-1} \right) = \frac{1}{2\lambda} \Pi_\lambda (I_{V_\lambda} + N_\lambda),
\end{equation}

\begin{equation*}
N_\lambda : \mathcal{H}_{\text{comp}} \rightarrow V_\lambda, \quad N_\lambda^{M_\lambda} = 0.
\end{equation*}

Since $I_{V_\lambda} + N_\lambda : V_\lambda \rightarrow V_\lambda$ is invertible (for instance by a finite Neumann series) it follows that

\begin{equation*}
\dim \left( \frac{1}{2\pi i} \oint_{\lambda} R(\zeta) d\zeta \right) (\mathcal{H}_{\text{comp}}) \geq \dim \Pi_\lambda(V_\lambda) = \dim \Pi_\lambda(\mathcal{H}_{\text{comp}}) = \text{rank} \Pi_\lambda.
\end{equation*}

The opposite inequality is clear from (4.2.24). The two give (4.2.21).
3. To obtain (4.2.22) we will use the proof of Theorem 4.4. In the notation of (4.2.10) we put \( Q(\zeta) := Q(\zeta, \lambda_0) \) and \( K(\zeta) := K(\zeta, \lambda_0) \) (since \( \lambda_0 \) is fixed in (4.2.11)). That gives
\[
R(\zeta) = Q(\zeta) - R(\zeta)K(\zeta).
\]
For \( \chi \in C_c^\infty(\mathbb{R}^n, [0, 1]) \), \( \chi \equiv 1 \) in a neighbourhood of \( B(0, R_0) \), we can choose \( \chi_j, j = 0, 1, 2 \) in the definition of \( K \) – see (4.2.8), (4.2.9) and (4.2.10) – so that \( \chi K(\zeta) = K(\zeta) \). Hence
\[
R(\zeta) = Q(\zeta) - R(\zeta)\chi K(\zeta).
\]
We now use the holomorphy of \( \zeta \mapsto Q(\zeta) \) to write
\[
\frac{1}{2\pi i} \oint_\lambda R(\zeta)2\zeta d\zeta = -\frac{1}{2\pi i} \oint_\lambda R(\zeta)\chi K(\zeta)2\zeta d\zeta
\]
\[
= \Pi_\lambda \chi \left( \frac{1}{2\pi i} \sum_{k=1}^{M_\lambda} \oint_\lambda \frac{(P - \lambda^2)k-1K(\zeta)}{(\zeta^2 - \lambda^2)^k} d\zeta \right)
\]
\[
= \Pi_\lambda \chi K_2(\lambda) = \left( \frac{1}{2\pi i} \oint_\lambda R(\zeta)\chi 2\zeta d\zeta \right) K_2(\lambda),
\]
where
\[
K_2(\lambda) := -\frac{1}{2\pi i} \sum_{k=1}^{M_\lambda} \oint_\lambda \frac{(P - \lambda^2)k-1K(\zeta)}{(\zeta^2 - \lambda^2)^k} d\zeta.
\]
Since \( \partial_\lambda^k K(\lambda) : \mathcal{H}_{\text{comp}} \rightarrow \mathcal{H}_{\text{comp}} \), the residue calculus implies that \( K_2(\lambda) : \mathcal{H}_{\text{comp}} \rightarrow \mathcal{H}_{\text{comp}} \).
Using (4.2.21) it follows that
\[
m_R(\lambda) \leq \dim \left( \oint_\lambda R(\zeta)\chi 2\zeta d\zeta \right) (\mathcal{H}_{\text{comp}})
\]
\[
\leq \dim \left( \oint_\lambda R(\zeta) 2\zeta d\zeta \right) (\mathcal{H}_{\text{comp}}) = m_R(\lambda),
\]
completing the proof of (4.2.22).

**DEFINITION 4.8 (Resonant states).** Let \( \lambda \neq 0 \) be a resonance of \( P \) and let \( \Pi_\lambda \) be given by (4.2.20). Then, in the notation of (4.2.19), an element of \( \mathcal{D}_{\text{loc}} \)
\[
u \in \Pi_\lambda(\mathcal{H}_{\text{comp}}), \quad (P - \lambda^2)u = 0,
\]
is called a resonant state. Also,
\[
v \in \Pi_\lambda(\mathcal{H}_{\text{comp}})
\]
is call a generalized resonant state.

We note that in the notation of (4.2.19) \( (P - \lambda^2)^{M_\lambda}v = 0 \).
4.2. MEROMORPHIC CONTINUATION

THEOREM 4.9 (Characterization of resonant states). A vector $u \in D_{\text{loc}}$ is a resonant state corresponding to $\lambda \in \mathbb{C}\setminus\{0\}$ if and only if $(P-\lambda^2)u = 0$ and there exist $g \in L^2_{\text{comp}}(\mathbb{R}^n)$ and $R > 0$ such that

$$u|_{\mathbb{R}^n \setminus B(0,R)} = R_0(\lambda)g|_{\mathbb{R}^n \setminus B(0,R)}.$$  

**INTERPRETATION.** The theorem provides a stationary characterization of resonant states as outgoing functions. A dynamical interpretation of the outgoing property (4.2.25) in the spirit of Lax–Phillips [LP68] is given as follows.

Suppose that

$$(4.2.26) \quad u_0 := R_0(\lambda)f, \quad f \in D'(\mathbb{R}^3), \quad \text{supp } f \in B(0,R), \quad \lambda \in \mathbb{C},$$

and that $u(t,x) \in C(\mathbb{R}; D'(\mathbb{R}^n), n \geq 3, \text{odd}$ satisfies

$$\square u(x,t) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$(4.2.27) \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = -i\lambda_0u_0(x), \quad x \in \mathbb{R}^n \setminus B(0,R),$$

Then

$$(4.2.28) \quad \text{supp } u \subset \{(t,x) : t < |x| + R\}.$$ 

This means that outgoing initial data in the sense of (4.2.25) gives outgoing solutions of the free wave equation in the sense of (4.2.28) – see Exercise 4.3 for the outline of the proof.

**Proof.** 1. Suppose that $\chi_1 \in C^\infty_c(\mathbb{R}^n)$ is equal to 1 in $B(0,R_0)$ and $\chi \in C^\infty_c(\mathbb{R}^n)$ is equal to 1 on supp $\chi_1$. Then for $\text{Im } \zeta > 0$,

$$(\Delta - \zeta^2)(1 - \chi)R(\zeta)\chi_1 = (P - \zeta^2)(1 - \chi)R(\zeta)\chi_1$$

$$= (1 - \chi)\chi_1 - [P,\chi]R(\zeta)\chi_1$$

$$= -[P,\chi]R(\zeta)\chi_1.$$ 

This implies that for $\text{Im } \zeta > 0$

$$(4.2.29) \quad R_0(\zeta)[P,\chi]R(\zeta)\chi_1 = -(1 - \chi)R(\zeta)\chi_1.$$ 

By analytic continuation (4.2.29) holds for $\zeta \in \mathbb{C}$. 

2. Using (4.2.20) we have from (4.2.29)

\[(1 - \chi)\Pi\lambda\chi_1 = \frac{1}{2\pi i} \int_{\lambda} R_0(\zeta) [P, \chi] R(\zeta) \chi_1 2\zeta d\zeta \]

(4.2.30)

\[= \frac{1}{2\pi i} \int_{\lambda}^{M_\lambda} \sum_{j=1}^{M_\lambda} \frac{R_0(\zeta)}{(\zeta^2 - \lambda^2)^j} 2\zeta d\zeta \left[ P, \chi \right] (P - \lambda^2)^{j-1} \Pi\lambda\chi_1 \]

\[= \sum_{j=1}^{M_\lambda} T_j(\lambda) [P, \chi] (P - \lambda^2)^{j-1} \Pi\lambda\chi_1, \]

where

\[T_j(\lambda) := \frac{1}{(j-1)!} (2\zeta)^{-1}\partial_\zeta (\zeta^2 - \lambda^2)^{j-1} R_0(\zeta) \big|_{\zeta = \lambda}, \quad T_1(\lambda) = R_0(\lambda). \]

3. Now suppose that \( u = \Pi\lambda v, \ v \in \mathcal{H}_{\text{comp}}, \) is a resonant state. Choose \( \chi \) and \( \chi_1 \) in Step 1 so that \( \chi_1 v = v \). Since \( (P - \lambda^2)^j u = 0 \) for \( j > 1 \), (4.2.30) gives

\[(1 - \chi)u = (1 - \chi)\Pi\lambda \chi v = (1 - \chi)\Pi\lambda\chi_1 v = T_0(\lambda) [P, \chi] \Pi\lambda\chi_1 v. \]

Since \( T_0(\lambda) = R_0(\lambda) \) we obtain (4.2.25) with \( g := [P, \chi] \Pi\lambda v \).

4. We now assume that \( u|_{\mathbb{R}^n \setminus B(0, R)} = (R_0(\lambda)g)|_{\mathbb{R}^n \setminus B(0, R)} \) for some \( g \in L^2_{\text{comp}}, \) and that \( (P - \lambda^2)u = 0 \). By applying \(-\Delta - \lambda^2\) to both sides we see that \( \text{supp } g \subset B(0, R) \).

We have \( R(\zeta)(P - \zeta^2)\chi = \chi \), again by meromorphic continuation. Hence, using \( (P - \lambda^2)u = 0 \),

\[\chi u = R(\zeta)(P - \lambda^2 + \lambda^2 - \zeta^2)\chi u \]

(4.2.31)

\[= R(\zeta)[-\Delta, \chi] u + (\lambda^2 - \zeta^2) R(\zeta) \chi u. \]

To analyse \( (1 - \chi)u \) we note that by reversing the roles of \( P \) and \(-\Delta\) in the derivation of (4.2.29) we obtain

\[R(\zeta)[\Delta, \chi] R_0(\zeta) \chi_1 = (1 - \chi) R_0(\zeta) \chi_1. \]

In particular by choosing \( \chi_1 = 1 \) on a neighbourhood of \( B(0, R) \) we see that

\[R(\zeta)[\Delta, \chi] R_0(\zeta) g = (1 - \chi) R_0(\zeta) g. \]

Hence,

\[(1 - \chi) R_0(\zeta) g = R(\zeta)[\Delta, \chi] R_0(\zeta) g \]

(4.2.32)

\[= R(\zeta)[\Delta, \chi] (R_0(\lambda) g + R_0(\zeta) g - R_0(\lambda) g) \]

\[= R(\zeta)[\Delta, \chi] u + R(\zeta)[\Delta, \chi] (R_0(\zeta) g - R_0(\lambda) g). \]
4.3. UPPER BOUNDS ON THE NUMBER OF RESONANCES

We then define
\[ u(\zeta) := (1 - \chi)R_0(\zeta)g + \chi u, \quad u(\lambda) = u. \]

Adding (4.2.31) and (4.2.32) gives
\[ u(\zeta) = R(\zeta) (|\Delta, \chi| (R_0(\zeta) - R_0(\lambda)) g + (\lambda^2 - \zeta^2)\chi u). \]

Dividing by \( \zeta^2 - \lambda^2 \) and integrating over a small positively oriented circle centered at \( \lambda \) we obtain
\[ u = u(\lambda) = \frac{1}{2\pi i} \oint_{\lambda} \frac{u(\zeta)}{\zeta^2 - \lambda^2} 2\zeta d\zeta \]
\[ = \frac{1}{2\pi i} \oint_{\lambda} R(\zeta) \left( \chi u - |\Delta, \chi| \frac{R_0(\zeta) - R_0(\lambda)}{\zeta^2 - \lambda^2} g \right) 2\zeta d\zeta \]
\[ = \Pi_{\lambda} \frac{1}{2\pi i} \oint_{\lambda} \sum_{k=1}^{M_{\lambda}} (P - \lambda^2)^{k-1} \left( \chi u - |\Delta, \chi| \frac{R_0(\zeta) - R_0(\lambda)}{\zeta^2 - \lambda^2} g \right) 2\zeta d\zeta \]
\[ = \Pi_{\lambda} v, \]

where
\[ v := \chi u - \sum_{k=1}^{M_{\lambda}} \frac{1}{k!} (-\Delta - \lambda^2)^{k-1} |\Delta, \chi| \left( (2\lambda)^{-1}\partial\right)^k R_0(\lambda) g \in \mathcal{H}_{\text{comp}}. \]

This proves the claim that \( u \) is a resonant state in the sense of Definition 4.8.

4.3. UPPER BOUNDS ON THE NUMBER OF RESONANCES

In this section we will obtain a far reaching generalization of Theorem 3.27 which gave upper bounds on the number of resonances for \( P = -\Delta + V, \quad V \in L^\infty_{\text{comp}}(\mathbb{R}^n), n \text{ odd}. \)

To formulate the result in the black box setting we introduce a reference operator \( P^\#(h) \). It is defined as follows. Let
\[ \mathbb{T}_{R_1}^n := \mathbb{R}^n / R_1 \mathbb{Z}, \quad R_1 > R_0. \]

In the notation of (4.1.1) we then put
\[ H_{R_1}^\# := \mathcal{H}_{R_0} \oplus L^2(\mathbb{T}_{R_1}^n \setminus B(0, R_0)), \]
where we identified \( B(0, R_0) \subset \mathbb{R}^n \) with its image under the projection \( \mathbb{R}^n \to \mathbb{T}_{R_1}^n \). (We will use the same convention for \( B(0, R_1) \) as well.) The corresponding orthogonal projections are denoted by
\[ u \mapsto 1_{B(0,R_0)} u = u|_{B(0,R_0)}, \quad u \mapsto 1_{\mathbb{T}_{R_1}^n \setminus B(0,R_0)} u = u|_{\mathbb{T}_{R_1}^n \setminus B(0,R_0)}. \]
If \( P(h) \) is a black box Hamiltonian in the sense of §4.1 with domain \( \mathcal{D} \) we define

\[
\mathcal{D}^{\#}_{R_1} := \{ u \in \mathcal{H}^{\#}_{R_1} : \chi \in C_c^\infty(B(0, R_1)), \chi = 1 \text{ near } B(0, R_0) \Rightarrow \chi u \in \mathcal{D}, (1 - \chi)u \in H^2(\mathbb{T}^n_{R_1}) \}
\]

and, for any \( \chi \) with the property in (4.3.2),

\[
P_{R_1}^{\#}(h) : \mathcal{D}^{\#}_{R_1} \to \mathcal{H}^{\#}_{R_1},
\]

\[
P_{R_1}^{\#}(h)u := P(h)(\chi u) + (-h^2 \Delta)((1 - \chi)u).
\]

Assumptions (4.1.4) and (4.1.5) show that this definition is independent of the choice of \( \chi \).

**DEFINITION 4.10 (The reference operator).** For a black box Hamiltonian \( P(h) \) the operator \( P_{R_1}^{\#}(h) \) is called a reference operator. Once we fix \( R_1 > R_0 \) we use notation

\[
P^{\#}(h) : \mathcal{H}^{\#} \to \mathcal{D}^{\#}.
\]

The spaces \( \mathcal{D}^{\#}_{h}^{\alpha} \) are defined as in (4.1.8) and (4.1.9):

\[
\|u\|_{\mathcal{D}^{\#}_{h}^{\alpha}} = \|(P^{\#}(h) + i)^{\alpha}u\|_{\mathcal{H}^{\#}}.
\]

**REMARK.** There are many possible choices for a reference operator. For instance, instead of (4.3.1), (4.3.2) we can take

\[
\mathcal{H}^{\#} := \mathcal{H}_{R_0} \oplus L^2(B(0, R_1) \setminus B(0, R_0)),
\]

\[
\mathcal{D}^{\#} := \{ u \in \mathcal{H}^{\#} : \chi \in C_c^\infty(B(0, R_1)), \chi = 1 \text{ near } B(0, R_0) \Rightarrow \chi u \in \mathcal{D}, (1 - \chi)u \in H^2(B(0, R_1)) \cap H^1_0(B(0, R_1)) \}.
\]

This means that we introduce the Dirichlet boundary condition on at \( \partial B(0, R_1) \). This will be useful in Example 2 below.

**LEMMA 4.11 (Properties of the reference operator).** Suppose \( P^{\#}(h) \) is a reference operator defined by (4.3.3) for some \( R_1 > R_0 \). Then, with \( \mathcal{H}^{\#} \) given by (4.3.1),

\[
P^{\#}(h) : \mathcal{H}^{\#} \rightarrow \mathcal{H}^{\#},
\]

is a self-adjoint operator with domain given by \( \mathcal{D}^{\#} \) defined in (4.3.2).

The resolvent \((P^{\#}(h)+i)^{-1}\) is compact and hence the spectrum of \( P^{\#}(h) \) is discrete.
Proof. In the proof we consider $P = P(1)$ as $h > 0$ is a harmless parameter.

1. The symmetry of $P^\#$ will follow from the definition (4.3.3) and from (4.1.5). To see it, choose $\chi_j \in C_c^\infty(B(0, R_1))$, $\chi_j = 1$ near $B(0, R_0)$, so that, with $\chi$ from (4.3.2),

$\chi_1 \equiv 1$ near supp $\chi$, $\chi \equiv 1$ near supp $\chi_2$.

For $u, v \in D^\#$,

$$\langle P^#u, v \rangle_{H^\#} = \langle P(\chi u), \chi_1 v \rangle_H + \langle -\Delta((1 - \chi)u), (1 - \chi_2)v \rangle_{L^2(T^n)}$$

$$= \langle u, P(\chi_1 v) \rangle_H + ((1 - \chi)u, -\Delta((1 - \chi_2)v))_{L^2(T^n)}$$

$$= \langle u, P(\chi_1 v) - \Delta((1 - \chi_1)v) \rangle_{H^\#} - ((1 - \chi)u, P(\chi_1 v))_{H^\#}$$

$$- \langle \chi u, -\Delta((1 - \chi_2)v) \rangle_{H^\#} + \langle u, -\Delta(1 - \chi_2)v \rangle_{H^\#}$$

$$= \langle u, P^#v \rangle_{H^\#} + \langle u, Qv \rangle_{H^\#},$$

where, using (4.3.7) (and our convention of multiplication operators),

$$Q := (1 - \chi)\Delta(1 + \chi \Delta(1 - \chi_2) - \Delta(1 - \chi_2)$$

$$(\chi_1 - \chi)\Delta + [\Delta, \chi_1] + (\chi - \chi_2)\Delta - [\Delta, \chi_2]$$

$$- (\chi_1 - \chi_2)\Delta - [\Delta, \chi_1] + [\Delta, \chi_2] \equiv 0,$$

It follows that for $u, v \in D^\#$, $\langle P^#u, v \rangle_{H^\#} = \langle u, P^#v \rangle_{H^\#}$, that is $P^#$, is symmetric.

2. According to Theorem B.5 and the definitions following it, self-adjointness of $P^#$ will follow from showing that $D((P^#)^*) \subset D^#$. Hence, suppose that $v \in H^\#$ and that for all $u \in D^#$,

$$\langle P^#u, v \rangle_{H^\#} \leq C\|u\|_{H^\#},$$

That is a characterization of $v \in D((P^#)^*)$ and it implies that there exists $w \in H^\#$ such that $\langle u, w \rangle_{H^\#} = \langle P^#u, v \rangle_{H^\#}$ for all $u \in D^#$. We then have $w := (P^#)^*v \in H^\#$.

Taking $u \in C_c^\infty(T^n \setminus \overline{B(0, R_0)})$, we have $\langle u, w \rangle_{H^\#} = \langle -\Delta u, v \rangle_{H^\#}$, and hence

$$L^2(T^n \setminus B(0, R_0)) \ni w|_{T^n \setminus B(0, R_0)} = -\Delta(v|_{T^n \setminus B(0, R_0)}).$$

It follows that $v|_{T^n \setminus B(0, R_0)} \in H^2(T^n \setminus B(0, R_0))$. With $\chi$ as in step 1 this gives

$$\langle 1 - \chi \rangle v \in H^2(T^n).$$

3. From (4.3.8) we also see that if $\chi_2v_1 = 0$ ($\chi_j$ are as in (4.3.7)) and $(1 - \chi_2)v_1 \in H^2(T^n)$ then $v_1 \in D((P^#)^*)$. As $\chi_2(1 - \chi)v = 0$, (4.3.9) shows

$$v \in D((P^#)^*) \implies v_1 := \chi v = v - (1 - \chi)v \in D((P^#)^*).$$
We now apply (4.3.8) with this \( v_1 \) (noting that \( v_1 \in \mathcal{H} \)) and \( u \in \chi_1 \mathcal{D} \subset \mathcal{D}^\# \): 
\[
\langle Pu, v_1 \rangle_{\mathcal{H}} = \langle P^\# u, v_1 \rangle_{\mathcal{H}} \leq \|u\|_{\mathcal{H}}.
\]
This implies that \( v_1 \in \mathcal{D}(P^*) = \mathcal{D}(P) \) showing that 
\[
\chi v = v_1 \in \mathcal{D}.
\]
Combined with (4.3.9) and the definition (4.3.2) we obtain \( v \in \mathcal{D}^\# \). We proved that \( \mathcal{D}((P^\#)^*) \subset \mathcal{D}^\# \) and as \( P^\# \) is symmetric (step 1), it follows that \( P^\# \) is self-adjoint.

4. To prove compactness of \( (P^\# + i)^{-1} \) we consider a bounded sequence \( u_j \in \mathcal{H}^\# \) and \( v_j := (P^\# + i)^{-1} u_j \in \mathcal{D}^\# \),
\[
\|v_j\|_{\mathcal{H}^\#} + \|P^\# v_j\|_{\mathcal{H}^\#} \leq C.
\]
It follows that \((1 - \chi)v_j \) is a bounded sequence in \( H^2(\mathbb{T}^n) \) which then (see Theorem B.3) has a convergent subsequence in \( L^2(\mathbb{T}^n) \), and hence in \( \mathcal{H}^\# \).

On the other hand, \( \chi v_j \in \mathcal{D} \) and
\[
(P + i) \chi v_j = [-\Delta, \chi] v_j + \chi (P^\# + i)(P^\# + i)^{-1} u_j =: w_j + \chi u_j.
\]
The \( w_j \)'s can be considered as elements of \( L^2(B(0, R_1) \setminus B(0, R_0)) \) and, since \((1 - \chi)v_j \)'s are bounded in \( H^2(\mathbb{T}^n) \), \( w_j \) are bounded in \( L^2(B(0, R_1) \setminus B(0, R_0)) \), and hence in \( \mathcal{H} \). Also \( \chi u_j \) form a bounded sequence in \( \mathcal{H} \). Hence,
\[
\chi v_j = \mathbf{1}_{B(0,R_1)} \chi v_j = \mathbf{1}_{B(0,R_1)}(P + i)^{-1}(w_j + \chi u_j).
\]
Lemma 4.2 shows that \( \mathbf{1}_{B(0,R_1)}(P + i)^{-1} \) is a compact operator which shows that \( \chi v_j \) has a convergent subsequence in \( \mathcal{H} \). It follows that \( v_j \) has a convergent subsequence in \( \mathcal{H}^\# \). \(\square\)

The lemma shows that the spectrum of \( P^\#(h) \) is discrete. To count resonances we make the following assumption about the counting function for the eigenvalues of \( P^\#(h) \):
\[
\text{(4.3.10)} \quad |\text{Spec}(P^\#(h)) \cap [-r^2, r^2]| \leq C_0 r^{n^\#} h^{-n^\#}, \quad r \geq 1, \quad 0 < h < h_0,
\]
for some \( n^\# \geq n \).

**Remark.** The upper bound in terms \( t \mapsto t^{n^\#} \) can be replaced by a more general bound \( t \mapsto \Phi(t) \) where \( \Phi \) is increasing and satisfies some natural conditions. The conclusion (4.3.17) below then holds with \( t^{n^\#} \) replaced by \( \Phi(t) \) – see [SZ91], [Sj02], and [Vo92].
EXAMPLES. 1. Elliptic perturbations of the semiclassical Laplacian. Suppose that
\[ P(h)u := \sum_{i,j=1}^{n} hD_{x_j}(a_{ij}(x)hD_{x_i}u) + c(x)u, \]
where \( a_{ij} - \delta_{ij}, b_j, c_j \in C_c^\infty(B(0,R_0), \mathbb{R}) \),
\[ \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geq c_0|\xi|^2, \quad c_0 > 0, \]
and \( \mathcal{H} = L^2(\mathbb{R}^n) \) and \( \mathcal{D} = H^2(\mathbb{R}^n) \). This operator satisfies all the black box assumptions from §4.1. Theorem 4.4 then provides the meromorphic continuation of the resolvent.

The reference operator \( P^\#(h) \) is the same elliptic operator acting on \( \mathcal{D}^\# = H^2(\mathbb{T}_{R_1}^\#, R_1 > R_0) \). Standard estimates for the number of eigenvalues of elliptic operators (see for instance [Zw12, Theorem 14.11]) show that (4.3.10) holds with \( n^\# = n \).

2. Pseudo-Laplacian for finite volume surfaces. In Example 3 of §4.1 (see (4.1.22)) the Hilbert spaces is given by \( \mathcal{H} = \mathcal{H}_a \oplus L^2([a,\infty)) \) In that case it is more useful to consider the reference operator defined using (4.3.5). For \( b > a \) we put
\[ \mathcal{H}_b^\# := \mathcal{H}_a \oplus L^2([a,b]). \]

The operator \( P_b^\# \) is then pseudo-Laplacian of Lax–Phillips and Colin de Verdière [CdV83]. We claim that (4.3.10) holds with \( n^\# = 2 \) (note that now \( n = 1 \) and we do not have a semiclassical parameter):
\[ |\text{Spec}(P_b^\#) \cap [0,r^2]| \leq Cr^2, \quad r > 1. \]
Since \( P_b^\# \geq 0 \) this is the same as (4.3.10).

REMARK. One can improve (4.3.11) to obtain an asymptotic formula for the number of eigenvalues of \( P_b^\# \) – see [CdV83, §4]. That proceed through an improved version of the following lemma – see [CdV83, Lemma 4.2].

To prove the bound (4.3.11) we use

**LEMMA 4.12.** Let \((C_{\alpha,\beta}, g_0)\) be the cylinder \([\alpha,\beta] \times S^1, S^1 = \mathbb{R}/\mathbb{Z}, \alpha > 0, \)
equipped with the metric \( g_0 = dr^2 + e^{-2r}d\theta^2 \). We say that \( \varphi \in H^2(C_{\alpha,\beta}) \) is a Neumann eigenfunction of the Laplacian on \( C_{\alpha,\beta} \) with eigenvalue \( E \) if
\[ -\Delta \varphi = E \varphi, \quad E \leq k^2, \quad \partial_y \varphi(\alpha, \theta) = \partial_y \varphi(\beta, \theta) = 0, \]
\[ \int_{0}^{2\pi} \varphi(y, \theta)d\theta = 0, \quad \alpha < y < \beta. \]
Let \( N_{\alpha,\beta}(k) \) be the number of independent Neumann eigenfunctions with \( E \leq k^2 \). Then, with \( C \) independent of \( \alpha \) and \( \beta \),

\[
N_{\alpha,\beta}(k) \leq \frac{e^\beta - e^\alpha}{e^{2\alpha}} k^2 + \frac{C}{e^\alpha} k. \tag{4.3.13}
\]

**Proof.** 1. An explicit calculation gives the result for the counting function \( M_\ell(k) \) for the same class of eigenfunctions but for the cylinder \( K_{\ell,A} := [A, A + \ell] \times S^1, A > 0 \), with the metric \( g_1 = dr^2 + d\theta^2 \): the eigenfunctions are given by

\[
\varphi(r, \theta) = \cos \left( \frac{m\pi (y - A)}{\ell} \right) e^{in\theta}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{Z} \setminus \{0\}, \quad E = \left( \frac{m\pi}{\ell} \right)^2 + n^2.
\]

The counting function of \( E \leq k^2 \) satisfies

\[
M_\ell(k) \leq \ell k^2 + C k. \tag{4.3.14}
\]

2. We now introduce a diffeomorphism \( \kappa : C_{\alpha,\beta} \rightarrow K_{\ell,A}, \ell = e^\beta - e^\alpha, A = e^\alpha \), given by \( \kappa(r, \theta) = (e^r, \theta) \). Then

\[
\kappa^* g_1 = e^{2r} g_0, \quad \kappa^* d\text{Vol}_{g_1} = e^{2r} d\text{Vol}_{g_0}, \tag{4.3.15}
\]

and

\[
\int_{K_{\ell,A}} |df|_{g_1}^2 d\text{Vol}_{g_1} = \int_{C_{\alpha,\beta}} |d\kappa^* f|_{\kappa^* g_0}^2 \kappa^* d\text{Vol}_{g_1} = \int_{C_{\alpha,\beta}} |d\kappa^* f|_{g_0}^2 d\text{Vol}_{g_0}. \tag{4.3.16}
\]

(We note that \( |df(r, \theta)|_{g_1}^2 \) is calculated using the dual metric on \( T_{(r,\theta)}^* K_{\ell,M} \) so that the two factors \( e^{2r} \) cancel.)

From (4.3.15) we see that

\[
d\text{Vol}_{g_0} |_{C_{\alpha,\beta}} = e^{-2r} \kappa^* d\text{Vol}_{g_1} |_{C_{\alpha,\beta}} \leq e^{-2\alpha} \kappa^* d\text{Vol}_{g_1} |_{C_{\alpha,\beta}},
\]

which combined with (4.3.16) gives

\[
\frac{\int_{C_{\alpha,\beta}} |d\kappa^* f|_{g_0}^2 d\text{Vol}_{g_0}}{\int_{C_{\alpha,\beta}} |\kappa^* f|^2 d\text{Vol}_{g_0}} \geq e^{2\alpha} \frac{\int_{K_{\ell,A}} |df|_{g_1}^2 d\text{Vol}_{g_1}}{\int_{K_{\ell,A}} |f|^2 d\text{Vol}_{g_1}}.
\]

This provides a comparison for Rayleigh quotients used in the min-max characterization of eigenvalues (see Theorem B.11): from (B.1.18) we see that eigenvalues of each metric satisfy \( \lambda_k(g_0) \geq e^{2\alpha} \lambda_k(g_1) \). This leads to a comparison of the counting function (see (4.3.12)):

\[
N_{\alpha,\beta}(k) \leq M_{e^\beta - e^\alpha}(k/e^\alpha).
\]
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Using (4.3.14) we then obtain (4.3.13). □

Proof of (4.3.11). The operator $P^#_b$ with domain given by (4.3.5) is obtained (just as $P$ was) from the quadratic form (4.1.17) restricted to the domain (in the notation of (4.1.17))

$$D^#_Q = \{ u \in H^1(X) : a_0(r) = 0 \text{ for } r > b \}.$$ 

Here we used the fact that

$$\{ u \in H^1((\infty, \infty)) : u(r)|_{r>b} = 0 \} \ni u \mapsto u|_{r\leq b} \in H^1_0((\infty,b)),$$

is an isomorphism.

We now see that

$$D^#_Q \subset H^1(X_0) \oplus \bigoplus_{p=0}^{\infty} \left\{ u \in H^1(C_{b+p,b+p+1}), \int_0^{2\pi} u(r,\theta)d\theta \equiv 0 \right\}.$$ 

Let $N_{X_0}(k)$ be the counting function for eigenvalues less than $k^2$ of the Neumann realization of $-\Delta_g$ on $X_0$. By the standard Weyl law we have

$$N_{X_0}(k) \simeq \frac{\text{Vol}(X_0)}{4\pi} k^2.$$ 

It now follows from Theorem (B.1.18) and Lemma 4.12 that, for $r \geq 1$,

$$| \text{Spec}(P^#_b) \cap [0,r^2] | \leq N_{X_0}(r) + \sum_{p=0}^{\infty} N_{C_{b+p,b+p+1}}(r) \leq C r^2 + C \sum_{p=0}^{\infty} r^2 e^{-p} \leq C r^2,$$

which is (4.3.11). □

We now come to the main result of this section. As remarked after (4.3.10) a more general counting functions are possible.

THEOREM 4.13 (Upper bounds on the number of resonances). Suppose that $P(h)$ is a semiclassical black box Hamiltonian with $n \geq 1$ odd, and that (4.3.10) hold. Then for some constant $C_1$

$$\sum \{ m_R(\lambda) : |\lambda| \leq r \} \leq C_1 r^{n^#} h^{-n^#}, \ r > 1.$$

REMARK. When $n$ is even or when perturbations have long range, global bounds are more complicated – see § 4.7. Semiclassical bounds in compact sets away from 0, for compactly supported perturbations, can be proved by the methods presented here – see Theorem 7.4 for a simple version.
Before starting the proof of Theorem 4.13 we modify some notation from the proof of Theorem 4.4. First, we define the semiclassical free resolvent

\[(4.3.18) \quad R_0(\lambda, h) = (-h^2\Delta - \lambda^2)^{-1} = h^{-2}R_0(\lambda/h), \quad \text{Im} \lambda > 0,\]

and the meromorphic extension of the resolvent of \(P(h)\).

\[R(\lambda, h) = (P(h) - \lambda^2)^{-1}, \quad \text{Im} \lambda > 0.\]

Step 4 in the proof Theorem 4.4 shows that

\[R(\lambda, h)\chi = (Q_0(\lambda, h) + Q_1(h))(I + K(\lambda, h))^{-1},\]

where

\[Q_0(\lambda, h) := (1 - \chi_0)R_0(\lambda, h)(1 - \chi_1), \quad Q_1(h) := \chi_2(P(h) - \lambda_0^2)^{-1}\chi_1, \quad K(\lambda, h) := K_0(\lambda, h) + K_1(\lambda, h),\]

\[(4.3.19) \quad K_0(\lambda, h) := [-h^2\Delta, \chi_0]R_0(\lambda, h)(1 - \chi_1)\chi, \quad K_1(\lambda, h) := (\lambda_0^2 - \lambda^2)\chi_2(P(h) - \lambda_0^2)^{-1}\chi_1\]

\[+ [-h^2\Delta, \chi_2](P(h) - \lambda_0^2)^{-1}\chi_1.\]

We use the cut-off functions (4.2.6) and \(\chi = \chi_3\).

Step 2 in the proof of Theorem 4.4 shows that \(\lambda_0\) can be chosen independently of \(h\). We recall that \(\lambda_0\) has to be chosen so that \(I + K(\lambda_0, h)\) is invertible. With \(\lambda_0\) given by (4.2.11), \(\lambda_0 = e^{i\pi/4}\mu\), we apply Lemma 4.3 to obtain

\[\|[\chi_2(P(h) - \lambda_0^2)^{-1}\chi]\|_{\mathcal{H} \to \mathcal{H}} \leq C\mu,\]

and

\[\|[\chi_0]R_0(\lambda_0, h)(1 - \chi_1)\chi\|_{\mathcal{H} \to \mathcal{H}} \leq C\mu.\]

Since \(0 < h < 1\), we can take \(\mu\) fixed and large and conclude that

\[K(\lambda_0, h) \leq \frac{1}{2};\]

obtaining invertibility of \(I + K(\lambda_0, h)\).

We have two lemmas related to \(K(\lambda, h)\):

**LEMMA 4.14 (Estimates on singular values).** With the notation of (4.3.19) and for \(\lambda_0 = e^{i\pi/4}\mu\) with \(\mu \geq 1\) fixed and large, we have the following characteristic value estimates:

\[s_j (\chi_2(P(h) - \lambda_0^2)^{-1}\chi) \leq C \left(\mu^2 + (hj^{1/n^*})^2\right)^{-1},\]

\[s_j (\chi_0]R_0(\lambda_0, h)(1 - \chi_1)\chi\|_{\mathcal{H} \to \mathcal{H}} \leq C\mu.\]

Since \(0 < h < 1\), we can take \(\mu\) fixed and large and conclude that

\[K(\lambda_0, h) \leq \frac{1}{2};\]

obtaining invertibility of \(I + K(\lambda_0, h)\).

We have two lemmas related to \(K(\lambda, h)\):
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Proof. 1. For \( P(h) \) replaced by \( P^\#(h) \) the first estimate in (4.3.21) follows from (4.3.10). To see that let \( \{\mu_j^2\}_{j=0}^\infty, 0 \leq \mu_j \leq \mu_{j+1}, \) be the eigenvalues of \( (P(h))^2 \). Then (4.3.10) means that for \( 0 < h < h_0, \)

\[
\mu_j < r^2 \implies j < C_0(\max(r, 1)/h)^n^#.
\]

This implies that

\[
j = C_0(r/h)^n^# \quad \text{for} \quad r \geq 1 \implies \mu_j \geq r^2 = ((j/C_0)^{1/n^#} h)^2.
\]

If \( ((j/C_0)^{1/n^#} h)^2 < 1 \) then we only conclude \( \mu_j \geq 0 \). Thus for all \( j, \)

\[
\mu_j \geq ((j/C_0)^{1/n^#} h)^2 - 1.
\]

Since \( \mu \geq 1, \)

\[
\mu_j + \mu^2 \geq \frac{1}{2} \mu^2 + (h(j/C_0)^{1/n^#})^2,
\]

and

\[
s_j \left( (P^\#(h))^{-1} - \lambda_0^2 \right) \leq C/(\mu^2 + (h(j/C_0)^{1/n^#})^2).
\]

Since we are taking singular values of self-adjoint operators we also have

(4.3.22) \[
\quad s_j \left( (P^\#(h))^{-1} - \lambda_0^2 \right) \leq C/(\mu^2 + (h(j/C_0)^{1/n^#})^2)^{-\frac{1}{2}}.
\]

We then consider multiplication by \( \chi \) and \( \chi_2 \) as bounded function on \( L^2(\mathbb{T}^n) \) and obtain the first estimate in (4.3.21) with \( P^\#(h) \) in place of \( P(h) \).

2. To obtain the second estimate in (4.3.21) we note that

\[
s_j \left( (-h^2\Delta, \chi_2)(P^\#(h))^{-1} \right) \leq C h \|(1 - \chi_0)(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{H^\# \rightarrow H^{\frac{1}{2}}(\mathbb{T}^n)} s_j \left( (P^\#(h))^{-1} - \lambda_0^2 \right),
\]

where the last factor on the right is estimated by (4.3.22). The estimate the first term we write

\[
\|(1 - \chi_0)(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{H^\# \rightarrow H^{\frac{1}{2}}(\mathbb{T}^n)} \leq C \|(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{H^\# \rightarrow H^{\frac{1}{2}}}
\]

\[
= C \|(P^\#(h) + i)^{\frac{1}{2}}(P^\#(h) - \lambda_0^2)^{-\frac{1}{2}}\|_{H^\# \rightarrow H^{\#}}
\]

\[
= C \|(I + (\lambda_0^2 + i)(P^\#(h) - \lambda_0^2)^{-1})^{\frac{1}{2}}\|_{H^\# \rightarrow H^{\#}} \leq C',
\]

where we used definition (4.3.4) and (4.1.10) and the fact that \( \|(P^\#(h) - \lambda_0^2)^{-1}\| = \|(P^\#(h) + \mu^2i)^{-1}\| = \mu^{-2}. \)

We reiterate the conclusions of Steps 1 and 2:

(4.3.23) \[
\begin{align*}
& s_j \chi_2(P^\#(h) - \lambda_0^2)^{-1} \chi \leq C(\mu^2 + (h(j/C_0)^{1/n^#})^2)^{-1}, \\
& s_j \left( [-h^2\Delta, \chi_2](P^\#(h) - \lambda_0^2)^{-1} \chi \right) \leq C h(\mu^2 + (h(j/C_0)^{1/n^#})^2)^{-1/2}.
\end{align*}
\]
3. We now compare the resolvents of $P^\#(h)$ and $P(h)$. We claim that for $\chi \in C_c^\infty(B(0, R_1))$ which is equal to 1 in a neighbourhood of $B(0, R_0)$ we have, for any $N$,

(4.3.24) $\chi(P^\#(h) - \lambda_0^2)^{-1}\chi - \chi(P(h) - \lambda_0^2)^{-1}\chi = O_N(h^{\infty})_{\mathcal{H} \rightarrow \mathcal{D}^N}.$

We note that although operators $P^\#$ and $P$ act on different spaces, the cut-off functions produce operators acting on $\mathcal{H}$ or $\mathcal{H}^\#$.

To see (4.3.24) we choose $\chi_4 \in C_c^\infty(B(0, R_1))$ such that $\chi_4 = 1$ in $\text{supp } \chi$. The estimate (4.3.24) follows from the same estimate for

(4.3.25) $Q(h) := \chi_4(P^\#(h) - \lambda_0^2)^{-1}\chi - \chi_4(P(h) - \lambda_0^2)^{-1}\chi.$

Since $\chi_4 P^\# = \chi_4 P$ (see (4.1.5)) we have

$$(P(h) - \lambda_0^2)Q(h) = [-h^2\Delta, \chi_4](A(h) - B(h)),
A(h) := \psi(P^\#(h) - \lambda_0^2)^{-1}\chi, \quad B(h) := \psi(P(h) - \lambda_0^2)^{-1}\chi,$$

where

$\psi \in C_c^\infty(B(0, R_1)), \quad [-\Delta, \chi_4] \psi = [-\Delta, \chi_4], \quad \text{supp } \psi \cap \text{supp } \chi = \emptyset.$

4. Since

$$Q(h) = (P(h) - \lambda_0^2)^{-1}[-h^2\Delta, \chi_4](A(h) - B(h)),$$

the estimate for $Q(h)$, and hence (4.3.24), follow from

(4.3.26) $A(h), B(h) = O_N(h^{2N})_{\mathcal{H} \rightarrow \mathcal{D}^N}.$

To establish this we choose $\varphi_j \in C_c^\infty(B(0, R_1))$, $j = 1, \ldots, N$, with the following properties

$$\varphi_1 \chi = 1, \quad \varphi_j|_{\text{supp } \varphi_{j-1}} = 1, \quad \varphi_{2N} \psi = 0.$$

That is possible since the supports of $\psi$ and $\chi$ are disjoint. Note that $\varphi_j$ are equal to 1 near $B(0, R_0)$ as $\chi$ is equal to 1 there.

The properties of the supports give

$$\psi(P(h) - \lambda_0^2)^{-1}\varphi_{2N}(P(h) - \lambda_0^2) = \psi(P(h) - \lambda_0^2)^{-1}[\varphi_{2N}, P(h)],$$

and

$$[\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}\varphi_{j-1}(P(h) - \lambda_0^2) = [\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}[\varphi_{j-1}, P(h)].$$

These can be re-written as

$$\psi(P(h) - \lambda_0^2)^{-1}\varphi_{2N} = \psi(P(h) - \lambda_0^2)^{-1}[\varphi_{2N}, P(h)](P(h) - \lambda_0^2)^{-1},$$

$$[\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}\varphi_{j-1} = [\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1}[\varphi_{j-1}, P(h)](P(h) - \lambda_0^2)^{-1}.$$
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Hence
\[ B(h) = \psi(P(h) - \lambda_0^2)^{-1} \chi = \psi(P(h) - \lambda_0^2)^{-1} \varphi_1 \chi \]
\[ = \psi(P(h) - \lambda_0^2)^{-1} [\varphi_2, P(h)](P(h) - \lambda_0^2)^{-1} \]
\[ [\varphi_{2N-1}, P(h)](P(h) - \lambda_0^2)^{-1} \cdots [\varphi_1, P(h)](P(h) - \lambda_0^2)^{-1} \chi. \]

With the norms defined in (4.1.9) we have
\[ [\varphi_j, P(h)](P(h) - \lambda_0^2)^{-1} = O(h) : \mathcal{D}^{k+\frac{1}{2}}_h \rightarrow \mathcal{D}^N_h. \]

The proof for \( A(h) \) is similar. We note that \( \mathcal{D}_h \) can be replaced by \( \mathcal{D}^\#_h \) in the estimate. This gives (4.3.26). By going back to (4.3.25) we obtain (4.3.24).

5. We can now prove (4.3.21) using (4.3.23) and (4.3.24): with \( Q(h) \) defined by (4.3.25) we have (see (B.3.5))
\[ s_j(\chi_2 (P(h) - \lambda_0^2)^{-1} \chi) \leq s_{[j/2]}(\chi_2 (P^\#(h) - \lambda_0^2)^{-1} \chi) + s_{[j/2]}(Q(h)) \]
\[ \leq \frac{C}{\mu^2 + (hj^{1/n^\#})^2} \]
\[ + s_{[j/2]}((P^\#(h) - \lambda_0^2)^{-N}) ||Q(h)||_{\mathcal{H} \rightarrow \mathcal{D}^N_h} \]
\[ \leq \frac{C}{\mu^2 + (hj^{1/n^\#})^2} + \frac{C'h^{2N}}{(\mu^2 + (hj^{1/n^\#})^2)^N} \]
\[ \leq \frac{C'}{\mu^2 + (hj^{1/n^\#})^2}. \]

The argument for the second estimate in (4.3.21) is similar. \( \square \)

**Lemma 4.15** (Upper bound on multiplicities of resonances). With the notation of (4.3.19) and for \( \lambda_0 = e^{i\pi/4} \mu \) with \( \mu > 0 \) fixed and large, \( K(\lambda, h)^{n^\# + 1} \in \mathcal{L}^1(\mathcal{H}, \mathcal{H}) \), and

\[ m_R(\lambda) \leq m_H(\lambda) := \frac{1}{2\pi i} \oint_{\lambda} \frac{\partial \zeta H(\zeta, h)}{H(\zeta, h)} d\zeta, \]
\[ H(\zeta, h) := \det(I - (-K(\zeta, h))^{n^\# + 1}). \]

where \( m_R(\lambda) \) is the multiplicity of the resonance at \( \lambda \) given by (4.2.18) and the integral is over a positively oriented circle containing no other resonances than \( \lambda \).
Proof. 1. The trace class property of $K(\lambda, h)^{n^\# + 1}$ follows from the estimates on the singular values: from Proposition B.15 we have

\[
\begin{align*}
\sigma_j(K(\lambda, h)^{n^\# + 1}) & \leq \left( \sigma_{\lfloor j/(n^\# + 1) \rfloor}(K(\lambda, h)) \right)^{n^\# + 1}, \\
\sigma_\ell(K(\lambda, h)) & \leq (|\lambda|^2 + \mu^2)\sigma_{\ell/3}(\chi_2(P(h) - \lambda_0^2)^{-1}) \\
& + \sigma_{\ell/3}([-h^2 \Delta, \chi_2](P(h) - \lambda_0^2)^{-1}) \\
& + \sigma_{\ell/3}([-h^2 \Delta, 0]R_0(\lambda, h)).
\end{align*}
\]

(4.3.28)

To estimate the last term we consider

\[[-h^2 \Delta, 0]R_0(\lambda, h) = O_{\lambda, h}(1) : L^2(\mathbb{T}^n) \to H^1(\mathbb{T}^n), \quad \mathbb{T}^n := \mathbb{R}^n/(R_2\mathbb{Z})^n,\]

for some large $R_2$. Hence, using (B.3.6),

\[
s_k([-h^2 \Delta, 0]R_0(\lambda, h)) \leq \sigma_k((-\Delta + i)^{-1/2}|[-h^2 \Delta, 0]R_0(\lambda, h)\chi|_{L^2(\mathbb{T}^n)\to H^1(\mathbb{T}^n)} \leq C_{\lambda, h} k^{-1/n}.
\]

Using this and (4.3.21) in (4.3.28) we conclude that for $j \geq 1$,

\[
s_j(K(\lambda, h)) \leq C'_{\lambda, h}(j^{-2/n^\#} + j^{-1/n^\#} + j^{-1/n} n^\# + 1) \leq C''_{\lambda, h} j^{-(n^\# + 1)/n^\#}.
\]

Thus the singular values are summable and definition (B.4.2) shows that $K(\lambda, h)$ is of trace class.

(The main effort in the proof of Theorem 4.13 will be to improve this rough estimate on characteristic values.)

2. The proof of (4.3.27) is based on the Gohberg-Sigal theory reviewed in §C.4 and the identity based (4.2.13) and $(1 - q)^{-1} = (1 + q + \cdots + q^{n^\#})/(1 - q^{n^\# + 1})^{-1}$:

\[
R(\lambda, h)\chi = Q(\lambda, h)\chi W(\lambda, h)(I - (-K(\lambda, h))^{n^\# + 1})^{-1}(I + K(\lambda, h)(1 - \chi)).
\]

\[
W(\lambda, h) := (I - K(\lambda, h) + \cdots + (-1)^{n^\#} K(\lambda, h)^{n^\#}).
\]

Theorems 4.7 (specifically (4.7)) and C.8 (apply (C.4.3) with $A(\lambda) = I + (-K(\lambda, h))^{n^\# + 1}$) give the estimate on the multiplicities. \qed

Proof of Theorem 4.13. 1. In view of (4.3.27) we need to estimate the number of zeros of $H$ in the disc of $D(0, r)$, $r > 1$. We can assume that $r$ is large enough so that $r \gg |\lambda_0|$. The Jensen formula (D.1.6) gives the estimate (D.1.8):

\[
\sum_{|\lambda| \leq r} \{m_H(\lambda) : |\lambda| \leq r\} \leq C \max_{|\lambda| \leq 2r} \log |H(\lambda, h)| - C \log |H(\lambda_0, h)|.
\]

(4.3.29)
2. We start with the upper bound on \( H(\lambda, h) \). The estimate (4.3.28) and the Weyl inequalities of Proposition B.24 show that

\[
(4.3.30) \quad \log |H(\lambda, h)| \leq C \left( \log P_1 + \log P_2 + \log P_3 \right),
\]

where

\[
P_1 := \prod_{j=0}^{\infty} \left( 1 + s_j \left( (|\lambda|^2 + |\lambda_0|^2)\chi_2(P(h) - \lambda_0^2) - 1 \right)^{n\# + 1} \right),
\]

(4.3.31)
\[
P_2 := \prod_{j=0}^{\infty} \left( 1 + s_j \left( [-h^2 \Delta, \chi_2](P(h) - \lambda_0^2) - 1 \right)^{n\# + 1} \right),
\]

\[
P_3 := \prod_{j=0}^{\infty} \left( 1 + s_j \left( [-h^2 \Delta, \chi_0]R_0(\lambda, h) \right)^{n\# + 1} \right).
\]

To estimate \( P_1 \) and \( P_2 \) we use (4.3.21): since \( \lambda_0 = e^{\pi i/4} \mu \) is fixed we drop the dependence on \( \mu \):

\[
\log P_1 \leq \sum_{j=0}^{\infty} \log \left( 1 + \left( \frac{C(\lambda)^2}{1 + (h^{j/\#})^2} \right)^{n\# + 1} \right)
\]

\[
\leq C \log \langle \lambda \rangle + \sum_{j=2}^{\infty} \log \left( 1 + \left( \frac{C(\lambda)h^{-1/n\#}}{1 + (h^{j-1/\#})^{2(n\# + 1)}} \right)^2 \right)
\]

\[
\leq C \log \langle \lambda \rangle + \int_{1}^{\infty} \log \left( 1 + C(\lambda)(h^{-1/n\#}x^{-1/n\#})^{2(n\# + 1)} \right) dx
\]

\[
\leq C \log \langle \lambda \rangle + C'h^{-n\#} \langle \lambda \rangle^{n\#} \int_{1}^{\infty} \log \left( 1 + y^{2(n\# + 1)/n\#} \right) dy
\]

\[
\leq C'h^{-n\#} \langle \lambda \rangle^{n\#}.
\]

(The integral comparison is justified as \( x \mapsto \log(1 + \alpha x^{-\beta}) \), \( \alpha, \beta > 0 \), is a decreasing function.) A similar argument shows that \( \log P_2 \leq C'h^{-n\#} \): using the second estimate in (4.3.21) and dropping the dependence on the fixed constant \( \mu \),

\[
\log P_2 \leq \sum_{j=0}^{\infty} \log \left( 1 + C'h \left( 1 + (h^{j-1/\#})^{2} \right)^{-\frac{1}{2}(n\# + 1)} \right)
\]

\[
\leq C + \int_{1}^{\infty} \log \left( 1 + C'h(h^{-1}x^{-1/n\#})^{(n\# + 1)} \right) dx
\]

\[
\leq C + C'h^{-n\#} \langle \lambda \rangle^{n\#} \int_{1}^{\infty} \log \left( 1 + y^{-(n\# + 1)/n\#} \right) dy
\]

\[
\leq C''h^{-n\#}.
\]
3. We now need to estimate $\log P_3$ in (4.3.31). We will use the following estimate proved in Steps 6 and 7 below:

$$
\begin{align*}
  s_j([-h^2 \Delta, \chi_0] R_0(\lambda, h) \chi) &\leq C \langle \lambda \rangle h^{-1} j^{-1/n} \\
  &\quad + \exp \left( \langle \lambda \rangle / h - j^{1/n-1}/C \right).
\end{align*}
$$

(4.3.32)

Assuming (4.3.32) we obtain,

$$
\begin{align*}
  \log P_3 &\leq C \sum_{j=1}^{\infty} \left( \log \left( 1 + C \langle \lambda \rangle h^{-1} j^{-1/n} \right)^{n#+1} \right) \\
  &\quad + \log \left( 1 + e^{\langle \lambda \rangle / h - j^{1/n-1}/C} \right) \\
  &\leq C \langle \lambda \rangle / h^n + \sum_{j=1}^{\infty} \langle \lambda \rangle / h + \sum_{C(\lambda)/h}^{\infty} e^{-j^{1/n-1}/C'} \\
  &\leq C \langle \lambda \rangle / h^n.
\end{align*}
$$

Here we used the same argument as in Step 2 to estimate the sum of $\log(1 + C \langle \lambda \rangle j^{1/n})^{n#+1}$. This estimate and the estimates on $\log P_1$ and $\log P_2$ in Step 2, show that (see (4.3.31)) that

$$
(4.3.33) \quad \log H(\lambda, h) \leq C \langle \lambda \rangle / h^n
$$

4. Going back to (4.3.29) we need a lower bound on $\log |H(\lambda_0, h)|$. For that we write

$$
H(\lambda_0, h)^{-1} = \det \left( (I - (-K(\lambda_0, h))^{n#+1})^{-1} \right)
$$

Now,

$$
(I - (-K(\lambda_0, h))^{n#+1})^{-1} = I - (I - (-K(\lambda_0, h))^{n#+1})^{-1} K(\lambda_0, h)^{n#+1}.
$$

From (4.3.20) we see that

$$
\|(I - (-K(\lambda_0, h))^{n#+1})^{-1}\|_{H \to H} \leq 2,
$$

and hence, using Weyl inequalities again (see Proposition B.24), we see that

$$
| \det \left( (I - (-K(\lambda_0, h))^{n#+1})^{-1} \right) | \leq \prod_{j=0}^{\infty} \left( 1 + (s_{j/(n#+1)}(K(\lambda_0, h)))^{n#+1} \right)
\leq \prod_{j=0}^{\infty} \left( 1 + (s_j(K(\lambda_0, h)))^{n#+1} \right)^{n#+1}.
$$

From the estimates in Steps 2 and 3 we see that

$$
|H(\lambda_0, h)^{-1}| \leq Ch^{-n#}.
$$

This, (4.3.33) and (4.3.29) show that

$$
\sum \{ m_H(\lambda) : |\lambda| \leq r \} \leq Cr^{n#} h^{-n#}, \quad r > 1.
$$
In view of (4.3.27) this proves the theorem.

5. It remains to establish (4.3.32). Since $R_0(\lambda, h) = h^{-2}R_0(\lambda/h)$ (see (4.3.18)) it is enough to show that

\begin{equation}
(4.3.34) \quad s_j([\Delta, \chi_0]R_0(\lambda)\chi) \leq C(\lambda)j^{-1/n} + \exp(C(\lambda) - j^{1/n-1}/C).
\end{equation}

(We remark that the proof of Theorem 3.28 contains a slightly simpler version of this estimate.)

We start with the estimate for $\text{Im} \lambda \geq 0$. Then considering the operator $[\Delta, \chi_0]R_0(\lambda)\chi$ as acting on $L^2(\mathbb{T}^n)$, $\mathbb{T}^n := \mathbb{R}^n/R_2\mathbb{Z}^n$ for some large $R_2$,

\begin{equation}
(4.3.35) \quad s_j([\Delta, \chi_0]R_0(\lambda)\chi) \leq s_j((-\Delta_{\mathbb{T}^n} + I)^{-1/2}||\chi R_0(\lambda)\chi||_{L^2(\mathbb{T}^n)} \to H^2(\mathbb{T}^n)) \leq Cj^{-1/n}(\lambda),
\end{equation}

where we used the estimate (3.1.12) for the norm of $\chi R_0(\lambda)\chi$. This gives the estimate (4.3.34) for $\text{Im} \lambda \geq 0$.

6. To obtain estimates for $\text{Im} \lambda < 0$ we use Stone’s formula (3.1.19) and write

\begin{equation}
\chi(R_0(\lambda) - R_0(-\lambda))\chi = a_n\lambda^{n-2}E_{\chi}(\lambda)^*E_{\chi}(\lambda),
\end{equation}

\begin{equation}
(4.3.36) \quad E_{\chi}(\lambda)u(\omega) := \int_{\mathbb{R}^n} e^{-i\lambda(\omega,y)}\chi(y)u(y)dy,
\end{equation}

$E_{\rho}(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{S}^{n-1})$.

Then (B.3.5) gives

\begin{equation}
(4.3.37) \quad s_j([\Delta, \chi_0]R_0(\lambda)\chi) \leq C|\lambda|^{n-2}||[-\Delta, \chi_0]E_{\rho}(\lambda)||s_{[j/2]}(E_{\chi}(\lambda)) + s_{[j/2]}([-\Delta, \chi_0]R_0(-\lambda)\chi) \leq C \exp(C(\lambda))s_{[j/2]}(E_{\chi}(\lambda)) + C\langle \lambda \rangle j^{-1/n}.
\end{equation}

7. To estimate $s_j(E_{\chi}(\lambda))$ we repeat the argument of Step 4 of the proof of Theorem 3.28. If $-\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on $\mathbb{S}^{n-1}$ then (B.3.6) gives

\begin{equation}
(4.3.38) \quad s_j(E_{\chi}(\lambda)) \leq s_j((-\Delta_{\mathbb{S}^{n-1}} + 1)^{-1/2} \leq C\ell^{1/2}||(-\Delta_{\mathbb{S}^{n-1}} + 1)^{\ell}E_{\chi}(\lambda)|| \leq C\ell^{1/2}\exp(C_1|\lambda|)(2\ell)!.
\end{equation}

An optimization of this estimate in $\ell$ gives

\begin{equation}
(4.3.39) \quad s_j(E_{\chi}(\lambda)) \leq C_2 \exp\left(C_2|\lambda| - j^{1/4}/C_2\right).
\end{equation}

Combined with (4.3.37) this gives (4.3.32) completing the proof of theorem.

□
Theorem 4.13 applies to Examples 1 and 2 presented earlier in this section. In the case of Example 2 the fact that \( n^# > n \) gives asymptotics for the number of resonances. We conclude this section with a classical scattering problem in which the first polynomial, and optimal, bound was given by Melrose [Me84b]:

**EXAMPLE.** Consider scattering by an obstacle \( \mathcal{O} \) in odd dimensions (Example 2 in §4.1) with any self-adjoint boundary condition. Let \( m_\mathcal{O}(\lambda) \) be the multiplicity of a resonance \( \lambda \) of the corresponding Laplacian. Eigenvalue counting estimates for the Laplacian on a compact manifold with boundary \( \mathbb{T}^n \setminus \mathcal{O} \), \( \mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n \) show that (4.3.10) holds with \( n^# = n \). Hence, Theorem 4.13 applied with \( h = 1 \) gives

\[
\sum \{ m_\mathcal{O}(\lambda) : |\lambda| \leq r \} \leq C r^n, \quad r > 1.
\]

This bound is optimal as shown by the example of the sphere – see Stefanov [St06] and references given there.

### 4.4. PLANE WAVES AND THE SCATTERING MATRIX

In this section we define the scattering matrix for a black box operator. Since we will deal with exact formulas and not asymptotic properties we only consider the case \( h = 1 \). All the formulas remain valid for operator \( P(h) \) with the obvious rescaling:

\[
-h^2 \Delta \rightsquigarrow -\Delta, \quad P(h) \rightsquigarrow h^{-2} P(h), \quad \lambda \rightsquigarrow \lambda / h.
\]

Our presentation will be close to that in §§3.6 and 3.7. As did previous section of this chapter it will also depend on the properties of the free resolvent from §3.1.

#### 4.4.1. Outgoing solutions.

The plane waves,

\[
e_0(\lambda, \omega) = e_0(\lambda, \omega, x) = e^{-i\lambda \langle x, \omega \rangle}, \quad (-\Delta - \lambda^2) e_0 = 0,
\]

are important from both physical and mathematical points of view – see for instance the spectral decomposition of \(-\Delta\) in (3.4). Motivated by Fig. 3.4 we now want to consider solutions to \((P - \lambda^2)w = 0\) which are sums of plane wave (away from the black box) and of an outgoing wave.

In the notation of §4.1 let \( \chi \in C_\infty^0(\mathbb{R}^n, [0, 1]) \) be equal to 1 near \( B(0, R_0) \). Let

\[
R(\lambda) : \mathcal{H}_{\text{comp}} \rightarrow \mathcal{D}_{\text{loc}}
\]

be the extension of \((P - \lambda^2)^{-1}\) from \( \text{Im} \lambda > 0 \), given in Theorem 4.4.
For \( \lambda \in \mathbb{R} \setminus \{0\} \) we then define *distorted plane waves*

\[
\begin{align*}
  e(\lambda, \omega) &= (1 - \chi)e_0(\lambda, \omega) + w, \\
  w &= R(\lambda) \left( [-\Delta, \chi]e^{-i\lambda\langle \cdot, \omega \rangle} \right) \in \mathcal{D}^\infty_{\text{loc}}.
\end{align*}
\]

(4.4.1)

Note that in principle \( e \) could have poles in \( \lambda \) at places where \( R(\lambda) \) has real poles. As we will see that cannot happen. The definition (4.4.1) should be compared to the definition (3.8.1) in the case of potential scattering.

The regularity of \( w \) comes from (4.1.6):

\[
(P + i)^N R(\lambda) \left( [\Delta, \chi]e^{-i\lambda\langle \cdot, \omega \rangle} \right) = R(\lambda) \left( (-\Delta + i)^N [\Delta, \chi]e^{-i\lambda\langle \cdot, \omega \rangle} \right) \in R(\lambda)(\mathcal{H}_{\text{comp}}).
\]

(4.4.2)

For \( \lambda \in \mathbb{R} \setminus (\{0\} \cup \text{Res}(P)) \) (and, as we will see for \( \lambda \in \mathbb{R} \setminus \{0\} \)) we have

\[
(P - \lambda^2)e(\lambda, \omega) = 0, \quad e(\lambda, \omega) - (1 - \chi)e_0(\lambda, \omega) \quad \text{is outgoing.}
\]

Here the meaning of outgoing is the same as in definition (3.32) and Theorem 3.37 modified to the black box setting.

**DEFINITION 4.16 (Outgoing solutions).** Suppose \( P \) is a black box Hamiltonian in the sense of Definition (4.1). For \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( f \in \mathcal{H}_{\text{comp}} \), a solution to

\[
(P - \lambda^2)u = f,
\]

(4.4.3)

is called outgoing if and only if there exists \( g \in L^2_{\text{comp}}(\mathbb{R}^n) \) and \( R > R_0 \) such that

\[
u |_{\mathbb{R}^n \setminus B(0, R)} = (R_0(\lambda)g) |_{\mathbb{R}^n \setminus B(0, R)},
\]

where \( R_0(\lambda) \) is the free outgoing resolvent given in 3.1.

A solution \( u \) to (4.4.3) is called incoming if there exist \( g_1 \in L^2_{\text{comp}}(\mathbb{R}^n) \) and \( R > 0 \) such that,

\[
u |_{\mathbb{R}^n \setminus B(0, R)} = (R_0(-\lambda)g_1) |_{\mathbb{R}^n \setminus B(0, R)}.
\]

**INTERPRETATION.** From the asymptotics of the free resolvent given in (3.1.20) if \( u \) is outgoing then

\[
u (x) = \frac{e^{i\lambda|x|}}{|x|^\frac{n-1}{2}} \left( h \left( \frac{x}{|x|} \right) + \mathcal{O} \left( \frac{1}{|x|} \right) \right),
\]

(4.4.4)

with a full asymptotic expansion. Theorem 3.37 is easily adapted to the black box case and it gives equivalent conditions for being an outgoing solution.
The next result is the black box version of the Rellich uniqueness theorem. The proof is an adaptation of the proof of Theorem 3.35 (and that proof can be read independently of the rest of §3.6 – see Exercise 4.4).

**THEOREM 4.17 (Rellich uniqueness theorem III).** Suppose that $P$ is a black box Hamiltonian in the sense of Definition 4.1. Suppose that $\lambda \in \mathbb{R} \setminus \{0\}$ and that $u \in \mathcal{D}_{\text{loc}}$ satisfies

\[(P - \lambda^2)u = 0, \quad \lim_{R \to \infty} \int_{\partial B(0, R)} |(\partial_r - i\lambda)u|^2 dS = 0.\]

where the last integral makes sense for $R > R_0$ as $u|_{\mathbb{R}^n \setminus B(0, R_0)} \in H^2_{\text{loc}}(\mathbb{R}^n \setminus B(0, R_0))$.

Then

\[(4.4.6) \quad u|_{\mathbb{R}^n \setminus B(0, R_0)} \equiv 0\]

We conclude from Theorem 4.17 that outgoing solutions to the homogeneous equation have to compactly supported and that the resonances on $\mathbb{R} \setminus \{0\}$ are given by embedded eigenvalues.

We say that $E > 0$ is an embedded eigenvalue of $P$ if

\[(4.4.7) \quad (P - E)v = 0, \quad v \in \mathcal{D}, \quad v \neq 0.\]

The multiplicity of that eigenvalue is the dimension of the spaces of solutions of (4.4.7).

**THEOREM 4.18 (Outgoing solutions at positive energies).** Suppose that $P$ is a black box Hamiltonian and $\lambda \in \mathbb{R} \setminus \{0\}$. We have:

(i) if $u$ satisfies (4.4.3) with $f = 0$ and is outgoing then $u|_{\mathbb{R}^n \setminus B(0, R_0)} \equiv 0$;

(ii) if $\lambda$ is a pole of $R(\lambda)$ then $\lambda^2$ is an embedded eigenvalue of $P$ and $m_R(\lambda)$ is the multiplicity of that eigenvalue;

(iii) the distorted plane waves defined in (4.4.1) are defined for all $\lambda \in \mathbb{R} \setminus \{0\}$ and the map

\[\mathbb{R} \setminus \{0\} \times \mathbb{S}^{n-1} \ni (\lambda, \omega) \mapsto e(\lambda, \omega) \in \mathcal{D}_{\text{loc}}^\infty\]

is real analytic.

**Proof.** 1. Part (i) is an immediate consequence of Theorem 4.17: if $u = R_0(\lambda)f$ then (4.4.4) holds and the condition (4.4.5) is satisfied.

2. Self-adjointness of $P$ shows that

\[R(\zeta) = O(1/\Im \zeta)_{\mathcal{H} \to \mathcal{H}}, \quad \text{for} \ \Im \zeta > 0, \ |\Re \zeta| > c > 0.\]

Hence the pole at $\lambda$ must be simple, that is $(P - \lambda^2)\Pi_{\lambda} = 0$. Theorem 4.9 shows that all resonant states are outgoing and hence compactly supported.
4.4. PLANE WAVES AND THE SCATTERING MATRIX

More precisely, for \( \chi_0 \in C_c^\infty(\mathbb{R}^n) \) equal to 1 near \( B(0, R_0) \) and \( \chi \in C_c^\infty(\mathbb{R}^n) \) equal to 1 on \( \text{supp} \chi_0 \) we use (4.2.30) to write
\[
(1 - \chi) \Pi_\lambda \chi_0 = R_0(\lambda)[P, \chi] \Pi_\lambda \chi_0.
\]
This means that every element of \( \Pi_\lambda(\mathcal{H}_{\text{comp}}) \) is outgoing and hence by part (i) compactly supported. We conclude that
\[
\Pi_\lambda(\mathcal{H}_{\text{comp}}) \subset D,
\]
and each element is an eigenvector of \( P \). The rank of \( \Pi_\lambda \) is the the dimension of the eigenspace.

3. Part (iii) of the theorem follows from part (ii) and (4.4.1): the singular part \( R(\lambda) \) near a pole \( \lambda_0 \in \mathbb{R} \setminus \{0\} \) is the projection onto an eigenspace and all eigenvectors vanish in \( \mathbb{R}^n \setminus B(0, R_0) \). Hence, if
\[
R(\lambda) = \frac{\Pi_{\lambda_0}}{\chi^2 - \lambda^2_0} + A(\lambda)
\]
where \( A(\lambda) \) is holomorphic near \( \lambda_0 \). It follows that
\[
R(\lambda) \left( [-\Delta, \chi]e^{-\lambda(\cdot, \omega)} \right) = A(\lambda) \left( [-\Delta, \chi]e^{-\lambda(\cdot, \omega)} \right)
\]
is well behaved in \( \lambda \).

EXAMPLES. 1. **Obstacle scattering.** Suppose \( P \) is the Dirichlet (or Neumann) realization of \(-\Delta\) on a connected set \( \mathbb{R}^n \setminus \mathcal{O} \), where \( \mathcal{O} \) is a bounded open set with a smooth boundary. Theorem 4.18 shows that there are no resonances in \( \mathbb{R} \setminus \{0\} \): unique continuation for \(-\Delta - \lambda^2\) (see Lemma 3.34 for a proof which can be adapted to this situation) shows that any resonant state would have to vanish on \( \mathbb{R}^n \setminus \mathcal{O} \). Unlike in potential scattering (see §3.3) there is also no resonance at 0 but that requires another argument:

**THEOREM 4.19 (No zero resonance in obstacle scattering).** Suppose that \( P = -\Delta_g \) is the Dirichlet or Neumann Laplacian for a metric \( g \) on a connected set \( \mathbb{R}^n \setminus \mathcal{O} \), \( n \) odd, where \( \mathcal{O} \) is bounded and has a smooth boundary. We assume that \( g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n \setminus \mathcal{O}) \), that is \( P \) is a compact metric perturbation of the Euclidean Laplacian.

Then the meromorphic extension of
\[
R(\lambda) = (P - \lambda^2)^{-1} : L^2_{\text{comp}}(\mathbb{R}^n \setminus \mathcal{O}) \to H^2_{\text{loc}}(\mathbb{R}^n \setminus \mathcal{O})
\]
is holomorphic near 0. In other words, 0 is not a resonance in obstacle scattering.

**Proof.** 1. Let \( R(\lambda) := (\Delta_g - \lambda^2)^{-1} \) where \( -\Delta_g \) is the Dirichlet or Neumann realization of the Laplacian on \( \mathbb{R}^n \setminus \mathcal{O} \). We will consider the Neumann case, the other one being similar.
Since
\[ \|R(\lambda)\|_{L^2 \to L^2} = \frac{1}{d(\lambda^2, (0, \infty))}, \quad \text{Im} \lambda > 0, \]
we see that the pole 0 can have at most order 2:
\[ R(\lambda) = \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + A(\lambda), \]
where \( A_j \) are finite rank operators,
\[ A_j : L^2_{\text{comp}}(\mathbb{R}^n \setminus O) \to (H^2_{\text{loc}} \cap H^1_{0,\text{loc}})(\mathbb{R}^n \setminus O), \quad -\Delta A_j = 0, \]
and
\[ A(\lambda) : L^2_{\text{comp}}(\mathbb{R}^n \setminus O) \to (H^2_{\text{loc}} \cap H^1_{0,\text{loc}})(\mathbb{R}^n \setminus O) \]
is holomorphic near 0.

2. We now recall (4.2.29),
\[ (1 - \chi)R(\lambda)\chi_0 = R_0(\lambda)[\Delta, \chi]R(\lambda)\chi, \]
where \( \chi_0, \chi \in C_c^\infty(\mathbb{R}^n) \), \( \chi_0 = 1 \) near \( O \) and \( \chi = 1 \) on \( \text{supp} \chi \). Hence comparing the singular parts on each side we obtain
\[ (1 - \chi)A_2\chi_0 = R_0(0)[\Delta, \chi]A_2\chi_0 \]
and
\[ (1 - \chi)A_1\chi_0 = R_0(0)[\Delta, \chi]A_1\chi_0 + \partial_\lambda R_0(0)[\Delta, \chi]A_2\chi_0. \]

We note that if \( A_j \neq 0 \) then \( [\Delta, \chi]A_j\chi_0 \neq 0 \) by the unique continuation property of \( -\Delta \) (see Lemma 3.34).

We conclude that if 0 is a pole of \( R(\lambda) \) then there exists \( u \in H^2_{\text{loc}}, u \neq 0 \), (obtained using the first equation in (4.4.8) if \( A_2 \neq 0 \), and the second one if \( A_2 = 0 \)) such that
\[ -\Delta_g u = 0, \quad \partial_r u|_{\partial \mathcal{O}} = 0, \]
\[ u(x) = \mathcal{O}(r^{2-n}), \quad \partial_r u(x) = \mathcal{O}(r^{1-n}), \quad r := |x| \to \infty, \]
where \( \partial_r u \) is the outward normal derivative of \( u \) for \( O \). The behaviour as \( r \to \infty \) comes from the asymptotics of \( R_0(0) \) – see Theorem 3.3.

3. We now apply the divergence theorem to \( \bar{u}\nabla u \): writing \( B_R = B(0, R) \),
\[ \int_{B_R \setminus \mathcal{O}} |\nabla u|^2 \, dx = -\int_{B_R \setminus \mathcal{O}} \Delta u \bar{u} - \int_{\partial \mathcal{O}} \partial_r u \bar{u} dS + \int_{\partial B_R} \partial_r u \bar{u} dS \]
\[ = \mathcal{O}(R^{3-2n}) \int_{\partial B_R} dS = \mathcal{O}(R^{2-n}) \to 0, \quad R \to \infty. \]

Since \( n \geq 3 \), \( \nabla u \equiv 0 \) and as \( u \to 0, r \to \infty, u \equiv 0 \). (In the case of the Dirichlet boundary condition we could simply invoke the maximum principle.)
2. **Scattering on finite volume surfaces.** This is the case of scattering on $X$ given by (4.1.22). Then $n = 1$ and outside of the black box $P = -\partial_s^2$ on $L^2([a, \infty))$ – see (4.1.13). It is clear that for $\lambda^2 > 0$ any solution to $(-\partial_s^2 - \lambda^2)a_0(s) = 0$ (we use the notation preceding (4.1.14)), $a_0 \in L^2([a, \infty))$ has to be identically 0. Hence, in agreement with Theorem 4.18, $1_{[a, \infty)} u = 0$.

The next result is a an adaptation of Theorem 3.47 to the black box setting. The proof is left as Exercise 4.5.

For simplicity we make the reality assumption (4.4.11). To formulate it we assume that there exists an involution of $H$ (see (4.1.1)), $u \mapsto \bar{u}$ such that

$$\tag{4.4.10} \Xi u = \bar{z}\bar{u}, \quad (\bar{u}|_{\mathbb{R}^n \setminus B(0, R_0)} = \bar{u}|_{\mathbb{R}^n \setminus B(0, R_0)}, \quad \langle \bar{u}, \bar{v} \rangle_H = \langle v, u \rangle_H.$$  

**THEOREM 4.20 (Stone’s formula for black box Hamiltonians).** Suppose that $P$ is a black box Hamiltonian satisfying

$$\tag{4.4.11} P(\bar{u}) = \overline{P u},$$

where $u \mapsto \bar{u}$ is an involution satisfying (4.4.10).

For $\lambda \in \mathbb{R} \setminus \{0\}$ and $\omega \in S^{n-1}$ define $e(\lambda, \omega)$ by (4.4.1) (see (iii) of Theorem 4.18). Then $e(\lambda, \omega) = e(-\lambda, \omega)$, and for $f \in H_{\text{comp}},$

$$\tag{4.4.12} (R(\lambda) - R(-\lambda))f = \frac{i}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{S^{n-1}} e(\lambda, \omega) \langle f, e(\lambda, \omega) \rangle d\omega.$$  

The spectral measure of $P$ corresponding the continuous spectrum is given by

$$\tag{4.4.13} \langle dE_{\lambda} f, g \rangle = \frac{\lambda^{n-1}}{(2\pi)^n} \int_{S^{n-1}} \langle e(\lambda, \omega), g \rangle \langle e(\lambda, \omega), f \rangle d\omega,$$

and

$$P = \sum_{k=K_-}^{K_+} E_k u_j(\bullet, u_j) + \int_0^\infty \lambda^2 dE_{\lambda}, \quad I = \sum_{k=1}^{K} u_k(\bullet, u_k) + \int_0^\infty dE_{\lambda},$$

where $u_k$’s are normalized eigenfunctions of $P$ corresponding to eigenvalues $E_k, E_k \leq E_{k+1}$, where $K_{\pm}$ can take values $\pm \infty$.

**4.4.2. The Fermi Golden Rule.** We now present a result about perturbation of embedded eigenvalues in the black box setting. Our principal example is that of scattering on finite volume surfaces (4.1.22) which is a black box perturbation of $-\partial_s^2$ on a a half-line. Other examples can be constructed using hypoelliptic operators, see also Exercise 4.1.
We consider a smooth family of black box Hamiltonians acting on a fixed Hilbert space $H$ and self-adjoint with the same domain $D$:
\begin{equation}
P(s) \in C^\infty((-s_0, s_0); \mathcal{L}(D, H)),
\end{equation}

\begin{equation}
P(s)^* = P(s), \quad P(s)(\bar{u}) = (P(s)u).
\end{equation}
Here $u \mapsto \bar{u}$ is an involution satisfying (4.4.10). We will denote $P := P(0)$, $\dot{P} := \partial_s P(s)|_{s=0} : D \to \mathcal{H}_{R_0}$.
The fact that the image of the derivative is in $\mathcal{H}_{R_0}$ follows from (4.1.5).

We start with a lemma which will be proved in §4.5.5:

**LEMMA 4.21 (Smoothness of simple resonances).** Suppose that $P(s)$ is a family of black box Hamiltonians (4.4.14) and that $\lambda \in \mathbb{R}\setminus\{0\}$ is a simple eigenvalue of $P = P(0)$. Let $u$ be the eigenstate corresponding to $\lambda$. Then there exist $s_1 > 0$, $\epsilon_1 > 0$ and $u(s) \in C^\infty((-s_1, s_1); \mathcal{D}_{\text{loc}})$, $\lambda(s) \in C^\infty((-s_1, s_1); \mathbb{C})$, $u(0) = u$, $\lambda(0) = \lambda$,
\begin{equation}
\end{equation}

such that $\lambda(s)$, is the unique resonance of $P(s)$ in $D(\lambda, \epsilon_1)$ and $u(s)$ is a corresponding resonance state.

**REMARK.** The statement remains true for resonances $\lambda \neq 0$ – see the proof in §4.5.5 and [St94] for a yet more general version. One can also study the perturbation of the zero resonance or zero eigenvalue but that involves more analysis – see §3.3.

We now present a condition which guarantees its dissolution to a resonance under a perturbation:

**THEOREM 4.22 (The Fermi Golden Rule for embedded eigenvalues).** Suppose that $\lambda^2 > 0$ is an embedded eigenvalue of $P = P(0)$ where $s \mapsto P(s)$ satisfies (4.4.14) and $(P - \lambda^2)u = 0$, $\|u\|_\mathcal{H} = 1$.

Then in the notation of (4.4.15),
\begin{equation}
\text{Im}\tilde{\lambda} = -\frac{\lambda^{n-3}}{4(2\pi)^{n-1}} \int_{S^{n-1}} |\langle \dot{P}u, e(\lambda, \omega) \rangle|^2 d\omega,
\end{equation}

where $e(\lambda, \omega)$ is the distorted plane wave defined in (4.4.1).

**Proof.** 1. Let $\mathcal{H}_R := \mathcal{H}_{R_0} \oplus L^2(B(0, R) \setminus B(0, R_0))$ for some $R > R_0$. Let $z(s) = \lambda(s)^2$ and $u(s)$ be given by Lemma 4.21. Then
\begin{equation}
\text{Im} z(s) \|u(s)\|_{\mathcal{H}_R}^2 = -\text{Im} \int_{\partial B(0, R)} \overline{u(s)} \partial_r u(s) dS.
\end{equation}

We have already seen this in (2.8.17) for one dimensional problems. In the black box case, choose $\chi \in C_c^\infty(B(0, R))$ such that $\chi = 1$ near $B(0, R_0)$. 


Dropping the dependence on \( s \) and using self-adjointness of \( P \) we see that for any \( v \in \mathcal{D}_{\mathrm{loc}} \),

\[
\begin{align*}
\text{Im}(\langle (P - z)v, v|_{B(0,R)} \rangle) &= \text{Im}(\langle Pv, v|_{B(0,R)} \rangle) - \text{Im} z \|v\|^2_{\mathcal{H}_R} \\
&= \text{Im}(\langle P\chi v, \chi v \rangle) + \text{Im}(\langle (P\chi v, (1 - \chi)v) + \langle (1 - \chi)v, \chi v \rangle \rangle) \\
&+ \text{Im}(\langle \partial_r(v, (1 - \chi)v) \rangle_{B(0,R)}) - \text{Im} z \|v\|^2_{\mathcal{H}_R} \\
&= \text{Im}(\langle P(1 - \chi)v, (1 - \chi)v|_{B(0,R)} \rangle) - \text{Im} z \|v\|^2_{\mathcal{H}_R} \\
&= -\text{Im} \int_{B(0,R)} \Delta((1 - \chi)v)(1 - \chi)\bar{v} dx - \text{Im} z \|v\|^2_{\mathcal{H}_R} \\
&= -\text{Im} \int_{\partial B(0,R)} \partial_r v \bar{v} dS - \text{Im} z \|v\|^2_{\mathcal{H}_R}.
\end{align*}
\]

(4.4.18)

Putting \( v = u \) gives (4.4.17) since the left hand side vanishes.

2. Since \( \text{Im} z(s) \leq 0 \) it follows that \( \text{Im} \dot{z} = 0 \) (\( \dot{z} := \partial_s z(s)|_{s=0} \) and \( \text{Im} z(0) = 0 \)). Also, from Theorem 4.18,

\[
\text{Im} z(0) = \text{Im} \dot{z}(0) = 0, \quad u(0)|_{\mathbb{R}^n \setminus B(0,R_0)} = 0, \quad \|u(0)\|_{\mathcal{H}_R} = \|u(0)\|_{\mathcal{H}} = 1.
\]

We now differentiate (4.4.17) twice with respect to \( s \). Using \( \text{Im} z(0) = \text{Im} \dot{z}(0) = 0, \ u(0)|_{\mathbb{R}^n \setminus B(0,R_0)} = 0 \) and (4.4.18) with \( v = \dot{u} \), we obtain (at \( s = 0 \)),

\[
\begin{align*}
\text{Im} \ddot{z} &= \text{Im} \frac{d^2}{ds^2} \|u(s)\|^2_{\mathcal{H}_R}|_{s=0} + 2 \text{Im} \dot{z} \partial_s \|u(s)\|^2_{\mathcal{H}_R}|_{s=0} \\
&- \text{Im} \int_{\partial B(0,R)} (2\partial_r \dot{u}\bar{u} + \partial_r \ddot{u} + \partial_s \dot{u}\bar{u}) dS \\
&= -2 \text{Im} \int_{\partial B(0,R)} \partial_r \dot{u}\bar{u} dS \\
&= 2 \text{Im}(\langle (P - z)\dot{u}, \dot{u}|_{B(0,R)} \rangle).
\end{align*}
\]

(4.4.20)

Also, differentiating \( (P(s) - z(s))u(s) = 0 \) gives

\[
(P - z)\ddot{u} = \dot{z}u - \dot{P}u.
\]

(4.4.21)

Since \( u \in \mathcal{H}_{R_0} \) and \( \dot{P} : \mathcal{D} \to \mathcal{H}_{R_0} \) we see that the right hand side in (4.4.21) is in \( \mathcal{H}_{R_0} \). In particular, we can drop the restriction to \( B(0, R) \) on the right hand side of (4.4.20).

3. We now claim that \( \dot{u} \) is outgoing in the sense of Definition 4.16. To see that we observe from (4.2.30) (applied with \( M_{\chi(s)} = 1 \) as our resonances are simple) that

\[
(1 - \chi)u(s) = R_0(\lambda)[P, \chi]u(s), \quad z = \lambda^2,
\]
where \( \chi \) as in (4.4.18). From (4.4.19) we see that \([P, \chi]u(0) = 0 \) and \([\hat{P}, \chi]u(0) = 0 \). Hence

\[
(1 - \chi) \dot{u} = R_0(\lambda)[P, \chi] \dot{u}.
\]

To find an expression for \( \dot{u} \) we first observe that \( \dot{z}u - \dot{\hat{P}}u = 0 \) is orthogonal to \( u \). In fact, if \( \chi \) is as in (4.4.18) then \( \chi \dot{u} \in \mathcal{D} \) and \( \chi \equiv 1 \) near \( \text{supp } u \). Then,

\[
\langle \dot{z}u - \dot{\hat{P}}u, u \rangle = \langle (P - z)\dot{u}, u \rangle
\]

(4.4.22)

\[
= \langle (P - z)(\chi \dot{u}), u \rangle
\]

\[
= \langle \chi \dot{u}, (P - z)u \rangle = 0.
\]

In view of this orthogonality property we define \( v := R(\lambda)(\dot{z}u - \dot{\hat{P}}u) \) which is another outgoing solution to (4.4.21). Theorem 4.18 shows that \( v - \dot{u} \) must be compactly supported and from the simplicity of the eigenvalue it follows that \( v - \dot{u} = \alpha u, \alpha \in \mathbb{C} \). Because of (4.4.22) this means that we can replace \( \dot{u} \) with \( v \) in (4.4.20) which gives

\[
\text{Im } \ddot{z} = 2 \text{Im} \langle \dot{z}u - \dot{\hat{P}}u, R(\lambda)(\dot{z}u - \dot{\hat{P}}u) \rangle, \quad z = \lambda^2.
\]

Noting that

\[
(R(\lambda) - R(-\lambda))u = 0, \quad R(\lambda)^* = R(-\lambda),
\]

we obtain

\[
\text{Im} \langle \dot{z}u, R(\lambda) \dot{z}u \rangle = |\dot{z}|^2 \langle u, (R(\lambda) - R(-\lambda))u \rangle / 2i = 0.
\]

Since \( \text{Im } \ddot{z} = 0 \) we also have

\[
\text{Im} (\langle \dot{z}u, R(\lambda) \dot{\hat{P}}u \rangle + \langle \dot{\hat{P}}u, \dot{z}R(\lambda)u \rangle) = \dot{z} \text{Im} (-\langle R(\lambda) \dot{\hat{P}}u, u \rangle + \langle \dot{\hat{P}}u, R(\lambda)u \rangle)
\]

\[
= \dot{z} \text{Im} (\dot{\hat{P}}u, (R(\lambda) - R(-\lambda)u)
\]

\[
= 0.
\]

Hence (4.4.12) gives

\[
2\lambda \text{Im } \ddot{\lambda} = 2 \text{Im} \langle \dot{\hat{P}}u, R(\lambda) \dot{\hat{P}}u \rangle
\]

\[
= \frac{1}{i} \left( \langle \dot{\hat{P}}u, R(\lambda) \dot{\hat{P}}u \rangle - \langle \dot{\hat{P}}u, R(-\lambda) \dot{\hat{P}}u \rangle \right)
\]

\[
= -\frac{1}{2} \frac{\lambda^{n-2}}{(2\pi)^{n-1}} \int_{S^{n-1}} |\langle \dot{\hat{P}}u, e(\lambda, \omega) \rangle|^2 d\omega.
\]

This proves (4.4.16).

\[\square\]

**REMARK.** Formally (4.4.20) follows from differentiating \( 0 = \langle (P(s) - z(s))u(s), u(s) \rangle \) but we have to be careful as for \( s \neq 0 \), \( u(s) \) will not, typically, be in \( \mathcal{H} \). The use of (4.4.17) (and of its derivation (4.4.18) based on Green’s formula) remedies this problem. The complex scaling method described in
§4.5 allows a more direct argument which can be adapted to the case of long range perturbations.

**EXAMPLE.** Suppose that \((X, g)\) is the manifold given by (4.1.22). As explained before that definition, we put the scattering problem for \(-\Delta_g\) in the black box formalism with \(n = 1\) (and \(\mathbb{R}^n\) replaced by a finite union of half lines). We then define the analogues of \(e(\lambda, \omega)\) (the sphere at infinity, \(S^{n-1}\), is replaced by \(\mathbb{Z}/N\mathbb{Z}\)), \(e_j(\lambda, x), j = 1, \ldots, N, \)

\[
(-\Delta_g - \frac{1}{4} - \lambda^2)e_j(\lambda, x) = 0,
\]

(4.4.23)

\[
\frac{1}{b_\ell} \int_0^{b_\ell} e_j|_{X_\ell} d\theta = e^{r/2} \left( e^{-i\lambda r} \delta_{\ell j} + s_{\ell j}(\lambda) e^{i\lambda r} \right).
\]

Let us consider a conformal change of the metric which results in the family

\[
P(t) = e^{\frac{1}{2} t f}(-\Delta_g - \frac{1}{4}) e^{\frac{1}{2} t f}, \quad f \in C_0^\infty(X_0; \mathbb{R}).
\]

The conjugation of the Laplacian, \(e^{-\frac{1}{2} t f} (e^{t f} \Delta_g) e^{\frac{1}{2} t f}\), was introduced to fix the Hilbert space on which the operators \(P(t)\) are self-adjoint. We then see that the assumption (4.4.14) are satisfied.

Suppose that \(E = \lambda^2 + \frac{1}{4} > \frac{1}{4}\) is an embedded eigenvalue of \(-\Delta_g\) and \(u\) is the corresponding normalized eigenfunction. Then (4.4.16) gives

\[
\text{Im} \hat{\lambda} = -\frac{1}{4\lambda^2} \sum_{\ell=1}^N \left| \langle \frac{1}{2} (f(\Delta_g + \frac{1}{4}) + (\Delta_g + \frac{1}{4}) f) u, e_\ell(\lambda) \rangle \right|^2
\]

(4.4.24)

\[
= -\frac{1}{4} \lambda^2 \sum_{\ell=1}^N |\langle f u, e_\ell(\lambda) \rangle|^2.
\]

This can be used to show that for a generic \(f \in C_0^\infty(\Omega; \mathbb{R})\), where \(\emptyset \neq \Omega \subset X_0\) is an open set, there are no embedded eigenvalues. First one shows that for a generic \(f\) all the embedded eigenvalues are simple and that follows from showing generic simplicity of eigenvalues for the reference operator, that is for the pseudo-Laplacian in Example 2 in §4.3. Then (4.4.24) can be used to show that any finite number of eigenvalues become resonances under a perturbation – see [CdV83] for details (a less direct argument than (4.4.24) is used there). See also the proof Theorem 2.23 for an example of the scheme for proving generic results.

**4.4.3. Definition of the scattering matrix.** We now use the plane waves (4.4.1) to define the scattering matrix and to obtain its representation. This is very similar to what has been done in §3.7 and we leave the proofs as exercises for the reader.
The first result is an adaptation of the boundary pairing result of Theorem 3.39. The proof is left as an exercise.

**THEOREM 4.23 (Boundary pairing for black box Hamiltonians).**

Let $P$ be a black box Hamiltonian in the sense of Definition 4.1. Suppose that $u_\ell \in D_{\text{loc}}$, $\ell = 1, 2$ satisfy
\[
(P - \lambda^2)u_\ell = F_\ell \in H_{\text{comp}}, \quad \lambda \in \mathbb{R} \setminus \{0\},
\]
with $f_\ell, g_\ell \in C^\infty(S^{n-1})$, and the expansion valid also for derivatives with respect to $\partial_r$. Then
\[
2i\lambda \int_{S^{n-1}} (g_1 \bar{g}_2 - f_1 \bar{f}_2) \, d\omega = \langle F_1, u_2 \rangle_H - \langle u_1, F_2 \rangle_H.
\]

The interesting case comes from considering $F_\ell \equiv 0$ in which case the incoming or outgoing data can be prescribed. The proof again follows directly from the proof of Theorem 3.42.

**THEOREM 4.24 (Prescribing incoming data in black box scattering).** Let $P$ be a black box Hamiltonian in the sense of Definition 4.1. Then for $\lambda \in \mathbb{R} \setminus \{0\}$ and any $g \in C^\infty(S^{n-1})$ there exist unique $f \in C^\infty(S^{n-1})$ and $v \in D_{\text{loc}}$ such that
\[
(P - \lambda^2)v = 0,
\]
with $f_\ell, g_\ell \in C^\infty(S^{n-1})$, and the expansion valid also for derivatives with respect to $\partial_r$. Then
\[
2i\lambda \int_{S^{n-1}} (g_1 \bar{g}_2 - f_1 \bar{f}_2) \, d\omega = \langle F_1, u_2 \rangle_H - \langle u_1, F_2 \rangle_H.
\]

Using this two theorems we can now defined the absolute and relative scattering matrices for black box scattering – see Definition 3.40 for more motivation.

**DEFINITION 4.25 (Scattering matrix).** In the notation of Theorem 4.24, the map
\[
S_{\text{abs}}(\lambda) : C^\infty(S^{n-1}) \to C^\infty(S^{n-1}), \quad S_{\text{abs}}(\lambda) : g \mapsto f,
\]
is called the absolute scattering matrix. By Theorem 4.23 it extends to a unitary transformation
\[
S_{\text{abs}}(\lambda) : L^2(S^{n-1}) \to L^2(S^{n-1}).
\]
The scattering matrix is defined as
\[
S(\lambda) = i^{n-1}S_{\text{abs}}(\lambda)J, \quad Jg(\theta) := g(-\theta).
\]

Representation of the Schwartz kernel of $S(\lambda)$ in terms of the resolvent is again the same as in §3.7:
THEOREM 4.26 (Description of the scattering matrix). Let $P$ be a black box Hamiltonian. For $\rho \in C_c^\infty(\mathbb{R}^n)$ define
\[ E_\rho : L^2(\mathbb{R}^n) \to L^2(S^{n-1}), \quad E_\rho(\lambda)(x,\omega) := \rho(x)e^{-i\lambda(x,\omega)}. \]
Choose $\chi_i \in C_c^\infty(\mathbb{R}^n; [0,1]), i = 1, 2, 3,$ such that for some $R_1 > R_0,$
\[ \chi_i|_{B(0,R_1)} = 1, \quad \chi_i|_{\text{supp } \chi_i} = 1, \quad i = 1, 2. \]

Then the scattering matrix is given by
\[ (4.4.29) \quad S(\lambda) = I + a_n\lambda^{n-2}E\chi_3(\lambda)[\Delta,\chi_1]R(\lambda)[\Delta,\chi_2]E\chi_3(\tilde{\lambda})^*, \]
where $R(\lambda)$ is the extension of $(P - \lambda^2)^{-1}$ and $a_n := (2\pi)^{-n+1}/2i.$

EXAMPLES. 1. Obstacle scattering. Suppose $P = -\Delta_O,$ the Dirichlet Laplacian on a connected set $\mathbb{R}^n \setminus O$ where $O$ is bounded and $\partial O$ is smooth. Then comparison of (4.4.29) with (4.4.1) (used with $\chi = \chi_2$ and noting that $(1 - \chi_2)[\Delta,\chi_1] \equiv 0$ shows that
\[ S_O(\lambda) = I + A_O(\lambda), \]
\[ (4.4.30) \quad A_O(\lambda,\omega,\theta) = a_n\lambda^{n-2} \int_{\mathbb{R}^n} e^{-i\lambda(x,\omega)}[-\Delta,\chi_1]e(-\lambda,\theta,x)dx, \]
where $e(\lambda,\theta,x)$ is the unique function satisfying
\[ (-\Delta - \lambda^2)e(\lambda,\theta,x) = 0, \quad e(\lambda,\theta,\bullet)|_{\partial O} = 0, \]
\[ e(\lambda,\theta,x) = e^{-i\lambda(x,\theta)} + \frac{e^{i\lambda|\nu|}}{|\nu|^{(n-1)/2}} \left( h(\frac{x}{|\nu|}) + O(\frac{1}{|\nu|}) \right). \]

We now apply Green’s formula in (4.4.30) noting that, because of support properties of $\chi_1,$ we can change the domain of integration to $\mathbb{R}^n \setminus O$:
\[ \int_{\mathbb{R}^n \setminus O} e^{-i\lambda(x,\omega)}[-\Delta,\chi_1]e(-\lambda,\theta,x)dx \]
\[ = \int_{\mathbb{R}^n \setminus O} e^{-i\lambda(x,\omega)}(\chi_1(x)(\Delta e(-\lambda,\theta,x)) - \Delta(\chi_1(x)e(-\lambda,\theta,x))dx \]
\[ = \int_{\mathbb{R}^n \setminus O} \left( -\lambda^2 e^{-i\lambda(x,\omega)}\chi_1 e(-\lambda,\theta,x) - e^{-i\lambda(x,\omega)}\Delta(\chi_1(x)e(-\lambda,\theta,x)) \right)dx \]
\[ = \int_{\mathbb{R}^n \setminus O} \Delta e^{-i\lambda(x,\omega)}\chi_1 e(-\lambda,\theta,x) - e^{-i\lambda(x,\omega)}\Delta(\chi_1(x)e(-\lambda,\theta,x))dx. \]

Applying Green’s formula gives the following representation of $A_O$ in terms of the normal derivative of the plane wave at the boundary:
\[ A_O(\lambda,\omega,\theta) = \frac{i\lambda^{n-2}}{2(2\pi)^{n+1}} \int_{\partial O} e^{-i\lambda(x,\omega)}\partial_\nu e(-\lambda,\theta,x)dS(x). \]
2. **Surfaces with cusps.** We now consider the case of scattering on surfaces (4.1.22). We will also compute the scattering explicitly in the case of scattering on the modular surface $X = SL_2(\mathbb{Z}) \backslash \mathbb{H}^2$.

In the case of surface with $N$ cusps the scattering matrix appears in (4.4.23). If we compare that expansion with the expansion in Theorem 4.24 we see that we need to replace $S^{n-1}$ with the discrete set of points, $\mathbb{Z}/N\mathbb{Z}$: the boundaries of the cusps at infinity. Then the scattering matrix acts on $L^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ is in fact a matrix:

\[
\begin{align*}
S(\lambda) : \mathbb{C}^N &\to \mathbb{C}^N, \\
S(\lambda)u_j &= \sum_{\ell=1}^{N} s_{j\ell}(\lambda)u_{\ell},
\end{align*}
\]

with $s_{j\ell}(\lambda)$’s given by (4.4.23).

3. **The Modular surface.** The discrete group

\[
SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \ ad - cb = 1 \right\}
\]

acts on

\[
(\mathbb{H}^2, g) := \left\{ (x, y) : x, y \in \mathbb{R}, y > 0 \right\}, \quad \frac{dx^2 + dy^2}{y^2},
\]

by linear fractional transformations

\[
x + iy = z \mapsto \frac{az + b}{cz + d}.
\]
4.4. PLANE WAVES AND THE SCATTERING MATRIX

The action of $SL_2(\mathbb{Z})$ is then generated by two transformations

\[(4.4.32)\]
\[
S(z) := -\frac{1}{z}, \quad T(z) := z + 1,
\]

and the famous fundamental domain of its action is given by

\[(4.4.33)\]
\[
\{(x, y) : -\frac{1}{2} < x < 0, (1 - x^2)^{\frac{1}{2}} < y\} \cup \\
\{(x, y) : 0 \leq x \leq \frac{1}{2}, (1 - x^2)^{\frac{1}{2}} \leq y\},
\]

see Fig. 4.2 and [Ah78, §7.2] for an elementary presentation. The surface $X$ is given by $SL_2(\mathbb{Z}) \setminus \mathbb{H}^2$. (The compact part $X_0$ is not smooth as there are two conic singularities but that does not cause trouble in the analysis and we neglect this point.) Functions on $X$ can be identified with functions on $\mathbb{H}^2$ invariant under the action of $SL_2(\mathbb{Z})$. That only needs to be checked for the two generators (4.4.32).

We now want to give a (relatively) explicit construction of the generalized plane waves $e(x, \lambda)$ given by (4.4.1) in general (with no dependence on $\omega$) and (4.4.23) (with $N = 1$) for the case of surfaces with cusps. In keeping with traditional notation we will put

\[(4.4.34)\]
\[
s = \frac{1}{2} - i\lambda, \quad y^s = e^{r/2}e^{-i\lambda r}, \quad y^{1-s} = e^{r/2}e^{i\lambda r}, \quad y = e^r.
\]

We then define, for $c, d \in \mathbb{Z},$

\[
B(z, s) := \sum_{(c, d) \neq (0, 0)} \frac{y^s}{|cz + d|^{2s}}, \quad z = x + iy \in \mathbb{H}^2.
\]

We first note that the sum converges for $\text{Re} \gg 1$ and that

\[
\frac{y^s}{|cz + d|^{2s}} = \gamma^s(y^s), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Since $-\Delta_y(y^s) = -y^2 \partial_y^2(y^s) = s(s - 1)y^s$, it follows that

\[-\Delta_y B(z, s) = s(1 - s)B(z, s), \quad s \gg 1.\]

We also see that

\[B(z + 1, s) = B(z, s), \quad B(-1/z, s) = B(z, s).\]

In fact, the first identity is obvious and for the second we note that

\[
B(-1/z, s) = \sum_{(c, d) \neq (0, 0)} \frac{(\text{Im}(1/z))^s}{|\text{Re}(1/z)|^2s} = \sum_{(c, d) \neq (0, 0)} \frac{y^s}{|dz - c|^{2s}} = B(z, s).
\]

Hence, $B(z, s)$ defines a function on $X = SL_2(\mathbb{Z}) \setminus SL_2(\mathbb{R})$.

We now look at $B(z, s)$ in

\[X_0 \simeq \{(x, y) : -\frac{1}{2} < x \leq \frac{1}{2}, \quad y > 2\}\]
We write
\[ B(z, s) = \sum_{d \neq 0} d^{-2s} y^s + \sum_{c \neq 0} \frac{y^s}{|cz + d|^{2s}} =: 2\zeta(2s)y^s + w(x, y, s). \]

As in (4.4.23), we calculate the “scattering” component of \( w \):
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} w(x, y, s)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{c \neq 0} \frac{y^s}{|cz + d|^{2s}} dx
= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{c \neq 0} \sum_{k \in \mathbb{Z}} \sum_{\ell=0}^{\lfloor c \rfloor-1} \frac{y^s}{((c(x + k) + \ell)^2 + c^2y^2)^s}
= 2c \sum_{c \neq 0} \frac{1}{|c|^{2s-1}y^{1-s}} \int_{\mathbb{R}} \frac{1}{(1 + x^2)^s} dx
= 2\zeta(2s - 1) \frac{\pi^{\frac{s}{2}} \Gamma(s - \frac{1}{2})}{\Gamma(s)} y^{1-s}.
\]

Here we used some facts about beta functions: with the substitution \( t = x^2 / (1 + x^2) \), \( dx = \frac{1}{2} t^{-1/2} (1 - t)^{-3/2} dt \),
\[
\int_{\mathbb{R}} \frac{1}{(1 + x^2)^s} dx = 2 \int_{0}^{\infty} \frac{1}{(1 + x^2)^s} dx = \int_{0}^{1} t^{-\frac{s}{2}} (1 - t)^{-\frac{1}{2} + s} dt
= \frac{\Gamma(s)}{\Gamma(s - \frac{1}{2})} \frac{\pi^{\frac{1}{2}} \Gamma(s - \frac{1}{2})}{\Gamma(s)},
\]
see for instance [HöI, (3.4.8),(3.4.9)].

Putting,
\[ e(z, s) := \frac{1}{2\zeta(2s)} B(z, s), \quad \text{Re } s \gg 1 \]
we have found a function on \( X \) such that
(4.4.36)
\[ (-\Delta_g - s(1 - s)) e(z, s) = 0, \quad \text{Re } s \gg 1, \]
\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} e(z, s)|X|dx = y^s + \frac{\zeta(2s - 1) \pi^{\frac{s}{2}} \Gamma(s - \frac{1}{2})}{\zeta(2s)} y^{1-s}. \]

The second term on the right hand side of the last equality is in \( L^2(X_1, d\text{Vol}_g) \),
\( d\text{Vol}_g = y^{-2} dx \, dy \) for \( \text{Re } s \gg 1 \). If \( \chi \in C_c^\infty([0, \infty)) \) is equal to 1 for \( y < 2 \) then proceeding as in (4.4.35) we see that
\[ e(z, s) - (1 - \chi(y))y^s \in L^2(X, d\text{Vol}_g), \quad \text{Re } s \gg 1. \]
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But, as in the construction of \( e(y, \lambda) \) in (4.4.1) this means that

\[
e(z, s) = (1 - \chi(y))y^s + (\Delta_g - s(1 - s))^{-1}([-\Delta_g, \chi]y^s).
\]

The meromorphic continuation of \((\Delta_g - s(1 - s))^{-1} = (\Delta_g - \frac{1}{4} - \lambda^2)^{-1}\) shows that \( e(z, s) \) is meromorphic in \( \mathbb{C} \) with no poles for \( \text{Re} \ s = \frac{1}{2}, \ s \neq \frac{1}{2} \).

Returning to (4.4.23) and (4.4.31) (and keeping in mind the change of convention (4.4.34)) we obtain the scattering matrix for the modular surface – in this case a number:

\[
S(\lambda) = \pi^{\frac{1}{2}} \frac{\Gamma(-i\lambda)\zeta(2i\lambda)}{\Gamma(\frac{1}{2} - i\lambda)\zeta(1 - 2i\lambda)}.
\]

The unitarity (modulus one) on the real axis follow from general scattering theory or from the properties of the Riemann zeta function and the Gamma function.

Theorem 4.27 below shows that the resonances of \( P = -\Delta_g - \frac{1}{4} \) are given by \( \lambda \) such that

\[
\lambda^2 \in \text{Spec}_{pp}(P) \quad \text{or} \quad \Gamma(\frac{1}{2} - i\lambda)\zeta(1 - 2i\lambda) = 0.
\]

The latter condition means that \( 1 - 2i\lambda \) is a non-trivial zero of the zeta function. Hence, the Riemann hypothesis states that all resonances which do not come from eigenvalues lie on the line

\[
\text{Im} \ \lambda = -\frac{1}{4}.
\]

4.4.4. Resonance multiplicities. The representation of the scattering matrix given in Theorem 4.26 and the meromorphy of \( R(\lambda) \) show that \( S(\lambda) \) forms a meromorphic family of operators on \( \mathbb{C} \) (we assume here, as elsewhere, that \( n \) is odd – otherwise we have to work with the logarithmic plane). The unitarity relation shows that

\[
S(\lambda)^{-1} = S(\bar{\lambda})^*, \quad \lambda \in \mathbb{C}.
\]

We note that unitarity of \( S(\lambda) \) is valid for \( \lambda \in \mathbb{R} \setminus \{0\} \) by Theorem 4.23 and remains valid at \( \lambda = 0 \) by continuity which follows from the fact that \( S(\lambda) \) is meromorphic.

Definition 4.25 gives \( S_{\text{abs}}(\lambda) = S_{\text{abs}}(-\lambda)^{-1} \) and hence (see Theorem 3.43)

\[
S(\lambda)^{-1} = JS(-\lambda)J, \quad Jf(\theta) = f(-\theta), \quad \lambda \in \mathbb{C}.
\]

The set of poles is contained in the set of poles of \( R(\lambda) \) and the multiplicity is defined using the Gohberg–Sigal theory in (4.4.37) below. The precise relation between the multiplicities of the poles of \( S(\lambda) \) and \( R(\lambda) \) is given as follows:
THEOREM 4.27 (Equivalence of multiplicities). Suppose $P$ is a black box Hamiltonian and $S(\lambda)$ is its scattering matrix. Then

$$m_S(\lambda) = \tilde{m}_R(\lambda) - \tilde{m}_R(-\lambda),$$

(4.4.37)

$$m_S(\lambda) := -\frac{1}{2\pi} \oint_{\lambda} \text{tr} \partial_\zeta S(\zeta) S(\zeta)^{-1} d\zeta,$$

$$\tilde{m}_R(\lambda) = \text{rank} \mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} \Pi_{\lambda},$$

where the integral is over a positively oriented circle enclosing $\lambda$ and not other pole of $S(\lambda)$ or $S(\lambda)^{-1}$, and $\Pi_{\lambda}$ is given by (4.2.20).

REMARKS. 1. As always high multiplicities cause problems in the analysis. We will prove the theorem under the assumption that $m_R(\lambda) = 1$ when $\lambda \not\in \mathbb{R}$. To get the general statement we can use the approach from Theorem 3.14 to perturb resonances or Theorem 4.39 in §4.5. A direct proof using the method of complex scaling can be found in Nedelec [Ne04].

2. We note that $\tilde{m}_R(\lambda) = m_R(\lambda)$ for $\lambda^2 \not\in \mathbb{R}$ – any compactly supported resonant states have to be eigenfunctions. (See also Exercise 4.7). Hence (4.4.37) really means that

$$m_S(\lambda) = \begin{cases} m_R(\lambda) & \text{Im} \lambda < 0, \\ -m_R(\lambda) & \text{Im} \lambda > 0, \end{cases}, \quad \lambda^2 \not\in \mathbb{R},$$

and the difference is relevant only on the imaginary axis.

An artificial example in which $\tilde{m}_R(\lambda) < m_R(\lambda)$, $\lambda \in i(0, \infty)$ is given as follows. Consider the Dirichlet realization of $P = -\Delta + V$ on $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, where we assume that $\Omega$ has two connected components, $\Omega_1$, $\Omega_2$, with $\Omega_1$ unbounded, $\Omega_2$ bounded and $\partial \Omega_2$ smooth. We can find $V \in C_\infty^c(\Omega_2)$ such that $-\Delta + V$ with Dirichlet boundary conditions has an eigenvalue $-t^2$, $t > 0$. Then $0 = \tilde{m}_R(it) < m_R(it)$.

Proof. 1. Since $S(\lambda)$ is unitary for $\lambda \in \mathbb{R}$ we have $m_S(\lambda) = 0$ for $\lambda \in \mathbb{R}$. On the other hand for $\lambda^2 > 0$ the only poles of $R(\lambda)$ come from embedded eigenvalues – see part (ii) of Theorem 4.18 and hence $\tilde{m}_R(\lambda) = \tilde{m}_R(\lambda) = 0$.

This means that we only need to establish (4.4.37) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

2. We now prove the theorem under the assumption that $m_R(\lambda) = 1$ – see Remark 1 after the statement of the theorem on how to remove that hypothesis.

3. From Theorem C.8 we recall that

$$m_S(\lambda) = N_\lambda(S^{-1}) - N_\lambda(S),$$
where, near $\lambda$, with invertible and holomorphic $U(\zeta), V(\zeta)$,

$$S(\zeta) = U(\zeta)(P_0 + \sum_{m=1}^{M} (\zeta - \lambda)^{km} P_m)V(\zeta),$$

(4.4.38) \hspace{1cm} P_m P_\ell = \delta_{km}, \quad \text{rank}(I - P_0) = M, \quad \text{rank} P_m = 1, \quad m \neq 0,

$$N_\lambda(S^{-1}) = -\sum_{k_\ell < 0} k_\ell, \quad N_\lambda(S) = \sum_{k_\ell > 0} k_\ell.$$

Hence to prove (4.4.37) it is enough to show that

(4.4.39) \hspace{1cm} N_\lambda(S^{-1}) = \tilde{m}_R(\lambda).

From the discussion in Step 2, we can assume that $\tilde{m}_R(\lambda) \leq 1$. From (4.4.29) and (4.4.38) we see that, under the simplicity assumption $\tilde{m}_R(\lambda) \geq N_\lambda(S^{-1})$.

4. To see that $\tilde{m}_R(\lambda) = 1$ implies that $S(\zeta)$ has a pole at $\lambda$ we write

$$1 \| R_n \setminus B(0,R_0) \| \| R_n \setminus B(0,R_0) \| \| \varphi, v \| R_n \setminus B(0,R_0) \| H, \quad \varphi \in H_{\text{comp}},$$

where $(P - \lambda^2)u = 0, (P^* - \bar{\lambda}^2)v = 0$. From (4.4.29) we see that

$$S(\zeta) = a_n \zeta^{n-2} \frac{F \otimes G}{\zeta^2 - \lambda^2} + T(\zeta),$$

where $T$ is holomorphic near $\lambda$ and

$$F(\omega) = \hat{f}(\lambda \omega), \quad f(x) := [\Delta, \chi_1]u(x),$$

$$G(\theta) = \hat{g}(-\bar{\lambda} \theta), \quad g(x) := [\Delta, \chi_2]v(x).$$

We also note that

$$(-\Delta - \lambda^2)(1 - \chi_1)u = f, \quad (-\Delta - \bar{\lambda}^2)(1 - \chi_2)v = g.$$

5. To prove that $S(\zeta)$ has a pole at $\zeta = \lambda$ we need to show that $F \neq 0$ and $G \neq 0$. We note that $f \in C_c^\infty(\mathbb{R}^n)$ and hence $\hat{f}$ is an entire function on $\mathbb{C}^n$. If $F \equiv 0$ then

$$f(e^{i\theta} \eta) \equiv 0 \quad \text{for } \eta \in \Sigma \cap \mathbb{R}^n,$$

where

$$\lambda = e^{i\theta} |\lambda|, \quad \Sigma := \{ \eta \cdot \eta = |\lambda|^2 \} \subset \mathbb{C}^n, \quad \eta \cdot \eta = \sum_{j=1}^{n} \eta_j^2, \quad \eta \in \mathbb{C}^n.$$

Since $\Sigma \cap \mathbb{R}^n$ is a totally real submanifold of the complex variety $\Sigma$, $f(e^{i\theta} \eta) \equiv 0$ for $\eta \in \Sigma$. This implies that

$$W(\xi) = \frac{f(\xi)}{\xi \cdot \xi - \lambda^2} = \frac{e^{-2i\theta} f(e^{i\theta} \eta)}{\eta \cdot \eta - |\lambda|^2}, \quad \xi = e^{i\theta} \eta, \quad \xi \in \mathbb{C}^n,$$
is an entire function. Paley-Wiener theorem as applied in [H61, Theorem 7.3.2] shows that $W = \hat{w}$, $w \in C_c^\infty(\mathbb{R}^n)$. By Theorem 4.9

$$(\Delta - \lambda^2)((1 - \chi_1)u - w) = 0,$$

$$((1 - \chi_1)u - w)|_{\mathbb{R}^n \setminus B(0,R)} = (R_0(\lambda)g)|_{\mathbb{R}^n \setminus B(0,R)} ,$$

where $g \in L^2_{\text{comp}}(\mathbb{R}^n \setminus B(0,R_0))$. Applying the opposite implication in Theorem 4.9 with $P = -\Delta$ we conclude that $(1 - \chi_1)u - w$ would be a resonant state for $-\Delta$. Hence $(1 - \chi_1)u = w \in C_c^\infty(\mathbb{R}^n)$ which contradicts the fact that $u$ is a nontrivial element of $\mathbb{I}_{\mathbb{R}^n \setminus B(0,R_0)} \Pi_\lambda(H_{\text{comp}})$.

We conclude that $F \neq 0$.

A similar argument shows that $G \neq 0$ and hence $\tilde{m}_R(\lambda) = 1$ implies that $N_\lambda(S^{-1}) = 1$, completing the proof. □

4.5. COMPLEX SCALING

So far resonances for black box perturbations were defined as poles of the meromorphic continuation of the resolvent. The structure of that continuation near the poles given in Theorem 4.7 shows that it behaves like a resolvent of a non-self-adjoint operator. The method of complex scaling allows an equivalent definition: instead continuing the resolvent, the Hamiltonian $P$ is deformed to a non-self-adjoint Hamiltonian $P_\theta$ and the resonances $\lambda^2$ with $\arg \lambda^2 > -2\theta$ are the eigenvalues of $P_\theta$: for $-2\theta < \arg z < 2\pi - 2\theta$, $P_\theta - z$ is a Fredholm operator on suitable spaces. The advantage of this method is both practical and theoretical. It allows numerical calculation of resonances by discretizing $P_\theta$ and it provides access to method from spectral theory of non-self-adjoint operators. That will become particulary apparent in Part 3.

In the simple but instructive setting of dimension one the method was described in Section 2.7.

4.5.1. The complex scaled operator. For $0 \leq \theta < \pi$, let $\Gamma_\theta \subset \mathbb{C}^n$ be the following deformation of $\mathbb{R}^n \subset \mathbb{C}^n$:

$$\Gamma_\theta \cap B_{\mathbb{C}^n}(0,R_1) = B_{\mathbb{R}^n}(0,R_1) ,$$

$$\Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,R_2) = e^{i\theta} \mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0,R_2) ,$$

$$\Gamma_\theta = f_\theta(\mathbb{R}^n), \quad f_\theta : \mathbb{R}^n \to \mathbb{C}^n \text{ is injective.}$$

(4.5.1)

Here we take $R_0 < R_1 < R_2$ where $R_0$ is the same as in (4.1.1). Since no deformation is performed in $B(0,R_1)$ we can consider $\Gamma_\theta \setminus B(0,R_1)$ as a deformation of $\mathbb{R}^n \setminus B(0,R_1)$.

The next definition and lemma will allow us to define the deformation of $P$ to an operator on $C^\infty(\Gamma_\theta)$ in a coordinate free way.
DEFINITION 4.28 (Totally real submanifolds). (i) An n-dimensional smooth submanifold \( M \) of \( \mathbb{C}^n \cong \mathbb{R}^{2n} \) is called (maximally) totally real if for any \( m \in M \),
\[
T_m M \cap iT_m M = \{0\},
\]
where we identify the tangent space \( T_m M \) with a subspace of \( \mathbb{R}^{2n} \). (ii) If \( u \in C^\infty(M) \) then \( \tilde{u} \in C^\infty(\mathbb{C}^n) \) is called an almost analytic extension of \( u \) if
\[
\bar{\partial}_j \tilde{u}(z) = O(\text{dist}(z, M)^\infty), \quad z \in \mathbb{C}^n.
\]

We first see that it is easy to find totally real submanifolds of the form (4.5.1):

LEMMA 4.29 (Totally real deformations of \( \mathbb{R}^n \)). Suppose that \( \Gamma_\theta \subset \mathbb{C}^n \) is given by (4.5.1). Then it is totally real if and only if
\[
\det(\partial_x f_\theta) \neq 0.
\]
In particular, if \( 0 \leq \theta < \pi/2 \) and
\[
f_\theta(x) = x + i \partial_x f_\theta : \mathbb{R}^n \to \mathbb{C}^n
\]
where \( f_\theta : \mathbb{R}^n \to \mathbb{R} \) is a convex function then \( \Gamma_\theta \) is totally real.

Proof. 1. Since (4.5.4) is a statement about derivatives we can assume that \( f_\theta =: A \in M_{n \times n}(\mathbb{C}) \) is a linear function and \( M := \Gamma_\theta \) is a (real) linear subspace of \( \mathbb{C}^n \cong \mathbb{R}^{2n} \), \( M = A(\mathbb{R}^n) \).

2. The condition that \( M \cap iM \neq 0 \) is equivalent to the existence of an invertible \( B \in M_{n \times n}(\mathbb{C}) \) (a complex linear transformation) such that \( B(M) = \mathbb{R}^n \subset \mathbb{C}^n \). Indeed, if \( M \) is totally real and \( e_1, \ldots, e_n \) is a real basis of \( M \), then it is a complex basis of \( \mathbb{C}^n \). We then define by \( B \) as mapping \( e_1, \ldots, e_n \) to the canonical basis of \( \mathbb{C}^n \). Since \( B(ix) = iB(x) \) the converse is clear.

3. Hence, if \( \det A \neq 0 \) we can take \( B = A^{-1} \) and \( M \) is totally real. If \( M \) is totally real then \( B \circ A : \mathbb{R}^n \to \mathbb{R}^n \) is an invertible (real) matrix, and in particular \( \det A \neq 0 \).

For simplicity we will restrict our attention to the case of \( \Gamma_\theta \) given by
\[
0 \leq \theta < \pi/2, \quad f_\theta \text{ given by (4.5.5) where } F_\theta \text{ is convex.}
\]

For scaling for angles greater than \( \pi/2 \) (which, as we will see, would recover all resonances in odd dimension case) see [SZ91].
EXAMPLE. We can construct a convex $F_\theta$ so that $\Gamma_\theta = f_\theta(\mathbb{R}^n)$ with $f_\theta$ given by (4.5.5) satisfies (4.5.1) as follows. Take $g \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$g(t) = 0 \text{ for } t \leq R_1, \quad g(t) = \frac{1}{2}t^2 \text{ for } t \geq 2R_1, \quad g''(t) \geq 0.$$ 

We note that these assumptions imply that $g'$ is non-decreasing and hence $g' \geq 0$. We then put

$$F_\theta(x) = \tan \theta g(|x|), \quad 0 \leq \theta < \frac{1}{2}\pi.$$ 

It follows that

$$\frac{\partial^2}{\partial x} F_\theta(x) = \tan \theta \left( \frac{g'(|x|)}{|x|^3} (|x|^2 I - x \otimes x) + \frac{g''(|x|)}{|x|^2} x \otimes x \right)$$

is positive definite and $\frac{\partial^2}{\partial x} F_\theta = \tan \theta I$ for $|x| > 2R_1$.

We now show that smooth functions on totally real submanifolds have almost analytic extensions which can use to define restrictions of holomorphic differential operators:

**LEMMA 4.30 (Totally real submanifolds and almost analytic extensions).** Suppose that $M$ is a totally real submanifold of $\mathbb{C}^n$. Then every $u \in C^\infty(M)$ has an almost analytic extension to $\mathbb{C}^n$ in the sense of (4.5.3).

If $\widetilde{P} = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_z^\alpha$ is a holomorphic differential operator near $M$ ($a_\alpha$ are holomorphic in $\mathbb{C}^n$ near $M$) then $\widetilde{P}$ defines a unique differential operator $P_M$ whose action on $C^\infty(M)$ is given by

$$(4.5.7) \quad P_M u = \left. \widetilde{P}(\widetilde{u}) \right|_M.$$

**Proof.** 1. We recall that for any $v \in C^\infty(\mathbb{R}^n)$ and an open set $V \subset \mathbb{C}^n$, supp $v \subset V$, we can find $\widetilde{v} \in C^\infty(\mathbb{C}^n)$ such that

$$\text{supp } \widetilde{v} \subset V,$$

$$\text{supp } v \subset V,$$

$$\frac{\partial}{\partial z_j} \widetilde{v}(z) = \mathcal{O}(|\text{Im } z|^{\infty}), \quad z = x + iy, \quad x, y \in \mathbb{R}^n,$$

see for instance [DS99, (8.1),(8.2)]. That gives almost analytic extensions in the case of $M = \mathbb{R}^n$.

2. Using a partition of unity we only need to construct extensions of $u \in C^\infty_c(U)$, where $U \subset M$ is given by $U = f(B(0, r))$, $B(0, r) = \{ x \in \mathbb{R}^n : |x| < r \}$, $f : \mathbb{R}^n \rightarrow \mathbb{C}^n$. From Lemma 4.29 we see that $\partial_x f(x)$ is non-degenerate for $x$ in a neighbourhood of $B(0, r)$ (we can decrease $r$). Let $\widetilde{f} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be an almost analytic extension of $f$. Then, $\partial_z \widetilde{f}|_{B(0, r)} = \partial_x f$ is non-degenerate. Hence for some neighbourhoods $\widetilde{V}, \widetilde{U} \subset \mathbb{C}^n$ of $B(0, r)$ and $U$ respectively, $\widetilde{f} : \widetilde{V} \rightarrow \widetilde{U}$ is a diffeomorphism. In addition, $\widetilde{f}^{-1}$ is almost analytic as well.
To see this we write $z = x + iy$, $\zeta = \xi + i\eta$, $x, y, \xi, \eta \in \mathbb{R}^n$, $z = \bar{f}(\zeta, \bar{\zeta})$ and $\zeta = \bar{g}(z, \bar{z})$, $\bar{g} := \tilde{f}^{-1}$. Then, since $d(z, M) \sim |\operatorname{Im}\zeta|$

$$\frac{\partial z}{\partial \zeta} \begin{bmatrix} \partial_z \zeta & \partial_z \bar{\zeta} \\ \partial_{\bar{\zeta}} \zeta & \partial_{\bar{\zeta}} \bar{\zeta} \end{bmatrix}^{-1} = \begin{bmatrix} \partial_z z & O(|\operatorname{Im}\zeta|) \\ O(|\operatorname{Im}\bar{\zeta}|) & \frac{1}{\partial_z \bar{\zeta}} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (\partial_z z)^{-1} O(|\operatorname{Im}\zeta|) \\ O(|\operatorname{Im}\bar{\zeta}|) (\partial_z \bar{\zeta})^{-1} \end{bmatrix} \cdot \begin{bmatrix} \partial_z \zeta & O(d(M, z)^\infty) \\ O(d(M, z)^\infty) & \partial_{\bar{\zeta}} \bar{\zeta} \end{bmatrix}$$

3. Using (4.5.8) we now define

$$\tilde{u} = \bar{v} \circ \tilde{f}^{-1}, \quad v := u \circ f \in C^\infty_c(B(0, r)), \quad \supp \bar{v} \subset \bar{V}.$$

Using almost analyticity of $\bar{v}$ and $\tilde{f}^{-1}$ we obtain

$$\partial_{\bar{z}_j} \tilde{u} = \frac{\partial \zeta}{\partial \bar{z}_j} \cdot \frac{\partial \bar{v}}{\partial \zeta} + \frac{\partial \bar{\zeta}}{\partial \bar{z}_j} \cdot \frac{\partial \bar{v}}{\partial \bar{\zeta}} = O(d(M, z)^\infty),$$

which shows that $\tilde{u}$ is an almost analytic continuation of $u$.

4. Arguing as in Steps 2 and 3, we can assume that $M = \mathbb{R}^n$ and that the coefficients of $\tilde{P}$ satisfy $\partial_z a_\alpha(z) = O(|\operatorname{Im} z|^\infty)$. To define $P R^n$, we then to show that if $\tilde{u}$ is almost analytic and $u|_{\mathbb{R}^n} = 0$ then $\bar{P}\tilde{u}|_{\mathbb{R}^n} = 0$. But this follows by induction from showing that $\partial_z \tilde{u}|_{\mathbb{R}^n} = 0$, which is obvious since $\partial_z \tilde{u} = \partial_z \tilde{u} + O(|\operatorname{Im} z|^\infty)$.

We can now define the complex scaled operator, $P_\theta$:

**DEFINITION 4.31 (The complex scaled operator).** Suppose that $\Gamma_\theta$ is given by (4.5.1) with $f_\theta$ satisfying (4.5.6). Suppose that $P$ is a black box Hamiltonian in the sense of Definition 4.1. With $\chi \in C^\infty_c(B(0, R_1))$ equal to 1 on $B(0, R_0)$, (so that $1 - \chi$ is a smooth function on $\mathbb{R}^n$ and $\Gamma_\theta$), define

$$\mathcal{H}_\theta = \mathcal{H}_{R_0} \oplus L^2(\Gamma_\theta \setminus B(0, R_0)),$$

$$\mathcal{D}_\theta = \{u \in \mathcal{H}_\theta : \chi u \in \mathcal{D}, \quad (1 - \chi) u \in H^2(\Gamma_\theta), \quad P_\theta u = P(\chi u) + (-\Delta_\theta)((1 - \chi) u), \}$$

where

$$\Delta_\theta := \Delta_{\Gamma_\theta}$$

is defined using (4.5.7).
Using $F_\theta$ in (4.5.6) we calculate $-\Delta_\theta$. First we note that for $z = x + i\partial_x F_\theta(x)$ we have

$$\frac{\partial}{\partial z} = \left( \frac{\partial x}{\partial z} \right)^T \frac{\partial}{\partial x} = (I + iF''_\theta(x))^{-1} \frac{\partial}{\partial x},$$

where $F''_\theta(x) := \partial^2 F_\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric matrix.

Since $-\Delta_\theta = \partial_z \cdot \partial_z$ (see (4.5.7)) we obtain, in the coordinates $x \in \mathbb{R}^n$ on $\Gamma_\theta$,

$$-\Delta_\theta u = \left((I + iF''_\theta(x))^{-1} \partial_x \right) \cdot \left((I + iF''_\theta(x))^{-1} \partial_x u\right), \quad u \in C^\infty(\Gamma_\theta).$$

The symbol of this operator is given by

$$\sigma(\Delta_\theta)(x, \xi) = \left((I + iF''_\theta(x))^{-1} \xi\right) \cdot \left((I + iF''_\theta(x))^{-1} \xi\right).$$

Here we used $(x, \xi)$ as coordinates on $T^* \Gamma_\theta \simeq T^* \mathbb{R}^n$.

We now have

**THEOREM 4.32 (Ellipticity of $\Delta_\theta$).** The operator $\Delta_\theta$ defined in (4.5.11) is an elliptic differential operator of order two:

$$|\xi|^2/C \leq |\sigma(\Delta_\theta)(x, \xi)| \leq C|\xi|^2$$

*Proof.* By homogeneity in $\xi$ we need to show that for $\xi \neq 0$,

$$\left((I + iF''_\theta(x))^{-1} \xi\right) \cdot \left((I + iF''_\theta(x))^{-1} \xi\right) \neq 0.$$

Noting that $I + (F''_\theta(x))^2$ is invertible ($F''_\theta(x)$ is real and symmetric) it is enough to show that for $\eta \neq 0$,

$$\left((I - iF''_\theta(x))\eta\right) \cdot \left((I - iF''_\theta(x))\eta\right) \neq 0, \quad \eta := (I + (F''_\theta(x))^2)^{-1} \xi \neq 0.$$

The left hand side is equal to

$$|\eta|^2 - |F''_\theta(x)\eta|^2 - 2i\langle F''_\theta(x)\eta, \eta\rangle.$$

Since $F''_\theta(x)$ is positive semidefinite (our convexity assumption),

$$\langle F''_\theta(x)\eta, \eta\rangle = 0 \implies F''_\theta(x)\eta = 0.$$

This concludes the proof as then the real part is equal to $|\eta|^2$. \[\square\]

Using invertibility of

$$-e^{-2i\theta} \Delta - \lambda^2 = e^{-2i\theta}(-\Delta - (e^{i\theta}\lambda)^2) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

for $\text{Im}(e^{i\theta}\lambda) > 0$ and the ellipticity of $\Delta_\theta$ it is easy to show that

$$-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$$

is a Fredholm operator for $\text{Im}(e^{i\theta}\lambda) > 0$,

see Exercise 4.8. Instead we will show that $-\Delta_\theta - \lambda^2$ is invertible for this range of $\lambda$’s and describe the inverse using the kernel of $R_\theta(\lambda)$. 

4.5.2. The resolvent of $-\Delta_\theta$. We will construct explicitly the inverse of $-\Delta_\theta - \lambda^2$ where $\Delta_\theta$ is given by (4.5.11):

$$R_\theta(\lambda) := (-\Delta_\theta - \lambda^2)^{-1} : L^2(\Gamma_\theta) \to H^2(\Gamma_\theta), \quad \text{Im}(e^{i\theta} \lambda) > 0.$$ 

The operators $R_\theta(\lambda)$ will then form a holomorphic family of operators in the half plane $-\theta < \arg \lambda < \pi - \theta$.

The idea is to extend $|(x - y)\cdot(x - y)|^{\frac{1}{2}}$ holomorphically from $\mathbb{R}^n \times \mathbb{R}^n$ to a complex neighbourhood of

$$\bigcup_{0 \leq \theta \leq \theta_0} \Gamma_\theta \times \Gamma_\theta \subset \mathbb{C}^n \times \mathbb{C}^n.$$ 

Hence we start with:

**LEMMA 4.33 (Well defined square root)**. Suppose that $\Gamma_\theta$ is given by (4.5.1) with $f_\theta$ given by (4.5.6), $0 \leq \theta < \pi/2$. Then for $z, w \in \Gamma_\theta$,

$$\text{Im}(z - w) \cdot (z - w) \geq 0,$$

where for $v, v' \in \mathbb{C}^n$, $v \cdot v' := v_1v'_1 + \cdots + v_nv'_n$. Moreover,

$$\langle A(x,y) : \mathbb{R} \to \mathbb{R} \rangle ((z - w) \cdot (z - w))^{\frac{1}{2}} \geq 0, \quad z, w \in \Gamma_\theta.$$

**Proof.** 1. For $z = x + i\partial_x F_\theta(x)$ and $w = y + i\partial_x F(y)$

$$\text{Im}(z - w) \cdot (z - w) = 2\langle F'_\theta(x) - F'_\theta(y), x - y \rangle.$$

2. Define

$$A(x, y) := \int_0^1 F''_\theta(tx + (1 - t)y)dt,$$

which, by the convexity assumption on $F_\theta$ is a positive semidefinite matrix. Since $F'_\theta(x) - F'_\theta(y) = A(x, y)(x - y)$,

$$\langle F'_\theta(x) - F'_\theta(y), x - y \rangle = \langle A(x, y)(x - y), x - y \rangle \geq 0,$$

proving (4.5.17).

3. To see (4.5.18) we note that $\langle A(x, y)(x - y), x - y \rangle = 0$ implies $A(x, y)(x - y) = 0$, that is $F'_\theta(x) - F'_\theta(y) = 0$. Hence $0 = (z - w) \cdot (z - w) = \langle x - y, x - y \rangle$, implies $x = y$. 

We can now proceed with the following definition:
**DEFINITION 4.34 (Complex scaling of the free resolvent).** For $\text{Im}(\lambda e^{i\theta}) > 0$, $0 \leq \theta < \pi/2$, we define the following operator $C_c^\infty(\Gamma_{\theta}) \to C^\infty(\Gamma_{\theta})$:

$$R_{\theta}(\lambda) \varphi(z) = \int_{\Gamma_{\theta}} R_0(\lambda, z - w) \varphi(w) dw, \quad \varphi \in C_c^\infty(\Gamma_{\theta}),$$

where, in the notation of (3.1.16) and (4.5.19),

$$R_0(\lambda, w) = \frac{e^{i\lambda (w \cdot w)^{\frac{1}{2}}} P_n(\lambda (w \cdot w)^{\frac{1}{2}})}{((w \cdot w))^{n-2}} d w.$$ 

**INTERPRETATION.** The element of integration, $d w$, is given by

$$d w = d w_1 \wedge \cdots d w_n = \text{det}(I + i F_0'(y)) dy,$$

where $w = y + i \partial_y F_0(y) \in \Gamma_{\theta}$. The integral in (4.5.22) should then be considered as a contour integral – see [Zw12, §13.2] for a down-to-earth review. For $u, v \in C_c^\infty(\Gamma_{\theta})$, let $\tilde{u}, \tilde{v} \in C_c^\infty(\mathbb{C}^n)$ be their almost analytic extensions. We have

$$\partial_{w_j} \tilde{u}(w) d w_1 \wedge \cdots d w_n = (-1)^{j-1} d(\tilde{u}(w) d w_1 \cdots d w_{j-1} \wedge d w_{j+1} \cdots d w_n)$$

$$- \sum_{k=1}^{n} \tilde{\partial}_{w_k} \tilde{u} d w_k \wedge d w_1 \cdots d w_{j-1} \wedge d w_{j+1} \cdots d w_n,$$

and the second term on the right vanishes on $\Gamma_{\theta}$. Hence by Stokes’s theorem

$$\int_{\Gamma_{\theta}} \partial_{w_j} \tilde{u}(w) v(w) dw = - \int_{\Gamma_{\theta}} u(w) \partial_{w_j} \tilde{v}(w) dw.$$

The next result shows that $R_{\theta}(\lambda)$ is in fact the inverse of $-\Delta_{\theta} - \lambda^2$:

**THEOREM 4.35 (Resolvent of $\Delta_{\theta}$).** For $\text{Im}(e^{i\theta} \lambda) > 0$, $0 \leq \theta < \pi/2$, $R_{\theta}(\lambda) : C_c^\infty(\Gamma_{\theta}) \to C^\infty(\Gamma_{\theta})$ given by (4.5.22) extends to an operator

$$R_{\theta}(\lambda) : L^2(\Gamma_{\theta}) \to H^2(\Gamma_{\theta}),$$

which is the two sided inverse of $-\Delta_{\theta} - \lambda^2$. Moreover, for $\delta > 0$

$$\text{Im} \lambda > \delta \text{ Re} \lambda \geq 0 \implies R_{\theta}(\lambda) = \mathcal{O}(\text{Im} \lambda^{-2}) : L^2(\Gamma_{\theta}) \to L^2(\Gamma_{\theta}).$$

**Proof.** 1. We first refer to Theorem 3.3 for the properties of $P_n$ appearing in (4.5.23). They give

$$|R_0(\lambda, w)| \leq C e^{\text{Re}(\lambda (w \cdot w)^{\frac{1}{2}})} ((w \cdot w)^{\frac{1}{2}})^{2-n} + \lambda^{\frac{n-3}{2}} |(w \cdot w)^{\frac{1}{2}}|^{\frac{n+1}{2}}.$$
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In fact,
\[ |\zeta^{2-n} P_n(\lambda \zeta)| \leq a_n|\zeta|^{2-n}(1 + |\lambda \zeta| + \cdots + |\lambda \zeta|^{n-1}) \]
\[ \leq a_n^{n-1}|\zeta|^{2-n}(1 + |\lambda \zeta|^{n-1}) \leq b_n(|\zeta|^{2-n} + |\lambda|^{n-1} |\zeta|^{-n-1}). \]

2. Let \( \delta := \Im(e^{i\theta} \lambda)/|\lambda| \). Since \( \Gamma_\theta \cap \mathbb{C}^n \setminus B(0, R_2) = e^{i\theta} \mathbb{R}^n \setminus B(0, R_2) \),
\[ \Re \left( i\lambda ((z - w) \cdot (z - w))^{1/2} \right) = \Re \left( i\lambda (e^{i\theta} + \mathcal{O}((1 + |z| + |w|)^{-1})) \right) |z - w| \]
\[ = - \Im(e^{i\theta} \lambda) |z - w| + \mathcal{O}((1 + |z| + |w|)^{-1}) |\lambda| |z - w| \]
\[ = - \Im(e^{i\theta} \lambda) (1 + \mathcal{O}(\delta^{-1} (1 + |z| + |w|)^{-1})) |z - w|, \quad z, w \in \Gamma_\theta. \]

From (4.5.18) and again from the fact that \( \Gamma_\theta \) agrees with \( e^{i\theta} \mathbb{R} \) outside of a ball in \( \mathbb{C}^n \), we deduce that
\[ \left| (z - w) \cdot (z - w) \right| \geq |z - w|/C, \quad z, w \in \Gamma_\theta. \]

We also trivially have
\[ \Re \left( i\lambda ((z - w) \cdot (z - w))^{1/2} \right) \leq |\lambda||z - w|. \]

Using these three inequalities in (4.5.23) and (4.5.28) gives, with \( C_j \)'s depending on \( \lambda \) and \( \delta \),
\[ \int_{\Gamma_\theta} |R_0(\lambda, z - w)||dw| \leq C_0 \int_{\Gamma_\theta \cap \{|z-w|\geq C_0\}} e^{-|z-w|/C_0} |z - w|^{n-1}|dw| \]
\[ + C_0 \int_{\Gamma_\theta \cap \{|z-w|\leq C_0\}} |z - w|^{2-n}|dw| \]
\[ \leq C_1 \int_0^{\infty} e^{-r/C_0} r^{n-1} dr + C_1 \leq C_2, \]

where \(|dw| = |\det(I + iF''_\theta(y))|dy\). The same estimate holds when we integrated with respect to \(|dz|\). Hence the boundedness on \( L^2(\Gamma_\theta) \) follows from Schur's criterion (A.5.3). Since Theorem 4.32 gave us ellipticity of \( \Delta_\theta \), (4.5.26) follows.

3. Now suppose that for \( 0 < \delta \leq 1, \Im \lambda > \delta \Re \lambda \geq 0 \). Then for \( \zeta \in \mathbb{C} \) satisfying \( \Re \zeta, \Im \zeta \geq 0 \),
\[ \Re(i\lambda \zeta) = - \Im \lambda \Re \zeta - \Re \lambda \Im \zeta \]
\[ \leq - \Im \lambda (\Re \zeta + \delta \Im \zeta) \leq - \delta \Im \lambda |\zeta|. \]

In view of (4.5.20) we can apply this inequality with \( \zeta := ((z - w) \cdot (z - w))^{1/2} \) to obtain
\[ \Re(i\lambda ((z - w) \cdot (z - w))^{1/2}) \leq - \delta \Im \lambda |((z - w) \cdot (z - w))^{1/2}|, \quad z, w \in \Gamma_\theta. \]
Hence, under our assumption on $\lambda$, this and (4.5.29) give

$$\exp(i\lambda((z-w) \cdot (z-w))^{\frac{1}{2}}) \leq \exp(-\delta \Im \lambda |z-w|/C), \quad z, w \in \Gamma_\theta.$$  

We now proceed as in Step 2 and use (4.5.28). However we keep track of the dependence on $\lambda$ with the constants depending on $\delta$ and changing from line to line:

$$\int_{\Gamma_\theta} |R_0(\lambda, z-w)|^2 dw \leq C |\Im \lambda|^{\frac{n-3}{2}} \int_{\Gamma_\theta} \frac{e^{-\Im \lambda|z-w|/C}}{|z-w|^{\frac{n-1}{2}}} dw$$

$$+ C \int_{\Gamma_\theta} \frac{e^{-\Im \lambda|z-w|/C}}{|z-w|^{2-n}}|dw|$$

$$\leq C |\Im \lambda|^{\frac{n-3}{2}} \int_{\Gamma_\theta} e^{-\Im \lambda |z-w|^2/C} + C \int_{\Gamma_\theta} e^{-\Im \lambda r} r^{n-1} dr$$

$$= C |\Im \lambda|^{-2}.$$  

This proves (4.5.27).

4. Suppose that $\varphi, \psi \in C_c^\infty(\Gamma_\theta)$, with $\tilde{\varphi}$ and $\tilde{\psi}$ being the corresponding almost analytic extensions.

Since

$$\Delta_\theta \psi = \left( \sum_{j=1}^{n} \partial^2_{x_j} \tilde{\psi} \right)|_{\Gamma_\theta},$$

integration by parts using (4.5.25) shows that

(4.5.30) $$\int_{\Gamma_\theta} \Delta_\theta \varphi(w)\psi(w) dw := \int_{\Gamma_\theta} \varphi(w)\Delta_\theta \psi(w) dw.$$  

Hence to show that $R_\theta(\lambda)$ given by (4.5.22) is the left and right inverse of $-\Delta_\theta - \lambda^2$ it is enough to show that for $\varphi \in C_c^\infty(\Gamma_\theta)$ we have

(4.5.31) $$\int_{\Gamma_\theta} ((-\Delta_\theta)_z - \lambda^2) R_0(\lambda, z-w) \varphi(w) dw = \varphi(z),$$

$$\int_{\Gamma_\theta} ((-\Delta_\theta)_w - \lambda^2) R_0(\lambda, z-w) \varphi(w) dw = \varphi(z).$$

Since $(\Delta_\theta)_z R_0(\lambda, z-w) = (\Delta_\theta)_w R_0(\lambda, z-w)$, we only need to prove the first identity.

5. We first show that

(4.5.32) $$((-\Delta_\theta)_z - \lambda^2) R_0(\lambda, z-w) = 0 \quad z \neq w, \ z, w \in \Gamma_\theta.$$  

In fact, let $\Omega$, be an open set such that

$$\bigcup_{0 \leq \theta' \leq \theta} \Gamma_{\theta'} \times \Gamma_{\theta'} \setminus \Delta(C^n) \subset \Omega \subset \mathbb{C}^n \times \mathbb{C}^n \setminus \Delta(C^n),$$
where \( \Delta(\mathbb{C}^n) := \{(z, z) : z \in \mathbb{C}^n\} \). By taking \( \Omega \) small enough we can assume that its connected components intersect \( \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta(\mathbb{R}^n) \subset \mathbb{C}^n \times \mathbb{C}^n \setminus \Delta(\mathbb{C}^n) \) and that \( R_0(\lambda, z-w) \) for \( (z, w) \in \Omega \).

We note that \( \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta(\mathbb{R}^n) \subset \Omega_\theta \setminus \Delta(\mathbb{C}^n) \) is (maximally) totally real and
\[
(-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2) R_0(\lambda, z-w) |_{\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta(\mathbb{R}^n)} = 0.
\]

This implies that the Taylor series vanishes on that submanifold and we have
\[
(-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2) R_0(\lambda, z-w) |_{\Omega} = 0.
\]

Since \((-\Delta_\theta)z - \lambda^2) R_0(\lambda, z, w) = (-\sum_{j=1}^n \partial_{z_j}^2 - \lambda^2) R_0(\lambda, z-w) |_{z,w \in \Gamma_\theta} \), the equations (4.5.32) follows.

6. For a fixed \( z \in \Gamma_\theta \), (4.5.23) shows that \( R_0(\lambda, z-w) \in H^{-n/2+2}_loc(\Gamma_\theta)_w \).

On the other hand (4.5.32) shows that the support of

\[
((-\Delta_\theta)z - \lambda^2) R_0(\lambda, z-w) = ((-\Delta_\theta)_w - \lambda^2) R_0(\lambda, z-w) \in H^{-n/2+2}_loc((\Gamma_\theta)_w),
\]

is contained in \( \{z\} \). Since as a function of \( z \in \Gamma_\theta \), \( R_0(\lambda, z-w) \) is a smooth family of distributions in \( w \), Schwartz’s Lemma [HöI, Theorem 2.3.4] shows that

\[
((-\Delta_\theta)z - \lambda^2) R_0(\lambda, z,w) = c_\theta(z) \delta_z(w), \quad c_\theta \in L^\infty(\Gamma_\theta),
\]

\[
\delta_z \in \mathcal{S}'(\Gamma_\theta), \quad \int_{\Gamma_\theta} \varphi(w) \delta_z(w) dw = \varphi(z), \quad \varphi \in \mathcal{S}(\Gamma_\theta).
\]

We need to show that \( c_\theta(z) \equiv 1 \). For that put

\[
\varphi(z) := \exp(-e^{-i2\theta}z \cdot z), \quad z \in \mathbb{C}^n, \quad \varphi' := \varphi|_{\Gamma'\theta}, \quad 0 \leq \theta' < \pi/2.
\]

Then \( \varphi' \in \mathcal{S}(\Gamma') \) for \( |\theta' - \theta| < \pi/4, 0 \leq \theta' < \pi/2 \).

A contour deformation and the decay and holomorphy of \( \varphi \) show that
\[
\int_{\Gamma_\theta} R_0(\lambda, z-w) \varphi(w) dw = \int_{-z+\Gamma_\theta} R_0(\lambda, w) \varphi(z+w) dw
\]
\[
= \int_{\Gamma_\theta} R_0(\lambda, w) \varphi(z+w) dw.
\]

(We use Stokes’s theorem – see for instance [Zw12, §13.2.1] – and note that \( \Gamma_\theta \) can be deformed to \( \Gamma_\theta - z \) using totally real submanifolds all passing through \( w = 0 \) on which \( R_0(\lambda, w) \) is holomorphic for \( w \neq 0 \).) Hence

\[
z \mapsto \int_{\Gamma_\theta} R_0(\lambda, z-w) \varphi(w) dw,
\]
and
\[ z \mapsto G(z) := (-\sum_{j=1}^{n} \partial w \varphi(w)dw, \]
are holomorphic in a neighbourhood \( V \) of \( \bigcup_{0<\theta<\pi/2} \Gamma_{\theta} \).

On the other hand \( G(z) = c_{\theta}(z)\varphi(z) \) for \( z \in \Gamma_{\theta} \). It follows that \( c_{\theta} = c|_{\Gamma_{\theta}} \) where \( c \) is a holomorphic function in \( V \). Since \( c|_{\Gamma_{\theta}} = c_{\mathbb{R}^n} = 1 \) it follows that \( c \equiv 1 \) proving (4.5.31).

7. We have shown that \( R_{\theta}(\lambda) : L^2(\Gamma_{\theta}) \rightarrow L^2(\Gamma_{\theta}) \) and that \( (\Delta_{\theta} - \lambda^2)R_{\theta}(\lambda) = I_\theta \). Theorem 4.32 and the fact that \( -\Delta_{\theta} = -e^{-2i\theta}\Delta_x, \ z = x + iF_{\theta}(x) \in \Gamma_{\theta} \), show that \( R_{\theta} : L^2(\Gamma_{\theta}) \rightarrow H^2(\Gamma_{\theta}) \) completing the proof. \( \square \)

4.5.3. Fredholm properties of \( P_{\theta} \). Using the invertibility of \( -\Delta_{\theta} - \lambda^2 \), for \( \text{Im}(e^{i\theta} \lambda) > 0 \) established in Theorem 4.35 we now show that for the same range of \( \lambda \)’s, \( P_{\theta} - \lambda^2 \) is a Fredholm operator:

**THEOREM 4.36 (Fredholm property of the scaled operator).** Let \( P_{\theta}, D_{\theta} \text{ and } H_{\theta}, 0 \leq \theta < \pi/2 \), be given in Definition 4.31.

If \( \text{Im}(e^{i\theta} \lambda) > 0 \) then

\[ P_{\theta} - \lambda^2 : D_{\theta} \rightarrow H_{\theta}, \]

is a Fredholm operator of index 0. In particular the spectrum of \( P_{\theta} \) in \( \mathbb{C} \setminus e^{-2i\theta}[0, \infty) \) is discrete.

**REMARK.** The condition for the Fredholm property is formulated in terms of \( \lambda \) though of course in terms of \( \lambda^2 \) it means that \( \lambda^2 \in \mathbb{C} \setminus e^{-2i\theta}[0, \infty) \).

**Proof.** The strategy of the proof is to construct \( Q_{\theta}(\lambda) \) and \( S_{\theta}(\lambda) \) such that for \( \lambda^2 \notin e^{-2i\theta}[0, \infty) \),

\[ (P_{\theta} - \lambda^2)Q_{\theta} = I + K_{\theta}(\lambda), \quad S_{\theta}(\lambda)(P_{\theta} - \lambda^2) = I + L_{\theta}(\lambda), \]

where \( K_{\theta}(\lambda) : H_{\theta} \rightarrow H_{\theta}, L_{\theta}(\lambda) : D_{\theta} \rightarrow D_{\theta} \) are compact operators. That will show the Fredholm property (see Remark 1 at the end of §C.3). To see that the index is 0 we will show that for some \( \lambda_0, I + K_{\theta}(\lambda_0) \) and \( I + L_{\theta}(\lambda_0) \) are invertible. That means that \( P - \lambda_0^2 : D_{\theta} \rightarrow H_{\theta} \) is invertible. Hence, for all \( \lambda \notin e^{-2i\theta}[0, \infty) \) the index is 0 – see Theorem C.4.

1. To find \( Q_{\theta} \) We follow the proof of Theorem 4.4. Let \( R_j, j = 0, 1 \) be as in (4.5.1) and let \( \chi_j \in C_c^\infty(B(0, R_1)) \) satisfy \( \chi_0 \equiv 1 \) on \( B(0, R_0 + \epsilon) \), \( \epsilon > 0 \),

\[ \chi_j(x) \equiv 1 \text{ for } x \in \text{supp} \chi_{j-1}, \quad j = 1, 2. \]
For \( \lambda \) and \( \lambda_0 \) satisfying \( \text{Im}(e^{i\theta} \lambda_0) > 0 \), \( \text{Im} \lambda_0 > 0 \) and \( \text{Im}(e^{i\theta} \lambda) > 0 \) (\( \lambda_0 \) will be chosen later) we put

\[
Q_\theta(\lambda, \lambda_0) := (1 - \chi_0) R_\theta(\lambda)(1 - \chi_1) + \chi_2(P - \lambda_0^2)^{-1} \chi_1.
\]

Then

\[
(P_\theta - \lambda^2) Q_\theta(\lambda, \lambda) = I + K_\theta(\lambda, \lambda_0),
\]

\[
K_\theta(\lambda, \lambda_0) := [\Delta, \chi_0] R_\theta(\lambda)(1 - \chi_1) - [\Delta, \chi_2](P - \lambda_0^2)^{-1} \chi_1 + (\lambda_0^2 - \lambda^2) \chi_2(P - \lambda_0^2)^{-1} \chi_1.
\]

2. We now show that \( K_\theta(\lambda, \lambda_0) : \mathcal{H}_\theta \to \mathcal{H}_\theta \) forms a family of compact operators and we do this by analysing individual terms.

First, compactness of the terms involving \( (P - \lambda_0^2)^{-1} \) follows from Lemma 4.14: the singular values of these operators go to 0. Theorem 4.35 shows that

\[
(\Delta, \chi_0) R_\theta(\lambda)(1 - \chi_1) : \mathcal{H}_\theta \to \mathcal{H}^1(B(0, R_1) \setminus B(0, R_0))
\]

Since \( \mathcal{H}^1(B(0, R_1) \setminus B(0, R_0)) \) embeds compactly in \( L^2(B(0, R_1) \setminus B(0, R_0)) \subset \mathcal{H}_\theta \) (see Theorem B.3) the term involving \( R_\theta \) is also compact.

3. It remains to show that we can find \( \lambda_0 \) satisfying \( \text{Im} \lambda_0 > 0 \) and \( \text{Im}(e^{i\theta} \lambda_0) > 0 \) for which \( I + K_\theta(\lambda_0, \lambda_0) \) is invertible. For that we note that if \( \lambda_0 = e^{\pi i/4} \mu \), \( \mu \gg 1 \), then the conditions are satisfied and in addition \( \text{Re} \lambda > 0 \). Hence the argument in Step 2 of the proof of Theorem 4.4 applies using (4.5.27) in Theorem 4.35. That completes the analysis of \( Q_\theta \).

4. The construction of \( S_\theta \) is similar and we use the same notation:

\[
S_\theta(\lambda, \lambda_0) := (1 - \chi_1) R_\theta(\lambda)(1 - \chi_0) + \chi_1(P - \lambda_0^2)^{-1} \chi_2,
\]

so that

\[
S_\theta(\lambda, \lambda)(P_\theta - \lambda^2) = I + L_\theta(\lambda, \lambda_0),
\]

\[
L_\theta(\lambda, \lambda_0) := (1 - \chi_1) R_\theta(\lambda)[\Delta, \chi_0] - \chi_1(P - \lambda_0^2)^{-1}[\Delta, \chi_2] + (\lambda_0^2 - \lambda^2) \chi_1(P - \lambda_0^2)^{-1} \chi_2.
\]

Since \( S_\theta(\lambda, \lambda_0) : \mathcal{H}_\theta \to \mathcal{D}_\theta \), we see that \( L_\theta(\lambda, \lambda_0) : \mathcal{D}_\theta \to \mathcal{D}_\theta \).

5. Same argument as in Step 2 shows that \( L_\theta(\lambda, \lambda_0) \) is compact as an operator from \( \mathcal{H}_\theta \) to \( \mathcal{D}_\theta \). This implies that it is enough to show that

\[
(P_\theta - \lambda_0^2)L_\theta(\lambda, \lambda_0) : \mathcal{D}_\theta \to \mathcal{H}_\theta \quad \text{is compact.}
\]
We expand the operator on the left (using the support properties of $\chi_j$’s given in Step 1):

\[(\ref{eq:4.5.37})\]
\[\bar{L}_\theta(\lambda,\lambda_0) := [\Delta, \chi_1 R_\theta(\lambda)] [\Delta, \chi_0] + [\Delta, \chi_1](P - \lambda_0^2)^{-1}[\Delta, \chi_2] - (\lambda_0^2 - \lambda^2)[\Delta, \chi_1](P - \lambda_0^2)^{-1}\chi_2.\]

Arguing as in Step 2 shows that $\tilde{L}_\theta(\lambda,\lambda_0) : D_\theta \hookrightarrow H_\theta \rightarrow H_\theta$, is compact. It remains to show that multiplication by $\chi_1$ from $D_\theta \rightarrow H_\theta$ is compact. But Definition 4.31 gives
\[
\chi_1 D_\theta = \chi_1 D = \chi_1 (P + i)^{-1} H \hookrightarrow \chi_1 H = \chi_1 H_\theta,
\]
and the second inclusion is compact by (4.1.12).

6. It remains to show that for $\lambda_0$ chosen in Step 3, $(I + L_\theta(\lambda_0,\lambda_0))^{-1} : D_\theta \rightarrow D_\theta$ exists. Same argument as in Step 3 shows that the inverse exists as a maps $H_\theta \rightarrow H_\theta$. Then in the notation of (4.5.37),

\[(P_\theta - \lambda_0^2)(I + L_\theta(\lambda_0,\lambda_0))^{-1} = P_\theta - \lambda_0^2 + (P_\theta - \lambda_0^2)L_\theta(\lambda_0,\lambda_0)(I + L_\theta(\lambda_0,\lambda_0))^{-1}
\]

\[= P_\theta - \lambda_0^2 + \bar{L}_\theta(\lambda_0,\lambda_0)(I + L_\theta(\lambda_0,\lambda_0))^{-1}
\]

\[= \mathcal{O}(1)_{D_\theta \rightarrow H_\theta} + \mathcal{O}(1)_{H_\theta \rightarrow H_\theta} = \mathcal{O}(1)_{D_\theta \rightarrow H_\theta},\]

which concludes the proof: we have found left and right inverses of $P_\theta - \lambda_0^2 : D_\theta \rightarrow H_\theta$, and approximate inverses at any $\lambda$. \(\square\)

4.5.4. Resonances as eigenvalues of $P_\theta$. We established the fact that the spectrum of $P_\theta$ in $\mathbb{C} \setminus e^{-i\theta}[0,\infty)$ is discrete. It is remarkable that the spectrum does not depend on $\theta$ and coincides, with agreement of multiplicities, with the squares of resonances with $-\theta < \arg \lambda < \pi - \theta$ (that is satisfying $\text{Im}(e^{i\theta}\lambda) > 0$). In an equivalent convention we would simply say that the spectrum coincides with the resonances – see §2.1.

For $\text{Im}(e^{i\theta}\lambda) > 0$, Theorems 4.36 and C.6 show that

\[\Pi_\lambda^\theta := \frac{1}{2\pi i} \oint_\lambda (\zeta^2 - P_\theta)^{-1} 2\zeta d\zeta,\]

is a finite rank projection. The multiplicity of the eigenvalue of $P_\theta$ at $\lambda^2$ is given by the trace of this projection:

\[(\ref{eq:4.5.38})\]
\[m_\theta(\lambda) := \text{tr} \Pi_\lambda^\theta, \quad \text{Im}(e^{i\theta}\lambda) > 0.\]

The agreement with resonances follows from the agreement of resolvents in the interaction region:
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THEOREM 4.37 (Agreement of the resolvents away from scaling).
Let $R_0 < R_1$ be as in the definition of the black box Hamiltonian (§4.1) and $\Gamma_\theta$ (4.5.1).

If $\chi \in C_c^\infty(B(0, R_1))$ is equal to one near $B(0, R_0)$ then

\[
\chi(P - \lambda^2)^{-1} = \chi(P_\theta - \lambda^2)^{-1} \quad \text{for } \Im(e^{i\theta} \lambda) > 0.
\]

Proof. 1. From the proof of Theorem 4.36 we see that, if we choose $\chi_3$ (in the notation of (4.5.33)) so that

\[(1 - \chi_3)K_\theta(\lambda) = 0, \quad K_\theta(\lambda)\chi_3 = K_0(\lambda)\chi_3, \quad K_\theta(\lambda) := K_\theta(\lambda, \lambda_0),\]

where we used Definition 4.34. Then as in Step 3 of the proof of Theorem 4.4,

\[
(P_\theta - \lambda^2)^{-1} = Q_\theta(\lambda)(I + K_\theta(\lambda))^{-1}
\]
\[
= Q_\theta(\lambda)((I + K_\theta(\lambda))(1 - \chi_3))(I + K_\theta(\lambda)\chi_3))^{-1}
\]
\[
= Q_\theta(\lambda)(I + K_\theta(\lambda)\chi_3)^{-1}(I + K_\theta(\lambda)(1 - \chi_3))
\]
\[
= Q_\theta(\lambda)(I + K_0(\lambda)\chi_3)^{-1}(1 + K_\theta(\lambda)(1 - \chi_3)).
\]

We also recall from (4.2.14) in the proof of Theorem 4.4 that

\[
(P - \lambda^2)^{-1} = Q_0(\lambda)(I + K_0(\lambda)\chi_3)^{-1}(I + K_0(\lambda)(1 - \chi_3)).
\]

2. For $\chi$ in the statement of the theorem we choose $\chi_0 \in C_c^\infty(B(0, R_1))$ in the definition of $Q_\theta$ and $K_\theta$ so that $\chi_0 = 1$ (and hence $\chi_j = 1, j = 1, 2, 3$) on supp $\chi$.

For $|\lambda - \lambda_0| \ll |\lambda_0|^{-2}$ the Neumann series argument shows that

\[
\chi_3(I + K_0(\lambda))^{-1} = (I + K_0(\lambda))^{-1}\chi,
\]

and hence this holds for all $\lambda$. Also,

\[
\chi Q_\theta(\lambda)\chi_3 = \chi Q_0(\lambda)\chi_3, \quad (1 - \chi_3)\chi = 0.
\]

Applying $\chi$ on both sides of (4.5.40) and (4.5.41) and using these two formulas proves (4.5.39).

It is now easy to prove

THEOREM 4.38. The spectrum of $P_\theta$ in $\mathbb{C} \setminus e^{-2i\theta}[0, \infty)$ agrees with resonances satisfying $\Im(e^{i\theta} \lambda) > 0$. More precisely,

\[
m_R(\lambda) = m_\theta(\lambda), \quad \Im(e^{i\theta} \lambda) > 0,
\]

where $m_R(\lambda)$ is the multiplicity of the resonance at $\lambda$ given in (4.2.18) and $m_\theta(\lambda)$ is the multiplicity of the eigenvalue of $P_\theta$ at $\lambda^2$ – see (4.5.38).
Proof. 1. Since $\Pi^\theta_{\lambda^2}$ is a projection we see that
\[ m_\theta(\lambda) = \text{tr} \Pi^\theta_{\lambda^2} = \text{rank} \Pi^\theta_{\lambda^2}. \]
Arguing as in the proof of (4.2.22) we see that
\[ m_\theta(\lambda) = \text{rank} \Pi^\theta_{\lambda^2} \chi. \]
where $\chi \in C^\infty_c(B(0, R_0))$ is equal to 1 near $B(0, R_0)$.
2. We now claim that
\[ (4.5.43) \quad m_\theta(\lambda) = \text{rank} \chi \Pi^\theta_{\lambda^2} \chi. \]
Otherwise there would exist solutions $v \in \mathcal{D}_\theta$ to $(P_\theta - \lambda^2)^k v = 0$, $u := (P_\theta - \lambda^2)^{k-1} v \neq 0$, satisfying $\chi v = 0$. But that would mean that $u$ can be indetified with an element of $H^2(\Gamma_\theta)$ satisfying
\[ (-\Delta_\theta - \lambda^2) u = 0, \quad u|_{B(0, R_0)} \equiv 0. \]
Since $-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta)$ is invertible (Theorem 4.35), $u \equiv 0$.
3. Combining (4.5.43) with (4.5.39) we see that
\[ m_\theta(\lambda) = \text{rank} \int_\lambda \chi(P - \zeta^2)^{-1} \chi 2\zeta d\zeta. \]
We need to show that this is the same as
\[ m_R(\lambda) = \text{rank} \int \chi(P - \zeta^2)^{-1} \chi 2\zeta d\zeta. \]
We now argue as in Step 2, to see that otherwise we would have solutions to $(-\Delta - \lambda^2) u = 0$ equal to 0 in $B(0, R_0)$. But unique continuation results for second order elliptic differential equations show that $u \equiv 0$. \qed
4.5.5. Applications. As the first application we provide

**Proof of Lemma 4.21.** 1. Since \( \lambda \in \mathbb{R} \setminus \{0\} \) we can take any \( \theta > 0 \) and use (4.5.38) and Theorem 4.38 to see that

\[
\Pi(0) := \frac{1}{2\pi i} \text{tr} \int_{\gamma} (\zeta^2 - P_\theta(0))^{-1} 2\zeta \, d\zeta,
\]

\[
1 = m_0(\lambda) := \text{tr} \Pi(0), \quad \gamma : t \to \lambda + \epsilon e^{2\pi it}, \quad t \in [0,1),
\]

where we write \( P_\theta(s) := (P(s))_\theta \), for the complex scaled operators in our family.

2. We now pose a Grushin problem for the operator \( P_\theta(0) \) – see §C.1. For that let we write the rank one projection \( \Pi(0) \) as

\[
(4.5.44) \quad \Pi(0) f = w \langle f, \bar{w} \rangle_{\mathcal{H}_\theta}, \quad w, \bar{w} \in \mathcal{D}_\theta, \quad \langle w, \bar{w} \rangle_{\mathcal{H}_\theta} = 1,
\]

\[
(P_\theta(0) - \lambda^2) w = 0, \quad (P_\theta(0)^* - \bar{\lambda}^2) \bar{w} = 0.
\]

The equation for \( \bar{w} \) follows from the fact that \( \Pi(0) \) commutes with \( P_\theta(0) \) and hence for all \( f \in \mathcal{D}_\theta, \)

\[
0 = (P_\theta(0) - \lambda^2) \Pi(0) f = \Pi(0)(P_\theta(0) - \lambda^2) f = w \langle (P_\theta(0) - \lambda^2) f, \bar{w} \rangle = w \langle f, (P_\theta(0)^* - \bar{\lambda}^2) \bar{w} \rangle.
\]

We then define

\[
\mathcal{P}_\theta(s,\zeta) := \begin{pmatrix} P_\theta(s) - \zeta^2 & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}_\theta \times \mathbb{C} \to \mathcal{H}_\theta \times \mathbb{C},
\]

\[
R_- := u_- w, \quad R_+ u := \langle u, \bar{w} \rangle.
\]

Writing

\[
(P_\theta(0) - \zeta^2)^{-1} = \frac{\Pi(0)}{\lambda^2 - \zeta^2} + Q(\zeta),
\]

where \( Q(\zeta) \) is holomorphic near \( \zeta = \lambda \), we check that

\[
\mathcal{E}(0,\zeta) := \begin{pmatrix} Q(\zeta) E_+ \\ E_- \end{pmatrix}, \quad E_+ v := v_+ w, \quad E_- v := \langle v, \bar{w} \rangle,
\]

satisfies

\[
\mathcal{P}_\theta(0,\zeta) \mathcal{E}(0,\zeta) = \mathcal{I}_{\mathcal{H}_\theta \times \mathbb{C}}, \quad \mathcal{E}(0,\zeta) \mathcal{P}_\theta(0,\zeta) = \mathcal{I}_{\mathcal{D}_\theta \times \mathbb{C}}.
\]

The only computation that requires some reflection is checking that \( R_+ Q(\zeta) = 0_{\mathcal{D}_\theta \to \mathbb{C}} \). But using (4.5.44),

\[
R_+ Q(\zeta) v = \langle Q(\zeta) v, \bar{w} \rangle = \langle (P_\theta(0) - \zeta^2)^{-1} v, \bar{w} \rangle - (\lambda^2 - \zeta^2)^{-1} \Pi(0) v, \bar{w} \rangle = 0.
\]
3. The assumption (4.4.14) shows that for $|s| \leq \sigma_0 \ll 1$
\[
\| (P_0(s) - P_0(0)) Q(\zeta) \|_{\mathcal{H}_0 \to \mathcal{H}_0} \leq \| P(s) - P(0) \|_{\mathcal{D} \to \mathcal{H}_{R_0}} \| Q(\zeta) \|_{\mathcal{H}_0 \to \mathcal{D}} < 1,
\]
\[
\| Q(\zeta)(P_0(s) - P_0(0)) \|_{\mathcal{D} \to \mathcal{D}} \leq \| P(s) - P(0) \|_{\mathcal{D} \to \mathcal{H}_{R_0}} \| Q(\zeta) \|_{\mathcal{H}_0 \to \mathcal{D}} < 1,
\]
that is, the assumptions of Lemma C.2 are satisfied. We conclude that we have an inverse depending smoothly on $s$ (and holomorphic in $\zeta$ near $\lambda$):
\[
E_\theta(s,\zeta) = \begin{pmatrix} Q(s,\zeta) & E_+(s,\zeta) \\ E_-(s,\zeta) & E_-^+(s,\zeta) \end{pmatrix},
\]
\[
E_+(s,\zeta)v_+ := v_+ w(s,\zeta), \quad E_-v := \langle v, \tilde{w}(s,\bar{\zeta}) \rangle, \quad E_-^+(0,\zeta) = \zeta^2 - \lambda^2.
\]
4. Since $\partial_\zeta E_-(0,\zeta) \neq 0$, we see that for small $s$, $E_-(s,\zeta) = 0$ has a unique smooth solution $\zeta = \lambda(s)$, $\lambda(0) = 0$. The Schur complement formula (C.1.1) then shows that $\lambda(s)$ is the unique simple eigenvalue of $P_\theta(s)$ near $\lambda$. The eigenfunction is then given by $w(s) := w(s,\lambda(s))$ and it depends smoothly on $s$. In the notation of (4.5.1) the resonant state of $P(s)$, $u(s)$ satisfies $1_{B(0,R_1)} u(s) = 1_{B(0,R_1)} w(s,\lambda(s))$. Since we can choose $R_1$ arbitrarily large, the smoothness of $s \mapsto u(s) \in \mathcal{D}_{\text{loc}}$ follows. This completes the proof of Lemma 4.21.

The next application is the higher dimensional version of Theorem 2.23. By a generic set we again mean an intersection of open dense sets. The space of perturbations we will consider is
\[
\dot{C}^\infty(B(0,R_1) \setminus B(0,R_0); \mathbb{R}) := \left\{ u \in C^\infty(\mathbb{R}^n) : \text{supp } u \subset B(0,R_1) \setminus B(0,R_0) \right\}.
\]
(This definition is self explanatory but we mention that the general notation comes from [HöIII, §B.2].)

**THEOREM 4.39.** Suppose that $P$ is a black box Hamiltonian in the sense of Definition 4.1. Then for any $R_1 > R_0$ there exists a generic set $\mathcal{V} \subset \dot{C}^\infty(B(0,R_1) \setminus B(0,R_0); \mathbb{R})$ such that all resonances of $P + V$, $V \in \mathcal{V}$, with
\[
-\frac{\pi}{2} < \arg \lambda < 0 \quad \text{are simple.}
\]

**REMARKS.**
1. The condition $\arg \lambda > -\pi/2$ can be eliminated in odd dimensions using large angle complex scaling of [SZ91] (and to $\arg \lambda > -2\pi + \epsilon$ in odd dimensions.) One can replace complex scaling by Agmon’s theory of resonance perturbation [Ag98] as in Borthwick–Perry [BP02].
2. The reason for demanding that $\arg \lambda < 0$ is the fact that compactly supported embedded eigenvalues cannot be split using perturbations supported away from the black box. The positive eigenvalues are typically unstable.
and can be perturbed to become resonances – see Theorem 4.22 and the example after its proof.

3. It is not clear how to prove this theorem following the same strategy as that in the proof of Theorem 2.23 as the situation in Step 4 of the proof is more complicated in higher dimensions.

Before proving Theorem 4.39 we state the following lemma.

**Lemma 4.40.** For \(0 < \theta < \pi/2\) let \(P_\theta\) be the complex scaled operator of a black box Hamiltonian (see Definition 4.31). If \(z_0, \arg z_0 > -2\theta\) is an eigenvalue of \(P_\theta\) (that is \(z_0 = \lambda^2\) where \(\lambda\) is a resonance of \(P\)), then for some \(K\) and \(\epsilon\) sufficiently small,

\[
(P_\theta - z)^{-1} = \sum_{k=1}^{K} \frac{(P_\theta - z_0)^{k-1}\Pi}{(z_0 - z)^k} + G(z),
\]

\[
\Pi := \frac{1}{2\pi i} \int_{\gamma} (z - P)^{-1} dz, \quad \gamma(t) = z_0 + i\epsilon e^{it}, \quad t \in [0, 2\pi),
\]

where \(G(z) : \mathcal{H}_\theta \to \mathcal{D}_\theta\) is a holomorphic family of operators for \(z\) near \(z_0\).

In addition there exist \(w_j, \tilde{w}_j \in \mathcal{D}_\infty^\infty, j = 1, \cdots, N = \text{tr} \Pi\), such that

\[
\Pi v = \sum_{j=1}^{N} w_j \langle v, \tilde{w}_j \rangle, \quad \langle w_k, \tilde{w}_j \rangle = \delta_{kj},
\]

\[
(P_\theta - z)^K w_j = 0, \quad (P_\theta^* \tilde{z})^K \tilde{w}_j = 0.
\]

**Proof.**
1. The expansion (4.5.45) follows as in the proof of Theorem 4.7: everything is easier now as we deal with a genuine resolvent, \((P_\theta - z)^{-1}\).

2. We only need to check (4.5.46). Since \(\Pi\) is an operator of rank \(N\), there exist a basis \(\{w_j\}_{j=1}^{N}\) of \(\Pi \mathcal{H}_\theta\). Also, there exist \(\tilde{w}_j \in \mathcal{H}_\theta\) such that the first formula in (4.5.46) holds. The projection property, \(\Pi w_k = w_k\), then shows that \(\langle w_k, \tilde{w}_j \rangle = \delta_{jk}\).

3. Since \(P_\theta\) commutes with \(\Pi\) it follows that \(w_j \in \mathcal{D}_\infty^\infty\). Since \(P_\theta^*\) commutes with \(\Pi^*\) it also follows that \(\tilde{w}_j\) are in the domain of \((P_\theta^*)^k\) which is the same as the domain of \((P_\theta)^k\), for any \(k\). Since \((P_\theta - z)^K \Pi = 0\) the last part of (4.5.46) follows. \(\square\)

**Proof of Theorem 4.39.**
1. We identify resonances \(\lambda\) with \(0 > \arg \lambda > -\theta\) with eigenvalues of \(z \in \text{Spec}(P_\theta), 0 > \arg z > -2\theta, z = \lambda^2\). We recall (and rename) the definition of multiplicity:

\[
m_V(z) := \frac{1}{2\pi i} \text{tr} \oint_{z} (\zeta - (P_\theta + V))^{-1} d\zeta,
\]
where the integral is over a sufficiently small positively oriented circle around $z$. We then define

$$E_\theta^n := \{ W \in C_{R_0,R_1} : m_W(z) \leq 1, \ z \in \Gamma_r \},$$

(4.5.47)

$$C_{R_0,R_1} := \mathcal{C}^\infty(B(0,R_1) \setminus B(0,R_0); \mathbb{R}).$$

$$\Gamma_r := \{ z : -\theta + 1/r \leq \arg z \leq -1/r, \ 1/r \leq |z| \leq r \}.$$

We want to show that for $r > 0$, $E_\theta^n$ is open and dense. That will show that the set

$$E_\theta := \{ W \in C_{R_0,R_1} : m_W(z) \leq 1 \text{ for } \arg z > -\theta \} = \bigcap_{n \in \mathbb{N}} E_\theta^n,$$

is generic (and in particular, by the Baire category theorem, it has a nowhere dense complement). By taking

$$\mathcal{V} := \bigcap_{n \in \mathbb{N}} E_{\pi/2 - 1/n},$$

we obtain the generic set in the statement of the theorem.

2. Suppose that $P_\theta + W$ has exactly one resonance $z_0$ in $D(z_0,2r)$. For $\Omega := D(z_0,r)$ we then define

(4.5.48) \[ \Pi_W(\Omega) := \frac{1}{2\pi i} \int_{\partial\Omega} (\zeta - (P_\theta + W))^{-1} d\zeta, \quad m_W(\Omega) := \text{tr} \Pi_W(\Omega). \]

If $V \in C_{B_0,B_1}$ and $\|V\|_\infty$ is sufficiently small then for $\zeta \in \partial\Omega$,

$$\ (P_\theta + W + V - \zeta)^{-1} = (P_\theta + W - \zeta)^{-1}(I + V(P_\theta + W - \zeta)^{-1})^{-1},$$

exists and we can define $\Pi_{W+V}$. This also shows that if $\|V\|_\infty < \epsilon$ for sufficiently small $\epsilon$ then for $\zeta \in \partial\Omega$,

$$\ (P_\theta + W - \zeta)^{-1} - (P_\theta + W + V - \zeta)^{-1} = O_\epsilon(\|V\|_\infty)_{\mathcal{H}_{\theta} \to \mathcal{H}_{\theta}}.$$ 

Hence,

$$\|\Pi_W(\Omega) - \Pi_{W+V}(\Omega)\|_{\mathcal{H}_{\theta} \to \mathcal{H}_{\theta}} \leq C_\epsilon \|V\|_\infty.$$

If we take $\|V\|_\infty < \min(\epsilon, 1/C_\epsilon)$ then $\Pi_W(\Omega)$ and $\Pi_{W+V}(\Omega)$ have the same rank. (If for two finite rank projections $P_1$ and $P_2$, $\|P_1 - P_2\| < 1$ then the ranks are the same. Indeed, we have $\|P_1 - P_2 P_1\| < 1$ and hence $P_1 P_2 P_1$ is invertible on the range of $P_1$ and rank $P_1 \leq$ rank $P_2$.)

We restate this as follows:

(4.5.49) $m_{W+V}(\Omega)$ is constant for $\|V\|_\infty$ sufficiently small.

3. The statement (4.5.49) immediately implies that $E_\theta^n$ is open: if $z$ is a simple resonance then $m_W(\Omega) = 1$ and it stays constant under small changes in the potential. (We could have also used the argument in the proof of Lemma (4.21) to see that but (4.5.49) will be needed later.)
4. Now we want to show that $E^r_\theta$ is dense. This follows from the following statement

$$\forall W \in \mathcal{C}_{B_0,B_1}, \, \epsilon > 0 \ \exists V \in \mathcal{C}_{B_0,B_1} \quad W + V \in E^r_\theta, \quad \|V\|_\infty < \epsilon.$$  \hspace{1cm} (4.5.50)

Since the number of eigenvalues of $P_\theta + W$ in $\Gamma_r$ (see eq:Ethetar) is finite it is enough to prove a local statement as it can be applied successively to obtain (4.5.50) (once an eigenvalue is simple it stays simple for sufficiently small perturbations by Step 3).

Hence to obtain (4.5.50) it suffices to prove, in the notation of Step 2,

$$\forall W \in \mathcal{C}_{B_0,B_1}, \, \epsilon > 0 \ \exists V \in \mathcal{C}_{B_0,B_1} \quad \forall z \in \Omega \quad m_{W+V}(z) \leq 1, \quad \|V\|_\infty < \epsilon.$$  \hspace{1cm} (4.5.51)

5. To establish (4.5.51) we proceed by induction. One of the following cases has to occur:

$$\forall \epsilon > 0 \ \exists V \in \mathcal{C}_{R_0,R_1}, \, z \in \Omega \quad 1 \leq m_{W+V}(z) < m_{W+V}(\Omega), \quad \|V\|_\infty < \epsilon,$$  \hspace{1cm} (4.5.52)

or

$$\exists \epsilon > 0 \ \forall V \in \mathcal{C}_{R_0,R_1}, \|V\|_\infty < \epsilon \ \exists z = z(V) \in \Omega \quad m_{W+V}(z) = m_{W+V}(\Omega).$$  \hspace{1cm} (4.5.53)

The first possibility means that by adding an arbitrarily small $V$ to $W$ we can obtain at least two distinct eigenvalues of $P_\theta + V + W$. The second possibility means that for any small perturbation the maximal multiplicity persists.

We will show that (4.5.53) cannot occur.

Assuming this fact we prove (4.5.51) by induction on $m_W(z_0)$ (where $z_0$ is the unique eigenvalues of $P_\theta + W$ in $D(z_0, 2r)$, $\Omega := D(z_0, r) –$ see Step 2). If $m_W(z_0) = 1$ there is nothing to prove. Assuming that we proved (4.5.51) for $m_W(z_0) < M$ assume that $m_W(z_0) = M$. Using (4.5.52) we see that we can find $V_0$, $\|V_0\|_\infty < \epsilon/2$ such that $m_{W+V_0}(\Omega) = m_W(\Omega)$ (using (4.5.49)) and such all eigenvalues in $\Omega$, $z_1, \ldots, z_k$, satisfy $m_{W+V_0}(z_j) < M$. We now find $r_j$ such that,

$$D(z_j, 2r_j) \subset \Omega, \quad D(z_j, 2r_j) \cap D(z_k, 2r_k) = \emptyset, \quad j \neq k,$$

$$\{z_j\} = D(z_j, 2r_j) \cap \text{Spec}(P_\theta + W + V_0).$$

We put $\Omega_j := D(z_j, r_j)$ and apply (4.5.51) successively to $W + V_0 + \cdots V_{j-1}$, $j = 1, \ldots, k$, in $\Omega_j$ with $\|V_j\|_\infty < \epsilon/2^{j+1}$. That gives the desired $V = \sum_{j=0}^k V_j$. 


6. It remains to show that (4.5.53) is impossible. Hence, assume that 
\( m_W(z) = M \) and that (4.5.53) holds. For \( V \in C_{R_0,R_1}, \|V\|_\infty < \epsilon, \) put 
\[
k(V) := \min\{k : (P_\theta + W + V - z(V))^k \Pi_{W+V}(\Omega) = 0\}.
\]
Then \( 1 \leq k(V) \leq M \) and \( B_{CM}(0,\epsilon) \ni V \mapsto k(V) \) is a lower semi-continuous function. In fact, if \( \|V_j - V\|_{CM} \to 0 \) and then, from (4.5.48), we see that \( (P_\theta + W + V_j - z(V_j))^k \Pi_{W+V_j}(\Omega) = 0, \) then \( P_\theta + W + V - z(V))^k \Pi_{W+V}(\Omega) = 0. \)

Defining 
\[
k_0 := \max\{k(V) : V \in V_{R_0,R_1}, \|V\|_\infty < \epsilon/2\},
\]
we see that if \( k(V') = k_0, \) \( k(V + V') = k_0 \) for \( \|V\|_{CM} < \delta, \) with a sufficiently small \( \delta. \) Hence we can replace \( W \) by \( W + V', \) decrease \( \epsilon \) and assume that 
\[
(P_\theta + W + V - z(V))^k_0 \Pi_{W+V}(\Omega) = 0,
\]
(4.5.54) 
\[
(P_\theta + W + V - z(V))^{k_0 - 1} \Pi_{W+V}(\Omega) \neq 0,
\]
\[
m_W(V)(z(V)) = \text{tr} \Pi_{W+V} = M > 1, \quad \forall V, \quad \|V\|_{CM} < \epsilon.
\]

7. To see that (4.5.54) is impossible we first assume that \( k_0 > 1. \) Take 
\( V = V(t) = W + tV, \|V\|_{CM} < \epsilon, \ t \in [-1,1]. \) For \( h, g \in D_\theta \) we define (dropping \( \Omega \) in \( \Pi_{\bullet}(\Omega) \)) 
\[
w(t) := (P_\theta + W + tV - z(t))^{k_0 - 1} \Pi_{W+tV} h,
\]
\[
\tilde{w}(t) := (P_\theta^* + W + tV - z(t))^{k_0 - 1} \Pi_{W+tV}^* g.
\]

By our assumption (4.5.54) we can choose \( g \) and \( h \) so that 
(4.5.55) 
\[
w := w(0) \neq 0, \quad \tilde{w} := \tilde{w}(0) \neq 0.
\]

From Step 2 (or arguments presented in the proof of Lemma 4.40) we see that \( t \mapsto z(tV), \Pi_{W+tV}, w(t) \) are are smooth functions of \( t. \) We then differentiate 
\[
0 = \frac{d}{dt}(P_\theta + W + tV - z(t))^{k_0} \Pi_{W+tV} h
\]
\[
= V(P_\theta + W + tV - z(t))^{k_0 - 1} \Pi_{W+tV} h + (P_\theta + W + tV - z(t)) H(t)
\]
where \( H(t) \in D_{\theta}^\infty. \) We now put \( t = 0 \) and take the \( H_\theta \) inner product with \( \tilde{w}: \) the term with \( H(0) \) disappears as \( (P_\theta^* + W + tV)^{k_0} \Pi_{W}^* \equiv 0 \) and we obtain 
\[
\forall V \in C_{R_0,R_1} \ (V w, \tilde{w}) = 0.
\]

This shows that 
\[
\overline{w}|_{B(0,R_0) \setminus B(0,R_1)} \overline{\tilde{w}}|_{B(0,R_0) \setminus B(0,R_1)} \equiv 0.
\]

Since \( w \) and \( \tilde{w} \) solve \(( -\Delta_{\theta} - z) w = 0 \) and \(( -\Delta_{\theta}^* - z) \tilde{w} = 0 \) in \( B(0,R_0) \setminus B(0,R_1), \) the unique continuation property of the equations shows that \( w = \)
\[ \tilde{w} = 0 \text{ in } \Gamma \setminus B(0, R_0). \] But then \( z \) has to be an eigenvalue of \( P + W \) (there is no scaling near \( B(0, R_0) \)) and \( \arg z = 0 \), a contradiction.

8. It remains to consider the case of \( k_0 = 1 \) in (4.5.54). We then use the notation of Lemma 4.40 and have

\[
0 = \frac{d}{dt} (P_\theta + W + tV - z(tV))\Pi_{W + tV}w_k
\]

\[
= V\Pi_{W + tV}w_k - \frac{d}{dt} z(tV)\Pi_{W + tV}w_k + (P_\theta + W + tV)\frac{d}{dt} \Pi_{W + tV}w_k.
\]

We then put \( t = 0 \) and take an inner product with \( \tilde{w}_j \). That gives:

\[
\frac{d}{dt} z(tV)(0)\delta_{kj} = \langle Vw_k, \tilde{w}_j \rangle, \quad k, j = 1, \cdots, M.
\]

Taking \( k \neq j \) and arguing as at the end of Step 7 we again obtain a contradiction. \( \Box \)

### 4.6. SINGULARITIES AND RESONANCE FREE REGIONS

In this section we present an abstract non-trapping condition which guarantees the presence of resonance free regions for black box Hamiltonians. This generalizes Theorem 3.10 and provides an expansion of scattered waves for “non-trapping black boxes”. In §6.2 we will present a semiclassical version of the non-trapping resolvent estimates, see also §6.6.

To explain the idea of the proof we first present the result in the simpler case of potential scattering. It improves Theorem 3.10 in the case of smooth potential since the logarithmic region is now arbitrarily large:

**THEOREM 4.41** (Non-trapping estimates for smooth potentials). Suppose that \( V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \), \( n \geq 3 \), odd, and \( R_V(\lambda) := (-\Delta + V - \lambda^2)^{-1} \), \( \text{Im} \lambda > 0 \). For any \( M > 0 \) there exists \( C_0 \) such that \( R_V(\lambda) : L^2_{comp} \to L^2_{loc} \) continues holomorphically to

\[
\Omega_M := \{ \lambda \in \mathbb{C} : \text{Im} \lambda > -M \log |\lambda|, \ |\lambda| > C_0 \}.
\]

Moreover, for any \( \chi \in C^\infty_c(\mathbb{R}^n) \) there exist \( C_1 \) and \( T \) such that

\[
\| \chi R_V(\lambda) \chi \|_{L^2 \to L^2} \leq C_1 |\lambda|^{-1} e^{T(\text{Im} \lambda)^-}, \quad \lambda \in \Omega_M.
\]

**Proof.** 1. Let

\[
U_V(t) := \frac{\sin t\sqrt{-\Delta + V}}{\sqrt{-\Delta + V}},
\]

where functions of \(-\Delta + V\) are defined using the spectral theorem.

Then \( U_V(t) : L^2(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \) solves the wave equation

\[
\square_V U_V(t) = 0, \quad U_V(0) = 0, \quad \partial_t U_V(0) = I, \quad \square_V := \partial_t^2 - \Delta + V.
\]
We use the same notation for the Schwartz kernel of the operator $U_V(t)$:

$$U_V(t)f(x) = \int_{\mathbb{R}^n} U_V(t, x, y)f(y)dy, \quad f \in C_c^\infty(\mathbb{R}^n),$$

$$U_V \in C^\infty(\mathbb{R}_t, D'(\mathbb{R}_x^n \times \mathbb{R}_y^n)).$$

The sharp Huyghens principle in odd dimensions states

$$\text{supp} \ U_0(t, \bullet) = \{(x, y) : |x - y| = |t|\},$$

and $\text{singsupp} \ U_0(t) = \text{supp} \ U_0(t)$.

The Huyghens principle/finite speed of propagation holds for $U_V$:

$$\text{supp} \ U_V(t, \bullet) \subset \{(x, y) : |x - y| \leq |t|\}.$$

The key to the proof is the following result about the singular support (see Exercise E.15) of $U_V(t)$:

$$\text{singsupp} \ U_V(t, \bullet) = \{(x, y) : |x - y| = |t|\}.$$  \hspace{1cm} (4.6.4)

This follows from general results presented in Theorem E.49 but in this simple case can also be deduced from an explicit parametrix construction for $\partial_t^2 - \Delta + V$ – see for instance [MU, §§1,4].

2. Property (4.6.4) implies the following statement. Suppose $\psi_a \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies

$$\text{supp} \ \psi_a \subset \{(t, x) : |x| + T_a \leq t \leq |x| + T_a + 1\}, \quad T_a \geq a,$$

we can take $T_a = a$ but this generality will be useful in later) with $a$ large enough so that

$$\text{supp} \ V \subset B(0, a).$$

We claim that for $\chi \in C_c^\infty(B(0, a))$

$$\psi_aU_V(t)\chi \in C^\infty(\mathbb{R}; L^2(\mathbb{R}^2, H^1)).$$  \hspace{1cm} (4.6.5)

Since

$$t \geq |x| + T_a, \quad |x - y| = t \implies |y| \geq t - |x| \geq T_a \geq a,$$

we see that if $(t, x) \in \text{supp} \ \psi_a$ and $y \in \text{supp} \ \chi$ then $|x - y| \neq t$ so that (4.6.4) implies that

$$\psi_a(t, x)U_V(t, x, y)\chi(y) \in C^\infty(\mathbb{R}_t \times \mathbb{R}^2_{(x,y)}),$$

and that implies a stronger statement than (4.6.5).

3. Just as in (3.1.11), we can use spectral theorem to relate the propagator $U_V$ to the resolvent by the formula

$$R_V(\lambda) := \mathcal{F}_{t \rightarrow \lambda}(H(t)U_V(t)) := \int_0^\infty e^{i\lambda t}U_V(t)dt, \quad \text{Im} \ \lambda > 0,$$

where $H(t)$ is the Heaviside function. We will use related formulas to obtain an effective expression for $R_V(\lambda)$. 

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4. For $\zeta \in C^\infty(\mathbb{R})$ equal to 1 for $s \leq \frac{1}{3}$ and 0 for $s \geq \frac{2}{3}$ we put

\begin{equation}
\zeta_a(x,t) := \zeta(t - T_a - |x|),
\end{equation}

so that

\begin{align*}
\zeta_a(x,t) &= \begin{cases} 1 & t \leq |x| + T_a \\
0 & t \geq |x| + T_a + 1. \end{cases}
\end{align*}

Suppose that

\begin{align*}
\text{supp } \chi_a &\subset B(0,a), \quad \chi_a|_{\text{supp } V} \equiv 1
\end{align*}

and write

\begin{align}
\Box_V \zeta_a U_V(t) \chi_a &= [\Box_V, \zeta_a] U_V(t) \chi_a =: F_a(t), \nonumber \\
\zeta_a U_V(t) \chi_a|_{t=0} &= 0, \quad \partial_t (\zeta_a U_V(t) \chi_a)|_{t=0} = \chi_a.
\end{align}

From (4.6.5) we deduce that

\begin{equation}
F_a(t) \in C^\infty(\mathbb{R}; \mathcal{L}(L^2, L^2)).
\end{equation}

5. Putting

\begin{equation}
\Rhat_a(\lambda) g := \mathcal{F}^*_{t \to \lambda}(\zeta_a H(t) U_V(t) \chi_a g), \quad g \in L^2
\end{equation}
we obtain
\[(4.6.11) \quad (-\Delta + V - \lambda^2)\tilde{R}_a(\lambda)g = \chi_a g + \mathcal{F}_{t\rightarrow \lambda}(F_a(t))g.\]

We note that the Fourier transforms are well defined because of the \(\zeta_a(t, x)\) factor and (4.6.7).

6. We will modify \(\tilde{R}_a(\lambda)\) to obtain \(R_a^\#(\lambda)\) such that
\[(4.6.12) \quad (-\Delta + V - \lambda^2)R_a^\#(\lambda) = \chi_a(I + K_\lambda(\lambda)),\]
where \(K(\lambda)\) has small norm for \(\lambda \in \Omega_M\) (see (4.6.1) for the definition of \(\Omega_M\)) and \(R_a^\#(\lambda)\) satisfies the estimate (4.6.2). We will then have
\[(4.6.13) \quad R(\lambda)\chi_a = R_a^\#(\lambda)(I + K_\lambda(\lambda))^{-1},\]
from which (4.6.2) will follow.

7. The modification is achieved by solving the free wave equation:
\[(4.6.14) \quad \Box_0 W_a(t) = -F_a(t), \quad W_a(0) = 0, \quad \partial_t W_a(0) = 0.\]

We will use \(W_a(t)\) to cancel most of the second term on the right in (4.6.11).

We first observe that (4.6.9) shows that for any \(g \in L^2\),
\(F_a(t)g(x) \in C^\infty(\mathbb{R}_t, L^2(\mathbb{R}_x^n))\), \(\text{supp } F_a(t)g(x) \subset \{ (t, x) : |x| + T_a \leq |x| + T_a + 1 \}\).

Consequently, the Duhamel formula
\[(4.6.15) \quad [W_a(t)g](x) = -\int_0^t U_0(t - s, x, y)[F_a(s)g](y)dyds,\]
and (4.6.3) show that
\[(4.6.16) \quad [W_a(t)g](x) \in C^\infty(\mathbb{R}_t, L^2(\mathbb{R}_x^n)).\]

(In fact \([W_a(t)g(x)]\) is smooth but that is not important here.) We claim that
\[(4.6.17) \quad \text{supp } [W_a(t)g](x) \subset \{ (x, t) : |x| + T_a \leq t \leq |x| + C_a \}.\]

The lower bound on \(t\) in the support comes from the support properties of \(F_a\) and the finite speed of propagation. Hence we only need to check the upper bound.

8. To establish (4.6.17) we rewrite \(W_a(t)\) as follows
\[(4.6.18) \quad W_a(t) = \zeta_a H(t)U_V(t)\chi_a - H(t)U_0(t)\chi_a + Q_a(t).\]

The first two terms satisfy the support property (4.6.17) with \(C_a = T_a + 1\) and \(C_a = |a|\) respectively.
The last term satisfies $Q_a(t) \equiv 0$ for $t \leq 0$, and
\[
\begin{align*}
\square_0 Q_a(t) &= \square_0 W_a(t) - \square_V (\zeta_a H(t) U_V(t) \chi_a) + V \zeta_a H(t) U_V(t) \chi_a \\
&\quad - \square_0 (H(t) U_0(t) \chi_a) \\
&= - F_a(t) + [\square_V, \zeta_a] U_V(t) \chi_a + V \zeta_a H(t) U_V(t) \chi_0 \\
&= V \zeta_a H(t) U_V(t) \chi_a.
\end{align*}
\] (4.6.19)

Here we used the definition of $F_a(t)$ in (4.6.8) and the fact that
\[
\square_V (H(t) U_V(t) \chi_a g) = \square_0 (H(t) U_0(t) \chi_a g) = \delta(t) \chi_a g, \quad g \in L^2.
\]

In view of (4.6.7) the right hand side of (4.6.19) is compactly supported in both $x$ and $t$. The sharp Huyghens principle (4.6.3) and the Duhamela formula (see (4.6.15)) show
\[
Q_a(t) = - \int_0^t U_0(t-s) V \zeta_a U_V(s) \chi_a ds,
\]
then show that for any $g \in L^2$, (recall that supp $V \subset B(0, a)$)
\[
\text{supp}(Q_a(t)g(x)) \subset \{ (t, x) : \exists |y| \leq a, 0 \leq s \leq |y| + T_a + 1, |x-y| = t-s \} \subset \{ (t, x) : 0 \leq t \leq 2a + |x| + T_a + 1 \}.
\]

Hence (4.6.17) holds with $C_a = 2a + T_a + 1$.

In particular this means that $\mathcal{F}^\#_{t \to \lambda}(W_a(t)g)$ is well defined. (This is the crucial part of the argument: the compact support of the perturbation – $V \in C^\infty_c(\mathbb{R}^n)$ – forces the compact support in time of $W_a(t)$ which compares the truncated perturbed evolution with the free evolution.)

9. We now put
\[
R^\#_a(\lambda) g := \tilde{R}_a(\lambda) g + \mathcal{F}^*_{t \to \lambda}(W_a(t)g), \quad g \in L^2,
\]
where $\tilde{R}_a(\lambda)$ was defined in (4.6.10). If
\[
\chi_a = 1 \quad \text{on supp } V
\]
(which is possible since supp $V \subset B(0, a)$) then
\[
(-\Delta + V - \lambda^2) R^\#_a(\lambda) g = \chi_a g + V \mathcal{F}^*_{t \to \lambda}(W_a(t)g) = \chi_a (I + K_a(\lambda)) g,
\]
where $K_a(\lambda) := V \mathcal{F}^*_{t \to \lambda}(W_a(t))$.

10. In view of the definitions of $\tilde{R}_a$ and $R^\#_a(\lambda)$, (4.6.2) will follow once we show the following estimates
\[
\| \mathcal{F}^*_{t \to \lambda} \chi_a H(t) U_V(t) \chi_a \|_{L^2 \to L^2} \leq C_0(\lambda)^{-1} e^{C_0(\text{Im } \lambda)}, \quad \chi \in C^\infty_c(\mathbb{R}^n),
\]
(4.6.20)
\[
\| \chi \mathcal{F}^*_{t \to \lambda}(W_a(t)) \|_{L^2 \to L^2} \leq C_0(\lambda)^{-1} e^{C_0(\text{Im } \lambda)},
\]
(4.6.21)
\[
\| \mathcal{F}^*_{t \to \lambda}(V W_a(t)) \|_{L^2 \to L^2} \leq C_N(\lambda)^{-N} e^{C_1(\text{Im } \lambda)}.
\]
(4.6.22)
where $C_0$ depends on $\chi$ and $a$ and $C_1$ depends on $a$, and $N$ is arbitrary.

11. We start with the last two bounds. In fact, a stronger bound than (4.6.21) is valid: for any $\chi \in C_c^\infty(\mathbb{R}^n)$,

$$\|\chi \mathcal{F}_{t \to \lambda}^*(W_a(t))\|_{L^2 \to L^2} \leq C_N(\lambda)^{-N} e^{C_1(\text{Im} \lambda)^{-1}}.$$ 

In fact, the regularity property (4.6.16) and the support property (4.6.17) imply that $\chi W_a(t) \in C_c^\infty((0, \infty); \mathcal{L}(L^2, L^2))$, and the bound on the Fourier transform follows. This also implies (4.6.22) as we can take $\chi = V$.

12. To prove (4.6.20) we use an argument similar to that in the proof of Theorem 3.1. We first note that

$$U_V(t), \partial_t U_V(t) = O(\exp C|t|)_{L^2 \to L^2}$$

where the exponential growth is due to the possible presence of negative eigenvalues of $-\Delta + V$. Hence

$$\chi \mathcal{F}_{t \to \lambda}^*(\zeta_a H(t)U_V(t)\chi_a) = O(e^{C(\text{Im} \lambda)^{-1}})_{L^2 \to L^2},$$

where $C$ depends on $\chi$ and $a$. We also have (since $U_V(0) = 0$)

$$i\lambda \mathcal{F}_{t \to \lambda}^*(\zeta_a H(t)U_V(t)\chi_a) = -\mathcal{F}_{t \to \lambda}^*(\zeta_a H(t)\partial_t U_V(t)\chi_a)$$

$$= O(e^{C(\text{Im} \lambda)^{-1}})_{L^2 \to L^2},$$

from which (4.6.20) follows. This completes the proof of (4.41).

We now move to the general black box case. The proof follows the same strategy but we need more cut-off functions. All of them will be either identically 0 or 1 in a neighbourhood of the black box.

We also require an abstract condition which will replace (4.6.2). Let $P$ be a black box Hamiltonian in the sense of Definition 4.1. Since it is a self-adjoint operator we use the spectral theorem to define

$$(4.6.23) \quad U(t) := \sin t\sqrt{P}.$$ 

We assume here that $h = 1$.

**DEFINITION 4.42 (Non-trapping black box).** Suppose that $P$ is a black box Hamiltonian and that $U(t)$ is given by (4.6.23). We say that $P$ is non-trapping if

$$(4.6.24) \quad P \geq -C$$

for some $C$ and if the following condition holds:

$$(4.6.25) \quad \forall a > R_0 \quad \exists T_a \quad \forall \chi \in C_c^\infty(B(0, a)), \chi|_{B(0, R_0 + \epsilon)} \equiv 1, \chi U(t)\chi|_{t > T_a} \in C^\infty((T_a, \infty); \mathcal{L}(\mathcal{H}, \mathcal{D})),$$

where the space $\mathcal{D}$ is the domain of $P$. 

EXAMPLE. Suppose that $P = -\Delta_g$ where $g$ is a Riemannian metric on $\mathbb{R}^n$, $n$ odd, with the property that $g^{ij} - \delta_{ij} \in C^\infty_c(B(0, R_0))$. Suppose that the metric is classically non-trapping that is, 
\[
\forall (x, \xi) \in T^*\mathbb{R}^n \setminus 0 \quad \pi(\exp tH_p(x, \xi)) \to \infty, \quad t \to \pm\infty,
\]
\[
p(x, \xi) = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j.
\]
Here $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the natural projection and $H_p$ is the Hamilton vector field of $p$ -- see Proposition E.5.

In particular this implies that for any $a > 0$ there exists $T_a$ such that 
\[
|x| < a, \quad p(x, \xi) = 1, \quad |t| > T_a \implies |\pi(\exp tH_p(x, \xi))| > a.
\]

Hence the result on propagation of singularities – see Theorem E.49 – shows that for $\chi \in C^\infty_c(B(0, a))$ and any $N > 0$.
\[
\chi(\sin t\sqrt{-\Delta_g/\sqrt{-\Delta_g}})\chi \in C^\infty((T_a, \infty); L^2(\mathbb{R}^n), H^N(\mathbb{R}^n)).
\]
This means that classical non-trapping for the metric $g$ implies non-trapping for the propagator in the sense of Definition 4.42.

We can now state a theorem relating propagation of singularities to resonance region for general black box Hamiltonians in odd dimensions. The proof follows the same idea as the proof of Theorem 4.41 but with more cut-offs related to the abstract black box.

**THEOREM 4.43 (Non-trapping estimates for black box Hamiltonians).** Suppose that $P$ is a black box Hamiltonian in the sense of Definition 4.1 and $R(\lambda) : H_{\text{comp}} \to D_{\text{loc}}$ is the meromorphically continued resolvent.

If $P$ is non-trapping in the sense of Definition 4.42 then for any $M$ there exists $C_0$ such that $R(\lambda)$ is holomorphic in 
\[
\Omega_M := \{\lambda \in \mathbb{C} : \text{Im} \lambda > -M \log |\lambda|, \quad |\lambda| > C_0\}. \tag{4.6.26}
\]
Moreover, for any $\chi \in C^\infty_c(\mathbb{R}^n)$, $\chi = 1$ near $B(0, R_0)$ (the black box) there exist $C_1$ and $T$ such that 
\[
\|\chi R(\lambda)\chi\|_{H \to H} \leq C_1 |\lambda|^{-1} e^{T(\text{Im} \lambda)^-}, \quad \lambda \in \Omega_M. \tag{4.6.27}
\]

**REMARKS.** 1. The estimate (4.6.27) can be improved to an estimate valid between $H$ and $D^\alpha$, $\alpha = 0, \frac{1}{2}, 1$ – see (4.6.44). That is important for obtaining resonance expansions for data in natural spaces.
2. The condition (4.6.25) seems weaker than the condition
\[
\chi U(t)\chi \in C^\infty((T_a, \infty); L(H, D^N)), \tag{4.6.28}
\]
for all $N$. However, differentiation of (4.6.23) with respect to $t$ and changing the cut-offs $\chi$ shows that (4.6.27) implies the seemingly stronger statement (4.6.28).

3. As observed in [Sj02, §3] condition (4.6.25) can be weakened to demanding that

$$\chi U(t) \chi \in C^\infty((T_a, T_a + c); \mathcal{L}(L^2, D))$$

for some $c > 0$. That can already be seen in the proof of Theorem 4.41: we only use (4.6.5) and for that only smoothness for in $(T_a, T_a + c)$ for some $c$ is needed. See [Sj02, §3] for a slightly different argument.

4. As pointed out in [BW13] the proof below shows that $C^\infty$ in (4.6.27) can be replaced by $C^k$ for any $k \geq 0$. In the case of $k = 0$ we obtain a resonance free region of the form $\text{Im } \lambda \geq -C$, $|\lambda| > C$ for any $C$ – see Step 6 of the proof: the Fourier transform of a continuous compactly supported function is needed. See [Sj02] for all $\$.

Also, (4.6.30) gives

$$\chi_1 U_V(t) \chi_1 |_{t > T_a} \in C^\infty((T_a, \infty); \mathcal{L}(L^2, D)).$$

In particular, $u \mapsto \psi_a u$ is well defined as an operator from $C^\infty(\mathbb{R}, \mathcal{H}_{\text{loc}})$ to itself.

Using (4.6.25) we see that

$$(4.6.31) \quad \psi_a U(t) \chi_a \in C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}, D)).$$

Also, (4.6.30) gives

$$\square_0 ((1 - \chi_0)U_V(t)\chi_a) = -[\square_0, \chi_0] \chi_1 U_V(t) \chi_1 |_{t > T_a} =: G_a(t)$$

$$G_a(t)|_{t > T_a} \in C^\infty((T_a, \infty), \mathcal{L}(\mathcal{H}, H^1_{\text{comp}}(B(0, a) \setminus \overline{B(0, R_0)}))).$$
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Figure 4.5. Supports of $\psi_a$, $1-\chi_0$ and $G_a(t)$ showing the validity of (4.6.32). The cone represents the support of $U(t-s)$ in the Duhamel formula.

$$(4.6.33)$$
Using \( \zeta_a \) we put
\[
(4.6.34) \quad F_a(t) := [\Box, \zeta_a]U(t)\psi_a \in C^\infty(\mathbb{R}; \mathcal{L}(\mathcal{S}, \mathcal{D}^\frac{1}{2})),
\]
where the regularity comes from (4.6.30) and we change from \( \mathcal{D} \) to \( \mathcal{D}^\frac{1}{2} \) (in fact \( H^1 \) since the cut-offs move us outside of the “black box”) because of the first order operator \( [\Box, \zeta_a] \) – see (4.1.9) and (4.1.10).

3. Now choose \( \chi_b \in C_c^\infty(B(0, a)) \) such that \( \chi_b = 1 \) near \( B(0, R_0) \) and \( \chi_a \equiv 1 \) on \( \text{supp} \chi_b \) and solve
\[
(4.6.35) \quad \Box_b W_a(t) = -(1 - \chi_b)F_a(t), \quad W_a(t) \equiv 0, \quad t \leq 0.
\]
We then proceed as in Step 7 of Theorem 4.41 to see that for \( \chi_b \in C_c^\infty(B(0, a)) \),
\[
\text{supp}[W_a(t)g](x) \subset \{(x, t) : |x| + T_a \leq t \leq |x| + C_a\},
\]
where
\[
(4.6.36) \quad [W_a(t)g](x) \in C^\infty(\mathbb{R}_t, \mathcal{D}^\frac{1}{2}).
\]

The smoothness statement comes from (4.6.34) and to see the support property we write
\[
W_a(t) = (1 - \chi_b)\zeta_aH(t)U(t)\chi_a - H(t)U_0(t)(1 - \chi_b)\chi_a + Q_a(t),
\]
where \( H(t) \) is the Heaviside function and where \( Q_a(t) \) solves
\[
\Box_0 Q_a(t) = \Box_0 W_a(t) - (1 - \chi_b)\Box(\zeta_aH(t)U(t)\chi_a)
+ [\Box_0, \chi_b]\zeta_aH(t)U(t)\chi_a - \Box_0(U(t)H(t)(1 - \chi_b)\chi_a)
= -(1 - \chi_b)F_a(t) + (1 - \chi_b)(\chi_a + F_a(t))
+ [\Box_0, \chi_b]\zeta_aH(t)U(t)\chi_a(1 - \chi_b)\chi_a - \chi_a(1 - \chi_b)\chi_a
= -[\Delta, \chi_b]\zeta_aH(t)U(t)\chi_a.
\]
For any \( g \in \mathcal{H} \),
\[
[\Delta, \chi_b]\zeta_aH(t)U(t)\chi_a \in L^2(B(0, a) \setminus B(0, R_0)),
\]
that is the support is contained in a fixed compact subset of \( \mathbb{R}_t \times \mathbb{R}^n \). (The compactness in \( t \) comes from \( \zeta_a \).) The sharp Huyghens principle (see Step 8 of the proof of Theorem 4.41) then shows that
\[
\text{supp}[Q_a(t)g](x) \subset \{(t, x) : 0 \leq t \leq C_a + |x|\}.
\]
The lower bound on \( t \) in (4.6.36) follows from the support property of \( (1 - \chi_b)F_a(t) \) and the final speed of propagation.

5. The support property in (4.6.36) allows us to define the following approximation of the resolvent \( R(\lambda)\chi_a \): choose \( \chi_c \in C_c^\infty(B(0, a)) \) equal to 1 near \( B(0, R_0) \) and such that \( \chi_b = 1 \) on \( \text{supp} \chi_c \) and put
\[
(4.6.37) \quad R^\#_a(\lambda) = \mathcal{F}_{t \rightarrow \lambda}^*(\zeta_aH(t)U(t)\chi_a + (1 - \chi_c)W_a(t)).
\]
We note that \( R^\#_a(\lambda) : \mathcal{H} \rightarrow \mathcal{D} \) for \( \text{Im} \lambda > 0 \) except for poles given by a discrete set of eigenvalues – see Theorem 4.5, (4.6.24) and (4.6.6).
4. We calculate

\[(P - \lambda^2) R^\#_a(\lambda) = \chi_a(I + K_a(\lambda)),\]

where, using (4.6.35),

\[K_a(\lambda) := \mathcal{F}_{t \to \lambda}^* \left( \chi_b F_a(t) + [\Delta, \chi_c] W_a(t) \right).\]

We prove (4.6.25) by showing that

\[\chi R^\#_a(\lambda) = O(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)^{-1}_{H \to H},\]

(4.6.39)

and

\[K_a(\lambda) = O_N(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)^{-1}_{H \to H},\]

(4.6.40)

for any \(N\).

In fact, for \(\text{Im} \lambda > 0\), in view of mapping properties of \(R^\#_a(\lambda)\) discussed in Step 3, we have

\[R(\lambda) \chi_a = R^\#_a(\lambda)(I + K_a(\lambda))^{-1},\]

(4.6.41)

and by analytic continuation in holds in \(\Omega_M\): the bound (4.6.40) implies invertibility of \(I + K_a(\lambda)\) there.

5. To obtain (4.6.39) we proceed as in Step 12 of the proof of Theorem 4.41. Since \(P\) is bounded from below (see (4.6.24)) the functional calculus of self-adjoint operators implies that

\[U(t), \partial_t U(t) = O(\exp C|t|)_{H \to H}.\]

Hence

\[\chi \mathcal{F}_{t \to \lambda}^* (\zeta_a H(t) U_V(t) \chi_a) = O(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)_{H \to H},\]

where \(C\) depends on \(\chi\) and \(a\). We also have (since \(U(0) = 0\))

\[i\lambda \chi \mathcal{F}_{t \to \lambda}^* (\zeta_a H(t) U(t) \chi_a) = -\mathcal{F}_{t \to \lambda}^* (\zeta_a H(t) \partial_t U(t) \chi_a) = O(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)_{H \to H}.\]

The second term in \(R^\#_a(\lambda)\) satisfies an even better bound in view of (4.6.36):

\[\mathcal{F}_{t \to \lambda}^* (\chi W_a(t)) = O(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)_{H \to H}.\]

This proves (4.6.39).

6. It remains to show (4.6.40). Since the cut-offs \(\chi_b\) and \([\Delta, \chi_c]\) restrict \(t\) to a compact set smoothness in \(t\) – from (4.6.34) and (4.6.36) – we obtain the estimates \(O(e^{C(\text{Im} \lambda)} - \langle \lambda \rangle - N)_{H \to H}\). (It is here that we need \(D\) rather than \(D^{1/2}\) in (4.6.25) as we need to apply the first operator \([\Delta, \chi_c]\) to \(W_a(t)\).) That completes the proof. \(\square\)

We can state a theorem about the expansion of scattered waves. The proof follows the same lines as the proof of Theorem 2.7.
**THEOREM 4.44** (Resonance expansions for non-trapping black box Hamiltonians.) Suppose that $P$ is a non-trapping black box Hamiltonian in the sense of Definition 4.42 and that that $w(t)$ is the solution of

$$(\partial_t^2 + P)w(t) = 0, \quad w(0) = w_0 \in \mathcal{D}_{\text{comp}}^{\frac{3}{2}}, \quad \partial_t w(0) = w_1 \in \mathcal{H}_{\text{comp}}.$$  

Then, for any $A > 0$,

$$w(t) = \sum_{\lambda_j \in \text{Res}(P) \cap \text{Im} \lambda_j > -A} \sum_{\ell=0}^{m_R(\lambda_j) - 1} t^\ell e^{-i\lambda_j t} f_{j,\ell} + E_A(t),$$

where sum is finite,

$$(4.6.43) \quad \sum_{\ell=0}^{m_R(\lambda_j) - 1} t^\ell e^{-i\lambda_j t} f_{j,\ell} = \text{Res}_{\mu=\lambda_j} \left((iR(\mu)w_1 + \lambda R(\mu)w_0) e^{-i\mu t}\right),$$

and for any $K > 0$, such that $\text{supp } w_j \subset B(0, K)$, there exist constants $C_{K,A}$ and $T_{K,A}$

$$\|E_A(t)\|_{\mathcal{D}} \leq C_{K,A} e^{-tA} \left(\|w_0\|_{\mathcal{D}^{\frac{3}{2}}} + \|w_1\|_{\mathcal{H}}\right), \quad t \geq T_{K,A}.$$  

**Proof.** To repeat the proof of Theorem 2.7 we need the following improvement of the estimate (4.6.25):

$$(4.6.44) \quad \|\chi R(\lambda)\chi\|_{\mathcal{H} \rightarrow \mathcal{D}^*} \leq C(\lambda)^{\alpha/2 - 1} e^{C(\text{Im } \lambda)}, \quad \alpha = 0, 1, 2.$$  

From (4.6.41) we see that it is enough to prove these estimates with $R(\lambda)\chi$, supp $\chi \subset B(0, a)$, replaced by $R_a^\#(\lambda)$, where $R_a^\#(\lambda)$ is given by (4.6.37) with $\chi_a = \chi$. But this follow from the support and mapping properties of $\zeta_a U(t) \chi_a$ and $W_a(t)$ following the argument in the proof of (3.1.12) in Theorem 3.1 with $-\Delta$ replaced by $P$.  

**4.7. NOTES**

The black box formalism was introduced in [SZ91]. The presentation in §4.2 comes from that paper with additional improvements, including (4.2.13) and the proof of Theorem 4.7, from Sjöstrand [Sj02].

Theorem 4.13 comes essentially from [SZ91] but the proof follows Vodev [Vo92] and Petkov–Zworski [PZ01] and is based on the same ideas as the proof of Theorem 3.27. The case of global bounds in even dimensions is proved in Vodev [Vo94a], [Vo94b] while semiclassical bounds valid for a
large class of operators (including some long range perturbations of $-\hbar^2 \Delta$) are given in Sjöstrand [Sj96a].

The scattering theoretical interpretation of spectral theory on finite volume hyperbolic quotients goes back to Selberg and was made explicit by Fadeev–Pavlov [FP72] and Lax–Phillips [LP76]. The pseudo-Laplacian was introduced by Colin de Verdière [CdV83] and was used to show generic absence of embedded eigenvalues for variable curvature surfaces with cusps. The Fermi Golden rule in that setting was discovered by Phillips–Sarnak [PS85] and led to much work – see Hillairet–Judge [HJ18] for recent progress and references and [LZ16] for a version in which a boundary condition changes. For an early mathematical treatment of the the Fermi Golden rule see Simon [Si73] and for more recent developments, for instance Soffer–Weinstein [SW98] and Jensen–Nenciu [JN06]. For more general perturbations and continuity statements about resonances see Stefanov [St94].

The derivation of the scattering matrix for the modular surfaces combines the classical approach of Titchmarsh and Heath-Brown [Ti86, Notes for Chapter II] with the black box approach. That in essence is Colin de Verdière’s proof of the meromorphic continuation of Eisenstein series [CdV81b].

The method of complex scaling originated in the work of Aguilar-Combes [AC71], Balslev-Combes [BC71] and was developed by Simon [Si72],[Si73], [Si79a], Hunziker [Hu86], Helffer-Sjöstrand [HS86], Hislop–Sigal [HS89] (see also [HS96]) and other authors. For compactly supported black box perturbations (and large $\theta$) it was introduced in [SZ91] while an adaptation to the case of long range black box perturbations was provided in [Sj96a]. In our presentation we opted for a quick approach which benefits from the precise knowledge of the resolvent of the free Laplacian. A more systematic approach is based on the theory of differential operators with analytic coefficients – see [SZ91] and [Sj02]. The method has been extensively used in computational chemistry – see Reinhardt [Re07] for a review. As the method of perfectly matched layers it reappeared in numerical analysis – see Berenger [Be94].

Theorem 4.39 is a slight generalization of a result of Klopp–Zworski [KZ95]. It extends to the case of resonances the now classical result of Uhlenbeck [Uh76] for eigenvalues of the Laplacian on a compact manifold. Instead of complex scaling one could use Agmon’s theory of resonance perturbations [Ag98] as was done by Borthwick–Perry [BP02]. That extends applicability of the method to, for instance, scattering on asymptotically hyperbolic manifolds – see §5.1.

The presentation of resolvent estimates for non-trapping black boxes in §4.6 is based on Tang–Zworski [TZ00, §3] but the method is due to Vainberg
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See these references for the case of even dimensions. For a different presentation see Sjöstrand [Sj02, §3] and for some recent applications Baskin–Wunsch [BW13], Baskin–Spence–Wunsch [BSW16] and Galkowski [Ga16c]. Another point of view on linking propagation of singularities with resolvent estimates and energy decay is given by the Lax–Phillips theory – see [LP68] and for a “black box” presentation [SZ94]. One of the main applications is to obstacle problems where propagation of singularities was established by Andersson, Ivrii, Melrose, Lebeau, Sjöstrand and Taylor – see Hörmander [HöIII, Chapter 24] and references given there.

4.8. EXERCISES

Section 4.1

1. Suppose $O$ is bounded region with a smooth boundary $\Gamma := \partial O$. Suppose that $V : L^2(\Gamma) \to L^2(\Gamma)$ is a bounded operator. Show that

$$P := -\Delta + V \otimes \delta_\Gamma$$

satisfies the black box hypothesis. Construct $\Gamma$ and $V$ so that $P$ has embedded eigenvalues. (See Galkowski–Smith [GS15] for more information and deeper analysis.)

2. Define a Hilbert space $H := \bigoplus_{k=1}^{\infty} L^2([0, \infty))$ with the norm $\|\{b_k\}_{k=1}^{\infty}\|_H^2 := \sum_{k=1}^{\infty} \|b_k\|_{L^2([0, \infty))}^2$. For $G(s)$, a continuous function on $[0, \infty)$, define

$$H_G := \left\{ \{b_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} \int_0^\infty (|\partial_s b_k(s)|^2 + k^2 e^{G(s)} |b_k(s)|^2) ds < \infty \right\}.$$

Show that if $G(s) \to \infty$ as $s \to \infty$ then $H_G \hookrightarrow H$ is a compact inclusion.

In particular this proves the compactness of (4.1.21).

Section 4.2

3. Prove (4.2.28). Here are suggested steps, see also [HZ17, Proof of (4.3)].

(a) Show that for a fixed $f$ in (4.2.26), $\lambda \mapsto u \in C(\mathbb{R}_t; D'(\mathbb{R}^n))$ is a holomorphic function for $\lambda \in \mathbb{C}$. Conclude that it is enough to prove (4.2.28) for $\text{Im} \lambda > 0$.

(b) Use the Fourier transform to show that if $\tau \in \mathbb{R}$ (for $f \in C^\infty_c(\mathbb{R}^n)$ and in the distributional sense for $f \in \mathcal{E}'(\mathbb{R}^n)$)

$$\mathcal{F}_{t \mapsto \tau} u(\tau, x) = \frac{1}{2(2\pi)^{n-1}} \sum_{\pm} \int_{S^{n-1}} e^{\pm i \tau (\omega, x)} \hat{f}(\pm \tau \omega) \frac{1}{\tau^2 - \lambda^2} (1 - \lambda/\tau)(\pm \tau)^{n-1} d\omega,$$
4.8. EXERCISES

where \( \mathcal{F}_{t \to \tau} \) is the Fourier transform in \( t \). This formula holds in any dimension.

(c) Conclude that for \( n \geq 3 \) and odd

\[
\mathcal{F}_{t \to \tau} u(\tau, x) = \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} e^{i\tau \langle \omega, x \rangle} \hat{f}(\tau \omega) \frac{\tau}{\tau + \lambda}^{n-2} d\omega,
\]

and hence \( \tau \mapsto \mathcal{F}_{t \to \tau} u(\tau, x) \) extends to a holomorphic function in \( \text{Im} \tau > -\text{Im} \lambda \).

(d) Use the Paley–Wiener theorem \([\text{H6I}, \text{Theorem 7.3.1}]\) applied to \( f \) to prove that

\[
|\mathcal{F}_{t \to \tau} u(\tau, x)| \leq C(\tau)^M e^{i\text{Im} \tau |x| + R}, \quad \text{Im} \tau > -\text{Im} \lambda + \varepsilon
\]

for some \( C \) and \( M \) depending on \( \varepsilon > 0 \). Use the other direction of the Paley–Wiener theorem (applied to \( \mathcal{F}_{t \to \tau} u(\tau, x) \)) to obtain (4.2.28).

Section 4.4

4. Using the proof of Theorem 3.35 justify Theorem 4.17.

5. Prove Theorem 4.20 using the proof of Theorem 3.47 as a blueprint. That proof can be read independently of the rest of §3.9.

6. Suppose that (4.4.11) holds. For \( S(\lambda) \) given in Definition 4.25 and the generalized plane wave given by (4.4.1) show that

(4.8.1) \( S(\lambda) e(-\lambda, \cdot) = e(\lambda, \cdot) \),

in the sense that for \( f \in \mathcal{H}_{\text{comp}} \),

\( S(\lambda) (\langle e(-\lambda, \cdot), f \rangle) = \langle e(\lambda, \cdot), f \rangle, \quad \langle e(-\lambda, \cdot), f \rangle \in C^\infty(S^{n-1}) \).

7. Show that if \( \lambda^2 > 0 \) then \( m_R(\lambda) = m_R(-\lambda) \).

Section 4.5

8. Use Theorem 4.32 and invertibility of

\[
e^{-2i\theta} \Delta - \lambda^2 = e^{-2i\theta} (\Delta - (e^{i\theta} \lambda)^2) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \text{Im}(e^{i\theta} \lambda) > 0,
\]

to show directly (that is, without using the explicit calculations of §4.5.2) that

\[-\Delta_\theta - \lambda^2 : H^2(\Gamma_\theta) \to L^2(\Gamma_\theta),
\]

is a Fredholm operator for \( \text{Im}(e^{i\theta} \lambda) > 0 \).

9. Show (4.5.30) directly using (4.5.13) and the expression \( dw = \det(I + F^{\prime\prime}_\theta(y))dy \).
Chapter 5

SCATTERING ON HYPERBOLIC MANIFOLDS

5.1 Asymptotically hyperbolic manifolds
5.2 A motivating example
5.3 The modified Laplacian
5.4 Phase space dynamics
5.5 Propagation estimates
5.6 Meromorphic continuation
5.7 Applications to general relativity
5.8 Notes

In this chapter, we show meromorphic continuation of the resolvent of the Laplacian on complete asymptotically hyperbolic Riemannian manifolds. A basic example is given by the hyperbolic space $\mathbb{H}^n$, which can be viewed as the open unit ball in $\mathbb{R}^n$ with the metric

\begin{equation}
(5.0.1) \quad g = 4 \frac{dw^2}{(1 - |w|^2)^2}, \quad w \in B_{\mathbb{R}^n}(0, 1).
\end{equation}

A larger family of examples is provided by convex co-compact hyperbolic surfaces, which are complete two-dimensional Riemannian manifolds of constant sectional curvature $-1$ whose infinite ends are funnels, that is they have the form

\begin{equation}
(5.0.2) \quad [0, \infty)_v \times S^1_\ell, \quad S^1 = \mathbb{R}/\ell\mathbb{Z}, \quad \ell > 0; \quad g = dv^2 + (\cosh v)^2 d\theta^2.
\end{equation}
Convex co-compact hyperbolic surfaces can be viewed as the quotients of $\mathbb{H}^2$ by certain discrete subgroups of its isometry group $\text{PSL}(2, \mathbb{R})$, and have profound applications in algebra and number theory [Bo16]. Furthermore, they give fundamental examples of hyperbolic trapped sets and are a model object to study the effects of hyperbolic trapping on distribution of resonances.

We take a geometric approach to scattering on hyperbolic manifolds and consider more general asymptotically hyperbolic ends, whose metrics approach (5.0.2) in a certain sense as $s \to +\infty$, and satisfy an additional evenness assumption. We follow a modified version of the recent microlocal approach of Vasy [Va12] — see [Zw16a] for an introduction to the method in the non-semiclassical setting. Here we emphasize that in addition to establishing the meromorphic continuation of the resolvent, the method works well in the semiclassical limit. That makes the methods of Part 3 applicable in scattering on asymptotically hyperbolic manifolds and, as indicated in §5.7, black hole backgrounds.

On an asymptotically hyperbolic manifold of dimension $n$, the scattering resolvent is the meromorphic continuation of

$$\left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right)^{-1} : L^2(M) \to L^2(M), \quad \text{Im} \lambda > 0,$$

to the entire half-plane. Compared to Euclidean scattering, we have an additional $\frac{(n-1)^2}{4}$ term, which is related to the fact that the essential spectrum of $-\Delta_g$ is $\left( \frac{(n-1)^2}{4}, \infty \right)$. An explanation of this shift of the spectral parameter is provided below in (5.2.9).

This chapter is structured as follows:

- in §5.1, we define asymptotically hyperbolic manifolds and study their geometric properties;
- in §5.2, we use an example to motivate the use of the modified spectral family of the Laplacian $P(\lambda)$;
- in §§5.3–5.5, we introduce the operator $P(\lambda)$ and use microlocal techniques to prove estimates on this operator;
- in §5.6, we show that $P(\lambda)$ has a meromorphic inverse and use this to give a meromorphic continuation of (5.0.3);
- in §5.7, we apply the methods of the present chapter to wave decay on certain spacetimes.

### 5.1. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

An asymptotically hyperbolic manifold is a complete Riemannian manifold whose infinite ends are locally asymptotic to the infinity of the hyperbolic
space (5.0.1). A standard way to treat these manifolds is to define them as interiors of compact manifolds with boundary. For instance, (5.0.1) defines the hyperbolic space as the interior of the closed unit ball in $\mathbb{R}^n$. We will use the following general

**DEFINITION 5.1.** Let $M$ be a compact manifold with boundary $\partial M$ and interior $M$. A **boundary defining function** on $M$ is a $C^\infty$ function

$$y_1 : M \to [0, \infty)$$

such that $y_1 = 0$ on $\partial M$, $dy_1 \neq 0$ on $\partial M$, and $y_1 > 0$ on $M$.

Such functions always exist, as can be shown using local charts and a partition of unity. Any two boundary defining functions $y_1, \tilde{y}_1$ are multiples of each other:

$$\tilde{y}_1 = e^f y_1 \quad \text{for some } f \in C^\infty(M; \mathbb{R}).$$

Moreover, a neighborhood of the boundary has a **product structure**: namely, for any boundary defining function $y_1$ and $\varepsilon_1 > 0$ small enough, there exists a diffeomorphism

$$y_1^{-1}([0, \varepsilon_1]) \to [0, \varepsilon_1) \times \partial M, \quad x \mapsto (y_1(x), y'(x)), \quad y'|_{\partial M} = I.$$

We now define asymptotically hyperbolic manifolds as follows:

**DEFINITION 5.2.** An **asymptotically hyperbolic manifold** is a complete Riemannian manifold $(M, g)$ such that:

1. $M$ is the interior of a compact manifold with boundary, denoted $\overline{M}$;
2. for a boundary defining function $y_1$, the metric $y_1^2 g$ extends to a smooth Riemannian metric on $\overline{M}$;
3. we have $|dy_1|_{y_1^2 g} = 1$ on $\partial \overline{M}$.
Note that properties (2) and (3) do not depend on the choice of the boundary defining function, as follows immediately from (5.1.1). In particular, the invariance of property (3) follows from the identity
\[ |d(e^f y_1)|_{e^{2f} y_2^2 g} = e^{-f}|d y_1|_{y_2^2 g} = |d y_1|_{y_2^2 g} \quad \text{on } \partial M. \]
Manifolds which satisfy properties (1) and (2) from Definition 5.2 are called conformally compact. Property (3) has the additional effect that the sectional curvatures on \( M \) converge to \(-1\) at infinity; see [GL91, §2].

The boundary metric \( (y_2^2 g)|_{\partial M} \) depends on the choice of the boundary defining function \( y_1 \). However, by (5.1.1), a different choice of \( y_1 \) multiplies the boundary metric by a conformal factor. Therefore, to each asymptotically hyperbolic manifold corresponds a conformal class of Riemannian metrics on \( \partial M \):
\[
(5.1.3) \quad [g]_{\partial M} = \{ (y_1^2 g)|_{\partial M} : y_1 \text{ is a boundary defining function} \}.
\]

**EXAMPLES.** 1. The hyperbolic space (5.0.1) is asymptotically hyperbolic, where we may take for instance \( y_1 = 1 - |w|^2 \).
2. Another example is the hyperbolic cylinder
\[
(5.1.4) \quad M = \mathbb{R}_v \times S^1, \quad S^1 = \mathbb{R}/(\ell \mathbb{Z}), \quad g = dv^2 + (\cosh v)^2 d\theta^2.
\]
The hyperbolic cylinder is the union of two funnels (5.0.2) along a geodesic neck of length \( \ell \). The two infinite ends are given by \( v = \pm \infty \); the compactification \( \overline{M} \) is obtained by attaching a circle to each infinite end and using the defining function \( y_1 = (\cosh v)^{-1} \). See Figure 5.1.
3. More generally, one may consider convex co-compact hyperbolic manifolds, which are asymptotically hyperbolic manifolds of constant sectional curvature \(-1\). These manifolds can always be written as quotients \( \Gamma \backslash \mathbb{H}^n \) of the hyperbolic space by a discrete group of isometries. A detailed study of this setting is outside the scope of this book, however we refer the reader to Borthwick [Bo16] for an extensive overview of classical and recent results in the case of surfaces. See Figure 5.2 for an example.

**5.1.1. Canonical product structures.** We now show existence of canonical product structures near the boundary of an asymptotically hyperbolic manifold, in which the metric takes the form of a stretched product. They are defined as follows:

**Definition 5.3 (Canonical product structures).** Let \((M, g)\) be an asymptotically hyperbolic manifold. A boundary defining function \( y_1 \) is called canonical if
\[ |d y_1|_{y_1^2 g} = 1 \quad \text{in a neighborhood of } \partial M. \]
5.1. ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

Figure 5.2. (a) An example of a convex co-compact hyperbolic surface: three-funnel surface with geodesic neck lengths \( \ell_1, \ell_2, \ell_3 \). (b) Resonances for the three-funnel surface with \( \ell_1 = \ell_2 = \ell_3 = 7 \), computed numerically by Borthwick–Weich [BW14]. See [Bo16, Chapter 16] for more numerical results.

For such a function \( y_1 \), a product structure \((y_1, y')\) from (5.1.2) is called canonical, if the pushforward of the metric \( g \) under (5.1.2) has the form

\[
g = \frac{dy_1^2 + g_1(y_1, y', dy')}{y_1^2} \quad \text{in a neighborhood of } \partial M
\]

where \( g_1(y_1, y', dy') \) is a family of Riemannian metrics on \( \partial M \) depending smoothly on \( y_1 \in [0, \varepsilon) \).

**EXAMPLES.**

1. On the hyperbolic space (5.0.1), a canonical coordinate system on \( \{ w \neq 0 \} \) is given by

\[
y_1 = \frac{1 - |w|}{1 + |w|} \in [0, 1), \quad \theta = \frac{w}{|w|} \in S^{n-1},
\]

and the metric in the \((y_1, \theta)\) coordinates takes the form

\[
g = \frac{dy_1^2 + g_1}{y_1^2}, \quad g_1 = \frac{(1 - y_1^2)^2}{4} g_S(\theta, d\theta)
\]

where \( g_S \) denotes the standard metric on \( S^{n-1} \).

2. On the hyperbolic cylinder (5.1.4), a canonical coordinate system on \( \{ \pm v > 0 \} \) is given by

\[
y_1 = \exp(\mp v) \in [0, 1), \quad \theta \in S^1,
\]
and the metric in the \((y_1, \theta)\) coordinates takes the form

\[
g = \frac{dy_1^2 + g_1}{y_1^2}, \quad g_1 = \frac{(1 + y_1^2)^2}{4}d\theta^2.
\]

The following theorem shows existence and uniqueness of canonical coordinates, if one fixes the representative of the conformal class at the boundary:

**THEOREM 5.4 (Existence of canonical product structures).** Let \((M, g)\) be an asymptotically hyperbolic manifold and fix \(g_0 \in [g]_{\partial M}\). Then there exists a canonical product structure \((y_1, y')\) on \(M\) such that

\[
g_0 = (y_1^2g)|_{\partial M}.
\]

**Proof.** 1. We first construct \(y_1\). Fix a boundary defining function \(\tilde{y}_1\) such that \(g_0 = (\tilde{y}_1^2g)|_{\partial M}\) and denote \(\bar{g} := \tilde{y}_1^2g\). Then

\[
y_1 := e^{f\tilde{y}_1}, \quad f \in C^\infty(M; \mathbb{R}), \quad f|_{\partial M} = 0,
\]

is a canonical boundary defining function if and only if near \(\partial M\),

\[
|d\tilde{y}_1 + \tilde{y}_1df\bar{g}|^2 = 1.
\]

This is equivalent to the eikonal equation

\[
F(x, df(x)) = 0 \quad \text{for all } x \text{ near } \partial M,
\]

with the function \(F \in C^\infty(T^*M; \mathbb{R})\) given by

\[
F(x, \xi) = \frac{|d\tilde{y}_1(x) + \tilde{y}_1(x)\xi|_{\bar{g}(x)}^2 - 1}{\tilde{y}_1(x)}
\]

\[
= \frac{|d\tilde{y}_1(x)|_{\bar{g}(x)}^2 - 1}{\tilde{y}_1(x)} + 2\langle d\tilde{y}_1(x), \xi\rangle_{\bar{g}(x)} + \tilde{y}_1(x)|\xi|_{\bar{g}(x)}^2.
\]

With \(H_F\) denoting the Hamiltonian flow of \(F\), we compute

\[
H_F\tilde{y}_1(x, \xi) = 2|d\tilde{y}_1(x)|_{\bar{g}(x)}^2 = 2 \quad \text{for } x \in \partial M.
\]

Therefore, the equation (5.1.11) is noncharacteristic with respect to the hypersurface \(\{\tilde{y}_1 = 0\} \subset T^*M\). The restriction \(f|_{\partial M} = 0\) together with (5.1.11) lead to the initial condition

\[
df(x) = \frac{1 - |d\tilde{y}_1(x)|_{\bar{g}(x)}^2}{2\tilde{y}_1(x)}d\tilde{y}_1(x) \quad \text{for } x \in \partial M.
\]

The existence of a solution to (5.1.11), (5.1.13) satisfying \(f|_{\partial M} = 0\) follows from the local existence and uniqueness theorem for first order partial differential equations, see for instance [TaI, Theorem 1.15.3] or [Ev98, Theorem 3.2].
2. Having obtained $y_1$, we now construct a canonical product structure $(y_1, y')$. The gradient vector field $\nabla y_1 g_{y_1}$ is inward pointing at the boundary, therefore for $\varepsilon_1 > 0$ small enough the map
\begin{equation}
\Phi : [0, \varepsilon_1) \times \partial M \to \overline{M}, \quad \Phi(s, y') = \exp(s \nabla y_1 g_{y_1})(y')
\end{equation}
is a diffeomorphism onto a neighborhood of $\partial M$. Since $y_1$ is a canonical boundary defining function, we have $y_1(\Phi(s, y')) = s$. Then any tangent vector to $\partial M$ is mapped by $\partial y_{y'} \Phi$ to a vector tangent to a level set of $y_1$, and thus orthogonal to $\nabla y_1 g_{y_1} = \partial_s \Phi$. It follows that the inverse $(y_1, y') := \Phi^{-1}$ is a canonical product structure. \hfill $\square$

5.1.2. Even metrics. We next study the subclass of even asymptotically hyperbolic manifolds. These are the manifolds for which meromorphic extension is established in this chapter. Without the evenness assumption, the resolvent (5.0.3) does not admit a meromorphic extension to $\mathbb{C}$ with poles of finite rank as shown by Guillarmou [Gu05, Theorem 1.4].

**DEFINITION 5.5 (Even asymptotically hyperbolic metric).** An asymptotically hyperbolic manifold $(M, g)$ is called even, if there exists a canonical product structure $(y_1, y')$ from Definition 5.3 such that the corresponding metric $g_1$ satisfies
\begin{equation}
\partial^{2k+1} y_1 g_1(0, y', dy') = 0, \quad k \in \mathbb{N}_0.
\end{equation}
In other words, $g_1(0, y', dy')$ can be extended as an even function of $y_1$ past the boundary $\{y_1 = 0\}$.

It is easy to see that the examples (5.1.7) and (5.1.9) are even metrics.

The evenness of a metric does not depend on the choice of the canonical product structure, as shown by

**THEOREM 5.6 (Invariance of the evenness condition).** Assume that $(M, g)$ is an asymptotically hyperbolic manifold $(y_1, y')$, $(\tilde{y}_1, \tilde{y}')$ are two canonical product structures on $\overline{M}$, and $g_1, \tilde{g}_1$ the corresponding metrics. Assume that (5.1.15) holds for $\tilde{g}_1$. Then it also holds for $g_1$, and
\begin{align}
\partial^{2k} y_1 &= 0 \quad \text{on } \partial M, \quad k \in \mathbb{N}_0, \\
\partial^{2k+1} (\psi \circ y') &= 0 \quad \text{on } \partial M, \quad k \in \mathbb{N}_0, \quad \psi \in C^\infty(\partial M).
\end{align}

**Proof.** 1. We first show (5.1.16). Since both $y_1$ and $\tilde{y}_1$ are canonical boundary defining functions, we have $y_1 = e^f \tilde{y}_1$ where $f \in C^\infty(\overline{M}; \mathbb{R})$ solves the eikonal equation (5.1.11). In coordinates $(\tilde{y}_1, \tilde{y}')$ the eikonal equation takes the form
\begin{equation}
2 \partial_{\tilde{y}_1} f + \tilde{y}_1 \left( (\partial_{\tilde{y}_1} f)^2 + |\partial_{\tilde{y}'} f|^2_{\tilde{g}_1(\tilde{y}_1, \tilde{y}')} \right) = 0.
\end{equation}
Differentiating (5.1.18) \(2k\) times in \(\tilde{y}_1\), we obtain for each \(k \in \mathbb{N}_0\),
\[
(5.1.19) \quad \partial^2_{\tilde{y}_1} f \big|_{\tilde{y}_1=0} = -k \partial^{2k-1}_{\tilde{y}_1} \left( (\partial_{\tilde{y}_1} f)^2 + (\partial_{\tilde{y}} f)_{\tilde{y}_1}(\tilde{y}_1,\tilde{y}) \right) \big|_{\tilde{y}_1=0}.
\]
By induction using the Leibniz rule and the fact that \(\tilde{g}_1\) satisfies (5.1.15), we see that \(\partial^2_{\tilde{y}_1} f \big|_{\tilde{y}_1=0} = 0\) for all \(k\), implying (5.1.16).

2. We next show (5.1.17). By (5.1.14) the function \(\psi \circ y'\) is characterized by the conditions \((\psi \circ y')|_{\partial M} = \psi\) and
\[
\langle d(\psi \circ y'), \nabla_{\tilde{y}_1} y_1 \rangle = 0.
\]
In coordinates \((\tilde{y}_1, \tilde{y}')\) the latter equation takes the form
\[
\partial_{\tilde{y}_1} (\psi \circ y') + \tilde{y}_1 (\partial_{\tilde{y}_1} f \cdot \partial_{\tilde{y}_1} (\psi \circ y') + \langle \partial_{\tilde{y}_1} f, \partial_{\tilde{y}_1} (\psi \circ y') \rangle_{\tilde{y}_1}) = 0.
\]
Differentiating this expression \(2k\) times in \(\tilde{y}_1\) at \(\partial M\) and arguing by induction as in step 1, we obtain (5.1.17).

3. We finally show that \(g_1\) satisfies (5.1.15). It is enough to prove that for each \(\psi \in C^\infty(\partial M)\), we have
\[
(5.1.20) \quad \partial^2_{\tilde{y}_1} \left( |d(\psi \circ y')|^2_{\tilde{y}_1(g_1)} \right) \big|_{\tilde{y}_1=0} = 0, \quad k \in \mathbb{N}_0.
\]
By (5.1.16) it suffices to prove this statement with \(y_1\) replaced by \(\tilde{y}_1\):
\[
(5.1.20) \quad \partial^2_{\tilde{y}_1} \left( |d(\psi \circ y')|^2_{\tilde{y}_1(g_1)} \right) \big|_{\tilde{y}_1=0} = 0, \quad k \in \mathbb{N}_0.
\]
Now (5.1.20) follows from (5.1.17) and the fact that \(\tilde{g}_1\) satisfies (5.1.15). \(\square\)

5.1.3. The even extension. We finally describe an extension of an even asymptotically hyperbolic manifold, which is the underlying manifold for the analysis of the rest of this chapter:

DEFINITION 5.7. Let \((M,g)\) be an even asymptotically hyperbolic manifold and fix a canonical product structure \((y_1,y') \in [0,\varepsilon_1) \times \partial M\). Consider the diffeomorphism
\[
(5.1.21) \quad M \cap \{y_1 < \varepsilon_1\} \to (0,\varepsilon_1^2) \times \partial M, \quad x \mapsto (x_1,x') := (y_1^2, y').
\]
We define:

- the even compactification \(\overline{M}_{\text{even}}\) of \(M\) to be the manifold with boundary obtained by gluing \(M\) with \([0,\varepsilon_1^2) \times \partial M\) using the map (5.1.21);
- the even extension \(X = X_\varepsilon\) of \(M\) to be the manifold with boundary obtained by gluing \(M\) with \([\varepsilon,\varepsilon_1^2) \times \partial M\) using the map (5.1.21). Here \(\varepsilon > 0\) is a given constant.

See Figure 5.3 below. Note that by (5.1.16) and (5.1.17), the compactifications \(\overline{M}_{\text{even}}\) arising from different choices of canonical product structures are diffeomorphically equivalent.
5.2. A MOTIVATING EXAMPLE

We also note that a smooth function on $C^\infty(\overline{M}_{\text{even}})$ can be thought of a smooth function on $C^\infty(\overline{M})$ which admits an even extension in the $y_1$ variable across the boundary.

We will suppress the dependence of $X$ on $\varepsilon$; we fix $\varepsilon$ small enough for the construction to work.

**EXAMPLES.**

1. For the hyperbolic space (5.0.1), we may take

\begin{equation}
X = B_{\mathbb{R}^n}(0, 2), \quad M_{\text{even}} = B_{\mathbb{R}^n}(0, 1),
\end{equation}

where, using polar coordinates $r, \theta$ on $\overline{M}_{\text{even}}$, and with $y_1$ defined by (5.1.6),

\[
\begin{align*}
    r &= \frac{2|w|}{1 + |w|^2} = \frac{1 - y_1^2}{1 + y_1^2} \in [0, 1], \\
    \theta &= \frac{w}{|w|} \in S^{n-1}.
\end{align*}
\]

The metric $g$ in the $(r, \theta)$ coordinates takes the form

\begin{equation}
    g = \frac{dr^2}{(1 - r^2)^2} + \frac{r^2 g_S(\theta, d\theta)}{1 - r^2}.
\end{equation}

2. For the hyperbolic cylinder (5.1.4), we may take

\begin{equation}
X = [-2, 2]_r \times S^1_\theta, \quad M_{\text{even}} = [-1, 1]_r \times S^1_\theta,
\end{equation}

where, with $y_1$ defined in (5.1.8),

\[
    r = \tanh v; \quad 1 - r^2 = \frac{4y_1^2}{(1 + y_1^2)^2}.
\]

The metric $g$ in the $(r, \theta)$ coordinates takes the form

\begin{equation}
    g = \frac{dr^2}{(1 - r^2)^2} + \frac{d\theta^2}{1 - r^2}.
\end{equation}

5.2. A MOTIVATING EXAMPLE

Before proceeding with the general construction of the scattering resolvent, we present a simple example of an asymptotically hyperbolic manifold, and use it as motivation for introducing the modified Laplacian in §5.3 below. We will not provide proper proofs for most claims made in this section; instead they will follow from the general construction presented in the rest of this chapter.

Consider an asymptotically hyperbolic manifold $(M, g)$ with a boundary defining function $y_1$ and a product structure

\begin{equation}
(y_1, y') : \overline{M} \cap \{y_1 < 1\} \to [0, 1) \times N,
\end{equation}

in which the metric has the form

\[
g = \frac{dy_1^2 + g_0(y', dy')}{{y_1}^2}.
\]
5. A MOTIVATING EXAMPLE

Here $N = \partial M$ has dimension $n - 1$ and $g_0$ is a Riemannian metric on $N$.

On the domain \( \{ y_1 < 1 \} \) of (5.2.1), we calculate the volume form
\[
dVol_g = y_1^{-n} dy_1 dVol_{g_0}
\]
and the Laplacian (here $\Delta_{g_0}$ is the Laplacian in the $y'$ variables)
\[
\Delta_g = y_1^2 \partial_{y_1}^2 + (2 - n) y_1 \partial_{y_1} + y_1^2 \Delta_{g_0}.
\]

The scattering resolvent of $M$ is a meromorphic family of operators
\[
R_g(\lambda) : C_0^\infty(M) \to C^\infty(M), \quad \lambda \in \mathbb{C},
\]
such that for all $f \in C_0^\infty(M)$,
\[
(5.2.2) \quad \left( - \Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) R_g(\lambda) f = f;
\]
\[
(5.2.3) \quad R_g(\lambda) f \in L^2(M; dVol_g) \quad \text{for} \ \text{Im} \lambda > 0.
\]
The equation
\[
(5.2.4) \quad \left( - \Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) u = f
\]
has infinitely many solutions for every given $f$. However, the solution $u = R_g(\lambda) f$ can be identified uniquely (except at resonances) if we know enough information about the asymptotic behavior of $u$ at the infinite end of $M$, which in our case corresponds to $y_1 \to 0$.

Given (5.2.3), it makes sense to first consider the case $\text{Im} \lambda > 0$ and understand the asymptotic behavior of the $L^2$ solutions of (5.2.4). In this section, we do it using separation of variables. Let $(\beta_k)_{k \in \mathbb{N}}$ be the spectrum of the Laplacian $-\Delta_{g_0}$ on $N$, and let $v_k \in C^\infty(N)$ be the corresponding basis of eigenfunctions. Assume that
\[
(5.2.5) \quad u \in C^\infty(M) \cap L^2(M; dVol_g)
\]
is a solution to (5.2.4) and write the Fourier series in $\{y_1 < 1\}$,
\[
u(y_1, y') = \sum_k u_k(y_1) v_k(y'), \quad u_k \in C^\infty((0,1)).
\]
The right-hand side $f$ is supported in $\{y_1 \geq \delta\}$ for some $\delta > 0$. For simplicity of notation, we assume that $\text{supp} \ f \subset \{y_1 \geq 1\}$. Then (5.2.4) implies that $u_k$ solves the ordinary differential equation in $(0,1)$
\[
(5.2.6) \quad \left( - y_1^2 \partial_{y_1}^2 + (n-2) y_1 \partial_{y_1} - \lambda^2 - \frac{(n-1)^2}{4} + y_1^2 \beta_k \right) u_k(y_1) = 0.
\]
This is Bessel's equation, and (unless $\lambda \in i\mathbb{Z}$ where a bit of extra care is needed), it has two solutions asymptotic to $y_1^{\alpha^+}$ and $y_1^{\alpha^-}$, where $\alpha_{\pm} \in \mathbb{C}$,
5.2. A MOTIVATING EXAMPLE

called the *indicial roots*, are solutions to the equation

\[ I(\alpha) = 0, \quad I(\alpha) := -\left(\alpha - \frac{n-1}{2}\right)^2 - \lambda^2. \]

Indeed, we have

\[
\left( - y_1^2 \partial_{y_1}^2 + (n-2)y_1 \partial_{y_1} - \lambda^2 - \frac{(n-1)^2}{4} + y_1^2 \beta_k \right) y_1^\alpha = I(\alpha) y_1^\alpha + \beta_k y_1^{\alpha+2};
\]

then (see Exercise 5.5) the two solutions to (5.2.4) can be found in the form of power series

\[ y_1^{\alpha_{\pm}} \sum_{j=0}^{\infty} a_{j,\pm} y_1^{2j} \]

for some coefficients \( a_{j,\pm} \in \mathbb{C}, \, a_{0,\pm} \neq 0 \).

In our case, the indicial roots are

\[ \alpha_{\pm} = \frac{n-1}{2} \pm i\lambda. \]

This explains the shift of the spectral parameter by \( \frac{(n-1)^2}{4} \) in (5.0.3) – without this, \( \alpha_{\pm} \) will not be holomorphic functions of \( \lambda \).

Recall that we are looking for \( u \in L^2(M; dV_{g}) \), thus we need

\[ y_1^{-n/2} u_k \in L^2((0,1); dy_1). \]

Since \( \text{Im} \, \lambda > 0 \), (5.2.10) holds for \( y_1^{\alpha_{-}} \), but not for \( y_1^{\alpha_{+}} \). Therefore, \( y_1^{-\alpha_{-}} u_k \) has to be the sum of a power series in \( y_1^2 \). This leads to the following statement, whose full proof is a corollary of Theorem 5.32 below:

**PROPOSITION 5.8.** Assume that \( u \in C^\infty(M) \cap L^2(M; dV_{g}) \) solves the equation (5.2.4), for some \( f \in C^\infty_0(M) \) and \( \text{Im} \, \lambda > 0 \). Then

\[ u \in y_1^{\frac{n-1}{2} - i\lambda} C^\infty(M_{\text{even}}) \]

where the even compactification \( M_{\text{even}} \) was introduced in Definition 5.7.

Returning to meromorphic continuation of the resolvent to \( \text{Im} \, \lambda \leq 0 \), the main idea is to define \( R_g(\lambda)f \) as the solution to (5.2.4) satisfying the outgoing condition (5.2.11). For this, the outgoing condition should be strong enough to rule out all other solutions. In practice, we replace \( C^\infty(M_{\text{even}}) \) by the Sobolev space \( H^s(X) \), where \( X \) is the even extension introduced in Definition 5.7 and \( s \) is large enough depending on \( \lambda \).

To see why an outgoing solution to (5.2.4) is uniquely defined (for \( \lambda \) not a resonance), we consider the conjugated operator

\[ y_1^{i\lambda - \frac{n-1}{2}} \left( - \Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) y_1^{\frac{n-1}{2} - i\lambda} \]
on the space $H^s(X)$. In the product coordinates (5.2.1), it takes the form
\[ -y_1^2 \partial^2_{y_1} - y_1^2 \Delta_{g_0} + (2i\lambda - 1)y_1 \partial_{y_1}. \]

Passing to the product structure $(x_1, x') = (y_2^2, y')$ on the even compactification $\overline{M}_{even}$ and extending the resulting operator to $\overline{X} = \{ x_1 \geq -\varepsilon \}$, we obtain
\[ -4x_1^2 \partial^2_{x_1} - x_1 \Delta_{g_0} + 4(i\lambda - 1)x_1 \partial_{x_1}. \]

The result can be divided on the left by $x_1$, obtaining the final operator on $\overline{X}$ that will be the focus of this chapter (with the previous calculation showing that the two operators below coincide in $\{ 0 < x_1 < 1 \}$):
(5.2.13) \[ P(\lambda) = \begin{cases} \frac{i\lambda}{x_1^2} - \frac{n-1}{4} \left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right)x_1^{\frac{n+1}{4} - \frac{i\lambda}{2}} & \text{on } \{ x_1 > 0 \}, \\ -4x_1^2 \partial^2_{x_1} - \Delta_{g_0} + 4(i\lambda - 1)x_1 \partial_{x_1} & \text{on } \{ x_1 < 1 \}. \end{cases} \]

The operator $P(\lambda)$ has nondegenerate principal part for $x_1 \neq 0$, and the degeneracy at $x_1 = 0$ is minimal as only the coefficient of $\partial^2_{x_1}$ vanishes. This minimal degeneracy, which is the result of passing to the space $\overline{M}_{even}$ and dividing the conjugated operator by $x_1$, makes it possible to prove the following key statement (see Theorem 5.30 below for the general case):

**PROPOSITION 5.9.** Consider the space $\mathcal{X}^s := \{ u \in \bar{H}^s(X) \mid P(0)u \in \bar{H}^{s-1}(X) \}$; see Definition E.21 for Sobolev spaces on domain with boundary. Then $P(\lambda) : \mathcal{X}^s \to \bar{H}^{s-1}(X)$, $\text{Im } \lambda > \frac{1}{2} - s$ is a Fredholm operator. Moreover, it is invertible for some values of $\lambda$.

To show the Fredholm property of $P(\lambda)$, we need to show its invertibility modulo a smoothing operator. In $M = \{ x_1 > 0 \}$, this property follows from the fact that $P(\lambda)$ is elliptic. In $\overline{X} \setminus \overline{M}_{even} = \{ x_1 < 0 \}$, $P(\lambda)$ is hyperbolic and one may use standard energy estimates. Finally, on $\{ x_1 = 0 \}$ the operator $P(\lambda)$ degenerates and we use radial points estimates, which is where the condition $\text{Im } \lambda > \frac{1}{2} - s$ becomes important.

Given Proposition 5.9, we see by Theorem C.5 that $P(\lambda)^{-1} : \bar{H}^{s-1}(X) \to \mathcal{X}^s$ is a meromorphic family of operators. The scattering resolvent is then defined as
\[ R_g(\lambda) = \frac{i\lambda}{x_1^2} - \frac{n-1}{4} \mathbf{1}_M P(\lambda)^{-1} \mathbf{1}_M x_1^{\frac{n+1}{4} - \frac{i\lambda}{2}}, \]
which coincides with the $L^2$ resolvent for $\text{Im } \lambda > 0$ since the outgoing condition (5.2.11) implies that $u \in L^2(M; d\text{Vol}_g)$. 


5.3. THE MODIFIED LAPLACIAN

In this section introduce the modified spectral family of the Laplacian, generalizing (5.2.13). As explained in §5.2, the Fredholm property of this operator on appropriately chosen spaces will ultimately give meromorphic continuation of the resolvent (5.0.3).

Let \((M, g)\) be an even asymptotically hyperbolic manifold, see Definition 5.5. Fix a canonical boundary defining function \(y_1\) and let \((y_1, y') \in [0, \varepsilon_1) \times \mathbb{N}\) be the corresponding canonical product structure; see Definition 5.3. Let \(\overline{M}_{\text{even}}\) be the even compactification of \(M\) and \(\overline{X} = \overline{X}_\varepsilon\) be its even extension, see Definition 5.7. We denote by \(X\) the interior of \(\overline{X}\).

Put

\[
x_1 := y_1^2,
\]

so that \(x_1\) is a boundary defining function of \(\overline{M}_{\text{even}}\), and \(\overline{X} = \{x_1 \geq -\varepsilon\}\). We write (see Figure 5.3)

\[
(5.3.1) \quad \overline{X} = M \cup \overline{Y}, \quad M = \{x_1 > 0\}, \quad \overline{Y} := \{-\varepsilon \leq x_1 < \varepsilon_1^2\},
\]

where \(\{y_1 < \varepsilon_1\}\) is the domain of the product structure \((y_1, y')\). On \(\overline{Y}\), we have the product structure

\[
(5.3.2) \quad (x_1, x') \in [-\varepsilon, \varepsilon_1^2) \times \partial M, \quad x' := y'.
\]

Since \((M, g)\) is an even metric, we can write on \(M \cap Y\) in the coordinates (5.3.2),

\[
(5.3.3) \quad g = \frac{dx_1^2}{4x_1^2} + \frac{g_1(x_1, x', dx')}{x_1}
\]
where \( g_1 \) is smooth in \( x_1 \in (0, \varepsilon^2) \) up to \( x_1 = 0 \). We fix an extension of \( g_1 \) to \( x_1 \in [-\varepsilon, \varepsilon^2] \) as a smooth family of Riemannian metrics on \( \partial M \):

\[
(5.3.4) \quad g_1 \in C^\infty([-\varepsilon, \varepsilon^2] \times \partial M; \otimes^2 T^* \partial M).
\]

Following (5.2.13), consider the differential operator on \( M \),

\[
(5.3.5) \quad x_1^{\frac{n+1}{4}} \left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) x_1^{\frac{n-1}{4} - \frac{i}{2}}.
\]

**Lemma 5.10.** On \( M \cap Y = \{ 0 < x_1 < \varepsilon^2 \} \), the operator (5.3.5) has the following form under (5.3.2):

\[
-4 x_1 \partial_{x_1}^2 + 4 (i \lambda - 1) \partial_{x_1} - \gamma (4 x_1 \partial_{x_1} + n - 2 i \lambda) - \Delta g_1,
\]

where \( \Delta g_1 \) is the Laplace–Beltrami operator of \( g_1 \) in the \( x' \) variables and

\[
(5.3.6) \quad \gamma(x_1, x') := J^{-1} \frac{\partial J}{\partial x_1} \in C^\infty(M; \mathbb{R}), \quad J := (\det(g_1^{ij}))^{1/2}
\]

is the logarithmic derivative of the Jacobian of the metric \( g_1 \), which is independent of the choice of local coordinates on \( \partial M \).

**Proof.** We compute from (5.3.3)

\[
\Delta_g = 4 x_1^{\frac{n+1}{4}} \partial_{x_1} x_1^{\frac{3-n}{2}} \partial_{x_1} + 4 x_1^2 \gamma \partial_{x_1} x_1 \Delta g_1.
\]

Using the identity \( x_1 \partial_{x_1} x_1^\alpha = x_1^\alpha (x_1 \partial_{x_1} + \alpha), \alpha \in \mathbb{C} \), we compute (5.3.5) as

\[
-\frac{1}{x_1} \left( 2 x_1 \partial_{x_1} + 1 - \frac{n}{2} - i \lambda \right) \left( 2 x_1 \partial_{x_1} + \frac{n-1}{2} - i \lambda \right)
\]

\[
- \gamma (4 x_1 \partial_{x_1} + n - 2 i \lambda) - \Delta g_1 - \frac{1}{x_1} \left( \lambda^2 + \frac{(n-1)^2}{4} \right)
\]

\[
= -4 x_1 \partial_{x_1}^2 + 4 (i \lambda - 1) \partial_{x_1} - \gamma (4 x_1 \partial_{x_1} + n - 2 i \lambda) - \Delta g_1,
\]

finishing the proof. \( \square \)

Using Lemma 5.10, we continue the operator (5.3.5) to \( X \):

**Definition 5.11.** Define the **extended modified Laplacian** as the second order differential operator \( P(\lambda) \) on \( X \) given by (5.3.5) on \( M \) and by the following formula on \( Y \):

\[
(5.3.7) \quad P(\lambda) = -4 x_1 \partial_{x_1}^2 + 4 (i \lambda - 1) \partial_{x_1} - \gamma (4 x_1 \partial_{x_1} + n - 2 i \lambda) - \Delta g_1
\]

where \( g_1 \) is defined in (5.3.4) and \( \gamma \in C^\infty(Y; \mathbb{R}) \) is defined in (5.3.6).

To facilitate the study of the high frequency limit \( \Re \lambda \to \infty \) in Part 3 of the book we introduce the semiclassically rescaled version of the operator \( P(\lambda) \):

\[
(5.3.8) \quad P_h(\omega) := h^2 P(h^{-1} \omega), \quad 0 < h \ll 1, \ \omega \in \mathbb{C}.
\]
We calculate

\[
P_h(\omega) = x_1^{\frac{i\omega}{\pi}} \left( -h^2 \Delta_g - \omega^2 - \frac{(n - 1)^2}{4} h^2 \right) x_1^{\frac{n-1}{2} - \frac{i\omega}{2}}
\]

on \( M \) and, denoting \( D = \frac{1}{i} \partial \),

\[
P_h(\omega) = 4x_1(hDx_1)^2 - 4(\omega + ih)hDx_1 - h^2 \Delta_{g_1} - ih \gamma(4x_1hDx_1 + i(1 - n)h - 2\omega)
\]

on \( Y \). It follows from (5.3.9) and (5.3.10) that \( P_h(\omega) \) is a second order semiclassical differential operator on \( X \) (see §5.1) with coefficients smooth up to the boundary.

Consider the semiclassical principal symbol of \( P_h(\omega) \),

\[
p = p(x, \xi; \omega) := \sigma_h(P_h(\omega)) \in \text{Poly}^2(T^*X).
\]

We compute on \( M \),

\[
p(x, \xi; \omega) := \frac{1}{x_1} \left( |\xi - \omega d\mu_{1}|^2 - \omega^2 \right)
\]

and on \( Y \),

\[
p(x_1, x', \xi_1, \xi; \omega) = 4x_1 \xi_1^2 - 4\omega \xi_1 + p_1, \quad p_1 := |\xi_1|^2_{g_1(x_1, x')}.
\]

The behavior of \( p \) for any \( \omega \in \mathbb{R} \) can be reduced to the cases \( \omega = 0 \) and \( \omega = 1 \), as follows from the scaling relation

\[
p(x, s\xi; s\omega) = s^2 p(x, \xi; \omega), \quad s \in \mathbb{R}.
\]

**REMARK.** In (5.3.5), we used the function \( x_1 = y_1^2 \), where \( y_1 \) was a canonical boundary defining function on \( M \). However, one may consider a more general function \( e^{-2\psi}x_1 \), where \( \psi \in C^\infty(\overline{X}; \mathbb{R}) \), and study the resulting operators

\[
P_\psi(\lambda) := e^{(\frac{n+3}{2} - i\lambda)\psi} P(\lambda)e^{(i(\frac{n-1}{2})\psi)}, \quad P_{\psi,h}(\omega) := h^2 P(h^{-1}\omega).
\]

The operator \( P_\psi(\lambda) \) has similar properties to \( P(\lambda) \); in particular, it satisfies Theorem 5.30. This more general form is used in the examples below and for applications to general relativity in §5.7.

**EXAMPLES.** 1. For the hyperbolic space (see (5.1.22), (5.1.23)), we choose \( \psi \) so that \( e^{-2\psi}x_1 = 1 - r^2 \). Then on \( M \),

\[
P_\psi(\lambda) = (1 - r^2) \frac{\lambda - \frac{n+3}{2}}{2} \left( -\Delta_g - \lambda^2 - \frac{(n - 1)^2}{4} (1 - r^2) \right) \left( 1 - r^2 \right) \frac{n-1}{2} - \frac{i\omega}{2},
\]
and on the entire $\mathcal{X}$ we have
\begin{equation}
P_\psi(\lambda) = -(1 - r^2)\partial^2_r + (n + 1 - 2i\lambda) r \partial_r + \frac{1 - n}{r} \partial_r
\end{equation}
\begin{equation}
- \lambda^2 - ni\lambda + \frac{n^2 - 1}{4} - \frac{1}{r^2} \Delta_S,
\end{equation}
with $\Delta_S$ the Laplacian on the sphere. The principal symbol of $P_{\psi,h}(\omega)$ is
\begin{equation}
p_\psi(r, \theta, \xi_r, \xi_\theta; \omega) = (1 - r^2)\xi_r^2 + 2\omega r \xi_r - \omega^2 + \frac{|\xi_\theta|^2}{r^2}.
\end{equation}
If we instead use the defining function $x_1 = y_1^2$ where $y_1$ is given by (5.1.8), then we have
\begin{equation}
P(\lambda) = -4x_1 \partial^2_{x_1} + 4(i\lambda - 1) \partial_{x_1}
\end{equation}
\begin{equation}
+ \frac{n - 1}{1 - x_1} (4x_1 \partial_{x_1} + n - 1 - 2i\lambda) - \frac{4}{(1 - x_1)^2} \Delta_{g_S}.
\end{equation}
2. For the hyperbolic cylinder (see (5.1.24), (5.1.25)), we also choose $\psi$ so that $e^{-2\psi}x_1 = 1 - r^2$. Then on $M$, $P_\psi(\lambda)$ has the form (5.3.16) and on the entire $\mathcal{X}$ we have
\begin{equation}
P_\psi(\lambda) = -(1 - r^2)\partial^2_r + 2(1 - i\lambda) r \partial_r - \lambda^2 - i\lambda + \frac{1}{4} - \partial^2_{\theta}.
\end{equation}
The principal symbol of $P_{\psi,h}(\omega)$ is
\begin{equation}
p_\psi(r, \theta, \xi_r, \xi_\theta; \omega) = (1 - r^2)\xi_r^2 + 2\omega r \xi_r - \omega^2 + \xi_\theta^2.
\end{equation}
If we instead use the defining function $x_1 = y_1^2$ where $y_1$ is given by (5.1.8), then we have
\begin{equation}
P(\lambda) = -4x_1 \partial^2_{x_1} + 4(i\lambda - 1) \partial_{x_1}
\end{equation}
\begin{equation}
- \frac{1}{1 + x_1} (4x_1 \partial_{x_1} + 1 - 2i\lambda) - \frac{4}{(1 + x_1)^2} \partial^2_{\theta}.
\end{equation}

The following statement is a direct corollary of (5.3.12) and (5.3.13):

**PROPOSITION 5.12.** Let $p_0(x, \xi)$ be the quadratic in $\xi$ part of the symbol $p$, that is the quadratic form obtained by putting $\omega := 0$ in $p$. Then the operator $P(\lambda)$, considered as a nonsemiclassical differential operator, is
- **elliptic** in $M = \{x_1 > 0\}$, in the sense that $p_0$ is positive definite;
- **hyperbolic** in $X \setminus \overline{M}_{\text{even}} = \{x_1 < 0\}$, in the sense that $p_0$ has signature $(n - 1, 1)$.

We finally discuss the imaginary part of the operator $P_h(\omega)$. Define a volume form $d \text{Vol}$ on $\overline{X}$ as follows:
\begin{equation}
d \text{Vol} = \begin{cases}
2x_1^{\frac{n+1}{2}} d \text{Vol}_g & \text{on } M; \\
dx_1 d \text{Vol}_{g_1} & \text{on } \overline{Y}.
\end{cases}
\end{equation}
A direct calculation shows that
\[(5.3.24) \quad P_h(\omega)^* = P_h(\bar{\omega})\]
where \(P_h(\omega)^*\) denotes the formal adjoint of \(P_h(\omega)\) in \(L^2(X; d\text{Vol})\). It follows that for \(\omega \in \mathbb{R}\), the operator \(P_h(\omega)\) is symmetric. For general values of \(\omega\), we have the following statement:

**Proposition 5.13.** Assume that \(\omega = \omega_R + i\omega_I\), where \(\omega_R, \omega_I \in \mathbb{R}\) vary in a compact set, and put
\[
\text{Im} P_h(\omega) = \frac{P_h(\omega) - P_h(\omega_R)}{2i}.
\]
Then \(h^{-1}P_h(\omega)\) is a first order semiclassical differential operator (see §E.1.1) and its principal symbol is
\[(5.3.25) \quad \sigma_h(h^{-1} \text{Im} P_h(\omega)) = \omega_I \partial_{\omega} p.
\]

**Proof.** By (5.3.24), we have
\[
\text{Im} P_h(\omega) = \text{Im} (P_h(\omega) - P_h(\omega_R)).
\]
By (5.3.9) and (5.3.10), \(P_h(\omega)\) is a quadratic polynomial in \(hD_x, \omega\) with coefficients smooth in \(x\). It follows that \(h^{-1}(P_h(\omega) - P_h(\omega_R))\) is a first order semiclassical differential operator with principal symbol \(i\omega_I \partial_{\omega} p\); (5.3.25) follows. \(\square\)

### 5.4. PHASE SPACE DYNAMICS

We now study the zero set and the Hamiltonian flow of the principal symbol \(p\) of the conjugated operator \(P_h(\omega)\) (see (5.3.11)), in preparation for the propagation estimates of §5.5. In this section, we always consider the case of \(\omega \in \mathbb{R}\).

To understand the behavior of \(p\) both for bounded \(\xi\) and for \(|\xi|\) going to infinity, we use the fiber-radially compactified cotangent bundle \(\overline{T^*X}\), see §E.1.2. The values of \(|\xi| = \infty\) correspond to the boundary \(\partial \overline{T^*X}\), which is isomorphic to the sphere bundle on \(X\).

Since \(p\) is a classical symbol of order 2, the rescaled symbol \(\langle \xi \rangle^{-2} p\) and the rescaled flow \(\langle \xi \rangle^{-1} H_p\) extend smoothly to \(\overline{T^*X}\). Therefore, we consider the flow
\[(5.4.1) \quad \exp(t\langle \xi \rangle^{-1} H_p) \quad \text{on} \quad \{\langle \xi \rangle^{-2} p = 0\} \subset \overline{T^*X}.
\]
Note that the symbol \(p\) depends on the choice of \(\omega\).
5. PHASE SPACE DYNAMICS

We will often use the following coordinates on the fibers of $T^*Y \setminus 0$, where $Y$ is defined in (5.3.1):

\[
\rho = (\xi_1^2 + |\xi'|_{g_1})^{-1/2} \in [0, \infty), \quad \hat{\xi} = (\hat{\xi}_1, \hat{\xi}') = \rho \xi \in S^*_x Y.
\]

where $S^*_Y$ denotes the cosphere bundle:

\[
S^*_Y = \{(x, \hat{\xi}_1, \hat{\xi}') \in \pi_x Y | \xi_1^2 + |\xi'|^2_{g_1} = 1\}.
\]

Using (5.3.13), we calculate in the coordinates $(x_1, \xi_1, \rho, \hat{\xi})$,

\[
\rho^2 p = 4x_1 \hat{\xi}_1^2 - 4\rho \omega \xi_1 + |\hat{\xi}'|^2_{g_1}.
\]

5.4.1. Characteristic set. We first study the characteristic set

\[
\{\langle \xi \rangle^{-2} p = 0\} \subset T^* M.
\]

We will show that it splits into two components. To define them, we use an auxiliary function

\[
\varphi = \varphi(x_1) \in C^\infty(\overline{X}; \mathbb{R}),
\]

such that, with $\varepsilon_1 > 0$ defined in (5.3.1)

\[
\varphi(x_1) = \frac{1}{2} \log x_1 \quad \text{for } x_1 \geq \varepsilon_1^2;
\]

\[
\varphi'(x_1) > 0, \quad x_1 \varphi'(x_1) < 1 \quad \text{for all } x_1.
\]

We note the following corollary of (5.3.3) and (5.4.5), (5.4.6):

\[
\left| d\varphi - \frac{dx_1}{2x_1} \right|_{g} < 1 \quad \text{on } M.
\]

**Lemma 5.14.** There exist closed $\omega$-dependent sets $\Sigma_{\pm} \subset \overline{T^* X}$ such that (see Figure 5.4)

\[
\{\langle \xi \rangle^{-2} p = 0\} = \Sigma_+ \cup \Sigma_-, \quad \omega \in \mathbb{R} \setminus \{0\};
\]

\[
\{\langle \xi \rangle^{-2} p = 0\} \cap \partial T^* X = \check{\Sigma}_+ \cup \check{\Sigma}_-, \quad \omega \in \mathbb{R};
\]

where $\check{\Sigma}_{\pm} := \Sigma_{\pm} \cap \partial T^* X$ are independent of $\omega$. Moreover,

\[
\pm \langle \xi \rangle^{-1} (\partial_\omega p + \partial_{\xi} p \cdot \partial_\varphi) > 0 \quad \text{on } \Sigma_{\pm},
\]

where $\varphi$ was defined in (5.4.5), (5.4.6). Finally,

\[
\Sigma_{\pm} \cap \partial T^* M = \emptyset \quad \text{for } \omega \in \mathbb{R};
\]

\[
\Sigma_{\pm} \cap \overline{T^* M} = \emptyset \quad \text{for } \pm \omega > 0.
\]
Proof. 1. Define the function
\[ q := \partial_\omega p + \partial_\xi p \cdot \partial_\varphi. \]
We put
\[ (5.4.13) \Sigma_\pm := \{ \langle \xi \rangle^{-2} p = 0 \} \cap \{ \pm \langle \xi \rangle^{-1} q \geq 0 \}. \]
Clearly, \( \Sigma_+ \) and \( \Sigma_- \) are closed and their union equals the characteristic set. To show that \( \Sigma_+ \cap \Sigma_- = \emptyset \), as well as (5.4.10), we need
\[ (5.4.14) \{ \langle \xi \rangle^{-2} p = 0 \} \cap \{ \langle \xi \rangle^{-1} q = 0 \} = \emptyset \quad \text{for} \quad \omega \in \mathbb{R} \setminus \{0\}. \]
The splitting (5.4.9) follows from (5.4.8) and the fact that \( \Sigma_\pm \cap \partial T^* X \) are independent of \( \omega \).

2. We first show (5.4.14) on \( T^* M = \{ x_1 > 0 \} \). By (5.12),
\[ \{ \langle \xi \rangle^{-2} p = 0 \} \cap \partial T^* M = \emptyset; \]
this proves (5.4.11). In the interior \( T^* M \), we have by (5.3.12),
\[ p = 0 \implies \left| \xi - \omega \frac{dx_1}{2x_1} \right| g = |\omega|, \]
\[ q = 2 \frac{x_1}{\xi} \left( \left\langle \xi + \frac{dx_1}{2x_1} \right, \partial_\varphi \left( x_1 \right) g \right) - \omega \right). \]
By (5.4.7) and the Cauchy–Schwarz inequality for the metric \( g \), for \( \omega \in \mathbb{R} \setminus 0 \) the function \( q \) has the same sign as \( -\omega \) on \( \{ p = 0 \} \cap T^* M \). This gives (5.4.12) and finishes the proof of (5.4.14) on \( M \).

3. It remains to show (5.4.14) on \( T^*(X \setminus M) = \{ x_1 \leq 0 \} \). In the interior \( T^*(X \setminus M) \cap \{ p = 0 \} \cap \{ q = 0 \} \), we have by (5.3.13),
\[ 4(x_1 \xi - \omega) \xi_1 + |\xi'|^2_{\xi_1} = 0, \]
\[ (5.4.16) (2x_1 \xi - \omega) \varphi'(x_1) = \xi_1. \]
Since \( \omega \neq 0 \), we have \( \xi_1 \neq 0 \). Solving (5.4.16) for \( \omega \) and substituting it into (5.4.15), we get a contradiction with (5.4.6).

On the fiber infinity \( \partial T^*(X \setminus M) \cap \{ \langle \xi \rangle^{-2} p = 0 \} \cap \{ \langle \xi \rangle^{-1} q = 0 \} \), we have in the coordinates (5.4.2),
\[ 4x_1 \hat{\xi}_1^2 + |\hat{\xi}'|^2_{\xi_1} = 0, \]
\[ (5.4.17) \hat{\xi}_1 (2x_1 \varphi'(x_1) - 1) = 0. \]
Since \( x_1 \leq 0 \), (5.4.6) implies that \( 2x_1 \varphi'(x_1) - 1 < 0 \). Therefore, (5.4.18) implies that \( \hat{\xi}_1 = 0 \), giving a contradiction with (5.4.17) and finishing the proof. \( \square \)
5.4.2. Hamiltonian flow. We now analyse the Hamiltonian flow (5.4.1), concentrating on its behavior on $T^*Y$ (see (5.3.1)). Using (5.3.13), we calculate in the coordinates induced by (5.3.2),

$$H_p = 4(2x_1\xi_1 - \omega)\partial_{x_1} - 4\xi_1^2\partial_{\xi_1} + H_{p_1},$$

where $H_{p_1}$ generates the rescaled geodesic flow of the metric $g_1$.

In the coordinates (5.4.2) near the fiber infinity $\partial T^*Y$, we compute

$$(5.4.19) \quad \rho H_p = 4(2x_1\dot{\xi}_1 - \rho\omega)\partial_{x_1} + 4\dot{\xi}_1^2(\dot{\xi}_1\rho\partial_{\rho} - |\dot{\xi}|^2 g_{\dot{\xi}}\partial_{\dot{\xi}}) + \dot{\xi}_1\dot{\xi}_1' \cdot \partial_{\dot{\xi}} + \rho H_{p_1};$$

this flow is a reparametrization of (5.4.1) on the characteristic set.

Let $\hat{\Sigma}_\pm$ be given by Lemma 5.14. Using the coordinates (5.4.2), define

$$L_\pm := \{ \rho = 0, \ x_1 = 0, \ \dot{\xi}_1 = 0, \ \dot{\xi}_1' = \mp 1 \} = \hat{\Sigma}_\pm \cap \{ x_1 = 0 \};$$

note that $L_+ \cup L_-$ is the intersection of the conormal bundle to $\{ x_1 = 0 \}$ with the fiber infinity. The importance of the sets $L_\pm$ comes from the following...
LEMMA 5.15. \( L_\pm \) are invariant under the Hamiltonian flow (5.4.1). Moreover, \( L_+ \) is a radial sink and \( L_- \) is a radial source for the flow (5.4.1) in the sense of Definition E.52, for all \( \omega \in \mathbb{R} \). See Figure 5.4.

Proof. We argue in a neighborhood of \( L_+ \sqcup L_- \), using the coordinates (5.4.2) and replacing the flow (5.4.1) by its rescaling \( \exp(t \rho H_p) \). Using the function \( \rho^2 p_1(x, \xi) = |\hat{\xi}|^2_{g_1} \), we write
\[
L_+ \sqcup L_- = \{ \rho = 0, \ x_1 = 0, \ \rho^2 p_1 = 0 \}.
\]
Using (5.4.19), we compute
\[
(5.4.21) \quad \rho H_p \rho = (4 \hat{\xi}_1^3 + H_p) \rho, \\
\rho H_p x_1 = 8 \hat{\xi}_1 x_1 - 4 \omega \rho, \\
\rho H_p (\rho^2 p_1) = (8 \hat{\xi}_1^3 + 2 H_p) \rho^2 p_1.
\]
Here \( H_p, \rho \) is a symbol of order 0 and thus extends smoothly to the fiber infinity. It follows that (5.4.21) vanish on \( L_+ \sqcup L_- \), thus \( L_\pm \) are invariant under the flow \( \exp(t \rho H_p) \).

We next have \( \rho^{-2} = \xi^2 + p_1 \), thus \( H_p (\rho^{-2}) = -2 \xi_1 \partial_{x_1} p_1 \). Since \( p_1 \) is a quadratic form in \( \xi' \), so is \( \partial_{x_1} p_1 \), and we obtain
\[
(5.4.22) \quad H_{p_1} \rho = \hat{\xi}_1 \rho^2 \partial_{x_1} p_1 = O(\rho^2 p_1) \text{ near } L_+ \sqcup L_-.
\]
By (5.4.21) and (5.4.22), in a neighborhood of \( L_\pm \) we have
\[
(5.4.23) \quad \mp \rho H_p (x_1^2 + \rho^2 p_1) \geq 4(x_1^2 + \rho^2 p_1) - C_0 \rho |x_1|, \\
(5.4.24) \quad \mp \rho H_p \rho \geq 2 \rho
\]
where the constant \( C_0 > 0 \) depends on \( \omega \). It follows that
\[
\mp \rho H_p f \geq 2 f \text{ near } L_\pm, \\
f := x_1^2 + \rho^2 p_1 + C_0^2 \rho^2,
\]
where we have used the inequality \( 2 C_0 \rho |x_1| \leq x_1^2 + C_0^2 \rho^2 \). Since \( f \) is a quadratic defining function of \( L_\pm \), we get uniformly in \( (x, \xi) \) in a neighborhood of \( L_\pm \),
\[
e^{\pm t \rho H_p} (x, \xi) \to L_\pm \text{ as } t \to \infty,
\]
\[
\rho(e^{\pm t \rho H_p} (x, \xi)) \leq C e^{-2 t} \rho(x, \xi) \text{ for } t \geq 0,
\]
where for the second inequality we used (5.4.24). This shows that \( L_+ \) is a radial sink and \( L_- \) is a radial source. \( \square \)

We next study the global behavior of the flow on \( T^* X \), using

LEMMA 5.16. For \( \omega \in \mathbb{R} \setminus 0 \), we have (see Figure 5.4)
\[
(5.4.25) \quad \pm (\xi)^{-1} H_p x_1 > 0 \quad \text{on} \quad \Sigma_\pm \cap \{ x_1 \leq 0 \} \setminus L_\pm.
\]
5. PHASE SPACE DYNAMICS

\[ \xi_1 - \frac{\omega \varphi'(x_1)}{1 - 2x_1 \varphi'(x_1)} \]

Figure 5.5. The graph of the left-hand side of (5.4.26), as a function of \( \xi_1 \).

Proof. 1. We first consider the case of \( x_1 < 0 \) and finite \( \xi \). Fix \( x_1 < 0, x', \xi' \).

We write the equation \( p = 0 \) using (5.3.13):

\[ 4x_1 \xi_1^2 - 4\omega \xi_1 + |\xi'|^2 g_1 = 0. \]

This is a quadratic equation in \( \xi_1 \) with discriminant \( 16(\omega^2 - x_1 |\xi'|^2 g_1) > 0 \), therefore it has two roots \( \xi_1^- < \xi_1^+ \). See Figure 5.5.

By (5.4.13) and (5.4.6), \( \Sigma_{\pm} \) are characterized by the inequalities

\[ \pm \left( \xi_1 + \frac{\omega \varphi'(x_1)}{1 - 2x_1 \varphi'(x_1)} \right) < 0. \]

Substituting \( \xi_1 = -\omega \varphi'(x_1)/(1 - 2x_1 \varphi'(x_1)) \) into the left-hand side of (5.4.26), we obtain a positive number. Therefore,

\[ (x_1, x', \xi_1^-, \xi') \in \Sigma_{\pm}. \]

On the other hand, \( H_p x_1 = \partial_{\xi_1} p \) is positive at \( \xi_1^- \) and negative at \( \xi_1^+ \); (5.4.25) follows.

2. The case of the fiber infinity \( \partial T^* X \cap \{ x_1 < 0 \} \) is considered similarly, writing the equation \( (\xi)^{-2} p = 0 \) in the coordinates (5.4.2) as

\[ 4x_1 \xi_1^2 + |\xi'|^2 g_1 = 0 \]

and solving it in \( \xi_1 \).

By (5.4.20), the remaining case is \( x_1 = 0 \) and \( \xi \) finite. It is handled similarly to the first case, with (5.4.26) now a linear equation whose only root lies in \( \Sigma_{\pm} \) for \( \pm \omega > 0 \). \( \square \)

Since \( X \) is a manifold with boundary, some Hamiltonian trajectories may exit \( T^* X \) through the boundary \( \partial X \). This is formalized in

DEFINITION 5.17. We say that a trajectory \( \gamma(t) = e^{t(\xi)^{-1}H_p(x, \xi)}, (x, \xi) \in T^* X \), exits \( T^* X \) at time \( t_0 \in \mathbb{R} \), if the projection of \( \gamma(t_0) \) to the base lies in \( \{ x_1 = -\varepsilon \} = \partial X \), with \( \varepsilon \) fixed in the beginning of §5.3.

Together with Lemma 5.15 and (5.4.11), (5.4.25) gives the structure of the flow on the fiber infinity (see the proof of Lemma 5.19 below for a detailed argument):
LEMMA 5.18. Assume that \((x, \xi) \in \hat{\Sigma}_\pm\) and put \(\gamma(t) = e^{t \langle \xi \rangle - 1} H_p(x, \xi)\). Then:

1. \(\gamma(t) \to L_\pm\) as \(t \to \pm \infty\).
2. If \((x, \xi) \notin L_\pm\), then \(\gamma\) exits \(T^*X\) at some time \(t_0\), \(\pm t_0 \leq 0\).

The situation on the entire \(T^*X\) is more complicated because trajectories may become trapped inside the manifold \(M\) (see also Figure 5.4):

LEMMA 5.19. There exists \(\delta > 0\) such that for all \(\omega \in \mathbb{R} \setminus \{0\}\) and \((x, \xi) \in T^*X\), \(\gamma(t) := e^{t \langle \xi \rangle - 1} H_p(x, \xi)\),

1. If \((x, \xi) \in \Sigma_\pm\), then either
   (a) \(\gamma(t) \to L_\pm\) as \(t \to \pm \infty\), or
   (b) there exists \(t_0 \geq 0\) such that \(\gamma(t) \in \{x_1 > \delta\}\) for all \(t, \pm t \geq t_0\).
2. If \((x, \xi) \in \Sigma_\pm \setminus L_\pm\), then there exists \(t_0 \geq 0\) such that either
   (a) \(\gamma(t)\) exits \(T^*X\) at time \(\mp t_0\), or
   (b) \(\gamma(t) \in \{x_1 > \delta\}\) for all \(t, \mp t \geq t_0\).

Proof. 1. We concentrate on the case \((x, \xi) \in \Sigma_+\); the case \((x, \xi) \in \Sigma_-\) can be handled similarly, reversing the direction of the flow.

By Lemma 5.15, there exists a neighborhood \(U_+\) of \(L_+\) such that
\[
e^{t \langle \xi \rangle - 1} H_p(x, \xi) \to L_+ \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad (x, \xi) \in U_+.
\]
By Lemma 5.16, there exists \(\delta > 0\) such that
\[
\langle \xi \rangle - 1 H_p x_1 \geq \delta \quad \text{on} \quad \Sigma_+ \cap \{x_1 \leq \delta\} \setminus U_+.
\]
Indeed, if (5.4.28) failed for each \(\delta\), we could find a sequence of counterexamples; the limit of any its convergent subsequence would give a contradiction to (5.4.25). Moreover, \(\delta\) can be chosen independent of \(\omega\), as follows from (5.3.14).

2. We first prove part 1. The trajectory \(\gamma(t)\) cannot exit \(T^*X\) for positive times due to (5.4.25). Next, if \(\gamma(t) \in U_+\) for some \(t \geq 0\), then case 1(a) holds. On the other hand, if \(\gamma(t) \in \Sigma_+ \setminus U_+\) for all \(t \geq 0\), then it follows from (5.4.28) that \(\gamma(t) \in \{x_1 > \delta\}\) for all \(t \geq 0\) large enough, thus case 1(b) holds.

3. We now prove part 2. Assume that case 2(a) does not hold; then \(\gamma(t) \in \Sigma_+\) is well-defined for all \(t \leq 0\). Since \((x, \xi) \notin L_+\), we may take a neighborhood \(V_+\) of \(L_+\) such that \((x, \xi) \notin V_+\). By (5.4.27), there exists \(T > 0\) such that
\[
e^{t \langle \xi \rangle - 1} H_p(U_+) \subset V_+ \quad \text{for all} \quad t \geq T.
\]
Since \((x, \xi) \notin V_+\), it follows that
\[
\gamma(t) \in \Sigma_+ \setminus U_+ \quad \text{for all} \quad t \leq -T.
\]
By (5.4.28), \(\gamma(t) \in \{ x_1 > \delta \}\) for all \(t \leq -T\), thus case 2(b) holds.

We finally study the relation of Hamiltonian trajectories of \(p\) inside \(M\) to geodesics on the original Riemannian manifold \((M, g)\), which will be used in §6.2.3. Define the smooth map
\[
(5.4.29) \quad j : T^*M \setminus 0 \to T^*M, \quad j(x, \xi) := (x, \xi + |\xi|_g \frac{dx_1}{2x_1}).
\]

Then (5.3.12) implies that for each \(\omega > 0\), the characteristic set of \(p\) intersected with \(T^*M\) is the image of a rescaled cosphere bundle under \(j\):
\[
(5.4.30) \quad \{ p = 0 \} \cap T^*M = j ( \{ (x, \xi) \in T^*M : |\xi|_g = \omega \}).
\]

On \(M\), geodesics are the trajectories of the Hamiltonian flow \(\exp(tH_{|\xi|^2}_g)\), and they give rise to trajectories of \(H_p\) on the characteristic set as follows:

**Lemma 5.20.** Fix \(\omega > 0\) and let \((x, \xi) \in T^*M, |\xi|_g = \omega\). Then we have
\[
(5.4.31) \quad j(\exp(tH_{|\xi|^2}_g)(x, \xi)) = \exp(s(t)\langle \xi \rangle - H_p(j(x, \xi)))
\]

for some smooth function \(s : \mathbb{R} \to \mathbb{R}\), depending on \((x, \xi)\) and such that \(s(0) = 0, s'(t) > 0\) for all \(t\). Moreover, \(\lim_{t \to -\infty} s(t) = -\infty\).

**Proof.** Put \((x(t), \xi(t)) = \exp(tH_{|\xi|^2}_g)(x, \xi)\). Then \(|\xi(t)|_g = \omega\) for all \(t\), thus
\[
\gamma(x(t), \xi(t)) = j_\omega(x(t), \xi(t)), \quad t \in \mathbb{R},
\]
where \(j_\omega : T^*M \to T^*M\) is the symplectomorphism defined by
\[
j_\omega(x, \xi) = (x, \xi + \omega \frac{dx_1}{2x_1}).
\]

By (5.3.12), we have on \(T^*M\)
\[
(x_1p) \circ j_\omega = |\xi|_g^2 - \omega^2.
\]

Since symplectomorphisms preserve Hamiltonian flows, we have
\[
x_1(t)H_p(j_\omega(x(t), \xi(t))) = dj_\omega(x(t), \xi(t))H_{|\xi|^2}_g(x(t), \xi(t))
\]
\[
= dj_\omega(x(t), \xi(t))
\]
where \(x_1(t) := x_1(x(t))\). Now (5.4.31) follows with
\[
s(t) := \int_0^t x_1(r)\langle \xi(r) + \omega \frac{dx_1}{2x_1}(r) \rangle \, dr.
\]

To show that \(s(t) \to -\infty\) as \(t \to -\infty\), we argue by contradiction, assuming that
\[
s_0 := \lim_{t \to -\infty} s(t) \in (-\infty, 0).
\]
5.5. PROPAGATION ESTIMATES

Put \((x', \xi') := \exp(s_0(\xi)^{-1}H_p)(j(x, \xi))\). Then \((x', \xi') \in \Sigma_+ \cap \{x_1 = 0\}\). Moreover, \(\exp(s(\xi)^{-1}H_p)(j(x, \xi)) \in \{x_1 > 0\}\) for \(s \in (s_0, 0]\), thus
\[
(x', \xi') \notin L_-, \quad \langle \xi \rangle^{-1}H_p x_1(x', \xi') \geq 0.
\]
We obtain a contradiction with Lemma 5.16.

5.5. PROPAGATION ESTIMATES

In this section, we prove estimates for the operator \(P_h(\omega)\) introduced in (5.3.8), as well as its formal adjoint \(P_h(\omega)^*\) defined using the density (5.3.23). For that we combine the properties of the semiclassical principal symbol \(p = \sigma_h(P_h(\omega))\) established in §5.4 with the propagation of singularities and radial points estimates of Appendix E.5. The resulting Propositions 5.27 and 5.28 will be the key components of the proof of meromorphic continuation of the scattering resolvent in §5.6. We will freely use the notation of Appendix E.

We first discuss the functional spaces used. Recall from §5.3 that the even extension \(\bar{X}\) is a compact manifold with boundary \(\partial \bar{X} = \{x_1 = -\varepsilon\}\) and interior \(X = \{x_1 > -\varepsilon\}\). Consider the semiclassical Sobolev spaces (see Definition E.21)
\[
\bar{H}_h^s(X), \quad \bar{H}_h^s(\bar{X}), \quad s \in \mathbb{R}.
\]
Recall that \(\bar{H}_h^s(X)\) and \(\bar{H}_h^{-s}(\bar{X})\) are dual to each other with respect to the standard \(L^2\) pairing. The norms of these spaces depend on \(h\), but the underlying Hilbert spaces \(\bar{H}^s(X), \bar{H}^s(\bar{X})\) are \(h\)-independent. We will later assume that \(s\) is large enough depending on \(\omega\) – see (5.5.9).

The operator \(P_h(\omega)\) is a second order differential operator and thus defines a map \(\bar{H}^s(X) \to \bar{H}^{s-2}(X)\). However, our propagation estimates bound the norm \(\|u\|_{\bar{H}_h^s(X)}\) in terms of \(\|P_h(\omega)u\|_{\bar{H}_h^{s-1}(X)}\), rather than the weaker norm \(\|P_h(\omega)\|_{\bar{H}_h^{s-2}(X)}\). For that reason, we will apply \(P_h(\omega)\) to functions \(u\) satisfying
\[
(5.5.1) \quad u \in \bar{H}^s(X), \quad P_h(\omega)u \in \bar{H}^{s-1}(X).
\]
Similarly, we will apply the adjoint operator \(P_h(\omega)^*\) to functions \(v\) satisfying
\[
(5.5.2) \quad v \in \bar{H}^{1-s}(\bar{X}), \quad P_h(\omega)^*v \in \bar{H}^{-s}(\bar{X}).
\]
Our estimates are uniform in \(\omega\) as long as
\[
(5.5.3) \quad |\text{Re} \omega| \leq C_0, \quad |\text{Im} \omega| \leq C_0 h,
\]
where \(C_0 > 0\) is any fixed constant. In terms of the original spectral parameter \(\lambda = h^{-1}\omega\), (5.5.3) corresponds to \(|\text{Re} \lambda| \leq C_0 h^{-1}, \quad |\text{Im} \lambda| \leq C_0\). In particular, by taking \(h\) small enough we can handle arbitrarily large values of \(\lambda\), as long as \(\text{Im} \lambda\) is bounded.
By (5.5.3), the operators $P_{\hbar}(\omega)$ and $P_{\hbar}(\text{Re}\omega)$ differ by an element of $h\Psi^1_{\hbar}$, thus they have the same principal symbols. We will use $\text{Re}\omega$ instead of $\omega$ in the definition of the principal symbol $p$, so that the results of §5.4 apply. The adjoint operator $P_{\hbar}(\omega)^*$ has principal symbol $p$ as well.

5.5.1. Microlocal estimates. We now prove microlocal propagation estimates for the operators $P_{\hbar}(\omega), P_{\hbar}(\omega)^*$. See Figure 5.6 below for a phase space illustration of these estimates. We use the following notation:

DEFINITION 5.21. Assume that $V \subset \mathcal{T}^\omega X$ is an open set, fix $\omega \in \mathbb{R}$, and let $p$ be given by (5.3.11). We say that a point $(x, \xi) \in \mathcal{T}^\omega X$ is controlled by $V$, and write $(x, \xi) \in \text{Con}_p(V)$, if either

(a) $(x, \xi) \notin \{\langle \xi \rangle^{-2}p = 0\}$, or

(b) there exists $t \in \mathbb{R}$ such that $e^{t\langle \xi \rangle^{-1}H_{\hbar}(x, \xi)} \in V$.

Note that $\text{Con}_p(V)$ is an open subset of $\mathcal{T}^\omega X$.

In this section, we will use pseudodifferential operators in $\Psi^0_{\hbar}(X)$ which are compactly supported; that is, their Schwartz kernels do not intersect the boundary of $\overline{X} \times X$. Our first estimate is a combination of elliptic bounds and propagation of singularities:

LEMMA 5.22. Assume that $A, B \in \Psi^0_{\hbar}(X)$ are compactly supported and (5.5.4) $\text{WF}_{\hbar}(A) \subset \text{Con}_p(\text{ell}_{\hbar}(B))$.

Then for $\omega$ satisfying (5.5.3), all $s, N$, and $u, v$ satisfying (5.5.1), (5.5.2),

$$
\|Au\|_{H^s_{\hbar}} \leq Ch^{-1}\|P_{\hbar}(\omega)u\|_{\dot{H}^{1-s}_{\hbar}(X)} + C\|Bu\|_{H^s_{\hbar}} + O(h^{\infty})\|u\|_{\dot{H}^{-N}_{\hbar}(X)},
$$

(5.5.5)

$$
\|Av\|_{\dot{H}^{1-s}_{\hbar}} \leq Ch^{-1}\|P_{\hbar}(\omega)^*v\|_{\dot{H}^{-s}_{\hbar}(\overline{X})} + C\|Bv\|_{\dot{H}^{1-s}_{\hbar}} + O(h^{\infty})\|v\|_{\dot{H}^{-N}_{\hbar}(\overline{X})}.
$$

(5.5.6)

Proof. 1. We will show (5.5.5); the estimate (5.5.6) is proved in exactly the same way. We first claim that for all $u \in C^\infty(X)$,

$$
\|Au\|_{H^s_{\hbar}} \leq Ch^{-1}\|\chi P_{\hbar}(\omega)u\|_{\dot{H}^{1-s}_{\hbar}(X)} + C\|Bu\|_{H^s_{\hbar}} + O(h^{\infty})\|\chi u\|_{\dot{H}^{-N}_{\hbar}},
$$

(5.5.7)

where $\chi \in C^\infty_0(X)$ is some cutoff function depending on $A, B$. For that we consider the following cases:

- $\text{WF}_{\hbar}(A) \cap \{(\xi)^{-2}p = 0\} = \emptyset$: (5.5.7) follows by the semiclassical elliptic estimate, Theorem E.32;
• for all \((x, \xi) \in \text{WF}_h(A)\), there is \(t \leq 0\) with \(e^{t\langle \xi \rangle^{-1}H_p(x, \xi)} \in \text{ell}_h(B)\): (5.5.7) follows by semiclassical propagation of singularities, Theorem E.49;

• for all \((x, \xi) \in \text{WF}_h(A)\), there is \(t \geq 0\) with \(e^{t\langle \xi \rangle^{-1}H_p(x, \xi)} \in \text{ell}_h(B)\): (5.5.7) follows by Theorem E.49 applied to \(-P_h(\omega)\);

• a general \(A\) satisfying (5.5.4) can be written as a sum of operators falling into the above three cases, by a pseudodifferential partition of unity.

2. We next claim that (5.5.7) holds for each \(u\) satisfying (5.5.1). Indeed, choose \(\chi\) above so that \(A = A\chi, B = B\chi\). By Lemma E.47, there exists a sequence \(u_j \in C^\infty(X), \|\chi(u_j - u)\|_{\text{H}^s_h} \to 0, \|\chi(P_h(\omega)u_j - P_h(\omega)u)\|_{\text{H}^{s-1}_h} \to 0\).

Applying (5.5.7) to \(u_j\), we obtain in the limit (5.5.7) for \(u\). Finally,

\[\|\chi P_h(\omega)u\|_{\text{H}^{s-1}_h} \leq C\|P_h(\omega)u\|_{\tilde{\text{H}}^{s-1}_h(X)}, \quad \|\chi u\|_{\text{H}^{-N}_h} \leq C\|u\|_{\tilde{\text{H}}^{-N}_h(X)}\]

Therefore, (5.5.7) implies (5.5.5). \(\square\)

The next statement uses radial estimates to bound \(u\) microlocally near the sets \(L_{\pm}\), assuming \(s\) is large enough, and then propagates this bound to a neighborhood of the fiber infinity \(\partial T^* X\):

**Lemma 5.23.** Let \(A \in \Psi^0_h(X)\) be compactly supported. Then there exists \(A_0 \in \Psi^0_h(X)\) compactly supported such that

\[\text{WF}_h(A_0) \cap \partial T^* X = \emptyset\]

and for \(\omega\) satisfying (5.5.3), \(s\) satisfying

\[s > \frac{1}{2} - \frac{\text{Im} \, \omega}{h},\]

all \(N\), and \(u\) satisfying (5.5.1),

\[\|Au\|_{\text{H}^s_h} \leq Ch^{-1}\|P_h(\omega)u\|_{\tilde{\text{H}}^{s-1}_h(X)} + C\|A_0u\|_{\text{H}^s_h} + O(h^\infty)\|u\|_{\tilde{\text{H}}^{-N}_h(X)}\]

Proof. 1. Recall the radial sets \(L_{\pm}\) introduced in (5.4.20); by Lemma 5.15, \(L_+\) is a radial sink and \(L_-\) is a radial source for the flow \(\exp(t\langle \xi \rangle^{-1}H_p)\). Let
\( \rho \) be defined in (5.4.2). Using (5.3.25), (5.4.3), and (5.4.21), we compute
\[
\pm (\xi)^{-1} \left( \sigma_h(h^{-1} \text{Im} P_h(\omega)) + \left( s - \frac{1}{2} \right) \frac{H_p(\xi)}{\xi} \right) \bigg|_{L^\pm}
\]
\[
= \pm \rho \left( \sigma_h(h^{-1} \text{Im} P_h(\omega)) + \left( \frac{1}{2} - s \right) \rho^{-1} H_p \rho \right) \bigg|_{L^\pm}
\]
\[
= 4 \left( \frac{\Im \omega}{\rho} + s - \frac{1}{2} \right) > 0
\]
by (5.5.9); here we used that \( \langle \xi \rangle = \rho^{-1} \sqrt{1 + \rho^2} \).

We apply the high regularity radial estimate, Theorem E.54, to \( L_\pm \) and the operator \( \mp P_h(\omega) \). It follows that there exist compactly supported \( A_\pm \in \Psi^0_h(X) \) such that \( L_\pm \subset \ell h(A_\pm) \) and
\[
\| A_\pm u \|_{\ell h^{-1}} \leq C h^{-1} \| P_h(\omega) u \|_{\ell h^{-1}(X)} + O(h^\infty) \| u \|_{\ell h^{-1}(X)},
\]
where we used an argument similar to step 2 of the proof of Lemma 5.22.

2. By (5.4.9) and part 1 of Lemma 5.18,
\[
\partial T^* X \subset \text{Con}_p \left( \ell h(A_+) \cup \ell h(A_-) \right).
\]
Take compactly supported \( A_0 \in \Psi^0_h(X) \) satisfying (5.5.8) and elliptic on the compact set \( \text{WF}_h(A) \setminus \text{Con}_p(\ell h(A_+) \cup \ell h(A_-)) \subset T^* X \). Then
\[
\text{WF}_h(A) \subset \text{Con}_p \left( \ell h(A_0) \cup \ell h(A_+) \cup \ell h(A_-) \right).
\]
Applying Lemma 5.22 with \( B := A_0^* A_0 + A_+^* A_+ + A_-^* A_- \), we get
\[
\| Au \|_{\ell h^1} \leq C h^{-1} \| P_h(\omega) u \|_{\ell h^{-1}(X)} + C \| A_0 u \|_{\ell h^1} + C \| A_+ u \|_{\ell h^1} + C \| A_- u \|_{\ell h^1} + O(h^\infty) \| u \|_{\ell h^{-1}(X)}.
\]
Estimating \( \| A_\pm u \|_{\ell h^1} \) by (5.5.12), we obtain (5.5.10). \( \square \)

For the adjoint operator \( P_h(\omega)^* \), we cannot obtain an a priori bound microlocally near the radial sets of the form (5.5.12), since Theorem E.54 does not hold in the low regularity space \( H^{1-s} \). We will instead put on the right-hand side the norm of \( u \) near the boundary \( \partial X \). Fix
\[
\chi \in C^\infty_0(X), \quad \chi = 1 \quad \text{near } \overline{M_{\text{even}}} = \{ x_1 \geq 0 \}.
\]
Then we have

**Lemma 5.24.** Assume that \( A \in \Psi^0_h(X) \) is compactly supported. Then there exists \( A'_0 \in \Psi^0_h(X) \) compactly supported such that (5.5.8) holds and for \( \omega \) satisfying (5.5.3), \( s \) satisfying (5.5.9), all \( N \), and \( v \) satisfying (5.5.2),
\[
\| Av \|_{\ell h^{-1-s}} \leq C h^{-1} \| P_h(\omega)^* v \|_{\ell h^{-1}(\overline{X})} + C \| A'_0 v \|_{\ell h^{-1-s}}
\]
\[
+ C \| (1 - \chi_1) v \|_{\ell h^{-1-s}(\overline{X})} + O(h^\infty) \| v \|_{\ell h^{-1}(\overline{X})}.
\]
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Proof. 1. We calculate similarly to (5.5.11), under the condition (5.5.9)

\[ \mp \langle \xi \rangle^{-1} \left( \sigma_h (h^{-1} \text{Im } P_h^* (\omega)) + \left( \frac{1}{2} - s \right) \frac{H_p (\xi)}{\langle \xi \rangle} \right) \bigg|_{L^\pm} \]

\[ = 4 \left( \text{Im } \frac{\omega}{h} + s - \frac{1}{2} \right) > 0 \]

We apply the low regularity radial estimate, Theorem E.56, to \( L^\pm \) and the operator \( \pm P_h (\omega)^* \). It follows that there exist compactly supported \( A'_\pm, B_\pm \in \Psi^0_h (X) \) such that

\[ L^\pm \subset \text{ell}_h (A'_\pm), \quad \text{WF}_h (B_\pm) \cap (L_+ \sqcup L_-) = \emptyset, \]

and the following estimate holds:

\[ \| A'_\pm v \|_{H^{1-s}_h} \leq C h^{-1} \| P_h (\omega)^* v \|_{H^{1-s}_h (X)} + C \| B_\pm v \|_{H^{1-s}_h} + \mathcal{O} (h^\infty) \| v \|_{H^{1-s}_h (X)}, \tag{5.5.16} \]

where we used an argument similar to step 2 of the proof of Lemma 5.22.

2. By (5.4.9) and part 2 of Lemma 5.18,

\[ \partial T^\infty X \setminus (L_+ \sqcup L_-) \subset \text{Con}_p (\{1 - \chi_1 \neq 0\}). \]

Therefore,

\[ \partial T^\infty X \subset \text{Con}_p (\text{ell}_h (A'_+) \cup \text{ell}_h (A'_-) \cup \{1 - \chi_1 \neq 0\}). \]

Arguing as in the proof of Lemma 5.23, we construct \( A'_0 \) such that (5.5.8) holds and

\[ \text{WF}_h (A) \subset \text{Con}_p (\text{ell}_h (A'_0) \cup \text{ell}_h (A'_+) \cup \text{ell}_h (A'_-) \cup \{1 - \chi_1 \neq 0\}), \]

\[ \text{WF}_h (B_\pm) \subset \text{Con}_p (\text{ell}_h (A'_0) \cup \{1 - \chi_1 \neq 0\}). \]

By Lemma 5.22, this implies the estimates

\[ \| Av \|_{H^{1-s}_h} \leq C h^{-1} \| P_h (\omega)^* v \|_{H^{1-s}_h (X)} + C \| A'_0 v \|_{H^{1-s}_h} + C \| A'_+ v \|_{H^{1-s}_h} + C \| A'_- v \|_{H^{1-s}_h} + C \| (1 - \chi_1) v \|_{H^{1-s}_h (X)} + \mathcal{O} (h^\infty) \| v \|_{H^{1-s}_h (X)}, \tag{5.5.17} \]

\[ \| B_\pm v \|_{H^{1-s}_h} \leq C h^{-1} \| P_h (\omega)^* v \|_{H^{1-s}_h (X)} + C \| A'_0 v \|_{H^{1-s}_h} + C \| (1 - \chi_1) v \|_{H^{1-s}_h (X)} + \mathcal{O} (h^\infty) \| v \|_{H^{1-s}_h (X)}. \tag{5.5.18} \]

Substituting (5.5.16) into (5.5.17) and combining the result with (5.5.18), we obtain (5.5.15). \( \square \)
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5.5.2. Hyperbolic estimates and global regularity. The microlocal estimates proved above are only valid away from the boundary \( \partial \mathcal{X} \). To estimate the functions \( u, v \) near the boundary, we use semiclassical hyperbolicity of \( P_h(\omega) \) in \( \{ x_1 < 0 \} \) when \( \omega \) is away from 0:

**Lemma 5.25.** Let \( \chi_1 \) satisfy (5.5.14) and assume that \( \omega \) satisfies the following strengthening of (5.5.3) for some fixed \( C_0 \):

\[
C_0^{-1} \leq |\text{Re} \omega| \leq C_0, \quad |\text{Im} \omega| \leq C_0 h.
\]

Then for all \( s \) and \( u, v \) satisfying (5.5.1), (5.5.2),

\[
\|u\|_{H^s_h(\mathcal{X})} \leq Ch^{-1}\|P_h(\omega)u\|_{\dot{H}^{-s-1}_h(\mathcal{X})} + C\|\chi_1 u\|_{H^s_h},
\]

\[
\|(1 - \chi_1)v\|_{\dot{H}^{-s}_h(\mathcal{X})} \leq Ch^{-1}\|P_h(\omega)^*v\|_{\dot{H}^{-s}_h(\mathcal{X})}.
\]

**Proof.** Consider the defining function \( t := 1 + x_1/\varepsilon \) on \( \overline{\mathcal{X}} \) and the product structure \((t, x')\) on \( \{ t < 1 \} = \{ x_1 < 0 \} \). By (5.3.13), the operators \( P_h(\omega), P_h(\omega)^* \) are semiclassically hyperbolic with respect to \( t \) on \( \{ t < 1 \} \), in the sense of Definition E.57. Then (5.5.20), (5.5.21) follow immediately from Theorem E.59. \( \square \)
When \( \omega = 0 \), the operator \( P_h(\omega) \) is not semiclassically hyperbolic (taking \( \xi' = 0 \)) and the constants in (5.5.20), (5.5.21) may depend on \( h \):

**Lemma 5.26.** Let \( \chi_1 \) satisfy (5.5.14) and assume that \( \omega \) satisfies (5.5.3). Then for all \( s \) and \( u, v \) satisfying (5.5.1), (5.5.2),

\[
\| u \|_{\dot{H}^s(X)} \leq C \| P_h(\omega)u \|_{\dot{H}^{s-1}(X)} + C \| \chi_1 u \|_{\dot{H}^s},
\]

(5.5.22)

\[
\|(1 - \chi_1)v\|_{\dot{H}^{1-s}(X)} \leq C \| P_h(\omega)^* v \|_{\dot{H}^{s-1}(X)},
\]

(5.5.23)

with the constants in (5.5.22), (5.5.23) depending on \( h \).

**Proof.** Consider the product structure \((t, x')\) as in the proof of Lemma 5.25. By (5.3.13), the operators \( P_h(\omega), P_h(\omega)^* \) are hyperbolic with respect to \( t \) on \( \{ t < 1 \} \) in the sense of Definition E.57. Then (5.5.22), (5.5.23) follow immediately from Theorem E.58. \( \square \)

Combining Lemmas 5.23, 5.24, and 5.26, we arrive to the following statement, which is used in §5.6 below to prove the Fredholm property of \( P_h(\omega) \):

**Proposition 5.27.** For \( \omega \) satisfying (5.5.3), \( s \) satisfying (5.5.9), all \( N \), and \( u, v \) satisfying (5.5.1), (5.5.2), we have the estimates

\[
\| u \|_{\dot{H}^s(X)} \leq C \| P_h(\omega)u \|_{\dot{H}^{s-1}(X)} + C \| \chi_1 u \|_{\dot{H}^s},
\]

(5.5.24)

\[
\| v \|_{\dot{H}^{1-s}(X)} \leq C \| P_h(\omega)^* v \|_{\dot{H}^{s-1}(X)} + C \| v \|_{\dot{H}^s},
\]

(5.5.25)

with the constants in (5.5.24), (5.5.25) depending on \( h \).

**Proof.** Let \( \chi_1 \) satisfy (5.5.14). To see (5.5.24), it suffices to substitute (5.5.10) with \( A := \chi_1 \) into (5.5.22). Here \( \| A_0 u \|_{\dot{H}^s} \leq C \| u \|_{\dot{H}^{-N}(X)} \) since \( A_0 \) is compactly supported on \( X \) and \( \text{WF}_h(A_0) \subset T^*M \).

To see (5.5.25), we first substitute (5.5.23) into (5.5.15) with \( A := \chi_1 \), to estimate \( \| \chi_1 v \|_{\dot{H}^{1-s}_h(X)} \). Combining the result with (5.5.23), we obtain an estimate on \( \| v \|_{\dot{H}^{1-s}_h(X)} \). \( \square \)

### 5.5.3. Invertibility in the upper half-plane.

The Fredholm property of \( P_h(\omega) \), following from Proposition 5.27, is not enough to conclude that \( P_h(\omega)^{-1} \) is meromorphic. Indeed, it could happen that \( P_h(\omega) \) is not invertible for any \( \omega \), for instance if it were a family of Fredholm operators of nonzero index. Here we show that this is not the case, by proving that \( P_h(\omega) \) is invertible for some \( \omega \) in the upper half-plane:

**Proposition 5.28.** Fix \( s \in \mathbb{R} \). Then there exists \( \beta > 0 \) such that \( s > \frac{1}{2} - \beta \) and for

\[
\omega := 1 + ih\beta,
\]
small enough $h$, and $u,v$ satisfying (5.5.1), (5.5.2), we have the estimates

\begin{align}
\|u\|_{H^s_h(X)} &\leq Ch^{-1}\|P_h(\omega)u\|_{H^{s-1}_h(X)}, \\
\|v\|_{H^{s-1}_h(X)} &\leq Ch^{-1}\|P_h(\omega)^*v\|_{H^{s-1}_h(X)}.
\end{align}

\textbf{Proof.} 1. Fix $\chi_1$ satisfying (5.5.14). We first claim that it is enough to prove the following estimates for some $\chi \in C_0^\infty(X)$ and all $u,v \in C_0^\infty(X)$:

\begin{align}
\|\chi_1 u\|_{H^s_h(X)} &\leq Ch^{-1}\|\chi P_h(\omega)u\|_{H^{s-1}_h(X)} + Ch^{1/2}\|\chi u\|_{H^{s-1/2}_h(X)}, \\
\|\chi_1 v\|_{H^{s-1}_h(X)} &\leq Ch^{-1}\|\chi P_h(\omega)^*v\|_{H^{s-1}_h(X)} + C\|\chi(1 - \chi_1)v\|_{H^{1-s}_h(X)}.
\end{align}

Indeed, using Lemma E.47 as in the proof of Lemma 5.22, we see that (5.5.28), (5.5.29) hold for all $u,v$ satisfying (5.5.1), (5.5.2). Combining these estimates with (5.5.20), (5.5.21) similarly to the proof of Proposition 5.27, and taking $h$ small enough to remove the $O(h^{1/2})$ remainder, we obtain (5.5.26), (5.5.27).

2. To show (5.5.28), (5.5.29), we use Lemma E.51, which is a basic positive commutator estimate. Let $\Sigma_{\pm}$ be defined in Lemma 5.14, where we put $\omega = 1$. Take cutoff functions

$$
\psi_{\pm} \in C_0^\infty(\mathbb{T}^s X \setminus \Sigma_{\pm}; [0,1])
$$

with the following properties:

$$
\psi_{\pm} = 1 \quad \text{near } \Sigma_{\pm} \cap \text{supp } \chi_1, \quad \pm \langle \xi \rangle^{-1} H_p \psi_{\pm} \geq 0 \quad \text{near } \Sigma_{\pm}.
$$

The existence of such functions follows from (5.4.25), where we make $\psi_{\pm}$ be increasing functions of $x_1$ near $\Sigma_{\pm}$. With $\beta > 0$ to be chosen later, we put

$$
f_{\pm} := e^{\beta \varphi} \psi_{\pm}, \quad \tilde{f}_{\pm} := e^{-\beta \varphi} \psi_{\pm},
$$

where $\varphi \in C^\infty(\mathbb{R})$ is the function introduced in (5.4.4).

3. Using (5.3.25) and (5.4.10), we compute

$$
\pm \langle \xi \rangle^{-1}(H_p f_{\pm} + \sigma_h (h^{-1} \text{Im } P_h(\omega)) f_{\pm})
\begin{align}
&= \pm e^{\beta \varphi} \langle \xi \rangle^{-1}(H_p \psi_{\pm} + \beta (H_p \varphi + \partial_\omega p)\psi_{\pm}) \\
&\leq -2\delta \beta f_{\pm} \quad \text{near } \Sigma_{\pm}
\end{align}
$$

for some $\delta > 0$ independent of $\beta$.

Fix $\beta$ large enough depending on $s$ such that near $\Sigma_{\pm}$,

$$
\pm \langle \xi \rangle^{-1}(H_p f_{\pm} + \sigma_h (h^{-1} \text{Im } P_h(\omega)) f_{\pm} + \left(s - \frac{1}{2}\right) H_p(\xi) f_{\pm}) \leq -\delta f_{\pm}.
$$
Using (5.4.8), take compactly supported \( \hat{A}_\pm, \hat{B} \in \Psi_h^0(X) \) such that
\[
\text{supp } \chi_1 \subset \text{ell}_h(\hat{A}_+) \cup \text{ell}_h(\hat{A}_-) \cup \text{ell}_h(\hat{B}),
\]
(5.5.32)
\[
\text{WF}_h(\hat{A}_\pm) \subset \{ f_\pm > 0 \}, \quad \text{WF}_h(\hat{B}) \cap \{ |\xi|^{-2} p = 0 \} = \emptyset,
\]
and (5.5.31) holds in a neighborhood of \( \bar{T}^* X \setminus \text{ell}_h(\hat{B}) \).

By Lemma E.51 applied to \( \bar{\chi}_\pm \) and (5.5.33), we compute
\[
\| \hat{A}_\pm u \|_{H_h^s} \leq C h^{-1} \| \chi P_h(\omega) u \|_{H_h^{s-1}}
\]
(5.5.33)
\[+ C \| \hat{B} u \|_{H_h^s} + C h^{1/2} \| \chi u \|_{H_h^{s-1/2}}.
\]
By the elliptic estimate, Theorem E.32, \( \| \hat{B} u \|_{H_h^s} \) is bounded in terms of
\( \| \chi P_h(\omega) u \|_{H_h^{s-2}} \). By a pseudodifferential partition of unity and Theorem E.32,
\( \| \chi_1 u \|_{H_h^s} \) is bounded in terms of \( \| \hat{A}_\pm u \|_{H_h^s} \) and \( \| \hat{B} u \|_{H_h^s} \). Combining these estimates with (5.5.33), we obtain (5.5.28).

4. Similarly to (5.5.30), we compute
\[
\pm (\xi)^{-1} \left( H_p \bar{f}_\pm + \sigma_h(h^{-1} \text{Im } P_h(\omega)^*) \bar{f}_\pm \right)
\]
(5.5.34)
\[= \pm e^{-\beta \varphi} (\xi)^{-1} \left( H_p \psi_\pm - \beta (H_p \varphi + \partial_x p) \psi_\pm \right)
\]
\[\leq -2 \delta \beta \bar{f}_\pm \text{ near } \Sigma_\pm \cap \text{supp } \chi_1,
\]
where we used that \( H_p \psi_\pm = 0 \) near \( \Sigma_\pm \cap \text{supp } \chi_1 \). Arguing as in step 3, we see that for \( \beta > 0 \) large enough, there exist \( \hat{A}_\pm, \hat{B} \) satisfying (5.5.32) and such that near \( \bar{T}^* X \setminus (\{ \chi_1 \neq 1 \} \cup \text{ell}_h(\hat{B})) \),
\[
\pm (\xi)^{-1} \left( H_p \bar{f}_\pm + \sigma_h(h^{-1} \text{Im } P_h^*(\omega)) \bar{f}_\pm + \left( \frac{1}{2} - s \right) \frac{H_p(\xi)}{\langle \xi \rangle} \bar{f}_\pm \right) \leq -\delta \bar{f}_\pm.
\]
By Lemma E.51 applied to \( \bar{f}_\pm \), the operators \( \pm P_h(\omega)^* \), and \( B := \hat{B}^* \hat{B} + \chi(1 - \chi_1) \) with a correct choice of \( \chi \), we obtain
\[
\| \hat{A}_\pm v \|_{H_h^{1-s}} \leq Ch^{-1} \| \chi P_h(\omega)^* v \|_{H_h^{-s}} + C \| \chi(1 - \chi_1) v \|_{H_h^{1-s}}
\]
\[+ C \| \hat{B} v \|_{H_h^{1-s}} + Ch^{1/2} \| \chi v \|_{H_h^{1/2-s}}.
\]
Arguing as in step 3, we see that this estimate implies (5.5.29). \( \square \)

5.6. MEROMORPHIC CONTINUATION

In this section, we use the estimates proved in §5.5 to show that the family of operators \( P(\lambda) \) (see Definition 5.11) has a meromorphic inverse \( P(\lambda)^{-1} \). We next use \( P(\lambda)^{-1} \) to show meromorphic continuation of the scattering resolvent (5.0.3).
The main results of §5.5, Propositions 5.27 and 5.28, estimate \( \|u\| \tilde{H}_h^s(X) \) in terms of \( \|P_h(\omega)u\| \tilde{H}^{s-1}_h(X) \). However, \( P(\lambda) \), as a second order differential operator, is not bounded \( \tilde{H}^s(X) \to \tilde{H}^{s-1}(X) \). To resolve this issue, we consider the domain of \( P(\lambda) \),

\[
\mathcal{X}^s = \{ u \in \tilde{H}^s(X) \mid P(0)u \in \tilde{H}^{s-1}(X) \},
\]

endowed with the norm

\[
\|u\|_{\mathcal{X}^s} = (\|u\|_{\tilde{H}_h^s(X)}^2 + \|P(0)u\|_{\tilde{H}^{s-1}_h(X)}^2)^{1/2}.
\]

It is easy to see that \( \mathcal{X}^s \) is a Hilbert space, by identifying it with the closed subspace

\[
\{ (u, u_1) \mid P(0)u = u_1 \} \subset \tilde{H}_h^s(X) \oplus \tilde{H}^{s-1}_h(X).
\]

Moreover, the norms \( \| \cdot \|_{\mathcal{X}^s} \) for different \( h \) are equivalent, with constants depending on \( h \). Since \( P(\lambda) - P(0) \) is a first order differential operator, it is bounded \( \tilde{H}^s(X) \to \tilde{H}^{s-1}(X) \); therefore

\[
P(\lambda) : \mathcal{X}^s \to \tilde{H}^{s-1}(X)
\]

is a family of bounded operators holomorphic in \( \lambda \).

Before we prove meromorphy of \( P(\lambda)^{-1} \), let us introduce notation for the kernel, cokernel, and kernel of the adjoint of (5.6.2):

\[
(5.6.3) \quad \ker^s P(\lambda) = \{ u \in \tilde{H}^s(X) \mid P(\lambda)u = 0 \},
\]

(5.6.4) \( \operatorname{coker}^s P(\lambda) = \{ v \in \tilde{H}^{1-s}(X) \mid \langle P(\lambda)u, v \rangle_{L^2} = 0 \text{ for all } u \in \mathcal{X}^s \}, \)

(5.6.5) \( \ker^{1-s} P(\lambda)^* = \{ v \in \tilde{H}^{1-s}(X) \mid P(\lambda)^*v = 0 \}. \)

Here we recall that \( \tilde{H}^{1-s}(X) \) is the dual to \( \tilde{H}^{s-1}(X) \) with respect to the \( L^2 \) inner product.

**Lemma 5.29.** For all \( s, \lambda \), we have \( \operatorname{coker}^s P(\lambda) = \ker^{1-s} P(\lambda)^* \).

**Proof.** 1. By (5.5.23), we have

\[
(5.6.6) \quad v \in \ker^{1-s} P(\lambda)^* \implies \text{supp } v \subset \mathcal{M}_{\text{even}}.
\]

In particular, \( v \) is compactly supported inside \( X \). Therefore, by Lemma E.48, for all \( u \in \mathcal{X}^s \) and \( v \in \ker^{1-s} P(\lambda)^* \),

\[
\langle P(\lambda)u, v \rangle_{L^2} = \langle u, P(\lambda)^*v \rangle_{L^2} = 0.
\]

This implies that \( \ker^{1-s} P(\lambda)^* \subset \operatorname{coker}^s P(\lambda) \).

2. Assume now that \( v \in \operatorname{coker}^s P(\lambda) \). Then we have

\[
\langle u, P(\lambda)^*v \rangle_{L^2} = \langle P(\lambda)u, v \rangle_{L^2} = 0, \quad u \in C^\infty(\overline{X}).
\]

This implies that \( P(\lambda)^*v = 0 \) and thus \( \operatorname{coker}^s P(\lambda) \subset \ker^{1-s} P(\lambda)^* \). \( \square \)
THEOREM 5.30. Fix $s \in \mathbb{R}$. Then (5.6.2) is a Fredholm operator of index zero for $\text{Im} \lambda > \frac{1}{2} - s$, and it has a meromorphic inverse with poles of finite rank,

$$P(\lambda)^{-1} : \tilde{H}^{s-1}(X) \to \mathcal{X}^s.$$  

Proof. 1. By Proposition 5.27, recalling the definition (5.3.8) of the semiclassically rescaled operator $P_h(\omega)$, we have for $\text{Im} \lambda > \frac{1}{2} - s$ and all $u \in \mathcal{X}^s, v \in \tilde{H}^{1-s}(X), P(0)v \in \tilde{H}^{-s}(X),$

$$\|u\|_{\mathcal{X}^s} \leq C\|P(\lambda)u\|_{\tilde{H}^{s-1}(X)} + C\|u\|_{\tilde{H}^{s-1}(X)},$$  

$$\|v\|_{\tilde{H}^{1-s}(X)} \leq C\|P(\lambda)^{*}v\|_{\tilde{H}^{-s}(X)} + C\|v\|_{\tilde{H}^{-s}(X)},$$

where the constant $C$ depends on $s$ and $\lambda$. By Rellich’s theorem the embeddings $\mathcal{X}^s \to \tilde{H}^{s-1}(X), \tilde{H}^{1-s}(X) \to \tilde{H}^{-s}(X)$ are compact operators. We now follow a standard argument from functional analysis to establish the Fredholm property of $P(\lambda)$.

2. We first show the following statement: if $u_j$ is a bounded sequence in $\mathcal{X}^s$ such that $P(\lambda)u_j$ converges in $\tilde{H}^{s-1}(X)$, then $u_j$ has a subsequence which converges in $\mathcal{X}^s$. Indeed, since the embedding $\mathcal{X}^s \to \tilde{H}^{s-1}(X)$ is compact and $\|u_j\|_{\mathcal{X}^s}$ is bounded, by passing to a subsequence we may assume that $u_j$ converges in $\tilde{H}^{s-1}(X)$. Applying (5.6.8) to the differences $u_j - u_k$, we see that $u_j$ is a Cauchy sequence in $\mathcal{X}^s$; therefore, it converges.

It follows immediately that ker$^s P(\lambda)$ is finite dimensional. Indeed, otherwise there exists an $\mathcal{X}^s$-orthonormal sequence $u_j \in \ker^s P(\lambda); P(\lambda)u_j = 0$ converges and $\|u_j\|_{\mathcal{X}^s}$ is bounded, yet $u_j$ has no convergent subsequence.

3. We next show that the image of (5.6.2) is a closed subspace of $\tilde{H}^{s-1}(X)$. Take a convergent sequence

$$f_j \in P(\lambda)(\mathcal{X}^s), \quad f_j \to f_\infty \quad \text{in} \quad \tilde{H}^{s-1}(X).$$

We may write $f_j = P(\lambda)u_j$, where $u_j \in \mathcal{X}^s$ is $\mathcal{X}^s$-orthogonal to ker$^s P(\lambda)$.

Assume first that $\|u_j\|_{\mathcal{X}^s}$ is bounded. Then by step 2, by passing to a subsequence we can make $u_j$ converge to some $u_\infty$ in $\mathcal{X}^s$. It follows that $f_\infty = P(\lambda)u_\infty$ lies in the image of (5.6.2).

If on the contrary $\|u_j\|_{\mathcal{X}^s}$ is not bounded, then by passing to a subsequence we may assume that $\|u_j\|_{\mathcal{X}^s} \to \infty$. Put

$$\tilde{u}_j := \frac{u_j}{\|u_j\|_{\mathcal{X}^s}}, \quad \tilde{f}_j = P(\lambda)\tilde{u}_j = \frac{f_j}{\|u_j\|_{\mathcal{X}^s}}.$$  

Then $\|\tilde{u}_j\|_{\mathcal{X}^s} = 1$ and $\tilde{f}_j \to 0$ in $\tilde{H}^{s-1}(X)$. By step 2, passing to a subsequence, we may assume that $\tilde{u}_j$ converges to some $\tilde{u}_\infty$ in $\mathcal{X}^s$. We have $\|\tilde{u}_\infty\|_{\mathcal{X}^s} = 1$, $P(\lambda)\tilde{u}_\infty = 0$, and $\tilde{u}_\infty$ is $\mathcal{X}^s$-orthogonal to ker$^s P(\lambda)$; this gives a contradiction.
4. To finish the proof of the Fredholm property of (5.6.2), it remains to show that its image has finite codimension. Since this image is closed, by Lemma 5.29 it suffices to show that \( \ker^{1-s} P(\lambda)^* \) is finite dimensional. To do this, we may argue as in step 2, using (5.6.9) instead of (5.6.8).

5. To show that (5.6.2) has a meromorphic inverse, we apply Analytic Fredholm Theory, Theorem C.5. For that, we need to show that (5.6.2) is invertible for some choice of \( \lambda \), \( \im \lambda > \frac{1}{2} - s \). This statement together with continuity of index of Fredholm operators will also imply that \( P(\lambda) \) has index zero.

Take \( \lambda := h^{-1} + i\beta \), where \( h \) is small enough and \( \beta > 0 \) is fixed in Proposition 5.28. Then by (5.5.26), \( \ker^{s} P(\lambda) \) is trivial; by (5.5.27), \( \coker^{s} P(\lambda) \) is trivial. Together with the Fredholm property, these imply that (5.6.2) is invertible as needed. \( \square \)

The next proposition shows that for each \( \lambda \), the inverse \( P(\lambda)^{-1} \) defined in (5.6.7) does not depend on the choice of \( s \). Moreover, it maps \( \bar{H}^{s-1}(X) \to \bar{H}^{s}(X) \) for all \( s > \frac{1}{2} - \im \lambda \) and thus also defines an operator

\[
P(\lambda)^{-1} : C^{\infty}(\overline{X}) \to C^{\infty}(\overline{X}).
\]

**Proposition 5.31.** Assume that \( s < t \), and let \( P^{(s)}(\lambda)^{-1}, P^{(t)}(\lambda)^{-1} \) be the inverses of \( P(\lambda) \) as an operator \( X^{s} \to \bar{H}^{s-1}(X) \) and \( X^{t} \to \bar{H}^{t-1}(X) \) respectively. Then we have for \( \im \lambda > \frac{1}{2} - s \),

\[
P^{(s)}(\lambda)^{-1} f = P^{(t)}(\lambda)^{-1} f, \quad f \in \bar{H}^{t-1}(X).
\]

**Proof.** By analytic continuation, it suffices to prove (5.6.11) when \( \lambda \) is not a pole of either \( P^{(s)}(\lambda)^{-1} \) or \( P^{(t)}(\lambda)^{-1} \). Then

\[
u := P^{(t)}(\lambda)^{-1} f \in X^{t} \subset X^{s}.
\]

Since \( P(\lambda)u = f \), we see that \( u = P^{(s)}(\lambda)^{-1} f \) as needed. \( \square \)

We are now ready to establish the meromorphic continuation of the resolvent. Following (5.0.3), define for

\[
\im \lambda > 0, \quad \lambda \notin i\left[0, \frac{n-1}{2}\right],
\]

the holomorphic \( L^{2} \) resolvent

\[
\left(-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}\right)^{-1} : L^{2}(M) \to L^{2}(M).
\]

**Theorem 5.32.** The family (5.6.13) admits a meromorphic continuation

\[
R(\lambda) : L^{2}_{\text{comp}}(M) \to H^{2}_{\text{loc}}(M), \quad \lambda \in \mathbb{C}.
\]
Moreover, for each $s > \frac{1}{2} - \text{Im} \lambda$, $R(\lambda)$ can be extended to

\begin{equation}
R(\lambda) : x_1^{\frac{n+3}{4} - \frac{i\lambda}{2}} H^{s-1}(M_{\text{even}}) \rightarrow x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} H^s(M_{\text{even}}),
\end{equation}

where $x_1 = y_1^2$ is a boundary defining function of the even compactification $M_{\text{even}}$ introduced in Definition 5.7. In particular,

\begin{equation}
R(\lambda) f \in x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} C^\infty(M_{\text{even}}), \quad f \in C_0^\infty(M).
\end{equation}

**REMARKS.**

1. It follows from Theorem 5.32 that $-\Delta_g$ can only have discrete spectrum in $[0, \frac{(n-1)^2}{4})$.

2. We define resonances of $(M, g)$ as the poles of (5.6.14). The operator $P(\lambda)^{-1}$ from (5.6.10) can have a larger set of poles, see Exercises 5.8, 5.9.

**Proof.**

1. We first consider the case of the right-hand side $f \in C_0^\infty(M)$. Using the operator (5.6.10), define

\begin{equation}
R(\lambda) f := \left( x_1^{\frac{n-1}{4} - \frac{i\lambda}{2}} P(\lambda)^{-1} x_1^{\frac{i\lambda}{2}} - \frac{n+3}{4} f \right)_{|M}.
\end{equation}

Here we make $x_1^{\frac{i\lambda}{2}} - \frac{n+3}{4} f$ into an element of $C_0^\infty(X)$ by extending it by zero outside of $M$, and $R(\lambda)f$ satisfies (5.6.16).

It follows from Definition 5.11 that $R(\lambda)f$ solves the equation

\[ \left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) R(\lambda)f = f. \]

Moreover, it follows from (5.3.3) and (5.6.16) that $R(\lambda)f \in L^2(M)$ for $\text{Im} \lambda > 0$. Since $-\Delta_g - \lambda^2 - \frac{(n-1)^2}{4}$ is invertible on $L^2(M)$ for $\lambda$ satisfying (5.6.12), we see that (5.6.17) is indeed a meromorphic continuation of (5.6.13).

For $s > \frac{1}{2} - \text{Im} \lambda$ and $f \in x_1^{\frac{n+3}{4} - \frac{i\lambda}{2}} H^{s-1}(M_{\text{even}})$, we can also define $R(\lambda)f$ by (5.6.17), using (5.6.7) and extending $x_1^{\frac{i\lambda}{2}} - \frac{n+3}{4} f$ to an element of $H^{s-1}_{\text{comp}}(X)$; the mapping property (5.6.15) follows immediately.

2. It remains to show an extension of $R(\lambda)$ to an operator with mapping properties (5.6.14). Such extension will necessarily be unique since $C_0^\infty(M)$ is dense in $L^2_{\text{comp}}(M)$.

We will handle the case of $f \in L^2_{\text{comp}}(M)$ using an elliptic parametrix. Take properly supported $A \in \Psi^0_h(M)$ such that

\[ \text{WF}_h(A) \cap \partial_t T^* M = \emptyset, \quad \text{WF}_h(I - A) \cap \{ |\xi|_g \leq 1 \} = \emptyset. \]

The operator $h^2( - \Delta_g - \lambda^2 - \frac{(n-1)^2}{4} ) \in \Psi^2_h(M)$ is elliptic on $\text{WF}_h(I - A)$ as long as $|\lambda| \leq h^{-1}$. By Proposition E.31, there exists properly supported
$Q \in \Psi^{-2}_h(M)$ such that
\[ I - A = h^2 \left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) Q, \]
where we removed the $O(h\infty)_{\Psi^{-\infty}}$ remainder by putting it into $A$.

By (5.6.17), we have for each $\lambda$ not a pole of $R$,
\[ u = R(\lambda) \left( -\Delta_g - \lambda^2 - \frac{(n-1)^2}{4} \right) u, \quad u \in C_0^\infty(M). \]
Applying this to $u := h^2 Qf$, we get
\[ R(\lambda)(I - A)f = h^2 Qf, \quad f \in C_0^\infty(M). \]
It follows that
\[ R(\lambda)f = h^2 Qf + R(\lambda)Af, \quad f \in C_0^\infty(M). \]

We may now define $R(\lambda)f \in H^2_{\text{loc}}(M)$ for $f \in L^2_{\text{comp}}(M)$ using (5.6.18).

**EXAMPLES.** 1. For the hyperbolic space (5.0.1), there are no resonances when $n$ is odd and there are resonances
\[ \lambda_k = -i \left( k + \frac{n-1}{2} \right), \quad k \in \mathbb{N}, \]
when $n$ is even – see [GZ95b, §2]. Moreover, for $n$ even the values $\lambda_k$ are poles of the operator $P(\lambda)^{-1}$ on $X$ – see Exercise 5.9.

2. For the hyperbolic cylinder (5.1.4), the resonances are given by (see [Bo16, Proposition 5.3] and Figure 5.1(b))
\[ \lambda_{j,k} = \frac{2\pi k}{\ell} - \left( j + \frac{1}{2} \right) i, \quad k \in \mathbb{Z}, \quad j = 0, 1, 2, \ldots \]

We finally give a few applications of the construction of this chapter to high frequency asymptotics, using the semiclassically rescaled operator $P_h(\omega) = h^2 P(h^{-1}\omega)$. Recall the class of operators $\Psi^\text{comp}_h$ introduced in Definition E.29, the symbol $p$ defined in (5.3.11), and the components $\Sigma_\pm$ of the characteristic set defined in Lemma 5.14. We start with constructing an approximate inverse for $P_h(\omega)$:

**THEOREM 5.33.** Fix a compact subset $\Omega \subset (0, \infty)$ and constants $C_0 > 0$, $s > \frac{1}{2} + C_0$. Assume that
\[ Q \in \Psi^\text{comp}_h(M), \quad \sigma_h(Q) \geq 0, \]
controls trapping in the following sense: for each $\omega \in \Omega$ and $(x, \xi) \in \Sigma_-$, $\gamma(t) := e^{t(\xi)^2}H_p(x, \xi)$, either $\gamma(t) \to L_-$ as $t \to -\infty$ or $\gamma(t_0) \in \text{ell}_h(Q)$ for some $t_0 \leq 0$. Let the space $X^s$ be defined in (5.6.1).
Then for $h$ small enough and $\text{Re } \omega \in \Omega$, $|\text{Im } \omega| \leq C_0 h$, the operator $P_h(\omega) - iQ : X^s \to \tilde{H}^{s-1}(X)$ is invertible and

$$
(5.6.20) \quad \|(P_h(\omega) - iQ)^{-1}\|_{H^{s-1}_h(X) \to X^s} \leq Ch^{-1}.
$$

**REMARKS.** 1. An operator $Q$ with above properties always exists. Indeed, by part 1 of Lemma 5.19, it is enough to make $Q$ elliptic on $\Sigma_+ \cap \{x_1 \geq \delta\}$, and the latter set is compact by Lemma 5.4.11.

2. In Chapter 6, we will further explore high frequency estimates for the resolvent under additional assumptions on the set of trapped geodesics on $M$. In the special case when $M$ has no trapped geodesics, we may take $Q = 0$ in Theorem 5.33 – see Theorem 6.13.

**Proof.** 1. We first claim that it suffices to prove the following estimate for all compactly supported $A \in \Psi^0_h(X)$ and all $u \in X^s$:

$$
(5.6.21) \quad \|Au\| \leq Ch^{-1}\|(P_h(\omega) - iQ)u\|_{\tilde{H}^{s-1}_h(X)} + O(h^\infty)\|u\|_{\tilde{H}^{-N}_h(X)}
$$

Indeed, let $\chi_1$ satisfy (5.5.14). Combining (5.6.21) for $A := \chi_1$ with (5.5.20) (whose proof applies to $P_h(\omega) - iQ$ since $Q$ is supported in $\{x_1 > 0\}$), we obtain

$$
(5.6.22) \quad \|u\|_{\tilde{H}^s_h(X)} \leq Ch^{-1}\|(P_h(\omega) - iQ)u\|_{\tilde{H}^{s-1}_h(X)} + O(h^\infty)\|u\|_{\tilde{H}^{-N}_h(X)}.
$$

Since $Q$ is compactly microlocalized, it is a smoothing operator on $X$, and thus a compact operator $X^s \to \tilde{H}^{s-1}(X)$. Then by Theorem 5.30, $P_h(\omega) - iQ$ is a Fredholm operator of index zero $X^s \to \tilde{H}^{s-1}(X)$. By (5.6.22), for $h$ small enough this operator has trivial kernel and thus is invertible, and (5.6.20) holds.

2. To show (5.6.21) we follow the proof of Lemma 5.23. The radial estimates (5.5.12) still hold for $P_h(\omega) - iQ$, since $\text{WF}_h(Q) \cap L_\pm = \emptyset$. Therefore, it remains to prove (5.5.13). For that, we follow the proof of Lemma 5.22, using a partition of unity to reduce to the situation when $\text{WF}_h(A)$ is contained in a small neighborhood of some point $(x, \xi) \in T^*X$, and considering the following cases for the trajectory $\gamma(t) = e^{t(\xi)}^{-1}H_p(x, \xi)$:

1. $(x, \xi) \in \text{ell}_h(P_h(\omega) - iQ) = \{\langle \xi \rangle^{-2}p \neq 0\} \cup \text{ell}_h(Q)$: then $\|Au\|$ can be controlled using the semiclassical elliptic estimate, Theorem E.32;

2. $(x, \xi) \in \Sigma_+$: by (5.4.12) and Lemma 5.19, $\gamma(t)$ converges to $L_+$ as $t \to +\infty$ and does not pass through $T^*M \supset \text{WF}_h(Q)$. Therefore, $\|Au\|$ can be controlled by $\|A_+ u\|$ using propagation of singularities, Theorem E.49;

3. $(x, \xi) \in \Sigma_-$ and $\gamma(t)$ converges to $L_-$ as $t \to -\infty$: $\|Au\|$ can be controlled by $\|A_- u\|$ using propagation of singularities and the fact that $\sigma_h(Q) \geq 0$;
We follow the proof of Lemma 5.23. First of all, since (5.6.23) can be controlled by \( \|Au\| \) using propagation of singularities, for some \( A' \in \Psi^\comp_h(M) \) with \( \WF_h(A') \subset \ell h(Q) \), and \( \|A'u\| \) can be controlled using case (1).

We next show a semiclassically outgoing property for the operators \( P_h(\omega) \) and \( P_h(\omega) - iQ_h \):

**THEOREM 5.34.** Fix a compact subset \( \Omega \subset (0,\infty) \) and constants \( C_0 > 0 \), \( s > \frac{1}{2} + C_0 \), and assume that \( Q \in \Psi^\comp_h(M) \), \( \sigma_h(Q) \geq 0 \). Consider compactly supported operators

\[
A \in \Psi^0_h(M), \quad B \in \Psi^0_h(X), \quad L_\pm \cup \WF_h(A) \subset \ell h(B)
\]

such that for each \( (x, \xi) \in \WF_h(A) \cap \Sigma_- \), the trajectory \( e^{i(\xi)^{-1}H_p(x, \xi)} \) converges to \( L_- \) as \( t \to -\infty \) and stays in \( \ell h(B) \) for all \( t \leq 0 \).

Then for all \( \omega \in \Omega + i[-C_0, C_0] \), and \( N \) and \( u \in X^s \), we have

\[
(5.6.23) \quad \|Au\|_{H^s_h} \leq C h^{-1} \|B(P_h(\omega) - iQ)u\|_{H^{s-1}_h} + \mathcal{O}(h^\infty) \|u\|_{H^{-N}_h(X)}.
\]

**Proof.** We follow the proof of Lemma 5.23. First of all, since \( B \) is elliptic on \( L_- \) and \( \WF_h(Q) \cap L_- = \emptyset \), Theorem E.54 gives the following strengthening of (5.5.12):

\[
(5.6.24) \quad \|A_-u\|_{H^s_h} \leq C h^{-1} \|B(P_h(\omega) - iQ)u\|_{H^{s-1}_h} + \mathcal{O}(h^\infty) \|u\|_{H^{-N}_h(X)},
\]

for some \( A_- \in \Psi^0_h(X) \) elliptic on \( L_- \).

Since \( \WF_h(A) \) lies in \( \overline{T^* M} \), by (5.4.12) it does not intersect \( \Sigma_+ \). Thus by a pseudodifferential partition of unity we may reduce to the following cases:

(1) \( \WF_h(A) \cap \{ (\xi)^{-2} = 0 \} = \emptyset \): by the elliptic estimate, Theorem E.32, we get

\[
\|Au\|_{H^s_h} \leq C \|B(P_h(\omega) - iQ)u\|_{H^{s-2}_h} + \mathcal{O}(h^\infty) \|u\|_{H^{-N}_h(X)}.
\]

(2) \( e^{i(\xi)^{-1}H_p(\WF_h(A))} \) is contained in \( \ell h(A_-) \) for \( -t \geq 0 \) large enough and in \( \ell h(B) \) for all \( t \leq 0 \): by propagation of singularities, Theorem E.49, we get

\[
\|Au\|_{H^s_h} \leq C \|A_-u\|_{H^s_h} + C h^{-1} \|B(P_h(\omega) - iQ)u\|_{H^{s-1}_h} + \mathcal{O}(h^\infty) \|u\|_{H^{-N}_h(X)},
\]

and the first term on the right-hand side is estimated by (5.6.24). \( \square \)
5.7. APPLICATIONS TO GENERAL RELATIVITY

We now discuss applications of the methods of this chapter to quasi-normal modes and wave decay on black hole spacetimes. We focus on spacetimes which correspond to asymptotically hyperbolic manifolds. Other spacetimes to which the methods developed here apply include the Schwarzschild–de Sitter spacetime, considered in Exercise 5.23, and more general Kerr–de Sitter spacetimes and their stationary perturbations, studied in the original work of Vasy [Va13].

Our starting point is the following procedure, associating a pseudo-Riemannian metric to a family of second order differential operators:

**DEFINITION 5.35.** Let $X$ be an $n$-dimensional manifold and $P(\lambda)$ a family of second order differential operators on $X$ of the form

$$P(\lambda) = -\sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k + 2i \sum_{j=1}^{n} a_{0j} \lambda \partial_j + a_{00} \lambda^2 + B$$

where $(a_{jk})_{j,k=0}^{n}$ is an invertible real-valued symmetric matrix, $a_{jk} \in C^\infty(X)$, and $B = b_0 \lambda + \sum_{j=1}^{n} b_j \partial_j + c$ for some $b_j, c \in C^\infty(X)$. Then we define the associated pseudo-Riemannian manifold $(\tilde{X}, \tilde{g})$ as follows:

$$\tilde{X} := \mathbb{R}_t \times X, \quad \tilde{g}^{-1} (dx_j, dx_k) = a_{jk},$$

where $x_0 := t$.

We apply Definition 5.35 to the modified Laplacian introduced in §5.3. As in that section, let $(M, g)$ be an even asymptotically hyperbolic manifold. We consider the even extension $\tilde{X}$, fix a product structure $(x_1, x')$ as in (5.3.2) and extend to $\tilde{X}$ the metric $g_1(x_1, x', dx')$ as in (5.3.4). We fix

$$\psi = \psi(x_1) \in C^\infty(\tilde{X})$$

and consider the operator $P_\psi(\lambda)$ defined in (5.3.15).

Let $\tilde{g}$ be the metric on $\tilde{X} := \mathbb{R}_t \times X$ associated to $P_\psi(\lambda)$ according to Definition 5.35. We can write the inverse metric $\tilde{g}^{-1}$ on covectors in terms of the symbol $p(x, \xi; \omega)$ defined in (5.3.11):

$$(5.7.1) \quad |\tilde{\xi}|_{\tilde{g}(t,x)}^2 = e^{2\psi} p(x, \xi - \xi_t d\psi; -\xi_t).$$

Here we write covectors on $\tilde{X}$ as $\tilde{\xi} = (\xi_t, \xi)$, with $\xi \in T^* M$. Note that $\tilde{g}$ is stationary, that is it is independent of $t$.

By (5.3.12), we have on $\mathbb{R} \times M$, 

$$\tilde{g}^{-1}(t, x, \xi_t, \xi) = \frac{e^{2\psi}}{x_1} \left( g^{-1}(x, \xi + \left( \frac{dx_1}{2x_1} - d\psi \right) \xi_t) - \xi_t^2 \right);$$
by (5.3.13), we have on $\mathbb{R} \times Y$,
\[
\tilde{g}^{-1}(t, x_1, x', \xi_t, \xi, \xi') = e^{2\psi} \left( 4(\xi_1 - \psi'(x_1)\xi_1)(x_1\xi_1 + \xi_t(1 - x_1\psi'(x_1))) \\
+ g_t^{-1}(x_1, x', \xi') \right).
\]
It follows that $\tilde{g}^{-1}$ is Lorentzian, that is it has signature $(n, 1)$. We refer the reader to [TaI, §2.7] for an introduction to Lorentzian geometry.

On $\mathbb{R} \times M$, we may write $\tilde{g}$ as a product metric which is singular at the boundary $\{x_1 = 0\}$, by using the change of variables
\[
\Phi : (t, x) \mapsto \left( t - \psi + \frac{1}{2} \log x_1, x \right).
\]
Indeed, we compute
\[
\Phi^* \tilde{g}^{-1} = \frac{e^{2\psi}}{x_1} (g^{-1}(x, \partial x) - \partial_t^2),
\]
therefore on $\mathbb{R} \times M$,
\[
(5.7.2) \quad \Phi^* \tilde{g} = x_1 e^{-2\psi} (-dt^2 + g(x, dx)).
\]

5.7.1. The geometry of spacetime. We now study the properties of the spacetime $(\tilde{X}, \tilde{g})$. Note that for each $\omega \in \mathbb{R}$, (5.7.1) gives a correspondence
\[
(5.7.3) \quad (\xi_t, \xi) \in \mathcal{C} \cap \{\xi_t = -\omega\} \mapsto \xi - \xi_t d\psi \in \{p = 0\}
\]
between the characteristic set $\{p = 0\}$ and the intersection of the dual light cone
\[
\mathcal{C} = \{(t, x, \tilde{\xi}) \in T^* \tilde{X} \mid |\tilde{\xi}|^2_{\tilde{g}(t, x)} = 0\}
\]
with the hypersurface $\{\xi_t = -\omega\}$.

We impose the conditions (5.4.6), (5.4.7) on $\psi$:
\[
\left| \frac{dx_1}{2x_1} \right|_g < 1 \quad \text{on } M,
\]
\[
\psi'(x_1)(1 - x_1\psi'(x_1)) > 0 \quad \text{on } Y.
\]
Then the surfaces $\{t = \text{const}\}$ are spacelike; that is, $|dt|^2_{\tilde{g}} < 0$. The cone $\mathcal{C}$ can be split into the positively and negatively oriented light cones $\mathcal{C}_\pm$ as follows:
\[
\mathcal{C}_\pm := \mathcal{C} \cap \{ \pm (\hat{\xi}, dt)_{\tilde{g}} < 0 \}.
\]
We see from (5.4.13) (where we put $\varphi := \psi$) and (5.7.1) that under (5.7.3), $\mathcal{C}_\pm$ correspond to the components $\Sigma_\pm$ of the characteristic set.

Next, the surface $\{x_1 = 0\} \subset \tilde{X}$, which we call the event horizon, is null in the sense that $|dx_1|^2_{\tilde{g}} = 0$ on $\{x_1 = 0\}$. We call $\{x_1 > 0\} = \mathbb{R} \times M$ the domain of outer communications and $\{x_1 < 0\}$ the black hole region. If $\gamma(s) : \mathbb{R} \to \tilde{X}$ is a curve which is timelike (that is, $|\dot{\gamma}|^2_{\tilde{g}} < 0$) and future time
oriented (that is, $\langle dt, \dot{\gamma} \rangle > 0$), and $\gamma(0)$ lies on $\{x_1 = 0\}$, then for $s \geq 0$, $\gamma(s)$ lies in $\{x_1 < 0\}$. That is, if an observer crosses the event horizon at some point, he can never escape the black hole region and will be pushed farther into this region as time goes on – see Figure 5.7.1.

We finally define the *red-shift trajectories*, which are the null geodesics on $\tilde{X}$ on the event horizon $\{x_1 = 0\}$ given in the phase space $T^{*}\tilde{X}$ by

\[ \begin{align*}
5.7.4 & \quad x_1 = 0, \quad \xi_1 = \mp e^{-t}, \quad t = \ln(\mp 4e^{2\psi(0)}s), \quad \xi_t = 0, \quad x' = \text{const}, \quad \xi' = 0,
\end{align*} \]

where $s, \mp s > 0$, is the geodesic parameter. We call (5.7.4) red-shift trajectories because the frequency $\xi$ decays exponentially along them as $t \to \infty$.

Writing the trajectories (5.7.4) using the correspondence (5.7.3), we obtain the following Hamiltonian trajectories of $e^{\omega H_p}$ on $T^{*}X$ for $\omega = 0$:

\[ \begin{align*}
5.7.5 & \quad x_1 = 0, \quad \xi_1 = \frac{1}{4e^{2\psi(0)}s}, \quad x' = \text{const}, \quad \xi' = 0.
\end{align*} \]

As $s \to 0 \mp$, this trajectory converges in $T^{*}X$ to the radial sets $L_{\pm}$ defined in (5.4.20). In other words, $L_{\pm}$ correspond to limits of the red-shift trajectories as $t \to -\infty$.

**EXAMPLES.**

1. For the hyperbolic space (see (5.3.17)), we have

\[ \begin{align*}
\tilde{X} = \mathbb{R}_t \times X, \quad X = B_{\mathbb{R}^n}(0,2),
\end{align*} \]

and with $(r, \theta)$ denoting polar coordinates on $X$ and $g_S$ the standard metric on $S^{n-1}$,

\[ \begin{align*}
5.7.5 & \quad \tilde{g}^{-1} = (1 - r^2)\partial_r^2 - 2r \partial_r \partial_r - \partial_t^2 + \frac{1}{r^2}g_S^{-1}(\theta, \partial_\theta);
\end{align*} \]

\[ \begin{align*}
\tilde{g} = -(1 - r^2)dt^2 - 2r dr dt + dr^2 + r^2g_S(\theta, d\theta).
\end{align*} \]

On $\mathbb{R} \times M$, $M = B_{\mathbb{R}^n}(0,1)$, we use (5.1.23) together with the formula $x_1e^{-2\psi} = 1 - r^2$ to see that the pullback metric (5.7.2) is

\[ \begin{align*}
\Phi^*\tilde{g} = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2g_S(\theta, d\theta).
\end{align*} \]

Thus $(\tilde{X}, \tilde{g})$ is a subset of the de Sitter spacetime – see [Va13, (4.1)].

2. For the hyperbolic cylinder (see (5.3.20)), we have

\[ \begin{align*}
\tilde{X} = \mathbb{R}_t \times X, \quad X = [-2,2]_r \times S^1_{\theta};
\end{align*} \]

\[ \begin{align*}
5.7.6 & \quad \tilde{g}^{-1} = -\partial_t^2 - 2r \partial_r \partial_r + (1 - r^2)\partial_t^2 + \partial_\theta^2;
\end{align*} \]

\[ \begin{align*}
\tilde{g} = -(1 - r^2)dt^2 - 2r dr dt + dr^2 + d\theta^2.
\end{align*} \]
5. APPLICATIONS TO GENERAL RELATIVITY

On $\mathbb{R} \times M$, $M = B_{\mathbb{R}^n}(0,1)$, the pullback metric (5.7.2) is (using (5.1.25) and the formula $x_1 e^{-2\psi} = 1 - r^2$)

$$\Phi^* \tilde{g} = -(1 - r^2) dt^2 + \frac{dr^2}{1 - r^2} + d\theta^2.$$  

This spacetime has two event horizons, $\{r = 1\}$ and $\{r = -1\}$; see Figure 5.7.1.

5.7.2. Resonance expansions. We finally present an application of the results of this chapter to wave decay on the spacetime $(\tilde{X}, \tilde{g})$. We only provide an outline, see [Va13, §§3, 6] or [Dy12, §1.1] for details.

Let $\Box \tilde{g}$ be the d’Alembert–Beltrami operator on $\tilde{X}$. Consider a future solution to the inhomogeneous wave equation,

$$-\Box \tilde{g} u = f, \quad \text{supp } u \subset \{t > 0\},$$

where the right-hand side $f$ satisfies

$$f \in C_0^\infty(\tilde{X} \cap \{0 < t < 1\}).$$

Such solution exists and is unique, see [TaII, §7.7]. The proof of this fact uses the following energy estimate, valid for some constant $C_2$:

$$\|u(t)\|_{\dot{H}^1(X)} + \|\partial_t u(t)\|_{L^2(X)} \leq C e^{C_2 t} \|f\|_{L^2_{t,x}(\tilde{X})}.$$  

The energy estimate is obtained by integration by parts (see [TaI, Proposition 2.8.1] or [Dy11, §1.1]) using that the surfaces $\{t = \text{const}\}$ and $\{|r| = 2\}$ are spacelike and if $V$ is a positively time oriented timelike vector field in the sense that $\tilde{g}(V, V) < 0$ and $\langle dt, V \rangle > 0$, then $V$ is pointing outside of the region $\{|r| \leq 2\}$. Higher derivatives of $u$ can be estimated in the same way, see [TaI, §6.5].
We want to understand the decay properties of solutions to (5.7.7) as $t \to +\infty$. For that, we use the Fourier–Laplace transform,

$$
\hat{u}(\lambda) := \int_{\mathbb{R}} e^{i\lambda t} u(t) \, dt \in \bar{H}^1(X), \quad \text{Im} \lambda > C_2 \gg 1.
$$

The integral converges exponentially thanks to (5.7.8).

Fourier transforming the wave equation, we obtain

$$
\tilde{P}(\lambda) \hat{u}(\lambda) = \hat{f}(\lambda), \quad \text{Im} \lambda > C_2,
$$

where $\tilde{P}(\lambda)$ is the second order differential operator on $\bar{X}$ obtained from $-\Delta_{\tilde{g}}$ by replacing $\partial_t$ with $-i\lambda$.

By Definition 5.35, we see that $\tilde{P}(\lambda) - P_\psi(\lambda)$ is a first order polynomial in $\lambda$ and $\partial_x$. In particular, the semiclassically rescaled operator

$$
\tilde{P}_h(\omega) = h^2 \tilde{P}(h^{-1}\omega)
$$

satisfies

$$
\tilde{P}_h(\omega) = P_{\psi,h}(\omega) + O(h)\psi_1^h(X).
$$

**THEOREM 5.36.** Fix $s \in \mathbb{R}$. Then for $\text{Im} \lambda > \frac{1}{2} - s$,

$$
\tilde{P}(\lambda) : \{u \in \bar{H}^s(X) \mid P(0)u \in \bar{H}^{s-1}(X)\} \to \bar{H}^{s-1}(X)
$$

is a Fredholm operator of index zero and has a meromorphic inverse

$$
\tilde{P}(\lambda)^{-1} : \bar{H}^{s-1}(X) \to \bar{H}^s(X).
$$

**Proof.** Recalling (5.3.15), we see that it is enough to prove the statement for the operator

$$
e^{(i\lambda - \frac{n+3}{2})}\psi \tilde{P}(\lambda) e^{(\frac{n-1}{2}-i\lambda)\psi}.
$$

The corresponding semiclassical rescaling is equal to $P_h(\omega) + O(h)\psi_1^h(X)$. Then the proof of Theorem 5.30 applies. The only part of the proof that used the subprincipal symbol is Proposition 5.13; it is still true since $\tilde{P}(\lambda)$ is self-adjoint for real-valued $\lambda$ with respect to the density induced on the slices \{t = \text{const}\} by the metric $\tilde{g}$. \hfill \Box

The poles of (5.7.11) are called quasi-normal modes of the spacetime $(\bar{X}, \tilde{g})$.

Note that

$$
\hat{u}(\lambda) = P(\lambda)^{-1} \hat{f}(\lambda).
$$

Indeed, since $\hat{f} \in C_0^\infty(X)$, it suffices to consider the case $s = 1$ – see Proposition 5.31. Then both sides of (5.7.12) are solutions to the equation (5.7.9) in $\bar{H}^1(X)$. For $\lambda$ which is not a pole of $P(\lambda)^{-1}$, (5.7.12) follows from the
invertibility of (5.7.10); by analytic continuation, (5.7.12) holds also at the poles of $R_\tilde{g}$.

The right-hand side of (5.7.12) provides a meromorphic continuation of $\hat{u}(\lambda)$ to $\{\text{Im } \lambda > \frac{1}{2} - s\}$ for every $s$, and thus to the entire complex plane. By the Fourier inversion formula in $t$ applied to $e^{-(C_2+1)t}u(t)$, we have

$$(5.7.13) \quad u(t) = \frac{1}{2\pi} \int_{\text{Im } \lambda = C_2+1} e^{-i\lambda t} P(\lambda)^{-1} f(\lambda) \, d\lambda.$$  

If we have a polynomial resolvent bound

$$(5.7.14) \quad \|P(\lambda)^{-1}\|_{\dot{H}^{-1}_h(X) \to \dot{H}^1(X)} \leq C|\lambda|^{N-1},$$

$$|\text{Re } \lambda| \gg 1, \quad \text{Im } \lambda \in [-\nu, C_2 + 1]$$

for some constants $N, \nu$, then as in the proof of Theorem 2.7, we can shift the contour of integration to $\{\text{Im } \lambda = -\nu\}$ for some $\nu > 0$. That gives a resonance expansion similar to (2.3.3), (2.3.4).

In terms of the rescaled operator $\tilde{P}_h(\omega)$ the bound (5.7.14) is

$$(5.7.15) \quad \|\tilde{P}_h(\omega)^{-1}\|_{\dot{H}^{-1}_h(X) \to \dot{H}^1(X)} \leq C h^{-1-N},$$

$$|\text{Re } \omega| \in \left[\frac{1}{2}, 1\right], \quad \text{Im } \omega \in [-\nu h, (C_2 + 1) h], \quad 0 < h \ll 1.$$  

**EXAMPLES.** 1. For the spacetime (5.7.5) corresponding to the hyperbolic space, we compute

$$-\Box_\tilde{g} = -(1 - r^2)\partial_t^2 + 2r \partial_r \partial_t + \partial_r^2 - \frac{1}{r^2} \Delta S$$

$$+ (n + 1)r \partial_r - \frac{n - 1}{r} \partial_r + n \partial_t,$$

$$\tilde{P}(\lambda) = -(1 - r^2)\partial_t^2 + (n + 1 - 2i\lambda)r \partial_r + \frac{1-n}{r} \partial_r$$

$$- \lambda^2 - ni\lambda - \frac{1}{r^2} \Delta S.$$  

[TODO quasi-normal modes]

[TODO refer to the proof of the bound (5.7.15)]

2. For the spacetime (5.7.6) corresponding to the hyperbolic cylinder, we compute

$$-\Box_\tilde{g} = \partial_t^2 + 2r \partial_r \partial_t - (1 - r^2)\partial_t^2 - \partial_r^2 + \partial_t + 2r \partial_r,$$

$$\tilde{P}(\lambda) = -(1 - r^2)\partial_t^2 + 2(1 - i\lambda)r \partial_r - \lambda^2 - i\lambda - \partial_r^2.$$  

[TODO quasi-normal modes]

[TODO refer to the proof of the bound (5.7.15)]
5.8. NOTES

TODO.

Borthwick [Bo16] Vasy [Va13, Va12]

5.9. EXERCISES

Section 5.1

1. Show that any two canonical product structures satisfying (5.1.10) coincide on a neighborhood of $\partial M$.

2. Let $(M, g)$ be an asymptotically hyperbolic manifold and $\varphi_t : S^*M \to S^*M$ the geodesic flow. Fix a canonical product structure $(y_1, y')$ and let $(y_1, y', \eta_1, \eta')$ be the corresponding product structure for $T^*M$. The following three exercises explore the behaviour of geodesics on $M$ near $\partial M$.

(a) Denote $\mu_1 := y_1 \eta_1$. Show that trajectories of $\varphi_t$ solve the evolution equations

$$\dot{y}_1 = y_1 \mu_1, \quad \dot{\mu}_1 = \mu_1^2 - 1 - \frac{y_1^3}{2} \langle \eta', \eta' \rangle \partial_{y_1, \eta_1(y_1, y')}, \quad (y'_1, \eta'_1) = y_1^2 G_{y_1}(y', \eta')$$

where $G_{y_1}$ is the generator of the geodesic flow of the metric $g_1(y_1, \cdot)$.

(b) Let $\gamma : \mathbb{R} \to S^*M$ be a geodesic such that $y_1(\gamma(0)) < \varepsilon_1, \eta_1(\gamma(0)) \leq 0$. Show that for $\varepsilon_1$ small enough, we have as $t \to \infty$ along the geodesic $\gamma(t)$

$$y_1(t) = O(e^{-t}), \quad \mu_1(t) = 1 + O(e^{-2t}), \quad (y'_1(t), \eta'_1(t)) = (y'_1, \eta'_1) + O(e^{-2t})$$

for some $(y'_1, \eta'_1) \in T^*\partial M$.

3. Let $\Gamma_1^\pm \subset S^*M$ be the union of geodesics which stay in a compact set as $t \to \pm \infty$, see (6.1.4).

(a) Show that the limits

$$B_\pm(x, \xi) = \lim_{t \to \pm \infty} (y'(\varphi_t(x, \xi)), \eta'(\varphi_t(x, \xi))), \quad (x, \xi) \in S^*M \setminus \Gamma_1^\pm,$$

exist and define surjective smooth maps

$$B_\pm : S^*M \setminus \Gamma_1^\pm \to T^*\partial M.$$ 

(b) Show that the differential $dB_\pm$ is surjective at every point and $B_\pm(x, \xi) = B_\pm(\bar{x}, \bar{\xi})$ if and only if $(x, \xi)$ and $(\bar{x}, \bar{\xi})$ lie on the same geodesic.
4. Define the scattering relation as follows:

\[ S: B_-(S^*M \setminus (\Gamma_+ \cup \Gamma_-)) \to B_+(S^*M \setminus (\Gamma_+ \cup \Gamma_-)), \]

\[ S(B_-(x, \xi)) = B_+(x, \xi), \quad (x, \xi) \in S^*M \setminus (\Gamma_+ \cup \Gamma_-). \]

(a) Show that the domain and range of \( S \) are open subsets of \( T^*M \) and \( S \) is a canonical transformation, i.e. a diffeomorphism preserving the symplectic form of \( T^*M \).

(b) Compute the scattering relation for the hyperbolic space and the hyperbolic cylinder, using the product structures from the examples following Definition 5.3.

Section 5.2

5. Show that for \( \lambda \not\in i\mathbb{Z} \), the equation (5.2.6) has two solutions of the form (5.2.8), with

\[ a_{j,\pm} := (-\beta_k)^j \prod_{\ell=1}^j \frac{1}{I(\alpha_\pm + 2\ell)} \]

and \( I(\alpha) \) defined in (5.2.7), \( \alpha_\pm \) defined in (5.2.9). Show that the series (5.2.8) converges for all \( y_1 > 0 \).

6. On the hyperbolic plane \( \mathbb{H}^2 \) (see (5.0.1)), the Schwartz kernel of the resolvent

\[ R(\lambda) = \left( -\Delta_g - \lambda^2 - \frac{1}{4} \right)^{-1} : L^2(\mathbb{H}^2) \to L^2(\mathbb{H}^2), \quad \text{Im} \lambda > 0, \]

is given by (see e.g. [Bo16, Proposition 4.1])

\[ R(\lambda)(x, y) = \frac{1}{4\pi} \int_0^1 \left( t(1-t) \right)^{-\frac{1}{2}-i\lambda} (\sigma(x, y) - t)^{-\frac{1}{2}+i\lambda} dt, \]

where, with \( d_{\mathbb{H}^2} \) denoting the hyperbolic distance,

\[ \sigma(x, y) = \cosh^2 \left( \frac{d_{\mathbb{H}^2}(x, y)}{2} \right) = 1 + \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}. \]

Use this formula to prove Proposition 5.8 for \( \mathbb{H}^2 \).

Section 5.3

7. Verify that the formulas (5.3.12) and (5.3.13) for \( p \) agree on \( M \cap Y \).

8. Show that there exist \( f_0, \ldots, f_{k-1} \in \mathcal{D}'(\partial M) \), \( f_{k-1} \neq 0 \), satisfying

\[ (5.9.1) \quad P(\lambda)^* \left( \sum_{j=0}^{k-1} \delta^{(j)}(x_1)f_j(x') \right) = 0 \]
5.9. EXERCISES

if and only if \( \lambda = -ik \) and \( f_j \) solve the equations for \( j = 0, \ldots, k - 1 \)

\[
4(j-k)f_{j-1} + \sum_{\ell=j}^{k-1} (-1)^{\ell-j}(\ell) (-\Delta_{\ell-j} + (4\ell + 5 - n - 2k)\gamma_{\ell-j}) f_\ell = 0
\]

where \( f_{-1} := 0 \), the second order differential operator \( \Delta_j \) on \( \partial M \) is obtained
by differentiating the coefficients of \( \Delta_{g_{1(x_1,\cdot)}} \) \( j \) times in \( x_1 \) at \( x_1 = 0 \), and
\( \gamma_j := \partial_{x_1}^j \gamma |_{x_1=0} \in C^\infty(\partial M; \mathbb{R}) \).

Note that if \( f_{k-1} \) is fixed, then the equations (5.9.2) for \( j = 1, \ldots, k-1 \)
determine \( f_0, \ldots, f_{k-2} \) uniquely, and the \( j = 0 \) equation is satisfied if and
only if \( D_k f_{k-1} = 0 \) where \( D_k \) is an order 2 \( k \) differential operator on \( \partial M \)
with principal part \( (-\Delta_0)^k \).

9. (a) Using (5.3.19) multiplied by \( (1 - x_1)^2 \), show that for the hyperbolic
space \( \mathbb{H}^n \), there exist \( f_0, \ldots, f_{k-1}, f_{k-1} \neq 0 \) satisfying (5.9.1) if and only if
\( \lambda = -ik \) and \( f_j \) solve the equations

\[
(j-k)f_{j-1} + (a_j - \Delta_{g_{S}})f_j + (j+1)f_{j+1} = 0,
\]

\[
a_j := 2(j+1)(j+1-k) + \frac{(n-1)(n-5-4j+2k)}{4},
\]

\[
b_j := (j+2)(j+2-k) + \frac{(n-1)(n-9-4j+2k)}{4}
\]

for all \( j = 0, \ldots, k-1 \), where we put \( f_{-1} = f_{k} = 0 \).

(b) Using [TODO], show that (5.9.3) has a nontrivial solution if and only if
\( -\Delta_{g_{S}} \) has an eigenvalue \( (r + \frac{3-n}{2})(r + \frac{n-1}{2}) \) for some \( r = 0, \ldots, k-1 \). Using
the known spectrum of \( -\Delta_{g_{S}} \), conclude that for \( n \) even, (5.9.3) never has
a nontrivial solution, and for \( n \geq 3 \) odd, it has a nontrivial solution if and
only if \( k \geq \frac{n-1}{2} \).

10. Let \( P_{\psi,h}(\omega) \) be defined in (5.3.15). Show that \( P_{\psi,h}(\omega) \) lies in \( \text{Diff}^2_h(X) \)
and its principal symbol is
\[
p_{\psi}(x, \xi; \omega) = e^{2\psi}p(x, \xi + \omega d\psi).
\]

11. This and several following exercises give a perspective on Euclidean
scattering based on a Fredholm problem similar to the one studied in this
chapter.

(a) In the spacetime \( \mathbb{R}_t \times \mathbb{R}^n_x \), consider the polar coordinates
\[
r = \sqrt{t^2 + |x|^2}, \quad (\tau, \theta) = \frac{1}{r}(t, x) \in S^n.
\]

Show that in these coordinates, the d’Alembertian \( \Box := \partial_t^2 - \Delta_x \) satisfies
\[
\Box(r^{-i\lambda}u(\tau, \theta)) = r^{-2-i\lambda}p(\lambda)u(\tau, \theta),
\]
where $\mathcal{P}(\lambda)$ is the following differential operator on $\mathbb{S}^n$:

$$\mathcal{P}(\lambda) = \text{TODO}$$

(b) Show that the operator $\mathcal{P}_h(\omega) := h^2 \mathcal{P}(h^{-1}\omega)$ lies in $\text{Diff}_h^2(\mathbb{S}^n)$ and its principal symbol is given by [TODO].

Section 5.4

12. Show that (5.4.8) and (5.4.25) are no longer true when $\omega = 0$.

13. Show that the vector field $\langle \xi \rangle^{-1} H_p$ vanishes on the sets $L_{\pm}$.

14. TODO the phase space picture of $\mathcal{P}(\lambda)$ for the Minkowski space.

Section 5.5

15. Using Exercises E.28, E.33, E.34, and [TODO hyperbolic estimate], show the following strengthening of Lemma 5.27:

(a) if $s > m > \frac{1}{2} - h^{-1} \text{Im} \omega$ and $u \in \dot{H}^m(X)$, $P_h(\omega)u \in \dot{H}^{s-1}(X)$, then $u \in \dot{H}^s(X)$ and (5.5.24) holds;

(b) if $s > \frac{1}{2} - h^{-1} \text{Im} \omega$, $v \in \mathcal{D}'(X)$ can be extended beyond $\partial X$ to a distribution supported on $\bar{X}$, and $P_h(\omega)^*v \in \dot{H}^{-s}(\bar{X})$, then $v \in \dot{H}^{1-s}(\bar{X})$ and (5.5.25) holds.

16. TODO estimates on $\mathcal{P}(\lambda)$ for the Minkowski space.

Section 5.6

17. Arguing as in the proof of Proposition 5.28, show that (5.5.27) also holds for fixed $s$ and $\beta$ large enough and negative. Why doesn’t this imply that the poles of the operator $\mathcal{P}(\lambda)^{-1}$ defined in (5.6.10) are confined to a strip?

18. Show that for $s > \frac{1}{2} - \text{Im} \lambda$, the spaces $\ker^s P(\lambda)$, $\text{coker}^s P(\lambda)$ defined in (5.6.3), (5.6.4) do not depend on $s$. (Hint: one solution uses the fact that $P(\lambda)$ is a Fredholm operator of index zero and $\ker^{s'} P(\lambda) \subset \ker^s P(\lambda)$, $\text{coker}^{s'} P(\lambda) \supset \text{coker}^s P(\lambda)$ for $s' > s$. Another solution is based on Exercise 5.15.)

We henceforth drop $s$ in $\ker^s P(\lambda)$, $\text{coker}^s P(\lambda)$ and remark that $\ker P(\lambda) \subset C^\infty(\bar{X})$, $\text{coker} P(\lambda) \subset \dot{H}^s(\bar{X})$ for all $s < \frac{1}{2} + \text{Im} \lambda$.

19. Assume that $u \in \ker P(\lambda)$. Using [TODO], show that if $u|_M = 0$, then $u \equiv 0$.

20. Using Exercise 5.19, show that $\lambda$ is a pole of $P(\lambda)^{-1}$ in $\{\text{Im} \lambda > 0\}$ if and only if $\lambda^2 + \frac{(n-1)^2}{4}$ is an $L^2$ eigenvalue for $-\Delta_g$. 
21. Using Exercises 5.8, 5.18, and 5.19 together with (5.6.6), show that if the equation (5.9.2) has no nontrivial solutions for all \( k \leq m \), then the poles of \( P(\lambda)^{-1} \) in \( \{ \text{Im} \lambda > -m \} \) coincide with the resonances of \( M \) (that is, all resonant and coresonant states do have a presence inside the physical region \( M \)).

22. Show that the poles of the operator \( P(\lambda)^{-1} \) do not depend on the choice of the extension of the metric \( g_1 \) in (5.3.4). (Hint: using (5.6.6), show that any element of \( \text{coker} \, P(\lambda)^* \) for one extension will also lie in \( \text{coker} \, P(\lambda)^* \) for the other extension.)

Section 5.7

23. Define the Schwarzschild–de Sitter spacetime by

\[ M_{\text{SdS}} := (r_-, r_+) \times S^2, \quad \tilde{g}_{\text{SdS}} = -Gdt^2 + G^{-1}dr^2 + r^2g_S(y, dy), \]

where \( g_S \) is the standard metric on \( S^2 \), the function \( G \) is given by

\[ G(r) = 1 - \frac{Ar^2}{3} - \frac{2M_0}{r}, \quad M_0 > 0, \quad 0 < \Lambda < \frac{1}{9M_0^2}, \]

and \( r_- < r_+ \) are the two positive roots of the equation \( G(r) = 0 \), with \( G > 0 \) on \((r_- , r_+)\).

[TODO revise]

(a) Use the similarity of \( \tilde{g}_{\text{SdS}} \) with (??) to find a function \( F \) such that \( \Phi^*_F \tilde{g}_{\text{SdS}} \) continues smoothly past \( \{ r = r_{\pm} \} \) and the surfaces \( \{ t = \text{const} \} \) are spacelike.

(b) Arguing as in \( \S 5.7.2 \), show that for each solution \( u \) to (5.7.7), the Fourier–Laplace transform \( \hat{u}(\lambda) \) continues meromorphically to \( \lambda \in \mathbb{C} \), and define resonances of the Schwarzschild–de Sitter spacetime as the poles of \( \hat{u}(\lambda) \).
Part 3

RESONANCES IN THE SEMICLASSICAL LIMIT
Chapter 6

RESONANCE FREE REGIONS

6.1 Geometry of trapping
6.2 Resonances in strips
6.3 Normally hyperbolic trapping
6.4 Logarithmic resonance free regions
6.5 Lower bounds on resonance widths
6.6 Notes

In this section, we study existence of resonance free regions at high energies or in the semiclassical limit. We have already seen logarithmic resonances free regions in potential scattering in Theorems 2.8, 3.10, 4.41. Theorem 4.43 provided such resonance free region for black box perturbations under an abstract non-trapping assumption given in Definition 4.42. A semiclassical version of these results will be discussed in §6.4.

In this chapter we will discuss resonance free regions including resonance free strips in some trapping situations.

A basic setting is given by the semiclassical Schrödinger operator on $\mathbb{R}^n$

$$P = P(h) = -h^2 \Delta + V(x), \quad V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}).$$

Here $h > 0$ is a constant called the semiclassical parameter. We consider the semiclassical régime, using the operator $P - z$ where

$$z \in [\alpha, \beta] + i[-\nu(h), \infty), \quad 0 < \alpha \leq \beta, \quad \nu(h) \in (0, 1), \quad h \to 0.$$

1For technical simplicity we present the case of $n$ odd but that restriction is irrelevant for the results of this chapter.
The meromorphic continuation of the scattering resolvent

\[ R(z, h) = (P - z)^{-1} : \begin{cases} L^2 \rightarrow H^2, & z \in [\alpha, \beta] + i(0, \infty]; \\ L^2_{\text{loc}} \rightarrow H^2_{\text{comp}}, & z \in [\alpha, \beta] + i[-\nu(h), 0] \end{cases} \]

follows by rescaling from Theorem 3.8.

We say that \( P \) has a resonance free region of size \( \nu(h) \) in the energy range \( [\alpha, \beta] \), if there exist \( N, h_0 > 0 \) such that for all \( h \in (0, h_0) \), \( \chi \in C^\infty_c(\mathbb{R}^n) \) the following cutoff resolvent estimate holds:

\[ \| \chi R(z, h) \chi \|_{L^2 \rightarrow L^2} \leq C \chi h^{-N}, \quad z \in [\alpha, \beta] + i[-\nu(h), \infty). \]

In particular, this implies that there are no resonances in \( [\alpha, \beta] + i[-\nu(h), \infty) \), for \( h < h_0 \). The estimate (6.0.3) implies decay of solutions to the wave equation at high frequency, with the rate depending on \( \nu(h) \) – see the table below.

The classical objects associated to the operator \( P \) are its semiclassical principal symbol

\[ p(x, \xi) = |\xi|^2 + V(x), \quad (x, \xi) \in \mathbb{R}^{2n} = T^*\mathbb{R}^n, \]

and its Hamiltonian flow

\[ \exp(tH_p) : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n, \]

\[ H_p := \sum_{j=1}^n \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}. \]

The size, \( \nu(h) \), of the resonance free region depends on the structure of the trapped set \( K_{[\alpha, \beta]} \), consisting of trajectories of \( \exp(tH_p) \) in the energy slab \( p^{-1}([\alpha, \beta]) \) which do not escape to infinity in either direction; see §6.1.

In case when trapping is present, we treat the interaction region in a way which is decoupled from the analysis at infinity. To illustrate this, and to broaden the class of examples to which the results of this chapter apply, we consider the following two settings, the first of which generalizes (6.0.1):

- \((\mathbb{R}^n, g)\) is a compact metric perturbation of the Euclidean space, \( V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \), and

\[ P(h) = -h^2 \Delta_g + V(x); \tag{6.0.4} \]

- \((M, g)\) is an even asymptotically hyperbolic manifold in the sense of Definitions 5.2 and 5.5, and

\[ P(h) = -h^2 \Delta_g. \tag{6.0.5} \]

Our presentation will be centered around case (6.0.4), indicating what changes are necessary to cover case (6.0.5). The restriction on the supports of \( g_{ij} - \delta_{ij} \)
and $V$ can be relaxed significantly – see [Sj96a] for a general treatment. The key is the structure of the trapped set.

The results of this chapter are summarized in the following table:

<table>
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<th>Trapping assumptions</th>
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<tr>
<td>§6.4</td>
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<td>§6.3</td>
<td>Normally hyperbolic</td>
<td>(6.0.4), (6.0.5)</td>
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<td>$e^{-\xi/h}$</td>
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Remarks about generalizations and improvements, in particular in the case of “no assumptions”, will be given in §6.6

### 6.1. GEOMETRY OF TRAPPING

In this section, we define and investigate the incoming/outgoing tails and the trapped set associated to the Hamilton flow $\exp(tH_p)$. Here $p(x, \xi)$ is the semiclassical principal symbol of $P$ given by (6.0.4) or (6.0.5), and $\exp(tH_p)$ is the Hamiltonian flow of $p$ on the cotangent bundle.

#### 6.1.1. The Euclidean case.

We start with the Euclidean case (6.0.4), where

$$ p(x, \xi) = |\xi|^2 + V(x), \quad (x, \xi) \in T^*\mathbb{R}^n; \quad V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) $$

and the Riemannian metric $g$ on $\mathbb{R}^n$ is a compactly supported perturbation of the Euclidean metric. We fix a constant $r_0 > 0$ such that

$$ \text{supp } V, \text{ supp}(g_{ij} - \delta_{ij}) \subset B(0, r_0). $$

The classical trajectories $(x(t), \xi(t)) = e^{tH_p}(x, \xi)$ solve Hamilton’s equations. In $\{|x| \geq r_0\}$, we have $p(x, \xi) = |\xi|^2$. and Hamilton’s equations take the form

$$ \dot{x}(t) = 2\xi(t), \quad \dot{\xi}(t) = 0, $$

so the corresponding trajectories are straight lines.

**DEFINITION 6.1.** Let $(x, \xi) \in T^*\mathbb{R}^n$ and $(x(t), \xi(t)) = e^{tH_p}(x, \xi)$ be the corresponding trajectory. We say that $(x, \xi)$ **escapes** as $t \to +\infty$ (respectively as $t \to -\infty$) if

$$ x(t) \to \infty \quad \text{as } t \to +\infty \quad \text{(respectively } t \to -\infty). $$
We define the **incoming tail** $\Gamma^-$ and the **outgoing tail** $\Gamma^+$ to be the sets of trajectories which do not escape as $t \to +\infty$, respectively as $t \to -\infty$:

$$\Gamma^\pm = \{(x, \xi) \mid x(t) \not\to \infty \text{ as } t \to \mp \infty\}. $$

The **trapped set** is defined as the set of points which do not escape in either time direction, namely

$$K := \Gamma^+ \cap \Gamma^-.$$ 

For a set $J \subset \mathbb{R}$, we may consider the incoming/outgoing tails and the trapped set at energies in $J$:

$$\Gamma^\pm_J := \Gamma^\pm \cap p^{-1}(J), \quad K_J := K \cap p^{-1}(J).$$

For $E \in \mathbb{R}$, we put $\Gamma^\pm_E := \Gamma^\pm \{E\}$ and $K_E := K \{E\}$. 

It follows from the definition that $\Gamma^\pm, K, \Gamma^\pm_J, K_J$ are invariant under the flow $\exp(tH_p)$. Moreover,

$$K_{(-\infty,0]} = p^{-1}((-\infty,0]),$$

that is escaping trajectories only exist at positive energies. See Figure 6.1.
6.1. GEOMETRY OF TRAPPING

We can lift \( r \) to a function on \( T^*\mathbb{R}^n \), putting \( r(x, \xi) := r(x) \). The derivatives of \( r \) along the Hamiltonian flow are

\[
H_p r(x, \xi) = 2\langle x, \xi \rangle / r(x), \quad H_p^2 r(x, \xi) = 4(|\xi|^2 |x|^2 - \langle x, \xi \rangle)^2 / r(x)^3 \quad \text{for} \ r(x) \geq r_0.
\]

This and (6.1.2) imply that every point \((x, \xi)\) satisfying the inequalities

\[
(6.1.6) \quad r(x) > r_0, \quad \pm H_p r(x, \xi) > 0
\]

escapes as \( t \to \pm\infty \), and conversely every trajectory escaping as \( t \to \pm\infty \) eventually satisfies (6.1.6).

In fact, if \( \pm H_p r > 0, \ r(x) > r_0 \) then by (6.1.6) \( \pm \partial_t r(x)^2 = \pm 4 \langle x, \xi \rangle > 0 \), \( \partial_t \xi = 0 \) and hence \( r(x) \to \infty, t \to \pm\infty \). On the other hand if a trajectory escapes to infinity as \( t \to +\infty \) or \( t \to -\infty \) then \( \xi \neq 0 \) and eventually \( H_p^2 r(x, \xi) > 0 \). That implies \( H_p r(x, \xi) \) is strictly increasing, which gives \( \pm H_p(r, \xi) > 0 \) for \( \pm t \gg 1 \).

We also note that if \( H_p r(x_0, \xi_0) = 0, \ r(x_0) \geq r_0 \) and \( \xi_0 \neq 0 \) then \( H_p^2 r(x_0, \xi_0) > 0 \) which means that \( H_p r(x, \xi) \) is strictly increasing on the trajectory through \((x_0, \xi_0)\) and \( r(x) \geq r_0 \).

We record this in

**LEMMA 6.2.** Let \((x, \xi) \in T^*\mathbb{R}^n \) and \((x(t), \xi(t)) = e^{tH_p}(x, \xi) \). Then:

1. If \( r(x) \geq r_0, \xi \neq 0, \) and \( \pm H_p r(x, \xi) \geq 0 \), then \((x, \xi) \notin \Gamma^\pm \) and \((x(t), \xi(t))\) satisfies (6.1.6) for all \( t > 0 \).
2. If \((x, \xi) \notin \Gamma^\pm \), then \((x(t), \xi(t))\) satisfies (6.1.6) for \( \pm t \gg 1 \).

Using Lemma 6.2, we establish topological properties of \( \Gamma^\pm \) and \( K \):

**PROPOSITION 6.3.** The sets \( \Gamma^\pm \) are closed and for each compact \( J \subset \mathbb{R} \setminus \{0\}, K_J \) is compact. Moreover,

\[
(6.1.7) \quad K_{\mathbb{R}\setminus\{0\}} \subset \{ r < r_0 \}.
\]

Finally, if \( K_E = \emptyset \) for some \( E \in \mathbb{R} \), then \( K_{[E-\delta, E+\delta]} = \emptyset \) for some \( \delta > 0 \).

**Proof.** 1. We show that \( \Gamma^- \) is closed with the case of \( \Gamma^+ \) handled similarly. Let \((x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \Gamma^- \). By part 2 of Lemma 6.2, there exists \( T \geq 0 \) such that \( e^{TH_p}(x_0, \xi_0) \) satisfies (6.1.6) with a \( + \) sign. Since that is an open condition, it is satisfied by \( e^{TH_p}(x, \xi) \) for all \((x, \xi)\) in some neighbourhood \( U \) of \((x_0, \xi_0)\). Now, by part 1 of Lemma 6.2, \( U \cap \Gamma^- = \emptyset \). It follows that \( T^*\mathbb{R}^n \setminus \Gamma^- \) is open and thus \( \Gamma^- \) is closed.
2. To see \((6.1.7)\), take \((x, \xi) \in \{r \geq r_0\} \) with \(\xi \neq 0\). If \(H_{\rho_1}r(x, \xi) \geq 0\), then by part 1 of Lemma 6.2, \((x, \xi) \notin \Gamma^–\). If \(H_{\rho_1}r(x, \xi) \leq 0\), then similarly \((x, \xi) \notin \Gamma^+\). In either case \((x, \xi) \notin K\).

3. Since \(K\) is an intersection of two closed sets, it is closed. Combining this fact with \((6.1.7)\), we see that \(K_j = K \cap p^{-1}(J)\) is bounded, hence compact, for each compact \(J \subset (0, \infty)\).

4. To show the last statement we argue by contradiction, assuming that there exists a sequence \(E_j \to E, E_j \in [E_1, E_2],\) such that \(K_{E_j} \neq \emptyset\). Take \((x_j, \xi_j) \in K_{E_j}\). Since \((x_j, \xi_j)\) lie in the compact set \(K_{[E_1, E_2]}\), we may pass to a subsequence to make \((x_j, \xi_j)\) converge to some point \((x_\infty, \xi_\infty)\). Since \(K\) is closed, \((x_\infty, \xi_\infty) \in K\), moreover \(p(x_\infty, \xi_\infty) = E\). Thus \((x_\infty, \xi_\infty) \in K_E\), a contradiction.

**Proposition 6.4.** Let \((x, \xi) \in \Gamma^+_E\) for some \(E \in \mathbb{R}\) and \((x(t), \xi(t)) = e^{tH_\rho(x, \xi)}\). Then

\[
(6.1.8) \quad (x(t), \xi(t)) \to K_E \quad \text{as} \quad t \to \mp \infty
\]

in the sense that the distance from \((x(t), \xi(t))\) to \(K_E\) converges to zero. In particular, if \(K_E = \emptyset\), then \(\Gamma^+_E = \emptyset\) as well.

**Proof.** 1. Again, we consider only the case of \((x, \xi) \in \Gamma^-_E\). We may assume that \(E > 0\), since otherwise \(\Gamma^-_E = K_E\).

We first show that \((x(t), \xi(t))_{t \geq 0}\) is precompact. More precisely, with \(\rho(t) := r(x(t))\),

\[
(6.1.9) \quad (x(t), \xi(t)) \in p^{-1}(E) \cap \{r \leq \max(r_0, \rho(t))\}, \quad t \geq 0.
\]

Indeed, if \((6.1.9)\) does not hold, then there exists \(T > 0\) such that \(\rho(t) > r_0\) and \(\rho(t) > \rho(0)\). Let \(t_0 \in [0, T]\) be the point where the function \(r\) achieves its maximal value, then \(t_0 > 0, \rho(t_0) > r_0,\) and

\[H_{\rho_1}r(x(t_0), \xi(t_0)) = \dot{\rho}(t_0) = 0.\]

By part 1 of Lemma 6.2, we have \((x(t), \xi(t)) \notin \Gamma^-\), thus \((x, \xi) \notin \Gamma^-\), a contradiction.

2. We now prove \((6.1.8)\) by contradiction. Indeed, if it does not hold, then there exists a sequence \(t_j \to +\infty\) and a neighbourhood \(U\) of \(K_E\) such that \((x(t_j), \xi(t_j)) \notin U\) for all \(j\). By \((6.1.9)\), we may pass to a subsequence to make

\[
(x(t_j), \xi(t_j)) \to (x_\infty, \xi_\infty) \quad \text{for some} \quad (x_\infty, \xi_\infty) \notin K_E = \Gamma^+_E \cap \Gamma^-_E.
\]

Since \(\Gamma^-_E\) is closed, we have \((x_\infty, \xi_\infty) \in \Gamma^-_E\). It follows that \((x_\infty, \xi_\infty) \notin \Gamma^+_E\). Then \(r(e^{tH_\rho(x_\infty, \xi_\infty)}) \to \infty\) as \(t \to -\infty\), thus there exists \(T > 0\) such that

\[r(e^{-TH_\rho(x_\infty, \xi_\infty)}) > \max(r_0, \rho(0)).\]
Since \((x(t_j - T), \xi(t_j - T)) = e^{-TH_p}(x(t_j), \xi(t_j))\) converges to \(e^{-TH_p}(x_\infty, \xi_\infty)\), we have for \(j\) large enough,

\[\rho(t_j - T) > \max(r_0, \rho(0)).\]

But since \(t_j \to +\infty\) this contradicts (6.1.9). \(\square\)

The next result has the flavor of Poincaré's recurrence theorem and to formulate we recall the definition of the Liouville measure on a non-degenerate energy surface \(p^{-1}(E)\). The cotangent bundle \(T^*\mathbb{R}^n\) is equipped with a canonical measure obtained from the symplectic form:

\[m = \omega^n/n!, \quad \omega := d\sigma, \quad \sigma := \xi dx.\]

If \(dp|_{p^{-1}(E)} \neq 0\) then \(p^{-1}(E)\) is a smooth hypersurface in \(T^*\mathbb{R}^n\) and a natural measure on \(p^{-1}(E)\) is given by

\[dp \wedge \mathcal{L}_E := m.\]

Since \(p\) and \(m\) are preserved by the flow of \(H_p\), so is \(\mathcal{L}_E\).

**Proposition 6.5.** Denote by \(m\) the canonical measure on \(T^*\mathbb{R}^n\). Then

(6.1.10) \[m(\Gamma^\pm \setminus K) = 0.\]

Also, if \(E\) satisfies \(dp|_{p^{-1}(E)} \neq 0\) and \(\mathcal{L}_E\) is the Liouville measure on \(p^{-1}(E)\), then

(6.1.11) \[\mathcal{L}_E(\Gamma^+_E \setminus K_E) = 0.\]

**Proof.** We show (6.1.11), with (6.1.10) proved similarly. By (6.1.9), we have

\[e^{tH_p}(\Gamma^-_E \cap \{r \leq r_0\}) \subset \Gamma^-_E \cap \{r \leq r_0\}, \quad t \geq 0.\]

For \(j \in \mathbb{Z}\), put \(A_j := e^{jH_p}(\Gamma^-_E \cap \{r \leq r_0\})\). Then,

\[A_{j+1} \subset A_j, \quad \bigcup_{j \in \mathbb{Z}} A_j = \Gamma^-_E, \quad \bigcap_{j \in \mathbb{Z}} A_j = K_E,\]

where the last statement follows from (6.1.8).

Since \(A_j\) is compact, \(\mathcal{L}_E(A_j) < \infty\). It follows that

\[\mathcal{L}_E(K_E) = \lim_{j \to +\infty} \mathcal{L}_E(A_j), \quad \mathcal{L}_E(\Gamma^-_E) = \lim_{j \to -\infty} \mathcal{L}_E(A_j).\]

However, \(\mathcal{L}_E\) is invariant under the flow \(\exp(tH_p)\), thus \(\mathcal{L}_E(A_j) = \mathcal{L}_E(A_0)\) for all \(j\). Therefore

\[\mathcal{L}_E(\Gamma^-_E) = \mathcal{L}_E(K_E) = \mathcal{L}_E(A_0),\]

and it follows that \(\mathcal{L}_E(\Gamma^-_E \setminus K_E) = 0\). Similarly \(\mathcal{L}_E(\Gamma^+_E \setminus K_E) = 0\). \(\square\)
6.1.2. The asymptotically hyperbolic case. We now consider the second case discussed in the introduction, with the Hamiltonian given by (6.0.5). The classical Hamiltonian is

\[ p(x, \xi) = |\xi|^2_g, \quad (x, \xi) \in T^* M \]

and \((M, g)\) in an asymptotically hyperbolic manifold in the sense of Definition 5.2.

The replacement of the function \(r\) defined by (6.1.5) is

\[ r := y_1^{-1}, \]

where \(y_1 : \overline{M} \to [0, \infty)\) is any fixed canonical boundary defining function in the sense of Definition 5.3. Note that the level sets \(r^{-1}([0, C])\) are compact for each \(C\). As in the Euclidean case, we lift \(r\) to a function on \(T^* M\). We use Definition 6.1 to introduce the incoming/outgoing tails \(\Gamma^\pm\) and the trapped set \(K\), and (6.1.4) to define the sets \(\Gamma^+_J, K_J\); all of these are subsets of \(T^* M\).

As before, we have \(K((-\infty, 0]) = p^{-1}((-\infty, 0])\).

The behavior of the flow \(\exp(tH_p)\) near the infinity of \(M\) is less straightforward than in the Euclidean case. To establish the properties of \(\Gamma^\pm\) and \(K\), we use the following

**Lemma 6.6 (Convexity near infinity).** There exists \(r_0 > 0\) such that for each \((x, \xi) \in T^* M\),

\[ r(x) \geq r_0 \implies H^2_p r(x, \xi) \geq 2p(x, \xi). \]

**Proof.** Using Theorem 5.4, fix a canonical coordinate system on \(M\),

\[(y_1, y') \in (0, \varepsilon_1) \times \partial M.\]

Let \(\eta_1, \eta'\) be the momenta corresponding to \(y_1, y'\), then by (5.1.5)

\[ p(y_1, y', \eta_1, \eta') = y_1^2 (\eta_1^2 + |\eta'|^2 g_1(y_1, y')). \]

We compute

\[ H_p r = -y_1^{-2} H_p y_1 = -2\eta_1, \]

\[ H^2_p r = 4y_1^{-1} p + 2y_1^2 \langle \eta', \eta' \rangle_{\partial_{y_1} g_1(y_1, y')} \]

Since \(g_1(y_1, y')\) is smooth up to \(y_1 = 0\), there exists a constant \(C\) such that for small enough \(y_1 \in (0, \varepsilon_1)\),

\[ y_1^2 \langle \eta', \eta' \rangle_{\partial_{y_1} g_1(y_1, y')} \leq C p. \]

Therefore,

\[ H^2_p r \geq (4r - 2C) p \]

and (6.1.13) follows for \(r \geq r_0\) and \(r_0\) large enough. \(\square\)

We now show the analogue of Lemma 6.2 in the asymptotically hyperbolic setting:
LEMMA 6.7. Let \((x, \xi) \in T^* M\) and \((x(t), \xi(t)) = \exp(tH_\rho)(x, \xi)\). Then:

1. If \(r(x) \geq r_0, \xi \neq 0, \text{ and } H_\rho r(x, \xi) \geq 0\), then \((x, \xi) \notin \Gamma^-\) and

\[
(6.1.14) \quad r(x(t)) > r_0, \quad H_\rho r(x(t), \xi(t)) > 0 \quad \text{for all } t > 0.
\]

2. If \((x, \xi) \notin \Gamma^+\), then

\[
(6.1.15) \quad r(x(t)) > r_0, \quad H_\rho r(x(t), \xi(t)) > 0 \quad \text{for large enough } t > 0.
\]

Same is true for propagation in the negative time direction, replacing \(\Gamma^-\) by \(\Gamma^+\) and changing the sign of \(H_\rho r\).

Proof. 1. By rescaling we may assume that \(p(x(t), \xi(t)) = 1\). We put \(\rho(t) := r(x(t))\) and note that

\[
\dot{\rho}(t) = H_\rho r(x(t), \xi(t)), \quad \ddot{\rho}(t) = H_\rho^2 r(x(t), \xi(t)).
\]

Assume first that \(r(x) \geq r_0\) and \(H_\rho r(x, \xi) \geq 0\). In our new notation that means \(\rho(0) \geq r_0\) and \(\dot{\rho}(0) \geq 0\). By (6.1.13), we have

\[
\ddot{\rho}(t) \geq 2 \quad \text{for all } t \text{ such that } \rho(t) \geq r_0.
\]

If the maximal value of \(\rho\) on an interval \([0, t]\) is taken at \(0 < t_0 < t\) then \(\rho(t_0) \geq r_0, \dot{\rho}(t_0) = 0, \ddot{\rho}(t_0) \geq 2\), a contradiction. Since \(\rho\) is non-decreasing at 0 the maximal value is taken at \(t\), showing that \(\rho(t) \geq r_0\) for all \(t \geq 0\). Then \(\ddot{\rho}(t) \geq 2\) for all \(t \geq 0\) and

\[
\dot{\rho}(t) \geq 2t, \quad \rho(t) \geq r_0 + t^2 \quad \text{for } t \geq 0,
\]

which implies (6.1.14). Moreover, \(\rho(t) \to \infty\) as \(t \to \infty\), thus \((x, \xi) \notin \Gamma^-\).

2. Assume now that \((x, \xi) \notin \Gamma^-\). Then \(\rho(t) \to \infty\) as \(t \to +\infty\), thus there exists \(T_0 > 0\) such that

\[
\rho(T_0) > r_0, \quad \dot{\rho}(T_0) > 0.
\]

Arguing as in Step 1 of the proof we see that same is true for all \(t \geq T_0\), giving (6.1.15).

With Lemma 6.7 in place Propositions 6.3–6.5 apply to asymptotically hyperbolic manifolds. Indeed, their proofs only used Lemma 6.2, which in the asymptotically hyperbolic case should be replaced by Lemma 6.7. In fact, these statements hold under more general assumptions near infinity – see Exercises 6.1–6.8.
6.2. RESONANCES IN STRIPS

In most of this section, we study the properties of the scattering resolvent of an operator $P$ of the form (6.0.4) or (6.0.5) where the spectral parameter $h$ lies in an $h$-sized strip

$$\text{Re} z \in [\alpha, \beta] \subset (0, \infty), \quad \text{Im} z \in [-C_0 h, C_0 h],$$

$\alpha, \beta, C_0$ are fixed and the semiclassical parameter $h$ tends to zero. We show that in the nontrapping case, the strip (6.2.1) has no resonances for $h$ small enough – see Theorems 6.10 and 6.13. We also prove existence of semiclassical defect measures associated to sequences of resonant states and show that they are supported on the outgoing tail – see Theorems 6.11 and 6.14. In fact, one way to establish existence of resonance free strips is by using these defect measures and that will be exploited in a trapping situation in §6.3.

We assume that $P(h)$ is a second order semiclassical differential operator on a manifold $M$ of the form (6.0.4) or (6.0.5). The $L^2$ resolvent

$$R(z, h) = (P(h) - z)^{-1} : L^2(M) \to H^2(M), \quad \text{Im} z > 0$$

admits a meromorphic continuation to the region (6.2.1),

$$R(z, h) : L^2_{\text{comp}}(M) \to H^2_{\text{loc}}(M), \quad z \in [\alpha, \beta] + i[-C_0 h, C_0 h].$$

Indeed, for the case (6.0.4) this follows from Theorem 4.4 since the operator $P(h)$ satisfies the black box assumptions of §4.1, see Example 1 preceding Lemma 4.12. For the case (6.0.5) this follows from Theorem 5.32.

Fix $h$ and assume that $z$ is a resonance, that is a pole of $R(z, h)$, in the region (6.2.1). Then

$$R(w, h) = \sum_{j=1}^{J} \frac{B_j}{(w - z)^j} + B(w, z),$$

where $w \mapsto B(w, z)$ is holomorphic near $z$.

The space of resonant states at $z$ is the range of the operator $B_J$ – see Theorems 4.7 and 4.9. It is a finite-dimensional subspace of $C^\infty(M)$ and each resonant state $u$ solves the equation

$$(P(h) - z)u = 0.$$  

6.2.1. The Euclidean case. We start with the case (6.0.4), where $P = P(h)$ is a Schrödinger operator on a metric perturbation of $\mathbb{R}^n$, with $n$ odd.

To study the resolvent at high energies, we will use the method of complex scaling as presented in §4.5. More precisely, fix a constant $r_1 > r_0$, with
6.2. RESONANCES IN STRIPS

$r_0$ given by (6.1.1), take a scaling angle $\theta \in (0, \pi/2)$, and define the contour $\Gamma_\theta \subset \mathbb{C}^n$ by

$$\Gamma_\theta = f_\theta(\mathbb{R}^n); \quad f_\theta : \mathbb{R}^n \to \mathbb{C}^n, \quad f_\theta(x) = x + i\partial_x F_\theta(x),$$

where $F_\theta : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function satisfying

$$(6.2.4) \quad F_\theta(x) = 0 \text{ near } B(0, r_1); \quad F_\theta(x) = \tan \frac{\theta}{2} |x|^2, \quad x \geq 2r_1.$$  

Consider the complex scaled operator $P_\theta = P_\theta(h)$, which is a semiclassical differential operator on $\Gamma_\theta$ defined as follows:

- on $f_\theta(B(0, r_1)) = B(0, r_1) \subset \mathbb{R}^n$, $P_\theta(h) = P(h) = -h^2 \Delta_\theta + V$;
- on $f_\theta(\mathbb{R}^n \setminus B(0, r_0))$, $P_\theta(h) = -h^2 \Delta_{\Gamma_\theta}$, with $\Delta_{\Gamma_\theta}$ defined by (4.5.7).

The two definitions agree in the transition region by (6.1.1). We use the map $f_\theta$ to identify $\Gamma_\theta$ with $\mathbb{R}^n$; then

$$(6.2.5) \quad P_\theta - z : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

By Theorem 4.36, (6.2.5) is a Fredholm operator for $z$ in the region $C \setminus e^{-2i\theta}[0, \infty)$. We henceforth assume that $h$ is small enough so that this region contains (6.2.1). Moreover, (6.2.5) has a meromorphic inverse

$$(P_\theta - z)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n).$$

By Theorem 4.38, resonances in the region (6.2.1) coincide with the poles of the inverse of (6.2.5).

We record here some basic facts about the semiclassical principal symbol of $P_\theta$:

**Lemma 6.8.** Suppose that $p_\theta = \sigma_h(P_\theta)$ is the semiclassical principal symbol of $P_\theta$. Then

1. $\text{Im } p_\theta(x, \xi) \leq 0$ everywhere.
2. For $E \in \mathbb{R}$,

$$(6.2.6) \quad \{\langle \xi \rangle^{-2}(p_\theta - E) = 0\} \subset p^{-1}(E).$$

Moreover, if $0 < \alpha < \beta$; there exists $\delta > 0$ such that for $E \in [\alpha, \beta]$ and $|x| \geq 2r_1$ (see (6.2.4))

$$(6.2.7) \quad |p_\theta(x, \xi) - E| \geq \delta |\xi|^2.$$

3. If

$$(6.2.8) \quad \varphi_t := \exp(t\langle \xi \rangle^{-1}H_{\text{Re } p_\theta}) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n,$$

then for each $(x, \xi)$ and $t_0 \leq t_1$,

$$(6.2.9) \quad \varphi_t(x, \xi) \in \{\langle \xi \rangle^{-2} \text{Im } p_\theta = 0\} \text{ for all } t \in [t_0, t_1] \quad \implies \quad \varphi_t(x, \xi) = \exp(t\langle \xi \rangle^{-1}H_\theta)(x, \xi) \text{ for all } t \in [t_0, t_1].$$
Proof. 1. For \( r_1 > r_0 \) introduced before (6.2.4)

\[
p_\theta(x, \xi) = \begin{cases} 
(1 + i \nabla^2 F(x))^{-2} \xi, & |x| \geq r_0; \\
p(x, \xi) = |\xi|_g^2 + V(x), & |x| \leq r_1.
\end{cases}
\]

Since \( F \) is convex, \( \nabla^2 F(x) \) defines a nonnegative real quadratic form. Diagonalizing it, we derive the following properties of \( p_\theta \):

\[
\text{Im} \ p_\theta(x, \xi) \leq 0 \text{ everywhere}; \tag{6.2.11}
\]

\[
\text{Im} \ p_\theta(x, \xi) = 0 \implies \nabla^2 F(x) \xi = 0. \tag{6.2.12}
\]

This gives part (1) of the lemma.

Since \( \nabla^2 F(x) \) is nonnegative, for each \( (x, \xi) \) and \( j \) we have

\[
\nabla^2 F(x) \xi = 0 \implies \langle \partial_{x_j} \nabla^2 F(x) \xi, \xi \rangle = 0.
\]

This implies that for each \( (x, \xi) \) such that \( \text{Im} \ p_\theta(x, \xi) = 0 \),

\[
p_\theta(x, \xi) = p(x, \xi), \quad \nabla p_\theta(x, \xi) = \nabla p(x, \xi). \tag{6.2.13}
\]

2. From (6.2.13) we obtain (6.2.6). Moreover, for \( \alpha > 0 \), it follows from (6.2.4) and (6.2.10) that there exists \( \delta > 0 \) such that

\[
|p_\theta(x, \xi) - \text{Re} \ z| \geq \frac{1}{2} \left( |\text{Re} \ p_\theta(x, \xi) - \text{Re} \ z| + |\text{Im} \ p_\theta(x, \xi)| \right)
\geq \delta |\xi|^2 \quad \text{for } |x| \geq 2r_1.
\]

That gives (6.2.7).

3. We next relate the rescaled Hamiltonian flow associated to \( \text{Re} \ p_\theta \) given by (6.2.8) to the corresponding objects for the unscaled symbol \( p \). By (6.2.13), \( \varphi_t \) agrees with the flow generated by \( p \) as long as \( \text{Im} \ p_\theta = 0 \): that gives (6.2.9). \( \Box \)

We now show that at high frequency, \( L^2 \) solutions to the equation \( (P_\theta - z) u = f \) are controlled away from the outgoing tail. We also show that a positive fraction of their mass is localized near the trapped set. The resulting statement essentially reduces the analysis of resonance free regions inside (6.2.1) to a neighbourhood of the trapped set. We use the notion of semiclassical pseudodifferential operators, their wavefront sets \( \text{WF}_h \), and their elliptic sets \( \text{ell}_h \), see §§E.1,E.2.

PROPOSITION 6.9 (Semiclassical outgoing estimates). Fix \( 0 < \alpha \leq \beta \) and \( C_0 > 0 \). Then the following estimates hold for all \( z \) satisfying (6.2.1) and \( u \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( f := (P_\theta - z) u \in L^2(\mathbb{R}^n) \), with constants independent of \( u, z, h \):

1. Let \( A \in \Psi^0_h(\mathbb{R}^n) \) be compactly supported and \( \text{WF}_h(A) \cap \Gamma^+_{[\alpha, \beta]} = \emptyset \). Then

\[
\|Au\|_{L^2} \leq C h^{-1} \|f\|_{L^2} + O(h^\infty) \|u\|_{L^2}. \tag{6.2.14}
\]
2. Let \( B \in \Psi^0_h(\mathbb{R}^n) \) be compactly supported and \( K_{[\alpha, \beta]} \subset \text{ell}_h(B) \). Then for \( h \) small enough,

\[
\|u\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|f\|_{L^2}.
\]

Proof. 1. We start by remarking that \( \text{Im} z = O(h) \) and (6.2.6) imply that

\[
\sigma(P_\theta - z) = p_\theta - \text{Re} z, \quad \text{Re} z \in [\alpha, \beta],
\]

\[
\{|\xi|^{-2}(p_\theta - z) = 0\} \subset p^{-1}([\alpha, \beta]).
\]

To prove (6.2.14) we use the following fact: for each \((x, \xi) \in WF_h(A)\), there exists \( T \geq 0 \) such that (with \( \phi \) defined in (6.2.8))

\[
\varphi_{-T}(x, \xi) \in \text{ell}_h(P_\theta - z) = \{|\xi|^{-2}(p_\theta - \text{Re} z) \neq 0\}.
\]

Indeed, we have \((x, \xi) \notin \Gamma_{[\alpha, \beta]}^{+}\) if \((x, \xi) \notin p^{-1}([\alpha, \beta])\), then \((x, \xi) \in \text{ell}_h(P_\theta - z)\) by (6.2.6). Otherwise, \((x, \xi) \notin \Gamma^{+}\), therefore there exists \( T_1 \geq 0 \) such that \( e^{-T_1(|\xi|^{-1}H_p(x, \xi)} \in \{r \geq 2r_1\}\). By (6.2.7) and (6.2.9), we see that (6.2.16) holds for some \( T \in [0, T_1]\).

The estimate (6.2.14) follows from applying propagation of singularities, Theorem E.49, to the operator \( P = P_\theta - z\), with the controlling operator (denoted by \( B \) in Theorem E.49) equal to \( P_\theta - z \) as well. Here the sign condition (E.5.10) is satisfied by Lemma 6.8 and the control condition (E.5.11) is satisfied by (6.2.16).

2. We now prove (6.2.15). Fix a cutoff function

\[
\chi \in C^\infty_c(\mathbb{R}^n), \quad \chi = 1 \text{ near } \{r \leq 2r_1\}.
\]

We first show the estimate

\[
\|\chi u\|_{L^2} \leq C\|Bu\|_{L^2} + Ch^{-1}\|f\|_{L^2} + O(h^\infty)\|u\|_{L^2}.
\]

We use the following fact: there exists \( T \geq 0 \) such that for each \((x, \xi) \in T^{n+1}\), \( x \in \text{supp} \chi \),

\[
(6.2.18) \quad \varphi_{-T}(x, \xi) \in \text{ell}_h(P_\theta - z - iB^*B) = \text{ell}_h(P_\theta - z) \cup \text{ell}_h(B).
\]

Indeed, if \((x, \xi) \notin \Gamma_{[\alpha, \beta]}^{+}\), this follows from (6.2.16). If \((x, \xi) \in \Gamma_{[\alpha, \beta]}^{+}\), then by Lemma 6.4, and since \( K_{[\alpha, \beta]} \subset \text{ell}_h(B) \), there exists \( T_1 \geq 0 \) such that \( e^{-T_1(|\xi|^{-1}H_p(x, \xi)} \in \text{ell}_h(B) \). By (6.2.9), we see that (6.2.18) holds for some \( T \in [0, T_1]\).

To show (6.2.17), it remains to use propagation of singularities, Theorem E.49, with the controlling operator \( P_\theta - z - iB^*B \), together with the inequality \((B^* \in \Psi^0_h) \) is uniformly bounded on \( L^2\)

\[
\|(P_\theta - z - iB^*B)u\|_{L^2} \leq \|f\|_{L^2} + C\|Bu\|_{L^2}.
\]

In fact, we replace \( A\) with \( \chi \), \( B\) with \( P_\theta - z - iB^*B \in \Psi^0_h(\mathbb{R}^n) \) and \( B_1 = I\) in (E.5.12) to obtain (6.2.17). (See also Remark 1 after Theorem E.49.)
3. To conclude the proof of \((6.2.15)\) it remains to show the estimate
\[
\|(1 - \chi)u\|_{L^2} \leq C\|f\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.
\]
The operator \(P_0 - z\) can be written in terms of the standard quantization \((E.1.12)\) as follows:
\[
P_\theta - z = \text{Op}_h(\tilde{p}), \quad \tilde{p} = p_\theta - \text{Re} \, z + \mathcal{O}(h)_{S^1_0} \in S_1^2,
\]
where the classes \(S^k_{1,0} = S^k_{1,0}(T^*\mathbb{R}^n)\), defined in \((E.1.10)\), require uniform control on derivatives as \(x \to \infty\).

Using \((6.2.7)\) \((1 - \chi)\) is supported in the region \(|x| > 2r_1\) and the elliptic parametrix construction from the proof of Proposition \(E.31\) for the \(S^k_{1,0}\) calculus reviewed in §\(E.1.4\), we construct an operator
\[
Q = \text{Op}_h(S^{-2}_{1,0}), \quad 1 - \chi = Q(P_0 - z) + \text{Op}_h(S^{-\infty}_{1,0}).
\]
By Proposition \(E.18\), \(\|Q\|_{L^2 \to L^2} \leq C\), and the remainder in \((6.2.20)\) is \(\mathcal{O}(h^\infty)_{L^2 \to L^2}\). Applying \((6.2.20)\) to \(u \in L^2\), we get \((6.2.19)\).

Combining \((6.2.17)\) and \((6.2.19)\), and taking \(h\) small enough, we obtain \((6.2.15)\).

An immediate application of Proposition 6.9 is a resonance free strip of size \(C_0 h\) for arbitrary \(C_0\) when there is no trapping at the energies in \([\alpha, \beta]\):

**THEOREM 6.10 (Nontrapping estimate in strips).** Suppose that \(P\) is given by \((6.0.4)\) and \(P_0\) is its complex scaled version \((6.2.5)\). Fix \(0 < \alpha \leq \beta\), \(C_0 > 0\), \(\chi \in C_c^\infty(\mathbb{R}^n)\), and assume that
\[
K_{[\alpha, \beta]} = \emptyset.
\]
Then the following estimates hold for \(h\) small enough, all \(s \geq 0\), and all \(z \in [\alpha, \beta] + i([-C_0 h, C_0 h])\):
\[
\|(P_0 - z)^{-1}\|_{H^s_h \to H^{s+2}_h} \leq C h^{-1},
\]
\[
\|\chi R(z, h)\chi\|_{H^s_h \to H^{s+2}_h} \leq C h^{-1}.
\]
In particular we have a resonance free region
\[
\text{Res}(P) \cap ([\alpha, \beta] + i([-C_0 h, C_0 h])) = \emptyset.
\]

**Proof.** 1. Since \(C^\infty_c(\mathbb{R}^n)\) is dense in \(H^s(\mathbb{R}^n)\), the estimate \((6.2.21)\) follows from showing that for each
\[
f \in C^\infty_c(\mathbb{R}^n), \quad u := (P_0 - z)^{-1} f \in H^2(\mathbb{R}^n),
\]
we have
\[
\|u\|_{H^{s+2}_h} \leq C h^{-1} \|f\|_{H^s_h}.
\]
Next, (6.2.22) follows immediately from (6.2.21) and Theorem 4.37, if we choose the constant $r_1$ in the construction of the complex scaling contour $\Gamma_\theta$ such that $\text{supp} \chi \subset B(0, r_1)$.

2. The operator $P_\theta - z$ is elliptic near fiber ($\xi$) infinity, that is, there exists a constant $C_1 > 0$ such that

$$|p_\theta(x, \xi) - \text{Re} z| \geq \langle \xi \rangle^2 / C_1$$

for $|\xi| \geq C_1$.

Take $\chi \in C^\infty_c(\mathbb{R}^n)$ with $\chi = 1$ near $B(0, C_1)$. Using an elliptic parametrix as in Step 5 of the proof of Proposition 6.9, we get

$$\| (1 - \chi(hD_x))u \|_{H^{k+2}_h} \leq C\|f\|_{H^k_h} + O(h^\infty)\|u\|_{L^2}.$$ 

On the other hand,

$$\| \chi(hD_x)u \|_{H^{k+2}_h} \leq C\|u\|_{L^2}.$$ 

Adding these estimates, we get

$$\|u\|_{H^{k+2}_h} \leq C\|f\|_{H^k_h} + C\|u\|_{L^2}.$$ 

(6.2.25)

3. We will now use the fact that $K_{[\alpha, \beta]} = \emptyset$. This means that the estimate (6.2.15) holds with $B \equiv 0$, giving

$$\|u\|_{L^2} \leq Ch^{-1}\|f\|_{L^2}.$$ 

Combining this with (6.2.25), we get (6.2.24), finishing the proof. \hfill $\Box$

6.2.2. Semiclassical defect measures. One way to understand families of functions depending on $h \to 0$ is by associating them to measures on the cotangent bundle. Properties of these measures capture classical properties in a rougher way than propagation estimates – see §E.4.

In this section we study semiclassical measures associated to resonant states. These measures can be used to obtain resonance free regions as will be shown in §6.3 for the case of normally hyperbolic trapping.

We keep working in the Euclidean setting (6.0.4) and use the complex scaled operator $P_\theta(h)$, see §6.2.1. We endow $\mathbb{R}^n$ with the volume form induced by the metric $g$, so that the operator $P$ is symmetric.

Assume that we have sequences $h_j \to 0$, $z_j$ satisfying (6.2.1), and

$$u_j \in H^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$$

is a family of $L^2$ normalized $o(h)$-quasimodes to the operator $P_\theta - z$:

(6.2.26) \quad $\|u_j\|_{L^2} = 1,$

(6.2.27) \quad $\|(P_\theta(h_j) - z_j)u_j\|_{L^2} = o(h_j).$

In particular, one can take $z_j$ to be a sequence of resonances of $P$ and let $u_j$ be elements of the kernel of $P_\theta(h_j) - z_j$; on $\{r \leq r_1\}$, $u_j$ coincide with resonant states of $R(z, h)$. 


By Theorem E.44, passing to a subsequence we may assume that \( u_j \) converges to some nonnegative Borel measure \( \mu \) on \( T^*\mathbb{R}^n \) in the following sense:

\[
(6.2.28) \quad \langle A(h_j)u_j, u_j \rangle \to \int_{T^*\mathbb{R}^n} \sigma_h(A) \, d\mu \quad \text{as } j \to \infty, \quad A \in \Psi^\text{comp}_h(\mathbb{R}^n).
\]

By \( (6.2.1) \), we may moreover pass to a subsequence satisfying

\[
(6.2.29) \quad \Re z_j \to E \in [\alpha, \beta], \quad \frac{\Im z_j}{h_j} \to \nu \in [-C_0, C_0] \quad \text{as } j \to \infty.
\]

Henceforth we will suppress the dependence of \( h, z, u \) on \( j \) in notation.

The main result of this section is the following

**THEOREM 6.11.** (Measures associated to resonant states) Under the assumptions \( (6.2.26) - (6.2.29) \), the measure \( \mu \) has the following properties:

1. \( \mu \) is supported on \( \Gamma^+ + E \), that is \( \mu(T^*\mathbb{R}^n \setminus \Gamma^+_E) = 0 \).

2. For each neighbourhood \( U \) of \( K_E \), we have \( \mu(U) > 0 \).

3. For each \( U \subset T^*\mathbb{R}^n \cap \{ r \leq r_1 \} \) and each \( t \geq 0 \),

\[
(6.2.30) \quad \mu(e^{-tH_p}(U)) = e^{2\nu t} \mu(U).
\]

**REMARKS.**

1. The condition \( (6.2.30) \) can also be written in the form

\[
\mathcal{L}_{H_p}\mu = 2\nu\mu \quad \text{on } \{ r \leq r_1 \}
\]

where \( \mathcal{L}_{H_p} \) denotes the Lie derivative, \( \frac{d}{dt}\big|_{t=0}(e^{tH_p})^*\mu \).

2. By \( (6.1.9) \) (strictly speaking, its analog for \( \Gamma^+ \)), we have

\[
(6.2.31) \quad e^{-tH_p} (\Gamma^+ \cap \{ r \leq r_1 \}) \subset \Gamma^+ \cap \{ r \leq r_1 \}, \quad t \geq 0,
\]

therefore \( (6.2.30) \) implies that \( \nu \leq 0 \). This is consistent with the fact that there are resonances in the upper half \( z \)-plane.

3. Theorem 6.11 implies the resonance free strip of Theorem 6.10; indeed, \( K_{[\alpha, \beta]} = \emptyset \) implies \( \Gamma^+_E = \emptyset \) by Lemma 6.4 and thus parts 1 and 2 of Theorem 6.10 contradict each other. Of course, both Theorem 6.10 and Theorem 6.11 rely on Proposition 6.9, where most of the hard work is done.

**Proof.** 1. By Theorem E.45 applied to the operator \( P_0 - E \), we have

\[
\mu(T^*\mathbb{R}^n \setminus p^{-1}(E)) = 0.
\]

Moreover, for each \( A \in \Psi^\text{comp}_h(T^*\mathbb{R}^n) \) with \( \text{WF}_h(A) \cap \Gamma^+ = \emptyset \), we have by \( (6.2.14), (6.2.26), \) and \( (6.2.27) \)

\[
\|Au\|_{L^2} \leq Ch^{-1}\| (P_0 - z) u \|_{L^2} + O(h^\infty) \to 0 \quad \text{as } j \to \infty.
\]
It follows from (6.2.28) that for each \( a \in C_c^\infty(T^*\mathbb{R}^n) \),
\[
\text{supp } a \cap \Gamma^+ = \emptyset \quad \implies \quad \int_{T^*\mathbb{R}^n} a \, d\mu = 0.
\]
Therefore, \( \mu(T^*M \setminus \Gamma^+) = 0 \), finishing the proof of the first claim.

2. Let \( U \) be a neighbourhood of \( K_E \) and take \( B \in \Psi_h^{\text{comp}}(\mathbb{R}^n) \) such that \( K_E \subset \text{ell}_h(B) \). Then by (6.2.14), (6.2.26), and (6.2.27), we have
\[
\|Bu\|_{L^2} \geq C^{-1}\|u\|_{L^2} - h^{-1}\|(P_\theta - z)u\|_{L^2} \geq \frac{1}{2C} \quad \text{for large } j.
\]
Applying (6.2.28) to the expression \( \|Bu\|_{L^2}^2 = \langle Bu, u \rangle \), we see that
\[
\int_{T^*\mathbb{R}^n} |\sigma_h(B)|^2 \, d\mu > 0,
\]
which implies that \( \mu(U) > 0 \), proving the second claim.

3. To prove the last claim, take a cutoff function
\[
\chi \in C^\infty_c(\mathbb{R}^n), \quad \chi = 1 \quad \text{near } \{ r \leq r_1 \}, \quad \text{supp } \chi \cap \text{supp } F_\theta = \emptyset,
\]
where \( F_\theta \) is the function defining the complex scaling contour, see (6.2.4).

By (6.2.27) we have
\[
\|\chi(P_\theta - z)u\|_{L^2} = o(h).
\]
By (6.2.10), we have
\[
\sigma_h(\chi(P_\theta - z)) = \chi p.
\]
By Theorem E.46 with \( P := \chi(P_\theta - \text{Re } z - i\nu h) \) (whose proof still applies despite the fact that \( \text{Re } z \) depends on \( h \), since it does not influence the right-hand side of (E.4.9)), we get for each \( a \in C_c^\infty(T^*\mathbb{R}^n) \) with \( \text{supp } a \subset \{ r \leq r_1 \} \),
\[
\int_{T^*\mathbb{R}^n} H_\rho a - 2\nu a \, d\mu = 0.
\]
Since \( \mu \) is supported on \( \Gamma^+_E \) and by (6.2.31), we see that
\[
\int_{T^*\mathbb{R}^n} a \circ e^{tH_\rho} \, d\mu = e^{2\nu t} \int_{T^*\mathbb{R}^n} a \, d\mu \quad \text{when } \text{supp } a \subset \{ r \leq r_1 \}, \ t \geq 0.
\]
This implies (6.2.30), finishing the proof. \( \square \)

6.2.3. The asymptotically hyperbolic case. We now consider the case of asymptotically hyperbolic Laplacian (6.0.5).

We fix a canonical boundary defining function \( y_1 \) on \( M \), put \( x_1 := y_1^2 \), and use the extended modified Laplacian introduced in §5.3, which is a differential operator on the even extension \( \overline{X} \) of \( M \). Since that operator is denoted \( P(\lambda) \), we first fix some notation to avoid conflicts with Chapter 5.
Let $P_h(\omega)$ be the semiclassical version of the extended modified Laplacian introduced in (5.3.8). For $z$ in (6.2.1), put

$$\tilde{P}(z) = \tilde{P}(z; h) := P_h(\omega)$$

where $\omega \in \mathbb{C}$ is uniquely defined by

$$z = \omega^2 + \frac{(n - 1)^2}{4} h^2, \quad \text{Re} \omega > 0;$$

by (6.2.1), we have

$$\omega = \sqrt{\text{Re} z} + i \frac{\text{Im} z}{2\sqrt{\text{Re} z}} + O(h^2).$$

Then $\tilde{P}(z)$ is a second order semiclassical differential operator on $X$. It is related to the operator $P = -\hbar^2 \Delta_g$ from (6.0.5) by the following corollary of (5.3.9):

$$\tilde{P}(z) = x_1^{\frac{n}{2n-4}} (P - z) x_1^{\frac{n-1}{2n}}$$

on $M \subset X$.

Fix

$$s > C_0 + \frac{1}{2},$$

where $C_0$ is the constant from (6.2.1). By Theorem 5.30,

$$\tilde{P}(z) : \mathcal{X}^s \to \bar{H}^{s-1}(X), \quad z \in [\alpha, \beta] + i[-C_0 h, C_0 h]$$

is a holomorphic family of Fredholm operators, and it has a meromorphic inverse $\tilde{P}(z)^{-1}$. Here the space $\mathcal{X}^s \subset \bar{H}^s(X)$ is defined in (5.6.1) and $\bar{H}^s(X)$ denote Sobolev spaces on $X$ as a manifold with boundary.

The $L^2$ resolvent of $P$ can be continued from the upper plane meromorphically to (6.2.1) by Theorem 5.32. The resulting resonances are contained in the set of poles of $\tilde{P}(z)^{-1}$; in fact, by (5.6.17),

$$R(z, h) f = (x_1^{\frac{n-1}{2n}} \bar{P}(z)^{-1} x_1^{\frac{n+3}{2n}} f) \big|_M, \quad f \in C^\infty_c(M).$$

We denote by $\tilde{p}_z$ the semiclassical principal symbol of $\tilde{P}(z)$,

$$\tilde{p}_z(x, \xi) = \sigma_h(\tilde{P}(z)).$$

(Note: in §5.3, we use the letter $p$ for the principal symbol of $P_h(\omega)$; this would be in conflict with the use of letter $p$ in the present section, which follows (6.1.12).)

The analog of Proposition 6.9 in the asymptotically hyperbolic case is given by
PROPOSITION 6.12. Let \( j : T^*M \setminus 0 \to T^*M \) be the map defined in (5.4.29). Then the following estimates hold for all \( z \) satisfying (6.2.1) and \( u \in \dot{H}^s(X) \), \( f := \tilde{P}(z)u \in \dot{H}^{s-1}(X) \), with constants independent of \( u \), \( z \), \( h \in (0, 1) \):

1. Let \( A \in \Psi^0_h(M) \) be compactly supported and \( \text{WF}_h(A) \cap j(\Gamma^+_{[\alpha, \beta]}) = \emptyset \). Then

\[
\|Au\|_{\dot{H}^s_h(X)} \leq C h^{-1} \|f\|_{\dot{H}^{s-1}_h(X)} + O(h^\infty) \|u\|_{\dot{H}^s_h(X)}.
\]

2. Let \( B \in \Psi^0_h(X) \) be compactly supported and \( j(K_{[\alpha, \beta]}) \subset \text{ell}_h(B) \). Then for \( h \) small enough,

\[
\|u\|_{\dot{H}^s_h(X)} \leq C \|Bu\|_{\dot{H}^s_h(X)} + C h^{-1} \|f\|_{\dot{H}^{s-1}_h(X)}.
\]

REMARK. It is possible to make \( x_1 \) equal to 1, and thus \( j \) the identity map, on an arbitrary fixed compact subset of \( M \).

Proof. 1. For (6.2.35), we follow the proof of Lemma 5.23. Let \( A_{\pm} \in \Psi^0_h(X) \) be the operator constructed in Step 1 of that proof. We claim that for each \( z \in [\alpha, \beta] \) and \( (x, \xi) \in \text{WF}_h(A) \cap \{ \langle \xi \rangle^{-2} \tilde{p}_z = 0 \} \), there exists \( T \geq 0 \) such that

\[
\exp(-t \langle \xi \rangle^{-1} H_{\tilde{p}_z})(x, \xi) \in \text{ell}_h(A_{\pm}).
\]

Indeed, since \( \tilde{p}_z \) is elliptic at fiber infinity, \( \xi \) must be finite. Then by (5.4.30) we have \( (x, \xi) = j(\tilde{x}, \tilde{\xi}) \) for some \( (\tilde{x}, \tilde{\xi}) \in T^*M \), \( |\tilde{\xi}|^2_g = z \). Since \( \text{WF}_h(A) \cap j(\Gamma^+_{[\alpha, \beta]}) = \emptyset \), we have \( (\tilde{x}, \tilde{\xi}) \notin \Gamma^+ \).

Let \( \delta > 0 \) be the constant from Lemma 5.19. Since the geodesic \( e^{tH_p}(\tilde{x}, \tilde{\xi}) \) escapes as \( t \to -\infty \), there exists \( \tilde{T} \geq 0 \) such that

\[
e^{-tH_p}(\tilde{x}, \tilde{\xi}) \in \{ x_1 < \delta \} \quad \text{for all } t \geq \tilde{T}.
\]

By Lemma 5.20, there exists \( T \geq 0 \) such that

\[
\exp(-t \langle \xi \rangle^{-1} H_{\tilde{p}_z})(x, \xi) \in \{ x_1 < \delta \} \quad \text{for all } t \geq T.
\]

We have \( (x, \xi) \in \Sigma_{\pm} \) by (5.4.12). Therefore by Lemma 5.19,

\[
\exp(t \langle \xi \rangle^{-1} H_{\tilde{p}_z})(x, \xi) \to L_{\pm} \quad \text{as } t \to -\infty
\]

and (6.2.37) follows since \( L_{\pm} \subset \text{ell}_h(A_{\pm}) \).

Now, arguing as in Step 2 of the proof of Lemma 5.23, replacing \( A_{\pm} \) and \( A_0 \) with zero, we obtain (6.2.35).

2. For (6.2.36), we use the proof of (5.5.24) with \( A_0 := B \). For that we have to show that the proof of Lemma 5.23 still applies to arbitrary \( A \) and
A_0 := B. This is true as long as for each \((x, \xi) \in \Sigma_{\pm}\), one of the two following statements is true:

\[(6.2.38) \quad \exp(t(\xi)^{-1}H_{\tilde{p}_z})(x, \xi) \to L_{\pm} \quad \text{as } t \to \pm \infty;\]
\[(6.2.39) \quad \exp(-t(\xi)^{-1}H_{\tilde{p}_z})(x, \xi) \in \text{ell}_h(B) \quad \text{for some } t \geq 0.\]

When \((x, \xi) \in \Sigma_+\), (6.2.38) holds by Lemma 5.19 and (5.4.12). Now, assume that \((x, \xi) \in \Sigma_-\). Then by Lemma 5.19, either (6.2.38) holds or there is \(t_0 \geq 0\) such that

\[(6.2.40) \quad \exp(-t(\xi)^{-1}H_{\tilde{p}_z})(x, \xi) \in \{x_1 > \delta\} \quad \text{for all } t \geq t_0;\]

replacing \((x, \xi)\) by the result of its propagation to time \(-t_0\), we may assume that \(t_0 = 0\). In the case (6.2.40), by (5.4.30) we have \((x, \xi) = j(\tilde{x}, \tilde{\xi})\) for some \((\tilde{x}, \tilde{\xi}) \in T^*M, |\tilde{\xi}|^2_g = z\). By Lemma 5.20, we have \(e^{-tH_p}(\tilde{x}, \tilde{\xi}) \in \{x_1 > \delta\}\) for all \(t \geq 0\), therefore \((\tilde{x}, \tilde{\xi}) \in \Gamma^{+}_{[\alpha, \beta]}\). By Lemma 6.4 and since \(j(K_{[\alpha, \beta]}) \subset \text{ell}_h(B)\), we have \(j(e^{-tH_p}(\tilde{x}, \tilde{\xi})) \in \text{ell}_h(B)\) for some \(t \geq 0\); by Lemma 5.20, this implies (6.2.39), finishing the proof.

Armed with Proposition 6.12, we prove the asymptotically hyperbolic version of the nontrapping estimate, Theorem 6.10:

**THEOREM 6.13.** Fix \(0 < \alpha \leq \beta, C_0 > 0, \chi \in C^\infty_c(M),\) and assume that \(K_{[\alpha, \beta]} = \emptyset.\)

Then the following estimates hold for \(h\) small enough, \(s > \frac{1}{2} + C_0,\) and all \(z \in [\alpha, \beta] + i[-C_0h, C_0h]\):

\[(6.2.41) \quad \|\tilde{P}(z)^{-1}\|_{H^{s-1}_h(X) \to H^s_h(X)} \leq Ch^{-1},\]
\[(6.2.42) \quad \|\chi R(z, h)\chi\|_{H^{s-1}_h \to H^s_h} \leq Ch^{-1}.\]

**Proof.** Take \(f \in \tilde{H}^{s-1}(X)\) and put \(u := \tilde{P}(z)^{-1}f \in \tilde{H}^s(X)\). By (6.2.36) with \(B = 0\), we have

\[\|u\|_{\tilde{H}^s(X)} \leq Ch^{-1}\|f\|_{\tilde{H}^{s-1}_h(X)},\]

implying (6.2.41).

Next, (6.2.42) follows directly from (6.2.41), (6.2.34), and the fact that we may choose the defining function \(x_1\) to be equal to 1 on \(\text{supp } \chi\).

We finally study semiclassical measures associated to \(o(h)\) quasimodes for the operator \(\tilde{P}(z)\), see §6.2.2. Namely, assume that we have sequences \(h_j \to 0, z_j\) satisfying (6.2.1), and \(u_j \in \mathcal{A}^s\) satisfies

\[(6.2.43) \quad \|u_j\|_{\tilde{H}^{s}_h(X)} = 1,\]
Then the properties of semiclassical measures associated to $u_j$ are given by the following analog of Theorem 6.11 (whose proof applies to the asymptotically hyperbolic case, substituting Proposition 6.12 in place of Proposition 6.9):

**Theorem 6.14.** Under the assumptions (6.2.43), (6.2.44), (6.2.28) (for all $A \in \Psi^\text{comp}_h(M)$), and (6.2.29), the semiclassical measure $\mu$ on $T^*M$ has the following properties:

1. $\mu$ is supported on $j(\Gamma_E^+)$, that is $\mu(T^*M \setminus \Gamma_E^+) = 0$.
2. For each neighbourhood $U$ of $K_E$, we have $\mu(U) > 0$.
3. If $U \subset T^*M$ is bounded and $x_1 = 1$ on $K \cup e^{-tH}(\Gamma_E^+ \cap U)$ for all $t \geq 0$, then

\[
\mu(e^{-tH}(U)) = e^{2\nu t} \mu(U).
\]

### 6.3. Normally Hyperbolic Trapping

In this section, we consider systems whose trapped trajectories form a normally hyperbolic set – see the definition below. This gives a class of examples with nonempty trapped sets where it is typically impossible to describe individual resonances yet one can make precise statements about the distribution of resonances. That is due the fine structure of the trapped set. One setting where normally hyperbolic sets appear is Kerr-(de Sitter) black holes. Another comes from molecular dynamics.

The main result of this section is a resonance free strip given in Theorem 6.16.

We work in the setting of Euclidean Schrödinger operators (6.0.4) or asymptotically hyperbolic Laplacians (6.0.5). To formulate the assumptions on the trapped set, we use the material of §6.1, in particular the sets $\Gamma_{j}^\pm$, $K_J \subset T^*M$ defined in (6.1.4) using the Hamiltonian flow $e^{tH}$ (here $M = \mathbb{R}^n$ in the Euclidean case).

Assume that

\[
[\alpha', \beta'] \subset (\alpha, \beta) \subset (0, \infty)
\]

are such that:

- \textbf{(A1)} $\Gamma_{(\alpha, \beta)}^\pm \subset T^*M$ are $C^\infty$ orientable hypersurfaces intersecting transversely, that is

\[
T_{(x, \xi)}(T^*M) = T_{(x, \xi)}\Gamma_{(\alpha, \beta)}^+ + T_{(x, \xi)}\Gamma_{(\alpha, \beta)}^- \quad \text{for all } (x, \xi) \in K_{(\alpha, \beta)};
\]
(A2) $K_{(\alpha,\beta)}$ is symplectic, that is the restriction $\omega|_{TK_{(\alpha,\beta)}}$ is a nondegenerate 2-form, where $\omega$ is the standard symplectic form on $T^*M$.

We next want to make a hyperbolicity assumption on the flow near the trapped set. Roughly speaking, it states that every Hamiltonian trajectory in $p^{-1}([\alpha',\beta'])$ converges exponentially fast to $\Gamma^\pm$ as $t \to \pm\infty$. However, it is more convenient to use the linearization of the flow. By (A1), there exist defining functions $\varphi_{\pm}$ of $\Gamma^\pm$ in some neighbourhood $U \subset p^{-1}((\alpha,\beta))$ of $K_{[\alpha',\beta']}$, namely

(6.3.2) $\varphi_{\pm} \in C^\infty(U; \mathbb{R})$;
(6.3.3) $\{\varphi_{\pm} = 0\} = \Gamma^\pm \cap U$;
(6.3.4) $d\varphi_{\pm} \neq 0$ on $\Gamma^\pm \cap U$.

Note that for $(x, \xi) \in \Gamma^\pm \cap U$, $T_{(x,\xi)}\Gamma^\pm$ is the kernel of $d\varphi_{\pm}(x, \xi)$. Therefore, for each

$(x, \xi) \in K \cap U$, \quad v \in T_{(x,\xi)}(T^*M)$,

the quantity $|\langle d\varphi_{\pm}(x, \xi), v \rangle|$ measures the distance from $v$ to $T_{(x,\xi)}\Gamma^\pm$. The distance from the propagated vector $de^{tH_p}(x, \xi)v$ to $T_{e^{tH_p}(x,\xi)}\Gamma^\pm$ is given by the quantity

(6.3.5) $|\langle d\varphi_{\pm}(e^{tH_p}(x, \xi)), e^{tH_p}(x, \xi)v \rangle| = |\langle d(\varphi_{\pm} \circ e^{tH_p})(x, \xi), v \rangle|$.

Since $\Gamma^\pm$ are invariant under the flow $e^{tH_p}$, so are their tangent spaces. Therefore, $d(\varphi_{\pm} \circ e^{tH_p})(x, \xi)$ is a positive multiple of $d\varphi_{\pm}(x, \xi)$, thus it makes sense to divide these vectors by each other. The following assumption says that (6.3.5) decays exponentially as $t \to \pm\infty$:

(A3) There exists constant $C, \nu > 0$ such that for all $(x, \xi) \in K \cap U$,

(6.3.6) $\frac{d(\varphi_{\pm} \circ e^{\pm tH_p})(x, \xi)}{d\varphi_{\pm}(x, \xi)} \leq Ce^{-\nu t}$, \quad $t \geq 0$.

It is easy to see that the constant $\nu$ in assumption (A3) is independent of the choice of the defining functions $\varphi_{\pm}$. Define the minimal expansion rate

$\nu_{\text{min}} > 0$

as the supremum of all values of $\nu$ for which there exists a constant $C$ such that (6.3.6) holds.

**DEFINITION 6.15.** We say that the trapping is normally hyperbolic near $p^{-1}([\alpha',\beta'])$ if assumptions (A1)–(A3) hold for some $\alpha, \beta$ satisfying (6.3.1).

**REMARKS.** 1. For normally hyperbolic trapped sets, we have the following stable/unstable decomposition:

(6.3.7) $T_{(x,\xi)}(T^*M) = T_{(x,\xi)}K \oplus E_+(x, \xi) \oplus E_-(x, \xi)$, \quad $(x, \xi) \in K_{[\alpha',\beta']}$,
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Figure 6.2. A barrier-top potential and the corresponding phase space dynamics, with the sets $\Gamma^\pm_{[1-\delta,1+\delta]}$ shown by solid lines.

where the spaces $E_\pm(x,\xi)$ are defined by

$$E_\pm(x,\xi) := \mathbb{R}H_{\phi_\pm}, \quad T_{(x,\xi)}\Gamma^\pm = T_{(x,\xi)}K \oplus E_\pm(x,\xi).$$

Since the flow $e^{tH_p}$ consists of symplectomorphisms, by (6.3.6) its differential expands vectors in $E_+$ and contracts vectors in $E_-:$ for some $\nu > 0$

$$|de^{tH_p}(x,\xi)v| \leq Ce^{-\nu|t||v|}, \quad v \in E_\pm(x,\xi), \quad \forall t \geq 0. \quad (6.3.8)$$

The properties (6.3.7), (6.3.8), with $E_\pm$ of possibly higher dimensions, define a more general concept of normally hyperbolic trapping where $\Gamma^\pm$ need not be smooth, see [NZ15] and the references there.

2. Under an additional assumption, called $r$-normal hyperbolicity (roughly speaking, the expansion rates in the directions transversal to $K$ are at least $r$-fold bigger than the expansion rates along $K$, here $r \geq 1$), normally hyperbolic structures are stable under smooth perturbations, with the resulting submanifolds $\Gamma^\pm$ of class $C^r$ – see [HPS77].

EXAMPLES. 1. Consider the Schrödinger operator (6.0.1) on $\mathbb{R}$ such that the potential $V \in C^\infty_c(\mathbb{R};\mathbb{R})$ has the following properties:

$$V(0) = 1, \quad V'(0) = 0, \quad V''(0) < 0, \quad V(x) < 1 \quad \text{for} \ x \neq 0.$$

See Figure 6.2. The trapping is normally hyperbolic for energies in $[1-\delta,1+\delta]$, if $\delta > 0$ is small enough. In fact, the sets $\Gamma^\pm_{[1-\delta,1+\delta]}$ and $K_{[1-\delta,1+\delta]}$ are
given by
\[ \Gamma_{[1-\delta,1+\delta]}^\pm = \{ \xi = \pm \text{sgn} x \sqrt{1-V(x)} \}, \quad K_{[1-\delta,1+\delta]} = \{(0,0)\}. \]

Note that \( \Gamma_E^\pm = \emptyset \) for \( 0 < |E-1| \leq \delta \). Consider the following defining functions of \( \Gamma^\pm \):
\[ \varphi_{\pm}(x,\xi) = \xi \mp \text{sgn} x \sqrt{1-V(x)}. \]

Since \( p = \varphi_+ \varphi_- + 1 \), we find
\[ H_p \varphi_{\pm} = \mp \{ \varphi_+, \varphi_- \} \varphi_{\pm}; \quad \{|\varphi_+, \varphi_-\}|(0,0) = \sqrt{-2V''(0)}. \]

We see that (6.3.6) is satisfied, and
\[ \nu_{\text{min}} = \sqrt{-2V''(0)}. \]

2. Consider the operator (6.0.5) where the manifold \((M,g)\) is the hyperbolic cylinder studied in (5.1.4). Denote by \( \xi, \eta \) the momenta corresponding to the coordinates \((v, \theta)\), then
\[ p = \xi^2 + \frac{\eta^2}{\cosh^2 v}. \]

The incoming/outgoing tails and the trapped set at positive energies are
\[ \Gamma_{(0,\infty)}^\pm = \{ \xi = \pm |\eta| \tanh s, \eta \neq 0 \}, \quad K_{(0,\infty)} = \{ s = \xi = 0, \eta \neq 0 \}. \]

Consider the following defining functions of \( \Gamma^\pm \):
\[ \varphi_{\pm} = \xi \mp |\eta| \tanh s. \]

Since \( p = \varphi_+ \varphi_- + |\eta|^2 \) and \( \{|\eta|^2, \varphi_{\pm}\} = 0 \), we have
\[ H_p \varphi_{\pm} = \mp \{ \varphi_+, \varphi_- \} \varphi_{\pm}; \quad \{|\varphi_+, \varphi_-\}|_{K_{(0,\infty)}} = 2\sqrt{p}. \]

We see that (6.3.6) is satisfied on \( p^{-1}((\alpha, \beta)) \) for \( \alpha > 0 \), and
\[ \nu_{\text{min}} = 2\sqrt{\alpha}. \]

We now state the main result of this section, which is a resonance free strip of size just below \( \nu_{\text{min}}/2 \):

**THEOREM 6.16 (Spectral gap for normally hyperbolic trapping).**
Assume that the operator \( P \) is given by (6.0.4) or (6.0.5) and has normally hyperbolic trapping near \( p^{-1}([\alpha', \beta']) \). Fix \( \varepsilon, C_0 > 0, \chi \in C^\infty_c(M) \). Then for \( h \) small enough, the following estimates hold:

\[ \| \chi R(z,h) \chi \|_{L^2 \rightarrow L^2} = o(h^{-2}), \quad z \in [\alpha', \beta'] + ih \left[ -\frac{\nu_{\text{min}}}{2} + \varepsilon, C_0 \right]; \]
\[ (6.3.9) \]

\[ \| \chi R(z,h) \chi \|_{L^2 \rightarrow L^2} \leq C \frac{\log(1/h)}{h}, \quad z \in [\alpha', \beta']. \]
\[ (6.3.10) \]
REMARKS. 1. The estimate (6.3.10) is optimal as shown by Theorem 7.1 in the following chapter.
2. The proof of Theorem 6.16 in fact gives bounds of the form (6.3.9) on the inverse of the complex scaled operator $P_\theta - z$ (in the Euclidean setting) or the modified Laplacian $\tilde{P}(z)$ (in the asymptotically hyperbolic setting) – see (6.3.21) and (6.3.23) below.

The starting point of the proof of Theorem 6.16 is the following construction of defining functions of $\Gamma^\pm$ adapted to the flow:

**Lemma 6.17.** (Adapted defining functions) Fix $\varepsilon > 0$ and let the assumptions (A1)–(A3) above hold. Then there exists a neighbourhood $U \subset T^* M$ of $K[\alpha',\beta']$ and functions $\varphi_\pm$ satisfying (6.3.2)–(6.3.4) such that:

1. for $\delta$ small enough, the set $U_\delta$ is compactly contained in $U$, where
   \begin{equation}
   U_\delta := \{ |\varphi_+| < \delta, \ |\varphi_-| < \delta, \ p \in (\alpha' - \delta, \beta' + \delta) \};
   \end{equation}

2. there exist $c_\pm \in C^\infty(U; \mathbb{R})$ such that
   \begin{equation}
   H_p \varphi_\pm = \mp c_\pm \varphi_\pm, \quad c_\pm \geq \nu_{\min} - \varepsilon \quad \text{on } U.
   \end{equation}

3. $\{ \varphi_+, \varphi_- \} \geq 1$ on $U$.

**Proof.** 1. We show existence of $\varphi_+$, the case of $\varphi_-$ considered similarly. Fix a function $\tilde{\varphi}_+$ on some neighbourhood $U$ of $K[\alpha',\beta']$ satisfying (6.3.2)–(6.3.4).

Since $\Gamma^+$ is invariant under the flow, the function $H_p \tilde{\varphi}_+$ vanishes on $\Gamma^+$. Therefore, we may write for some $\tilde{c}_+ \in C^\infty(U; \mathbb{R})$

\begin{equation}
H_p \tilde{\varphi}_+ = -\tilde{c}_+ \tilde{\varphi}_+ \quad \text{on } U.
\end{equation}

We have on $K \cap U$,

\begin{equation}
\partial_t d(\tilde{\varphi}_+ \circ e^{TH_p}) = d((H_p \tilde{\varphi}_+) \circ e^{TH_p}) = -(\tilde{c}_+ \circ e^{TH_p})d(\tilde{\varphi}_+ \circ e^{TH_p}).
\end{equation}

Therefore, for all $T > 0$ we have on $K \cap U$,

\begin{equation}
-\frac{1}{T} \log \frac{d(\tilde{\varphi}_+ \circ e^{TH_p})}{d\tilde{\varphi}_+} = \langle \tilde{c}_+ \rangle_T := \frac{1}{T} \int_0^T \tilde{c}_+ \circ e^{TH_p} \, ds.
\end{equation}

2. By (6.3.6), we have for $T$ large enough,

\begin{equation}
\langle \tilde{c}_+ \rangle_T \geq \frac{\nu T - \log C}{T} > \nu_{\min} - \varepsilon \quad \text{on } K \cap U.
\end{equation}

Take $f_+ \in C^\infty(U; \mathbb{R})$ such that

\begin{equation}
f_+ = \frac{1}{T} \int_0^T (T - t)(\tilde{c}_+ \circ e^{TH_p}) \, dt \quad \text{on } K \cap U,
\end{equation}

then

\begin{equation}
H_p f_+ = \langle \tilde{c}_+ \rangle_T - \tilde{c}_+ \quad \text{on } K \cap U.
\end{equation}
3. Now, put
\[ \varphi_+ := e^{-f+} \tilde{\varphi}_+. \]
Then
\[ H_p \varphi_+ = -c_+ \varphi_+, \quad c_+ = \tilde{c}_+ + H_p f_. \]
On \( K \cap U \), we have \( c_+ = \langle \tilde{c}_+ \rangle_T > \nu_{\min} - \varepsilon \). Shrinking \( U \), we can make sure that this inequality holds on the whole \( U \), proving (6.3.12). Shrinking \( U \) further, we see that (6.3.11) holds as well.

Finally, by assumption (A2) we see that \( \{ \varphi_+, \varphi_- \} \neq 0 \) on \( K_{[\alpha', \beta']} \). Shrinking \( U \) and multiplying \( \varphi_+ \) by a constant, we obtain \( \{ \varphi_+, \varphi_- \} \geq 1 \) on \( U \).

□

We now construct an auxiliary pseudodifferential operator. (See Exercise 6.16 for the intuition behind the construction, in the model case.) Fix \( \delta > 0 \) such that the set \( U_{\delta} \) defined in (6.3.11) is compactly contained in \( U \).

We will henceforth argue in the Euclidean setting (6.0.4), with \( M = \mathbb{R}^n \); at the end of this section, we explain what changes are necessary for the asymptotically hyperbolic setting. Let \( P_{\theta} \) be the complex scaled operator, see §6.2.1. We choose the constant \( r_1 \) in (6.2.4) so that \( U \subset \{ r < r_1 \} \); then
\[ P_{\theta} = P + \mathcal{O}(h^\infty) \] microlocally on \( U \).

The key component of the proof is a bound on \( \Theta_+ u \) for quasimodes \( u \) of the operator \( P_{\theta} - z \). That bound implies that \( \| \Theta_+ u \| = \mathcal{O}(h)\| u \| \) microlocally on \( U_{\delta/2} \supset K_{[\alpha', \beta']} \) when \( (P_{\theta} - z)u = 0 \). That refines (6.2.14) in our current setting as we can get closer to \( \Gamma_+ \):

**Lemma 6.18 (Auxiliary pseudodifferential bound).** Assume that \( u \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( (P_{\theta} - z)u = f \in L^2(\mathbb{R}^n) \), and
\[ z \in [\alpha', \beta'] + i h [ - (\nu_{\min} - 2\varepsilon), C_0 ] . \]
Then for each \( A \in \Psi^\text{comp}_h(\mathbb{R}^n) \) such that \( \text{WF}_h(A) \subset U_{\delta/4} \) (see (6.3.11)),
\[ \| A \Theta_+ u \|_{L^2} \leq Ch^{-1}\| f \|_{L^2} + Ch\| u \|_{L^2} , \]
where the constant \( C \) is independent of \( u, z, h \).

**Remark.** The region (6.3.13) is larger than the one in (6.3.9), since \( \nu_{\min} \) is not divided by 2. In fact, under additional assumptions one may establish a second resonance free strip deeper than the one in (6.3.9) – see Exercise [TODO].
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Proof. 1. Fix an operator

\[ Z_+ \in \Psi_h^\text{comp}(\mathbb{R}^n), \quad Z_+^* = Z_+, \quad \sigma_h(Z_+) = c_+ \quad \text{on } U_\delta. \]

By (6.3.12), we have for some \( R_+ \in \Psi_h^\text{comp}(\mathbb{R}^n), \)

\[ [P_\theta - z, \Theta_+] = ihZ_+ \Theta_+ + h^2 R_+ + O(h^\infty) \quad \text{microlocally on } U_\delta. \]

Thus we have microlocally on \( U_\delta \)

\[
(P_\theta - z)\Theta_+ u = \Theta_+ f + ihZ_+ \Theta_+ u + h^2 R_+ u + O(h^\infty)\|u\|_{L^2}. 
\]

Therefore, for each \( B_1 \in \Psi_h^\text{comp}(\mathbb{R}^n) \) such that \( \WF_h(B_1) \subset U_\delta, \) we have

\[
(6.3.15) \quad \|B_1(P_\theta - ihZ_+ - z)\Theta_+ u\|_{L^2} \leq C\|f\|_{L^2} + C\delta^2\|u\|_{L^2}. 
\]

2. We will apply Lemma E.51, which is a positive commutator estimate, to the equation (6.3.15) to estimate \( \Theta_+ u. \) The part of the right-hand side localized away from \( \Gamma^+ \) will be estimated by Proposition 6.9.

3. We first construct the escape function \( g. \) Take cutoff functions

\[
\chi_1 \in C_\infty^\alpha((-\delta, \delta); [0, 1]), \quad \chi_1 = 1 \quad \text{near } [-\delta/2, \delta/2];
\]

\[
\chi_2 \in C_\infty^\alpha((\alpha' - \delta, \alpha' + \delta); [0, 1]), \quad \chi_2 = 1 \quad \text{near } [\alpha' - \delta/2, \alpha' + \delta/2],
\]

and define

\[
g = \chi_1(\varphi_+)\chi_1(\varphi_-)\chi_2(p) \in C_\infty^\alpha(U_\delta; [0, 1]), \quad g = 1 \quad \text{on } U_{\delta/2}.
\]

We additionally require that \( t\chi_1'(t) \leq 0. \) Then

\[
H_\rho g = c_- \varphi_+ \chi_1'(\varphi_-)\chi_2(p) \leq 0 \quad \text{near } U \cap \{\varphi_+ \leq \delta/2\}.
\]

Since \( U \cap \Gamma_+ = U \cap \{\varphi_+ = 0\}, \) there exists an operator

\[
B \in \Psi_h^\text{comp}(\mathbb{R}^n), \quad \WF_h(B) \subset U_\delta \setminus \Gamma^+,
\]

\[
(6.3.16) \quad H_\rho g \leq 0 \quad \text{in a neighbourhood of } T^*M \setminus \ell_h(B).
\]

We also fix an operator

\[
B_1 \in \Psi_h^\text{comp}(\mathbb{R}^n), \quad \WF_h(B_1) \subset U_\delta, \quad \text{supp } g \subset \ell_h(B_1).
\]

4. Put \( P_\theta := P_\theta - ihZ_+ - z. \) Then by (6.3.12) and (6.3.13)

\[
\sigma_h(h^{-1}\text{Im } P) = -c_+ - \frac{\text{Im } z}{h} \leq -\varepsilon \quad \text{on } U_\delta.
\]

Combining this with (6.3.16), we get

\[
H_\rho g + \sigma_h(h^{-1}\text{Im } P)g \leq -\varepsilon g \quad \text{in a neighbourhood of } T^*M \setminus \ell_h(B).
\]

This implies the condition (E.5.27), where we put \( s = 1/2 \) for convenience; the choice of \( s \) does not matter since our operators \( A, B, B_1 \) are compactly microlocalized.
Applying Lemma E.51 to (6.3.15), we obtain the following bound for each \( A \in \Psi_h^{\text{comp}}(\mathbb{R}^n) \) with \( \text{WF}_h(A) \subset U_{\delta/2}^{\text{comp}} \):

\[
\| A\Theta_+ u \|_{L^2} \leq C \| B\Theta_+ u \|_{L^2} + Ch^{-1} \| f \|_{L^2} + Ch^{1/2} \| B_1 \Theta_+ u \|_{L^2}.
\] (6.3.17)

5. We now upgrade the \( Ch^{1/2} \) term in (6.3.17) to \( Ch \) as follows. Repeating the argument in steps 2–3 with \( \delta/2 \) in place of \( \delta \), we see that there exist \( B', B'_1 \in \Psi_h^{\text{comp}}(\mathbb{R}^n) \) such that

\[
\text{WF}_h(B') \subset U_{\delta/2} \setminus \Gamma^+, \quad \text{WF}_h(B'_1) \subset U_{\delta/2},
\]

and (6.3.17) holds with \( B, B'_1 \) replaced by \( B', B'_1 \) and each \( A \in \Psi_h^{\text{comp}}(\mathbb{R}^n) \) such that \( \text{WF}_h(A) \subset U_{\delta/4}^{\text{comp}} \). Estimating \( \| B'_1 \Theta_+ u \|_{L^2} \) by the original estimate (6.3.17), we get

\[
\| A\Theta_+ u \|_{L^2} \leq C \| B'\Theta_+ u \|_{L^2} + Ch^{1/2} \| B\Theta_+ u \|_{L^2} + Ch^{-1} \| f \|_{L^2} + Ch \| u \|_{L^2}.
\]

Finally, by Proposition 6.9

\[
\| B'\Theta_+ u \|_{L^2} + \| B\Theta_+ u \|_{L^2} \leq Ch^{-1} \| f \|_{L^2} + \mathcal{O}(h^\infty) \| u \|_{L^2}.
\]

Combining the latter two estimates, we obtain (6.3.14).

The next step is to deduce from (6.3.14) partial regularity of semiclassical measures associated to \( \mathcal{O}(h^2) \) quasimodes of the operator \( P - z \):

**LEMMA 6.19.** Consider sequences

\[
h_j \to 0, \quad z_j \in [\alpha, \beta] + ih_j \left[ - (\nu_{\min} - 2\varepsilon), C_0 \right];
\]

(6.3.18) \[ u_j \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad f_j := (P_{\theta}(h_j) - z_j)u_j \in L^2(\mathbb{R}^n); \]

\[
\| u_j \|_{L^2} = 1, \quad \| f_j \|_{L^2} = \mathcal{O}(h_j^2)
\]

and assume that \( u_j \) converges to some measure \( \mu \) on \( T^*\mathbb{R}^n \) in the sense of (6.2.28). Then there exists a constant \( C \) such that, with \( U_\rho \) defined in (6.3.11),

(6.3.19) \[ \mu(U_\rho) \leq C \rho, \quad \rho > 0. \]

**Proof.** 1. Fix an operator \( A \in \Psi_h^{\text{comp}}(\mathbb{R}^n) \) with \( \text{WF}_h(A) \subset U_{\delta/4}^{\text{comp}} \) and \( A = I + \mathcal{O}(h^\infty) \) microlocally near \( U_{\delta/5}^{\text{comp}} \). By Lemma 6.18, we have

\[
\| A\Theta_+ u_j \|_{L^2} = \mathcal{O}(h_j).
\]

Applying Theorem E.46 to the operator \( P := A\Theta_+ \), we see that there exists a constant \( C \) such that

(6.3.20) \[ \left| \int_{T^*M} \{ \varphi_+, b \} \ d\mu \right| \leq C \sup |b| \quad \text{for all } b \in C^\infty_c(U_{\delta/5}). \]
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By (6.2.14) (see the proof of Theorem 6.11), we have

$$
\mu(T^* M \setminus \Gamma_{[\alpha', \beta']}^{+}) = 0.
$$

Therefore, (6.3.20) holds for each $$b \in C^\infty(U)$$ satisfying the condition

$$\text{supp } b \cap \Gamma_{[\alpha', \beta']}^{+} \subset U_{\delta/5}.$$

2. Since $$\|u_j\|_{L^2} = 1$$, we have $$\mu(U_\rho) \leq 1$$ for all $$\rho$$. Therefore, it suffices to prove (6.3.19) for $$\rho$$ small enough, in particular for $$\rho < \delta/10$$. Take a function (see Figure 6.3)

$$b_\rho \in C^\infty_c((-\delta/5, \delta/5)), \quad b_\rho(s) = s \text{ for } |s| \leq \rho; \quad \sup |b_\rho| \leq 2\rho, \quad b_\rho'(s) \geq -\frac{20\rho}{\delta}.$$

Applying (6.3.20) to $$b := b_\rho(\varphi_-)$$ and using the fact that $$\{\varphi_+, \varphi_-\} \geq 1$$, we obtain for some $$\rho$$-independent constant $$C$$,

$$C \rho \geq \int_{T^* M} \{\varphi_+, b_\rho(\varphi_-)\} \, d\mu$$

$$= \int_{U_\rho} b_\rho'(\varphi_-)\{\varphi_+, \varphi_-\} \, d\mu + \int_{T^* M \setminus U_\rho} b_\rho'(\varphi_-)\{\varphi_+, \varphi_-\} \, d\mu$$

$$\geq \mu(U_\rho) - C \rho,$$

proving (6.3.19). 

We are now ready to prove Theorem 6.16 in the Euclidean setting:

Proof of Theorem 6.16. 1. Choose the complex scaled operator $$P_\theta$$ as explained before Lemma 6.18. We first show the bound

$$
\|(P_\theta - z)^{-1}\|_{L^2 \rightarrow L^2} = o(h^{-2}, \quad z \in [\alpha', \beta'] + ih \left[ -\frac{\nu_{\min}}{2} + \varepsilon, C_0 \right].
$$

We argue by contradiction. Assume that (6.3.21) does not hold; then there exist sequences $$h_j, z_j, u_j, f_j$$ satisfying (6.3.18). By Theorem E.44,
we may pass to a subsequence to make $u_j$ converge to some measure $\mu$ in the sense of (6.2.28). Passing to a further subsequence, we can also make sure that (6.2.29) holds, i.e.

$$\text{Re } z_j \to E \in [\alpha', \beta'], \quad \frac{\text{Im } z_j}{h} \to \nu \in \left[-\frac{\nu_{\text{min}}}{2} + \epsilon, C_0\right].$$

2. By Theorem 6.11, we have for each $t \geq 0$,

$$(6.3.22) \quad \mu(T^*M \setminus \Gamma_E^+) = 0, \quad \mu(e^{-tH_p}(U_\delta)) = e^{2\nu t} \mu(U_\delta) > 0.$$

On the other hand, by (6.3.12),

$$e^{-tH_p}(U_\delta \cap \Gamma_E^+) \subset U_\delta \exp(-\nu_{\text{min}}\epsilon t).$$

Therefore, by Lemma 6.19

$$\mu(e^{-tH_p}(U_\delta)) \leq C e^{-(\nu_{\text{min}}\epsilon t)}.$$

Since $\nu \geq -\frac{\nu_{\text{min}}}{2} + \epsilon$, this contradicts (6.3.22) for sufficiently large $t > 0$, finishing the proof of (6.3.21).

3. The bound (6.3.9) follows immediately from (6.3.21) as explained in the proof of Theorem 6.10. Finally, to prove (6.3.10) we use the upper half-plane bound [TODO reference]

$$\|\chi_{R(z,h)}\|_{L^2 \to L^2} \leq \frac{C}{\gamma h}, \quad \text{Im } z = \gamma h > 0.$$

By Lemma D.1 applied to

$$\Omega = [\alpha', \beta'] + ih\left[-\frac{\nu_{\text{min}}}{2} + \epsilon, \gamma\right]$$

we get for a $\gamma$-independent constant $C$ and $z \in [\alpha', \beta']$,

$$\|\chi_{R(z,h)}\|_{L^2 \to L^2} \leq C h^{\theta - 2} \gamma^{-\theta}, \quad \theta = \frac{\nu_{\text{min}}/2 - \epsilon}{\nu_{\text{min}}/2 - \epsilon + \gamma}.$$

For $\gamma < 1$, since $\theta > 1 - C\gamma$, we get

$$\|\chi_{R(z,h)}\|_{L^2 \to L^2} \leq C h^{-1 - C\gamma} \gamma^{-1}.$$

Putting

$$\gamma := \frac{1}{\log(1/h)},$$

we obtain $\|\chi_{R(z,h)}\| \leq C h^{-1} \log(1/h)$ for $z \in [\alpha', \beta']$, proving (6.3.10). □

REMARK. For the asymptotically hyperbolic case (6.0.5), the proof of Theorem 6.16 should be modified as follows. Instead of the complex scaled operator $P_{\theta} - z$, we use the modified Laplacian $\tilde{P}(z)$ introduced in (6.2.32). We choose the defining function $x_1$ so that $x_1 = 1$ near $U$, so that by (6.2.33)

$$\tilde{P}(z) = P - z + O(h^\infty) \quad \text{microlocally on } U.$$
6.4. LOGARITHMIC RESONANCE FREE REGIONS

We replace Proposition 6.9 by Proposition 6.12 and Theorem 6.11 by Theorem 6.14. Repeating the argument of this section, we obtain the following bound for $s > (1 + \nu_{\min}/2)$ and $z$ satisfying the condition in (6.3.9):

\[(6.3.23) \| \tilde{P}(z)^{-1} \|_{\dot{H}^{s-1} \to \dot{H}^s} = o(h^{-2}).\]

As in the proof of Theorem 6.13, this implies the cutoff bound

\[(6.3.24) \| \chi_R(z, h) \|_{\dot{H}^{s-1} \to \dot{H}^s} = o(h^{-2}).\]

Using the fact that $P - z$ is elliptic at fiber infinity [TODO details?], we can upgrade (6.3.24) to a bound $H^{s-2} \to H^s$ for all $s \in \mathbb{R}$, in particular giving the $L^2$ bound (6.3.9). The bound (6.3.10) follows from (6.3.9) as in the Euclidean case.

6.4. LOGARITHMIC RESONANCE FREE REGIONS

In the Euclidean case we have a semiclassical analogue of the results of §4.6 which provide logarithmic resonance free regions. These are not currently available in the asymptotically hyperbolic setting since meromorphic continuation presented in §5.6 proceeds strip-by-strip. It is an interesting question if the same results hold in that case, possibly under some stronger assumptions about the structure of infinity.

The proof of existence of logarithmic resonance free regions uses the concept of an escape function, that is a function $G$ such that $H p G > 0$ on the energy surface $p^{-1}(E)$. In §6.2 we used such functions implicitly when citing propagation of singularities (Theorem E.49) – see Lemma E.50. The simplest example is given by the free Laplacian in which case

\[p = |\xi|^2, \quad G_0(x, \xi) = \chi(|\xi|/E)\langle x, \xi \rangle, \quad E > 0, \quad \chi \in C^\infty_c((1/2, 2); [0, 1]), \quad \chi(1) = 1.\]

Then clearly

\[H p G_0(x, \xi) = 2\chi(|\xi|/E)|\xi|^2 = 2E^2 > 0 \text{ on } p^{-1}(E).\]

We now show that we can construct such $G$ under the non-trapping condition:

**Lemma 6.20 (Construction of an escape function).** Suppose that $p(x, \xi) = |\xi|^2 + V(x)$, where $V, g_{ij} - \delta_{ij} \in C^\infty_c$ and $g_{ij}$ is positive definite. Suppose that at $E > 0$, $K_E = \emptyset$. Then for any $R > 0$ there exists $G \in C^\infty_c(T^*\mathbb{R}^n; \mathbb{R})$ and $\delta > 0$ such that

\[(6.4.1) H p G(x, \xi) \geq 1 \text{ for } |p(x, \xi) - E| \leq \delta, \quad |x| \leq R.\]
6. LOGARITHMIC RESONANCE FREE REGIONS

**REMARK.** It is convenient to take compactly supported \( G \) in which case \( H_p G \geq 1 \) can only hold in a compact set. The proof can be modified to show that we can construct \( G \in S_{1,0}^0(T^* \mathbb{R}^n; \mathbb{R}) \) such that \( H_p G \geq 1 \) is valid for all \( (x, \xi) \) satisfying \( |p(x, \xi) - E| \leq \delta \).

**Proof.**
1. From Proposition 6.3 we know that \( K_{[E - 3\delta, E + 3\delta]} = \emptyset \), for some \( \delta > 0 \). Proposition 6.4 then shows that \( \Gamma_{[E - 3\delta, E + 3\delta]} = \emptyset \) as well. Since the set \( \Sigma_2 \delta \),

\[
\Sigma_2 \delta := \{(x, \xi) : |p(x, \xi) - E| \leq 2\delta, |x| \leq R\},
\]

is compact this implies that for any sufficiently large \( R \) there exists \( T = T(R) \) such for

\[
|x(\exp(t H_p(x, \xi)))| > R, \quad |t| \geq T, \quad (x, \xi) \in \Sigma_2 \delta.
\]

2. For \( \rho = (x, \xi) \in \Sigma_2 \delta \) we first define \( G_\rho \in C^\infty_c(T^* \mathbb{R}) \), a local escape function supported in a neighbourhood of the segment

\[
I_\rho = \{\exp(t H_p) : t \in [-T, T]\},
\]

and which satisfies \( H_p G_\rho \geq 1 \) on the part of \( I_\rho \) lying over

\[
\pi^* \mathcal{B}(0, R) = \{(x, \xi) \in T^* \mathbb{R}^n : |x| \leq R\}
\]

3. For that, let \( \Gamma \) be a hypersurface through \( \rho \) which is transversal to \( H_p \). For a neighbourhood \( U_\rho \) of \( \rho \) we define

\[
V_\rho = \{\exp(t(U_\rho \cap \Gamma)) : t \in (-T - 1, T + 1)\}
\]

\[
\subset p^{-1}((E - 2\delta, E + 2\delta)).
\]

If \( U_\rho \) is sufficiently small \( V_\rho \) is a neighbourhood of \( I_\rho \) and we can identify it with the product

\[
V_\rho \simeq (-T - 1, T + 1) \times (U_\rho \cap \Gamma), \quad (x, \xi) = \exp(t H_p)(m),
\]

\[
(x, \xi) \in V_\rho, \quad t \in (-T - 1, T + 1), \quad m \in U_\rho \cap \Gamma.
\]

In view of (6.4.3)

\[
((-T - 1, -T) \cup (T, T + 1)) \times (U_\rho \cap \Gamma)) \cap \pi^* \mathcal{B}(0, R) = \emptyset.
\]

Now let \( \varphi_\rho \in C^\infty_c(U_\rho \cap \Gamma) \) be identically 1 near \( \rho \), and let

\[
\chi_T \in C^\infty_c((-T - 1, T + 1); [0, 1]), \quad \chi_T(t) = t, \quad t \in [-T, T].
\]

Let

\[
\psi \in C^\infty_c((E - 2\delta, E + 2\delta), [0, 1]), \quad \psi|_{[E - \delta, E + \delta]} = 1
\]

and, using the product coordinates, put

\[
G_\rho(\exp(t H_p)(m)) := \chi_T(t) \varphi_\rho(m) \psi(p(m)), \quad G_\rho \in C^\infty_c(V_\rho).
\]
6.4. LOGARITHMIC RESONANCE FREE REGIONS

It follows that
\[
H_\rho G_\rho(\exp(tH_\rho)(m)) = \chi_T'(t)\varphi_\rho(m)\psi(p(m)),
\]
satisfies
\[
H_\rho G_\rho = 1 \text{ on } V_\rho \cap \{|x| < R\}.
\]

4. Since \(\Sigma_\delta\) (see (6.4.2)) is compact, applying the previous argument for every \(\rho \in \Sigma_\delta\) gives a \(U_\rho\), and a \(U'_\rho \subset U_\rho\) on which \(\varphi_\rho = 1\). Since \(\{U'_\rho : \rho \in \Sigma_\delta\}\) covers \(\Sigma_\delta\), the compactness of \(K\) shows that we can pass to a finite subcover, \(\{U'_\rho : j = 1, \ldots, N\}\). We let
\[
G = \sum_{j=1}^{N} G_{\rho_j}.
\]
The construction of \(G_{\rho_j}\)'s now shows that
\[
H_\rho G(x, \xi) \geq 1, \quad (x, \xi) \in \Sigma_\delta, \quad G \in C_\infty^c(T^*\mathbb{R}^n),
\]
concluding the proof. \(\square\)

We will now use the function \(G\) constructed in Lemma 6.20 to prove

**THEOREM 6.21 (Logarithmic resonance free regions for non-trapping perturbations).** Suppose that \(P\) is given by (6.0.4) and that for some \(E > 0\) the trapped set at energy \(E\) defined by in Definition 6.1 is empty:
\[
K_E = \emptyset.
\]
Then there exists \(\delta > 0\) such that for each \(M > 0\) there exists \(h_M > 0\) so that for \(0 < h < h_M\),
\[
\text{Res}(P) \cap ([E - \delta, E + \delta] - i[0, Mh\log(1/h)]) = \emptyset.
\]
In addition, we have a bound on the truncated resolvent: for \(\chi \in C_\infty^c(\mathbb{R}^n)\),
\[
\|\chi R(z, h)\chi\|_{L^2 \to L^2} \leq \frac{C \exp(-C \text{Im } z/h)}{h},
\]
\(z \in [E - \delta, E + \delta] - i[0, Mh\log(1/h)], \ 0 < h < h_M\).

**REMARKS.** 1. The idea of the proof is to conjugate the operator \(P_\theta\) by an exponential weight to define
\[
e^{-t\text{Op}_h(G)/h} P_\theta e^{t\text{Op}_h(G)/h} = P_\theta - it\text{Op}_h(H_{\text{Re}p_0G}) + t\text{Op}_h(H_{\text{Im}p_0G}) + \mathcal{O}(t^2)_{H^s_{\text{loc}} \to L^2},
\]
with \(t = 2Mh\log(1/h)\). For \(z \in [E - \delta, E + \delta] - i[0, Mh\log(1/h)]\) the properties of \(G\) constructed in Lemma 6.20 show that
\[
\text{Im } \sigma \left(e^{-t\text{Op}_h(G)/h}(P_\theta - z)e^{t\text{Op}_h(G)/h}\right) \leq 0.
\]
Proof of Theorem 6.21. 1. Let $G \in C_c^\infty(T^*\mathbb{R}^n; \mathbb{R})$ be given by Lemma 6.20. For $t \in \mathbb{R}$ define the operator

$$P_\theta(t) := e^{-t \text{Op}_h(G)/h} P_\theta e^{t \text{Op}_h(G)/h} : H^2_h(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

The operator $\text{Op}_h(G)$ is bounded (uniformly in $h$) on $H^s_h(\mathbb{R}^n)$ for any $s$ and hence

$$e^{t \text{Op}_h(G)/h} = O(e^{C|t|/h}) : H^s_h(\mathbb{R}^n) \to H^s_h(\mathbb{R}^n).$$

Using the notation $\text{ad}_A B := [A, B]$ and the identity $e^{\text{ad}_A B} = e^A B e^{-A}$, Taylor’s formula shows that for any $N \geq 1$,

$$P_\theta(t) = \sum_{k=1}^N \frac{(-1)^k}{k!} (t/h)^k \text{ad}_{\text{Op}_h(G)} P_\theta + \frac{(-1)^{N+1}}{N! h^{N+1}} \int_0^1 (1 - \tau)^N \left( e^{-t \tau \text{Op}_h(G)/h} \text{ad}_{\text{Op}_h(G)} P_\theta \right) e^{t \tau \text{Op}_h(G)/h} d\tau.$$

Symbolic calculus gives,

$$\text{ad}_{\text{Op}_h(G)} P_\theta \in h^k \Psi^\text{comp}_h(\mathbb{R}^n), \quad k \geq 1.$$

Also, in view of (6.4.10), for $\tau \in [0, 1]$

$$e^{-t \tau \text{Op}_h(G)/h} \left( \text{ad}_{\text{Op}_h(G)} P_\theta \right) e^{t \tau \text{Op}_h(G)/h} = O(h^{N+1} e^{2C|t|/h})_{H^s_h \to H^s_h},$$

for any $r$ and $s$.

Since we can take $N$ arbitrarily large, we see that for $t$ satisfying

$$|t| \leq M_1 h \log(1/h)$$

and any $r$ and $s$ we have

(6.4.12) \( P_\theta(t) = P_\theta - (t/h) \text{ad}_{\text{Op}_h(G)} P_\theta + e(t), \quad e(t) = O(t^2)_{H^s_h \to H^r_h}. \)

2. We define the family of operators depending smoothly on $t \in \mathbb{R}$

$$\tilde{P}_\theta(t) := P_\theta - (t/h) \text{ad}_{\text{Op}_h(G)} P_\theta \in \Psi^2_h(\mathbb{R}^n),$$

$$P_\theta(t) = \tilde{P}_\theta(t) + e(t).$$

Let $\tilde{p}_\theta(t)$ be the principal symbol of $\tilde{P}_\theta(t)$. Then for $z \in \mathbb{C}$

(6.4.13) \( \text{Re}(\tilde{p}_\theta(t) - z) = \text{Re} p_\theta + tH \text{Im} p_\theta G - \text{Re} z, \)

$$\text{Im}(\tilde{p}_\theta(t) - z) = \text{Im} p_\theta - tH \text{Re} p_\theta G - \text{Im} z.$$
We now relate the parameters in the constructions of \( P_{\theta} \) and \( G \): we take \( R := 2r_1 \) in (6.4.1). Then there exists \( \varepsilon > 0 \) such that for

(6.4.14) \( \text{Re} \, z \in [E - \delta, E + \delta], \quad 0 \leq -2 \text{Im} \, z \leq t \leq \varepsilon \)

the symbol \( \tilde{p}_{\theta}(t) - z \) has the following properties:

(1) \( \text{Im}(\tilde{p}_{\theta}(t) - z) \leq 0 \) near the characteristic set

\[
\{ \langle \xi \rangle^{-2}(\tilde{p}_{\theta}(t) - z) = 0 \} = T^*\mathbb{R}^n \setminus \text{ell}_h(\tilde{P}_{\theta}(t) - z);
\]

(2) for each \( (x, \xi) \in T^*\mathbb{R}^n \), there exists \( T \geq 0 \) such that \( \varphi_{-T}(x, \xi) \in \text{ell}_h(\tilde{P}_{\theta}(t) - z) \), where \( \varphi_{-T} := \exp(-T\langle \xi \rangle^{-1}H_{\text{Re} \, \tilde{p}_{\theta}(t)}) \).

Indeed, for \( t = 0 \) (and thus \( \text{Im} \, z = 0 \)) these follow from Lemma 6.8 and (6.2.16). In the latter statement we used that \( \Gamma_{[E-\delta,E+\delta]} = \emptyset \) which follows from \( K_E = \emptyset \) for small enough \( \delta \). To deduce from here the case \( t > 0 \) we first note that by Lemma 6.8 for \( \text{Re} \, z \in [E - \delta, E + \delta] \)

\[
\{ \langle \xi \rangle^{-2}(p_{\theta} - \text{Re} \, z) = 0 \} \subset \{ |x| < 2r_1 \} \cap p^{-1}(\text{Re} \, z),
\]

\[
H_{\text{Re} \, p_{\theta}} G = H_p G \quad \text{on} \quad \{ \langle \xi \rangle^{-2}(p_{\theta} - \text{Re} \, z) = 0 \}.
\]

Thus by Lemma 6.20 we have

\[
H_{\text{Re} \, p_{\theta}} G \geq 1 \quad \text{on} \quad \{ \langle \xi \rangle^{-2}(p_{\theta} - \text{Re} \, z) = 0 \}.
\]

Since \( \tilde{p}_{\theta} - z = p_{\theta} - \text{Re} \, z + O(\varepsilon) \), for \( \varepsilon \) small enough each point \( (x, \xi) \) in \( \{ \langle \xi \rangle^{-2}(\tilde{p}_{\theta} - z) = 0 \} \) is close to the set \( \{ \langle \xi \rangle^{-2}(p_{\theta} - \text{Re} \, z) = 0 \} \), which implies \( H_{\text{Re} \, p_{\theta}} G(x, \xi) \geq \frac{1}{2} \). Since \( \frac{1}{2} + \text{Im} \, z \geq 0 \) we obtain property (1) above.

For property (2) it suffices to use that \( H_{\text{Re} \, \tilde{p}_{\theta}(t)} = H_{p_{\theta}} + O(\varepsilon) \) and thus

\[
\varphi_{-T}(x, \xi) = \exp(-T\langle \xi \rangle^{-1}H_{p_{\theta}})(x, \xi) + O(\varepsilon).
\]

3. Using properties (1)–(2) above, the elliptic estimate (Theorem E.32), and propagation of singularities (Theorem E.49) similarly to Steps 2–3 of the proof of Proposition 6.9 we get the following: for \( z, t \) satisfying (6.4.14) and all \( u \in C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) we have

(6.4.15) \( \|u\|_{L^2} \leq Ch^{-1}\|\tilde{P}_{\theta}(t) - z\|_{L^2} + O(h^\infty)\|u\|_{L^2} \)

where the constants are uniform in \( z, t \).

We now put \( t = \bar{t} := 2Mh \log(1/h) \) and assume that

\[
z \in [E - \delta, E + \delta] - i[0, Mh \log(1/h)].
\]

Applying (6.4.15) and using (6.4.12) we see that

(6.4.16) \( \|u\|_{L^2} \leq Ch^{-1}\|(P_{\theta}(\bar{t}) - z)u\|_{L^2} + O(h \log^2(1/h))\|u\|_{L^2} \).
For $h$ small enough we can absorb the $\mathcal{O}(h \log(1/h))$ term into the left-hand side, which gives invertibility of

$$P_\theta(\bar{t}) - z : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

4. The bound in (6.4.10) and (6.4.16) show that (with $\| \cdot \|$ denoting the $L^2 \to L^2$ operator norm)

$$\| (P_\theta - z)^{-1} \| = \| e^{\bar{t} \text{Op}_h(G)/h} (P_\theta(\bar{t}) - z)^{-1} e^{-\bar{t} \text{Op}_h(G)/h} \| \leq \| e^{\bar{t} \text{Op}_h(G)/h} \| \cdot \| e^{-\bar{t} \text{Op}_h(G)/h} \| \cdot \| (P_\theta(\bar{t}) - z)^{-1} \| = \mathcal{O}(h^{-4CM-1}).$$

(6.4.17)

In particular this implies the absence of resonances. We recall from Theorem 4.37 that $\chi R(z,h)\chi = \chi(P_\theta - z)^{-1}\chi$ provided that supp $\chi \subset B(0,r_1)$, where in the construction of $P_\chi$ we can take $r_1$ arbitrarily large. Hence the bounds (6.4.17) holds for the truncated resolvent.

To prove (6.4.9) we take $\varphi,\psi \in L^2(\mathbb{R}^n)$, $\| \varphi \|_{L^2} = \| \psi \|_{L^2} = 1$ we apply Lemma D.1 to

$$f(z) := e^{-iz/h} \langle \chi R(z,h)\chi \varphi, \psi \rangle, \quad \delta_- = Mh \log(1/h), \quad \delta_+ = 0.$$  

Then, in the notation of Lemma D.1, (6.4.17) gives $M_- = 1$ if we take a large enough $C$ in the definition of $f$. Increasing that $C$ if necessary, Theorem 6.10 gives $M_+ = C/h$. We check that other assumptions are satisfied and (D.1.4) gives

$$|f(z)| \leq Ce^{-C\text{Im}z/h}, \quad z \in [E - \delta/2, E - \delta/2] - i[0, Mh \log(1/h)].$$

Since $\varphi$ and $\psi$ were arbitrarily we obtain (6.4.9) (by changing $\delta$). \hfill \Box

**REMARK.** The bound (6.4.9) can be proved without invoking Theorem 6.10. In fact, we have

$$\chi R(z,h)\chi = \begin{cases} O(1/\text{Im}z), & \text{Im}z \geq 0, \\
O(h^{-4CM-1}), & \text{Im}z \leq 0 \end{cases} \quad z \in [E - \delta, E + \delta] - i[0, Mh \log(1/h)],$$

and a slightly more tricky application of Lemma D.1 gives (6.4.9).

### 6.5. LOWER BOUNDS ON RESONANCE WIDTHS

We will now show that the cut-off resolvent of a semiclassical Schrödinger operator with compactly supported potential is bounded by $\exp(C/h)$ on the real axis. When the cut-offs are supported outside of a ball containing the support of the potential the estimate is the same as the non-trapping estimate $C/h$. From this we deduce that the resonance width (imaginary parts) are bounded from below by $\exp(-C/h)$. In Theorem 2.30 we saw
6.5. LOWER BOUNDS ON RESONANCE WIDTHS

an elementary version of this result in the case of one dimension. As in indicated in §6.6 many generalizations are available.

We start with a uniform estimate of the resolvent in the upper half-plane. At this stage it is important to have weights rather than compactly supported cut-offs.

For $x \in \mathbb{R}^n$ we denote $r = |x|$ and $1_{\mathbb{R}^n \setminus B(0,R_0)}(x)$ the characteristic function of $\{ x \in \mathbb{R}^n : |x| \geq R_0 \}$.

**THEOREM 6.22 (Weighted resolvent estimates).** Suppose that for $E > 0$,

$$P = P_E := -h^2 \Delta + V - E, \quad V, \partial_r V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}), \quad n \geq 3.$$ 

For any $s > 1/2$ there exist $C, R_0, h_0 > 0$ such that

$$\| (1+r)^{-s} (P - i\varepsilon)^{-1} (1+r)^{-s} \|_{L^2(\mathbb{R}^n)} \rightarrow L^2(\mathbb{R}^n) \leq e^{C/h},$$

for all $\varepsilon > 0, h \in (0, h_0)$.

The proof is based on two lemmas. The first constructs a nondecreasing Carleman weight for $P$ which is constant outside of a compact set – see Lemma 3.34 for a simpler version of Carleman estimates. To formulate it we put

$$\delta := 2s - 1 > 0, \quad w = w_\delta(r) := 1 - (1+r)^{-\delta} > 0.$$ 

**LEMMA 6.23 (Construction of Carleman weights).** If $\delta > 0$ is small enough, there exist $R_0 > 0$ and $\varphi = \varphi(r) \in C^\infty([0, \infty))$ with $\varphi' \geq 0$ and supp $\varphi' \subset [0, R_0)$, such that

$$\partial_r (w(E - V + (\varphi')^2)) > \frac{Ew'}{4}, \quad r > 0.$$ 

**Proof.** 1. Fix $R > 0$ such that supp $V \subset B(0, R)$. We first construct a function $\psi = \psi_3(r)$ on $\mathbb{R}$ of the form

$$\psi := \begin{cases} 
A, & r \leq R, \\
B \frac{w(r)}{w(R)} - \frac{E}{4}, & R < r < R_0, \\
0, & r \geq R_0,
\end{cases}$$

where $A, B > 0, R_0 > R$ are chosen below so that $\psi$ is continuous and

$$\psi - V + (\psi' - \partial_r V) \frac{w}{w'} \geq -\frac{E}{2}, \quad r > 0, \quad r \neq R, \quad r \neq R_0.$$ 

We should think of $\psi$ as a prototype for $(\varphi')^2$. 

2. We first arrange it so that $\psi$ satisfies (6.5.5). For $R > R_0$, this is immediate as the left-hand side of (6.5.5) equals 0. For $R < r < R_0$ we compute this left-hand side as

$$\psi + \psi' \frac{w}{w'} = -\frac{E}{4}.$$  

Finally for $0 < r < R$ we observe that uniformly in $r \in [0, R]$

$$\frac{w}{w'} = (1 + r) \frac{(1 + r)^\delta - 1}{\delta} \to (1 + r) \log(1 + r),$$

as $\delta \to 0^+$. Putting

$$A := \max |V| + 2 \max |\partial_r V| (1 + R) \log(1 + R),$$

we obtain (6.5.5), provided $\delta$ is sufficiently small.

3. To make $\psi$ continuous at $r = R$, put

$$B := w(R)(A + E/4).$$

It remains to choose $R_0$ such that $\psi$ is continuous at $R_0$. For that we need

$$w(R_0) = 4B/E = w(R)(1 + 4A/E).$$

Since $w$ takes values in $[0, 1)$, this is possible only if

$$w(R) < \frac{1}{1 + 4A/E}.$$  

But since $w(R) \to 0$ as $\delta \to 0^+$, it is enough to take $\delta$ sufficiently small.

4. To obtain a smooth $\varphi$ satisfying (6.5.4) we fix $\rho \in C_0^\infty((0, \infty))$ with $\rho \geq 0$, $\int \rho = 1$. Take $\eta > 0$ and put

$$\varphi(r) := \int_0^r \chi(t) dt, \quad \chi := \rho_\eta \ast \sqrt{\psi}, \quad \rho_\eta(r) := \frac{1}{\eta} \rho \left( \frac{r}{\eta} \right).$$
The inequality (6.5.5) gives
\[ \frac{\partial_r (w(\chi^2 - V))}{w'} = \chi^2 - V + \frac{w(2\chi\chi' - \partial_r V)}{w'} > -\frac{E}{2} + O(\eta). \]
Since \( \varphi' = \chi \) this concludes the proof of (6.5.4) once we take \( \eta \) small enough.

We note that if \( h_0 \) is small enough we can modify (6.5.4) to obtain the inequality which will be used in our argument:

(6.5.6) \[ \partial_r \left( w(E - V + (\varphi')^2 - h\varphi'') \right) \geq Ew'/4, \]
for \( h \in (0, h_0] \).

The next lemma uses the weight constructed in Lemma 6.23 to prove a global Carleman estimate. Define \( m = m_\delta(r) \) by
\[ m := (1 + r^2)^{(1+\delta)/4} \sim (1 + r)^s. \]

**Lemma 6.24 (Weighted Carleman estimate).** Let \( \delta \) and \( \varphi = \varphi(r) \) be as in Lemma 6.23 and \( h_0 \) as in (6.5.6). Then there exists \( C > 0 \) such that

(6.5.7) \[ \|m^{-1}e^{\varphi/h}v\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C}{h^2}\|me^{\varphi/h}(P - i\varepsilon)v\|_{L^2(\mathbb{R}^n)}^2 \]
\[ + \frac{C\varepsilon}{h}\|e^{\varphi/h}v\|_{L^2(\mathbb{R}^n)}^2, \]
for all \( v \in C_0^\infty(\mathbb{R}^n) \), \( \varepsilon \geq 0 \), and \( h \in (0, h_0] \).

**Proof.** 1. Let
\[ P_\varphi := e^{\varphi/h}r^{(n-1)/2}(P - i\varepsilon)r^{-(n-1)/2}e^{-\varphi/h} \]
\[ = -h^2\partial_r^2 + 2h\varphi'\partial_r + \Lambda + V_\varphi - E - i\varepsilon, \]
where \( \Lambda \geq 0 \) a semi-definite operator
\[ \Lambda := h^2r^{-2}(-\Delta_{S^{n-1}} + (n - 1)(n - 3)/4), \]
and
\[ V_\varphi := V - \varphi'^2 + h\varphi''. \]

2. In this notation, (6.5.7) is equivalent to

(6.5.8) \[ \int_0^\infty \int_{S^{n-1}} w'|u|^2d\omega dr \leq \frac{C}{h^2} \int_0^\infty \int_{S^{n-1}} |P_\varphi u|^2 w' d\omega dr \]
\[ + \frac{C\varepsilon}{h} \int_0^\infty \int_{S^{n-1}} |u|^2 dr d\omega, \]
for \( u \in e^{\varphi/h}r^{(n-1)/2}C_0^\infty(\mathbb{R}^n) \).
We may assume $\varepsilon \leq h$, since $w' \leq 1$ makes (6.5.8) trivial for $\varepsilon > h$. We will prove

$$
\int_0^\infty \int_{S^{n-1}} \partial_r (w(E - V\phi)) |u|^2 d\omega dr \leq \frac{2}{h^2} \int_0^\infty \int_{S^{n-1}} \frac{|P\phi u|^2}{w'} d\omega dr + \frac{C\varepsilon}{h} \int_0^\infty \int_{S^{n-1}} |u|^2 d\omega dr,
$$

which, together with (6.5.6), implies (6.5.8).

3. To save space we write $\| \cdot \|_S$ and $\langle \cdot, \cdot \rangle_S$ for the norm and inner product in $L^2(S^{n-1})$. Then for $r > 0$ we put

$$
F(r) := \|h \partial_r u(r, \omega)\|_S^2 - \langle (\Lambda + V\phi(r, \omega) - E) u(r, \omega), u(r, \omega) \rangle_S.
$$

Note that

$$
\int_0^\infty (w(r)F(r))' dr \leq - \lim_{r \to 0} w(r) \lim \inf_{r \to 0} F(r) = 0.
$$

Using selfadjointness of $\Lambda + V\phi - E$ we compute the derivative of $F$ in terms of $P\phi$. With the notation $f' := \partial_r f$ this yields

$$
F' = 2 \text{Re} \langle h^2 u', u' \rangle_S - 2 \text{Re} \langle (\Lambda + V\phi - E) u, u \rangle_S
$$

$$
+ 2r^{-1} \langle \Lambda u, u \rangle_S - \langle V\phi' u, u \rangle_S
$$

$$
= -2 \text{Re} \langle P\phi u, u' \rangle_S + 4h \phi' \|u'\|_S^2 + 2\varepsilon \text{Im} \langle u, u' \rangle_S
$$

$$
+ 2r^{-1} \langle \Lambda u, u \rangle_S - \langle V\phi' u, u \rangle_S.
$$

Consequently

$$
wF' + w'F = -2w \text{Re} \langle P\phi u, u' \rangle_S + (4h^{-1}w\phi' + w') \|hu'\|_S^2
$$

$$
+ 2w \varepsilon \text{Im} \langle u, u' \rangle_S + (2wr^{-1} - w') \langle \Lambda u, u \rangle_S
$$

$$
+ \langle (w(E - V\phi))' u, u \rangle_S.
$$

(6.5.12)

4. The equality (6.5.12) and the inequalities

$$
w\phi' \geq 0, \quad w' > 0, \quad \Lambda \geq 0, \quad 2wr^{-1} - w' > 0,
$$

and

$$
-2 \text{Re} \langle a, b \rangle + \|b\|^2 \geq -\|a\|^2,
$$

gives the crucial inequality satisfied by $F$:

$$
(wF)' \geq - \frac{w^2}{h^2 w'} \|P\phi u\|_S^2 + 2w \varepsilon \text{Im} \langle u, u' \rangle_S + \langle (w(E - V\phi))' u, u \rangle_S.
$$

(6.5.13)
5. We can now return to the proof of (6.5.9). For that we combine (6.5.13) with (6.5.11) and use $w \leq 1$ we obtain

\begin{equation}
\int_0^\infty \int_{\mathbb{S}^{n-1}} (w(E - V_\phi))' |u|^2 \, d\omega \, dr \leq \frac{1}{h^2} \int_0^\infty \int_{\mathbb{S}^{n-1}} |P_\phi u|^2 \frac{w'}{w'} \, d\omega \, dr \\
+ 2 \varepsilon \int_0^\infty \int_{\mathbb{S}^{n-1}} |uu'| \, d\omega \, dr.
\end{equation}

(6.5.14)

On the other hand, for all $\gamma > 0$ there is $C_\gamma$ such that

\begin{align*}
\int_0^\infty \int_{\mathbb{S}^{n-1}} |hu'|^2 \, d\omega \, dr = \text{Re} \int_0^\infty \int_{\mathbb{S}^{n-1}} \bar{u}(P_\phi - 2h\varphi' \partial_r - \Lambda - V_\phi + E + i\varepsilon)u \, d\omega \, dr \\
\leq \int_0^\infty \int_{\mathbb{S}^{n-1}} |P_\phi u||u| \, d\omega \, dr + 2 \int_0^\infty \int_{\mathbb{S}^{n-1}} \varphi' |hu'||u| \, d\omega \, dr \\
+ \int_0^\infty \int_{\mathbb{S}^{n-1}} |E - V_\phi||u|^2 \, d\omega \, dr \\
\leq \int_0^\infty \int_{\mathbb{S}^{n-1}} |P_\phi u|^2 \, d\omega \, dr + C_\gamma \int_0^\infty \int_{\mathbb{S}^{n-1}} |u|^2 \, d\omega \, dr \\
+ \gamma \int_0^\infty \int_{\mathbb{S}^{n-1}} \varphi' |hu'|^2 \, d\omega \, dr.
\end{align*}

Choosing $\gamma = 1/(2 \max \varphi')$ gives

\begin{equation}
\int_0^\infty \int_{\mathbb{S}^{n-1}} |hu'|^2 \, d\omega \, dr \leq 2 \int_0^\infty \int_{\mathbb{S}^{n-1}} |P_\phi u|^2 \, d\omega \, dr \\
+ C \int_0^\infty \int_{\mathbb{S}^{n-1}} |u|^2 \, d\omega \, dr.
\end{equation}

(6.5.15)

Applying the inequality

\begin{equation}
2 \int_0^\infty \int_{\mathbb{S}^{n-1}} |uu'| \, d\omega \, dr \leq h^{-1} \int_0^\infty \int_{\mathbb{S}^{n-1}} |u|^2 \, d\omega \, dr + h^{-1} \int_0^\infty \int_{\mathbb{S}^{n-1}} |hu'|^2 \, d\omega \, dr
\end{equation}

to (6.5.14), and using (6.5.15) and $\varepsilon \leq h$, gives (6.5.9). \hfill \square

**Proof of Theorem 6.22.** 1. Put $C_0 = 2 \max \varphi$. Since $\varphi(r) = C_0$ for $r \geq R_0$, Lemma 6.24 gives

\begin{align*}
e^{-C_0/h} \|m^{-1} \mathbf{1}_{B(0,R_0)} v\|^2_{L^2} + \|m^{-1} \mathbf{1}_{\mathbb{R}^n \setminus B(0,R_0)} v\|^2_{L^2} \\
\leq e^{-C_0/h} \|m^{-1} e^{\varphi/h} v\|^2_{L^2} \\
\leq \frac{C}{h^2} \|m(P - i\varepsilon)v\|^2_{L^2} + \frac{C_1 \varepsilon}{h} \|v\|^2_{L^2}.
\end{align*}
Then using
\[ 2\varepsilon\|v\|_{L^2}^2 = -2\text{Im} \langle (P - i\varepsilon)v, v \rangle_{L^2} \]
\[ \leq \gamma^{-1} \|m 1_{\mathbb{R}^n \setminus B(0, R_0)}(P - i\varepsilon)v\|_{L^2}^2 + \gamma \|m^{-1} 1_{\mathbb{R}^n \setminus B(0, R_0)}v\|_{L^2}^2 \]
\[ + \gamma_0^{-1} \|m 1_{B(0, R_0)}(P - i\varepsilon)v\|_{L^2}^2 + \gamma_0 \|m^{-1} 1_{B(0, R_0)}v\|_{L^2}^2, \]
with \( \gamma = e^{-2C_0/h} \) and \( \gamma_0 = h/C_1 \) we conclude that, for \( h \) sufficiently small,
\[ e^{-C/h} \|m^{-1} 1_{B(0, R_0)}v\|_{L^2}^2 + \|m^{-1} 1_{\mathbb{R}^n \setminus B(0, R_0)}v\|_{L^2}^2 \leq \]
(6.5.16)
\[ e^{C/h} \|m 1_{B(0, R_0)}(P - i\varepsilon)v\|_{L^2}^2 + \|m 1_{B(0, R_0)}(P - i\varepsilon)v\|_{L^2}^2 \]
for all \( v \in C_0^\infty(\mathbb{R}^n) \).

2. We will deduce from (6.5.16) that, for any \( f \in L^2 \), we have
\[ e^{-C/h} \|m^{-1} 1_{B(0, R_0)}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 \leq \]
(6.5.17)
\[ + \|m^{-1} 1_{\mathbb{R}^n \setminus B(0, R_0)}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2}^2 \leq \]
\[ e^{C/h} \|m 1_{B(0, R_0)} f\|_{L^2}^2 + \frac{C}{h^2} \|m 1_{\mathbb{R}^n \setminus B(0, R_0)} f\|_{L^2}^2, \]
from which Theorem 6.22 follows.

3. To prove (6.5.17) we need the fact that, for fixed \( \varepsilon, h > 0 \),
\[ \frac{1}{C_{\varepsilon, h}} \|mv\|_{H^2} \leq \|m(P - i\varepsilon)v\|_{L^2} \leq C_{\varepsilon, h} \|mv\|_{H^2}, \quad mv \in H^2. \]
Assuming (6.5.18), fix \( f \in L^2 \), so
\[ (P - i\varepsilon)^{-1}m^{-1}f \in H^2. \]

Take \( v_k \in C_0^\infty \) with
\[ \|mv_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \to 0 \text{ as } k \to \infty. \]
Then in particular
\[ \|m^{-1}v_k - m^{-1}(P - i\varepsilon)^{-1}m^{-1}f\|_{L^2} \to 0, \]
and, by (6.5.18),
\[ \|m(P - i\varepsilon)v_k - f\|_{L^2} \leq C_{\varepsilon, h} \|mv_k - m(P - i\varepsilon)^{-1}m^{-1}f\|_{H^2} \]
\[ \to 0, \quad \text{as } k \to \infty. \]
Consequently (6.5.17) follows by applying (6.5.16) with \( v_k \) in place of \( v \), and
\[ \text{letting } k \to \infty. \]

4. It remains to prove (6.5.18). We have
\[ \|mv\|_{H^2_k} \leq (C/\varepsilon)(P - i\varepsilon)mv\|_{L^2} \leq (C'/\varepsilon)\|mv\|_{H^2}^{\varepsilon}, \]
(6.5.19)
for all \( v \) with \( mv \in H^2 \). On the other hand,
\[
[P, m] = -2h^2 m' \partial_r - h^2 m'' - h^2 (n-1) m'/r = \mathcal{O}(h) h^2 \to L^2,
\]
allowing us to deduce the second inequality in (6.5.18) from the second inequality in (6.5.19):
\[
\|m(P - i \varepsilon)v\|_{L^2} \lesssim \|mv\|_{H^2} + \|[P, m]v\|_{L^2} \lesssim \|mv\|_{H^2}.
\]
Similarly we deduce the first of (6.5.18) from the first of (6.5.19):
\[
\|mv\|_{H^2} \lesssim \|m(P - i \varepsilon)v\|_{L^2} + \|[P, m]v\|_{L^2} \lesssim \|m(P - i \varepsilon)v\|_{L^2}.
\]
\[\square\]

The weighted estimate immediately implies estimates for the cut-off resolvent on the real axis:

**THEOREM 6.25 (Estimates of the cut-off resolvent).** Suppose that \( V, \partial_r V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}) \), \( n \geq 3 \) and that
\[
R(z, h) := (-h^2 \Delta + V - z)^{-1}, \quad \text{Im} \, z > 0.
\]
Fix \( 0 < a < b \) and assume that \( E \in [a, b] \). Then there exists \( C_0 > 0 \) such that for any \( R > 0 \) and \( \chi \in C_c^\infty(B(0, R)) \) there exists \( C_1 \) and
\[
\|\chi R(E, h)\chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_1 \exp \frac{C_0}{h}.
\]
In addition, there exist \( R_0 \) such that for \( \chi \in C_c^\infty(B(0, R \setminus B(0, R_0))) \),
\[
\|\chi R(E, h)\chi\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq \frac{C_1}{h}.
\]

We now have an important conclusion about the minimal width of scattering resonances:

**THEOREM 6.26 (Lower bounds on resonance width).** Suppose that \( V \) satisfies the assumptions of Theorem 6.25 and that \( 0 < a < b \) are fixed. Then there exist constants \( C_0, h_0 \) such that for \( 0 < h < h_0 \),
\[
z \in \text{Res}(-h^2 \Delta + V), \quad \text{Re} \, z \in [a, b] \implies -\text{Im} \, z > e^{-C_0/h}.
\]

To prove this we will need the following lemma which provides a useful resolvent identity similar to the identities used in §4.2:

**LEMMA 6.27 (A resolvent identity).** For \( V \in L^\infty_c(\mathbb{R}^n; \mathbb{R}) \) and \( \text{Im} \, z > 0 \) put
\[
R(z) = R(z, h) := (-h^2 \Delta + V - z)^{-1},
\]
\[
R_0(z) = R_0(z, h) := (-h^2 \Delta - z)^{-1}.
\]
Suppose that \( \chi, \chi_0 \in C_c^\infty(\mathbb{R}^n) \), have the property that \( \chi_0 = 1 \) on \( \text{supp} \, V \) and \( \chi = 1 \) on \( \text{supp} \, \chi_0 \),

(6.5.20) \[ C = C(h) := [h^2 \Delta, \chi_0] = \mathcal{O}(h) : H_0^s(\mathbb{R}^n) \rightarrow H_0^{s-1}(\mathbb{R}^n). \]

and

(6.5.21) \[ R_\chi(z) := \chi R(z) \chi, \quad R_{0, \chi}(z) = \chi R_0(z) \chi. \]

Then

(6.5.22) \[ R_\chi(z) - R_\chi(z_0) = (z - z_0) R_\chi(z) \chi_0 (2 - \chi_0) R_\chi(z_0) + (1 - \chi_0 - R_\chi(z) C(h)) (R_{0, \chi}(z) - R_{0, \chi}(z_0)) \times (1 - \chi_0 + C(h) R_\chi(z_0)). \]

Proof. 1. Since \( \chi_0 = 1 \) on the support of \( V \),

\[ (-h^2 \Delta + V)(1 - \chi_0) = (-h^2 \Delta)(1 - \chi_0). \]

Hence,

(6.5.23) \[ R(z)(1 - \chi_0) - (1 - \chi_0) R_0(z) = R(z)(1 - \chi_0) (-h^2 \Delta - z) R_0(z) - R(z) (-h^2 \Delta - z)(1 - \chi_0) R_0(z) = R(z) [h^2 \Delta, \chi_0] R_0(z) \]

and similarly

(6.5.24) \[ (1 - \chi_0) R(z_0) - R_0(z_0)(1 - \chi_0) = R_0(z_0) [h^2 \Delta, \chi_0] R(z_0). \]

2. For \( \text{Im} \, z > 0 \) we also have

\[ R(z) - R(z_0) = (z - z_0) R(z) R(z_0) = (z - z_0) \left( R(z) \chi_0 (2 - \chi_0) R(z_0) + R(z)(1 - \chi_0)^2 R(z_0) \right). \]

We now use (6.5.23) and (6.5.24) to rewrite the second term on the right hand side. With the notation (6.5.20) this gives

\[ R(z)(1 - \chi_0)^2 R(z_0) = (R(z) C R_0(z) + (1 - \chi_0) R_0(z)) (1 - \chi_0) R(z_0) = R(z) C R_0(z_0) (1 - \chi_0) + R(z) C R_0(z_0) C R(z_0) + (1 - \chi_0) R_0(z_0) R_0(z_0) (1 - \chi_0) + (1 - \chi_0) R_0(z_0) R_0(z_0) C R(z) = (1 - \chi_0 + R(z) C) R_0(z_0) R_0(z_0)(1 - \chi_0 + C R(z_0)). \]

4. Since

\[ R_0(z) R_0(z_0) = (z - z_0)^{-1} (R_0(z) - R_0(z_0)) \]

and \( \chi \chi_0 = \chi_0, \chi C = C \) the combination of the two identities in Step 3 gives (6.5.22). \( \square \)
6.5. LOWER BOUNDS ON RESONANCE WIDTHS

Proof of Theorem 6.26. 1. We use the notation introduced in Lemma 6.27 and take \( z_0 \in [a, b] \) and \( z \) with \( \Re z = z_0, |\Im z| \leq h \). Theorem 6.26 shows that

\[
\| R_{\chi}(z_0) \| \leq C_1 \exp(C_2/h),
\]

for some constants \( C_1 \) and \( C_2 \) depending only on \( \chi \) and \( a, b \). The norm is the operator norm \( L^2 \to L^2 \).

2. The bound (3.1.12) for the free resolvent rescales to a semiclassical bound

\[
R_{0,\chi}(w) = \mathcal{O}(h^{-1}|\Im w|/h) : H^s_h(\mathbb{R}^n) \to H^{s+j}_h, \quad j = 0, 1, 2, \ w \in [a, b]+i[-1, 1].
\]

Cauchy inequalities show that we also have

\[
\partial_w R_{0,\chi}(w) = \mathcal{O}(h^{-2} : H^s_h(\mathbb{R}^n) \to H^{s+j}_h, \quad j = 0, 1, 2, \ w \in [a, b]+i[-h, h],
\]

and hence,

\[
\| R_{0,\chi}(z) - R_{0,\chi}(z_0) \| = (h^{-2}|\Im z|) : H^s_h \to H^{s+j}_h, \quad j = 0, 1, 2.
\]

3. We now use the resolvent identity (6.5.22), (6.5.26) and (6.5.25)

\[
\| R_{\chi}(z) \| \leq \| R_{\chi}(z_0) \| + |\Im z|\| R_{\chi}(z) \|\| R_{\chi}(z_0) \|
\]

\[
+ \sum_{j,k=0}^1 \| R_{\chi}(z) \|^j \| C(h)^j(R_{0,\chi}(z) - R_{0,\chi}(z_0))C(h)^k\|\| R_{\chi}(z_0) \|^k
\]

\[
\leq \| R_{\chi}(z_0) \| + |\Im z|\| R_{\chi}(z) \|\| R_{\chi}(z_0) \|
\]

\[
+ \sum_{j,k=0}^1 h^{k+j}\| R_{\chi}(z) \|^j \| R_{\chi}(z_0) \|^k\| R_{0,\chi}(z) - R_{0,\chi}(z_0) \|_{H^k_h \to H^{s+j}_h}
\]

\[
\leq C_3 e^{C_2/h} + |\Im z|C_3 h^{-2}e^{C_2/h}\| R_{\chi}(z) \|.
\]

The meromorphy of \( R_{\chi}(z) \) implies that \( R_{\chi}(z) \) is finite except on a discrete set and hence for

\[
|\Im z| < e^{-C_0/h} \leq (2C_3)^{-1}h^2e^{-C_2/h},
\]

we have

\[
\| R_{\chi}(z) \| \leq 2C_3 e^{C_2/h},
\]

which completes the proof. \( \square \)

REMARKS. 1. A more direct proof of Theorem 6.26 can given using a (much more complicated) version of the proof we presented in one dimension – see Theorem 2.30. The key element is the following inequality [Bu98, Proposition 2.2]: suppose that \( u \) is outgoing in the sense that

\[
u(h) := R_0(z, h)f(h), \quad f \in L^\infty_{\text{comp}}(B(0, R_0)),
\]
for $R_0$ fixed. Then $R_2 > R_1 > R_0$, $|\text{Im} \, z| \leq Ch$, $\text{Re} \, z \in (a, b)$, $0 < h < h_0$,

\begin{equation}
-\int_{\partial B(0,R_2)} \text{Im} \, h \partial_r u \overline{u} dS \geq c \int_{\partial B(0,R_2)} (|u|^2 + |hDu|^2) dS
- Ce^{-c/h} \int_{\partial B(0,R_1)} (|u|^2 + |hDu|^2) dS.
\end{equation}

2. Using (6.5.27) one can show that existence of resonance very close to real axis implies existence of localized quasimodes – see Stefanov [St00]. Then Theorem 6.26 can be proved using an argument by contradiction and the results of §7.3.

6.6. NOTES

For a presentation of classical scattering in a more general setting see [GS87, Appendix].

Semiclassical defect measures were used to show existence of resonance free strips (to which we will refer to as spectral gaps) by Burq [Bu02b] for obstacle problems. That provided a weaker results than logarithmic free strips obtained using propagation of singularities (see §§4.6,4.7) but was much simpler technically. Here we followed Dyatlov [Dy16a] and applied the semiclassical defect measures to get spectral gaps and optimal resolvent bounds for normally hyperbolic trapping. That improved earlier results of Gérard–Sjöstrand [GS88] in the analytic case and of Wunsch–Zworski [WZ11] in the smooth case. Under additional dynamical assumptions, existence of resonances and a counting law was given in Dyatlov [Dy15a]. One example of normally hyperbolic trapping with the assumptions of §6.3 satisfied is given by photon spheres of black holes – see [WZ11],[Dy11],[Dy12],[Dy15b],[HV14a] and [HV16]. (For a review see [DZ13].) This kind of trapping occurs also in molecular dynamics [G*10]. For a mathematical treatment under more general assumptions see Nonnenmacher–Zworski [NZ15].

When the trapped is hyperbolic in the sense of dynamical systems (see the references below) spectral gaps exist under a classical “topological pressure condition”. For obstacle problems that was initiated by Ikawa [Ik88] in mathematics and Gaspard–Rice [GR89] in physics, see also [PS10]. A semiclassical version was proved by Nonnenmacher–Zworski [NZ09],[NZ09b]. It is now suspected that quantum effects will produce spectral gaps for any hyperbolic trapped set. Evidence for that is provided by a recent result of Bourgain–Dyatlov [BD18] who proved a presence of a spectral gap for any convex co-compact Riemann surface. The proof was based on a fractal uncertainty principle and microlocal methods developed by Dyatlov–Zahl.
6.7. EXERCISES

Section 6.1

Exercises 6.1–6.9 explore a more general set of assumptions on the Hamiltonian \( p \) under which the results of §6.1 hold, which are as follows:

1. \( M \) is a manifold, \( p \in C^\infty(T^*M; \mathbb{R}) \), and the Hamiltonian flow \( \exp(tH_p) \) is defined on \( T^*M \) for all times;
(2) $r \in C^\infty(T^*M; \mathbb{R})$ is a function and $\alpha \leq \beta$ are numbers such that the sets

$$U_R = r^{-1}((-\infty, R]) \cap p^{-1}([\alpha, \beta])$$

are compact for each $R$;

(3) there exists a constant $r_0$ such that the following convexity assumption holds:

$$p(x, \xi) \in [\alpha, \beta], \quad r(x, \xi) \geq r_0, \quad H_p r(x, \xi) = 0 \implies H_p^2 r(x, \xi) > 0.$$  

We define the sets $\Gamma_{[\alpha, \beta]}^\pm$ as follows: $(x, \xi) \in \Gamma_{[\alpha, \beta]}^\pm$ if $p(x, \xi) \in [\alpha, \beta]$ and $r(e^{tH_p}(x, \xi))$ does not converge to infinity as $t \to \mp \infty$. Put $K_{[\alpha, \beta]} = \Gamma_{[\alpha, \beta]}^+ \cap \Gamma_{[\alpha, \beta]}^-$. 

1. Show that if assumptions (1)–(3) above hold, then they also hold with $r$ replaced by $e^r F(r)$, where $F : \mathbb{R} \to \mathbb{R}$ is any smooth function such that $F' > 0$ everywhere and $\lim_{r \to \pm \infty} F(r) = \pm \infty$, and $f \in C^\infty(M; \mathbb{R})$ is bounded.

2. Show that assumptions (1)–(3) above hold in each of the following cases:

(a) $M = \mathbb{R}^n$ and $p(x, \xi) = |\xi|^2 + V(x)$, where $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ satisfies

$$\limsup_{x \to \infty} V(x) < \alpha, \quad \limsup_{x \to \infty} (x, \nabla V(x)) \leq 0,$$

and $r(x, \xi) := |x|$ for $|x|$ large enough.

(b) $(M, g)$ is an asymptotically hyperbolic manifold in the sense of Definition 5.2, $p(x, \xi) = |\xi|^2 + \alpha > 0$, and $r = \gamma_1^{-1}$, where $\gamma_1$ is any boundary defining function of $\overline{M}$.

(c) $M = \mathbb{R}$, $p(x, \xi) = x^2$, and $r(x, \xi) = \sqrt{x^2 + \xi^2}$ for large enough $(x, \xi)$. What are $\Gamma_{[\alpha, \beta]}^\pm$ in this case?

3. Under assumptions (1)–(3) above, show that for each $R \geq r_0$, the set $U_R$ defined in (6.7.1) is convex with respect to the flow $\exp(tH_p)$: that is, if $e^{-tH_p}(x, \xi) \in U_R$, $e^{\gamma H_p}(x, \xi) \in U_R$ for some $\gamma \geq 0$, then $(x, \xi) \in U_R$. (Hint: find the maximal value of $r(e^{tH_p}(x, \xi))$ on the interval $t \in [-t^-, t^+]$.)

4. Under assumptions (1)–(3) above, take $(x, \xi)$ such that

$$p(x, \xi) \in [\alpha, \beta], \quad r(x, \xi) \geq r_0, \quad \pm H_p r(x, \xi) \geq 0.$$

(a) Show that

$$r(e^{tH_p}(x, \xi)) > r_0, \quad \pm H_p r(e^{tH_p}(x, \xi)) > 0 \quad \text{for all } t, \quad \pm t > 0.$$

(Hint: for the case $H_p r(x, \xi) \geq 0$ and each $T > 0$, find the maximum of the function $r(e^{tH_p}(x, \xi))$ on the interval $[0, T]$.)
6.7. EXERCISES

(b) Show that \((x, \xi) \notin \Gamma^\pm_{[\alpha, \beta]}\), that is \(r(e^{tH_p}(x, \xi)) \to \infty\) as \(t \to \pm \infty\). (Hint: argue by contradiction, taking for the case \(H_p r(x, \xi) \geq 0\) a sequence of \(t_j \to \infty\) such that \(r(e^{t_j H_p}(x, \xi))\) is bounded and extracting a convergent subsequence. Apply assumption (3) to the limiting point of this subsequence.)

5. Using the previous exercise and the proof of Proposition 6.3, show that the sets \(\Gamma^\pm_{[\alpha, \beta]}\) are closed and \(K_{[\alpha, \beta]} \subset \{r < r_0\}\) is compact.

6. Under assumptions (1)–(3) above, show that for each \(R\) and each neighbourhood \(U\) of \(K_{[\alpha, \beta]}\), there exists \(T > 0\) such that for all \(t \pm \geq T\),
\[
p(x, \xi) \in [\alpha, \beta], \quad r(e^{-t-H_p}(x, \xi)) \leq R, \quad r(e^{t+H_p}(x, \xi)) \leq R \implies (x, \xi) \in U.
\]

In other words, every trajectory that passes a long time in a bounded set has to have many points close to the trapped set. (Hint: argue by contradiction, taking a sequence \(t_j \to \infty\) and \((x_j, \xi_j) \in p^{-1}([\alpha, \beta]) \setminus U\) such that \(r(e^{-t_j H_p}(x_j, \xi_j)), r(e^{t_j+H_p}(x_j, \xi_j)) \leq R\). Using Exercise 6.3, take a subsequence of \((x_j, \xi_j)\) converging to some \((x_\infty, \xi_\infty) \notin K_{[\alpha, \beta]}\). Use the fact that \(e^{tH_p}(x_\infty, \xi_\infty)\) escapes in at least one time direction together with Exercise 6.4 to arrive to a contradiction.)

7. Use the previous exercise to show that for each \((x, \xi) \in \Gamma^\pm_{[\alpha, \beta]}\), the trajectory \(e^{tH_p}(x, \xi)\) converges to \(K_{[\alpha, \beta]}\) as \(t \to \pm \infty\), and the convergence is uniform for \((x, \xi)\) in a compact set.

8. Arguing as in the proof of Proposition 6.5, show that the sets \(\Gamma^\pm_{[\alpha, \beta]} \setminus K_{[\alpha, \beta]}\) have measure zero in \(T^*M\). Show that this is false with respect to the one-dimensional Lebesgue measure on \(p^{-1}(0)\) in the case of Exercise 6.2(c), and explain why this does not give a contradiction.

9. This exercise gives an example of a situation where the trapped set is not closed. Consider a Riemannian surface \((M, g)\) with one infinite end which is a cusp \([0, \infty)_r \times S^1_\theta\) with the metric \(g = dr^2 + e^{-2r}d\theta^2\) in the cusp. Let \(p(x, \xi) = |\xi|_g^2\) and take \(0 < \alpha \leq \beta\).

(a) Show that the function \(r\) does not satisfy (6.7.2).

(b) Show that a point \((x, \xi)\) with \(r = 0, \xi_r > 0, \xi_\theta = 0\) does not lie in \(\Gamma^-_{[\alpha, \beta]}\), but it lies in the closure of \(K_{[\alpha, \beta]}\). (Hint: for the latter part, show that nontrapped trajectories form a set of zero measure.)

Section 6.2

10. Deduce from Theorem 6.11 that:

(a) \(\nu \leq 0\);
(b) if $\nu = 0$, then $\mu$ is supported on $K$;
(c) if $\nu < 0$, then $\mu(K) = 0$.

11. In the case of (6.0.4) let $u$ be a resonant state with $z \in [\alpha, \beta] - i[0, Ch]$, $\alpha > 0$. Let $\chi_0 \in C_c^\infty(M)$ be equal to 1 on a sufficiently large ball. Prove that for $h$ small enough

\begin{equation}
\|\chi u\|_{L^2} \leq C(\chi, \chi_0)\|\chi_0 u\|_{L^2} \quad \text{for all } \chi \in C_c^\infty(M)
\end{equation}

**Hint:** Use Theorem 4.9 and properties of free resolvent.

12. Let $P = -h^2\Delta_g$ where $(M, g)$ has Euclidean infinite ends. Assume that $P$ has an essential spectral gap of size $\beta > 0$ with a polynomial resolvent bound: that is, there exist $h_0 > 0$ and $N > 0$ such that for all $\chi \in C_c^\infty(M)$

\begin{align*}
\|\chi R(z, h)\|_{L^2 \rightarrow L^2} &\leq C h^{-1-N}, \\
\text{Re } z = 1, \quad -\beta h \leq \text{Im } z \leq h, \quad 0 < h < h_0.
\end{align*}

Show that the wave equation on $\mathbb{R}^t \times M$ has a resonance expansion with loss of $N$ derivatives. What changes need to be made when $(M, g)$ is instead asymptotically hyperbolic?

13. Show that each measure $\mu$ from Theorem 6.11 is supported in $\{r \leq 2r_1\}$ where $r_1$ satisfies (6.2.4).

14. Prove Proposition 6.12 using the approximate inverse with complex absorption constructed in Theorem 5.33 together with its outgoing property proved in Theorem 5.34.

Section 6.3

15. For each $\varphi_\pm$ satisfying (6.3.2)–(6.3.3), assumptions (A1)–(A2) imply that $\{\varphi_+, \varphi_-\} \neq 0$ on $K_{[\alpha', \beta']}$. Use this to show that if (A3) holds for one of the functions $\varphi_\pm$, then it holds for the other one as well, and the constants $\nu_{\min}$ coming from $\varphi_+$ and $\varphi_-$ are the same.

16. TODO the model case $p = x\xi$.

[Exercise: hyperbolic cylinders in higher dimensions]

[Exercise: hyperbolic cylinder in 1D - obtain a full picture in the Vasy-setting.]

[Exercise: better bands under the $r$-normally hyperbolic assumption]

Section 6.4

17. Using the remark after the proof of Theorem 6.21 to prove (??) without invoking Theorem 6.10.

Section 6.5
Chapter 7

RESONANCES AND TRAPPING

7.1 Lower bounds on the resolvent
7.2 Semiclassical growth estimates
7.3 From quasimodes to resonances
7.4 The Sjöstrand trace formula
7.5 Resonance expansions for strong trapping
7.6 Notes

In this chapter we will discuss the effects of the presence of trapping on the distribution of resonances. We first present a result of Bony–Burq–Rammond which shows that a non-empty trapped set implies a lower bound the (cut-off) resolvent on the real axis. That lower bound differs from the non-trapping bound by a logarithmic factor. It is optimal as shown by Theorem 6.16.

We then discuss general bounds on the number of scattering poles and on the resolvent. These are applied to show how localized quasimodes (approximate eigenfunctions) imply existence of resonances close to the real axis. The general results of Tang–Zworski and Stefanov are presented in the special case of semiclassical Schrödinger operators but the method applies in great generality, for instance for obstacle problems.

We continue with Sjöstrand’s local trace formula and with his lower bounds on the number of resonances. In that case trapping is not explicitly discussed but comes from the presence of certain analytic singularities.
Finally, we present a result of Burq–Zworski on the expansion of solutions of evolution equations in terms of resonances close to the real axis, that is, resonances generated by strong trapping.

7.1. LOWER BOUNDS ON THE RESOLVENT

In §6.10 we have shown that the truncated resolvent satisfies

\[ K_E = \emptyset \implies \chi R(E, h) = O_{L^2 \to L^2}(1/h), \]

\[ R(E, h) := (P - E - i0)^{-1}, \quad P = -\hbar^2 \Delta_g + V, \]

\[ V, g_{ij} - \delta_{ij} \in C_\infty_c(\mathbb{R}^n; \mathbb{R}), \]

and this notation will be used throughout this section.

In this section we consider a lower bound on the norm of the resolvent in the case of arbitrary trapping.

**Theorem 7.1 (Lower bounds on resolvent for trapping perturbations).** Suppose that \( E_0 > 0 \) and that \( K_{E_0} \neq \emptyset \), and that \( \chi \in C_\infty_c(\mathbb{R}^n) \) is equal to 1 near \( \pi(K_{E_0}) \).

Then there exists \( C_0 = C_0(E_0) \) such that for any \( \delta > 0 \) there exists \( h_0 = h_0(\delta) \) so that

\[
\sup_{|E - E_0| < \delta} \| \chi R(E, h) \chi \|_{L^2 \to L^2} \geq \frac{\log(1/h)}{C_0 h},
\]

\( 0 < h < h_0. \)

**Remark.** Theorem 6.16 shows that this estimate is optimal. The point here is that for any trapping situation we cannot do better than (6.3.10).

Before giving the proof of Theorem 7.1 we need to present an older result, essentially due to Kato, relating resolvent estimates to local smoothing in Schrödinger propagation.

**Theorem 7.2 (Kato’s local smoothing).** Let \( E_0 > 0 \) and let \( K(h) \geq 1 \) be a function on \((0, 1)\).

Suppose that \( |E - E_0| < \delta \) and \( \chi \in L^\infty(\mathbb{R}^n) \) we have

\[
\| \chi R(E, h) \chi \|_{L^2 \to L^2} \leq \frac{K(h)}{h},
\]

Then for \( \varphi \in C_\infty_c((E - \delta, E + \delta); [0, 1]) \) and \( u \in L^2(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}} \| \chi \varphi(P) \exp(-itP/h)u \|_{L^2}^2 \, dt \leq CK(h)\| u \|_{L^2}^2,
\]

with \( C \) independent of \( h \).
INTERPRETATION. 1. If the integration in (7.1.3) takes place over a finite interval in time, \([0,T]\), then the estimate is obvious with \(CK(h)\) replaced by \(T\). The localization in space, \(\chi(x)\) and in energy, \(\varphi(P)\) are also not needed. Hence the point lies in having the integral over \(\mathbb{R}\). For that \(\chi\) for which (7.1.2) holds is needed. In our presentation we take \(\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)\) but finer weights, such as \(\langle x \rangle^{-1/2-\epsilon}\) also work – see [VZ00] and references given there.

2. When \(P = -h^2\Delta g\), where \(g\) is a metric, we can change variables in the the \(t\) integration in (7.1.3) to obtain
\[
\int_{\mathbb{R}} \|\chi \varphi(-h^2\Delta g) \exp(-it\Delta_g)u\|_{L^2}^2 dt \leq ChK(h)\|u\|_{L^2},
\]
where \(\varphi \in \mathcal{C}_c^\infty((0,\infty))\).

If \(K(h) = 1\), as is the case in (6.2.22) under non-trapping assumption, then
\[
\int_{\mathbb{R}} \|\chi \varphi_1(-h^2\Delta g)(-\Delta_g)^{1/4} \exp(-it\Delta_g)u\|_{L^2}^2 dt \leq C\|u\|_{L^2},
\]
where \(\varphi_1(\lambda) := \varphi(\lambda) / \lambda^{1/4} \in \mathcal{C}_c^\infty((0,\infty))\).

A dyadic decomposition (see for instance [Zw12, Section 7.5] for a presentation in semiclassical spirit) then shows that
\[
(7.1.4) \quad \int_{\mathbb{R}} \|\chi (1 - \psi)(-\Delta_g) \exp(-it\Delta_g)u\|_{H^{1/2}}^2 dt \leq C\|u\|_{L^2},
\]
where \(\psi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])\), \(\psi \equiv 1\) near 0. To control the term with \(\psi(-\Delta_g)\) one needs finer analysis of the bottom of the spectrum of \(-\Delta_g\) but a crude bound gives
\[
(7.1.5) \quad \int_{-T}^{T} \|\chi \exp(-it\Delta_g)u\|_{H^{1/2}}^2 dt \leq CT\|u\|_{L^2},
\]
This is the local smoothing estimate for non-trapping perturbations. In this formulation the smoothing character is clear: we gain 1/2 derivative when localizing in space and averaging in time.

3. Doi [Do96] showed that any trapping produces a loss in the \(H^{1/2}\) regularity. The proof of Theorem 7.1 uses Theorem 7.2 and a semiclassical and quantitative version of his argument to obtain the lower bound \(K(h) \geq \log(1/h)/C\).

Proof of Theorem 7.2. 1. We apply a \(TT^*\) argument which starts with defining
\[
T : u \mapsto \chi \varphi(P)e^{-itP/h},
\]
\[
T : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R} \times \mathbb{R}^n),
\]
so that (7.1.3) can be rewritten as
\[ \|Tu\|_{L^2_{tx}}^2 \leq CK(h)\|u\|_{L^2_x}^2, \]
which in turn is equivalent to
\[ \|T^*f\|_{L^2_{tx}}^2 \leq CK(h)\|f\|_{L^2_{tx}}^2. \]
This last inequality follows from showing that
\[ (7.1.6) \quad TT^* = O(K(h)): L^2(\mathbb{R} \times \mathbb{R}^n) \longrightarrow L^2(\mathbb{R} \times \mathbb{R}^n). \]

2. To obtain (7.1.6) we start by calculating the adjoint:
\[ T^*f = \int_{\mathbb{R}} e^{isP/h} \varphi(P) \chi f(s) ds, \]
first defined for \( f \in C_\infty^\infty(\mathbb{R} \times \mathbb{R}^n) \). We have
\[ TT^*f = \int_{\mathbb{R}} \chi e^{-i(t-s)P/h} \varphi(P)^2 \chi f(s) ds. \]
This we can rewrite as
\[ (7.1.7) \quad TT^*f = \left( \chi e^{-i\bullet P/h} \varphi(P)^2 \chi \right) * (f(\bullet))(t), \]
where \( * \) denotes the convolution in the \( t \) variable.

3. We now note that the inverse semiclassical Fourier transform,
\[ \mathcal{F}^*_h \psi(\lambda) := \frac{1}{\sqrt{2\pi h}} \int e^{it\lambda/h} \psi(t) dt, \]
of \( e^{-itP/h} \varphi(P) \) is, formally equal to
\[ h\delta(P - \lambda)\varphi(P) = h\delta(P - \lambda)\varphi(\lambda), \]
which can then be expressed using the Stone formula – see (B.1.14).

Returning to (7.1.7) and using the relation between the Fourier transforms and convolution (paying attention to the factor of \( \sqrt{h} \) because of the unitarity of \( \mathcal{F} \)) we see that
\[ TT^*f = (h/i)\mathcal{F}_\lambda \rightarrow t \left( \sum_\pm \chi(P - \lambda \pm i0)^{-1} \varphi(\lambda)^2 \chi \right) \mathcal{F}^*_0 \rightarrow \lambda \left( f(t) \right). \]
To conclude the proof we apply Plancherel’s formula:
\[ \|TT^*f\|_{L^2_{tx}} \leq h \left\| \left( \sum_{\pm} \pm \chi(P - \lambda \pm i0)^{-1}\varphi(\lambda)^2 \mathcal{F}_{t \rightarrow \lambda}(f(t)) \right) \right\|_{L^2_{\lambda x}} \leq 2h \sup_{\lambda} \|\varphi(\lambda)^2 \chi(P - \lambda - i0)^{-1}\chi\|_{L^2_{\lambda x} \rightarrow L^2_{\lambda x}} \|f\|_{L^2_{tx}} \]
Here we used hypothesis (7.1.2), the assumptions on \( \varphi \), and the basic fact that the norms of \( \chi(P - \lambda \pm i0)^{-1}\chi \) are the same. This proves (7.1.6) and consequently (7.1.3).

**Proof of Theorem 7.1.** 1. We will use Theorem 7.2. It shows that if for some nontrivial \( u_0 \in L^2(\mathbb{R}^n) \) and
\[ \varphi \in C_0^\infty((E_0 - \delta, E_0 + \delta); [0, 1]), \quad \varphi(E_0) = 1, \]
(7.1.8)
\[ \|\chi\varphi(P)\exp(-itP/h)u_0\|_{L^2_{tx}}^2 \geq K(h)\|u_0\|_{L^2_{tx}}, \]
then
\[ \sup_{|E - E_0| < \delta} \|\chi(P - E - i0)^{-1}\chi\|_{L^2 \rightarrow L^2} \geq \frac{K(h)}{Ch}. \]
Hence we need to show that for \( \chi \) satisfying
(7.1.9)
\[ \chi \in C_0^\infty(T^*\mathbb{R}^n), \quad \chi \equiv 1 \text{ near } \pi(K_{E_0}), \]
(7.1.8) holds with
\[ K(h) = c \log \frac{1}{h}, \]
where \( c \) is independent of \( \delta, 0 < h < h_0(\delta). \)

2. Functional calculus for pseudodifferential operators (see [DS99, Chapter 8] or [Zw12]) shows that
\[ \varphi(P(h))\chi(x)^2\varphi(P(h)) = a^w(x, hD), \quad a \in \mathcal{S}(T^*\mathbb{R}^n), \]
(7.1.10)
\[ a(x, \xi) = \chi(x)^2\varphi(p(x, \xi))^2 + O(h(x)^{-\infty}(\xi)^{-\infty}). \]
We put
\[ a^w_t(x, hD) := e^{itP/h}a^w(x, hD)e^{-itP/h}. \]
Theorem E.42 shows that for
(7.1.11)
\[ 0 < t < \alpha \log \frac{1}{h}, \]
with $\alpha$ sufficiently small, independent of $\delta$,

$$a_t \in S_\gamma(T^*\mathbb{R}^n), \quad 0 < \gamma < 1/2,$$

(7.1.12)

$$a_t - (\exp tH_p)^* a \in h^{2-3\gamma} S_\gamma(T^*\mathbb{R}^n),$$

with all the symbol estimates uniform for $t$ satisfying (E.3.2).

3. With this notation we have

$$\|\chi \varphi(P) \exp(-itP/h)u_0\|_{L^2_x}^2 = \int_\mathbb{R} \|\chi \varphi(P)e^{-itP/h}u_0\|_{L^2_x} dt \geq \int_0^{\alpha \log(1/h)} \|\chi \varphi(P)e^{-itP/h}u_0\|_{L^2_x} dt = \int_0^{\alpha \log(1/h)} \langle a^w_t(x, hD)u_0, u_0 \rangle_{L^2_x} dt.$$

(7.1.13)

Hence, it remains to find $u_0$ such that

$$\langle a^w_t(x, hD)u_0, u_0 \rangle \geq \frac{1}{2}, \quad \|u_0\|_{L^2(\mathbb{R}^n)} = 1,$$

uniformly for

$$0 < h < h_0, \quad 0 < t < \alpha \log(1/h).$$

4. To find $u_0$ satisfying (7.1.14) we choose $(x_0, \xi_0) \in K_{E_0}$ and take for $u_0$ a coherent state concentrated at $(x_0, \xi_0)$:

$$u_0(x) = (2\pi h)^{-n/4} \exp\left(\frac{i}{h}(\langle x - x_0, \xi_0 \rangle + i|x - x_0|^2/2)\right).$$

(7.1.15)

Since $K_{E_0}$ is invariant under the flow

$$\exp(tH_p)(x_0, \xi_0) \in K_{E_0}.$$

The assumption (7.1.9) and the fact that $\varphi(E_0) = 1$ show that

$$\langle \exp tH_p)^* a(x_0, \xi_0) = 1,$$

for all time.

Consequently, (7.1.12) gives

$$a_t(x_0, \xi_0) = 1 + O(h^{1/2}),$$

uniformly for $0 < t < \alpha \log 1/h$.

The properties of $\langle a^w_t(x, hD)u_0, u_0 \rangle$ are implied by the following lemma:
### Lemma 7.3

Suppose that $u_0$ is given by (7.1.15) and that $b \in S_\gamma$, $0 < \gamma < 1/2$. Then

\[
\langle b^u(x, hD)u_0, u_0 \rangle = b(x_0, \xi_0) + e(h),
\]

(7.1.16)

\[
|e(h)| \leq C_n h^\frac{\gamma}{2} \max_{|\alpha|=1} \sup_{T^* R^n} |\partial^\alpha b| \leq C_n(b) h^{1/2 - \gamma}.
\]

**Proof.** 1. Using the definition of $u_0$ (7.1.15) and making a change of variables $x = z + w$, $y = z - w$ we obtain

\[
\langle b(x, hD)u_0, u_0 \rangle = \frac{1}{(2\pi h)^n} \int_{R^n} \int_{R^n} b((x + y)/2, \xi) e^{\frac{i}{h}(\xi - y, \xi)} u_0(y) u_0(x) dyd\xi dx
\]

\[
= \frac{2^n}{(2\pi h)^n} \int_{R^n} \int_{R^n} \int_{R^n} b(z, \xi) e^{\frac{2i}{\pi}(w, \xi - \xi_0)} e^{-\frac{i}{h}(|z - x_0|^2 + |w|)^2} dwd\xi dz,
\]

2. For each fixed $z$ and $\xi$, the integral in $w$ is

\[
\int_{R^n} e^{\frac{2i}{\pi}(w, \xi - \xi_0)} e^{-\frac{i}{h}|w|^2} dw = e^{-\frac{i}{h}|\xi - \xi_0|^2} \int_{R^n} e^{-\frac{i}{h}|w + i(\xi - \xi_0)|^2} dw = 2^{-\frac{n}{2}} (2\pi h)^{\frac{n}{2}} e^{-\frac{i}{h}|\xi - \xi_0|^2}
\]

3. Therefore

\[
\langle b(x, hD)u_0, u_0 \rangle = \frac{2^n}{(2\pi h)^n} \int_{R^n} \int_{R^n} b(z, \xi) e^{-\frac{i}{h}(|z - x_0|^2 + |\xi - \xi_0|^2)} dxd\xi
\]

\[
= b_0(x_0, \xi_0) \frac{2^n}{(2\pi h)^n} \int_{R^n} \int_{R^n} e^{-\frac{i}{h}(|z - x_0|^2 + |\xi - \xi_0|^2)} dxd\xi + e(h)
\]

\[
= C_n(h) b(x_0, \xi_0) + e(h),
\]

where $e(h)$ satisfies the estimate of (7.1.16) and

\[
C_n(h) := \frac{2^n}{(2\pi)^n} \int_{R^n} \int_{R^n} e^{i(x, \xi)} e^{-\frac{i}{h}(|x|^2 + |\xi|^2)} dxd\xi.
\]

Taking $b \equiv 1$ and recalling that $\|u_0\|_{L^2} = 1$, we deduce that $C_n(h) = 1$. □

---

**End of proof of Theorem 7.1.** 5. We apply the lemma to $b = a_t$ which gives

\[
\langle a_t^u(x, hD)u_0, u_0 \rangle \to a_t(x_0, \xi_0) = 1,
\]

again uniformly in $t$. Hence (7.1.14) holds. Using (7.1.13) we obtain

\[
\|\chi \varphi(P) \exp(-itP/h)u_0\|_{L^2}^2 \geq \frac{\alpha}{2} \log \frac{1}{h},
\]

which is (7.1.8) with $K(h) = c \log(1/h)$, as needed for (7.1.1). □
7.2. SEMICLASSICAL GROWTH ESTIMATES

In this section we present local bounds on the resolvent and on the number of resonances for operators of the form

\[(7.2.1) \quad P = P(h) = -h^2 \Delta_g + V,\]

where \(g\) is a smooth Riemannian metric on \(\mathbb{R}^n\) satisfying \(g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n)\) and \(V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})\). If \(n\) is odd global bounds for compact black box perturbations were already presented in §4.3. The advantage of the argument here is that it applies to more general operators and to even dimensions – see §6.6 for pointers to the literature. We will also use some methods of the proof in §7.4 devoted to a local semiclassical trace formula for resonances.

**THEOREM 7.4.** Suppose that \(P(h)\) is given by (7.2.1) and that the set of resonances of \(PV\) is denoted \(\text{Res}(P(h))\). If \(\Omega \subset \{ \text{Re} z > 0 \}\) then

\[(7.2.2) \quad \# \text{Res}(P(h)) \cap \Omega \leq C_\Omega h^{-n}.\]

**Proof.** The proof is based on the characterization of resonances using the complex scaling method and then comparing the scaled operator \(P_\theta\) with an operator \(\tilde{P}_\theta\) such that \(\tilde{P}_\theta - z\) is invertible for \(z \in \Omega\).

1. We first choose \(0 < \theta < \pi/4\) such that \(\Omega \subset \{ \text{arg} z > -2\theta \}\). By Theorem 4.38 the resonances of \(P\) coincide with the eigenvalues of the scaled operator \(P_\theta\) (we recall that \(P_\theta - z\) is a Fredholm operator for \(z\) in a neighbourhood of \(\Omega\)). Let \(\chi \in C_c^\infty(\mathbb{R}^n; \mathbb{R})\) be equal to 1 on \(B(0, R_2)\), where \(R_2\) will be chosen sufficiently large. We then define

\[(7.2.3) \quad \tilde{P}_\theta := P_\theta - iM \chi(hD) \chi(x)^2 \chi(hD),\]

where to define \(\chi(hD)\) and \(\chi(x)\) we identified \(\Gamma_\theta\) with \(\mathbb{R}^n\) using (4.5.5).

2. We claim that if \(M\) and \(R_2\) in the definition of \(\tilde{P}_\theta\) are large enough then

\[(7.2.4) \quad (\tilde{P}_\theta - z)^{-1} = O(1) : L^2(\Gamma_\theta) \to L^2(\Gamma_\theta).\]

To see this we calculate the semiclassical symbol of \(\tilde{P}_\theta\) (see §E.1.1) using the identification of \(\Gamma_\theta\) with \(\mathbb{R}^n\):

\[\mathbb{R}^n \ni x \mapsto x + i\partial_x F_\theta(x),\]

where \(F_\theta : \mathbb{R}^n \to \mathbb{R}\) is a convex function in (4.5.6). We note that \(F_\theta(x) = 0\) for \(|x| \leq R_1\) and that \(\Gamma_\theta \setminus B_{\mathbb{C}^n}(0, 2R_1) = e^{2i\theta} \mathbb{R}^n\). With this notation the
symbol of \( \tilde{P}_\theta \), \( \sigma_h(\tilde{P}_\theta) \), is given by

\[
\begin{align*}
|x|^2 + V(x) - iM\chi(x)^2\chi(x)^2, & \quad |x| \leq R_1, \\
((I + iF''(x))^{-1}\xi) \cdot ((I + iF''(x))^{-1}\xi) - iM\chi(x)^2\chi(x)^2, & \quad R_1 \leq |x| \leq 2R_1, \\
e^{-2i\theta}|\xi|^2 - iM\chi(x)^2\chi(x)^2, & \quad |x| \geq 2R_1.
\end{align*}
\]

3. For \( z \in \Omega \in \{ \arg z > -2\theta \} \), \( 0 < \theta < \pi/4 \), we have

\[
|e^{-2i\theta}|\xi|^2 - z - iM\chi(x)^2\chi(x)^2| \geq (1 + |\xi|^2)/C.
\]

In fact, the left hand side cannot vanish since for \( \cos 2\theta|\xi|^2 = \text{Re} \ z > 0 \),

\[
\sin 2\theta|\xi|^2 + \text{Im} \ z + M\chi(x)^2\chi(x)^2 > \sin 2\theta|\xi|^2 - \text{Re} \ z \tan 2\theta = 0.
\]

Hence it is bounded from below for \( \xi \) in compact sets has the desired asymptotic behaviour.

To control the symbol for \( |x| \leq R_1 \) we choose \( R_2 > R_1 \) so that

\[
|\xi| > R_2/2 \implies |\xi|^2 + V(x) > 1 + \max_{x \in \Omega} \text{Re} \ z.
\]

If we choose \( M > \max_{x \in \Omega} (-\text{Im} \ z) \) then for \( z \in \Omega \) and \( |x| \leq R_1 \),

\[
||\xi|^2 + V(x) - iM\chi(x)^2\chi(x)^2 - z| \geq (1 + |\xi|^2)/C.
\]

It remains to consider the symbol for \( R_1 \leq |x| \leq 2R_1 \), where we assume that \( R_2 > 2R_1 \). For that we go back to \((4.5.16)\) in the proof of Theorem \( 4.32 \). It shows that with \( \eta = (I + (F''_\theta)^{-1}\xi, \text{and for } R_1 \leq |x| \leq R_2 \) (so that \( \chi(x) = 1 \)),

\[
-\text{Im} \ \sigma_h(\tilde{P}_\theta - z)(x, \xi) = 2\langle F''_\theta(x)\eta, \eta \rangle + M\chi(x)^2 + \text{Im} \ z.
\]

Since for \( |\xi| \leq R_2 \) the term \( M\chi(x)^2 \) provides a lower bound, it now suffices to show that for \( |\eta| \geq R_3 \) (we then choose \( R_2 \) large enough so that \( |\xi| > R_2 \) implies \( |\eta| > R_3 \)), we have

\[
|2\langle F''_\theta(x)\eta, \eta \rangle + \text{Im} \ z| \geq (1 + |\eta|^2)/C,
\]

for \( z \in \Omega \). But that is clear as the left hand side goes to \( \infty \) with \( |\eta| \to \infty \) for any fixed \( z \).

We conclude that \( \tilde{P}_\theta - z \) is elliptic as an element of \( \Psi^2_h(\Gamma_\theta) \). Theorem \( E.31 \) then shows that for \( h \) sufficiently small \((7.2.4)\) holds.

4. For \( z \in \Omega \) we can now write

\[
\begin{aligned}
P_\theta - z &= (\tilde{P}_\theta - z)(I + iM(\tilde{P}_\theta - z)^{-1}\chi(hD)\chi(x)^2\chi(hD)) \\
&= (\tilde{P}_\theta - z)(I + K(z)).
\end{aligned}
\]
Theorem C.8 then shows that
\[ \frac{1}{2\pi i} \text{tr} \oint (\zeta - P_\theta)^{-1} d\zeta = \frac{1}{2\pi i} \text{tr} \oint (I + K(\zeta))^{-1} K'(\zeta) d\zeta =: m_K(z), \]
where the integral is over a small circle around \( z \), not containing any eigenvalues of \( P_\theta \) other than possibly \( z \). This means that the eigenvalues of \( P_\theta \) coincide with multiplicities with the zeros of \( \det(I + K(z)) \). (The operator \( \chi(hD)\chi(x)^2\chi(hD) \) is of trace class as \( \chi \in C^\infty_c(\mathbb{R}^n) \)).

5. Hence the estimate (7.2.2) follows from the estimate on the number of zeros of
\[ k(z) := \det(I + K(z)), \]
and for that we use Jensen’s formula applied as in (D.1.8):
\[ \sum_{z \in \Omega} m_K(z) \leq C \sup_{z \in \Omega'} \log |k(z)| - C \log |k(z_0)|, \tag{7.2.6} \]
where \( \Omega \subseteq \Omega' \) where \( \Omega' \) is a bounded connected open set and \( z_0 \in \Omega' \). We take \( \Omega' \subset \{ -2\theta < \arg z < 2\pi - 2\theta \} \) with the property that (7.2.4) holds for \( z \in \Omega' \) and that we can find \( z_0 \in \Omega' \) with \( \text{Im} \; z_0 > \delta > 0 \). That is certainly possible following the argument in step 2.

6. To apply (7.2.6) we first estimates \( k(z) \) from above on \( \Omega' \) using (B.5.8):
\[ |\log k(z)| \leq \| \tilde{K}(z) \|_{L^1} \]
\[ \tag{7.2.7} \leq \| (\tilde{P}_\theta - z)^{-1} \| \chi(hD)\chi(x)^2\chi(hD) \|_{L^1} \]
\[ \leq C \text{tr} \left[ \chi(hD)\chi(x)^2\chi(hD) \right], \]
where the trace class norm equals to the trace as the operator is positive semidefinite. Pseudodifferential calculus [Zw12, Theorem 4.11] shows that \( \chi(hD)\chi(x)^2\chi(hD) = c(x, hD) \), with \( c \in \mathscr{S}(\mathbb{R}^{2n}) \). Hence,
\[ \text{tr} \left[ \chi(hD)\chi(x)^2\chi(hD) \right] = \frac{1}{(2\pi h)^n} \int \int c(x, \xi) dx d\xi = \mathcal{O}(h^{-n}). \]

We conclude that
\[ \log |k(z)| = \mathcal{O}(h^{-n}), \quad z \in \Omega'. \tag{7.2.8} \]

7. It remains to obtain a lower bound at \( z_0 \in \Omega' \) and we take \( z_0 \) with \( \text{Im} \; z_0 > \delta > 0 \). Arguing as in Step 2 we see that \( P_\theta - z_0 \) is elliptic as an element of \( \Psi^2_h(\Gamma_\theta) \). Hence for \( h \) small enough
\[ (P_\theta - z_0)^{-1} = \mathcal{O}(1) : L^2(\Gamma_\theta) \rightarrow L^2(\Gamma_\theta). \tag{7.2.9} \]
Using (7.2.5) we get

\[(I + K(z_0))^{-1} = (P_\theta - z_0)^{-1}(\tilde{P}_\theta - z_0) = I - iM(P_\theta - z_0)^{-1}(\chi(hD)\chi(x)\chi(hD)) := I + \tilde{K}(z_0)\]

Using (7.2.9) and arguing as in Step 5 we see that

\[\log |\det(I + \tilde{K}(z_0))| \leq C h^{-n}.\]

But that gives,

\[(7.2.10) \quad \log |k(z_0)| = -\log |\det(I + \tilde{K}(z_0))| \geq -C h^{-n}.\]

Inserting this and (7.2.8) into (7.2.6) gives the estimate (7.2.2). □

The next result provides a bound on the cut-off resolvent away from resonances. It will be crucial in showing that existence of a localized quasimode implies existence of a resonance nearby – see §7.3.

**THEOREM 7.5 (Exponential resolvent bounds).** Suppose that

\[P = -h^2\Delta_g + V\] where \(V \in C_\infty_c(\mathbb{R}^n; \mathbb{R})\) and \(g_{ij} - \delta_{ij} \in C_\infty_c(\mathbb{R}^n; \mathbb{R})\). Let \(R(z,h) := (P(h) - z)^{-1}, \text{Im } z > 0\) be the outgoing resolvent which continues meromorphically to \(\text{Im } z < 0\).

Suppose that \(\Omega \subset \{\text{Re } z > 0\}, \chi \in C_\infty_c(\mathbb{R}^n; \mathbb{R})\) and that \(h \mapsto \delta(h)\) is a positive function. Then there exist constants \(A = A(\Omega)\) and \(h_0 = h_0(\Omega)\) such that for \(0 < h < h_0\),

\[(7.2.11) \quad \|\chi R(z,h)\chi\|_{L^2 \to L^2} \leq A \exp \left( A h^{-n} \log \frac{1}{\delta(h)} \right), \quad \forall z \in \Omega \setminus \bigcup_{w \in \text{Res}(P(h))} D(w, \delta(h)).\]

**REMARKS.**
1. The proof of Theorem 7.5 combined with the methods of §4.3 gives the same result for black box Hamiltonians. The assumptions on \(P(h)\) can be weakened further and the bound holds in great generality.
2. If \(P = -\Delta_g + V\) (or a general black box Hamiltonian independent of \(h\)) then considering \(P(h) := h^2 P\) and rescaling provides the bound on the meromorphic continuation of the cut-off resolvent \((R(\lambda) = (P - \lambda^2)):\)

\[\|\chi R(\lambda)\chi\|_{L^2 \to L^2} \leq A \left( \exp A|\lambda|^2 \log \frac{1}{\delta(\lambda)} \right), \quad |\text{Re } \lambda| > 1, \lambda \notin \bigcup_{\zeta^2 \in \text{Res}(P)} D(\zeta, \delta(\zeta)).\]
Proof. 1. Theorem 4.37 shows that for \( \chi \in C_c^\infty(B(0,R_1)) \) (where \( R_1 \) is as in (4.5.1))

\[
\chi R(z,h)\chi = \chi(P_\theta - z)^{-1}\chi,
\]

and hence the estimate above follows from the estimate on the norm of \((P_\theta - z)^{-1}\), \(0 < \theta < \pi/4\), \(z \in \Omega \cap \{\arg z > -2\theta\}\).

2. Using (7.2.4) and (7.2.5) we write

\[
\| (P_\theta - z)^{-1} \|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} = \| (I + K(z))^{-1} \|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} \leq C\| (I + K(z))^{-1} \|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)}.
\]

To estimate the last norm we recall that \( K(z) \) is of trace class and hence we can apply (B.5.18):

\[
\| (I + K(z))^{-1} \|_{L^2(\Gamma_\theta) \to L^2(\Gamma_\theta)} \leq \det(I + K(z))^{-1}. \]

As in (7.2.7) we see that

\[
\det(I + [K(z)^*K(z)]^{1/2}) \leq C \exp(Ch^{-n}).
\]

Hence the estimate (7.2.11) will follow from the corresponding estimate on \( \det(I + K(z))^{-1} \).

3. From (7.2.7), (7.2.10) and (D.1.9) we now obtain that for

\[
z \in \Omega \setminus \bigcup_{w \in \text{Res}(P(h))} D(w,\delta) \log |\det(I + K(z))| \geq -Ch^{-n} \log \frac{1}{\delta},
\]

and, as we indicated above, that proves (7.2.11). \( \square \)

7. FROM QUASIMODES TO RESONANCES

In this section we will again consider operators of the form (7.2.1), \( P(h) = -h^2\Delta_g + V \), with \( V \in C_c^\infty(\mathbb{R}^n; \mathbb{R}) \) and \( g^{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}) \). The results hold in greater generality – see §6.6 for reference but the ideas behind the proofs are the same.

In the presence of classical trapping it is sometimes easy to construct approximate solutions to \((P(h) - E(h))u(h) = 0\), where \( E(h) \) is an energy level. That means finding \( u(h) \in C_c^\infty(\mathbb{R}^n) \) such that

\[
(P(h) - E(h))u(h) = \epsilon_0(h), \quad \| u \|_{L^2} = 1,
\]

where \( \epsilon(h) = O(h^\infty) \) or \( \epsilon(h) = O(e^{-S_0/h}) \), \( S_0 > 0 \).

**Example.** Suppose that for \( E > 0 \) the energy surface of \( P(h) = -h^2\Delta_g + V \),

\[
\Sigma_E = \{ (x,\xi) : |\xi|^2_g + V(x) = E \},
\]
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has at least two connected components,

\[
\Sigma_E = \Sigma_0 \cup \Sigma_\infty, \quad \Sigma_\bullet = \Sigma_\bullet, \quad \Sigma_0 \cap \Sigma_\infty = \emptyset, \\
\Sigma_0 \neq \emptyset, \quad \Sigma_\infty \cap T^*(\mathbb{R}^n \setminus B(0, R)) = \emptyset, \quad \text{supp} \, V \subset B(0, R).
\]

(7.3.2)

The fact that \(\Sigma_0\) and \(\Sigma_\infty\) are disjoint there exist open sets \(\Omega_0 \Subset \Omega_1 \Subset \mathbb{R}^n\) such that

\[
\pi(\Sigma_0) \subset \Omega_0, \quad V|_{\Omega_1 \setminus \Omega_0} > E.
\]

We can then find \(V_1 \in C_\infty^\infty(\mathbb{R}^n; \mathbb{R})\) with the following properties:

\[
V_1(x) = V(x), \quad x \in \Omega_1, \quad V_1(x) > E, \quad x \notin \Omega_0, \quad V_1(x) = \alpha |x|^2, \quad |x| > R,
\]

for some \(\alpha > 0\). The operator

\[
P_1(h) = -h^2\Delta_g + V_1
\]

is essentially self-adjoint and has discrete spectrum – see for instance [Zw12, 6.3]:

\[
(P_1(h) + V_1)u_j(h) = E_j(h), \quad \|u_j(h)\|_{L^2} = 1, \quad j \in \mathbb{N},
\]

and

\[
\{E_j(h)\}_{j=0}^\infty \cap [E-\delta, E+\delta] = \frac{1 + o(1)}{(2\pi h)^n} \int_{||\xi||^2 + V(x) - E \leq \delta} dxd\xi,
\]

see [Zw12, Theorems 6.8, 14.11].

The eigenfunctions of \(P_1(h)\) are exponentially small in the classically forbidden region \(V_1(x) > E + \epsilon, \epsilon > 0\) thanks for Agmon estimates – see [Zw12, Theorem 7.4]. In particular, \(\delta\) small enough,

\[
\|u_j(h)\|_{H^2_h(\mathbb{R}^n \setminus \Omega_0)} = \mathcal{O}(e^{-S_0/h}), \quad \text{for} \quad E_j(h) \in [E-\delta, E+\delta].
\]

Let \(\chi \in C_0^\infty(\mathbb{R}^n; [0, 1])\) satisfy

\[
\chi(x) = 1, \quad x \in \Omega_0, \quad \text{supp} \, \chi \subset \Omega_1.
\]

We then see from (7.3.4) and the fact that \(V\) coincides with \(V_1\) in \(\Omega_1\),

\[
(P(h) - E_j(h))(\chi u_j(h)) = \mathcal{O}(e^{-S_0/h})_{L^2},
\]

that is, (7.3.1) is satisfied with

\[
u(h) := \chi u_j(h)/\|\chi u_j(h)\|, \quad \epsilon(h) = \mathcal{O}(e^{-S_0/h}).
\]

Here we notice that (7.3.4) implies that \(\|\chi u_j\|_{L^2} = 1 + \mathcal{O}(e^{-S_0/h})\).

From (7.3.3) we see that in fact we have in fact obtained \(\sim h^{-n}\) quasi-modes. □
Figure 7.1. An example of a potential for which the assumption of the example in §7.3 are satisfied. It is a classical example from molecular dynamics: a cross-section of the energy surface of formaldehyde, H₂CO. That molecule, considered as a resonant state, has a very long lifetime at energy $E$ due to the strong barrier. That is not surprising considering the well known properties of formaldehyde.

If $P_1(h) = -h^2 \Delta_a + V$ is an operator on a compact manifold, or $P_1(h)$ is the operator in the example above, then the spectrum of $P_1(h)$ is discrete. Existence of a quasimode (7.3.1) and the spectral theorem immediately imply that there exists an eigenvalue of $P_1(h)$, $E_0(h)$, such that $|E(h) - E_0(h)| \leq \|\epsilon(h)\|_{L^2}$. If $E_0(h) > 0$ then we know Theorem 4.18 that we cannot have $L^2$ solutions to $(P(h) - E_0(h))u_0(h) = 0$ yet existence of a quasimode (7.3.1) should imply existence of a long living quantum state. That is indeed the case as shown in the next result:

**THEOREM 7.6 (From quasimodes to resonances).** Suppose that there exists a family $u(h) \in C^\infty_c(\mathbb{R}^n)$ such that $\text{supp } u \subset \Omega \subset \mathbb{R}^n$ where $\Omega$ is independent of $h$, and for some $E(h) = E_0 + o(1) > 0$,

\[(P(h) - E(h))u(h) = \epsilon_0(h), \quad \|u(h)\|_{L^2} = 1,\]

\[\epsilon_0(h) = O(h^\infty)_{L^2} \quad \text{or} \quad \epsilon_0(h) = O(e^{-S_0/h})_{L^2}, \quad S_0 > 0.\]

Then for $0 < h < h_0$ there exists $z(h) \in \text{Res}(P(h))$ such that

\[|z(h) - E(h)| \leq \epsilon(h),\]

\[\epsilon(h) = O(h^\infty) \quad \text{or} \quad \epsilon(h) = O(e^{-S/h}), \quad \forall S < S_0,\]

respectively.
REMARKS. 1. Under special conditions much finer estimates on the location of $z(h)$ are possible. One can already see this in the simple example presented in Theorem 2.25. For the analysis of the general “well-in-the-island” case see [HS86] and for recent advances and references see [FLM11].

2. The estimate in (7.3.7) can be improved and the localization of the imaginary part is much better than the localization of the real part [TZ98],[St99]. That can be seen from the proof below.

The following lemma will play a crucial role in the proof of Theorem 7.6:

**Lemma 7.7 (Semiclassical maximum principle).** Suppose $H$ is a Hilbert space and that $z \mapsto Q(z,h) \in \mathcal{L}(H)$ is a holomorphic family of operators for

\begin{equation}
\begin{aligned}
z &\in \Omega(h) := (2a(h), 2b(h)) + i(-\delta(h)h^{-L}, \delta(h)), \\
a(h) &< b(h), \quad 0 < \delta(h) < 1, \quad b(h) - a(h) \geq 20\delta(h)h^{-2L}.
\end{aligned}
\end{equation}

If

\begin{equation}
\begin{aligned}
\|Q(z,h)\| &\leq \exp(Ch^{-L}), \quad z \in \Omega, \\
\|Q(z,h)\| &\leq 1/\text{Im } z, \quad \text{Im } z > 0, \quad z \in \Omega,
\end{aligned}
\end{equation}

then for $0 < h < h_0$,

\begin{equation}
\|Q(E,h)\| \leq e^{C\delta(h)^{-1}}, \quad a(h) < E < b(h).
\end{equation}

**Proof.** We apply Lemma D.1 to the holomorphic family

$$F(z,h) := F_{f,g}(z,h) = \langle Q(z - \frac{a(h)+b(h)}{2}, h)g,f \rangle_H, \quad \|f\|_H = \|g\|_H = 1,$$

with

$$2R = b(h) - a(h), \quad \delta_+ = \delta(h), \quad \delta_- = \delta(h)h^{-L},$$

$$M = \exp(CH^{-L}), \quad M_+ = 1/\delta(h).$$

Then for $\text{Im } z = 0, \ |z| \leq R$

$$|F(z,h)| \leq e^{Ch^{-L}\delta_+/(\delta_+ + \delta_-)} \delta_-/(\delta_+ + \delta_-) = e^{C/(1+hL)}\delta(h)^{-1/(1+hL)} \leq e^{C\delta(h)^{-1}}.$$ 

Since $\|Q(z,h)\| = \sup_{f,g} |F_{f,g}(z,h)|$, the lemma follows.

**Proof of Theorem 7.6.** 1. We argue by contradiction. Suppose

$$\chi \in C_c^\infty(\mathbb{R}^n \setminus [0,1]), \quad \chi(x) = 1, \quad x \in \Omega.$$
Then for complex scaling starting outside of $\Omega$,
\[
\chi(P(h) - E(h))^{-1}\chi\varepsilon_0(h) = \chi(P_\theta(h) - E(h))^{-1}\chi(P(h) - E(h))u(h)
\]
\[
= \chi(P_\theta(h) - E(h))^{-1}(P_\theta(h) - E(h))u(h) = \chi u(h)
\]
Hence if we show that absence of resonances in $D(E, \varepsilon(h))$ implies
\[
(7.3.11) \quad \|\chi(P(h) - E(h))^{-1}\chi\|_{L^2 \to L^2} \ll \|\varepsilon_0(h)\|^{-1},
\]
we obtain a contradiction to $P_2$. To obtain (7.3.11) under the assumption that $\text{Res}(P(h)) \cap D(0, \varepsilon(h)) = \emptyset$ we will use Lemma 7.7.

For that let us put $L = n + 1$. Theorem 7.5 shows that for
\[
\varepsilon(h) \gg h^{-2L}\|\varepsilon_0(h)\|,
\]
(note that with this choice of $\varepsilon(h)$ we can satisfy (7.3.7)) we have, for some constant $C$,
\[
\chi R(z, h) \chi = O(e^{Ch^{-L}}), \quad z \in D(E_0, \delta) \setminus \bigcup D(E(h), 2\varepsilon(h)).
\]
Hence assuming that there is no resonance in $D(E(h), \varepsilon(h))$, we obtain
\[
\|\chi R(z, h)\|_{L^2 \to L^2} \leq Ce^{Ch^{-L}}, \quad z \in D(E(h), \varepsilon(h)),
\]
\[
\|\chi R(z, h)\chi\|_{L^2 \to L^2} \leq 1/\text{Im } z, \quad \text{Im } z > 0,
\]
where the last inequality comes from self-adjointness of $P(h)$:
\[
\|P(h) - z\|^{-1}_{L^2 \to L^2} = 1/\text{Im } z, \quad \text{Im } z > 0.
\]
We can now apply Lemma 7.7 with
\[
Q(z, h) = \chi R(z, h)\chi, \quad \delta(h) = h^{2L}\varepsilon(h)/M \gg \|\varepsilon_0(h)\|,
\]
\[
a(h) = E(h) - \varepsilon(h)/2, \quad b(h) = E(h) + \varepsilon(h)/2,
\]
where $M$ is a large constant (so that the assumption of the lemma are satisfied). This gives (7.3.11):
\[
\|\chi(P(h) - E(h))^{-1}\chi\| = \|\chi R(E(h), h)\chi\| \leq 2/\delta(h) \ll 1/\|\varepsilon_0(h)\|,
\]
completing the proof. \qed

**EXAMPLE.** Suppose that $P(h) = -h^2\Delta + V$ where $V$ satisfies the assumption (7.3.2) of the example in the beginning of this section. The construction of quasimodes (7.3.5) in that example, Theorem 7.6 and (7.3.3), show that there exist $E(h) = E + o(1)$, and
\[
z(h) \in \text{Res}(P(h)), \quad |E(h) - z(h)| \leq e^{-S/h}, \quad S > 0.
\]
In many cases very precise form of $E(h)$ can be given by using semiclassical spectral theory for $P_1(h)$ – see for instance [DS99, Chapter 3] and references given there.

Resonances obtained this way are sometimes called shape resonances and, as we indicated already, they can be analyzed more precisely under stronger assumptions – see [FLM11] and references given there.

**REMARKS.** 1. Other constructions can be use to obtain quasimodes and consequently resonances. For instance we can take $P(h) = -h^2 \Delta_g$ and construct quasimodes associated to elliptic geodesics – see Fig. 7.2 for an illustration and [TZ98],[St99] for references. In particular, the same construction can be used for obstacle problems.

2. Theorem 7.6 does not address the issue of multiplicities and hence we cannot immediately deduce from (7.3.3) that we have $\sim h^{-n}$ resonances close
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Figure 7.3. Examples of one-dimensional potentials illustrating condition (7.3.12): the component of infinity is non-trapping (that is (7.3.12) holds) in cases (a),(b),(c) but not (d).

to the real axis. That is remedied by Stefanov in [St99] and we outline his argument in Exercise 7.1.

We conclude this section by describing a dichotomy for imaginary parts of resonances in the case a potential barrier for which the component of infinity is non-trapping. This phenomenon of resonances splitting into those close to the real axis and those far away occurs in many other settings, see [TODO: references].

**THEOREM 7.8** (Dichotomy for resonance widths). Suppose that \( P(h) = -h^2 \Delta + V \) satisfies (7.3.2) for some \( E > 0 \) and that in addition,

\[
\forall (x, \xi) \in \Sigma_\infty \quad |\pi(\exp H_p(x, \xi))| \to \infty \quad \text{as} \quad t \to \pm \infty,
\]

where \( p = |\xi|^2 + V \).

Then there exist \( \delta > 0, S > 0 \) such that for every \( M \) there exists \( h_0 > 0 \), so that for \( 0 < h < h_0 \),

\[
(7.3.13) \quad z \in \text{Res}(P(h)), \quad |\text{Re } z - E| < \delta \implies \begin{cases} 
\text{Im } z > -e^{-S/h} \\
\text{or} \\
\text{Im } z < -Mh \log(1/h).
\end{cases}
\]

for \( 0 < h < h_0 \).

**REMARKS.** 1. Condition (7.3.12) means that the infinity component of the energy surface \( p = E \) is non-trapping – see Fig. 7.3.

2. We have a stronger conclusions than (7.3.13): all resonances with \( \text{Im } z > -e^{-S/h} \) come from quasimodes localized in \( p^{-1}(E) \setminus \Sigma_\infty \) and hence can be
7.3. FROM QUASIMODES TO RESONANCES

related to eigenvalues of the reference problem $P_1(h)$ presented in the first example – see Exercise 7.3.

Proof. 1. Let $P_\theta(h)$ be the complex scaled operator obtained by consider $P(h)$ as a black box Hamiltonian – see §4.5. We fix $M > 0$ and take $\theta = M_0 h \log(1/h)$ where $M_0 \gg M$ – see the proof of Theorem 6.21 for the motivation for this choice.

In particular, $P_\theta(h)$ coincides with $P(h)$ near $\text{supp} V$ and $\text{supp}(g^{ij} - \delta_{ij})$. If $z \in \text{Res}(P(h))$ we consider the corresponding resonant state

\[(7.3.14) \quad (P_\theta(h) - z) u_\theta = 0, \quad \|u_\theta\|_{L^2(\Gamma_\theta)} = 1.\]

2. Let $\chi \in C^\infty_c(\mathbb{R}^n; [0, 1])$ satisfy

\[(7.3.15) \quad \chi(x) = 1 \text{ near } \pi(\Sigma_0) \text{ and } \chi(x) = 0 \text{ near } \pi(\Sigma_\infty).\]

That is possible as $\Sigma_0$ and $\Sigma_\infty$ are closed and disjoint. Condition (7.3.15) implies that

\[(7.3.16) \quad V(x) > E \text{ for } x \in \text{supp } \nabla \chi.\]

We first claim that for $z$ and $u_\theta$ satisfying (7.3.14) and some $S > 0$ we have the following dichotomy for small enough $h$:

\[(7.3.17) \quad \|\chi u_\theta\| < e^{-S/h} \text{ or } \text{Im } z > -e^{-S/h}.\]

3. To prove (7.3.17) use self-adjointness of $P(h)$ to write

\[-2i \text{Im } z \|\chi u_\theta\|^2 = \langle (P_\theta(h) - z) \chi u_\theta, \chi u_\theta \rangle - \langle \chi u_\theta, (P_\theta(h) - z) \chi u_\theta \rangle = \langle [P_\theta(h), \chi] u_\theta, \chi u_\theta \rangle.\]

In view of (7.3.16), Agmon estimates (see [Zw12, Theorem 7.4]) show that

\[\|[P_\theta(h), \chi] u_\theta\| < e^{-2S/h} \|u_\theta\|\]

for $S > 0$ and $h$ small enough. Hence

\[|\text{Im } z| \|\chi u_\theta\| < e^{-2S/h},\]

and (7.3.17) follows.

4. We now show that for any $0 < M \ll M_0$ (where the angle of complex scaling is given by $\theta = M_0 h \log(1/h)$) there exists $h_0$ such that for $0 < h < h_0$,

\[(7.3.18) \quad \|\chi u_\theta\| < e^{-S/h} \implies \text{Im } z < -M h \log(1/h).\]

In view of (7.3.17) that will prove the theorem.

5. To establish (7.3.18) we use our assumption (7.3.2) to find $V_0 \in C^\infty_c(\mathbb{R}^n; \mathbb{R})$ such that

\[(7.3.19) \quad \text{supp}(V - V_0) \subset \text{supp } \chi, \quad V_0(x) > E, \quad x \notin \pi(\Sigma_\infty),\]
where $\chi$ satisfies (7.3.15). That means that $V_0$ “fills in” the finite components of the energy surface of $V$. (Its role is the opposite of that of $V_1$ in the first example.) Then

$$\{(x, \xi) : |\xi|^2 + V_0(x) = E\} = \Sigma_{\infty}$$

and the assumption (7.3.12) shows that

$$|\xi|^2 + V_0(x) = E \implies |\pi \exp(tH_{p_0})(x, \xi)| \to \infty, \ t \to \pm \infty,$$

where $p_0(x, \xi) = |\xi|^2 + V_0(x)$. That means that the energy level $E$ is non-trapping for $P_0$. If $P_{0,\theta}(h)$ is the corresponding scaled operator with $\theta = M_0 h \log(1/h)$. From (??) we see that

$$\|\left(P_{0,\theta}(h) - z\right)^{-1}\| \leq h^{-C_0 M}, \ z \in [E - \delta, E + \delta] - i[0, Mh \log(1/h)],$$

for $0 < h < h_0$.

6. Suppose that (7.3.18) is false that is $\text{Im} \ z \geq -M h \log(1/h)$.

Because of (7.3.19) we can choose $\tilde{\chi} \in C^\infty_c(\mathbb{R}^n; [0, 1])$ such that $\chi = 1$ on a neighbourhood of $\text{supp} \tilde{\chi}$ and

$$(P_\theta(h) - z)(1 - \tilde{\chi}) = (P_{0,\theta}(h) - z)(1 - \tilde{\chi}).$$

Hence

$$(1 - \tilde{\chi})u_\theta = (P_{0,\theta}(h) - z)^{-1}(P_{0,\theta}(h) - z)(1 - \tilde{\chi})u_\theta$$

$$= (P_{0,\theta}(h) - z)^{-1}(P_\theta(h) - z)(1 - \tilde{\chi})u_\theta$$

$$= (P_{0,\theta}(h) - z)^{-1}[P_\theta(h), \tilde{\chi}]u_\theta.$$  (7.3.21)

We can find open sets $U \subset W$ such that $\text{supp} \nabla \chi \subset U$ and $W \subset \{\chi = 1\}$. Semiclassical elliptic estimates (see [Zw12, Theorem 7.1]) and the assumption in (7.3.18) then give

$$\| [P_\theta(h), \tilde{\chi}] u_\theta \|_{L^2} \leq C h \| u_\theta \|_{H^1(U)}$$

$$\leq C h \| (P_\theta(h) - z) u_\theta \|_{L^2(W)} + C h \| u_\theta \|_{L^2(W)}$$

$$\leq C h \| \chi u_\theta \| \leq C e^{-S/h}.$$  

Combined with (7.3.20) and (7.3.21) this gives

$$1 = \| u_\theta \| \leq \|(1 - \tilde{\chi}) u_\theta \| + \| \tilde{\chi} u_\theta \|$$

$$\leq \|(1 - \tilde{\chi}) u_\theta \| + e^{-S/h} \leq 2 e^{-S/h}$$

which provides an obvious contradiction for $h$ small enough. \qed
7.4. THE SJÖSTRAND TRACE FORMULA

In this section we present a semiclassical local trace formula for resonances. It is different from the formula in §3.10 by involving only a finite number of resonances of a semiclassical black box operator.

We provide a complete proof in the simplified situation of $-\hbar^2\Delta + V$, and $V \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$. We then give an application by showing that for any potential $V \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$ there exist many energy levels $E$ such that for any $r > 0$,

\begin{equation}
|\text{Res}(-\hbar^2\Delta_g + V) \cap D(E, r)| \geq h^{-n}/C(r), \quad 0 < h < h_0(r).
\end{equation}

To formulate the theorem we introduce the following subsets of $\mathbb{C}$:

\begin{align}
\Omega := (a, b) + i(c, d), & \quad W := (a', b') + i(c', d), \\
0 < a < a' < b' < b, & \quad c < c' < 0 < d,
\end{align}

\begin{align}
\Omega_- := \Omega \cap \{\text{Im } z \leq 0\}, & \quad W_- := W \cap \{\text{Im } z \leq 0\}, \\
\Omega_\mathbb{R} = \Omega \cap \mathbb{R}, & \quad W_\mathbb{R} = W \cap \mathbb{R}.
\end{align}

The complex scaling method described in §4.5 will be crucial in the proof. Hence we choose $c, d$ small enough so that $\Omega \subset \{\text{arg } z > -\theta\}$ for some $\theta < \pi/2$. The regions are illustrated in Fig. 7.4

**THEOREM 7.9 (The Sjöstrand trace formula).** For $n = 3$ let

\[ P_V(h) := -\hbar^2\Delta + V(x), \]

where $V \in C_c^\infty(\mathbb{R}^3; \mathbb{R})$, $P_0(h) = -\hbar^2\Delta$, and suppose that $\Omega$ and $W$ are given by (7.4.2).

Let $f$ be a holomorphic function in a neighbourhood of $\Omega$ and satisfies

\begin{equation}
|f(z)| \leq 1, \quad z \in \Omega \setminus W,
\end{equation}

and let $\psi \in C_c^\infty(\Omega_\mathbb{R})$ be equal to 1 on $W_\mathbb{R}$.

Then

\begin{equation}
(\psi f)(P_V(h)) - (\psi f)(P_0(h)) \in L_1(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)),
\end{equation}

and

\begin{equation}
\text{tr}[\psi f(P_V(h)) - (\psi f)(P_0(h))] = \sum_{z \in \text{Res}(P_V(h)) \cap W_-} f(z) + \mathcal{O}(h^{-n}) + \mathcal{O}_f(h^\infty).
\end{equation}

**REMARKS.**

1. In the application of (7.4.5) we vary $f$ with a fixed $\psi$—hence we are not concerned about the dependence of the constants on $\psi$—only on $f$. 
2. The restriction to dimension $n = 3$ is due to better trace class properties used in the proof, just as in the proof of Theorem 3.51 for $n = 3$. The proof works also for $n = 1$. The statement remains valid for all $n$ but the proof needs to be modified: complex scaling needs to be presented for all dimensions and we need additional arguments to deal with trace class properties. Roughly, it amounts to replacing $(P - z)^{-1}$ in the proof by $(P - z)^{-1}(P - z_0)^{-m}$, $m \geq (n - 1)/2$ but that leads to (minor) algebraic complications.

3. Much weaker assumptions on $P(h)$ are needed and in particular the trace formula works for long range black box Hamiltonians. The formulation is also stronger: the error term $\mathcal{O}(\hbar^n)$ can be dropped and in (7.4.3) the bound is needed only in $\Omega_\setminus W$ – see the notes for references.

Before the proof we present two lemmas which provide quantitative estimates on trace class norms of resolvent differences.

**Lemma 7.10.** Suppose that $n = 3$. For $z \in \mathbb{C} \setminus \mathbb{R}$ we have

\begin{equation}
(P_V - z)^{-1} - (P_0 - z)^{-1} = \mathcal{O}(h^{-n} |\text{Im} z|^{-c_n} L^1(L^2(\mathbb{R}^n))),
\end{equation}

for some constant $c_n$ depending only on the dimension. If $\chi \in C_c^\infty(\mathbb{R}^n)$ and $\chi \equiv 1$ on supp $V$ then

\begin{equation}
(1 - \chi) ((P_V - z)^{-1} - (P_0 - z)^{-1}) = \mathcal{O}(h^N |\text{Im} z|^{-C_N}) : \langle x \rangle^N H^{-N}(\mathbb{R}^n) \to \langle x \rangle^{-N} H^N(\mathbb{R}^n).
\end{equation}

Proof. 1. For $z \in \mathbb{C} \setminus \mathbb{R}$, $P_V - z$ is a quantization of an elliptic symbol in $S(\langle \xi \rangle^2)$ where the symbol class associated to the weight $m(x, \xi) = \langle \xi \rangle^2$ is described in [Zw12, §4.4]. Hence the semiclassical version of Beals’s Lemma [Zw12, Theorem 8.3] shows that

\begin{equation}
(P_V - z)^{-1} = a_V(x, hD, z),
\end{equation}

in which $a_V(x, hD, z)$ is an integral operator with symbol $a_V(x, hD, z) = \chi(x) (P_V - z)^{-1} (P_0 - z)^{-1} a(x, hD, z)$ and $a(x, hD, z)$ is a smooth family of classical symbols.
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where $a_V$ satisfies
\[
\partial_x^\alpha \partial_\xi^\beta a_V(x, \xi, z) = O_{\alpha, \beta}(\langle \xi \rangle^{-2} |\text{Im } z|^{-M-2(|\alpha|+|\beta|)}).
\]

(We find it more convenient here to refer to the calculus based on weight functions – see [DS99, Chapter 7] and [Zw12, Chapter 4] rather than the calculus build for our application for scattering on manifolds in Appendix E. See [DS99, Chapter 8] and the proof [Zw12, Theorem 14.9] for similar arguments.)

2. Since $V \in S(\langle x \rangle^{-N})$ for any $N$,
\[
(P_V - z)^{-1} - (P_0 - z)^{-1} = (P_V - z)^{-1}V(P_0 - z)^{-1} = b(x, hD, z),
\]
where for $z \in \mathbb{C} \setminus \mathbb{R}$, $b \in S(\langle \xi \rangle^{-4}\langle x \rangle^{-N})$ for any $N$. Moreover,
\[
\partial_x^\alpha \partial_\xi^\beta b(x, \xi, z) = O_{\alpha, \beta, N}((\langle x \rangle^{-N}\langle \xi \rangle^{-4}) |\text{Im } z|^{-M'-2(|\alpha|+|\beta|)-CN}).
\]
This follows from the composition formula in [Zw12, Theorem 4.18] and the remainder estimates – see for instance [Zw12, (9.3.17)]. The trace class norm is bounded by (see for instance [DS99, Chapter 9])
\[
\|b(x, hD, z)\|_{L^1} \leq Ch^{-n} \sum_{|\alpha|+|\beta| \leq 2n+1} \int_{\mathbb{R}^{2n}} |\partial_x^\alpha \partial_\xi^\beta b(x, \xi, h)| dxd\xi
\]
\[
\leq C'h^{-n} |\text{Im } z|^{-cn},
\]
which is (7.4.6).

3. To see (7.4.7) we proceed similarly: for $z \in \mathbb{C} \setminus \mathbb{R}$ the pseudodifferential calculus shows that $(1 - \chi)b(x, hD, z) = O(h^\infty)_{\mathcal{S}_0 \to \mathcal{S}_0}$. In fact, all the terms in the expansion vanish and the remainder is in the residual class. Expanding the remainder explicitly as in [Zw12, (9.3.17)] shows a quantitative estimate:
\[
(1 - \chi)b(x, hD, z) = c(x, hD, z),
\]
\[
\partial_x^\alpha \partial_\xi^\beta c(x, \xi, z) = O_{\alpha, \beta, N}(h^N \langle x \rangle^{-N} \langle \xi \rangle^{-N} |\text{Im } z|^{-C_N-2(|\alpha|+|\beta|)}).
\]

This and [Zw12, Theorem 8.10] prove (7.4.7).

**LEMMA 7.11.** Suppose that $n = 3$ and that $\chi_0 \in C^\infty_c(B(0, R_1))$ is equal to 1 near $\text{supp } V$. (Here $R_1$ is the same as in (4.5.1)). Let $P_{V, \theta}$ and $P_{0, \theta}$ be the complex scaled operators $P_V$ and $P_0$, in the sense of Definition 4.31.

Then for $\text{Im } z \geq \delta$ and $\delta > 0$,
\[
(1 - \chi_0) \left( (P_{V, \theta} - z)^{-1} - (P_{0, \theta} - z)^{-1} \right) = O(h^\infty)_{L^1(L^2(\Gamma_\theta))}.
\]

**Proof.** As in the proof of Theorem 7.4 we see that $\text{Im } z > -\delta$, $P_{V, \theta} - z \in \Psi^2_h(\Gamma_\theta)$ is elliptic. Hence the estimate in the Lemma follows from the pseudodifferential calculus similarly to (7.4.7).
Proof of Theorem 7.9. 1. The starting point is the Helffer-Sjöstrand formula: for \( g \in C_c^\infty(\mathbb{R}) \) we write \( g(P_V) \) is given by

\[
g(P_V) := \frac{1}{\pi} \int_{\mathbb{C}} (P_V - z)^{-1} \partial_z \tilde{g}(z) dm(z),
\]

where \( \tilde{g} \in C_c^\infty(\mathbb{C}) \) is an almost analytic continuation of \( g \) – see §B.2 and [Zw12, Theorem 14.8].

Using the short hand notation,

\[
[F(P_\bullet)]_0^V := F(P_V) - F(P_0),
\]

we then have, with \( g = f\psi \),

\[
[(f\psi)(P_\bullet)]_0^V = \frac{1}{\pi} \int_{\mathbb{C}} [(P_\bullet - z)^{-1}]_0^V \partial_z \tilde{\psi}(z) f(z) dm(z).
\]

Lemma 7.10 shows that \( \|[(P_\bullet - z)^{-1}]_0^V \|_{L^1} = O(h^{-n}|\text{Im } z|^{-n-1}) \) and this gives (7.4.4).

2. We can arrange that \( \text{supp } \partial_z \tilde{\psi} \cap \{\text{Im } z < \delta\} \subset \Omega \setminus W \).

In particular

\[
\partial_z \tilde{\psi}(z)f(z) = O(1), \quad \text{Im } z < \delta,
\]

where the constant is independent of \( f \). This is where the assumption (7.4.3) is used.

The Cauchy–Green formula (D.1.1),

\[
2i \int_U \partial_z \varphi(z) dm(z) = \int_{\partial U} \varphi(z) dz, \quad \varphi \in C_c^\infty(\mathbb{C}),
\]

(\( U \) is an open set with a \( C^1 \) boundary) applied with the operator valued function

\[
\varphi(z) = [(P_\bullet - z)^{-1}]_0^V f(z) \tilde{\psi}(z),
\]

and \( U = \{\text{Im } z > \delta\} \) shows that

\[
\frac{1}{\pi} \int_{\mathbb{C}} [(P_\bullet - z)^{-1}]_0^V \partial_z f dm(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} [(P_\bullet - z)^{-1}]_0^V \tilde{\psi} f dz
\]

\[
+ \frac{1}{\pi} \int_{\text{Im } z < \delta} [(P_\bullet - z)^{-1}]_0^V \partial_z \tilde{\psi} f dm(z),
\]

where the line \( \Gamma_\delta := \mathbb{R} + i\delta \) is oriented negatively.

From (7.4.6) and (7.4.9) we see that the trace class norm of the second term on the right hand side is bounded by \( Ch^{-n} \) with a constant independent of \( f \).
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Hence,

\[ [(f\psi)(P_{\bullet})]_0^V = \frac{1}{2\pi i} \int_{\Gamma_3} [(P_{\bullet} - z)^{-1}]_0^V \tilde{\psi} f dz + O(h^{-n}) L^1. \]

3. For \( \chi \in C_c^\infty(\mathbb{R}^n) \) equal to 1 near \( \text{supp} V \) we apply (7.4.7) to obtain

\[ [(f\psi)(P_{\bullet})]_0^V = \frac{1}{2\pi i} \int_{\Gamma_3} \chi [(P_{\bullet} - z)^{-1}]_0^V \tilde{\psi} f(z) dz \]

\[ + O(h^{-n}) L^1 + O(f(h^\infty)) L^1. \]

Because of the cut-off \( \chi \) we can now use Theorem 4.37 and replace \( P_{\bullet} \)
by the complex scaled operators \( P_{\bullet,\theta} \). Since \( \text{Im} z = \delta > 0 \) in the integral, Lemma 7.11 shows that we can remove the cut-off function \( \chi \) at the expense of an error:

\[ \text{tr}[(f\psi)(P_{\bullet})]_0^V = \frac{1}{2\pi i} \text{tr} \int_{\Gamma_3} [(P_{\bullet,\theta} - z)^{-1}]_0^V \tilde{\psi} f(z) dz \]

\[ + O(h^{-n}) + O(f(h^\infty)) \]

We now recall the operators \( \tilde{P}_{\bullet,\theta} \) – see (7.2.3). Invertibility of \( \tilde{P}_{\bullet,\theta} - z \) for \( z \) near \( \Omega \) shows that

\[ \int_C [(\tilde{P}_{\bullet,\theta} - z)^{-1}]_0^V \partial_z \tilde{\psi}(z) f(z) dm(z) = 0. \]

Arguing as in Step 2 with \( (\tilde{P}_{\bullet,\theta} - z)^{-1} \) in place of \( (P_{\bullet} - z)^{-1} \) we obtain

\[ \text{tr}[(f\psi)(P_{\bullet})]_0^V = \frac{1}{2\pi i} \text{tr} \int_{\Gamma_3} [(P_{\bullet,\theta} - z)^{-1} - (\tilde{P}_{\bullet,\theta} - z)^{-1}]_0^V \tilde{\psi}(z) f(z) dz \]

\[ + O(h^{-n}) + O(f(h^\infty)), \]

4. The terms \( \bullet = 0, V \) can be treated separately as the differences are of trace class. Applying the Cauchy–Green formula again and taking the trace we obtain, noting that resonances are the poles of \( (P_{\bullet,\theta} - z)^{-1} - (\tilde{P}_{\bullet,\theta} - z)^{-1} \) (the second term is holomorphic):

\[ \text{tr}[(f\psi)(P_{\bullet})]_0^V = \sum_{z \in \text{Res}(P,V)} (f\tilde{\psi})(z) + O(h^{-n}) + O(f(h^\infty)) \]

\[ + \frac{1}{\pi} \int_{C_-} \text{tr} [(P_{\bullet,\theta} - z)^{-1} - (\tilde{P}_{\bullet,\theta} - z)^{-1}]_0^V \partial_z \tilde{\psi}(z) f(z) dm(z). \]

5. We now use the operators \( K_{\bullet} \) defined in (7.2.5) for \( P_{\theta} \) replaced by \( P_{\bullet,\theta} \). We note that

\[ (7.4.10) \quad (P_{\bullet,\theta} - z)^{-1} - (\tilde{P}_{\bullet,\theta} - z)^{-1} = -K_{\bullet}(z)(I + K_{\bullet}(z))^{-1}(\tilde{P}_{\bullet,\theta} - z)^{-1}, \]
and that
\[ (7.4.11) \quad \partial_z K\bullet(z) = (\bar{P}\bullet - z)^{-1} K\bullet(z). \]

These two identities show that
\[
\int_{\mathbb{C}_-} \text{tr} \left[ (P\bullet - z)^{-1} - (\bar{P}\bullet - z)^{-1} \right]^V \partial_z \tilde{\psi}(z) f(z) dm(z) = \\
\int_{\mathbb{C}_-} [\partial_z k\bullet(z)/k\bullet(z)]^V \partial_z \tilde{\psi}(z) f(z) dm(z),
\]

where \( k\bullet(z) := \text{det}(I + K\bullet(z)) \). The estimates on the determinant in the proof of Theorem 7.4 and the lower bound (D.1.9) show that for \( z \) near \( \Omega \),
\[
\frac{\partial_z k\bullet(z)}{k\bullet(z)} = H'\bullet(z) + \sum_{j=1}^f \frac{m_{k\bullet}(\zeta)}{|z - \zeta|}, \quad H'\bullet(z) = O(h^{-n}),
\]

\[
\sum_{\zeta \in \Omega} m_{k\bullet}(\zeta) = O(h^{-n}), \quad m_{k\bullet}(\zeta) \equiv 0.
\]

Since \( \mathbb{C}_- \), \( \tilde{\psi}(z)f(z) \) is bounded independently of \( f \) this give the bound \( O(h^{-n}) \) for the last integral. \( \Box \)

To state an application of Theorem 7.9 we need to review some basic facts about analytic singular support and wave front set. The standard references are [De92],[HöI, §9.3],[Ma02a, §3.3] and [Sj82].

**Definition 7.12.** Suppose that \( u \in S'(\mathbb{R}^n) \). Then the analytic singular support of \( u \) is the closed set \( \text{singsupp}_a u \), defined by the condition that \( x /\in \text{singsupp}_a u \) if and only if there exists a neighbourhood \( U \) of \( x \) such that \( u|_U \) is a real analytic function.

A useful characterization of the analytic singular support is given in terms of the analytic wave front set which is the analytic version of the wave front set defined in §E.2. One way to define the analytic wave front set is using the FBI (Fourier–Bros–Iagolnitzer) transform: for \( u \in \mathcal{S}'(\mathbb{R}^n) \),
\[ (7.4.12) \quad T_\lambda u(x,\xi) = \int_{\mathbb{R}^n} e^{-\lambda(x-y)^2/2+i\lambda\langle \xi, x-y \rangle} u(y) dy, \quad \lambda > 0, \]

where the integral is meant in the sense of a distributional pairing. We note \( |T_\lambda u(x,\xi)| \leq C\lambda^N \), for some \( N \). (The inequality follows from using a semi-norm of \( y \mapsto e^{-\lambda(x-y)^2/2+i\xi(x-y)} \) in \( \mathcal{S} \).

We then have

**Definition 7.13.** For \( u \in \mathcal{S}'(\mathbb{R}^n) \) the analytic wave front set of \( u \), \( \text{WF}_a(u) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) \), is defined as follows:
\[ (x,\xi) /\notin \text{WF}_a(u) \iff \exists U \subset \mathbb{R}^{2n} \text{ a neighbourhood of } (x,\xi) \text{ and } \delta > 0, \]
\[ |T_\lambda u(y, \eta)| \leq Ce^{-\delta\lambda}, \quad (y, \eta) \in U, \ \lambda \to \infty. \]

The result connecting the two objects is

(7.4.13) \[ \mathrm{singsupp}_a u = \pi(\mathrm{WF}_a(u)), \quad \pi(x, \xi) := x, \quad \pi : T^* \mathbb{R}^n \setminus 0 \to \mathbb{R}^n. \]

A lower bound on the number of resonances is now given in any neighbourhood of an energy level lying in the analytic singular support of the distribution function of the potential:

**THEOREM 7.14 (Lower bounds for the number of resonances).** Suppose that \( n = 3 \) and \( P_V = -\hbar^2 \Delta + V, \ V \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \). Let \( \lambda_V \) be the distribution function of \( V \):

\[ \lambda_V(t) := m(\{x : V(x) \geq t\}). \]

If \( 0 < E \in \mathrm{singsupp}_a \lambda_V \), then for any \( r > 0 \) there exists \( h_0, C \) and such that for \( 0 < h < h_0 \),

(7.4.14) \[ \sum_{z \in D(E, r)} m_V(z) \geq h^{-n}/C. \]

**REMARKS.**

1. For any potential \( V \), any critical value of \( V \) is in the \( C^\infty \) singular support of \( \lambda_V \) and hence in \( \mathrm{singsupp}_a \lambda_V \).

2. The theorem is true in all dimensions and for the same class potentials for which (7.4.5) holds – see [Sj96a] and [Sj96b].

**Proof.**

1. Suppose \( g \in C^\infty_c(\mathbb{R}) \). We then have \( g(P_V) = a_V(x, hD; h) \), where

\[ a_\bullet(x, \xi; h) = g(\xi^2 + V(x)) + ha_{1,V}(x, \xi, h), \]

\[ \partial_x^\alpha \partial_\xi^\beta (a_{1,V} - a_{1,0})(x, \xi, h) = O_{\alpha, \beta}(\langle x \rangle^{-\infty} \langle \xi \rangle^{-\infty}). \]

This follows from the functional calculus based on the Helffer–Sjöstrand formula with Lemma (7.10) provides the needed resolvent estimates.

It follows that

(7.4.15) \[ \mathrm{tr}(g(P_V) - g(P_0)) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} (g(\xi^2 + V(x)) - g(\xi^2))d\xi dx + O_g(h^{-n+1}). \]

2. We want to express the first term on the right hand side of (7.4.15) using the distribution function of \( V, \lambda_V \), or more precisely the distribution functions for \( V_\pm \):

\[ \lambda_{V_\pm}(t) = m(\{x : \pm V(x) \geq t\}), \quad t \geq 0. \]
For that we write
\[
\int_{\mathbb{R}^n} (g(\xi^2 + a) - g(\xi^2)) d\xi = 2^{-1} \text{Vol}(S^{n-1}) \int_0^\infty (g(r + a) - g(r)) r^{n-2} dr
\]
\[
= \pi^{n/2} (g * \chi^s_+ (a) - g * \chi^s_+ (0)),
\]
where
\[
\chi^s_+ = x^s_+ / \Gamma(s + 1), \quad x^s_+ = \begin{cases} x^s, & x > 0, \\ 0, & x \leq 0, \end{cases}
\]
first defined for Re \( s > -1 \) and then by analytic continuation for all \( s \in \mathbb{C} \).
(We assume that \( n = 3 \) but as the argument works for any dimension we proceed in that generality. The constant comes from the calculation of the volume of the sphere: \( \text{Vol}(S^{n-1}) = 2\pi^{n/2} / \Gamma(n/2) \).

We now recall that for \( F \in C^1 (\mathbb{R}) \) with \( F(0) = 0 \),
\[
\int_{\mathbb{R}^n} F(V(x)) dx = \int_0^\infty (F'(t) \lambda_{V_+} (t) - F'(-t) \lambda_{V_-} (t)) dt.
\]
Applying this with \( F(t) := g * \chi^{n-2}_+ (-t) - g * \chi^{n-2}_+ (0), \)
we see that
\[
\int_{\mathbb{R}^{2n}} (g(\xi^2 + V(x)) - g(\xi^2)) d\xi dx
\]
\[
= \pi^{n/2} \int_0^\infty \left( -g' * \chi^{n-2}_+ (-t) \lambda_{V_+} (t) + g' * \chi^{n-2}_+ (t) \lambda_{V_-} (t) \right) dt
\]
\[
= \pi^{n/2} \int_0^\infty g'(t) \left( -\chi^{n-2}_+ * \lambda_{V_+} (t) + \chi^{n-2}_+ * \lambda_{V_-} (-t) \right) dt
\]
\[
= \pi^{n/2} \langle u, g \rangle,
\]
where \( \langle \bullet, \bullet \rangle \) denotes distributional pairing and we used the fact that \( (\chi_+^s)' = \chi_+^{s-1} \) and
\[
(7.4.17) \quad u := \chi^{n-4}_+ * (\lambda_{V_+} (\bullet) - \lambda_{V_-} (-\bullet)).
\]

3. Since \( \text{singsupp}_a(\chi^s_+ ) = \{0\} \) and \( \chi_+^{-s-2} * \chi^s = \delta_0 \) we see that \( u \) defined in (7.4.17) satisfies
\[
\text{singsupp}_a(u) = \text{singsupp}_a(\lambda_{V_+} (\bullet) - \lambda_{V_-} (-\bullet))
\]
and for \( E > 0 \),
\[
E \in \text{singsupp}_a(u) \iff E \in \text{singsupp}_a(\lambda_{V}).
\]
Since \( u \) is real valued we conclude from (7.4.13) that
\[
E \in \text{singsupp}_a(u) \iff \forall \xi \in \mathbb{R} \setminus 0, \ (E, \xi) \in \text{WF}_a(u).
\]
7.4. THE SJÖSTRAND TRACE FORMULA

Hence suppose that $0 < E \in \text{singsupp}_a u$. Then $(E, 1) \in \text{WF}_a(u)$. Suppose that $\psi \in C_0^\infty(\mathbb{R}; [0, 1])$ is equal to 1 near $E$. Then Definition (7.13) of the analytic wave front set in terms of the FBI transform shows that there exist $t_j \to E$, $\tau_j \to 1$, $\epsilon_j \to 0$ and $\lambda_j \to +\infty$ such that

$$\left| \int_{\mathbb{R}} e^{-(t-t_j)^2/2+ i \tau_j(t-t_j)} \chi(t) u(t) dt \right| \geq e^{-\epsilon_j \lambda_j}. \quad (7.4.18)$$

4. We will apply Theorem 7.9 with

$$W = (E - b/2, E + b/2) + i(-a/2, a), \quad \Omega = (E - b, E + b) + i(-a, a),$$

$0 < b \ll 1$, $0 < a/b^2 \ll 1$, and

$$f_j(z) = M_j e^{\lambda_j (i \tau_j(t_j-z) - (t_j-z)^2)},$$

where

$M_j := \min_{z \in \Omega \setminus W} |f_j(z)|^{-1}$

$$= \min_{z \in \Omega \setminus W} e^{\lambda_j (i \tau_j \text{Im} z - (t_j - \text{Re} z)^2/2 + \text{Im} z^2/2)}$$

$$\geq e^{c_0 \lambda_j}, \quad c_0 > 0. \quad (7.4.19)$$

5. Applying (7.4.15) and (7.4.16) with $g = \psi f_j$ we obtain

$$\text{tr } [(\psi f_j)(P_V) - (\psi f_j)(P_0)]$$

$$= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} \left( (\psi f_j)(\xi^2 + V(x)) - (\psi f_j)(\xi^2) \right) d\xi dx + O_j(h^{1-n})$$

$$= \frac{1}{(2\pi h)^n} \langle u, \psi f_j \rangle + O_j(h^{1-n})$$

$$= \frac{1}{(2\pi h)^n} T_{\lambda_j}(\psi u)(t_j, \tau_j) + O_j(h^{1-n}).$$

The trace formula (7.4.5) and the lower bounds (7.4.18) (7.4.19) give

$$\left| \sum_{z \in W} f_j(z) \right| = \left| \frac{1}{(2\pi h)^n} T_{\lambda_j}(\psi u)(t_j, \tau_j) + O_j(h^{1-n}) + O(h^{-n}) \right|$$

$$\geq c_1 h^{-n} M_j e^{-\epsilon_j \lambda_j} - O_j(h^{1-n}) - O(h^{-n})$$

$$\geq c_1 h^{-n} e^{c_0 \lambda_j} - C_j h^{1-n} - C_0 h^{-n}.$$
But that implies that
\[ \max_{z \in W} |f_j(z)| \sum_{z \in W} m_V(z) \geq h^{-n}, \]
that is the number of resonances in \( W \) is bounded from below by \( h^{-n}/C \).
Since for any \( r \) we can choose \( W \) so that \( D(E, r) \subset W \) the estimates (7.4.14) follows.

7.5. RESONANCE EXPANSIONS FOR STRONG TRAPPING

In §§2.3 and 3.2.2 we saw that solutions of the wave equation \((-\partial_t^2 u - \Delta + V)u = 0\), \( V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{R}) \), \( n \) odd, with compactly supported initial data can be expanded in terms of resonances – see (3.2.11). The same arguments combined with Theorem 6.21 show that the same result holds for non-trapping metrics: if
\[ P := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C^\infty_c(\mathbb{R}^n), \quad V \in C^\infty_c(\mathbb{R}^n), \]
\[ \exp tH_p(x, \xi) \to \infty, \quad t \to \pm \infty, \quad p = |\xi|^2_g, \]
then as \( t \to +\infty \),
\[ u(x, t) = \sum_{\text{Im } \lambda_j < -A} e^{-i\lambda_j t} u_j(x) + E_A(t, x), \]
(7.5.1)
\[ \|E_A(t)\|_{H^N(K)} \leq C_K e^{-At} (\|u(x, 0)\|_{H^1} + \|\partial_t u(x, 0)\|_{L^2}), \]
for solutions of the wave equation,
\[ (\partial_t^2 - \Delta_g + V)u = 0, \quad u|_{t=0} \in H^1_{\text{comp}}, \quad \partial_t u|_{t=0} \in L^2_{\text{comp}}. \]
Here \( \lambda_j^2 \in \text{Res}(P) \), \( \text{Im } \lambda_j < 0 \). To make the statement clear we assumed for simplicity that resonances are semi-simple (algebraic and geometric multiplicities coincide) – see (3.2.11) for the general statement.

In this section we will consider the case of semiclassical Hamiltonians \( P(h) = -h^2\Delta_g + V \) with trapped sets which imply existence of resonances close to the real axis – see §7.3 for examples of that. In that case it is natural to expand \( u(t) \) given by the Schrödinger evolution \( u(t) = \exp(-itP(h)/h) u_0 \).

We start with some results in some cases where resonances are not close to the real axis. The analysis is based on results from §§6.3 and ??.

See Exercises ?? and 7.4 for applications of the same methods to the wave equation in odd and even dimensions respectively.
7.5. RESONANCE EXPANSIONS FOR STRONG TRAPPING

7.5.1. Schrödinger propagator in the case of resonance free regions. Suppose that for \( E > 0 \),

\[
P(h) := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n), \quad V \in C_c^\infty(\mathbb{R}^n),
\]

(7.5.2)

\[ p(x, \xi) = E \implies \exp tH_p(x, \xi) \to \infty, \quad t \to \pm \infty, \quad p = |\xi|^2 + V. \]

Theorem 6.21 shows that there exists \( \delta > 0 \) such that for any \( M \),

(7.5.3) \( \text{Res}(P(h)) \cap [E - \delta, E + \delta] - i[0, Mh \log(1/h)] = \emptyset, \quad 0 < h < h_0(M). \)

Moreover, for any \( \chi \in C_c^\infty(\mathbb{R}^n) \) we have the following bound on the meromorphically continued resolvent:

\[
\| \chi R(z, h) \chi \|_{L^2 \to L^2} \leq C \frac{\exp C \Im z/h}{h}, \quad z \in [E - \delta, E + \delta] - i[0, Mh \log(1/h)] = \emptyset, \quad 0 < h < h_0(M).
\]

(7.5.4)

This has the following consequence for the truncated Schrödinger propagator:

**THEOREM 7.15 (Schrödinger propagator at nontrapping energies).** Suppose that \( P(h) \) satisfies (7.5.2), \( \chi \in C_c^\infty(\mathbb{R}^n) \) and \( \psi \in C^\infty((E - \delta, E + \delta)) \) where \( \delta \) is the same as in (7.5.3) and \( E > 0 \).

Then there exists \( T_0 \) such that

(7.5.5) \( \chi \exp(-tP(h)/h)\psi(P(h))\chi = O\left(\left(\frac{(t - T_0)\pm}{h}\right)^{-\infty}\right)_{L^2 \to L^2} \)

**REMARK.** When the order of \( \chi \) and \( \psi(P(h)) \) is reversed on the left hand side of (7.5.5) we obtain an additional terms \( O(h^{\infty}) \) on the right hand side. That follows from the pseudodifferential calculus: if \( \chi_1 = 1 \) on \( \text{supp} \chi \) then \( (1 - \chi_1)\psi(P(h))\chi = O(h^{\infty})_{L^2 \to L^2} \) – see for instance [Zw12, §14.3].

**Proof.** 1. Let us write

(7.5.6) \( R_\pm(z, h) = (P(h) - z)^{-1}, \) analytic for \( \pm \Im z > 0 \),

using the same notation for the meromorphic continuation (this means that, in our standard notation, \( R(z, h) = R_+(z, h) \)). Using Stone’s formula (Theorem B.8) we write the spectral measure as

(7.5.7) \( dE_\lambda = (2\pi i)^{-1}(R_-(\lambda, h) - R_+(\lambda, h))d\lambda, \quad \lambda \in \mathbb{R}. \)

Using (7.5.7) the left hand side of (7.5.5) can be rewritten as

\[
\chi e^{-itP(h)/h}\psi(P(h))\chi = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-it\lambda/h} \chi(R_-(\lambda, h) - R_+(\lambda, h))\chi\psi(\lambda)d\lambda.
\]
2. Let \( \tilde{\psi} \in C^\infty_c(\mathbb{C}) \) be an almost analytic extension of \( \psi \) – see \( \S B.2 \). We can choose it so that 
\[
\text{supp } \tilde{\psi} \subset \{ z : \text{Re } z \in (E - \delta, E + \delta) \}.
\]
Green’s formula (D.1.1) then gives
\[
\chi e^{-itP(h)/h} \tilde{\psi}(P(h)) = A(h) + B(h),
\]
where
\[
A(h) := \frac{1}{2\pi i} \int_{\text{Im } z = -M_0 \log \frac{1}{h}} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h)) \chi \tilde{\psi}(z) dz
\]
and
\[
B(h) := \frac{1}{\pi} \iint_{-M_0 \log \frac{1}{h} < \text{Im } z < 0} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h)) \chi \tilde{\partial_z} \tilde{\psi}(z) dm(z).
\]
Using bound (7.5.4) on the analytic continuation of the resolvent we see that
\[
\| A(h) \|_{L^2 \rightarrow L^2} \leq C' h^{-(C-t)}
\]
and
\[
\| B(h) \|_{L^2 \rightarrow L^2} \leq C N h^{N-2MC(t)}.
\]
Since \( N \) is arbitrary we conclude that
\[
(7.5.9) \quad B(h) = \mathcal{O}(h^\infty(t)^{-N}).
\]
Since for all times the propagator is bounded this proves the theorem. \( \square \)

We now move to the case of \emph{normally hyperbolic trapping} presented in \( \S 6.3 \). We assume that \( P(h) = -h^2 \Delta_g + V(x) \) and that at \( E > 0 \), the trapped set \( K_E \neq \emptyset \) and that it satisfies the conditions (A1)–(A3) of \( \S 6.3 \). In that case Theorem 6.16 shows that
\[
(7.5.10) \quad \chi R(z) \chi = o_c(h^{-2})_{L^2 \rightarrow L^2}, \quad z \in [E - \delta, E + \delta] - i\hbar[0, \nu_{\min} - \epsilon, 0],
\]
for some \( \delta \) and any \( \epsilon > 0 \).

**Theorem 7.16 (Schrödinger propagator for normally hyperbolic trapping).** Suppose that \( P(h) \) is as in (7.5.2) but that at \( E > 0 \) the trapped is normally hyperbolic in the sense of Definition 6.15.
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**Theorem 7.15**

Theorem 7.16

Theorem 7.17

Theorem 7.18

Theorem 7.19

Theorem 7.20

**Theorem 7.21**

If \( \chi \in C_c^\infty(\mathbb{R}^n) \) and \( \psi \in C^\infty((E - \delta, E + \delta)) \) where \( \delta \) is the same as in (7.5.10), then for any \( \nu < \nu_{\text{min}} \),

\[
(7.5.11) \quad \chi \exp(-tP(h)/h)\psi(P(h))\chi = e^{-\nu t/2}o(h^{-2})_{L^2 \to L^2} + O(h^\infty(t)^{-\infty})_{L^2 \to L^2}.
\]

**Proof.** 1. We proceed as in the proof of Theorem 7.15 and use the same notation. For \( \nu < \nu_{\text{min}} \) (7.5.10) and Green’s formula now give

\[
\chi e^{-itP(h)/h}\psi(P(h))\chi = A(h) + B(h),
\]

\[
A(h) := \frac{1}{2\pi i} \int_{\text{Im } z = -\nu h/2} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h))\bar{\psi}(z)dz
\]

\[
B(h) := \frac{1}{\pi} \int_{-\nu h/2 < \text{Im } z < 0} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h))\bar{\psi}(z)\partial_z\bar{\psi}(z)dm(z).
\]

The \( B(h) \) term can be estimating in the same way as in (7.5.8) resulting in (7.5.9).

2. Using (7.5.10) we immediately see that \( A(h) = e^{-\nu t/2}o(h^{-2}) \) which concludes the proof. \( \square \)

**INTERPRETATION.** The localization in energy using the cut-off function \( \psi(P(h)) \) is necessary as different estimates are valid in different energy regimes – see Fig.7.5. At non-trapping energies, once time is large enough the localized propagator (in space and energy) is of size \( O(h^\infty t^{-\infty}) \), that is negligible.

In the case of normally hyperbolic trapping the localized propagator is semiclassically negligible (that is \( O(h^\infty(t)^{-\infty}) \)) once \( t \gg \log(1/h) \).
the intermediate region the exponentially decaying estimate provides a quantitative decay rate for the propagator.

7.5.2. Schrödinger propagator in the case of resonances converging to the real axis. We now consider the general case of

\[ P(h) := \Delta_g + V, \quad g_{ij} - \delta_{ij} \in C^\infty_c(\mathbb{R}^n), \quad V \in C^\infty_c(\mathbb{R}^n), \]

but the results are non-trivial only in the case of existence of resonances with \( \text{Im} z = O(h^\infty) \).

The difficulty in stating a general result is overlap of resonances. Since the results for the Schrödinger equation are semiclassical, Theorems 7.6 (see also and 7.14 show that resonances appear in dense (\( \sim h^{-n} \)) clouds and the possibility of overlap is inevitable. By overlap we mean resonances being very close to each other. The upper bound (7.2.2) shows that there are always some gaps between clouds of resonances. That motivates the following

**DEFINITION 7.17.** A family of rectangles

\[ h \mapsto W(h) := (a(h), b(h)) - i[0, c(h)), \quad 0 < c_0 < a(h) < b(h) < 1/c_0, \]

is called semiclassically admissible if

\[ d(\partial W(h) \setminus \mathbb{R}, \text{Res}(P(h))) > c_1 h^n, \quad c_1 h^n < c(h) < 1/c_0. \]

for some fixed constants \( c_0, c_1 > 0 \).

**REMARK.** The power \( n \) can be replaced by any larger power. Theorem 7.4 shows that near any fixed \( a, b \) and \( c \) we can find \( a(h), b(h) \) and \( c(h) \) so that (7.5.13) holds once \( c_1 \) is sufficiently small.

**THEOREM 7.18 (Resonance expansion for strong trapping).** Suppose that \( P(h) \) is given by (7.5.12), \( \psi \in C^\infty_c((0, \infty)), \) \( \text{supp} \psi = [a, b], \) \( \chi \in C^\infty_c(\mathbb{R}^n) \).

There exist a constant \( L_n \) depending only on the dimension such for any admissible \( W(h) \) with \( [a, b] \subseteq W(h) \),

\[ t > h^{-L_n} \]

implies

\[ \chi \exp(-tP(h)/h)\psi(P(h))\chi = \]

\[ \sum_{z \in W(h)} \chi \text{Res}_{w=z} \left( e^{-itw/h} R(w, h) \right) \chi \psi(P(h)) + O(h^\infty)_{L^2 \to L^2}. \]
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REMARKS. 1. The result states that for sufficiently large times the localized propagator can be expanded in terms of resonances close to the real axis with a semiclassical negligible error. From Theorem 6.26 (and its generalizations – see [CV02], [Da14] and references given there) we know the resonances in $W(h)$ satisfy $\text{Im } z > e^{-C/h}$. This means that the expansion is relevant for $h^{-2n-2} < t < e^{C/h}$.

2. The need for admissible rectangles comes from difficulties in estimating individual terms in the sum over resonances. Unless some, possibly very weak, separation between resonances is imposed we do not know how to estimate the residues.

3. If (7.5.13) $c_1 h^n$ is replaced by $h^M$ for some $N$ then (7.5.15) still holds once $L_n$ in (7.5.14) is changed into an $M$-dependent constant.

4. The constant in the estimate on $t$ in (7.5.14) is explicit in the proof but it is not clear what is the optimal general lower bound on $t$ guaranteeing validity of an expansion. In the case of a barrier – Theorem 7.8) – one only needs $t \geq T_0$ for some fixed $T_0$ [NSZ14, Main Theorem].

5. The order of $\chi$ and $\psi(P(h))$ on the left hand side of (7.5.15) does not matter since – see the remark following Theorem 7.15.

Proof. 1. Let $W(h)$ be an admissible rectangle in the sense of Definition 7.17. We assume that

$$a(h) < a < b < b(h)$$

. We choose $\psi_h \in C^\infty_c$ with satisfying

$$\text{supp } \psi_h \subset (a(h) - c_1 h^n, b(h) + c_1 h^n),$$

$$\psi_h(s) = 1 \text{ for } s \in [a(h), b(h)].$$

In particular, $\psi_h \equiv 1$ on the support of $\psi$. Functional calculus (see the remark following Theorem 7.15) then shows that

$$\chi \exp(-tP(h)/h)\psi(P(h))\chi = \chi \exp(-tP(h)/h)\psi_h(P(h))\psi(P(h))\chi$$
$$= \chi \exp(-tP(h)/h)\psi_h(P(h))\chi \psi(P(h)) + O(h^\infty)_{L^2 \to L^2}.$$  

This means that we can replace the left hand side of (7.5.15) by the first term in the last line.

3. To apply the same procedure as in step 2 of the proof of Theorems 7.15 we consider an almost analytic extension of $\psi_h$. By taking an almost extension of $s \mapsto \tilde{\psi}_h(h^n s)$ we obtain $\tilde{\psi}_h \in C^\infty(\mathbb{C})$ satisfying

$$\bar{\partial}_z \tilde{\psi}_h(z) = \begin{cases} O(|\text{Im } z|/h^n)^\infty, \\
O(h^{-n}), \\
0 \text{ if } |\text{Re } z - a(h)| > c_1 h^n, \\
0 \text{ if } |\text{Re } z + b(h)| > c_1 h^n. \end{cases}$$
Applying Green's formula (D.1.1) and using the notation (7.5.6) we obtain
\[ A(h) := \frac{1}{2\pi i} \int_{\text{Im } z = c(h)} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h))\bar{\psi}(z)dz \]
(7.5.17)
\[ B(h) := \frac{1}{\pi} \int_{W(h)} e^{-itz/h} \chi(R_-(z, h) - R_+(z, h))\bar{\psi}(z)dm(z), \]
\[ C(h) := \sum_{w \in W(h)} \chi \text{Res}_{z=w} \left(e^{-itz/h}R(z, h)\right)\chi.\]

We note that (7.5.16) show that we do not need to keep the terms \( \psi_h(w) \) in the formula for \( C(h) \).

4. To estimate \( A(h) \) we note that (7.5.13)
\[ d(\{\text{Im } z = c(h)\} \cap \text{supp } \bar{\psi}_h, \text{Res}(P(h))) > c_1h^n. \]
The estimate (7.2.11) in Theorem 7.5 then shows that
\[ \|\chi R_+(z, h)\chi\|_{L^2 \rightarrow L^2} \leq Ae^{Ah^{-n}\log \frac{1}{h}}, \]
for \( z \in \{\text{Im } z = c(h)\} \cap \text{supp } \bar{\psi}_h \). Since (7.5.6) shows that the other term, \( R_-(z, h) \), is bounded on \( \text{Im } z = c(h) \), we obtain
\[ \|A(h)\|_{L^2 \rightarrow L^2} \leq Ce^{-t\omega/2h}e^{Ah^{-n}\log \frac{1}{h}} \]
(7.5.19)
\[ = O(h^N), \quad t \gg Nh^{-n+1}\log \frac{1}{h}. \]

5. To estimate \( B(h) \) we split it into two terms:
\[ B(h) = B_1(h) + B_2(h) \]
where the integration is over \( \text{Im } z > -h^M \) and \( \text{Im } z < -h^M \) respectively, where \( M = M_n > n \) depending only on dimension \( n \) will be chosen later. We start with \( B_2 \) and we use the estimate (7.5.18) for both \( R_\pm \) (since we are now near the real axis) and the properties of \( \partial_z \bar{\psi}_h \) (see (7.5.16)):
\[ \|B_2(h)\|_{L^2 \rightarrow L^2} \leq C h^{-n} \int_{s=h^M}^{2\omega} e^{-ts/h}e^{Ah^{-n}\log \frac{1}{h}}ds \]
(7.5.20)
\[ \leq C h^{-n}e^{-th^{M-1}+Ah^{-n}\log \frac{1}{h}} \]
\[ = O(h^\infty), \quad t \gg h^{-n-M}. \]

6. To analyse \( B_1(h) \) we use the maximum principle in the form presented in Lemma 7.7 (see also Lemma D.1). The estimate (7.5.18) and estimates \( \|R_\pm(z, h)\|_{L^2 \rightarrow L^2} \leq 1/|\text{Im } z|, \quad \pm \text{Im } z > 0 \) show that there exist \( M = M_n > n \) and \( Q = Q_n \) such that
\[ \|\chi R_\pm(z, h)\chi\|_{L^2 \rightarrow L^2} \leq C h^{-Q_n}, \quad |\text{Im } z| \leq h^M. \]

7. RESONANCE EXPANSIONS FOR STRONG TRAPPING
This is the $M_n$ we choose in the splitting of $B$ into $B_1 + B_2$.

The bound (7.5.21) and (7.5.16) give the following estimate
\begin{equation}
\|B_2(h)\|_{L^2 \rightarrow L^2} \leq C_N \int_0^{h^M} e^{-st/h} h^{-Q} (s/h)^N ds
\end{equation}
Combining (7.5.19), (7.5.20) and (7.5.22) with (7.5.17) gives (7.5.15). □

7.5.3. Expansions of scattered waves for strong trapping. We now adapt the results of this chapter to study the wave equation. The results are valid for black box Hamiltonians of §4.1 and in particular for obstacle problems – see [BZ01]. To keep the presentation simple we restrict ourselves to the case of $P = -\Delta_g \geq 0$.

We obtain a resonance expansion of scattered waves in case of resonances converging to the real axis. It is much weaker than the expansion (7.5.1) valid in the non-trapping case but it addresses a more realistic situation.

In the case of the wave equation we replace admissible rectangles of Definition 7.17 by admissible contours:

**DEFINITION 7.19.** An admissible contour is the positively oriented contour given by
\[ \Gamma := \{ z = x - i\gamma(x), x \in \mathbb{R} \} \]
where $\gamma(x) > 0$ and for some $M$, $N$ and $c > 0$,
\begin{equation}
\begin{align*}
c(x)^{-M} &< \gamma(x) < 1/c, \\
|\gamma'(x)| &\leq 1/c, \\
d((x + i\gamma(x))^2, \text{Res}(P)) &> c(x)^{-N}.\end{align*}
\end{equation}
REMARK. The global bound on the number of resonances in Theorem 4.13 (or rescaling of the local semiclassical bound (7.2.2)) show that we can find admissible contours for any \( M \) and for \( N \geq n \).

**THEOREM 7.20** (Resonance expansions of scattered waves). Suppose that \( n \) is odd and \( P = -\Delta_g \), \( g_{ij} - \delta_{ij} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}) \) and put

\[
U(t) := \sin t\sqrt{P}/\sqrt{P}, \quad R(\lambda) := (P - \lambda^2)^{-1},
\]

with \( R(\lambda) \) first defined for \( \text{Im} \lambda > 0 \) and the continued meromorphically.

Let \( \chi \in C_c^\infty(\mathbb{R}^n) \) and let \( \Gamma \) be an admissible contour in the sense of Definition 7.19.

Then for any \( M > M_0 \) there exists \( \epsilon > 0 \) and a function \( t \to c(t) \), \(|c(t) - t'| \leq C\) such that

\[
(7.5.24) \quad \chi U(t) \chi = \sum_{\text{Im} \lambda > -\gamma(\text{Re} \lambda)} \chi i \text{Res}_{\xi = \lambda} \left( e^{-i \xi t} R(\xi) \right) \chi + E(\chi(t),
\]

where

\[
(7.5.25) \quad \|E(\chi(t))\|_{H^k \to L^2} \leq Ct^{-k(L-K_0)}
\]

where \( L \geq 0 \) is arbitrary and \( K_0 \) is a fixed constant.

**INTERPRETATION.** As in Theorems 2.7 and 3.11 the residues give expansions of solutions to \((\partial^2_t - \Delta_g) w = 0\) with compact initial data: for \( f \in C_c^\infty(\mathbb{R}^n) \),

\[
(7.5.26) \quad \text{Res}_{\xi = \lambda_j} \left( e^{-i \xi t} R(\xi) \right) f(x) = \sum_{\ell = 0} t^{\ell} e^{-i \lambda_j \ell} f_{\ell,j}(x).
\]

The error term \( E(\chi(t))f \) is then \( O(t^\infty) \) but each term in \( (7.5.26) \) eventually decays much faster. However, we include more terms \( t \to +\infty \) (\( |\text{Re} \lambda_j| \leq c(t) \sim t^\epsilon \)). Hence resonances with \( \text{Im} \lambda_j = O(\langle \lambda_j \rangle^{-\infty}) \) provide non-trivial contributions to the expansion.

**Proof.** 1. Theorem 4.19 shows that there is no eigenvalue or resonance at 0. We start as in the proof of Theorem 2.7 and obtain the analogue of (??):

\[
(7.5.27) \quad \chi U(t) \chi = \frac{1}{2\pi} \int_\mathbb{R} e^{-i \lambda t} \chi(R(\xi) - R(-\xi))\chi d\xi.
\]

2. Let \( c(t) \sim t^\epsilon \) be a function of \( t \) yet to be chosen and let \( \Gamma \) be an admissible contour. We define

\[
\Gamma_1 := \Gamma \cap \{|\text{Re} \lambda| \leq c(t)\}, \quad \Gamma_3 := \mathbb{R} \setminus (-c(t), c(t)),
\]

\[
\Gamma_2 := \{-c(t) - i\tau : \tau \in (0, \gamma(c(t)))\} \setminus \{c(t) - i\tau : \tau \in (0, \gamma(c(t)))\},
\]

...
with the natural orientation agreeing with the positive orientation of $\mathbb{R}$. By deforming the contour in (7.5.27) to $\Gamma_1 + \Gamma_2 + \Gamma_3$ (with the natural orientations)

$$\chi U(t) = \sum_{\text{Im}\lambda > -\gamma(\text{Re}\lambda)} \chi_i \text{Res}_{\zeta=\lambda} \left( e^{-i\zeta t} R(\zeta) \right) \chi + V_1(t) + V_2(t) + V_3(t),$$

$$V_j(t) := \frac{1}{2\pi} \int_{\Gamma_j} e^{-i\zeta t} (R(\zeta) - R(-\zeta)) \chi d\zeta.$$  

We need to show that we can choose $\epsilon > 0$ and $c(t) \sim t^\epsilon$ so that

(7.5.28)

$$V_j(t) = \mathcal{O}(t^{-\epsilon(L-K_0)})_{H^L \to L^2}.$$  

3. We start with $V_1$. The separation of $\Gamma_1$ from the set of resonances given in (7.5.23) and the estimate (7.2.12) show that

$$\chi(R(\zeta) - R(-\zeta) \chi = \mathcal{O}(e^{A|\zeta|^n \log |\zeta|}), \quad \zeta \in \Gamma_1.$$  

Hence

$$\|V_1(t)\|_{L^2 \to L^2} \leq C e^{-at} + C t^{1/2} e^{-t^\gamma(x)} e^{A x^n \log x} (1 + |\gamma'(x)|) dx$$

(7.5.29)

$$\leq C e^{-at} + C' t^\epsilon e^{-t^M + t^n \log t}$$

$$= \mathcal{O}(t^\infty), \quad \text{if} \quad \epsilon < \frac{1}{M+n}.$$  

Hence for $\epsilon$ small enough depending on $M$ (that is, on the admissible contour) we have (7.5.28) for $j = 1$.

4. To estimate $V_3(t)$ we write note that Stone’s formula (Theorem B.8) shows that

$$(R(\lambda) - R(-\lambda))(I - \Delta_g)^{L/2} = R(\lambda) - R(-\lambda) \langle \lambda \rangle^L, \quad \lambda > 1.$$  

Hence for $f \in H^L$

$$V_3(t) f = \int_{\mathbb{R} \setminus [-c(t),c(t)]} e^{-i\zeta t} \chi(R(\zeta) - R(-\zeta)) \langle \zeta \rangle^{-L} (I - \Delta_g)^{L/2} f d\zeta$$

We now use Stone’s formula (Theorem B.8) to see that

$$\int_{\mathbb{R}} (R(\zeta) - R(-\zeta)) \varphi(\zeta) d\zeta = \mathcal{O}(\sup |\varphi(\zeta)/\zeta|)_{L^2 \to L^2}.$$  

Applied with

$$\varphi(\zeta) := \langle \zeta \rangle^{-L} 1_{\mathbb{R} \setminus [-c(t),c(t)]} \langle \zeta \rangle,$$

we obtain

$$\|V_3(t) f\|_{L^2} \leq \langle c(t) \rangle^{-L-1} \leq C t^{-\epsilon L - \epsilon},$$
which gives (7.5.28) for $j = 3$.

5. We now come to estimating $V_2(t)$ and this is where a choice of $c(t)$ is essential. We choose $c(t)$ such that for some fixed

\[(7.5.30) \quad |c(t) - t^\epsilon| \leq 1, \quad D((\pm c(t) + i[0, 1/c])^2, \text{Res}(P)) > t^{-N},\]

for some $N$. This can always be accomplished for $N > n$ because of the upper bound on the number of resonances. As in Step 6 of the proof of Theorem 7.18, the bound (7.2.12) and Lemma 7.7 imply that there exist $M$ and $Q$ such that

\[
\|\chi R(\pm c(t) + iy)\chi\|_{L^2 \to L^2} \leq Cc(t)^M, \quad |y| \leq c(t)^{-Q}.
\]

Using this bound and the bound (7.2.12) (valid on $\Gamma_2$ because of (7.5.30)) we estimate $V_2$ as follows. Let $\chi_0 \in C^\infty_c$ be equal to 1 on $\text{supp} \chi$. Then for for $f \in H^L$,

\[
\|V_2(t)f\|_{L^2} \leq Ct^{-\epsilon L} \int_0^{\gamma(c(t))} \|\chi_0 R(c(t) + iy)\chi_1\|_{L^2 \to L^2} e^{-yt} dy \|\chi f\|_{H^L}
\]

\[
\leq C t^{-\epsilon L} \int_0^{t^\epsilon Q} t^M e^{-ty} dy + C t^{-\epsilon L} \int_{t^{-\epsilon Q}}^c e^{-ty + At^{\epsilon n} \log t} dy
\]

\[
\leq C t^{-\epsilon(L+M-Q)} + C' t^{-\epsilon L}, \quad \text{if} \quad \epsilon < \frac{1}{n+1}.
\]

This (7.5.28) for $j = 2$ with $K_0 = Q - M$. \qed

7.6. NOTES

Theorem 7.1 was proved by Bony, Burq and Ramond [BBR10]. The comment that $C$ is independent of $\delta$ was made by J.-F. Bony. For more connections between resolvent estimates and local smoothing for Schrödinger propagators see [Bu04], [Da09], and references given there.

The local upper bounds on the number resonances were first established by Sjöstrand [Sj90]. That paper also introduced geometric bounds in which the number of resonances was estimated using the dimension of the trapped set (6.1.3). For more recent developments and references see Datchev–Dyatlov [DD13], Nonnenmacher–Sjöstrand–Zworski [NSZ14], and [SZ07a]. For a physics perspective on counting resonances see Lu et al [LSZ03], Potzuweit et al [P*12] and references given there.

The observation that bounds on the number of resonances imply the bound on the resolvent was made in [Zw90]. It was used by Stefanov–Vodev [SV96] to show that a sequence of quasimodes with energies converging to infinity implies existence of resonances converging fast to the real axis. The argument is related classical works on the completeness of sets of eigenfunctions going back to Carleman [Ca36]. A quantitative generalization was
given by Tang–Zworski [TZ98] and that was refined further by Stefanov [St99] – Theorem 7.6 comes from that paper.

The local trace formula for resonances (7.4.5) was proved by Sjőstrand [Sj96a] in much greater generality and without the $O_f(h^\infty)$ error term. That term is irrelevant to our application to lower bounds which comes from Sjőstrand [Sj96b]. §7.4 is meant as an introduction to these two papers in a simpler setting. Refinements of the trace formula were given in Petkov–Zworski [PZ01] and by Bruneau–Petkov [BP03]. These papers were motivated by the closely related semiclassical version of the Breit–Wigner approximation for the derivative of the scattering phase (scattering shift). Other applications of the local trace formula can be found in Bony [Bo02], Bony–Sjőstrand [BS01] and Dimassi–Zerzeri [DZ03].

The presentation of resonance expansions in §7.5 comes from Burq–Zworski [BZ01] with some corrections and slight generalizations. Under specific assumptions which imply isolation of resonances better results are possible – see Gérard–Martinez [GM89], Merkli–Sigal [MS99] Nakamura–Stefanov–Zworski [NSZ14] and Stefanov [St01]. When some global conditions on separation of resonances are imposed stronger expansions for the wave equation were given by Tang–Zworski [TZ00]. The complex analytic techniques used for the resonance expansions have a longer tradition in the study of other non-self-adjoint problems such as damped wave equations – see Markus [Ma88].

7.7. EXERCISES

Section 7.3

1. We outline the steps for a refinement of Theorem 7.6 that is formulated as follows: suppose that we have $u_j(h), j = 1, \cdots, N(h)$ each satisfying the assumptions of Theorem 7.6 with $E_j(h) \in [a(h), b(h)], 0 < a_0 < a(h) \leq b(h) < b_0$:

$$(P(h) - E_j(h))u_j(h) = \epsilon_j(h), \quad \|u_j(h)\| = 1, \quad \text{supp} u_j(h) \subset \Omega \subset \mathbb{R}^n$$

where $\Omega$ is independent of $h$ and all $\epsilon_j(h) = O(h^\infty)$ or all $\epsilon_j(h) = O(e^{-S_0/h}), S_0 > 0$.

Suppose in addition that $u_j(h)$ are approximately orthogonal in the sense that

$$(7.7.1) \quad |\langle u_j(h), u_k(h) \rangle - \delta_{kj}| \leq \delta(h),$$

where $\delta(h) = O(h^\infty)$. Then there exists $\epsilon(h)$ satisfying (7.3.7)

$$(7.7.2) \quad |\text{Res}(P(h)) \cap [a(h) - \epsilon(h), b(h) + \epsilon(h)] - [0, \epsilon(h)]| \geq N(h).$$
We outline the steps of the proof (see [St99] for more details):

1. Show that

\[
N(h) = O(h^{-n}).
\]

**Hint:** construct a self-adjoint operator with a discrete spectrum for which \(u_j's\) are quasimodes; use (7.7.1) and the spectral theorem to show that the number of eigenvalues close to \(E_j(h)\)'s is at least \(N(h)\). Then use the bound (7.3.3).

2. With \(\epsilon(h)\) to be chosen (and satisfying (7.3.7)) let \(z_j, j = 1, \ldots, M(h)\) be the resonances of \(P(h)\) in \(\Omega(h) := (a(h) - \epsilon(h), b(h) + \epsilon(h)) - (0, \epsilon(h))\) including without multiplicities. Let

\[
\Pi_j := \frac{1}{2\pi i} \oint_{z_j} \chi R(z, h)\chi dz,
\]

where the integral is over a circle containing only \(z_j\). Let \(\Pi\) be the orthogonal projection of \(L^2\) onto

\[
\Pi_1 L^2 + \Pi_2 L^2 + \cdots + \Pi_M(h) L^2 \subset L^2.
\]

Then \((1 - \Pi)\chi R(z, h)\chi\) is holomorphic in \(\Omega(h)\). **(Hint:** use Theorem 4.7).

3. Choosing \(\epsilon(h)\) suitably apply Lemma 7.7 to \(Q(z, h) := (1 - \Pi)\chi R(z, h)\chi\) to obtain

\[
\|(1 - \Pi)\chi R(E, h)\chi\| \ll h^n/\max \|\epsilon_j(h)\|, \quad E \in [a(h), b(h)].
\]

4. As in Step 2 of the proof of Theorem 7.6 use (7.7.4) to show that

\[
\|(1 - \Pi)u_j(h)\| \ll h^n.
\]

From this and (7.7.1), (7.7.3) deduce that

\[
|\langle \Pi u_j(h), \Pi u_k(h) \rangle - \delta_{jk}| \ll 1/N(h).
\]

5. Show that if \(f_j \in L^2, j = 1, \ldots, N\) and \(|\langle f_j, f_k \rangle - \delta_{jk}| < 1/N\) then the set of \(f_j's\) is linearly independent. Deduce from (7.7.5) that the rank of \(\Pi\) is at least \(N(h)\). This proves (7.7.2).
2. Use (7.7.2) to show that for \( P(h) = -h^2 \Delta_g + V \) with \( V \) and \( E \) satisfying (7.3.2), we have for some \( S, \delta > 0 \),
\[
|\text{Res}(P(h)) \cap [E - \delta, E + \delta] - i[0, e^{-S/h}]| \geq h^{-n} / C.
\]
(In fact, using the Exercise 7.3 one can show the asymptotic formula (7.3.3) for resonances – see [NSZ03, Corollary, §5].)

3. Show the following stronger version of Theorem 7.8: under the same assumptions, suppose that \( u_\theta(h) \) satisfies \( (P_\theta(h) - z(h))u_\theta = 0, \|u_\theta\|_{L^2(\Gamma_\theta)} = 1 \), that is \( u_\theta(h) \) is a resonant state. (Here, as in the proof of Theorem 7.8, \( P_\theta(h) \) is a complex scaled operator with \( \theta = M_0 h \log(1/h) \).)

\[
\chi \in C^\infty_c(\mathbb{R}^n, [0, 1]) \quad \text{such that} \quad (7.3.15).
\]

If \( |\text{Re} z(h) - E| < \delta \) then either \( \text{Im} z > -e^{-S/h}, \|\chi u_\theta(h)\| = 1 + O(e^{-S/h}) \) or \( \text{Im} z < -Mh \log(1/h), \|\chi u_\theta(h)\| = O(e^{-S/h}) \), where \( S > 0 \) is a constant and \( M \) is arbitrarily large provided \( M_0 \) is chosen large and \( h \) is small enough. (Hint. From the proof of Theorem 7.8 we see that the only new result is the statement that \( \|\chi u_\theta\| = 1 + O(e^{-S/h}) \) when the resonance is close to the real axis. Hence suppose that \( \|(1 - \chi)u_\theta\| \geq e^{-S/h} \) and define \( v := u_\theta/\|(1 - \chi)u_\theta\| \). Hence, \( \|v\| \leq e^{S/h} \) and if \( S > 0 \) small enough tunneling estimates show that \( \|P(h)\chi_1\| v \| \leq e^{-S/h} \), where \( \chi_1 \in C^\infty_c, (1 - \chi_1)(1 - \chi) = 1 - \chi, \) and \( P_0(h) \) and \( P(h) \) coincide on the support of \( 1 - \chi_1 \). Applying the argument in Step 6 provides a contradiction.)

Section 7.5

4. This problem gives an analogue of (7.5.24) for \( n \) even.

1. Assume that \( P \) is as in Theorem 7.20 and that \( n \) is even and \( n \geq 4 \). The same argument as in the proof of Theorem 4.19 shows that there cannot be a resonance or an eigenvalue at \( 0 \). The cut-off resolvent then satisfies
\[
\chi R(\zeta)\chi = F(\zeta, \zeta^{n-2} \log \zeta), \zeta \in \mathbb{C} \setminus i(-\infty, 0],
\]
where \( F(\zeta, \omega) \) is holomorphic near \( (0, 0) \). (See Vodev [Vo94a].)

Use (7.7.6) to show that (7.5.24) holds with
\[
\|E_\chi(t)\|_{L^2 \to L^2} \leq Ct^{-n+1}, \quad t \to +\infty
\]
provided that \( L \) is large enough. (Hint: use the contour shown in Fig. 7.7 and then deform it near 0 to intervals along the negative imaginary axis.)

2. Let \( \Psi \in C^\infty(\mathbb{R}; [0, 1]) \) be an even function equal to 0 near 0 and 1 for \( s > 1 \). With the same assumptions as in part one, show that (7.5.24) and
(7.5.25) hold for $\chi U(t)\chi$ replaced by $\chi U(t)\Psi(\sqrt{P})\chi$. (Hint: use the contour deformation with the almost analytic extension of $\Psi$ shown in Fig. 7.8.)
Part 4

APPENDICES
NOTATION

A.1. BASIC NOTATION

$\mathbb{R}_+ = (0, \infty)$

$\mathbb{R}^n = n$-dimensional Euclidean space

$x, y$ denote typical points in $\mathbb{R}^n$: $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

$\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$

$z = (x, \xi), w = (y, \eta)$ denote typical points in $\mathbb{R}^n \times \mathbb{R}^n$:
$z = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n), \ w = (y_1, \ldots, y_n, \eta_1, \ldots, \eta_n)$

$\mathbb{T}^n = n$-dimensional flat torus $= \mathbb{R}^n/\mathbb{Z}^n$

$\mathbb{C} = \text{complex plane}$

$\mathbb{C}^n = n$-dimensional complex space

$U \subset V$ means $\bar{U}$ is a compact subset of $V$

$\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i = \text{inner product on } \mathbb{C}^n$

$|x| = \langle x, x \rangle^{1/2}$

$\langle x \rangle = (1 + |x|^2)^{1/2}$

$\mathcal{M}^{m \times n} = m \times n$-matrices

$S^n = n \times n$ real symmetric matrices
\[ A^T = \text{transpose of the matrix } A \]

\[ I \text{ denotes both the identity matrix and the identity mapping.} \]

\[ J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix} \]

\[ \sigma(z,w) = \langle Jz,w \rangle = \text{symplectic inner product} \]

\[ \# S = \text{cardinality of the set } S \]

\[ |E| = \text{Lebesgue measure of the set } E \subset \mathbb{R}^n \]

### A.2. FUNCTIONS DIFFERENTIATION

The support of a function is denoted “supp”, and a subscript “c” on a space of functions means those with compact support.

- Partial derivatives:
  \[ \partial_j := \frac{\partial}{\partial x_j}, \quad D_j := \frac{1}{i} \frac{\partial}{\partial x_j} \]

- Multiindex notation: A multiindex is a vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \), the entries of which are nonnegative integers. The size of \( \alpha \) is
  \[ |\alpha| := \alpha_1 + \cdots + \alpha_n. \]

We then write for \( x \in \mathbb{R}^n \):

\[ x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \]

where \( x = (x_1, \ldots, x_n) \).

Also

\[ \partial^{\alpha} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \]

and

\[ D^{\alpha} := \frac{1}{i^{|\alpha|}} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}. \]

(WARNING: Our use of the symbols “\( D \)” and “\( D^{\alpha} \)” differs from that in the PDE textbook [Ev98].)

If \( \varphi : \mathbb{R}^n \to \mathbb{R} \), then we write

\[ \partial \varphi := (\varphi_{x_1}, \ldots, \varphi_{x_n}) = \text{gradient}, \]
and
\[ \partial^2 \varphi := \begin{pmatrix} \varphi_{x_1x_1} & \cdots & \varphi_{x_1x_n} \\ \vdots & \ddots & \vdots \\ \varphi_{x_nx_1} & \cdots & \varphi_{x_nx_n} \end{pmatrix} = \text{Hessian matrix} \]

Also
\[ D\varphi := \frac{1}{i} \partial \varphi. \]

If \( \varphi \) depends on both the variables \( x, y \in \mathbb{R}^n \), we put
\[ \partial^2_x \varphi := \begin{pmatrix} \varphi_{x_1x_1} & \cdots & \varphi_{x_1x_n} \\ \vdots & \ddots & \vdots \\ \varphi_{x_nx_1} & \cdots & \varphi_{x_nx_n} \end{pmatrix} \]
and
\[ \partial^2_{x,y} \varphi := \begin{pmatrix} \varphi_{x_1y_1} & \cdots & \varphi_{x_1y_n} \\ \vdots & \ddots & \vdots \\ \varphi_{x_ny_1} & \cdots & \varphi_{x_ny_n} \end{pmatrix}. \]

- **Jacobians:** Let \( x \mapsto y = y(x) \) be a diffeomorphism, \( y = (y^1, \ldots, y^n) \). The Jacobian matrix is
\[ \partial y = \partial_x y := \begin{pmatrix} \frac{\partial y^1}{\partial x_1} & \cdots & \frac{\partial y^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x_1} & \cdots & \frac{\partial y^n}{\partial x_n} \end{pmatrix}_{n \times n}. \]

The absolute value of the determinant, \( |\det \partial y| \), which is the Jacobian factor in integration is denoted \( |\partial y| \).

- **Differentiation of determinants:** suppose \( t \mapsto A(t) \) is a function from \( \mathbb{R} \) to invertible \( N \times N \) matrices:
\[ A : \mathbb{R} \rightarrow GL(N, \mathbb{R}), \]

Then
\[ \frac{d}{dt} \det A(t) = \text{tr} \left( A(t)^{-1} \frac{dA(t)}{dt} \right) \det A(t), \]
and consequently
\[ \frac{d}{dt} |\det A(t)|^\alpha = \alpha \text{ tr} \left( A(t)^{-1} \frac{dA(t)}{dt} \right) |\det A(t)|^\alpha. \]
• Poisson bracket: If $f, g : \mathbb{R}^n \to \mathbb{R}$ are $C^1$ functions,

$$\{f, g\} := \langle \partial_\xi f, \partial_x g \rangle - \langle \partial_x f, \partial_\xi g \rangle = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j}.$$ 

### A.3. ELEMENTARY OPERATORS

- **Multiplication operator:** $M_\lambda f(x) = \lambda f(x)$
- **Translation operator:** $T_\xi f(x) = f(x - \xi)$
- **Reflection operator:** $Rf(x) := f(-x)$

### A.4. OPERATORS

- $A^*$ = adjoint of the operator $A$
- $[A, B] = AB - BA =$ commutator of $A$ and $B$
- $\sigma(A) =$ symbol of the pseudodifferential operator $A$
- $\text{spec}(A) =$ spectrum of $A$.
- $\text{tr}(A) =$ trace of $A$.

We say that a bounded operator $B$ is of **trace class** if

\[(A.4.1) \quad \|B\|_{\text{tr}} := \sum \sqrt{\lambda_j} < \infty,\]

where the $\lambda_j \geq 0$ are the eigenvalues of the self-adjoint operator $B^*B$.

- If $A : X \to Y$ is a bounded linear operator, we define the operator norm

$$\|A\| := \sup\{\|Au\|_Y \mid \|u\|_X \leq 1\}.$$ 

We will often write this norm as

$$\|A\|_{X \to Y}$$

when we want to emphasize the spaces between which $A$ maps.

The space of bounded linear operators from $X$ to $Y$ is denoted $L(X, Y)$; and the space of bounded linear operators from $X$ to itself is denoted $L(X)$. 
A.5. ESTIMATES

- We write
  \[ f = O(h^\infty) \quad \text{as } h \to 0 \]
  if for each positive integer \( N \) there exists a constant \( C_N \) such that
  \[ |f| \leq C_N h^N \quad \text{for all } 0 < h \leq 1. \]

- If we want to specify boundedness in the space \( X \), we write
  \[ f = O_X(h^N) \]
  to mean
  \[ \|f\|_X = O(h^N). \]

- If \( A \) is a bounded linear operator between the spaces \( X,Y \), we will often write
  \[ A = O_{X \to Y}(h^N) \]
  to mean
  \[ \|A\|_{X \to Y} = O(h^N). \]

- Grönwall’s inequality states that if \( B(s) \geq 0 \) and
  \[
  F(s) \leq A(s) + \int_0^t B(\sigma) F(\sigma) d\sigma, \quad 0 \leq s \leq t, \]
  then
  \[ F(t) \leq A(t) + \int_0^t B(s) A(s) e^{\int_s^t B(\sigma) d\sigma} ds. \]  \[ (A.5.1) \]

- Young’s Inequality: suppose that
  \[
  \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p,q,r \leq \infty. \]
  Then for \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n), \)
  \[ \|f * g\|_L^r \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \|h\|_{L^m} := \left( \int_{\mathbb{R}^n} |h(x)|^m dx \right)^{\frac{1}{m}} \]  \[ (A.5.2) \]

- Schur’s Estimate:
  If
  \[ Ku(x) = \int_Y K(x,y)u(y)dy, \quad x \in X \]
  then
  \[ \|K\|_{L^2 \to L^2}^2 \leq \sup_{x \in X} \int |K(x,y)| dy \times \sup_{y \in Y} \int |K(x,y)| dx. \]  \[ (A.5.3) \]
when the right hand side is finite.

- Interpolation estimate:
  For $m \leq \ell \leq p$, there exists $C$ such that

\[
(A.5.4) \quad \sup_{|\alpha| = \ell} \|\partial^\alpha f\|_{L^\infty} \leq C \left( \sup_{|\alpha| = m} \|\partial^\alpha f\|_{L^\infty} \right)^{\frac{p-\ell}{p-m}} \left( \sup_{|\alpha| = M} \|f\|_{L^\infty} \right)^{\frac{\ell-m}{p-m}}.
\]

### A.6. TEMPERED DISTRIBUTIONS

- The Schwartz space is
  \[
  \mathcal{S} = \mathcal{S}(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) \mid \sup_{\mathbb{R}^n} |x^\alpha \partial^\beta \varphi| < \infty \text{ for all multiindices } \alpha, \beta \right\}.
  \]

  We say
  \[
  \varphi_j \to \varphi \quad \text{in } \mathcal{S}
  \]
  provided
  \[
  \sup_{\mathbb{R}^n} |x^\alpha D^\beta (\varphi_j - \varphi)| \to 0
  \]
  for all multiindices $\alpha, \beta$.

  We write $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ for the space of tempered distributions, which is the dual of $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. That is, $u \in \mathcal{S}'$ provided $u : \mathcal{S} \to \mathbb{C}$ is linear and $\varphi_j \to \varphi$ in $\mathcal{S}$ implies $u(\varphi_j) \to u(\varphi)$.

  Sometimes the distributional pairing is denoted by

\[
(A.6.1) \quad u(\varphi) = \langle u, \varphi \rangle = \langle u(x), \varphi(x) \rangle.
\]

We say

\[
U_j \to U \quad \text{in } \mathcal{S}'
\]

provided

\[
U_j(\varphi) \to U(\varphi) \quad \text{for all } \varphi \in \mathcal{S}.
\]
A.7. DISTRIBUTIONS ON MANIFOLDS AND SCHWARTZ KERNELS

• Smooth functions:

Let $M$ be a manifold. We consider the space $C^\infty(M)$ of smooth functions and $C^\infty_0(M)$ of smooth compactly supported functions. The topology on $C^\infty$ is given by seminorms $\sup_K |\partial^\alpha \varphi|$, $\alpha \in \mathbb{N}^n$ and $K \subset M$. In other words

$$C^\infty(M) \ni \varphi_j \to 0 \iff \forall \alpha \in \mathbb{N}^n \forall K \subset M, \max_K |\partial^\alpha \varphi_j| \to 0.$$  

The topology on $C^\infty_c(M)$ is determined by demanding that

$$C^\infty_c(M) \ni \varphi_j \to 0 \iff \exists K \subset M, J \forall \alpha \in \mathbb{N}^n, j > J \text{ supp } \varphi_j \subset K, \max_K |\partial^\alpha \varphi_j| \to 0.$$  

• Distributions:

Denote by $D'(M)$ the space of distributions on $M$, that is the dual space to $C^\infty_0(M)$, and by $E'(M)$ the space of compactly supported distributions.

We use the notation (A.6.1) for the distributional pairing. When there is possibility of confusion with inner products on a Hilbert space $H$, we denote that inner product by $\langle u, v \rangle_H$.

• Schwartz kernels:

Let $M_1, M_2$ be two manifolds and fix some smooth density $dy$ on $M_2$. To each sequentially continuous operator

$$A : C^\infty_0(M_2) \to D'(M_1)$$

is associated its Schwartz kernel

$$(A.7.1) \quad K_A \in D'(M_1 \times M_2), \quad Af(x) = \int_{M_2} K_A(x, y)f(y) \, dy.$$  

Formally speaking, (A.7.1) means that

$$\langle Af, g \rangle = \langle K_A(x, y), g(x) \otimes f(y) \rangle$$

for each $f \in C^\infty_0(M_2), g \in C^\infty_0(M_1)$.

• Smoothing operators:

We say that $A$ is smoothing if it is sequentially continuous $E'(M_2) \to C^\infty(M_1)$. This is equivalent to

$$K_A \in C^\infty(M_1 \times M_2)$$

• Regular operators:
We say that $A$ is regular if it is sequentially continuous $C_0^\infty(M_2) \to C^\infty(M_1)$ and the adjoint $A^*$ is sequentially continuous $C_0^\infty(M_1) \to C^\infty(M_2)$; note that such $A$ maps $\mathcal{E}'(M_2) \to \mathcal{D}'(M_1)$.

- Compact supported operator:
  We say that $A$ is compactly supported if $K_A$ is compactly supported, that is $A$ maps $C^\infty(M_2) \to \mathcal{E}'(M_1)$.

- Properly supported operator:
  We say that $A$ is properly supported if for each $\chi_1 \in C_0^\infty(M_1)$, there exists $\chi_2 \in C_0^\infty(M_2)$ such that
  \[ \chi_1 A = \chi_1 A \chi_2 \]
  and same property holds for $A^*$; that is, $A$ maps $C^\infty(M_2) \to \mathcal{D}'(M_1)$ and $C_0^\infty(M_2) \to \mathcal{E}'(M_1)$. This condition can be formulated in terms of the support of $K_A$.

  If $A = A(h)$ is a family of operators depending on some parameter $h$, then the support properties is understood in the sense independent of $h$.

  If $A : C_0^\infty(M_2) \to \mathcal{D}'(M_1), B : C_0^\infty(M_3) \to \mathcal{D}'(M_2)$ are two operators and at least one of them is properly supported and regular, then the product $AB : C_0^\infty(M_3) \to \mathcal{D}'(M_1)$ is well-defined. In particular, regular properly supported operators on some manifold $M$ form an algebra.

- Locally finite collections of sets
  We say that a collection of open subsets $\{U_j \subset M\}$ is locally finite if for each compact set $K \subset M$, we have $U_j \cap K = \emptyset$ for all but finitely many indices $j$. We only work with paracompact manifolds, which implies that any locally finite collection is at most countable.
SPECTRAL THEORY

B.1. SPECTRAL THEORY OF SELF-ADJOINT OPERATORS

B.1.1. Bounded operators. Let $H$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. For a bounded operator, $A : H \to H$, we define the adjoint $A^* : H \to H$ using the inner product:

$\langle Au, v \rangle = \langle u, A^* v \rangle$.

An operator $A$ is self-adjoint if $A^* = A$.

THEOREM B.1 (Spectral theorem for bounded operators). Let $A$ be a bounded self-adjoint operator on $H$. Then there exist a measure space $(X, \mathcal{M}, \mu)$, a real-valued function $f \in L^\infty(X, \mu)$ and a unitary operator $U : H \to L^2(X, \mu)$ such that

$U^* M_f U = A$,

where $M_f$ is the multiplication operator:

$[M_f u](x) = f(x) u(x), \quad u \in L^2(X, \mu)$.

The same theorem applies to normal operators, that is, operators satisfying

$[A, A^*] = AA^* - A^* A = 0$.

In that case $f$ can be complex valued but otherwise the statement is the same.
DEFINITION. Suppose that $A$ is a bounded operator on $H$. Then the spectrum of $A$, $\text{Spec}(A) \subset \mathbb{C}$, is defined by

$$\text{Spec}(A) = \{ \lambda \in \mathbb{C} : (A - \lambda)^{-1} : H \to H \text{ exists}\}.$$ 

We say that $\lambda \in \text{Spec}(A)$ is an eigenvalue of $A$, if there exists $u \in H$ such that

$$Au = \lambda u.$$ 

Theorem B.1 implies that for a self-adjoint bounded operator $A$,

$$\text{Spec}(A) = \overline{\text{image}(f)} \subseteq \mathbb{R}.$$ 

The following important result concerns spectrum of compact operators: $A : H \to H$ is called compact if the image of $\{u : \|u\| \leq 1\}$ under $A$ is a pre-compact subset of $H$.

THEOREM B.2 (Spectra of compact operators). Suppose $A$ is a compact operator on $H$. Then

(i) Every $\lambda \in \text{Spec}(A) \setminus \{0\}$ is an eigenvalue of $A$.

(ii) For all nonzero $\lambda \in \text{Spec}(A) \setminus \{0\}$, there exist $N$ such that

$$\ker(A - \lambda)^N = \ker(A - \lambda)^{N+1}.$$

(iii) The eigenvalues can only accumulate at $0$.

(iv) $\text{Spec}(A)$ is countable.

(v) Every $\lambda \in \text{Spec}(A) \setminus \{0\}$ is a pole of the resolvent operator

$$\lambda \mapsto (A - \lambda)^{-1}.$$ 

(vi) Suppose in addition that $A$ is self-adjoint. Then there exists an orthonormal set $\{u_k\}_{k \in K} \subset H$, $K = \{0, 1, 2, \cdots, N\}$ or $K = \mathbb{N}$, such that

$$(B.1.2) \quad Au(x) = \sum_{k \in K} \lambda_k u_k(x) \langle u, u_k \rangle,$$

where $\lambda_0 \geq \lambda_1 \geq \cdots$ are the non-zero eigenvalues of $A$.

(vii) Conversely, if (B.1.2) holds with $\lambda_j \to 0$ then $A$ is compact.

One of the most frequently encountered classes of compact operators are inclusions between Hilbert spaces. Here is one which is used in this book:
THEOREM B.3 (Rellich-Kondrachov theorem for unbounded domains). Suppose that the Hilbert $H \subset L^2(\mathbb{R}^n)$ is defined by the norm

$$\|u\|_H^2 = \|\hat{\xi}^\alpha u\|_{L^2(\mathbb{R}^n)}^2 + \|a(x)^{-1}u\|_{L^2(\mathbb{R}^n)}^2,$$

where $\alpha > 0$, $a(x) > 0$, $\lim_{|x| \to \infty} a(x) = 0$, $\hat{\xi} u$ is the Fourier transform of $u$ and $a$ is continuous.

Then the inclusion

$$H \hookrightarrow L^2$$

is compact.


(i) Then $(A - \lambda)^{-1}$ exists and is a bounded linear operator on $H$ for $\lambda \in \mathbb{C} - \text{spec}(A)$, where $\text{spec}(A) \subset \mathbb{R}$ is the spectrum of $A$.

(ii) If $\text{spec}(A) \subset [a, \infty)$, then

$$(B.1.3) \quad \langle Au, u \rangle \geq a\|u\|^2 (u \in A).$$

B.1.2. Unbounded operators. We next review the more complicated theory for unbounded operators.

DEFINITIONS.

(i) An unbounded operator $A : H \to H$ is given by a subspace $\mathcal{D}(A) \subset H$ and a linear operator $A : \mathcal{D}(A) \to H$. We call $\mathcal{D}(A)$ the domain of $A$, and say that $A$ is densely defined if $\mathcal{D}(A)$ is dense in $A$.

(ii) The graph of $A$ is

$$\text{graph}(A) := \{(u, Au) \mid u \in \mathcal{D}(A)\} \subset H \times H.$$

(iii) If $A, B$ are unbounded operators on $H$, we say that $A \subseteq B$ if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and $Au = Bu$ for all $u \in \mathcal{D}(A)$.

(iv) The operator $A$ is closed if $\text{graph}(A)$ is a closed subspace of $H \times H$ equipped with the norm $\|(u, v)\|^2 = \|u\|^2 + \|v\|^2$.

(v) An unbounded operator $A$ is closable if there exists a closed unbounded operator $\bar{A}$ such that $A \subseteq \bar{A}$. The operator $\bar{A}$ is unique and is called the closure of $A$. 
THEOREM B.5 (Adjoint operator). Suppose $A : H \to H$ is an unbounded, densely defined operator. Then there exists an unbounded operator $A^* : H \to H$ defined by the rule

$$\langle A^*v, u \rangle := \langle v, Au \rangle$$

for all $v \in \mathcal{D}(A^*), u \in \mathcal{D}(A)$, where

$$\mathcal{D}(A^*) := \{v \in H \mid |\langle Au, v \rangle| \leq C(v)\|u\| \text{ for all } u \in \mathcal{D}(A)\}.$$

Here $C(v)$ is a constant depending on $v$.

The unbounded operator $A^*$ is always closed. If $A^*$ is densely defined, then $A$ is closable and $\bar{A} = (A^*)^*, \bar{A}^* = A^*$.

DEFINITIONS.

(i) An unbounded densely defined operator $A$ is called symmetric if

$$A \subseteq A^*.$$ (B.1.6)

Equivalently, $\langle Au, v \rangle = \langle u, Av \rangle$ for all $u, v \in \mathcal{D}(A)$.

(ii) An unbounded densely defined operator $A$ is called self-adjoint if

$$A = A^*.$$ (B.1.7)

(iii) A symmetric operator is called essentially self-adjoint if

$$\bar{A} = A^*.$$ (B.1.8)

THEOREM B.6 (Spectral Theorem for unbounded operators). Let $A$ be an unbounded self-adjoint operator on $H$. Then there exist a measure space $(X, \mathcal{M}, \mu)$, a real-valued measurable function $f$ and a unitary operator $U : H \to L^2(X, \mu)$ such that

$$x \in \mathcal{D}(A) \text{ if and only if } M_f(Ux) \in L^2(X, \mu)$$

and

$$x \in \mathcal{D}(A) \implies U(Ax) = M_f(Ux).$$

Here $M_f : x \mapsto fx$ denotes the unbounded multiplication operator on $X$.

As an immediate consequence we obtain this useful result valid for both bounded and unbounded operators:
THEOREM B.7 (Distance to spectrum).

(i) If $A$ is a self-adjoint operator, then
$$\text{Spec}(A) = \text{ess-image}(f) \subset \mathbb{R},$$
where $f$ is given in the Spectral Theorems B.6, B.1 and
$$\text{ess-image}(f) := \{ t \mid \mu(f^{-1}((t - \epsilon, t + \epsilon)) > 0 \text{ for all } \epsilon > 0 \}.$$

(ii) Furthermore, if $\lambda \in \mathbb{C} \setminus \text{Spec}(A)$, then
$$\| (A - \lambda)^{-1} \| = \frac{1}{\text{dist}(\lambda, \text{Spec}(A))}. \tag{B.1.11}$$

The spectral projector is given by boundary values of the resolvent in the following way:

THEOREM B.8 (Stone’s formula). Let $E = E(P)$ be the spectral measure of a self-adjoint operator $P$. For $a < b$
$$\frac{1}{2} (E((a, b)) + E([a, b])) = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_a^b \left( (P - t - i\epsilon)^{-1} - (P - t + i\epsilon)^{-1} \right) dt \tag{B.1.12}$$

If there exists a dense subspace $V$ such that for $f \in V$ the limit
$$\lim_{\epsilon \to 0^+} \langle (P - t - i\epsilon)^{-1} f, f \rangle =: \langle (P - t - i0)^{-1} f, f \rangle, \quad a < t < b$$
exists, then $\text{Spec}(P) \cap (a, b)$ is absolutely continuous and on $(a, b)$
$$dE_t(P) = \frac{1}{2\pi i} \left( (P - t - i0)^{-1} - (P - t + i0)^{-1} \right) dt. \tag{B.1.13}$$

REMARK. An informal but instructive way of writing (B.1.13) is
$$\delta(P - \lambda) = \frac{1}{2\pi i} \left( (P - \lambda - i0)^{-1} - (P - \lambda + i0)^{-1} \right) \tag{B.1.14}$$

There are many criteria determining if an operator is essentially self-adjoint and there are many subtleties in the subject. Here we only need the simplest one:

THEOREM B.9 (Criteria for essential self-adjointness). Suppose that $A : H \to H$ is symmetric. Then the following conditions are equivalent:

(i) $A$ is essentially self-adjoint.

(ii) For both signs, $(A^* \pm i)x = 0$, $x \in \mathcal{D}(A^*)$, implies $x = 0$.

(iii) For both signs, $\{(A \pm i)x \mid x \in \mathcal{D}(A)\}$ is dense in $H$. 
The Schrödinger propagators of self-adjoint operators are important for quantum dynamics:

**THEOREM B.10 (Stone’s Theorem).** Suppose that \( P : \mathcal{D}(A) \subseteq H \rightarrow H \) is a (possibly unbounded) self-adjoint operator.

(i) Then

\[
U(t) := e^{-itP} \quad (t \in \mathbb{R})
\]

defines a strongly continuous unitary group:

\[
\begin{cases}
U(t)U(s) = U(t+s), & U(t)^* = U(-t), \\
\lim_{t \to 0} \|U(t)u - u\|_H = 0 & (u \in H).
\end{cases}
\]

In addition,

\[
D_t(U(t)u) + U(t)Pu = 0 \quad (t \in \mathbb{R})
\]

for all \( u \in \mathcal{D}(P) \).

(ii) Conversely, if \( U(t) \) satisfies (B.1.16), then there exists a self-adjoint operator \( P \) such that (B.1.15) and (B.1.17) hold.

**THEOREM B.11 (Maximin and minimax principles).** Suppose that \( A : H \rightarrow H \) is self-adjoint and semibounded, meaning \( A \geq -c_0 \). Assume also that \( (A + 2c_0)^{-1} : H \rightarrow H \) is a compact operator.

Then the spectrum of \( A \) is discrete: \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \); and furthermore

(i)

\[
\lambda_j = \max_{V \subset H} \min_{\text{codim}V < j, v \neq 0} \frac{\langle Av, v \rangle}{\|v\|^2},
\]

(ii)

\[
\lambda_j = \min_{V \subset H} \max_{\text{dim}V \leq j, v \neq 0} \frac{\langle Av, v \rangle}{\|v\|^2}.
\]

In these formulas, \( V \) denotes a linear subspace of \( H \).

**DEFINITIONS.** (i) Let \( Q : H \rightarrow H \) be a bounded linear operator. We define the rank of \( Q \) to be the dimension of the range \( Q(H) \).

(ii) If \( A \) is an operator with real and discrete spectrum, we set

\[
N(\lambda) := \#\{\lambda_j \mid \lambda_j \leq \lambda\}
\]
to count the number of eigenvalues less than or equal to $\lambda$.

**THEOREM B.12 (Estimating $N(\lambda)$).** Let $A$ satisfy the assumptions of Theorem B.11.

(i) If

\[ (B.1.20) \begin{cases} 
\text{there exist } \delta > 0 \text{ and a self-adjoint operator } Q, \\
\text{with rank } Q \leq k, \text{ such that} \\
\langle Au, u \rangle \geq (\lambda + \delta)\|u\|^2 - \langle Qu, u \rangle \text{ for } u \in H, 
\end{cases} \]

then

$N(\lambda) \leq k$.

(ii) If

\[ (B.1.21) \begin{cases} 
\text{for each } \delta > 0, \text{ there exists a subspace } V \\
\text{with } \dim V \geq k, \text{ such that} \\
\langle Au, u \rangle \leq (\lambda + \delta)\|u\|^2 \text{ for } u \in V, 
\end{cases} \]

then

$N(\lambda) \geq k$.

**Proof.** 1. Set $W$ be the orthogonal complement of $Q(H)$, $W := Q(H)^\perp$. Thus codim $W = \text{rank } Q \leq k$. Therefore the maximin formula (B.1.18) implies

\[
\lambda_{k+1} = \max_{\text{codim } V < k} \min_{v \in V} \frac{\langle Av, v \rangle}{\|v\|^2} \geq \min_{v \in W} \min_{v \neq 0} \frac{\langle Av, v \rangle}{\|v\|^2} \\
= \min_{v \in W} \left( \lambda + \delta - \frac{\langle Qv, v \rangle}{\|v\|^2} \right) = \lambda + \delta,
\]

since $\langle Qv, v \rangle = 0$ if $v \in Q(H)^\perp$. Hence $\lambda < \lambda + \delta \leq \lambda_{k+1}$, and so

$N(\lambda) = \max \{ j \mid \lambda_j \leq \lambda \} \leq k$.

This proves assertion (i).

2. The minimax formula (B.1.19) directly implies that

\[
\lambda_k \leq \max_{v \in V} \min_{v \neq 0} \frac{\langle Av, v \rangle}{\|v\|^2} \leq \lambda + \delta.
\]

Hence $\lambda_k \leq \lambda + \delta$. This is valid for all $\delta > 0$, and so

$N(\lambda) = \max \{ j \mid \lambda_j \leq \lambda \} \geq k$.

This is assertion (ii).
B. SINGULAR VALUES

B.2. FUNCTIONAL CALCULUS

Almost analytic extensions
Helffer–Sjöstrand formula

B.3. SINGULAR VALUES

Let $H, H_1, H_2$ be separable Hilbert spaces and $B(H_1; H_2)$ denotes the space of bounded linear operators $H_1 \to H_2$.

For a compact self-adjoint operator $A : H \to H$, the Hilbert–Schmidt theorem says that there exists a complete orthonormal basis of eigenvectors $e_0, e_1, \ldots$ of $A$ with eigenvalues $\lambda_j(A) \to 0$. We can write

$$A = \sum_{j=0}^{\infty} \lambda_j(A)(e_j \otimes e_j),$$

where for $e \in H_1, f \in H_2$, we define their tensor product $e \otimes f \in B(H_1; H_2)$ by

$$(e \otimes f)u := \langle u, e \rangle f, \quad u \in H_1.$$ 

If $A$ is nonnegative, then we order $\lambda_j(A)$ so that $\lambda_j(A) \geq \lambda_{j+1}(A)$, and note that $\|A\| = \lambda_0(A)$.

For the case of a general (not necessarily self-adjoint) operator, the following decomposition holds:

**Proposition B.13.** Assume that $A : H_1 \to H_2$ is a compact operator. Then we can write

$$A = \sum_{j=0}^{\infty} s_j(e_j \otimes f_j),$$

where $s_0 \geq s_1 \geq \ldots$ is a sequence converging to 0 and $\{e_j \mid s_j \neq 0\} \subset H_1$ and $\{f_j \mid s_j \neq 0\} \subset H_2$ are orthonormal systems.

Moreover, the numbers $s_j = s_j(A)$, called the singular values of $A$, do not depend on the choice of the decomposition (B.3.3), and in fact $s_j(A) = \lambda_j(A^*A)^{1/2}$.

**Proof.** We first show the existence of the decomposition (B.3.3). The operator $A^*A : H_1 \to H_1$ is compact and self-adjoint, and moreover nonnegative; let $e_j$ be a Hilbert basis of $H_1$ composed of eigenvectors of $A^*A$ with eigenvalues $s_j(A)^2 = \lambda_j(A^*A)$. Put $f_j := s_j^{-1}Ae_j$ for $s_j \neq 0$ and $f_j = 0$ for
s_j = 0, so that (B.3.3) holds. It remains to note that for s_j \neq 0, s_k \neq 0, we have

\langle f_j, f_k \rangle = \frac{\langle Ae_j, Ae_k \rangle}{s_j s_k} = \frac{\langle A^*Ae_j, e_k \rangle}{s_j s_k} = \delta_{jk}.

Now, if A admits the decomposition (B.3.3) for some s_j, e_j, f_j, then we have

A^*A = \sum_{j=0}^{\infty} s_j^2 e_j \otimes e_j,

and it follows immediately that s_j^2 = \lambda_j(A^*A). \qed

We note that

\sup_{\|u\| \leq 1} \frac{\|Au\|^2}{\|u\|^2} = \sup_{\|u\| \leq 1} \frac{\langle A^*Au, u \rangle}{\|u\|^2} = \lambda_0(A^*A),

in other words,

(B.3.4) \quad \|A\|_{H_1 \to H_2} = s_0(A).

We also have s_j(A) = s_j(A^*) since the form of the decomposition (B.3.3) persists under taking adjoints.

The singular values can also be characterized as follows:

PROPOSITION B.14. We have for each n,

s_n(A) = \min\{\|A - K\|_{H_1 \to H_2} \mid K \in B(H_1; H_2), \text{ rank } K \leq n\}.

Moreover, the minimum is achieved by an operator K such that s_j(K) = s_j(A) for 0 \leq j < n and s_{j-n}(A - K) = s_j(A) for j \geq n.

Proof. First, assume that K \in B(H_1; H_2) and rank K \leq n; we will show that \|A - K\| \geq s_n(A). We take the decomposition (B.3.3) of A. Since e_0, \ldots, e_n are linearly independent, there exists a nontrivial linear combination u of these vectors such that K(u) = 0. We then have

\|A - K\| = \|Au\|^2 = \langle A^*Au, u \rangle \geq \lambda_0(A^*A),

and thus \|A - K\| \geq s_n(A) as needed.

Now, using the decomposition (B.3.3), for

K := \sum_{j=0}^{n-1} s_j(e_j \otimes f_j),

we have s_j(K) = s_j(A) for 0 \leq j < n, s_{j-n}(A - K) = s_j(A) for j \geq n, and in particular \|A - K\| = s_0(A - K) = s_n(A). \qed

Proposition B.14 lets us prove the following inequalities:
PROPOSITION B.15. For compact operators $A, B$,

\begin{align}
(B.3.5) & \quad s_{j+k}(A + B) \leq s_j(A) + s_k(B), \\
(B.3.6) & \quad s_{j+k}(AB) \leq s_j(A)s_k(B).
\end{align}

Same is true if $A$ is compact and $B$ is bounded then

\begin{equation}
(B.3.7) \quad s_j(AB), s_j(BA) \leq \|B\|s_j(A).
\end{equation}

Proof. Using Proposition B.14, we write $A = K_A + R_A$, $B = K_B + R_B$, where rank $K_A \leq j$, rank $K_B \leq k$, $\|R_A\| \leq s_j(A)$, $\|R_B\| \leq s_k(B)$. Then

\begin{align*}
A + B &= (K_A + K_B) + (R_A + R_B), \\
AB &= (K_AB + R_AR_B).
\end{align*}

and rank$(K_A + K_B)$, rank$(K_AB + R_AR_B) \leq j + k$, $\|R_A + R_B\| \leq s_j(A) + s_k(B)$, $\|R_AR_B\| \leq s_j(A)s_k(B)$, so it remains to apply Proposition B.14 one more time. □

The singular values of an operator are continuous in the norm topology:

PROPOSITION B.16. For compact operators $A, B : H_1 \to H_2$,

\begin{equation}
|s_j(A) - s_j(B)| \leq \|A - B\|.
\end{equation}

Proof. We have $s_j(A) \leq s_j(B) + \|A - B\|$ and $s_j(B) \leq s_j(A) + \|A - B\|$ by Proposition B.15, with $k = 0$. □

EXAMPLE. Suppose that $(M, g)$ is compact manifold $n$ dimensional Riemannian manifold and that $-\Delta_M$ is the Laplace-Beltrami operator on $M$. Then the Weyl law for eigenvalue asymptotics states that

\begin{align*}
|\{\lambda \geq 0 : \lambda^2 \in \text{Spec}(-\Delta_M), |\lambda| \leq r\}| &= c_n \text{vol}_g(M)r^n(1 + o(1)), \\
c_n &= \text{vol}(B_{R^n}(0, 1))/(2\pi)^n.
\end{align*}

If we order the eigenvalues of $-\Delta_M$ as $0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$, it then follows that

\begin{equation}
(B.3.8) \quad \lambda_j \geq (c_n \text{vol}_g(M))^{-\frac{1}{2}} j^{\frac{1}{2}} (1 - o(1)), \quad j \to \infty,
\end{equation}

and

\begin{equation}
(B.3.9) \quad s_j((-\Delta_M - 1)^{-s/2}) \leq C_M j^{-\frac{s}{2}}.
\end{equation}

Suppose now that $A : L^2(M) \to H^s(M)$, $s \in \mathbb{N}$. Then

\begin{align*}
&\quad s_j(A) \leq s_j((-\Delta_M - 1)^{-s/2})\|(-\Delta_M - 1)^{s/2}A\|_{L^2 \to L^2} \\
&\leq s_j((-\Delta_M - 1)^{-s/2})\|A\|_{L^2 \to H^2} \\
&\leq C_{AJ} j^{-\frac{s}{2}}.
\end{align*}
Hence, if $s > n$ then $A \in \mathcal{L}_1(L^2(M))$.

**B.4. THE TRACE CLASS**

We now discuss operators of trace class. First of all, if $A : H \to H$ is bounded and has finite rank, then the trace $\text{tr} A \in \mathbb{C}$ can be defined as the trace of the restriction of $A$ to any finite-dimensional subspace $V \subset H$ containing the range of $A$. In fact, if

$$A = \sum_{j=0}^{N-1} a_j (u_j \otimes v_j),$$

where $a_j \in \mathbb{C}$, $u_j, v_j \in H$, then

$$(B.4.1) \quad \text{tr} A = \sum_{j=0}^{N-1} a_j \langle v_j, u_j \rangle.$$  

This gives a linear functional on the space of all finite-dimensional operators, and moreover, as seen directly from (B.4.1),

$$\text{tr}(AB) = \text{tr}(BA), \quad A : H_1 \to H_2, \ B : H_2 \to H_1, \ \text{rank} A < \infty.$$  

To extend the notion of trace to infinite rank operators, we give the following

**DEFINITION.** Let $A : H_1 \to H_2$ be a compact operator. We say that $A$ is of trace class, and write $A \in \mathcal{L}_1(H_1; H_2)$, if the **trace norm**

$$(B.4.2) \quad \|A\|_1 := \sum_{j=0}^{\infty} s_j(A)$$

is finite.

Proposition B.13 easily gives the following alternative expression for the trace class norm:

$$(B.4.3) \quad \|A\|_1 = \max_{{\{e_k\}, \{f_\ell\}}} \sum_{{k, \ell}} (Ae_k, f_\ell),$$

where the maximum is taken over all pairs of orthonormal bases of $H_1$ and $H_2$.

Note that $\|A\|_{H_1 \to H_2} = s_0(A) \leq \|A\|_1$. By (B.4.1) applied to (B.3.3), we have

$$(B.4.4) \quad |\text{tr} A| \leq \|A\|_1.$$  

The partial sums in the definition of $\|A\|_1$ can be characterized as follows:
PROPOSITION B.17. Let $A : H_1 \to H_2$ be compact. Then for each $n$,
\[ \sum_{j=0}^{n-1} s_j(A) = \max\{ |\text{tr}(QA)| \mid \|Q\|_{H_2 \to H_1} \leq 1, \text{ rank } Q \leq n \}. \]

Proof. If $\|Q\|_{H_2 \to H_1} \leq 1$, then by (B.3.6), $s_j(QA) \leq s_j(A)$ for all $j$. On the other hand, since $\text{rank}(QA) \leq n$, we see that $s_j(QA) = 0$ for $j \geq n$. Therefore, by (B.4.4) we have
\[ |\text{tr}(QA)| \leq \sum_{j=0}^{n-1} s_j(QA) \leq \sum_{j=0}^{n-1} s_j(A). \]

Now, if we consider the decomposition (B.3.3) for $A$ and put
\[ Q := \sum_{j=0}^{n-1} f_j \otimes e_j, \]
then $\|Q\|_{H_2 \to H_1} \leq 1$, rank $Q \leq n$, and by (B.4.1),
\[ \text{tr}(QA) = \text{tr} \sum_{j=0}^{n-1} s_j(A)(e_j \otimes e_j) = \sum_{j=0}^{n-1} s_j(A). \]

To see that $\| \cdot \|_1$ is in fact a norm, it suffices to prove

PROPOSITION B.18. For $A, B : H_1 \to H_2$ compact operators, we have $\|A + B\|_1 \leq \|A\|_1 + \|B\|_1$.

Proof. It suffices to prove that for each $n$,
\[ \sum_{j=0}^{n-1} s_j(A + B) \leq \sum_{j=0}^{n-1} s_j(A) + \sum_{j=0}^{n-1} s_j(B). \]
This follows immediately from Proposition B.17, as for each $Q : H_2 \to H_1$ with $\|Q\|_{H_2 \to H_1} \leq 1$ and rank $Q \leq n$, we have $\text{tr}(QA + QB) = \text{tr}(QA) + \text{tr}(QB)$. □

The space $\mathcal{L}_1(H_1; H_2)$ equipped with the norm $\| \cdot \|_1$ is a Banach space, but we do not prove or use this fact here.

Finite rank operators are dense in $\mathcal{L}_1(H_1; H_2)$, since for each $A \in \mathcal{L}_1(H_1; H_2)$, using (B.3.3) we have
\[ (B.4.5) \quad \sum_{j=0}^{n-1} s_j(e_j \otimes f_j) \to A \quad \text{in } \mathcal{L}_1(H_1; H_2). \]

Another way of demonstrating this fact, which will be more convenient for families of operators later, is given by
**PROPOSITION B.19.** Assume that $A \in \mathcal{L}_1(H_1; H_2)$, $H_2$ is infinite dimensional, and $(u_j)_{j=1}^{\infty}$ is a Hilbert basis of $H_2$ respectively. Let $\Pi_N : H_2 \rightarrow H_2$ be the orthogonal projection onto the subspace spanned by $u_1, \ldots, u_N$. Then

$$\|A - \Pi_N A\|_1 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.$$ 

**Proof.** We know that $f(N) := \|(I - \Pi_N)A\|_{H_1 \rightarrow H_2}$ goes to 0 as $N \rightarrow \infty$, since $A$ is compact and $\Pi_N \rightarrow I$ in the strong operator topology. We have $s_j((I - \Pi_N)A) \leq f(N)$. Then by (B.3.6),

$$\sum_j s_j((I - \Pi_N)A) \leq \sum_j \min(s_j(A), f(N)) \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty$$

using the dominated convergence theorem. \hfill \Box

By (B.3.6) and the fact that $s_j(A^*) = s_j(A)$, we see that for $A \in \mathcal{L}_1$,

(B.4.6) \hspace{1cm} \|AB\|_1 \leq \|A\|_1 \|B\|_{H_1 \rightarrow H_2}, \quad \|A\|_1 = \|A^*\|_1.

In particular, $\mathcal{L}_1(H; H)$ is a two-sided ideal in the algebra of closed operators on $H$.

By (B.4.4), the trace functional on finite-dimensional operators extends uniquely to a bounded linear functional $\text{tr} : \mathcal{L}_1(H; H) \rightarrow \mathbb{C}$. Note that for each Hilbert basis $u_j$ of $H$, we have

(B.4.7) \hspace{1cm} \text{tr} A = \sum_j \langle Au_j, u_j \rangle, \quad A \in \mathcal{L}_1(H; H).

Indeed, this follows immediately from Proposition B.19, since $\text{tr}(\Pi_N A) = \sum_{j=1}^{N} \langle Au_j, u_j \rangle$.

By approximation by finite dimensional operators, we see that for each $A \in \mathcal{L}_1(H_1; H_2)$ and bounded $B : H_2 \rightarrow H_1$, we have

(B.4.8) \hspace{1cm} \text{tr}(AB) = \text{tr}(BA), \quad A \in \mathcal{L}_1(H_1; H_2), \ B \in B(H_2; H_1).

The fundamental example of trace class operators is given by the following proposition (see also the example at the end of §B.3).

**PROPOSITION B.20.** Let $X$ be a manifold of dimension $m$ and $A : H^s(X) \rightarrow H^{s'}(X)$ be bounded, where $s' > s + m$. Assume also that the Schwartz kernel of $A$ has compact support. Then $A \in \mathcal{L}_1(H^s(X); H^s(X))$.

**Proof.** Using coordinate charts and a partition of unity, we can reduce to the case when $X$ is the $m$-dimensional torus: $X = (\mathbb{R}/2\pi\mathbb{Z})^m$. By (B.4.6), it is enough to show that the inclusion operator $\iota : H^s(X) \rightarrow H^s(X)$ is
of trace class. An orthogonal basis of $H^s(X)$ and $H^{s'}(X)$ is given by $e^{ikx}$, where $k \in \mathbb{Z}^m$; we have $\iota(e^{ikx}) = e^{ikx}$ and
\[
\|e^{ikx}\|_{H^s} \leq C\langle k \rangle^{s-s'}\|e^{ikx}\|_{H^{s'}}.
\]
This gives a decomposition (B.3.3) of $\iota$, and we see that the singular value corresponding to $e^{ikx}$ is bounded by $C\langle k \rangle^{s-s'}$. It remains to note that
\[
\sum_{k \in \mathbb{Z}^m} \langle k \rangle^{s-s'} < \infty \quad \text{for } s' > s + m. \quad \square
\]

To compute the trace of an operator on $L^2$, the following formula is particularly useful:

**Proposition B.21.** Let $X$ be a manifold with a fixed volume form $d\text{Vol}$ (so that Schwartz kernels can be regarded as functions) and $A : \mathcal{D}'(X) \rightarrow C_c^\infty(X) \in C_c^\infty(X \times X)$. Then $A : L^2(X) \rightarrow L^2(X)$ is of trace class and
\[
\text{tr} \, A = \int K_A(x,x) \, d\text{Vol}(x). \quad (B.4.9)
\]

*Proof.* We start with the case when $A$ has the form
\[
A = \sum_{j=1}^N a_j(u_j \otimes v_j) \quad (B.4.10)
\]
where $u_j, v_j \in C_c^\infty(X)$ and $u_j \otimes v_j$ is defined by (B.3.2). Note that
\[
K_A(x,y) = \sum_{j=1}^N a_j v_j(x) \overline{u_j(y)}.
\]
Then by (B.4.1),
\[
\text{tr} \, A = \sum_{j=1}^N a_j \langle v_j, u_j \rangle = \sum_{j=1}^N a_j \int_X v_j(x) \overline{u_j(x)} \, d\text{Vol}(x) = \int_X K_A(x,x) \, d\text{Vol}(x)
\]
which proves (B.4.9).

For general $A$, using coordinate charts and a partition of unity, we reduce to the case when $X = (\mathbb{R}/2\pi\mathbb{Z})^m$. We write the Fourier series of $K_A$,
\[
K_A(x,y) = \sum_{\ell, r \in \mathbb{Z}^m} a_{\ell r} e^{i\ell x + i\ell r}
\]
and the series converges in $C^\infty$. If $e_\ell(x) = e^{i\ell x}$, then we can write
\[
A = \sum_{\ell, r \in \mathbb{Z}^m} a_{\ell r} e_{-r} \otimes e_\ell,
\]
and since the coefficients $a_{\ell r}$ are rapidly decreasing, the series converges in the trace class norm $\| \bullet \|_1$. Since the partial sum of the series has the form (B.4.10), the proof is finished by the continuity of the trace with respect to the trace class norm. □

**EXAMPLE.** Suppose that $(M, dx)$, and $(N, d\omega)$ are measure spaces and that $e_j(x, \omega) \in L^2(M \times N)$. Then

$$ Ku(x) := \int_M \int_N e_1(x, \omega) \overline{e_2(y, \omega)} u(y) d\omega dy, $$

defines

$$ K \in \mathcal{L}_1(M), \quad \text{tr} K = \int_{M \times N} e_1(x, \omega) \overline{e_2(x, \omega)} dxd\omega. $$

In fact, $K = E_1 E_2^*$ where $E_j : L^2(N) \to L^2(M)$ are defined by $E_j g(x) = \int_N e_j(x, \omega) g(\omega) d\omega$. If $\{ \psi_k \}$ is an orthonormal basis of $L^2(M, dx)$, then

$$ \sum_k \| E_j^* \psi_k \|^2_{L^2(N)} = \int_{M \times N} |e_j(x, \omega)|^2 dxd\omega. $$

Using (B.4.3),

$$ \| K \|_{L^1} = \max_{\{ \varphi_\ell \}, \{ \psi_k \}} \sum_{k, \ell} \langle K \varphi_\ell, \psi_k \rangle_{L^2(M)} $$

$$ \leq \max_{\{ \varphi_\ell \}, \{ \psi_k \}} \sum_k \| E_1^* \psi_k \|^2_{L^2(N)} \| E_2^* \varphi_\ell \|^2_{L^2(N)} = \| e_1 \|^2_{L^2(M \times N)} \| e_2 \|^2_{L^2(M \times N)}, $$

and

$$ \text{tr} K = \sum_k \langle E_2^* \psi_k, E_1^* \psi_k \rangle $$

$$ = \int_N \sum_k \langle \psi_k, e_2(\bullet, \omega) \rangle_{L^2(M)} \langle e_1(\bullet, \omega), \psi_k \rangle_{L^2(M)} d\omega = \langle e_1, e_2 \rangle_{L^2(M \times N)}, $$

which gives (B.4.11).

**REMARK.** The example above can be naturally generalized in different ways. It leads to another characterization of the trace class: $K \in \mathcal{L}_1(H)$ if and only if $K = E_1 E_2^*$ where $E_j : H_0 \to H$ and

$$ \| E_j^* \|_{\mathcal{L}_2(H, H_0)} := \sum_k \| E_j^* \psi_k \|_{H_0}^2 < \infty, $$

for any (or one) orthonormal basis of $H$. The operators in $\mathcal{L}_2(H, H_0)$ are called *Hilbert–Schmidt operators*. 

B. WEYL INEQUALITIES AND FREDHOLM DETERMINANTS

We now discuss the relation of singular values and the trace with the spectrum. Let \( A : H \to H \) be a compact operator and \( \lambda_0(A), \lambda_1(A), \ldots \) be its eigenvalues, listed according to multiplicity and ordered so that

\[
|\lambda_0(A)| \geq |\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots
\]

If \( A \) has only finitely many eigenvalues, we put the rest of \( \lambda_j(A) \) equal to zero. We always have \( \lambda_j(A) \to 0 \) and \( A - \lambda I \) is invertible unless \( \lambda = 0 \) or \( \lambda \) is an eigenvalue. The spectral projection corresponding to an eigenvalue \( \lambda \neq 0 \) of \( A \) is defined by

\[
\Pi_\lambda := \frac{1}{2\pi i} \oint_\lambda (zI - A)^{-1} \, dz,
\]

where the integral is taken over a contour enclosing \( \lambda \) but no other eigenvalues or zero. By the Cauchy formula and the identity

\[
(zI - A)^{-1}(wI - A)^{-1} = (w - z)^{-1}((zI - A)^{-1} - (wI - A)^{-1}), \quad z \neq w
\]

we see that \( \Pi_\lambda^2 = \Pi_\lambda \). Moreover, \( \Pi_\lambda \) is compact since \( A \) is compact and

\[
\Pi_\lambda = \frac{A}{2\pi i} \oint_\lambda z^{-1}(zI - A)^{-1} \, dz;
\]

therefore, \( \Pi_\lambda \) has finite rank. We define the (algebraic) multiplicity of \( \lambda \), \( m_\lambda(A) \), to be the rank of \( \Pi_\lambda \). Note that \( A \) commutes with \( \Pi_\lambda \) and thus \( A(Ran \Pi_\lambda) \subset Ran \Pi_\lambda \), and moreover \( \lambda \) is the only eigenvalue of \( A|_{Ran \Pi_\lambda} \), therefore

\[(A - \lambda I)^{m_\lambda(A)} \Pi_\lambda = 0.\]

We start with the following

**Proposition B.22.** For each \( n \), we have

\[
\prod_{j=0}^{n-1} |\lambda_j(A)| \leq \prod_{j=0}^{n-1} s_j(A).
\]

**Proof.** We may assume that \( \lambda_{n-1}(A) \neq 0 \). Let \( H_1 \subset H \) be the finite dimensional Hilbert space spanned by some linearly independent set \( u_0, \ldots, u_{n-1} \) of eigenvectors of \( A \) corresponding to \( \lambda_0(A), \ldots, \lambda_{n-1}(A) \). If

\[m(\lambda_j) > \dim \ker(A - \lambda_j I),\]

for some \( \lambda_j \), then, once we are out of eigenvectors of \( A \), we add to the list vectors in \( \ker((A - \lambda_j I)^2) \), then vectors in \( \ker((A - \lambda_j I)^3) \), and so on. Then \( A(H_1) \subset H_1 \) and the restriction \( A|_{H_1} \) has eigenvalues \( \lambda_0(A), \ldots, \lambda_{n-1}(A) \), counted with multiplicity. Denote by \( \iota : H_1 \to H \) the inclusion map and by
B.5. WEYL INEQUALITIES AND FREDHOLM DETERMINANTS

\[ \pi : H \to H_1, \] the orthogonal projector. Then the operator \( A_1 := \pi A \varepsilon : H_1 \to H_1 \) has eigenvalues \( \lambda_0(A), \ldots, \lambda_{n-1}(A) \) and by (B.3.6), we find \( s_j(A_1) \leq s_j(A) \) for all \( j \).

Now,
\[
\left| \prod_{j=0}^{n-1} \lambda_j(A) \right| = |\det(A_1)| = |\det(A_1^* A_1)|^{1/2} = \prod_{j=0}^{n-1} s_j(A_1) \leq \prod_{j=0}^{n-1} s_j(A)
\]

which proves (B.5.3).

We next need the following

**Lemma B.23.** Assume that \( \Phi \in C^\infty(\mathbb{R}; \mathbb{R}) \), \( \Phi' \geq 0 \), \( \Phi'' \geq 0 \), and \( \Phi(x) \to 0 \), \( x \Phi'(x) \to 0 \) as \( x \to -\infty \). Then for all \( a_1 \geq \cdots \geq a_n \), \( b_1 \geq \cdots \geq b_n \) such that
\[
\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j, \quad 1 \leq k \leq n,
\]
we have
\[
\sum_{j=1}^{k} \Phi(a_j) \leq \sum_{j=1}^{k} \Phi(b_j), \quad 1 \leq k \leq n.
\]

**Proof.** By Taylor’s formula, we have
\[
\Phi(x) = \Phi(y) + (x - y)\Phi'(y) + \int_{y}^{x} (x - t)\Phi''(t) \, dt.
\]

Letting \( y \to -\infty \), we see that
\[
\Phi(x) = \int_{-\infty}^{x} (x - t)\Phi''(t) \, dt = \int_{-\infty}^{\infty} (x - t)_+\Phi''(t) \, dt,
\]
where the integral converges absolutely as \( \Phi'' \geq 0 \). Then
\[
\sum_{j=1}^{k} \Phi(a_j) = \int_{-\infty}^{\infty} \left( \sum_{j=1}^{k} (a_j - t)_+ \right) \Phi''(t) \, dt;
\]
therefore, it suffices to prove that for each \( t \in \mathbb{R} \),
\[
\sum_{j=1}^{k} (a_j - t)_+ \leq \sum_{j=1}^{k} (b_j - t)_+.
\]

Fix \( t \) and choose the largest \( \ell \leq k \) such that \( a_\ell - t > 0 \). Then
\[
\sum_{j=1}^{k} (a_j - t)_+ = \sum_{j=1}^{\ell} (a_j - t) \leq \sum_{j=1}^{\ell} (b_j - t) \leq \sum_{j=1}^{k} (b_j - t)_+
\]
and the proof is finished. \( \square \)
Combining Proposition B.22 and Lemma B.23, we immediately get the following general Weyl inequality:

**Proposition B.24.** Assume that \( f \in C^\infty(0, \infty) \) is real-valued, \( f' \geq 0 \), \( \lim_{x \to +0} f(x) = 0 \), \( \lim_{x \to +0} f'(x)x \log x = 0 \), and \( f(e^t) \) is convex. Then for each compact operator \( A \) and each \( n \), we have

\[
\sum_{j=0}^{n-1} f(|\lambda_j(A)|) \leq \sum_{j=0}^{n-1} f(s_j(A)).
\]

**Proof.** Define \( \Phi(t) = f(e^t) \). It suffices to note that

\[
\sum_{j=0}^{k-1} \log |\lambda_j(A)| \leq \sum_{j=0}^{k-1} \log s_j(A)
\]

for all \( k \) and \( \Phi \) satisfies the conditions of Lemma B.23. \( \square \)

We write down two particularly useful special cases. Taking \( f(x) = x \) in Proposition B.24, we obtain for all \( n \),

\[
\sum_{j=0}^{n-1} |\lambda_j(A)| \leq \sum_{j=0}^{n-1} s_j(A),
\]

while taking \( f(x) = \log(1 + x) \), we get

\[
\prod_{j=0}^{n-1} (1 + |\lambda_j(A)|) \leq \prod_{j=0}^{n-1} (1 + s_j(A)).
\]

Note that (B.5.4) in particular implies

\[
\sum_{j=0}^{\infty} |\lambda_j(A)| \leq \|A\|_1.
\]

We now discuss determinants. Assume that \( A : H \to H \) is a finite rank operator and \( \lambda_0(A), \ldots, \lambda_{n-1}(A) \) are its eigenvalues ordered as in (B.5.1). Then for each finite dimensional subspace \( V \subset H \) containing \( \text{Ran} A \), the determinant of the restriction of \( I - A \) onto \( V \) is given by

\[
\det(I - A) = \prod_{j=0}^{n-1} (1 - \lambda_j(A)).
\]

By (B.5.5), we find

\[
|\det(I - A)| \leq \prod_{j=0}^{\infty} (1 + s_j(A)) \leq e^{\|A\|_1}.
\]
From the properties of the determinant on finite dimensional spaces, we have for finite rank operators $A, B$

\begin{equation}
\det((I - A)(I - B)) = \det(I - A) \det(I - B),
\end{equation}

when $\text{rank } A, \text{rank } B < \infty$, and

\begin{equation}
\det(I - AB) = \det(I - BA),
\end{equation}

when $\text{rank } A < \infty$. For the last identity we use the fact that the nonzero eigenvalues of $AB$ and $BA$ coincide with multiplicities – to see it, note that for each $\lambda \neq 0$ and each $j$, the maps

$$
\text{ker}((BA - \lambda I)^j) \xrightarrow{A} \text{ker}((AB - \lambda I)^j) \xrightarrow{B} \text{ker}((BA - \lambda I)^j)
$$

are injective. We also see from \(B.5.7\) that $I - A$ is invertible if and only if $\det(I - A) \neq 0$.

Another useful identity is the following: if $A_t$ is a $C^1$ family of finite rank operators depending on a parameter $t \in \mathbb{R}$, $\bigcup_t \text{Ran}(A_t)$ is contained in a fixed finite-dimensional subspace of $H$, and $I - A_t$ is invertible for all $t$, then

\begin{equation}
\partial_t \log \det(I - A_t) = -\text{tr}((I - A_t)^{-1}\partial_tA_t).
\end{equation}

We now prove that the map $A \mapsto \det(I - A)$ extends uniquely to a continuous (nonlinear) functional on $L_1(H; H)$; that is, we can define the determinant of $I - A$ when $A$ is of trace class. This follows from the fact that whenever $A \in L_1(H; H)$ and $B_k$ is a family of finite rank operators with $\|A - B_k\|_1 \to 0$, the sequence $\det(I - B_k)$ is a Cauchy sequence. To prove this, we consider two cases:

1. $I - A$ is invertible. Then for some constant $C$ and $k, \ell$ large enough we have for all $t \in [0, 1]$,

$$
\|(I - (tB_\ell + (1 - t)B_k))^{-1}\|_{H \to H} \leq C.
$$

Now, define $B(t) := tB_\ell + (1 - t)B_k$. By \(B.5.11\), for all $t \in [0, 1]

$$
|\partial_t \log \det(I - B(t))| = |\text{tr}((I - B(t))^{-1}(B_\ell - B_k))|
$$

$$
\leq \|(I - B(t))^{-1}(B_\ell - B_k)\|_1 \leq C\|B_\ell - B_k\|_1.
$$

Since $B_k$ is a Cauchy sequence in $L_1$, the proof is finished.

2. $I - A$ is not invertible. We then claim that $\det(I - B_k) \to 0$. Since for each $j$,

$$
|\det(I - B_k)| \leq |1 - \lambda_j(B_k)| \prod_r (1 + |\lambda_r(B_k)|) \leq |1 - \lambda_j(B_k)|e^{\|B_k\|_1},
$$

it suffices to show that for each $\varepsilon > 0$ and each $k$ large enough depending on $\varepsilon$, there exists $j$ such that $|1 - \lambda_j(B_k)| \leq \varepsilon$. Assume
the opposite, then, by passing to a subsequence, we may assume that exists \( \varepsilon > 0 \) such that \( |1 - \lambda_j(B_k)| \geq \varepsilon \) for all \( j, k \). We may also choose \( \varepsilon \) so that 1 is the only eigenvalue of \( A \) such that \( |1 - \lambda| \leq \varepsilon \). Then as \( k \to \infty \),

\[
0 = \oint_{|1 - \lambda| = \varepsilon} (z - B_k)^{-1} \, dz \to \oint_{|1 - \lambda| = \varepsilon} (z - A)^{-1} \, dz,
\]

however the right-hand side is nonzero applied to any eigenvector of \( A \) with eigenvalue 1, a contradiction.

This finishes the verification that \( \det(I - A) \) extends to a continuous functional on operators of trace class, \( A \). The identities (B.5.9) and (B.5.10) then extend to

\[
\det((I - A)(I - B)) = \det(I - A) \det(I - B),
\]

for \( A, B \in \mathcal{L}_1(H; H) \) and

\[
\det(I - AB) = \det(I - BA),
\]

for \( A \in \mathcal{L}_1(H_1; H_2), \ B \in \mathcal{B}(H_2; H_1) \).

**Proposition B.25.** If \( A \in \mathcal{L}_1(H; H) \), then \( I - A \) is invertible if and only if \( \det(I - A) \neq 0 \).

**Proof.** If \( I - A \) is not invertible, then \( \det(I - A) = 0 \) by the case (2) above. If \( I - A \) is invertible, then \( (I - A)^{-1} = (I - B) \), where \( B = A(B - I) \) is of trace class, and then by (B.5.9), \( 1 = \det(I - A) \det(I - B) \); it follows that \( \det(I - A) \neq 0 \). \( \square \)

The following estimates involving determinants will also be useful

**Proposition B.26.** Suppose that \( A, B \in \mathcal{L}^1 \). Then

\[
|\det(I + A)| \leq e^{\|A\|_1},
\]

and

\[
|\det(I + A) - \det(I + B)| \leq \|A - B\|_1 e^{1 + \|A\|_1 + \|B\|_1}.
\]

We finally discuss determinants of holomorphic families of operators. Assume that \( A(z) \) is a holomorphic family of operators in trace class for \( z \) in some domain \( \Omega \subset \mathbb{C} \). Using Proposition B.19, we write

\[
\det(I - A(z)) = \lim_{N \to \infty} \det(I - \Pi_N A(z)),
\]

and the limit so far is pointwise in \( z \). However, \( |\det(I - \Pi_N A(z))| \leq e^{\|\Pi_N A(z)\|_1} \leq e^{\|A(z)\|_1} \) is bounded locally uniformly in \( z \) as \( N \to \infty \); by compactness theorem, we see that the limit (B.5.16) is actually uniform on compact sets. As a corollary, the determinant \( \det(I - A(z)) \) is a holomorphic
function of $z$, and for $z$ such that $I - A(z)$ is invertible, we find using (B.5.11) and (B.5.16),
\begin{equation}
\partial_z \log \det(I - A(z)) = -\text{tr}((I - A(z))^{-1} \partial_z A(z)).
\end{equation}

In the case of matrices we know that $M^{-1}$ can expressed using Cramer’s rule and hence its norm can be estimated using $|\det M|^{-1}$. There is also an infinite dimensional version of this result:
\begin{equation}
\| (I - K)^{-1} \| \leq \frac{\det(I + (K^*K)^{\frac{1}{2}})}{\det(I - K^p)} , \ K \in \mathcal{L}_p.
\end{equation}

**B.6. LIDSKII’S THEOREM**

Let $A \in \mathcal{L}_1(H;H)$. In this section, we further explore the relation between the spectrum of $A$, the trace $\text{tr} A$, and the determinant $\det(I - A)$. Consider the holomorphic function
\[ D(z) := \det(I - zA), \quad z \in \mathbb{C}. \]

By (B.5.17), away from the zeroes of $D$ we have
\[ \partial_z \log D(z) = -\text{tr}((I - zA)^{-1} A). \]

In this section, we prove

**PROPOSITION B.27.** For $A \in \mathcal{L}_1(H;H)$,
\[ \det(I - zA) = \prod_j (1 - z\lambda_j(A)). \]

Before we start the proof, let us note that by taking the derivative at $z = 0$, we get the following theorem of Lidskiï:

**PROPOSITION B.28.** For $A \in \mathcal{L}_1(H;H)$, we have
\[ \text{tr} A = \sum_j \lambda_j(A). \]

The proof of Proposition B.27 starts with analyzing the zeroes of $D(z)$:

**PROPOSITION B.29.** The zeroes of $D(z)$ are exactly $\lambda_j(A)^{-1}$ for $\lambda_j(A) \neq 0$, and the multiplicity of $\lambda_j(A)^{-1}$ as a zero of $D(z)$ is equal to the (algebraic) multiplicity of $\lambda_j(A)$ as an eigenvalue of $A$.

**Proof.** The fact that the zeroes of $D(z)$ are exactly $\lambda_j(A)^{-1}$ follows immediately from Proposition B.25. To see that the multiplicities coincide, we fix
\( \lambda = \lambda_j(A) \) and note that the multiplicity of \( \lambda^{-1} \) as a zero of \( D(z) \) is equal to

\[
\frac{1}{2\pi i} \oint_{\lambda^{-1}} \partial_z \log D(z) \, dz = -\frac{1}{2\pi i} \text{tr} \oint_{\lambda^{-1}} (I - zA)^{-1} A \, dz
\]

where integration is over a contour containing \( \lambda^{-1} \) but no other zeroes of \( D(z) \). Let \( \Pi_\lambda \) be defined in (B.5.2), then using the change of variables \( z \mapsto z^{-1} \) we compute

\[
-\frac{1}{2\pi i} \oint_{\lambda^{-1}} (I - zA)^{-1} A \, dz = \frac{1}{2\pi i} \oint_{\lambda} z^{-1} (z - A)^{-1} A \, dz = \Pi_\lambda,
\]

thus (B.6.1) is equal to \( \text{tr} \, \Pi_\lambda \), that is, the multiplicity of \( \lambda \) as an eigenvalue of \( A \).

Proof of Proposition B.27. First we note that \( D(z) \) is of subexponential growth, namely for each \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that

\[
|D(z)| \leq C_\varepsilon e^{\varepsilon |z|}.
\]

Indeed, by (B.5.8) and approximation by finite rank operators, for each \( n \)

\[
|D(z)| \leq \prod_{j=0}^{\infty} (1 + |z|s_j(A)) \leq e^{\sum_{j=0}^{\infty} s_j(A)|z|} \prod_{j=0}^{n-1} (1 + |z|s_j(A)),
\]

and we have \( \sum_{j=0}^{\infty} s_j(A) \leq \varepsilon \) for \( n \) large enough.

Now, note that by (B.5.6), \( \sum_{j=0}^{\infty} \lambda_j(A) < \infty \). Then the function \( W(z) := \prod_{j=0}^{\infty} (1 - z\lambda_j(A)) \) is entire, satisfies the bound (B.6.2), and has the same zeros as \( D(z) \), counted with multiplicities. Therefore \( D(z) = e^{g(z)} W(z) \), where \( g \) is an entire function. In terminology of entire functions we have shown that \( D(z) \) is an entire of function of type 0. Hence, see §D.2, \( g = 0 \), proving Proposition B.27.

As an example of an application we record the following lemma which is useful in the text:

**Lemma B.30.** Suppose that

\[
H_s := \langle x \rangle^s L^2(\mathbb{R}^n), \quad ||u||_{H_s}^2 := \int_{\mathbb{R}^n} \langle x \rangle^{-2s} |u(x)|^2 \, dx,
\]

and that a linear operator \( A : H_0 \cap H_1 \to H_0 + H_1 \) extends to an operator \( A \in \mathcal{L}_1(H_0) \cap \mathcal{L}_1(H_1) \). Then

\[
\text{tr}_{H_0} A = \text{tr}_{H_1} A.
\]
**B.8. EXERCISES**

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**Proof.** 1. Lidskii’s theorem shows that the trace is the same for topologically equivalent inner products: the eigenvalues are independent of a specific choice as long as the Hilbert spaces are topologically the same.

2. Let \( B_0 = B(0, 1) \) and \( B_j = B(0, 2^j) \setminus B(0, 2^{j-1}) \), \( j \geq 1 \), and define the following equivalent norm on \( H_s \):

\[
\|u\|_{H_s}^2 := \sum_{j=1}^{\infty} 2^{-2sj} \|u\|_{L^2(B_j)}^2.
\]

Let \( \{e_{j,\ell}\}_{\ell=0}^{\infty}, \text{ supp } e_{j,\ell} \subset B_j \) be an orthonormal basis of \( L^2(B_j) \). Then \( \{e_{j,\ell}\}_{j,\ell=0}^{\infty} \) is an orthonormal basis of \( H_0 \) and \( \{f_{j,\ell}\}_{j,\ell=0}^{\infty}, f_{j,\ell} := 2^{js}e_{j,\ell} \) is an orthonormal basis of \( H_s \).

3. We now have

\[
\text{tr}_{H_0} A = \sum_{j,\ell=0}^{\infty} \langle Ae_{j,\ell}, e_{j,\ell} \rangle_{H_0} = \sum_{j,\ell=0}^{\infty} \int_{B_j} A e_{j,\ell}(x) e_{j,\ell}(x) dx
\]

\[
= \sum_{j,\ell=0}^{\infty} \int_{B_j} A e_{j,\ell}(x) \overline{e_{j,\ell}(x)} dx = \sum_{j=0}^{\infty} 2^{-2sj} \sum_{\ell=0}^{\infty} \int_{B_j} A f_{j,\ell}(x) \overline{f_{j,\ell}(x)} dx
\]

\[
= \sum_{j,\ell=0}^{\infty} \langle Af_{j,\ell}, f_{j,\ell} \rangle = \text{tr}_{H_s} A,
\]

completing the proof. \( \square \)

**B.7. NOTES**

Some of the presentation follows [Sj02, Chapter 5].

**B.8. EXERCISES**

Section B.5

1. Show that if \( A : H \to H \) satisfies \( \|A\| < 1 \) and \( A \in \mathcal{L}_1(H) \) then

\[
\det(I - A) = \exp \left( -\sum_{k=1}^{\infty} \frac{1}{k} \text{tr} A^k \right).
\]

**Hint:** Use the facts that the operator \( \log(I - A) \) is defined using the Taylor series and that \( \|A^k\|_{\mathcal{L}_1} \leq \|A\|^{k-1}\|A\|_{\mathcal{L}_1} \).

2. Show that if the operator \( f \otimes g \) is defined by \( f \otimes g(h) := f(h, g)_H \) then

\[
\det(I + f \otimes g) = 1 + \langle f, g \rangle.
\]
Hint: One solution uses Exercise B.1 applied to $A := zf \otimes \bar{g}$ and analytic continuation in $z$. Otherwise you can use Lidskii’s theorem.
FREDHOLM THEORY

In this appendix we will describe the role of the Schur complement formula in spectral theory, in particular in analytic Fredholm theory.

C.1. GRUSHIN PROBLEMS

Linear algebra. The Schur complement formula for two-by-two systems of matrices states that

\[
\begin{pmatrix}
P & R_- \\
R_+ & R_{+-}
\end{pmatrix}^{-1} = \begin{pmatrix}
E & E_+ \\
E_- & E_{-+}
\end{pmatrix},
\]

then \( P \) is invertible if and only if \( E_{+-} \) is invertible, with

\[
P^{-1} = E - E_+ E_{-+}^{-1} E_-, \quad E_{+-}^{-1} = R_{+-} - R_+ P^{-1} R_-.
\]

Generalization. We can generalize to problems of the form

\[
\begin{pmatrix}
P & R_- \\
R_+ & R_{+-}
\end{pmatrix} \begin{pmatrix}
u \\
u_-
\end{pmatrix} = \begin{pmatrix}
v \\
v_+
\end{pmatrix}
\]

where

\[
P : X_1 \to X_2, \quad R_+ : X_1 \to X_+, \quad R_- : X_- \to X_2, \quad R_{+-} : X_- \to X_+.
\]

are bounded operators on Banach spaces \( X_1, X_2, X_+, X_- \). If the operator (C.1.2) has a bounded inverse from \( X_2 \oplus X_+ \to X_1 \oplus X_- \) then, just as for matrices, invertibility of \( P \) is equivalent to the invertibility of \( E_{+-} \) and (C.1.1) holds.
DEFINITION. When $R_+ - R_- = 0$ we call (C.1.2) a Grushin problem:

(C.1.3) \[
\begin{pmatrix}
  P & R_-
  \\ R_+ & 0
\end{pmatrix}
\begin{pmatrix}
  u \\
  u_-
\end{pmatrix}
=
\begin{pmatrix}
  v \\
  v_+
\end{pmatrix},
\]

$P : X_1 \to X_2$, $R_+ : X_1 \to X_+$, $R_- : X_- \to X_2$,

In practice, we start with an operator $P$ and build a Grushin problem by choosing $R_\pm$, in which case it is normally sufficient to take $R_{+-} = 0$. Example above shows that in general that is not the case.

If the Grushin problem (C.1.3) is invertible, we call it well-posed and we write its inverse as follows:

(C.1.4) \[
\begin{pmatrix}
  u \\
  u_-
\end{pmatrix}
=
\begin{pmatrix}
  E & E_+
  \\ E_- & E_{-+}
\end{pmatrix}
\begin{pmatrix}
  v \\
  v_+
\end{pmatrix}
\]

for operators

$E : X_2 \to X_1$, $E_{-+} : X_+ \to X_-$, $E_+ : X_+ \to X_1$, $E_- : X_2 \to X_-.$

**LEMMA C.1 (The operators in a Grushin problem).** If (C.1.3) is well-posed, then the operators $R_+, E_-$ are surjective, and the operators $E_+, R_-$ are injective.

The next lemma is a result of a Neumann series calculation:

**LEMMA C.2 (Perturbation of a Grushin problem).** Suppose that (C.1.3) is well posed with the inverse given by (C.1.4). If $A : X_1 \to X_2$ is a bounded operator satisfying

(C.1.5) \[
\max(\| EA \|_{X_1 \to X_1}, \| AE \|_{X_2 \to X_2}) < 1.
\]

Then the Grushin problem

(C.1.6) \[
\begin{pmatrix}
  P + A & R_-
  \\ R_+ & 0
\end{pmatrix}
\]

is well posed with the inverse

(C.1.7) \[
\begin{pmatrix}
  F & F_+
  \\ F_- & F_{-+}
\end{pmatrix}
\]

where

(C.1.7) \[
F_{-+} = E_{-+} + \sum_{k=1}^{\infty} (-1)^k E_- A (EA)^{k-1} E_+.
\]
Proof. Since
\[
\begin{pmatrix}
  P & A \\
  R_+ & 0
\end{pmatrix}
= 
\begin{pmatrix}
  P & R_- \\
  R_+ & 0
\end{pmatrix}
\left(\begin{pmatrix}
  I_{X_1} & 0 \\
  0 & I_{X_-}
\end{pmatrix}
+ 
\begin{pmatrix}
  E & E_+ \\
  E_- & E_{-+}
\end{pmatrix}
\begin{pmatrix}
  A & 0 \\
  0 & 0
\end{pmatrix}
\right)
\begin{pmatrix}
  P & R_- \\
  R_+ & 0
\end{pmatrix}
\left(\begin{pmatrix}
  I_{X_1} & 0 \\
  0 & I_{X_-}
\end{pmatrix}
+ 
\begin{pmatrix}
  E & 0 \\
  0 & E\_A
\end{pmatrix}
\right)
\]
and
\[
\begin{pmatrix}
  A & 0 \\
  B & 0
\end{pmatrix}^k = \begin{pmatrix}
  A^k & 0 \\
  B A^{k-1} & 0
\end{pmatrix}
\]
we find the right inverse of (C.1.6) by a Neumann series if (C.1.5) holds. Similarly we obtain the left inverse. A simple calculation then gives (C.1.7).

\[\square\]

C.2. FREDHOLM OPERATORS

DEFINITIONS. (i) A bounded linear operator \( P : X_1 \to X_2 \) is called a Fredholm operator if the kernel of \( P \),
\[
\ker P := \{ u \in X_1 \mid Pu = 0 \},
\]
and the cokernel of \( P \),
\[
coker P := X_2/PX_1, \text{ where } PX_1 := \{ Pu \mid u \in X_1 \},
\]
are both finite dimensional. Here the cokernel of \( P \) is defined algebraically.

(ii) The index of a Fredholm operator is
\[
\text{ind } P := \dim \ker P - \dim \coker P.
\]

EXAMPLE. Many important Fredholm operators have the form
\[
P = I + K,
\]
where \( K \) a compact operator mapping a Banach space \( X \) to itself. See Example 2 later in this section for a simple proof when \( X \) is a Hilbert space.

Theorem C.4 below shows that the index does not change under continuous deformations of Fredholm operators (with respect to operator norm topology). Hence for operators of the form (C.2.1) the index is 0:
\[
\text{ind } P = \text{ind}(I + tK) = \text{ind } I = 0 \quad (0 \leq t \leq 1).
\]

The connection between Grushin problems and Fredholm operators is this:
THEOREM C.3 (Grushin problem for Fredholm operators). (i) Suppose that $P : X_1 \to X_2$ is a Fredholm operator. Then there exist finite dimensional spaces $X_\pm$ and operators $R_- : X_- \to X_2$, $R_+ : X_1 \to X_+$, for which the Grushin problem (C.1.3) is well posed. In particular, $PX_1 \subset X_2$ is closed.

(ii) Conversely, suppose that for some choice of spaces $X_\pm$ and operators $R_\pm$, the Grushin problem (C.1.3) is well posed. Then $P : X_1 \to X_2$ is a Fredholm operator if and only if $E_- : X_+ \to X_-$ is a Fredholm operator in which case

\[
\text{ind } P = \text{ind } E_-. \tag{C.2.3}
\]

Assertion (ii) is particularly useful when the spaces $X_\pm$ are finite dimensional.

Proof. 1. Assume $P : X_1 \to X_2$ is Fredholm. Let $n_+ := \dim \ker P$ and $n_- := \dim \ker P$, and write $X_+ := \mathbb{C}^{n_+}, X_- := \mathbb{C}^{n_-}$. If $\ker P$ is spanned by $x_j \in X_1$, $j = 1, \cdots, n_+$ then, by the Hahn-Banach theorem we can find $x_j^* : X_1 \to \mathbb{R}$, such that $x_j^*(x_i) = \delta_{ij}$ and $\|x_j^*\| \leq 1$. It follows that

\[
R_+ : X_1 \to \mathbb{C}^{n_+}, \quad R_+(x) := (x_1^*(x), \cdots, x_{n_+}^*(x)),
\]

has maximal rank, that is $\ker (R_+|_{\ker P}) = \{0\}$.

Now choose $y_j \in X_2$, $j = 1, \cdots, n_-$ so that $y_j + PX_1$ form a basis of $X_2 / PX_1$. Then define

\[
R_- : \mathbb{C}^{n_+} \to X_2, \quad R_-(a_1, \cdots, a_{n_-}) := \sum_{j=1}^{n_-} a_j y_j.
\]

The operator $R_-$ has maximal rank and $R_- X_- \cap PX_1 = \{0\}$.

We conclude that the operator

\[
\begin{pmatrix}
P & R_- \\
R_+ & O
\end{pmatrix} : X_1 \oplus \mathbb{C}^{n_-} \to X_2 \oplus \mathbb{C}^{n_+}
\]

has a trivial kernel and is onto. Hence it is invertible, and by the Open Mapping Theorem the inverse is continuous.

In particular, consider $P$ acting on the quotient space $X_1 / \ker P$, which is a Banach space since $\ker P$ is closed. We have $n_+ = 0$, and

\[
PX_1 = P(X_1 / \ker P) = \left( \begin{array}{c} P \\ R_- \end{array} \right) \left( \begin{array}{c} X_1 / \ker P \\ \{0\} \end{array} \right)
\]

is a closed subspace (the image of a closed subspace by the invertible operator $(P \ R_-)$).
2. Conversely, suppose that Grushin problem (C.1.3) is well-posed. According to Lemma C.1, the operators $R_+, E_-$ are surjective, and the operators $E_+, R_-$ are injective. We take $u_-=0$. Then

\[
\begin{align*}
\text{(C.2.4)} & \quad \begin{cases}
\text{the equation } Pu = v \text{ is equivalent to } \\
u = Ev + E_+v_+, \quad 0 = E_-v + E_+v_+.
\end{cases}
\end{align*}
\]

This means that $E_- : PX_1 \to E_-X_+$, and so we can define the induced map

$E^\# : X_2/PX_1 \to X_-/E_-X_+.$

Since $E_-$ is surjective, so is $E^\#$. Also, $\ker E^\# = \{0\}$. This follows since if $E_-v \in E_-X_+$, we can use (C.2.4) to deduce that $v \in PX_1$. Hence $E^\#$ is a bijection of the cokernels $X_2/PX_1$ and $X_-/E_-X_+$.

3. Next, we claim that $E_+ : \ker E_+ \to \ker P$ is a bijection. Indeed, if $u \in \ker P$, then $u = E_+v_+$ and $E_-v_+ = 0$. Therefore $E_+$ is onto; and this is all we need check, since $E_+$ injective.

We conclude that

\[
\dim \ker P = \dim \ker E_+ \quad \text{and} \quad \dim \coker P = \dim \coker E_+.
\]

In particular, the indices of $P$ and $E_+$ are equal. \hfill \square

**EXAMPLES.**

1. Suppose $X$ is a Banach space and $K : X \to X$. Then

\[
\text{(C.2.5)} \quad \dim KKX < \infty \implies I + K \text{ is a Fredholm operator.}
\]

Proof of (C.2.5). 1. A finite rank operator can be written as $K = \sum_{j=1}^J v_j w_j^*$ where $v_j \in X$, $w_j^* \in X^*$ (the dual space to $X$) and $J = \text{rank } K := \dim KKX$. In particular the sets

\[
\{v_j\}_{j=1}^J \subset X, \quad \{w_j^*\}_{j=1}^J \subset X^*,
\]

are linearly independent. This shows that $\ker K = \bigcap_{j=1}^J \ker w_j^*$ is a closed subspace of $X$ of codimension $J$.

2. Independence of $w_j^*$’s shows that can find $w_k \in X$, $k = 1, \ldots, J$ such that $w_j^*(w_k) = \delta_{jk}$. Then $\Pi := \sum_{j=1}^J w_j w_j^* : X \to X$ is a projection on a finite dimensional space and $\ker K = (I - P)X$.

3. We write

\[
I + K = I + K\Pi = I + \Pi KK\Pi + (I - \Pi)KK\Pi = (I + \Pi KK\Pi)(I + (I - \Pi)KK\Pi),
\]
where the second factor is invertible:

\[(I + (I - \Pi)K\Pi)^{-1} = I - (I - \Pi)K\Pi.\]

4. Hence we only need to check that the finite dimensionality of the kernel and cokernel of \(I + \Pi K\Pi\) and that follows from the properties of the finite dimensional operator \(I_{\Pi X} + \Pi K\Pi\). \(\square\)

2. Suppose now that \(X\) is a Hilbert space. Then

(C.2.6) \(K\) is a compact operator \(\implies I + K\) is a Fredholm operator.

The compactness of \(K\) means that \(KB_X(0,1) \subset X\), where \(B_X(0,1)\) is the unit ball in \(X\). (Property (C.2.6) holds in Banach spaces as well but a little more work is needed and in this book only Hilbert spaces are considered.)

Proof of (C.2.6). 1. Any compact operator \(K : X \to X\), where \(X\) is a Hilbert space can be approximated in norm by finite rank operators (this follows for instance from Theorem ??). This implies that there exists \(K_0 : X \to X\), \(\dim K_0 X < \infty\) such that \(\|K - K_0\| < \frac{1}{2}\). In particular \(I + K - K_0\) is invertible,

2. We write

\[I + K = (I + K - K_0)(I + (I + (K - K_0))^{-1}K_0).\]

The first fact is inverible and \((I + K - K_0)^{-1}K_0\) is a finite rank operator. Hence (C.2.6) follows from (C.2.5). \(\square\)

THEOREM C.4 (Invariance of the index under deformations). The set of Fredholm operators is open in \(L(X_1, X_2)\), and the index is constant in each component of that set.

Proof. When \(P\) is a Fredholm operator, we can use Theorem C.3 to obtain \(E_{-+} : \mathbb{C}^{n_+} \to \mathbb{C}^{n_-}\), with

(C.2.7) \(\text{ind } E_{-+} = n_+ - n_-\).

by the Rank-Nullity Theorem of linear algebra. The Grushin problem remains well-posed (with the same operators \(R_{\pm}\) ) if \(P\) is replaced by \(P'\), provided \(\|P - P'\| < \epsilon\) for some sufficiently small \(\epsilon > 0\). Hence the set of Fredholm operators is open.

Using (C.2.7) we see that the index of \(P'\) is the same as the index of \(P\). Consequently it remains constant in each connected component of the set of Fredholm operators. \(\square\)
**REMARKS.** 1. A bounded linear operator $P : X_1 \to X_2$ is a Fredholm operator if and only if there exists a bounded linear operator $E : X_2 \to X_1$ such that

$$PE = I_{X_2} + K_2, \quad EP = I_{X_1} + K_1,$$

where $K_j : X_j \to X_j$ are finite rank operators. (C.2.8)

In fact, if $P$ is a Fredholm operator then we use Theorem C.3 and obtain (C.2.8) with $K_2 = -R_-E_-$ and $K_1 = -E_+R_+$. On the other hand, if (C.2.8) holds then $\ker P \subset \ker(I + K_1)$ and $\text{coker} P \subset \text{coker}(I + K_2)$, hence both spaces are finite dimensional.

In particular (C.2.8) shows that adding a finite rank operator to a Fredholm operator, maintain the Fredholm property. Theorem C.4 shows that the index does not change.

2. If $P : X_1 \to X_2$ has index 0, Theorem C.3 shows that we can take $X_- = X_+ = \mathbb{C}^n$, for some $n$. In that case, we check easily that

$$P - R_-(I_{\mathbb{C}^n} - E_-)R_+ (E - E_+) = I_{X_2}.$$  

That means putting

$$K := -R_-(I - E_+)R_+, \quad \text{rank} K \leq n,$$

makes $P + K$ invertible.

We refer to Hörmander [H6II, Sect.19.1] for a comprehensive introduction to Fredholm operators.

---

**C.3. MEROMORPHIC CONTINUATION OF OPERATORS**

**DEFINITIONS.** 1. Let $\Omega \subset \mathbb{C}$ be a connected open set. If $X$ and $Y$ are Banach spaces then, $z \mapsto B(z) \in \mathcal{L}(X,Y)$ is holomorphic in $\Omega$ if for any $x \in X$ and $y^* \in Y^*$ (the dual of $Y$), $z \mapsto y^*(B(z)x)$ is a holomorphic function in $\Omega$. This is equivalent to the existence the holomorphic derivate of $B(z)$ in the norm topology – see [Ka80, Chapter III, Theorem 3.12].

2. We say that $z \mapsto B(z)$ is a meromorphic family of operators in $\Omega$ if for any $z_0 \in \Omega$ there exist operators $B_j$, $1 \leq j \leq J$, of finite rank and a family of operator $z \mapsto B_0(z)$, holomorphic near $z$, such that

$$B(z) = B_0(z) + \frac{B_1}{z - z_0} + \cdots \frac{B_J}{(z - z_0)^J}, \quad \text{near} \ z_0.$$
We say that $B(z)$ is a \textit{meromorphic family of Fredholm operators} if for every $z_0$, $B_0(z)$ is a Fredholm operator for $z$ near $z_0$. For nonsingular $z_0$, $B_0(z) = B(z)$.

**REMARK.** The Cauchy formula is valid for holomorphic families of operators:

$$B(\mu) = \frac{1}{2\pi i} \oint_\gamma \frac{B(\lambda)}{\lambda - \mu} d\lambda,$$

the integral is over a positive oriented curve enclosing $\mu$. Consequently, the Cauchy estimates hold:

(C.3.1) \[ \|\partial_\lambda B(\lambda)\|_{X \to Y} \leq \frac{1}{R} \max_{|\lambda - \zeta| \leq R} \|B(\zeta)\|_{X \to Y}. \]

The Grushin problem framework provides a proof of the following standard result:

**THEOREM C.5 (Analytic Fredholm Theory).** Suppose $\Omega \subset \mathbb{C}$ is a connected open set and \{\{A(z)\}\}_{z \in \Omega}$ is a holomorphic family of Fredholm operators.

If $A(z_0)^{-1}$ exists at some point $z_0 \in \Omega$, then the family $z \mapsto A(z)^{-1}$, $z \in \Omega$, is a meromorphic family of operators with poles of finite rank.

**REMARK.** The result also holds when the family $z \mapsto A(z)$ is meromorphic. We present the simpler holomorphic case here with the more general result following from the finer analysis in §C.4

**Proof.** 1. For any $w \in \Omega$ we produce a Grushin problem for $P = A(w)$, as described in the proof of Theorem C.3. The same operators $R_w^\pm$ also provide a well-posed Grushin problem for $P = A(z)$ for $z$ in some sufficiently small neighborhood $V(w)$ of $w$.

According to Theorem C.4

$$\text{ind } A(z) = \text{ind } A(z_0) = 0.$$

Consequently

$$n_+ = n_- = n,$$

and $E_{-+}^w(z)$ is an $n \times n$ matrix with holomorphic coefficients. The invertibility of $E_{-+}^w(z)$ is equivalent to the invertibility of $A(z)$.

2. It follows for any $w \in \Omega$ there exists a function $f_w(z) := \det E_{-+}^w(z)$, holomorphic in a neighbourhood of $w$ such that $A(z)$ is invertible if and only if $f_w(z) \neq 0$ Since $\Omega$ is connected and since $A(z_0)$ is invertible for at least one $z_0 \in \Omega$, none of $f_w$’s can be identically zero.
3. Since \( \det E_w^+(z) \) is not identically 0, \( E_w^+(z)^{-1} \) is a meromorphic family of matrices. Applying (C.1.1), we conclude that 
\[
A(z)^{-1} = E(z) - E^+(z)E_w^+(z)^{-1}E^-(z)
\]
is a meromorphic family of operators in the neighborhood \( w \). As \( w \) was arbitrary, \( A(z)^{-1} \) is meromorphic in all of \( \Omega \). \( \square \)

As a simple consequence we present

**Theorem C.6.** Let \( \Omega \subset \mathbb{C} \) be a connected open set. Suppose that \( X_1, X_2 \) are Banach spaces and \( X_1 \subset X_2 \) is a continuous inclusion. If for \( z \in \Omega \),
\[
P - z : X_1 \rightarrow X_2,
\]
is a Fredholm operator and for some \( z_0 \in \Omega \), \( P - z_0 \) is invertible then \( z \mapsto (P - z)^{-1} : X_2 \rightarrow X_1 \) is a meromorphic family of operators on \( \Omega \).

For \( z \in \Omega \) define
\[
\Pi_z := \frac{1}{2\pi i} \oint_{\gamma}(w - P)^{-1}dw,
\]
where the integral is over a positively oriented circle centered at \( z \) and including no poles of \( (w - P)^{-1} \) except possibly \( z \).

Then \( \Pi_z \) is a bounded projection of finite rank:
\[
\Pi_z^2 = \Pi_z, \quad \Pi_z : X_2 \rightarrow X_1 \subset X_2.
\]

**Proof.** In view of Theorem C.5 we only need to prove that \( \Pi_z^2 = \Pi_z \). For that we choose two circles 
\[
\gamma_j : t \mapsto z + r_je^{it}, \quad 0 \leq t \leq 2\pi, \quad 0 < r_1 < r_2 \ll 1.
\]

Then
\[
\Pi_z = \frac{1}{2\pi i} \int_{\gamma_j} (w_j - P)^{-1}dw_j, \quad j = 1, 2,
\]
and, using the resolvent identity,
\[
\Pi_z^2 = \frac{1}{2\pi i} \frac{1}{2\pi i} \int_{\gamma_2} \int_{\gamma_1} (w_1 - P)^{-1}(w_2 - P)^{-1}dw_1dw_2
\]
\[
= \frac{1}{2\pi i} \frac{1}{2\pi i} \int_{\gamma_2} \int_{\gamma_1} ((w_1 - P)^{-1} - (w_2 - P)^{-1}) \frac{dw_1dw_2}{w_2 - w_1}
\]
Since for \( w_2 \in \gamma_2, \int_{\gamma_1} dw_1/(w_2 - w_1) = 0 \) and for \( w_1 \in \gamma_1, \int_{\gamma_2} dw_2/(w_2 - w_1) = 2\pi i \), we see that
\[
\Pi_z^2 = \frac{1}{2\pi i} \frac{1}{2\pi i} \int_{\gamma_2} \int_{\gamma_1} (w_1 - P)^{-1} \frac{dw_1dw_2}{w_2 - w_1}
\]
\[
= \frac{1}{2\pi i} \int_{\gamma_1} (w_1 - P)^{-1}dw_1 = \Pi_z,
\]
completing the proof.

\[ \square \]

C.4. GOHBERG–SIGAL THEORY

Suppose \( A(\lambda) : X \to X, \lambda \in \Omega, \) is a meromorphic family of Fredholm operators with poles of finite rank acting on a Banach space \( X. \) Here \( \Omega \) a connected open subset of \( \mathbb{C}. \) From the definition in §C.3 this means that near any \( \mu \in \Omega, \) we have

\[(C.4.1) \quad A(\lambda) = \sum_{j=1}^{J} \frac{A_{j}(\lambda - \mu)^{k_{j}}}{\lambda - \mu} + A_{0}(\lambda), \]

where \( \lambda \mapsto A_{0}(\lambda) \) is holomorphic near \( \mu. \) If \( K = 0 \) that means that \( A(\lambda) = A_{0}(\lambda) \) is holomorphic near \( \mu. \)

The main result of this section is the following factorization theorem.

THEOREM C.7. Suppose that

\( \lambda \mapsto A(\lambda), \lambda \in \Omega, \)

is a meromorphic family of Fredholm operators with poles of finite rank (see §C.3). If \( A_{0}(\mu) \) in (C.4.1) has index 0 then there exist families of operators \( \lambda \mapsto U_{j}(\lambda), j = 1, 2, \) holomorphic and invertible near \( \mu, \) and operators \( P_{m}, 1 \leq m \leq M, \) such that, near \( \mu, \)

\[(C.4.2) \quad A(\lambda) = U_{1}(\lambda)(P_{0} + \sum_{m=1}^{M} (\lambda - \mu)^{k_{m}} P_{m})U_{2}(\lambda), \quad k_{\ell} \in \mathbb{Z} \setminus \{0\}, \]

\( P_{\ell}P_{m} = \delta_{\ell m}P_{m}, \quad \text{rank} P_{\ell} = 1, \quad \ell > 0, \quad \text{rank}(I - P_{0}) < \infty. \)

INTERPRETATION. 1. The inverse, \( A(\lambda)^{-1} \) exists, near \( \mu, \) as a meromorphic family of operators if and only if \( P_{0} + \sum_{m=1}^{M} P_{m} = I, \) in which case

\[(C.4.3) \quad A(\lambda)^{-1} = U_{2}(\lambda)^{-1}(P_{0} + \sum_{m=1}^{M} (\lambda - \mu)^{-k_{m}} P_{m})U_{1}(\lambda)^{-1}. \]

This shows that if \( A(\lambda_{0})^{-1} \) exists at some \( \lambda_{0} \in \Omega \) then, as \( \Omega \) is connected, \( A(\lambda)^{-1} \) is a meromorphic family of operators in \( \Omega. \) Hence Theorem C.7 implies the stronger version of Theorem C.5 in which we allow \( z \mapsto A(z) \) to be meromorphic.
2. The factorization of \( A(\lambda) \) provides a definition of a null multiplicity of \( A \) at \( \mu \): in the notation of (C.4.2),

\[
N_\mu(A) = \begin{cases} 
\sum_{k_\ell > 0} k_\ell, & \text{if } M = \text{rank}(I - P_0), \\
\infty, & \text{if } M < \text{rank}(I - P_0). 
\end{cases}
\]

When \( N_\mu(A) < \infty \) then \( A(\lambda)^{-1} \) is meromorphic and

\[
N_\mu(A^{-1}) = -\sum_{k_\ell < 0} k_\ell. 
\]

Theorem C.7 and the definitions (C.4.4), (C.4.5) gives the following result about multiplicities of poles and zeros of operators:

**THEOREM C.8.** Suppose that \( A(\lambda) \) and \( A(\lambda)^{-1} \) are meromorphic families of Fredholm operators. Then

\[
\frac{1}{2\pi i} \text{tr} \oint_{\mu} \partial_\lambda A(\lambda)A(\lambda)^{-1} d\lambda = N_\mu(A) - N_\mu(A^{-1}),
\]

where the integral is over a positively oriented circle which includes \( \mu \) and no other pole of \( \partial_\lambda A(\lambda)A(\lambda)^{-1} \).

In addition, if \( \sum_{j=1}^{p} s_j(A(\lambda))^p < \infty \) then, in the notation of \( \S \S \),

\[
\text{tr} \oint_{\mu} \partial_\lambda A(\lambda)A(\lambda)^{-1} d\lambda = \text{tr} \oint_{\mu} \partial_\lambda D(\lambda)D(\lambda)^{-1} d\lambda,
\]

\[
D(\lambda) := \det(I + A(\lambda)).
\]

In particular, when \( A(\lambda) = I + K(\lambda) \) where \( K(\lambda) \) is a meromorphic family of trace class operators then we obtain a formula for the multiplicity of zeros and poles of \( \det(I + K(\lambda)) \) given by the right hand side of (C.4.6):

\[
\frac{1}{2\pi i} \text{tr} \oint_{\mu} \frac{D'(\lambda)}{D(\lambda)} d\lambda = n_+(\mu) - n_-(\mu),
\]

\[
D(\lambda) := \det(I + K(\lambda)), \quad n_\pm(\mu) := N_\mu((I + K(\pm 1)).
\]

Another consequence is an operator valued version of Rouché’s theorem:

**THEOREM C.9 (Rouché’s Theorem for operator valued functions).** Suppose that \( A \) and \( B \) satisfy the assumptions of Theorem C.7 and that \( U \subset \Omega \) is a simply connected open set with a \( C^1 \) boundary \( \partial U \). If

\[
\|A(\lambda)^{-1}(A(\lambda) - B(\lambda))\|_{X \to X} < 1, \quad \lambda \in U,
\]

...
Then
\[
\sum_{\mu \in U} M_{\mu}(A) = \sum_{\mu \in U} M_{\mu}(B).
\]

**EXAMPLE.** Suppose that \(A(\lambda)\) and \(B(\lambda)\) are holomorphic families of matrices. Then \(\|A(\lambda)^{-1}(A(\lambda) - B(\lambda))\| < 1\) on \(\partial U\) implies that the number of zeros (counted with multiplicities) of \(\det A(\lambda)\) in \(U\) is the same as the number of zeros of \(\det B(\lambda)\) in \(U\).

The main step in the proof of Theorem C.7 is the following Lemma concerning matrix valued meromorphic functions. It is the finite dimensional version of Theorem C.7

**LEMMA C.10.** Suppose that \(\lambda \mapsto M(\lambda), \lambda \in D(0, r),\) is a family of \(n \times n\) matrices with meromorphic entries. Then there exist families of \(n \times n\) matrices, \(\lambda \mapsto E(\lambda), F(\lambda),\) holomorphic and invertible in \(D(0, \rho)\) for some \(\rho \leq r\) and such that

\[
(C.4.11) \quad M(\lambda) = E(\lambda) \begin{pmatrix}
\lambda^{k_1} & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \lambda^{k_2} & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ldots & \lambda^{k_N} & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ldots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0
\end{pmatrix} F(\lambda),
\]

where \(N \leq n\) and \(k_j \in \mathbb{Z}\).

**Proof.** 1. Since the entries of \(M(\lambda)\) are meromorphic near 0 we have

\[
M(\lambda) = (\lambda^{p_{ij}} a_{ij}(\lambda))_{1 \leq i, j \leq n},
\]

and \(a_{ij}\)'s are holomorphic in a neighbourhood of 0, \(a_{ij}(0) \neq 0\). We can choose \(\rho\) small enough so that

\[
|a_{ij}(\lambda)| > \epsilon > 0 \text{ in } D(0, \rho) \quad \text{or} \quad a_{ij} \equiv 0.
\]

2. By row and column operations, that is by multiplying \(M(\lambda)\) by invertible matrices on the left and on the right respectively, we can transform \(A(\lambda)\) to a matrix with \(p_{11} = \min_{1 \leq i, j \leq n} p_{ij}\), \(|a_{11}(\lambda)| > \epsilon\) in \(D(0, \rho)\). Then

\[
\frac{\lambda^{p_{ij}} a_{p_{ij}}(\lambda)}{\lambda^{p_{11}} a_{11}(\lambda)}, \quad 1 \leq i, j \leq q,
\]
are holomorphic in $D(0, \rho)$. Hence, further row and column operations depending holomorphically on $\lambda$ produce

$$M(\lambda) = E_1(\lambda) \begin{pmatrix} \lambda^{p_{11}} & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ 0 & & & M_1(\lambda) \end{pmatrix} F_1(\lambda),$$

where $M_1(\lambda)$ is now an $(n-1) \times (n-1)$ matrix with meromorphic coefficients, and $E_1(\lambda)$ and $F_1(\lambda)$ are invertible $n \times n$ matrices with holomorphic coefficients.

3. We can apply the same procedure to $M_1(\lambda)$ until we reach $M_n(\lambda) \neq 0$ or $M_N(\lambda) = 0_{(n-N) \times (n-N)}$. \hfill \Box

We also need two general facts:

**Lemma C.11.** Suppose that $X$ is a Banach space and $Y_0 \subset X$ is a finite dimensional subspace of $X$. Then there exists a closed subspace $X_0 \subset X$ such that $X_0 \cap Y_0 = \{0\}$, and $X_0 + Y_0 = X$.

**Proof.** 1. Let $y_1, \ldots, y_N$ be a basis of $Y_0$, and let $\tilde{y}_j : Y_0 \to \mathbb{R}$ be defined by $\tilde{y}_j(y) = \delta_{ij}$. By the Hahn-Banach Theorem we can extend $\tilde{y}_j$ to $y_j : X \to \mathbb{R}$ so that $\|y_j\| \leq \|\tilde{y}_j\| = 1$.

2. We then define a continuous linear transformation $\Pi : X \to X$ by $\Pi(x) = \sum_{j=1}^N y_j y_j^*$, so that $\Pi X = Y_0$, $\Pi^2 = \Pi$. Since $\ker \Pi = (I - \Pi)X$, putting $X_0 := \ker \Pi$ provides a closed subspace complementing $Y_0$ in $X$. \hfill \Box

**Lemma C.12.** Suppose $X$ is a Banach space and $X_1 \subset X$ is a closed subspace of $X$ satisfying $\dim X/X_1 < \infty$. Suppose also that a finite dimensional subset $Y_0 \subset X$ satisfies $Y_0 \cap X_1 = \{0\}$. Then there exists a finite dimensional subspace $Y_1$ such that $Y_0 \subset Y_1$, $Y_1 + X_1 = X$, $Y_1 \cap X_1 = \{0\}$.

**Proof.** 1. Let $\Pi$ be the projection constructed for $Y_0$ as in step 2 of the proof of Lemma C.11. The subspace $X_2 = X_1 + Y_0 \subset X$ has finite codimension as $X/(X_1 + Y_0) \to X/X_1$, $x + X_1 + Y_0 \mapsto (I - \Pi)x + X_1$ is injective. (Recall that $X_1 \cap Y_0 = \{0\}$.)

2. If $X_2 \neq X$ we now need to find a complement of the subspace $X_2$. For that we find a set $x_1, \ldots, x_J$ such that $x_j + X_2$ are a basis of $X/X_2$ and put $Y_2 := \text{Span}\{x_1, \ldots, x_J\}$. The desired space is then $Y_1 := Y_2 + Y_0$. \hfill \Box

**Proof of Theorem C.7.** 1. Without loss of generality we can assume that $\mu = 0$. Since we assumed that $A_0(0)$ is a Fredholm operator of index $0$,
Remark 2 in §C.2 shows that there exists a finite rank operator $C$ such that $A_0(0) + C$ is invertible. Consequently for $\lambda$ in a small neighbourhood of 0, $B(\lambda) := A_0(\lambda) + C$ is also invertible and

$$A(\lambda) = B(\lambda)(I + K(\lambda)), \quad K(\lambda) := B(\lambda)^{-1} \left( \sum_{j=1}^{J} A_j \lambda^{-j} - C \right).$$

We can now consider the Laurent series of $K(\lambda)$,

$$K(\lambda) = \sum_{j=1}^{J} K_j \lambda^{-j} + K_0(\lambda),$$

where $K_j$'s have finite rank and $K_0(\lambda)$ is holomorphic in $D(0, \rho)$.

2. We define

$$X_0 = \ker C \cap \bigcap_{j=1}^{J} \ker A_j \subset X,$$

which is a closed subspace of finite codimension. We note that (C.4.12)

$$K(\lambda)v = 0, \quad v \in X_0, \quad \lambda \in D(0, \rho).$$

Applying Lemma C.11 with

$$X = \bigcap_{j=1}^{J} \ker K_j, \quad Y_0 = \bigcap_{j=1}^{J} \ker K_j \cap \sum_{i=1}^{J} K_i X,$$

shows that there exists $Z_0$, a closed subspace of finite codimension satisfying

$$\bigcap_{j=1}^{J} \ker K_j = Z_0 + \bigcap_{j=1}^{J} \ker K_j \cap \sum_{i=1}^{J} K_i X, \quad Z_0 \cap \sum_{i=1}^{J} K_i X = \{0\}.$$

We then put

$$X_1 = X_0 \cap Z_0 \subset \bigcap_{j=1}^{J} \ker K_j,$$

which is a closed subspace of finite codimension. Because of the construction of $Z_0$,

$$X_1 \cap \sum_{j=1}^{J} K_j X = \{0\}.$$

Lemma C.12 used with $Y_0 = \sum_j K_j X$ shows that there exists a finite dimensional complement of $X_1$ invariant under $K_j$'s:

(C.4.13)

$$X_1 + Y_1 = X, \quad X_1 \cap Y_1 = \{0\}, \quad \dim Y_1 < \infty, \quad K_j|_{X_1} = 0, \quad K_j Y_1 \subset Y_1, \quad j = 1, \ldots, J, \quad K(\lambda)|_{X_1} = 0.$$
3. We define \( P : X \to Y_1 \) as the projection onto \( Y_1 \) with \( \ker P = X_1 \); since \( X_1 \) is closed the Closed Graph Theorem implies that \( P \) is continuous. The properties \((I - P)K_j P = 0 \) (invariance of \( Y_1 \) under \( K_j \)'s ), \( X_1 \subset X_0 \), and \((C.4.12)\) show that
\[
\]
The projection property, \( P(I - P) = 0 \), shows that the last factor is invertible:
\[
(I + (I - P)K_0(\lambda)P)^{-1} = I - (I - P)K_0(\lambda)P.
\]
Hence,
\[
A(\lambda) = B(\lambda)(I + PK(\lambda)P)C(\lambda), \quad C(\lambda) := I + (I - P)K_0(\lambda)P,
\]
and both \( B(\lambda) \) and \( C(\lambda) \) are invertible and holomorphic in \( D(0, \rho) \).

4. The operator \( P(I + PK(\lambda)P)P \) acts on the finite dimensional space \( Y_1 \) and hence we can apply Lemma C.10 to it:
\[
(C.4.15) \quad P(I + PK(\lambda)P)P = \iota E(\lambda) \left( \sum_{j=1}^{N} \lambda^{k_j} P'_j \right) F(\lambda)P,
\]
where \( P'_j : Y_1 \to Y_1 \) are one dimensional projections satisfying \( P'_j P'_i = \delta_{ij} P'_j \), \( N \leq \dim Y_0 \), and
\[
E(\lambda), F(\lambda) : PX \to PX \hookrightarrow X \text{ are holomorphic and invertible,}
\]
for \( \lambda \in D(0, \rho_1) \), where \( 0 < \rho_1 < \rho \).

5. Let us put denote \( \iota_P : PX \hookrightarrow X \) the inclusion map, and
\[
P_0 := I - P, \quad P_j = \iota_P P'_j P : X \to X, \quad P_j^2 = P_j, \quad \dim P_j X = 1.
\]
From \((C.4.15)\) we get
\[
I + PK(\lambda)P = P_0 + P(I + PK(\lambda)P)P
= P_0 + \iota_P E(\lambda) \left( \sum_{j=1}^{N} \lambda^{k_j} P'_j \right) F(\lambda)P
= (P_0 + \iota_P E(\lambda)) \left( P_0 + \sum_{j=1}^{N} \lambda^{k_j} P_j \right) (P_0 + \iota_P F(\lambda)P).
\]
The outside factors are invertible:
\[
(P_0 + \iota_P E(\lambda)P)^{-1} = P_0 + \iota_P E(\lambda)^{-1}P,
(P_0 + \iota_P F(\lambda)P)^{-1} = P_0 + \iota_P F(\lambda)^{-1}P,
\]
and hence (C.4.14) shows that (C.4.2) holds with
\[ U_1(\lambda) = B(\lambda)(P_0 + \iota P E(\lambda)P), \quad U_2(\lambda) = (P_0 + \iota P F(\lambda)P)C(\lambda). \]
This completes the proof of Theorem C.7. \qed

C.5. NOTES

For more about Grushin problems and connection to Feshbach reduction and other linear algebra constructions useful in spectral theory, see [SZ07b] and references given there.

The Gohberg–Sigal generalization of residue theory to operator valued meromorphic functions comes from the classical paper [GS70]. The proof of Lemma C.10 comes from [Vo94a, Appendix].

C.6. EXERCISES

1. Find the decomposition of Lemma C.10 for the matrix
\[ A(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \]

2. Assume that \( A(\lambda) : X \to X \) is a family of Fredholm operators of index zero on a Banach space \( X \) depending on \( \lambda \in \Omega, \Omega \ni 0 \), and there exist \( u_1,\ldots,u_N \in X, v_1,\ldots,v_N \in X^* \) such that
\[ A(0)u_j = 0, \quad A(0)^*v_j = 0, \quad \langle A'(0)u_k, v_j \rangle = \delta_{jk}. \]
Show that near \( \lambda = 0 \), \( A(\lambda)^{-1} \) has the expansion
\[ A(\lambda)^{-1} = \sum_{j=1}^N \frac{u_j \otimes v_j}{\lambda} + A_0(\lambda) \]
where \( A_0(\lambda) \) is holomorphic at 0.

3. Assume that \( A(\lambda) : X \to X \) is a family of Fredholm operators of index zero on a Banach space \( X \) depending on \( \lambda \in \Omega, \Omega \ni 0 \). For \( \ell \in \mathbb{N}_0 \), define the space \( V_\ell \) of polynomials \( p(\lambda) : \mathbb{C} \to X \) in \( \lambda \) of order no more than \( \ell \) such that \( A(\lambda)p(\lambda) = \mathcal{O}(\lambda^{\ell+1}) \) near \( \lambda = 0 \). In particular, \( V_0 \) is the kernel of \( A(0) \).
(a) Let \( T_\ell : V_\ell \to V_{\ell-1} \) be the homomorphism erasing the \( \lambda^\ell \) term; here \( V_{-1} := 0 \). Show that the kernel of \( T_\ell \) is canonically isomorphic to \( V_0 \). Consider also the injective homomorphism \( S_\ell : V_\ell \to V_{\ell+1} \) multiplying by \( \lambda \).

(b) Show that all the spaces \( V_\ell \) are finite dimensional and \( S_\ell \) is an isomorphism for \( \ell \) large enough.

(c) Show that the algebraic multiplicity of 0 as a pole of \( A(\lambda)^{-1} \) is equal to the limit \( \lim_{\ell \to \infty} \dim V_\ell \).
D.1. GENERAL FACTS

For a function \( f \) of two variables, \((x, y)\) we write
\[
\partial_z f = \frac{\partial f}{\partial z} := \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \partial_{\bar{z}} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right).
\]

A function is holomorphic in an open set \( \Omega \subset \mathbb{C} \) if and only if \( \partial_{\bar{z}} f \equiv 0 \), where the derivatives are taken in the sense of distributions.

If \( \Omega \) has a \( C^1 \) boundary \( \gamma \), positively oriented in the sense \( \Omega \) is always to the left of the direction on \( \gamma \), we have the following consequence of Green’s formula: for \( f \in C^1(\Omega) \),
\[
\int_{\gamma} f(w) dw = 2i \int_{\Omega} \frac{\partial f}{\partial \bar{w}}(w) dxdy, \quad w = x + iy,
\]
(D.1.1)
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw - \frac{1}{\pi} \int_{\Omega} \frac{1}{w - z} \frac{\partial f}{\partial \bar{w}}(w) dxdy.
\]

The following quantitative application of the maximum principle is useful in the study of resonances.

**Lemma D.1** (Three line theorem in a rectangle). Suppose that \( f(z) \) is holomorphic in a neighbourhood of \( \Omega := [-2R, 2R] + i[-\delta_-, \delta_] \). Suppose that, for \( M > 1 \), \( M_+, M_+ > 0 \), and \( 0 \geq \delta_+ < \delta_- < 1 \),
\[
|f(z)| \leq M_\pm, \quad \text{Im } z = \pm \delta_\pm, \quad |\text{Re } z| \leq 2R, \quad |f(z)| \leq M, \quad z \in \Omega.
\]
(D.1.2)
and that
\[(D.1.3) \quad R^2 > \delta_2^2(1 + 2 \log M).\]

Then for $\text{Im} z = 0$ and $|\text{Re} z| \leq R$,
\[(D.1.4) \quad |f(z)| \leq e^{M_0 \theta} M_1^{1-\theta}, \quad \theta := \frac{\delta_-}{\delta_+ + \delta_-}.\]

**Proof.**

1. If we replace $\Omega$ by $\Omega' := [-R, R] + i[-\delta_-, \delta_+]$ then it is enough to prove that $|f(0)| \leq e^{M_0 \theta} M_1^{1-\theta}$ under the assumption $(D.1.2)$ with $2R$ replaced by $R$ and $\Omega$ by $\Omega'$.

2. Putting $m_\pm = \log M_\pm$, $m = \log M > 0$, and $z = x + iy$, we consider the following subharmonic function defined in a neighbourhood of
   \[u(z) := \log |f(x + iy)| - \frac{\delta_- m_+ + \delta_+ m_- + y(m_+ - m_-)}{\delta_+ + \delta_-} - Kx^2 + Ky^2,\]
   where $K := 2m/(R^2 - \delta_-^2)$. From $(D.1.3)$ we obtain
   \[u(z) \leq \delta_-^2 K \leq 1, \quad \text{for } \text{Im} z = \pm \delta_\pm \text{ and } |\text{Re} z| \leq \rho.\]

Also,
\[u(z) \leq 2m - K(\rho^2 - \delta_-^2) \leq 0, \quad \text{for } |\text{Re} z| = \rho.\]

The maximum principle for subharmonic functions now shows that $\log |f(0)| - \theta m_+ - (1 - \theta)m_- \leq 1$ and that concludes the proof. \qed

The basic result relating the growth of a holomorphic function $f$ to the growth of the number of its zeros is the Jensen formula:

Suppose that $f(0) \neq 0$. Then
\[(D.1.5) \quad \int_0^r \frac{n(t)}{t} dt + \log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta}r)| d\theta ,\]
where $n(t)$ is the number of zeros of $f(z)$ with $|z| < t$.

From this we get an estimate on the number of zeros of $f$ in a disc of radius $r$:
\[(D.1.6) \quad n(r) \leq \frac{1}{\log 2} \int_r^{2r} \frac{n(t)}{t} dt \leq \frac{1}{\log 2} \left( \log \max_{|z|=2r} |f(z)| - \log |f(0)| \right).\]

If $f(0) = 0$ we apply the formula to $f(z)/z^p$ where $p$ is the order of vanishing of $f$ at 0.
We also use the Harnack inequality and the Borel-Carathéodory theorem: for \( f \) holomorphic in the closed disc \( D(0, R) \) and \( 0 < r < R \) we have

\[
\max_{|z| \leq r} |f(z)| \leq \frac{2r}{R-r} \max_{|z| \leq R} \text{Re } f(z) + \frac{R+r}{R-r} |f(0)|.
\]

A more general version of the upper bound on the number of zeros, which follows from the version for discs, can be stated as follows: suppose \( \Omega_0 \subset \subset \Omega_1 \) and that \( \Omega_1 \) is connected. If \( f \) is holomorphic in a neighbourhood of \( \Omega_1 \) and \( z_0 \in \Omega_0 \) then there exists a constant \( C_0 = C_0(\Omega_0, \Omega_1, z_0) \) such that the number of zero of \( f \) in \( \Omega_0 \), \( n_{\Omega_0} \), satisfies

\[
n_{\Omega_0} \leq C_0(\max_{z \in \Omega_1} \log |f(z)| - \log |f(z_0)|).
\]

An upper bound in \( \Omega_1 \) and a lower bound at a point \( z_0 \) also give lower bounds for the function away from zeros: suppose \( z_j, j = 1, 2, \ldots, \) are the zeros of \( f \) in \( \Omega_1 \). Then there exists \( C_1 = C_1(\Omega_0, \Omega_1, z_0) \) such that for any sufficiently small \( \delta > 0 \)

\[
\log |f(z)| \geq -C_1 \log \frac{1}{\delta} \left( \max_{z \in \Omega_1} \log |f(z)| - \log |f(z_0)| \right),
\]

\[
z \in \Omega_0 \setminus \bigcup_j D(z_j, \delta).
\]

We conclude this section with Carleman’s estimate for zeros of functions bounded in a halfplane [Ti39, §3.71]: suppose that \( f \) is holomorphic in \( \text{Im } z \geq 0 \) and that

\[
|f(z)| \leq C, \quad \text{Im } z \geq 0.
\]

If \( \{z_j\}_{j=0}^{\infty} \) are the zeros of \( f \) in \( \text{Im } z > 0 \) (included according to their multiplicities) then

\[
\sum_{j=0}^{\infty} \frac{\text{Im } z_j}{|z_j|^2} < \infty.
\]

The estimate (D.1.10) is a consequence of Carleman’s Theorem [Ti39, §3.7] which is a version of Jensen’s formula (D.1.5) for a half-plane.

D.2. ENTIRE FUNCTIONS

Suppose that \( f : \mathbb{C} \to \mathbb{C} \) is a holomorphic function. In other words, \( f \) is an entire function.

We can use canonical products to factorize \( f \). For that we recall the definition

\[
E_p(z) = (1 - z) \exp \left( z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right).
\]
If a sequence \( \{z_k\}_{k=1}^{\infty}, z_k \in \mathbb{C}, \) satisfies
\[
\sum_{k=1}^{\infty} \frac{1}{|z_k|^{p+1}} < \infty
\]
then the infinite product
\[
P(z) := \prod_{k=1}^{\infty} E_p(z/z_k)
\]
converges and
\[
m_P(z) := \frac{1}{2\pi i} \oint_{\gamma} \frac{P'(w)}{P(w)} dw = \# \{ k : z_k = z \}.
\]
Here the integral is over an “arbitrarily” small positively oriented circle around \( z. \)

Using the notation \( n(r) \) above we have the following estimate:
\[
\max_{|z| \leq r} \log |P(z)| \leq k_p r^p \left( \int_0^r \frac{n(t)}{t^{p+1}} dt + r \int_r^{\infty} \frac{n(t)}{t^{p+2}} dt \right).
\]

In particular, when
\[
n(r) \leq C r^p,
\]
we have
\[
\log |P(z)| \leq C |z|^p.
\]

A lower bound also holds and here is the case we use. When (D.2.4) is satisfied then for any \( \epsilon > 0 \) there existst \( r_0 \) such that
\[
\log |P(z)| \geq -|z|^{p+\epsilon}, \quad z \notin \bigcup_{m_P(w) > 0} D(w, \langle w \rangle^{-p-\epsilon}), \quad |z| \geq r_0.
\]

One consequence of these two bounds and (D.1.7) is a version of Hadamard’s factorization theorem: suppose that \( f \) is entire and that
\[
|f(z)| \leq C e^{C|z|^p}.
\]
If \( \{z_k\}_{k=1}^{\infty} \) are the zeros of \( f \) (included according to their multiplicities) then
\[
f(z) = e^{g(z)} P(z),
\]
where \( P(z) \) is given by (D.2.2) and \( g \) is a polynomial of degree less than or equal to \( p. \)

We say that \( f \) is of exponential type \( \tau \) if
\[
\limsup_{r \to \infty} \frac{\log \sup_{\lambda \leq \tau} |f(r)|}{r} = \tau.
\]
The type \( 0 < \tau < \infty \) is called normal.
The indicator function $f$ gives a more precise notion of order:

$$h(\theta) := \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}.$$  

The function $h$ is an indicator function of a convex set $K \subset \mathbb{C}$:

$$h(\theta) = \sup_{z \in K} (\cos \theta \Re z + \sin \theta \Im z).$$

The set $K$ is called the indicator diagram of $f$.

When $h(\theta)$ is a limit along a density one sequence of $r$’s (not just lim sup) and the convergence is uniform in $\theta$, the function $f$ is said to have completely regular growth. In that case we can describe the distribution of zeros in sectors using the indicator function – see [Le64].

Here we quote a specific result which is used in Section 2.5:

**THEOREM D.2 (Asymptotics of zeros).** If $f$ is of exponential type in $\mathbb{C}$ and if

$$\int_{\mathbb{R}} \frac{\log^+ |f(x)|}{1 + x^2} \, dx < \infty, \quad \log^+ r := \max(\log r, 0).$$

then $f$ has completely regular growth and the indicator diagram of $f$ is given by an interval $I_f \subset i\mathbb{R}$.

For $\epsilon > 0$ define $\Lambda_\epsilon := \{z = e^{i\theta} x : x \in \mathbb{R}, |\theta| < \epsilon\}$. Then, writing $m_f(z)$ for the multiplicity of a zero of $f$,

$$\lim_{r \to \infty} \frac{1}{r} \sum_{z \in \Lambda_\epsilon \cap D(0, r)} m_f(z) = 0,$$

$$\lim_{r \to \infty} \frac{1}{r} \sum_{z \in \Lambda_\epsilon \cap D(0, r)} m_f(z) = \frac{|I_f|}{2\pi},$$

$$\lim_{r \to \infty} \frac{1}{r} \sum_{\pi - \epsilon < \arg(\pm z) \leq \pi + \epsilon \atop |z| \leq r} m_f(z) = \frac{|I_f|}{2\pi}.$$

It is not difficult to check that if $f$ satisfies (D.2.8) and it has normal type $\tau$ then

$$|f(z)| \leq (1 + |z|)^N e^{\tau(\Im z)^-} \implies I_f = [-i\tau, 0].$$
In this appendix we present results from microlocal and semiclassical analyses. We start with the notion of a \textit{semiclassical pseudodifferential operator}. Roughly speaking, these operators have the form $a(x, hD_x)$ where $D_x = \frac{i}{h} \partial_x$ and $a(x, \xi)$ is a smooth function called the \textit{symbol} of the operator. The expression $Op_h(a) := a(x, hD_x)$ is called a \textit{quantization} of $a$. The small parameter $h > 0$ corresponds to the expected wavelength of the functions that we study; in applications to high frequency behaviour of resonances, we will often have $h \sim (\text{Re } \lambda)^{-1}$ where $\lambda$ is a resonance.

The emphasis is on aspects of the theory not easily accessible in recent texts. When easy references to [DS99] or [Zw12] are available we use them instead of referring to this appendix. Its most important application is to the material in Chapter 5.

\textbf{E.1. PSEUDODIFFERENTIAL OPERATORS}

The class of pseudodifferential operators includes all differential operators, however it is considerably more versatile. For instance, if $a$ is nonzero (or, more precisely, \textit{elliptic} – see §E.2.2 below), then we can consider the operator $(a^{-1})(x, hD_x)$ which will be an approximate inverse to $a(x, hD_x)$. One can also quantize symbols which are compactly supported in some set in the $(x, \xi)$ space; by applying the resulting operator to a function, we can \textit{microlocalize} this function to the corresponding set, and define the notion of \textit{wavefront set} – see §E.2. Other advantages of pseudodifferential calculus, such as propagation of singularities, will become apparent later in this appendix.
appendix. The price to pay is that semiclassical calculus will always yield
errors that are smoothing operators of norm $O(h^\infty)$. This means that the
$h \to 0$ semiclassical calculus is best suited to analysis at high frequencies,
while the fixed $h$ calculus only specifies where the location of $C^\infty$
singularities.

E.1.1. Differential operators. To motivate the construction that follows,
we first introduce the algebra of semiclassical differential operators $\text{Diff}^k_h(M)$
of order $k$ on a manifold $M$. In local coordinates, these have the form

$$A = \sum_{|\alpha| \leq k} \sum_{j=0}^{k-|\alpha|} h^j a_{\alpha j} (x) (hD_x)^{\alpha} : C^\infty(M) \to C^\infty(M)$$

where $\alpha$ is a multiindex, $a_{\alpha j}$ are smooth functions on $M$, and $D_x = \frac{1}{i} \partial_x$. In
this book, we will often consider the semiclassical operator

$$h^2 (-\Delta_g - \lambda^2)$$
on a Riemannian manifold $(M,g)$, and choose $h$ small enough so that $h\lambda$ is
bounded. The $h \to 0$ limit corresponds to $\lambda \to \infty$, and semiclassical calculus
is particularly suited to analysing high energy behaviour of resonances.

The class $\text{Diff}^k_h(M)$ is independent of the choice of coordinates, but the
individual coefficients $a_\alpha(x)$ are not. However, one can invariantly define the principal symbol

$$\sigma_h(A)(x,\xi) = \sum_{|\alpha| \leq k} a_{\alpha0}(x) \xi^\alpha \in \text{Poly}^k(T^*M),$$

where $T^*M$ is the cotangent bundle of $M$, $x$ is a coordinate system on $M$, $(x,\xi)$ is the induced coordinate system on $T^*M$, and $\text{Poly}^k(T^*M)$ stands for the class of smooth functions on $T^*M$ which are polynomials of degree at most $k$ on each cotangent space. Note that the kernel of the map $A \mapsto \sigma_h(A)$
on $\text{Diff}^k_h(M)$ is equal to $h \text{Diff}^{k-1}_h(M)$.

To justify the use of the cotangent bundle, rather than the tangent bundle, in the definition of $\sigma_h(A)$, it suffices to consider the case when $A = hX$, where $X$ is a vector field on $M$; then

$$\sigma_h(A)(x,\xi) = \langle \xi, X_x \rangle, \quad x \in M, \xi \in T^*_xM,$$

where $\langle \cdot, \cdot \rangle$ stands for the natural pairing between covectors and vectors. The principal symbol of the operator (E.1.2) is

$$\sigma_h(h^2 (-\Delta_g - \lambda^2))(x,\xi) = \langle \xi, \xi \rangle_g(x) - (h\lambda)^2.$$

The symbol map is multiplicative:

$$\sigma_h(AB) = \sigma_h(A)\sigma_h(B), \quad A \in \text{Diff}^k_h(M), B \in \text{Diff}^\ell_h(M).$$
Since multiplication of functions is commutative, this means that the commutator $[A, B]$ lies in $h \text{Diff}_h^{-1}(M)$. The principal symbol of this commutator is computed by the formula

$$\sigma_h(h^{-1}[A, B]) = \frac{1}{i} \{\sigma_h(A), \sigma_h(B)\},$$

where $\{\cdot, \cdot\}$ is the Poisson bracket.

**E.1.2. Symbols.** We start the construction of pseudodifferential calculus by specifying which functions on $T^* M$ can be quantized. In general, one can associate a pseudodifferential operator to a function $a(x, \xi; h)$ satisfying the following derivative bounds:

$$\sup_{x \in K, \xi \in T_x^* M} \langle \xi \rangle^{|\beta|-k} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| < \infty$$

for some $k \in \mathbb{R}$ (called the order of the symbol), all multiindices $\alpha, \beta$, and all compact subsets $K \subset M$. Here $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$ and $|\xi|$ denotes the length of the covector $\xi$ with respect to some continuous norm on the fibers of the cotangent bundle. The left-hand sides of (E.1.5) define a Fréchet space of symbols, which we denote $S^k_{1,0}(T^* M)$.

The class $S^k_{1,0}(T^* M)$ is independent of the choice of coordinates – see Exercise E.2.

In order to simplify exposition (in particular to avoid using quotient spaces for principal symbols), we further restrict ourselves to polyhomogeneous symbols, which have an asymptotic expansion in powers of $h$ and $\xi$. These symbols may have a complex order, so for notational convenience we define $S^k_{1,0} := S^k_{1,0}^{\text{Re} k}$ for $k \in \mathbb{C}$. The building blocks of polyhomogeneous symbols are given by

**DEFINITION E.1.** We say that $a \in C^\infty(T^* M)$ is positively homogeneous of order $k \in \mathbb{C}$, if $a(x, sa\xi) = s^k a(x, \xi)$ for all $s \geq 1$ and $|\xi| > F(x)$, where $F : M \to (0, \infty)$ is some continuous function.

It is easy to see that if $a$ is positively homogeneous of order $k$, then $a \in S^k_{1,0}(T^* M)$. To obtain general $h$-independent polyhomogeneous symbols, we use the following asymptotic expansion:

**DEFINITION E.2.** Let $a(x, \xi) \in S^k_{1,0}(T^* M)$. We say that

$$a(x, \xi) \sim \sum_{j=0}^\infty a_j(x, \xi)$$
for some $a_j$, $j = 0, 1, \ldots$ which are positively homogeneous of order $k - j$, if

$$a - \sum_{j=0}^{N-1} a_j \in S^{k-N}_{1,0}(T^*M) \quad \text{for all } N \in \mathbb{N}_0.$$  

If (E.1.6) holds for some $a_j$, then we say that $a$ is a polyhomogeneous symbol of order $k$ and denote $a \in S^k(T^*M)$.

Finally, our class of $h$-dependent symbols is given by

**DEFINITION E.3.** Let $b(x, \xi; h) \in S^k_{1,0}(T^*M)$, where the parameter $h$ takes values in $(0, h_0)$ for some constant $h_0$. We say that

(E.1.7) $$b(x, \xi; h) \sim \sum_{\ell=0}^{\infty} h^{\ell} b_\ell(x, \xi),$$

for some $b_\ell \in S^{k-\ell}_{1,0}(T^*M)$, $\ell = 0, 1, \ldots$ if

$$b - \sum_{\ell=0}^{N-1} h^{\ell} b_\ell \in h^N S^{k-N}_{1,0}(T^*M) \quad \text{for all } N \in \mathbb{N}_0.$$  

If (E.1.7) holds for some $b_\ell$, then we say that $b$ is a semiclassical polyhomogeneous symbol of order $k$ and denote $b \in S^k_h(T^*M)$.

A version of Borel’s Lemma states that for each $b_\ell \in S^{k-\ell}_{1,0}(T^*M)$, there exists $b \in S^k_h(T^*M)$ such that (E.1.7) holds, and same is true for (E.1.6) – see [Zw12, Theorem 4.15]. Moreover, the resulting symbols $a$ and $b$ are unique modulo an element of the residual class

$$S^{-\infty}(T^*M) = \bigcap_{k \in \mathbb{R}} S^k_{1,0}(T^*M)$$

for (E.1.6), and the class $h^{\infty} S^{-\infty}(T^*M)$ for (E.1.7). Note that

$$b \in h^{\infty} S^{-\infty}(T^*M)$$

$$\iff$$

$$\forall \alpha, \beta \in \mathbb{N}^n, N > 0, \quad \partial_x^\alpha \partial_\xi^\beta b(x, \xi; h) = O(h^N (\xi)^{-N}),$$

uniformly when $x$ varies in any compact subset of $M$.

To better understand the behaviour of symbols as $\xi \to \infty$, we consider them as functions on the fiber-radially compactified cotangent bundle $\overline{T^*M}$.

This bundle is a manifold with interior $T^*M$ and boundary diffeomorphic to the sphere bundle

$$\partial \overline{T^*M} \simeq S^*M = (T^*M \setminus \{0\})/\mathbb{R}^+,$$
where the group $\mathbb{R}^+$ acts on $T^*M \setminus 0 := \{(x, \xi) \in T^*M : \xi \neq 0\}$ by setting $s.(x, \xi) = (x, s\xi)$, $s \in \mathbb{R}^+$. Denote by

\begin{equation}
E.1.8 \quad \kappa : T^*M \setminus 0 \to \partial T^*M
\end{equation}

the natural projection map.

More precisely, if $g$ is a smooth Riemannian metric on $M$, then we can model $T^*M$ by the coball bundle $B^*M = \{(x, \xi) \in T^*M : |\xi|_g \leq 1\}$. An embedding $T^*M \to B^*M$ is given by $(x, \xi) \mapsto (x, \frac{\xi}{\sqrt{1 + |\xi|^2_g}})$, and the map $\kappa : T^*M \setminus 0 \to \partial B^*M$ is given by $(x, \xi) \mapsto (x, \frac{\xi}{|\xi|_g})$.

The smooth structure of $T^*M$ does not depend on the metric $g$. Moreover, the function $\rho(x, \xi) = \langle \xi \rangle^{-1}$ extends to a boundary defining function on $T^*M$ in the sense that $\rho = 0$ and $d\rho \neq 0$ on $\partial T^*M$, and $\rho > 0$ on $T^*M$.

Using the fiber-radial compactification, we characterize symbols as follows:

**Proposition E.4.** Let $a(x, \xi) \in C^\infty(T^*M)$. Then $a \in S^k(T^*M)$ if and only if $\langle \xi \rangle^{-k}a$ extends to a smooth function on $T^*M$.

Moreover, an appropriate rescaling of the Hamiltonian vector field of a classical symbol can be extended to $T^*M$:

**Proposition E.5.** For $a(x, \xi) \in S^k(T^*M)$, define the Hamiltonian vector field on $T^*M$:

$$H_a \in C^\infty(T^*M; T(T^*M)), \quad \omega(\cdot, H_a) = da, \quad \omega = d(\xi \cdot dx),$$

where $\xi \cdot dx$ is the invariant canonical one form on $T^*M$. Then $\langle \xi \rangle^{-1-k}H_a$ extends to a smooth vector field on $T^*M$ which is tangent to $\partial T^*M$.

**Remark.** The canonical one form is defined as $\xi \cdot dx = \eta$, $\eta_\rho(W) = \rho(\pi_*W)$, $\rho \in T^*M$, $W \in T_\rho(T^*M)$, $\pi_*W \in T_{\pi(\rho)}M$, where $\pi : T^*M \to M$ is the canonical projection. In local coordinates we have

$$H_a = \partial_\xi a \cdot \partial_x - \partial_x a \cdot \partial_\xi.$$

E.1.3. Method of stationary phase. The proofs of properties of pseudodifferential calculus rely on asymptotic expansions as $h \to 0$ of integrals of the form
\[ I_{\Phi,a}(h) = \int_M e^{i\Phi(x)}a(x) \, dx, \quad h > 0. \]
Here $M$ is an $n$-dimensional manifold, $dx$ is some smooth density on $M$, $a \in C^\infty_0(M)$, and $\Phi \in C^\infty(M; \mathbb{R})$ is a Morse function, as defined below:

**Definition E.6.** Let $\Phi \in C^\infty(M; \mathbb{R})$. We say that $x \in M$ is a critical point of $\Phi$, if $d\Phi(x) = 0$. For a critical point $x$, denote by $\nabla^2 \Phi(x) \in T^*_x M \otimes T^*_x M$, the Hessian of $\Phi$ at $x$. We say that $\Phi$ is a Morse function if $\nabla^2 \Phi(x)$ is nondegenerate for each critical point $x$.

**Remarks.**
1. At a critical point $x$, the Hessian is well defined: if $V \in C^\infty(M, TM)$ is a vector field on $M$ then $d(d\Phi(V)) \in C^\infty(M, T^*M)$ depends only on $V_x$ and hence defines a linear map $T_x M \to T^*_x M$. The corresponding element of $\nabla^2 \Phi(x) \in T^*_x M \otimes T^*_x M$ is the Hessian.
2. The critical points of Morse functions are necessarily isolated. If $x$ is a critical point, we denote by $\text{sgn} \nabla^2 \Phi(x)$ the signature of the corresponding Hessian, equal to $\delta_+ - \delta_-$, where $\delta_\pm$ are the maximal dimensions of subspaces on which $\pm \nabla^2 \Phi(x)$ is positive definite. Also, let $\det \nabla^2 \Phi(x)$ be the determinant of the matrix of $\nabla^2 \Phi(x)$ in any basis of $T_x M$ which has unit volume with respect to the density $dx$.

**Proposition E.7** (Method of stationary phase). Assume that $\Phi$ is a Morse function and let $x_1, \ldots, x_R$ be the critical points of $\Phi$ lying in $\text{supp} \, a$. Then for each $N \in \mathbb{N}_0$, we have
\[ I_{\Phi,a}(h) = \sum_{j=0}^{N-1} \sum_{k=1}^R e^{i\Phi(x_k)}h^{j+n/2}L_ja(x_k) + \mathcal{O}(h^{N+n/2})\|a\|_{C^{2N+n+1}} \]
for some differential operators $L_j$ of order $2j$. The operators $L_j$ and the constants in $\mathcal{O}(\cdot)$ depend on $\Phi$, but not on $a$. Moreover,
\[ L_0a(x_k) = (2\pi)^{n/2} \exp \left( \frac{i\pi}{4} \text{sgn} \nabla^2 \Phi(x_k) \right) |\det \nabla^2 \Phi(x_k)|^{-1/2}. \]

A proof can be found for instance in [HöI, §7.7] or [Zw12, §3.5].

In particular we have
\[ I_{\Phi,a}(h) = \mathcal{O}(h^\infty) \quad \text{if} \quad d\Phi \neq 0 \quad \text{on} \quad \text{supp} \, a. \]
E.1.4. Quantization on the Euclidean space. We now define pseudodifferential operators on \( \mathbb{R}^n \). For that, consider the class of symbols

\[(E.1.10) \quad \mathcal{S}_k^{1,0}(T^*\mathbb{R}^n) \subset \mathcal{S}_k^{1,0}(T^*\mathbb{R}^n) \]

defined using (E.1.5) but with \( K = \mathbb{R}^n \). In other words, the corresponding derivatives are bounded uniformly as \( x \to \infty \). We define the classes \( \mathcal{S}_h^k(T^*\mathbb{R}^n) \) and \( \mathcal{S}_h^k(T^*\mathbb{R}^n) \) similarly to the previous subsection. Note that these classes are not invariant under arbitrary diffeomorphisms of \( \mathbb{R}^n \).

**REMARK ON NOTATION.** Since in scattering theory we consider operators on non-compact manifolds the class \( \mathcal{S}_k^{1,0}(T^*X) \) cannot be defined using uniform growth estimates unless additional structure is introduced. In the case of \( \mathbb{R}^n \) use affine structure and that leads to the definition of \( \mathcal{S}_k^{1,0}(T^*\mathbb{R}^n) \): the bounds in (E.1.5) are invariant under symplectic lifts of affine transformations of \( \mathbb{R}^n \).

For \( a \in \mathcal{S}_k^{1,0}(T^*\mathbb{R}^n) \), define the operator

\[(E.1.11) \quad \text{Op}_h(a) = a(x,hD_x) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \]

calculating \( a \) by the following formula:

\[(E.1.12) \quad \text{Op}_h(a)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y,\xi)} a(x,\xi) f(y) \, dyd\xi. \]

The integral in (E.1.12) does not converge, however it can be calculated by first integrating in \( y \) and then using that the Fourier transform acts on \( \mathcal{S}(\mathbb{R}^n) \) to integrate in \( \xi \). It can also be defined using the concept of an oscillatory integral, see [Hö], §7.8] or [Zw, §3.6].

By the Fourier inversion formula, differential operators are quantizations of polynomials in \( \xi \):

\[ \text{Op}_h \left( \sum a_\alpha(x)\xi^\alpha \right) = \sum a_\alpha(x)(hD_x)^\alpha. \]

In particular, \( \text{Op}_h(1) \) is the identity operator.

The algebraic properties of (E.1.12) are given by

**PROPOSITION E.8.** Assume that \( a \in \mathcal{S}_k^{1,0}(T^*\mathbb{R}^n) \), \( b \in \mathcal{S}_\ell^{1,0}(T^*\mathbb{R}^n) \). Then:

1. We have \( \text{Op}_h(a) \text{Op}_h(b) = \text{Op}_h(a \# b) \), where \( a \# b \in \mathcal{S}_k^{k+\ell}(T^*\mathbb{R}^n) \) and

\[(E.1.13) \quad a \# b(x;\xi;h) \sim \sum_{j=0}^{\infty} \left( \frac{h}{\ell} \right)^j (\partial_\xi,\partial_y)^j (a(x,\xi)b(y,\eta)) \big|_{\eta=\xi}. \]
2. We have $\text{Op}_h(a^*) = \text{Op}_h(a^\#)$, where $a^\# \in \mathcal{S}_{1,0}^k(T^*\mathbb{R}^n)$ and
\begin{equation}
\tag{E.1.14}
a^\#(x,\xi;h) \sim \sum_{j=0}^{\infty} \left( \frac{h}{i} \right)^j \langle \partial_\xi, \partial_x \rangle^j a(x,\xi).
\end{equation}

The asymptotic expansions are understood in the sense of (E.1.7) in the classes $\mathcal{S}_{1,0}$.

For the proofs the reader is referred to [Zw12, Theorem 9.5], noting that the Weyl quantization used there is equivalent to the standard quantization for our class of symbols by [Zw12, Theorems 4.13 and 4.17].

We note the following corollaries of the expansions (E.1.13) and (E.1.14), the first two of which generalize (E.1.3) and (E.1.4):
\begin{equation}
\tag{E.1.15}
a^\#b = ab + O(h)_{\mathcal{S}_{1,0}^{k-1}(T^*\mathbb{R}^n)},
\end{equation}
\begin{equation}
\tag{E.1.16}
a^\#b - b^\#a = \frac{h}{i} \{a, b\} + O(h^2)_{\mathcal{S}_{1,0}^{k+\ell-2}(T^*\mathbb{R}^n)},
\end{equation}
\begin{equation}
\tag{E.1.17}
a^\# = \tilde{a} + O(h)_{\mathcal{S}_{1,0}^{k-1}(T^*\mathbb{R}^n)},
\end{equation}
\begin{equation}
\tag{E.1.18}
 a^\#b = O(h^\infty)_{\mathcal{S}^{-\infty}(T^*\mathbb{R}^n)}, \text{ if } \text{supp } a \cap \text{supp } b = \emptyset.
\end{equation}

**E.1.5. Quantization on general manifolds.** We now define pseudodifferential operators on a manifold. For that we use the local charts given by the following

**DEFINITION E.9.** Let $M$ be a manifold. A cutoff chart on $M$ is a pair $(\varphi, \chi)$ where $\varphi : U \to V$ is a diffeomorphism, $U \subset M, V \subset \mathbb{R}^n$ are open sets, and $\chi \in C_0^\infty(U)$. We define the lifted diffeomorphism
\begin{equation}
\tag{E.1.19}
\tilde{\varphi} : T^*U \to T^*V, \quad (x, \xi) \mapsto (\varphi(x), (d\varphi(x))^{-T}\xi).
\end{equation}

**REMARK.** The lifted diffeomorphism in (E.1.19) $\tilde{\varphi}$ is a symplectomorphism in the sense that $\tilde{\varphi}^*\omega_V = \omega_W$ where $\omega_U = d\eta_U$, $\omega_V = d\eta_V$ where $\eta_U$ and $\eta_V$ are canonical one forms on $T^*U$ and $T^*V$ respectively (see the remark after Proposition E.5).

The independence of the class of pseudodifferential operators on the choice of local charts follows from a change of variables statement for quantization on $\mathbb{R}^n$:

**PROPOSITION E.10.** Let $(\varphi, \chi)$ be a cutoff chart on $\mathbb{R}^n$. Then for each $a \in \mathcal{S}^k(T^*\mathbb{R}^n)$, there exists $b \in \mathcal{S}^k_h(T^*\mathbb{R}^n)$ such that
\[ \chi \varphi^* \text{Op}_h(a)(\varphi^{-1})^* \chi = \text{Op}_h(b), \]
and we have the following asymptotic expansion in the sense of (E.1.7) in the classes $\mathcal{S}_{1,0}$:

(E.1.20) \[ b \sim \sum_{j=0}^{\infty} h^j L_j(a \circ \varphi) \]

where $L_j$ are differential operators of order $2j$ on $T^*U$ with coefficients compactly supported in $x$, mapping $S^k(T^*U) \to S^{k-j}(T^*U)$, and $L_0 = \chi^2$.

Proof. We recover the symbol $b$ by oscillatory testing [Zw12, Theorem 4.19]:

\[ b(x, \xi) = e^{-\frac{i}{h}(x, \xi)} \chi \varphi^* \text{Op}_h(a)(\varphi^{-1})^* \chi(e^{\frac{i}{h}(\cdot, \xi)}) \]

\[ = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\varphi(x) - y', \eta) + (\varphi^{-1}(y') - x, \xi)} \chi(x) \chi(\varphi^{-1}(y')) a(\varphi(x), \eta) \, dy' \, d\eta \]

\[ = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(\varphi(x) - \varphi(y), \eta) + (y - x, \xi)} \chi(x) \chi(y) a(\varphi(x), \eta) J(y) \, dy \, d\eta \]

where we make the change of variables $y' = \varphi(y)$ and $J(y) = |\det d\varphi(y)|$ is the Jacobian.

To handle the case of large values of $\xi$, we put $\xi = r\xi'$ where $r = |\xi| \geq 1$, so that $|\xi'| \leq 1$. Making the change of variables $\eta = r\eta'$, we get

\[ b(x, r\xi') = (2\pi h')^{-n} \int_{\mathbb{R}^{2n}} e^{i\Phi'/h'} \chi(x) \chi(y) a(\varphi(x), r\eta') J(y) \, dy \, d\eta' \]

\[ \Phi = \langle \varphi(x) - \varphi(y), \eta' \rangle + \langle y - x, \xi' \rangle, \quad h' = h/r. \]

To obtain the expansion (E.1.20), we use the method of stationary phase (Proposition E.7). The critical points of the phase are given by the equations

\[ \varphi(x) = \varphi(y), \quad \xi' = (d\varphi(y))^T \eta'. \]

Therefore, for each $x \in U, \xi \in \mathbb{R}^n$ there exists unique critical point given by

\[ y = x, \quad \eta' = (d\varphi(x))^{-T} \xi'. \]

At the critical point we have $\Phi = 0$ and

\[ \nabla^2 \Phi = -\langle \nabla^2 \varphi(y), \eta \rangle - \langle d\varphi(y), d\eta \rangle. \]

Therefore, $\Phi$ is a Morse function and $\text{sgn} \, \nabla^2 \Phi = 0$ and $|\det \nabla^2 \Phi| = J(y)^2$ at the critical point.

The amplitude $\chi(x) \chi(y) a(\varphi(x), r\eta') J(y)$ is compactly supported in $y \in U$, but not necessarily in $\eta'$. We thus write it as a sum $a_1 + a_2$, where

\[ a_1 \in C^\infty_0(U \times \mathbb{R}^n), \quad \text{supp} \, a_2 \cap \{ \xi' = (d\varphi(y))^T \eta' \} = \emptyset. \]

We expand the integral featuring $a_1$ by Proposition E.7, obtaining (E.1.20); note that since the asymptotic parameter is $h' = h/r$, each next term in the expansion gains one power of $h$ and $|\xi|^{-1}$. As for the integral featuring
a_2, it is \( \mathcal{O}((h')^\infty) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty}) \) by repeated integration by parts in \( y \), as \( |\partial_y \Psi|^{-1} \leq C \langle \eta \rangle^{-1} \) on supp \( a_2 \). □

We define the residual class \( h^\infty \Psi^{-\infty} \) for pseudodifferential operators on a manifold as follows: \( A(h) \in h^\infty \Psi^{-\infty} \) if \( A \) is smoothing and each \( C^\infty(M \times M) \) seminorm of the Schwartz kernel of \( A \) is \( \mathcal{O}(h^\infty) \). We also use the notion of properly supported, compactly supported, and regular operators, see §A.7.

**Definition E.11.** Let \( M \) be a manifold and \( k \in \mathbb{R} \). Define the class of semiclassical pseudodifferential operators \( \Psi_k^h(M) \) as follows: a family of operators \( A(h) : C^\infty_0(M) \rightarrow C^\infty(M) \) lies in \( \Psi_k^h(M) \) if and only if it can be written as

\[
A = \sum_j \chi_j \varphi_j^* \text{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j + \mathcal{O}(h^\infty)_{\Psi^{-\infty}},
\]

for some cutoff charts \((\varphi_j, \chi_j)\) and symbols \( a_j \in S^k_{\mathbb{R}^n} \), where the domains of \( \varphi_j \) form a locally finite collection.

Note that all pseudodifferential operators are necessarily regular. An equivalent definition is given by the following

**Proposition E.12.** An operator \( A \) lies in \( \Psi_k^h(M) \) if and only if the following conditions hold:

1. For each \( \psi, \psi' \in C^\infty_0(M) \) such that \( \text{supp} \psi \cap \text{supp} \psi' = \emptyset \), we have \( \psi A \psi' \in h^\infty \Psi^{-\infty} \).

2. For each cutoff chart \((\varphi, \chi)\), there exists \( a_{\varphi, \chi} \in S^k_{\mathbb{R}^n} \) such that

\[
(\varphi^{-1})^* \chi A \varphi^* = \text{Op}_h(a_{\varphi, \chi}).
\]

**Proof.** Assume first that \( A \in \Psi_k^h(M) \). To verify property 1, we write

\[
\psi A \psi' = \sum_j \psi \chi_j \varphi_j^* \text{Op}_h(a_j)(\varphi_j^{-1})^* \chi_j \psi' + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}
\]

\[
= \sum_j \varphi_j^*((\psi \chi_j) \circ \varphi_j^{-1}) \text{Op}_h(a_j)((\psi' \chi_j) \circ \varphi_j^{-1})(\varphi_j^{-1})^* + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}
\]

and the sum above has finitely many nonzero terms due to the local finiteness condition. The supports of the functions \((\psi \chi_j) \circ \varphi_j^{-1}, (\psi' \chi_j) \circ \varphi_j^{-1} \in C^\infty_0(\mathbb{R}^n)\) do not intersect, therefore by \((E.1.18)\) we have \( \psi A \psi' \in h^\infty \Psi^{-\infty} \).

To verify property 2, we write

\[
(\varphi^{-1})^* \chi A \varphi^* = \sum_j \chi_j (\varphi_j')^* \text{Op}_h(a_j)((\varphi_j')^{-1})^* \chi_j + \mathcal{O}(h^\infty)_{\mathbb{R}^n \rightarrow \mathbb{R}^n}
\]
where \((\varphi'_j, \chi'_j)\) are the following cutoff charts on \(\mathbb{R}^n\):

\[
\varphi'_j = \varphi_j \circ \varphi^{-1}, \quad \chi'_j = (\chi \chi_j) \circ \varphi^{-1}.
\]

By Proposition E.10, and since operators in \(\mathcal{O}(h^\infty)_{\mathcal{S}' \to \mathcal{S}}\) are pseudodifferential with symbols in the class \(h^\infty_{\mathcal{S}}(\mathbb{R}^{2n})\), we have (E.1.22) with some \(a_{\varphi, \chi} \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\).

Now, assume that \(A\) satisfies properties 1 and 2 in the statement of this proposition; we write it in the form (E.1.21). Take a collection of cutoff charts \((\varphi_j, \chi_j)\) on \(M\) such that the domains \(U_j\) of \(\varphi_j\) cover \(M\) and \(\chi_j\) is a partition of unity, that is \(\sum_j \chi_j = 1\). Take also \(\chi'_j, \chi''_j \in C_0^\infty(U_j)\) such that \(\chi'_j = 1\) near \(\text{supp} \chi_j\) and \(\chi''_j = 1\) near \(\text{supp} \chi'_j\). We write

(E.1.24) \[ A = \sum_j \chi_j A = \sum_j \chi_j A \chi'_j + \sum_j \chi_j A(1 - \chi'_j). \]

By property 1, we see that the second term on the right-hand side is in \(h^\infty_{\Psi_{-\infty}}\). As for the first term, we write it as

(E.1.25) \[
\sum_j \chi_j A \chi'_j = \sum_j \chi'_j \varphi_j^* A_j (\varphi_j^{-1})^* \chi'_j, \quad A_j = (\chi_j \circ \varphi^{-1}_j)(\varphi_j^{-1})^* \chi''_j A \chi''_j \varphi^*_j.
\]

This has the form (E.1.21) as \((\varphi_j, \chi''_j)\) are cutoff charts and thus \(A_j = \text{Op}_h(a_j)\) for some \(a_j \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\) by property 2.

Proposition E.12 implies in particular that the class of semiclassical differential operators \(\text{Diff}^k_h(M)\) defined in §E.1.1 is contained in \(\Psi^k_h(M)\).

We now define the principal symbol of a pseudodifferential operator on a manifold. For \(a \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\), we say that \(a^0 \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\) is the principal part of \(a\) if it is the leading term in the expansion (E.1.7) of \(a\).

**Proposition E.13.** Let \(A \in \Psi^k_h(M)\). Then there exists a unique

\[ \sigma_h(A) \in S^k(T^*M), \]

called the **principal symbol** of \(A\), with the following properties:

1. For each representation (E.1.21) of \(A\), we have

(E.1.26) \[
\sigma_h(A) = \sum_j \chi_j(x)^2 a_j^0 \circ \varphi_j
\]

with \(\varphi_j\) defined in (E.1.19) and \(a_j^0 \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\) the principal part of \(a_j\).

2. For each cutoff chart \((\varphi, \chi)\) and \(a_{\varphi, \chi}\) defined in (E.1.22),

(E.1.27) \[
\chi(x)^2 \sigma_h(A)(x, \xi) = a_{\varphi, \chi}^0 \circ \varphi \quad \text{on} \ T^*U
\]

where \(a_{\psi, \chi}^0 \in \mathbb{S}^k_h(T^*\mathbb{R}^n)\) is the principal part of \(a_{\varphi, \chi}\).
Proof. Take a representation (E.1.21) of $A$ and define $\sigma_h(A)$ by (E.1.26). It follows from (E.1.23) and Proposition E.10 that (E.1.27) holds for each cutoff chart $(\varphi, \chi)$ and thus $\sigma_h(A)$ is independent of the choice of the representation (E.1.21). \[\square\]

We also define a (non-canonical) quantization procedure:

**Proposition E.14.** Let $(\varphi_j, \chi_j)$ be cutoff charts and $\chi'_j$ functions satisfying the conditions of the paragraph preceding (E.1.24), and $\tilde{\varphi}_j$ be defined by (E.1.19). For $a \in S^k(M)$, consider the operator

\begin{equation}
\text{Op}_h^M(a) = \sum_j \chi'_j \varphi_j^* \text{Op}_h \left( (\chi_j a) \circ \tilde{\varphi}_j^{-1} \right) (\varphi_j^{-1})^* \chi'_j.
\end{equation}

Then $A \in \Psi_k^h(M)$ is properly supported and

\begin{equation}
\sigma_h(\text{Op}_h^M(a)) = a.
\end{equation}

**Proof.** The fact that $A \in \Psi_k^h(M)$ follows immediately from Definition E.11, and (E.1.29) follows from (E.1.26). \[\square\]

When there is no risk of confusing the quantization map $\text{Op}_h^M$ defined above with the map $\text{Op}_h$ defined in (E.1.11), we denote $\text{Op}_h^M$ by $\text{Op}_h$.

**Proposition E.15.** The map

\begin{equation}
A \in \Psi_k^h(M) \mapsto \sigma_h(A) \in S^k(T^*M)
\end{equation}

is onto and its kernel is given by $h \Psi_h^{-1}(M)$. Moreover, for each $A \in \Psi_k^h(M)$ there exists $a \in S^k_h(M)$ such that

\begin{equation}
A = \text{Op}_h^M(a) + O(h^\infty)_{\Psi_{-\infty}}.
\end{equation}

**Proof.** The fact that (E.1.30) is onto follows from (E.1.29) and the fact that the kernel of (E.1.30) contains $h \Psi_h^{-1}(M)$ follows from (E.1.26). Now, assume that $A \in \Psi_k^h(M)$ and $\sigma_h(A) = 0$; we prove that $A \in h \Psi_h^{-1}(M)$. We write $A$ in the form (E.1.24), (E.1.25):

\begin{equation}
A = \sum_j \chi'_j \varphi_j^* \text{Op}_h(a_j)(\varphi_j^{-1})^* \chi'_j + O(h^\infty)_{\Psi_{-\infty}},
\end{equation}

and by (E.1.27), we see that $a_j = (\chi_j \sigma_h(A)) \circ \tilde{\varphi}_j^{-1} + O(h)_{S^k_h(T^*M)}$. Since $\sigma_h(A) = 0$, we have $a_j \in h \Psi_h^{-1}(T^*M)$. By (E.1.32), we have $A \in h \Psi_h^{-1}(M)$ as needed.

Finally, let $A \in \Psi_k^h(M)$; we find $a \in S^k_h(M)$ such that (E.1.31) holds. Let $a_0 := \sigma_h(A)$. Then by (E.1.29) and the first part of the current proposition,
we have $A - \text{Op}_h(a_0) \in h\Psi^k(M)$. Repeating this process, we construct symbols $a_j \in S^{k-j}(T^*M)$ by the formula

$$a_j = \sigma_h \left( h^{-j} \left( A - \sum_{\ell=0}^{j-1} \text{Op}_h^M(a_{\ell}) \right) \right).$$

(E.1.33)

It remains to take $a$ in the form (E.1.7): $a \sim \sum_{j=0}^{\infty} h^j a_j$. □

The following algebraic properties of principal symbols follow directly from (E.1.15)–(E.1.17) and (E.1.26):

**Proposition E.16.** 1. For properly supported $A \in \Psi^k(M), B \in \Psi^\ell(M)$, we have $AB \in \Psi^{k+\ell}(M)$ and

$$\sigma_h(AB) = \sigma_h(A)\sigma_h(B),$$

(E.1.34)

$$\sigma_h(h^{-1}[A,B]) = \frac{1}{i} \{\sigma_h(A),\sigma_h(B)\}.$$  

(E.1.35)

2. For $A \in \Psi^k(M)$, we have $A^* \in \Psi^k(M)$ (where the adjoint is taken with respect to any smooth density on $M$) and

$$\sigma_h(A^*) = \sigma_h(A).$$

**E.1.6. Sobolev spaces.** We now introduce Hilbert spaces on which semiclassical pseudodifferential operators act naturally. We start with the case of $\mathbb{R}^n$:

**Definition E.17.** For $s \in \mathbb{R}$, the semiclassical Sobolev space

$$H^s_h(\mathbb{R}^n), \quad \mathcal{S}(\mathbb{R}^n) \subset H^s_h(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

is defined as the Sobolev space $H^s(\mathbb{R}^n)$ with the $h$-dependent norm

$$\|u\|_{H^s_h} := \|\langle h\xi\rangle^s \hat{u}(\xi)\|_{L^2}.$$  

(E.1.36)

For notational convenience, we put $H^s_h(\mathbb{R}^n) := H^\text{Re}s_h(\mathbb{R}^n)$ for $s \in \mathbb{C}$.

Then pseudodifferential operators defined by (E.1.12) act on semiclassical Sobolev spaces:

**Proposition E.18 (Boundedness of pseudodifferential operators on $\mathbb{R}^n$).** Let $a \in S^k(T^*\mathbb{R}^n)$. Then for each $s$ the operator

$$\text{Op}_h(a) : H^s_h(\mathbb{R}^n) \to H^{s-k}_h(\mathbb{R}^n)$$

is bounded uniformly in $h$, and its operator norm is bounded by some fixed $S^k_{1,0}$ seminorm of $a$. 


This follows from the boundedness of $\langle hD \rangle^{s-k} \text{Op}_h(a) \langle hD \rangle^{-s}$ on $L^2(\mathbb{R}^n)$. From the pseudodifferential calculus in Proposition E.8 we see that the last operator is given by $\text{Op}_h(b)$ where $b \in \mathcal{S}^0_h(T^*\mathbb{R}^n)$. The operator $\text{Op}_h(b)$ is uniformly bounded on $L^2(\mathbb{R}^n)$ – see for instance [Zw12, Theorem 4.32] and for Hörmander’s simple proof for the class $\mathcal{S}^0$, Exercise E.6.

For each cutoff chart $(\varphi, \chi)$ on $\mathbb{R}^n$, there exists a constant $C$ such that

$$\|\chi(u \circ \varphi)\|_{H^s_h(\mathbb{R}^n)} \leq C\|u\|_{H^s_h(\mathbb{R}^n)}$$

for all $u \in H^s_h(\mathbb{R}^n)$. Indeed, we have

$$\text{(E.1.37)} \quad \|u\|_{H^s_h(\mathbb{R}^n)} = \|\text{Op}_h(a_s)u\|_{L^2(\mathbb{R}^n)}, \quad a_s(x, \xi) = \langle \xi \rangle^s.$$  

It remains to apply Propositions E.10 and E.18 and the fact that the space $L^2$ is invariant under changes of variables. We can now define Sobolev spaces on manifolds:

**Definition E.19.** Let $M$ be a manifold. For $s \in \mathbb{R}$, define the local semiclassical Sobolev space $H^s_{h,\text{loc}}(M)$, $C^\infty(M) \subset H^s_{h,\text{loc}}(M) \subset D'(M)$, as the Frechet space with Hilbert seminorms

$$\|(\chi u) \circ \varphi^{-1}\|_{H^s_h(\mathbb{R}^n)}$$

for all cutoff charts $(\varphi, \chi)$. Let $H^s_{h,\text{comp}}(M)$ consist of all elements of $H^s_{h,\text{loc}}(M)$ which are supported inside some $h$-independent compact subset of $M$.

Note that $H^s_{h,\text{comp}}(M)$ is dual to $H^{-s}_{h,\text{loc}}(M)$ under the standard $L^2$ pairing. For each $u \in H^s_{h,\text{comp}}$, we can define the norm $\|u\|_{H^s_h}$ by

$$\text{(E.1.38)} \quad \|u\|^2_{H^s_h} := \sum_j \|(\chi_j u) \circ \varphi_j^{-1}\|^2_{H^s_h(\mathbb{R}^n)}$$

where $(\varphi_j, \chi_j)$ is a collection of cutoff charts such that the domains of $\varphi_j$ form a locally finite covering of $M$ and $\sum_j |\chi_j| > 0$ everywhere on $M$. Moreover, two norms resulting from two different choices of $(\varphi_j, \chi_j)$ are equivalent with constants uniform in $h$ as long as we require that $u$ is supported in some fixed compact subset of $M$. In particular, when the manifold $M$ is compact, the norm (E.1.38) produces a canonically defined Hilbert space $H^s_h(M)$.

The spaces $H^s_{h,\text{loc}}$ for different values of $h$ consist of the same functions and the norms $\|u\|_{H^s_h}$ for different choices of $h$ are equivalent, with constants depending on $h$. We thus may use the $h$-independent notation (with $M$ compact in the third case below)

$$\text{(E.1.39)} \quad H^s_{h,\text{loc}}(M), \quad H^s_{h,\text{comp}}(M), \quad H^s(M)$$

for these spaces, where to define the norm we put $h := 1$. However, the $h$-dependent norms $\|\cdot\|_{H^s_h}$ will be used in the semiclassical estimates below, to ensure that the constants in these estimates are uniform in $h$. 
PROPOSITION E.20 (Interpolation inequality in Sobolev spaces). Let \( M \) be a manifold, \( V \subset M \) a compact set. Fix real numbers \( s_1 < r < s_2 \) and the corresponding norms (E.1.38). Then there exists a constant \( C \) such that for each \( \alpha > 0 \) and each \( u \in H^{s_2}_{\text{comp}}(M) \) with \( \text{supp} \ u \subset V \), we have

\[
\| u \|_{H^r_h} \leq \alpha \| u \|_{H^{s_2}_h} + C \alpha^{(s_1-r)/(s_2-r)} \| u \|_{H^{s_1}_h}.
\]

(E.1.40)

Proof. By (E.1.38), we reduce to the classes \( H^s_h(\mathbb{R}^n) \). Then (E.1.40) follows immediately from (E.1.36) and the following inequality:

\[
\langle h \xi \rangle^r \leq \alpha \langle h \xi \rangle^{s_2} + \alpha^{(s_1-r)/(s_2-r)} \langle h \xi \rangle^{s_1};
\]

the latter can be verified directly by multiplying both sides by \( a^{r/(s_2-r)} \) and using the inequality \( a^r \leq a^{s_2} + a^{s_1} \) for \( a := \alpha^{1/(s_2-r)} \langle h \xi \rangle \).

We next review Sobolev spaces on manifolds with boundary. We refer the reader to [HöIII, Appendix B.2] and [TaI, §§4.4,4.5] for a comprehensive treatment. Let \( \overline{M} \) be a compact manifold with boundary \( \partial M \) and interior \( M \). We embed \( \overline{M} \) into a compact manifold without boundary, denoted \( M_{\text{ext}} \). One way to do this is to let \( M_{\text{ext}} \) be the double space of \( \overline{M} \), obtained by gluing together two copies of \( \overline{M} \) along the boundary.

DEFINITION E.21 (Sobolev spaces on manifolds with boundary). Let \( M \subset \overline{M} \subset M_{\text{ext}} \) be as above. For \( s \in \mathbb{R} \), define the spaces

\[
\dot{H}^s_h(M) \subset \mathcal{D}'(M), \quad \dot{H}^s_h(\overline{M}) \subset \mathcal{D}'(M_{\text{ext}})
\]

as follows:

- \( \dot{H}^s_h(M) \) consists of restrictions to \( M \) of elements of \( H^s_h(M_{\text{ext}}) \);
- \( \dot{H}^s_h(\overline{M}) \) consists of elements of \( H^s_h(M_{\text{ext}}) \) whose supports are contained in \( \overline{M} \).

The space \( \dot{H}^s_h(\overline{M}) \) is a closed subspace of the Hilbert space \( H^s_h(M_{\text{ext}}) \) and inherits the norm of this ambient space. As for \( \dot{H}^s_h(M) \), we make it into a Hilbert space by identifying it with the orthogonal complement of \( H^s_h(M_{\text{ext}} \setminus M) \) in \( H^s_h(M_{\text{ext}}) \). We have the inclusions

\[
H^s_{h,\text{comp}}(M) \subset \dot{H}^s_h(\overline{M}), \quad \dot{H}^s_h(M) \subset H^s_{h,\text{loc}}(M).
\]

Similarly to (E.1.39), we use the notation

\[
\mathcal{H}^s(M), \quad \mathcal{H}^s(\overline{M})
\]

for the spaces \( \dot{H}^s_h(M), \dot{H}^s_h(\overline{M}) \) when the \( h \)-dependence of the norm is irrelevant. The space \( C^\infty(\overline{M}) \) of functions smooth up to the boundary is dense in \( \mathcal{H}^s(M) \) and \( C^\infty_0(M) \) is dense in \( \mathcal{H}^s(\overline{M}) \). The spaces \( \dot{H}^s_h(M) \) and \( \mathcal{H}^s(\overline{M}) \) are dual to each other with respect to the natural \( L^2 \) pairing.
Coming back to the case of manifolds without boundary, we now study the action of pseudodifferential operators on the Sobolev spaces $H^s_{h,\text{comp}}$ and $H^s_{h,\text{loc}}$. The following is a direct corollary of Proposition E.18:

**PROPOSITION E.22** (Boundedness of pseudodifferential operators on manifolds). Each $A \in \Psi^k_h(M)$ is bounded uniformly in $h$ on compact sets as an operator

$$A : H^s_{h,\text{comp}}(M) \to H^{s-k}_{h,\text{loc}}(M).$$

We also use the following version of the sharp Gårding inequality:

**PROPOSITION E.23.** Let $A \in \Psi^{2k+1}_h(M)$ be compactly supported, fix some smooth density on $M$, and assume that $(\xi)^{-2k-1} \Re \sigma_h(A) \geq 0$ everywhere. Then there exists a constant $C$ such that for each $u \in H^{k+1/2}(M)$,

$$(E.1.41) \quad \Re \langle Au, u \rangle_{L^2} \geq -Ch\|u\|^2_{H^k(M)}.$$

**Proof.** By a partition of unity and using cutoff charts (see Proposition E.15), we reduce to the case when $A = \text{Op}_h(a)$ for $a \in S^{2k+1}_2(T^*\mathbb{R}^n)$, $a \geq 0$, and Op$_h$ given by (E.1.12). Then (E.1.41) follows from the sharp Gårding inequality on $\mathbb{R}^n$, see [Zw12, Theorem 9.11].

As an application of Proposition E.23, we give the following improved bound on norms of pseudodifferential operators:

**PROPOSITION E.24.** Let $A \in \Psi^0_h(M)$ be compactly supported and fix some smooth density on $M$. Then there is a constant $C$ depending on $A$ such that

$$(E.1.42) \quad \|A\|_{L^2 \to L^2} \leq \sup_{(x,\xi) \in T^*M} |\sigma_h(A)(x,\xi)| + C \|\chi\|^2_{H^k(M)}.$$

**REMARK.** In fact, we have (see [Zw12, Theorem 13.13])

$$\|A\|_{L^2 \to L^2} = \sup_{(x,\xi) \in T^*M} |\sigma_h(A)(x,\xi)| + O(h).$$

**Proof.** Let $C_0 := \sup_{(x,\xi) \in T^*M} |\sigma_h(A)(x,\xi)|$. Take $\chi \in C^\infty_0(M)$ such that $|\chi| \leq 1$ everywhere and $\chi = 1$ near supp $\sigma_h(A)$. Then

$$\sigma_h((C_0\chi)^2 - A^*A) = |C_0\chi|^2 - |\sigma_h(A)|^2 \geq 0$$

By Proposition E.23, we have for each $u \in L^2(M)$,

$$C_0^2\|\chi u\|^2_{L^2} - \|Au\|^2_{L^2} = \langle (|C_0\chi|^2 - A^*A)u, u \rangle_{L^2} \geq -Ch\|u\|^2_{L^2}$$

and thus, using the fact that $\|\chi u\|_{L^2} \leq \|u\|_{L^2}$,

$$\|Au\|^2_{L^2} \leq C_0^2\|u\|^2_{L^2} + Ch\|u\|^2_{L^2},$$

which implies (E.1.42).
We finally show a convergence statement in strong operator topology. It will be used in the proof of Lemma E.47.

**Lemma E.25.** Let $\text{Op}_h^M$ be a quantization procedure from Proposition E.14. Assume that $a_j \in S^k(T^*M)$ is a sequence bounded uniformly in $j$ in the class $S^k_{1,0}$ and $a_j \to 0$ in the class $S^{k+1}_{1,0}$. Fix $h > 0$, $s \in \mathbb{R}$, $u \in H^s_{\text{loc}}(M)$, and $\chi \in C^\infty_0(M)$. Then

\[ \|\chi \text{Op}_h^M(a_j)u\|_{H^{s-k}_h} \to 0 \quad \text{as} \quad j \to \infty. \]

**Proof.** The quantization procedure (E.1.28) produces properly supported operators, and hence we may reduce to the case when $M = \mathbb{R}^n$, $\chi = 1$, $u \in H^s(\mathbb{R}^n)$, $\text{Op}_h^M = \text{Op}_h$ is given by (E.1.12), and $a_j \in S^k(T^*\mathbb{R}^n)$ are bounded uniformly in the class $S^k_{1,0}$ and converge to zero in the class $S^{k+1}_{1,0}$. Take $\psi \in C^\infty_0(\mathbb{R}^n)$ such that $\psi(0) = 1$, and for $\varepsilon > 0$ consider the operator

$\psi(\varepsilon h D_x) = \text{Op}_h(\psi\varepsilon), \quad \psi\varepsilon(x, \xi) = \psi(\varepsilon \xi).$

Put $u_\varepsilon := \psi(\varepsilon h D_x)u$. Since $a_j$ is bounded in $S^k_{1,0}$ uniformly in $j$, we have

$\|\text{Op}_h(a_j)(u - u_\varepsilon)\|_{H^{s-k}_h(\mathbb{R}^n)} \leq C\|u_\varepsilon - u\|_{H^s(\mathbb{R}^n)}$

where the constant $C$ does not depend on $\varepsilon$ and $j$. By the Dominated Convergence Theorem applied to the Fourier transform of $u$ we see that $u_\varepsilon \to u$ in $H^s_h(\mathbb{R}^n)$. Therefore,

\[ \lim_{\varepsilon \to 0} \limsup_{j \to \infty} \|\text{Op}_h(a_j)(u_\varepsilon - u)\|_{H^{s-k}_h(\mathbb{R}^n)} = 0. \]

Since $a_j$ converges to zero in $S^{k+1}_{1,0}$,

$\|\text{Op}_h(a_j)\|_{H^{s+1}_h(\mathbb{R}^n) \to H^{s-k}_h(\mathbb{R}^n)} \to 0, \quad j \to \infty.$

As $\psi\varepsilon$ is compactly supported, we have $u_\varepsilon \in H^{s+1}(\mathbb{R}^n)$ and therefore, for each $\varepsilon$

\[ \limsup_{j \to \infty} \|\text{Op}_h(a_j)u_\varepsilon\|_{H^{s-k}_h(\mathbb{R}^n)} = 0. \]

Combining (E.1.44) and (E.1.45), we get

\[ \lim_{\varepsilon \to 0} \limsup_{j \to \infty} \|\text{Op}_h(a_j)u\|_{H^{s-k}_h(\mathbb{R}^n)} = 0 \]

which implies (E.1.43). \qed
E.2. WAVEFRONT SETS AND ELLIPTICITY

We now define semiclassical wavefront sets, which consist of points in the phase space $T^*M$ where a pseudodifferential operator or a family of distributions is not $\mathcal{O}(h^\infty)$. To handle in a uniform way the case of large values of $\xi$, we embed wavefront sets into the fiber-radially compactified cotangent bundle $T^*M$ introduced in §E.1.2.

E.2.1. Wavefront sets of pseudodifferential operators.

**DEFINITION E.26.** Let $a \in S^k_h(M)$. We say that $a = O(h^\infty)$ near some point $(x_0, \xi_0) \in T^*M$, if there exists a neighbourhood $W$ of $(x_0, \xi_0)$ in $T^*M$ such that for all multiindices $\alpha, \beta$,

$$\partial^\alpha_x \partial^\beta_\xi a(x, \xi) = O(h^\infty \langle \xi \rangle^{-\infty})$$

$$(x, \xi) \in W.$$

**DEFINITION E.27.** Let $A \in \Psi^k_h(M)$. Define the wavefront set

$$\text{WF}_h(A) \subset T^*M$$

as follows: a point $(x_0, \xi_0) \in T^*M$ does not lie in $\text{WF}_h(A)$ if and only if for each cutoff chart $(\varphi, \chi)$ such that $x_0$ lies in the domain of $\varphi$, we have $a_{\varphi, \chi} = O(h^\infty)$ near $\widetilde{\varphi}(x_0, \xi_0)$, where $a_{\varphi, \chi}$ is defined by (E.1.22) and $\widetilde{\varphi}$ is defined by (E.1.19).

It follows from (E.1.23) and Proposition E.10 that if $a \in S^k_h(M)$ is $O(h^\infty)$ near some $(x_0, \xi_0) \in T^*M$ and $\text{Op}_h^M$ is a quantization procedure from Proposition E.14, then $(x_0, \xi_0) \notin \text{WF}_h(\text{Op}_h^M(a))$. Conversely, if $A \in \Psi^k_h(M)$ and $a$ is the symbol constructed after (E.1.33), then $(x_0, \xi_0) \notin \text{WF}_h(A)$ implies that $a = O(h^\infty)$ near $(x_0, \xi_0)$.

The set $\text{WF}_h(A)$ is closed in $T^*M$. We have $\text{WF}_h(A) = \emptyset$ if and only if $A \in h^\infty \Psi^{-\infty}$ and $\text{supp} \sigma_h(A) \subset \text{WF}_h(A)$. Moreover, we have the following corollaries of (E.1.15)–(E.1.17):

(E.2.1) $\text{WF}_h(A + B) \subset \text{WF}_h(A) \cup \text{WF}_h(B),$

(E.2.2) $\text{WF}_h(AB) \subset \text{WF}_h(A) \cap \text{WF}_h(B),$

(E.2.3) $\text{WF}_h(A^*) = \text{WF}_h(A)$.

We give two more useful definitions involving wavefront sets:

**DEFINITION E.28.** Let $A, B \in \Psi^k_h(M)$ and $U \subset T^*M$. We say that $A = B + O(h^\infty)$ microlocally on $U$ (if $U$ is open) or $A = B + O(h^\infty)$ microlocally near $U$ (if $U$ is closed), if $\text{WF}_h(A - B) \cap U = \emptyset$. 

DEFINITION E.29. A compactly supported operator $A \in \Psi^k_h(M)$ is called **compactly microlocalized** if $WF_h(A)$ is a compact subset of $T^*M$. We denote the class of such operators by $\Psi^\text{comp}_h(M)$; note that $\Psi^\text{comp}_h(M) \subset \Psi^\ell_h(M)$ for all $\ell$.

E.2.2. Ellipticity.

DEFINITION E.30. Let $A \in \Psi^k_h(M)$ and $(x_0, \xi_0) \in T^*M$. We say that $A$ is elliptic at $(x_0, \xi_0)$ in the class $\Psi^k_h$, if $\langle \xi \rangle^{-k}\sigma_h(A)(x_0, \xi_0) \neq 0$. Denote by $\text{ell}_h(A)$ the set of all points at which $A$ is elliptic.

Note that $\text{ell}_h(A)$ is an open subset of $T^*M$. The significance of ellipticity comes from our ability to invert $A$ microlocally near its elliptic points:

PROPOSITION E.31. Let $A \in \Psi^s_h(M)$, $B \in \Psi^k_h(M)$ be properly supported and satisfy $WF_h(A) \subset \text{ell}_h(B)$. Then there exist properly supported $Q, Q' \in \Psi^{s-k}_h(M)$ such that

$$A = BQ + O(h^\infty)_{\Psi^{-\infty}} = Q'B + O(h^\infty)_{\Psi^{-\infty}}.$$  

Proof. We construct $Q$; the operator $Q'$ is constructed similarly. By considering $\langle \xi \rangle^{-s}\sigma_h(A)$ and $\langle \xi \rangle^{-k}\sigma_h(B)$ as smooth functions on $T^*M$, we see that

$$q_0 := \frac{\sigma_h(A)}{\sigma_h(B)} \in S^{s-k}(M).$$

Fix a quantization procedure $\text{Op}_h = \text{Op}_h^M$ from Proposition E.14. Then

$$A = B\text{Op}_h(q_0) + hA_1, \quad A_1 \in \Psi^{s-1}_h(M), \quad WF_h(A_1) \subset \text{ell}_h(B).$$

Arguing by induction, we construct a sequence of operators $A_j \in \Psi^{s-j}_h(M)$ and symbols $q_j \in S^{s-k-j}(M)$, $j \in \mathbb{N}_0$, such that $A_0 = A$ and

$$A_j = B\text{Op}_h(q_j) + hA_{j+1}, \quad WF_h(A_j) \subset \text{ell}_h(B).$$

To obtain (E.2.4), it remains to take

$$Q := \text{Op}_h(q), \quad q \sim \sum_{j=0}^{\infty} h^j q_j,$$

where the asymptotic sum is understood in the sense of (E.1.7). \qed

An immediate corollary is the following

THEOREM E.32 (Elliptic estimate). Let $P \in \Psi^k_h(M)$ be properly supported and $p = \sigma_h(P)$. Assume that $A, B_1 \in \Psi^0_h(M)$ are compactly supported and

$$WF_h(A) \subset \text{ell}_h(P) \cap \text{ell}_h(B_1).$$
Fix $s, N \in \mathbb{R}$. Then for each $u \in \mathcal{D}'(M)$, if $B_1 Pu \in H^s_{comp}$, then $Au \in H^s_{comp}$ and
\[
\|Au\|_{H^s_h} \leq C \|B_1 Pu\|_{H^{s-k}_h} + \mathcal{O}(h^\infty) \|\chi u\|_{H^{-N}_h}.
\]
where the constant $C$ and the function $\chi \in C^\infty_0(M)$ do not depend on $u$ or $h$.

**REMARK.** The $\mathcal{O}(h^\infty)$ remainder term cannot be removed – see Exercise E.9.

**Proof.** We have $WF_h(A) \subset \text{ell}_h(B_1 P)$. By Proposition E.31, there exists $Q \in \Psi^{-k}_h(M)$ such that
\[
A = QB_1 P + R, \quad R \in h^\infty \Psi^{-\infty}.
\]
Since $Q, P$ are properly supported and $A, B_1$ are compactly supported, $R$ is compactly supported; therefore, $R = R\chi$ for some $\chi \in C^\infty_0(M)$.

Let $u \in \mathcal{D}'(M)$ and assume that $B_1 Pu \in H^{s-k}$. By Proposition E.22, $QB_1 Pu \in H^s$ and $\|QB_1 Pu\|_{H^s_h} \leq C \|B_1 Pu\|_{H^{s-k}_h}$. We also have $Ru \in H^s$ and $\|Ru\|_{H^s_h} = \mathcal{O}(h^\infty) \|\chi u\|_{H^{-N}_h}$ for all $N$. Adding these estimates together, we get (E.2.5). □

Another application of the elliptic parametrix construction is quantization of pseudodifferential partitions of unity:

**PROPOSITION E.33.** Assume that $\chi_1, \ldots, \chi_m \in S^0(T^*M)$ and
\[
\sum_{j=1}^m \chi_j = 1 \quad \text{near } V,
\]
for some closed set $V \subset \overline{T}^* M$. Then there exist properly supported operators $X_1, \ldots, X_n \in \Psi^0_h(M)$ such that $\sigma_h(X_j) = \chi_j$, $WF_h(X_j) \subset \text{supp } \chi_j$, and in the sense of Definition E.28,
\[
\sum_{j=1}^m X_j = I + \mathcal{O}(h^\infty) \quad \text{microlocally near } V.
\]
If moreover $\chi_j = \psi_j^2$ for some $\psi_j \in S^0(T^*M; \mathbb{R})$, then we can take $X_j = Y_j^2$ for some $Y_j \in \Psi^0_h(M)$ with $\text{supp } Y_j \subset \text{supp } \chi_j$, $\sigma_h(Y_j) = \psi_j$, and $Y_j^* = Y_j$.

**Proof.** Let the quantization procedure $\text{Op}_h$ be given by (E.1.28). The operator $\text{Op}_h(1) \in \Psi^0_h(M)$ is everywhere elliptic and hence by Proposition E.31, there exists $Z \in \Psi^0_h(M)$ such that
\[
Z \text{Op}_h(1) = I + \mathcal{O}(h^\infty)\Psi^{-\infty}.
\]
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Put \( X_j := Z \mathcal{O}_h(\chi_j) \). Then \( \text{WF}_h(X_j) \subset \text{supp} \chi_j \) and

\[
V \cap \text{WF}_h \left( I - \sum_j X_j \right) = V \cap \text{WF}_h \left( Z \mathcal{O}_h \left( 1 - \sum_j \chi_j \right) \right) = \emptyset,
\]

finishing the proof.

We now consider the case of \( \chi_j = \psi_j^2 \). We define a quantization procedure which sends real-valued functions to self-adjoint operators:

\[
a \mapsto \mathcal{O}_h'(a) := \mathcal{O}_h(a) + \mathcal{O}_h(a)^*.
\]

Put \( Y_j^0 := \mathcal{O}_h'(\psi_j) \), then \((Y_j^0)^* = Y_j^0\) and

\[
\sum_{j=1}^m (Y_j^0)^2 = I + hR_1 + \mathcal{O}(h^\infty) \quad \text{microlocally near } V
\]

for some \( R_1 \in \Psi^{-1}_h(M) \) with \( R_1^* = R_1 \). Put

\[
Y_j^1 := \mathcal{O}_h'(\psi_j^1), \quad \psi_j^1 := -\frac{1}{2}\psi_j\sigma_h(R_1),
\]

then \((Y_j^1)^* = Y_j^1\) and for some \( R_2 \in \Psi^{-2}_h(M) \),

\[
\sum_{j=1}^m (Y_j^0 + hY_j^1)^2 = I + h^2R_2 + \mathcal{O}(h^\infty) \quad \text{microlocally near } V.
\]

The proof is then finished by iteration and taking an asymptotic sum, similarly to Proposition E.31. □

Combining Proposition E.33 with partitions of unity in \( \mathcal{T}'M \), we obtain

**Proposition E.34.** Assume that \( V \subset \mathcal{T}'M \) is compact and

\[
U_1, \ldots, U_m \subset \mathcal{T}'M
\]

is an open cover of \( V \). Then there exist compactly supported \( X_1, \ldots, X_m \in \Psi^0_h(M) \) such that

\[
\text{WF}_h(X_j) \subset U_j; \quad \sum_{j=1}^m X_j = I + \mathcal{O}(h^\infty) \quad \text{microlocally near } V.
\]

Finally, we give a microlocal version of sharp Gårding inequality which will be used in the positive commutator estimates of §E.5:

**Proposition E.35.** Assume that \( A \in \Psi^{2s}_h(M), B, B_1 \in \Psi^0_h(M) \) are compactly supported (see §A.7) and

\[
\langle \xi \rangle^{-2s} \text{Re } \sigma_h(A) \geq 0 \quad \text{in a neighbourhood of } \mathcal{T}'M \setminus \ell_h(B),
\]

\[
\langle \xi \rangle^{-2s} \text{Re } \sigma_h(A) \geq 0 \quad \text{in a neighbourhood of } \mathcal{T}'M \setminus \ell_h(B),
\]

\[
\langle \xi \rangle^{-2s} \text{Re } \sigma_h(A) \geq 0 \quad \text{in a neighbourhood of } \mathcal{T}'M \setminus \ell_h(B),
\]

\[
\langle \xi \rangle^{-2s} \text{Re } \sigma_h(A) \geq 0 \quad \text{in a neighbourhood of } \mathcal{T}'M \setminus \ell_h(B),
\]
Then there exists a constant $C$ and $\chi \in C^\infty_0(M)$ such that for each $N$ and all $u \in C^\infty(M)$,

$$\text{Re} \langle Au, u \rangle \geq -C \| Bu \|^2_{H^2_h} - Ch\| B_1u \|^2_{H^{s-1/2}_h} - O(h^\infty) \| \chi u \|^2_{H^{-N}_h}.$$ 

Proof. The closure $W$ of the set $\{ \langle \xi \rangle^{-2s} \text{Re} \sigma_h(A) < 0 \} \subset T^*M$ is compact and contained in $\text{ell}_h(B) \cap \text{ell}_h(B_1)$. Take compactly supported $B_2 \in \Psi^*_h(M)$ such that

$$W \subset \text{ell}_h(B_2), \quad WF_h(B_2) \subset \text{ell}_h(B_1).$$ 

Then there exists a constant $C_0 > 0$ such that

$$\langle \xi \rangle^{-2s} (\text{Re} \sigma_h(A) + C_0 |\sigma_h(B_2B)|)^2 \geq 0 \quad \text{everywhere.}$$

Consequently for

$$Y := A + C_0 (B_2B)^* (B_2B) \in \Psi^{2s}_h(M),$$
we have

$$\langle \xi \rangle^{-2s} \text{Re} \sigma_h(Y) \geq 0.$$ 

In addition $WF_h(Y) \subset \text{ell}_h(B_1)$. By Proposition E.34, there exists compactly supported $X \in \Psi^0_h(M)$ such that

$$WF_h(Y) \cap WF_h(I - X) = \emptyset, \quad WF_h(X) \subset \text{ell}_h(B_1).$$

By Proposition E.23,

$$\text{Re} \langle Yu, u \rangle = \text{Re} \langle YXu, Xu \rangle + O(h^\infty) \| \chi u \|^2_{H^{-N}_h}$$

$$\geq -Ch\| Xu \|^2_{H^{s-1/2}_h} + O(h^\infty) \| \chi u \|^2_{H^{-N}_h}.$$ 

To finish the proof, it remains to note that

$$\text{Re} \langle Yu, u \rangle = \text{Re} \langle Au, u \rangle + C_0 \| B_2Bu \|^2_{L^2}, \quad \| B_2Bu \|_{L^2} \leq C \| Bu \|_{H^s_h},$$
and

$$\| Xu \|^2_{H^{s-1/2}_h} \leq C \| B_1u \|^2_{H^{s-1/2}_h} + O(h^\infty) \| \chi u \|^2_{H^{-N}_h}$$
where the last statement follows by Theorem E.32. \hfill \Box

### E.2.3. Wavefront sets of distributions

We now define semiclassical wavefront sets of $h$-dependent families of distributions and operators. For that, we need to impose polynomial growth assumptions:

**DEFINITION E.36.** 1. A family of distributions $u = u(h) \in \mathcal{D}'(M)$, $h \in (0, h_0)$, is called $h$-tempered, if for each $\chi \in C^\infty_0(M)$, there exist constants $C$ and $N$ such that $\| \chi u \|^2_{H^{-N}_h} \leq Ch^{-N}$.

2. A family of operators $B(h) : C^\infty_0(M_2) \to \mathcal{D}'(M_1)$ is called $h$-tempered, if the Schwartz kernels $K_{B(h)}$ form an $h$-tempered family in $\mathcal{D}'(M_1 \times M_2)$. 


The definition of wavefront set below is motivated by the following definition of the support of a distribution \( u \in \mathcal{D}'(M) \): a point \( x \in M \) does not lie in \( \text{supp} \, u \) if and only if there exists a neighbourhood \( U \) of \( x \) such that for each \( \chi \in C_0^\infty(M) \) with \( \text{supp} \, \chi \subset U \), we have \( \chi u = 0 \).

**Definition E.37.** 1. Let \( u = u(h) \in \mathcal{D}'(M) \) be an \( h \)-tempered family. The **semiclassical wavefront set** \( \text{WF}_h(u) \subset T^*M \) is defined as follows:

A point \( (x_0, \xi_0) \in T^*M \) does not lie in \( \text{WF}_h(u) \) if and only if there exists a neighbourhood \( U \) of \( (x_0, \xi_0) \) in \( T^*M \) such that for every properly supported \( A \in \Psi^k_h(M) \) with \( \text{WF}_h(A) \subset U \),

\[
Au = O(h^\infty)C^\infty.
\]

2. Let \( B(h) : C_0^\infty(M_2) \to \mathcal{D}'(M_1) \) be an \( h \)-tempered family of operators. Then \( \text{WF}'_h(B) \subset T^*(M_1 \times M_2) \) is defined as follows:

\[
\text{WF}'_h(B) = \{(x, \xi, y, \eta) : (x, \xi, y, -\eta) \in \text{WF}_h(\mathcal{K}_B)\}
\]

where \( \mathcal{K}_B(x, y) \in \mathcal{D}'(M_1 \times M_2) \) is the Schwartz kernel of \( B \).

Note that \( \text{WF}_h(u) \subset T^*M \) and \( \text{WF}_h(B) \subset T^*(M_1 \times M_2) \) are closed subsets. We have \( \text{WF}_h(u) = \emptyset \) if and only if \( u = O(h^\infty)C^\infty \) and \( \text{WF}_h(B) = \emptyset \) if and only if \( B = O(h^\infty)_{y^{-\infty}} \).

The switch of sign of \( \xi \) in the definition of \( \text{WF}'_h(B) \) is motivated by the identity

\[
\text{WF}_h(B^*)' = \{(y, \eta, x, \xi) : (x, \xi, y, \eta) \in \text{WF}_h(B)\}.
\]

This follows from the fact that \( \mathcal{K}_{B^*}(y, x) = \overline{\mathcal{K}_B(x, y)} \) and the following formula for the wavefront set of the complex conjugate:

\[
\text{WF}_h(\overline{u}) = \{(x, -\xi) : (x, \xi) \in \text{WF}_h(u)\}.
\]

To see (E.2.7), we use Definition E.37 and the following identity for the quantization formula (E.1.12):

\[
\text{Op}_h(a)u = \text{Op}_h(a')\overline{u}, \quad a'(x, \xi) = a(x, -\xi).
\]

A fundamental example of wavefront set calculation is given by

**Proposition E.38.** Assume that

\[
\varphi(x, \theta) \in C^\infty(U; \mathbb{R}), \quad U \subset M_x \times \mathbb{R}_\theta^m
\]

is a smooth function and

\[
a(x, \theta; h) \in C_0^\infty(U)
\]

is supported inside an \( h \)-independent compact set \( K_a \subset U \) and has all derivatives bounded uniformly in \( h \). Then the family of smooth functions

\[
u(x; h) := \int_{\mathbb{R}_\theta^m} e^{i \frac{1}{h} \varphi(x, \theta)} a(x, \theta; h) \, d\theta, \quad x \in M,
\]
is $h$-tempered and satisfies
\[ \WF_h(u) \subset \{(x, \partial_x \varphi(x, \theta)) : (x, \theta) \in K_a, \partial_y \varphi(x, \theta) = 0 \}. \]

**Proof.** By differentiation under the integral sign, we see immediately that $u$ is $h$-tempered, in fact each of its $C^\infty(M)$ seminorms is bounded polynomially in $h$. By a partition of unity applied to $a$, we reduce to the case when $M = \mathbb{R}^n$. Then it suffices to prove that for each $b \in \mathcal{S}^k(T^*\mathbb{R}^n)$ such that
\[ \text{(E.2.10)} \quad \text{supp } b \cap \{(x, \partial_x \varphi(x, \theta)) : (x, \theta) \in K_a, \partial_y \varphi(x, \theta) = 0 \} = \emptyset, \]
we have $\text{Op}_h(b)u = \mathcal{O}(h^\infty)_{C^\infty}$. We write
\[ \text{Op}_h(b)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{i\Phi/h} b(x, \xi) a(y, \theta; h) d\theta dy d\xi, \]
\[ \Phi = \langle x - y, \xi \rangle + \varphi(y, \theta) \]
The stationary points of the phase $\Phi$ in the $(\theta, y, \xi)$ variables are given by
\[ x = y, \quad \partial_y \varphi(x, \theta) = 0, \quad \xi = \partial_x \varphi(x, \theta). \]
By (E.2.10), all stationary points lie outside of the support of the amplitude $b(x, \xi) a(y, \theta; h)$. If $b(x, \xi)$ is compactly supported in $x, \xi$, then the method of nonstationary phase (E.1.9) gives $\text{Op}_h(b)u = \mathcal{O}(h^\infty)_{C^\infty}$. For the case of general $b$, the part of the integral coming from large $\xi$ is $\mathcal{O}(h^\infty)$ by repeated integration by parts in $y$, using the inequality $|\partial_y \Phi|^{-1} \leq C|\xi|^{-1}$ valid when $|\xi| \gg 1$. \hfill $\square$

We next study the behaviour of wavefront sets under pseudodifferential operators:

**PROPOSITION E.39.** Let $u(h) \in \mathcal{D}'(M)$ be $h$-tempered and $A \in \Psi^0_h(M)$ properly supported. Then
\[ \text{(E.2.11)} \quad \WF_h(Au) \subset \WF_h(A) \cap \WF_h(u), \]
\[ \text{(E.2.12)} \quad \WF_h(u) \subset \WF_h(Au) \cup (T^*M \setminus \ell h(A)). \]

**Proof.** 1. To see (E.2.11), take first $(x_0, \xi_0) \notin \WF_h(A)$. Define the open set $U := T^*M \setminus \WF_h(A)$. Then for each properly supported $B \in \Psi^0_h(M)$ such that $\WF_h(B) \subset U$, we have $BA = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$ and thus $B(Au) = \mathcal{O}(h^\infty)_{C^\infty}$ since $u$ is $h$-tempered; this implies that $(x_0, \xi_0) \notin \WF_h(Au)$.

Now, assume instead that $(x_0, \xi_0) \notin \WF_h(u)$. Then there exists an open neighbourhood $V$ of $(x_0, \xi_0)$ in $T^*M$ such that for each $B \in \Psi^0_h(M)$ with $\WF_h(B) \subset V$, we have $Bu = \mathcal{O}(h^\infty)_{C^\infty}$. Since $\WF_h(BA) \subset V$, we have $BAu = \mathcal{O}(h^\infty)_{C^\infty}$ as well, which implies that $(x_0, \xi_0) \notin \WF_h(Au)$. This finishes the proof of (E.2.11).
2. To prove (E.2.12), take 

\((x_0, \xi_0) \in \text{ell}_h(A) \setminus \text{WF}_h(Au)\).

Take an open neighbourhood \(U \subset T^*M\) of \((x_0, \xi_0)\) and properly supported \(A_1 \in \Psi^0_k(M)\) such that 

\(U \subset \text{ell}_h(A_1A), \quad \text{WF}_h(A_1) \cap \text{WF}_h(Au) = \emptyset.\)

Let \(B \in \Psi^k(M)\) be properly supported and \(\text{WF}_h(B) \subset U\). By Proposition E.31, there exists properly supported \(Q \in \Psi^{k-s}(M)\) such that 

\(B = QA_1A + \mathcal{O}(h^\infty)\).

Then \(Bu = \mathcal{O}(h^\infty)C^\infty\), as follows from (E.2.11) and the fact that 

\(\text{WF}_h(QA_1) \cap \text{WF}_h(Au) = \emptyset.\)

Therefore, \((x_0, \xi_0) \notin \text{WF}_h(u)\) as required. \(\square\)

We give the analog of Proposition E.39 for operators which we localize inside the cotangent bundle:

**Proposition E.40.** Let \(B(h) : C^\infty_0(M_2) \to \mathcal{D}'(M_1)\) be an \(h\)-tempered family of operators and \(A_1 \in \Psi^0_k(M_1), A_2 \in \Psi^0_k(M_2)\) be properly supported. Assume moreover that 

\(|WF_h(A_1) \cap \partial T^* M = WF_h(A_2) \cap \partial T^* M' = \emptyset.\)

Then:

1. \(WF'_h(A_1BA_2) \subset WF'_h(B) \cap (WF_h(A_1) \times WF_h(A_2)).\)
2. \(WF'_h(B) \cap (\text{ell}_h(A_1) \times \text{ell}_h(A_2)) \subset WF'(A_1BA_2).\)

**Proof.** The Schwartz kernel of \(A_1BA_2\) is given by 

\(K_{A_1BA_2}(x,y) = (A_1 \otimes A_2^T)K_B(x,y)\)

where \(A_1 \otimes A_2^T : \mathcal{D}'(M_1 \times M_2) \to \mathcal{D}'(M_1 \times M_2)\) is the tensor product of \(A_1\) and the operator 

\(A_2^T : \mathcal{D}'(M_2) \to \mathcal{D}'(M_2), \quad \overline{A_2^Tv} = A_2^T\overline{v}.\)

By (E.2.8), we see that \(A_2^T \in \Psi^0_k(M_2)\) and \(WF_h(A_2^T), ell_h(A_2^T)\) are obtained from \(WF_h(A_2), ell_h(A_2)\) by flipping the sign of \(\eta\).

Since \(WF_h(A_1), WF_h(A_2)\) do not intersect the fiber infinity, and the quantization procedure (E.1.12) satisfies 

\(Op_h(a \otimes b) = Op_h(a) \otimes Op_h(b), \quad a \in C^\infty_c(T^*R^n), b \in C^\infty_c(T^*R^n),\)

we have \(A_1 \otimes A_2^T \in \Psi^0_k(M_1 \times M_2),\)

\(WF_h(A_1 \otimes A_2^T) = \{(x, \xi, y, -\eta) : (x, \xi) \in WF_h(A_1), (y, \eta) \in WF_h(A_2)\}\)
Proposition E.39 to the Schwartz kernel $K$ and a similar statement is true for the elliptic set. It remains to apply Proposition E.40. □

As an application of Propositions E.39 and E.40, we show composition formulae for wavefront sets:

**Proposition E.41.** 1. Let $B(h) : C^\infty_0(M_2) \to \mathcal{D}'(M_1)$ and $u(h) \in \mathcal{D}'(M_2)$ be $h$-tempered, and $Q \in \Psi^\text{comp}_h(M_2)$. Then

$$\WF_h(BQu) \cap T^*M_1 \subset \{(x, \xi) : \exists (y, \eta) \in \WF_h(u) \cap \WF(Q) : (x, \xi, y, \eta) \in \WF_h(B)\}. \tag{E.2.13}$$

2. Let $B_1(h) : C^\infty_0(M_2) \to \mathcal{D}'(M_1)$ and $B_2(h) : C^\infty_0(M_3) \to \mathcal{D}'(M_2)$ be $h$-tempered, and $Q \in \Psi^\text{comp}_h(M_2)$. Then

$$\WF'_h(B_1QB_2) \cap T^*(M_1 \times M_3) \subset \{(x, \xi, y, \eta) : \exists (y, \eta) \in \WF'_h(Q) : (x, \xi, y, \eta) \in \WF'_h(B_1), (y, \eta, z, \eta) \in \WF'_h(B_2)\}. \tag{E.2.14}$$

*Proof.* We give a proof of part 2; part 1 is proved in a similar way. Assume that $(x_0, \xi_0, z_0, \zeta_0) \in T^*(M_1 \times M_3)$ does not lie in the right-hand side of (E.2.14). This implies that $V_1 \cap \WF_h(Q) = \emptyset$, where

$$V_1 = \{(y, \eta) \in T^*M' : (x_0, \xi_0, y, \eta) \in \WF'_h(B_1)\},$$

$$V_2 = \{(y, \eta) \in T^*M' : (y, \eta, z_0, \zeta_0) \in \WF'_h(B_2)\}$$

are closed sets. By quantizing a partition of unity, we write

$$Q = Q_1 + Q_2, \quad Q_j \in \Psi^\text{comp}_h(M_2), \quad \WF_h(Q_j) \cap V_j = \emptyset.$$

Then there exist $A_1 \in \Psi^\text{comp}_h(M_1), A_2 \in \Psi^\text{comp}_h(M_3)$ such that

$$(x_0, \xi_0) \in \ell_h(A_1), \quad (\WF_h(A_1) \times \WF_h(Q_1)) \cap \WF'_h(B_1) = \emptyset;$$

$$(z_0, \zeta_0) \in \ell_h(A_2), \quad (\WF_h(Q_2) \times \WF_h(A_2)) \cap \WF'_h(B_2) = \emptyset.$$

By part 1 of Proposition E.40, we have $\WF'_h(A_1B_1Q_1) = \emptyset$ and thus $A_1B_1Q_1 = O(h^\infty)_{\Psi^{-\infty}}$. Similarly $Q_2B_2A_2 = O(h^\infty)_{\Psi^{-\infty}}$. Then

$$A_1B_1QB_2A_2 = (A_1B_1Q_1)B_2A_2 + A_1B_1(Q_2B_2A_2) = O(h^\infty)_{\Psi^{-\infty}}.$$

By part 2 of Proposition E.40, we have $(x_0, \xi_0, z_0, \zeta_0) \notin \WF'_h(B_1QB_2)$ as required. □
E.3. PROPAGATORS AND EGOROV’S THEOREM

**THEOREM E.42 (Egorov’s theorem up to Ehrenfest time).** Suppose that \( m \geq 1 \) is an order function \( P = \text{Op}(p) \), \( p \in S(m) \), \( p = p_0 + O_{S(m)}(h^2) \), and \( p_0 \geq m/C - C \), for some \( C > 0 \).

Suppose also that \( a \in S \) satisfies
\[
\text{supp } a \subset \{ (x, \xi) : p_0(x, \xi) \leq R \},
\]
for some \( R > 0 \), and define
\[
\Gamma_R := \lim_{t \to \infty} \frac{1}{t} \sup_{p_0 \leq R} \log \| \partial \varphi_t \|, \quad \varphi_t = \exp tH_{p_0}.
\]

For any \( \gamma > \Gamma_R, T \geq 0 \), and \( \delta \in [0, 1/2) \), if
\[
0 \leq t \leq T + \frac{\delta}{\gamma} \log \frac{1}{h},
\]
then
\[
e^{itP/h}a^w(x, hD)e^{-itP/h} = a^w_t(x, hD),
\]
\[
a_t \in S_\delta(m^{-\infty}), \quad a_t - \varphi^*_t a \in h^{2-3\delta}S_\delta(m^{-\infty}),
\]
with symbolic estimates uniform in \( t \).

**INTERPRETATION.** This theorem estimates the length time on which we know that the classical/quantum correspondence remains valid. These correspondences refer to the correspondence between classical and quantum flows:
\[
t \mapsto e^{itP/h}a^w(x, hD)e^{-itP/h}
\]
is the quantum evolution of the quantum observable \( a^w(x, hD) \).
\[
t \mapsto \varphi^*_t a
\]
is the classical evolution of the classical observable \( a(x, \xi) \).

The statement that \( a_t - \varphi^*_t a = O(h^{2-3\delta}) \) means that the quantum evolution of \( a^w \) given by the conjugation with \( \exp(-itP/h) \) is well approximated by the classical evolution up to the time \( \delta/\gamma \log(1/h) \). Till that time we also know that the quantum evolved operator is a quantization of a slightly exotic (\( \delta > 0 \)) classical observable \( a_t \). When we allow \( p = p_0 + O_{S(m)}(h^2) \) then the error becomes \( O(h^{1-4}) \). The assumption \( p = p_0 + O_{S(m)}(h^2) \) with \( p_0 \) independent of \( h \) is natural as the term \( p_0 \) is (under further assumptions) invariantly defined up to \( O(h^2) \).
E.4. SEMICLASSICAL DEFECT MEASURES

In this section, we review the concept of semiclassical defect measures. We refer the reader to [Zw12, Chapter 5] for a comprehensive introduction. We use the class $\Psi^\text{comp}_h$ introduced in Definition E.29.

DEFINITION E.43 (Semiclassical measures). Let $M$ be a manifold with a fixed volume form and consider sequences $h_j \to 0$, $u_j \in \mathcal{D}'(M)$.

Let $\mu$ be a nonnegative Borel measure on $T^* M$. We say that $u_j$ converges to $\mu$ in the sense of semiclassical measures, if for each $A = A(h_j) \in \Psi^\text{comp}_h(M)$,

(E.4.1) \[ \langle A(h_j)u_j, u_j \rangle_{L^2(M)} \to \int_{T^* M} \sigma_h(A) d\mu. \]

THEOREM E.44 (Existence of semiclassical measures). Assume that $h_j \to 0$, $u_j \in \mathcal{D}'(M)$ are sequences such that for some $N$,

(E.4.2) \[ \| \chi u_j \|_{H^{-N}_{h_j}} \leq C \quad \text{for all} \quad \chi \in C^\infty_c(M) \]

with the constant $C$ depending on $\chi$ but not on $j$. Then there exists a subsequence $\{j_k\}$ such that $u_{j_k}$ converges to some measure $\mu$ on $T^* M$.

Proof. TODO cite [Zw12] and generalize the diagonal argument there a bit? \qed

Now, we assume that $u_j$ satisfies (E.4.2) and converges to a measure $\mu$ and

(E.4.3) \[ P(h_j)u_j = f_j \]

where $P = P(h) \in \Psi^h(M)$ is properly supported.

THEOREM E.45 (Support of semiclassical measures). Assume that (E.4.1), (E.4.2), (E.4.3) hold and for some $N$,

(E.4.4) \[ \| \chi f_j \|_{H^{-N}_{h_j}} = o(1) \quad \text{as} \quad j \to \infty, \quad \chi \in C^\infty_c(M). \]

Then the support of the measure $\mu$ is contained in $\{\sigma_h(P) = 0\}$, that is

(E.4.5) \[ \mu(\{\sigma_h(P) \neq 0\}) = 0. \]

Proof. We have for each $A \in \Psi^\text{comp}_h(M)$, by (E.4.2) and (E.4.4),

\[ \langle A P u_j, u_j \rangle_{L^2} = \langle A f_j, u_j \rangle_{L^2} \to 0 \quad \text{as} \quad j \to \infty. \]

On the other hand,

\[ \langle A P u_j, u_j \rangle_{L^2} \to \int_{T^* M} \sigma_h(A) \sigma_h(P) d\mu. \]
E.4. PROPAGATION ESTIMATES

It follows that for each \(a \in C_c^\infty(T^*M)\),
\[
\int_{T^*M} \sigma_h(P)a \, d\mu = 0,
\]
which immediately implies (E.4.5).

\[\square\]

THEOREM E.46 (Semiclassical measures and commutators). Assume that (E.4.1), (E.4.2), (E.4.3) hold and for some \(N\),
\[
\|\chi f_j\|_{H^N_{h_j}} = O(h_j) \quad \text{as } j \to \infty, \quad \chi \in C_c^\infty(M).
\]
Assume also that \(p = \sigma_h(P)\) is real-valued. Then
\[
\int_{T^*M} H_p a \, d\mu \leq C \sup |a| \quad \text{for all } a \in C_c^\infty(T^*M)
\]
for some constant \(C\) independent of \(a\).

Under the stronger assumption that \(\|\chi f_j\| = o(h_j)\), we have
\[
\int_{T^*M} (H_p a + 2ba) \, d\mu = 0 \quad \text{for all } a \in C_c^\infty(T^*M),
\]
where \(b := \sigma_h(h^{-1}\text{Im } P)\).

Proof. 1. It is enough to handle the case of real-valued \(a\). Take \(A \in \Psi^\comp_h(M)\) such that \(a = \sigma_h(A)\) and \(A^* = A\). We compute
\[
\frac{\text{Im} \langle f_j, Au_j \rangle}{h_j} = \frac{\langle APu_j, u_j \rangle - \langle P^*Au_j, u_j \rangle}{2ih_j} - \frac{\langle (\text{Im } P)Au_j, u_j \rangle}{h_j}.
\]
We have \([A, P], \text{Im } P \in h\Psi^\comp_h(M)\) and
\[
\sigma_h(h^{-1}[A, P]) = iH_p a, \quad \sigma_h(h^{-1}(\text{Im } P)A) = ba,
\]
therefore by (E.4.1), the right-hand side of (E.4.9) converges as \(j \to \infty\) to
\[
\int_{T^*M} \frac{H_p a}{2} + ba \, d\mu.
\]

2. By Proposition E.24, and since \(A\) is compactly microlocalized, we have
\[
\|A\|_{H^N_{h_j} \to H^N_{h_j}} \leq C \sup |a| + Ch^{1/2},
\]
therefore by (E.4.6)
\[
\limsup_{j \to \infty} |h_j^{-1} \text{Im} \langle f_j, Au_j \rangle| \leq C \sup |a|
\]
and (E.4.7) follows.

If instead \(\|\chi f_j\| = o(h_j)\), then the left-hand side of (E.4.9) converges to zero and (E.4.8) follows. \[\square\]
E.5. PROPAGATION ESTIMATES

In this section, we consider general equations of the form
\[(E.5.1)\quad Pu = f, \quad u \in H^s_{\text{loc}}(M).\]
Here \(P \in \Psi^k_h(M)\) is a properly supported semiclassical pseudodifferential operator on a manifold \(M\). We use the boldface notation for \(P\) because the corresponding principal symbol can be complex valued:
\[(E.5.2)\quad p := \sigma_h(P) = p - iq, \quad p, q \in S^k(T^*M; \mathbb{R}).\]

Our goal is to prove estimates of the form
\[(E.5.3)\quad \|Au\|_{H^s_h} \leq C \|Bu\|_{H^s_h} + Ch^{-1} \|B_1 f\|_{H^{s-k+1}_h} + O(h^\infty) \|\chi u\|_{H^{-N}_h}\]
where \(A, B, B_1 \in \Psi_0^0(M)\) are compactly supported and satisfy certain dynamical conditions, \(\chi \in C_0^\infty(M)\) is some function depending on the supports of \(A, B, B_1, P\), and \(N\) can be any number.

The estimate (E.5.3) should be compared with the elliptic estimate (E.2.5), which is stronger in the sense that the term \(\|Bu\|\) is absent and a weaker norm \(\|B_1 f\|_{H^{s-k+1}_h}\) is used. However, (E.2.5) only holds when \(\text{WF}_h(A) \subset \text{ell}_h(P)\), while the propagation estimates of this section are valid on the characteristic set \(\{\langle \xi \rangle^{-k} \sigma_h(P) = 0\}\) as well.

In the non-elliptic case, the appearance of the norm \(\|B_1 f\|_{H^{s-k+1}_h}\) on the right-hand side of (E.5.3) shows that a stronger regularity assumption on \(f\) is needed to obtain a propagation estimate. Equation (E.5.1) with \(u \in H^s_{\text{loc}}(M)\) implies only \(f \in H^{s-k}_{\text{loc}}(M)\). In other words \(\|B_1 f\|_{H^{s-k+1}_h}\) might be infinite for \(u \in H^s_{\text{loc}}(M)\).

The following approximation lemma shows that if \(Pu \in H^{s-k+1}_{\text{loc}}(M)\) then it is enough to verify (E.5.3) for the case of smooth functions:

**LEMMA E.47 (Approximation lemma).** Let \(P \in \Psi^k_h(M)\) be properly supported. Assume that
\[(E.5.4)\quad u \in H^s_{\text{loc}}(M), \quad Pu \in H^{s-k+1}_{\text{loc}}(M).\]
Fix \(h > 0, \chi \in C_0^\infty(M)\). Then there is a sequence \(u_j \in C^\infty(M)\) such that
\[(E.5.5)\quad \|\chi(u_j - u)\|_{H^s_h} \to 0, \quad \|\chi(Pu_j - Pu)\|_{H^{s-k+1}_h} \to 0.\]

**Proof.** Assume first that \(u \in H^{s+1}_{\text{loc}}(M)\), so that (E.5.4) is trivially satisfied. Since \(C^\infty(M)\) is dense in the Sobolev space \(H^{s+1}_{\text{loc}}\), there exists a sequence \(u_j \in C^\infty(M)\) such that \(u_j \to u\) in \(H^{s+1}_{\text{loc}}\); it follows that (E.5.5) holds. Therefore, for the general case it suffices to find functions \(u_j\) which are in \(H^{s+1}_{\text{loc}}(M)\) rather than \(C^\infty(M)\).
Let $\text{Op}_h = \text{Op}_h^M$ be a quantization procedure from Proposition E.14. Take $\psi \in C_0^\infty(\mathbb{R})$ which is equal to 1 near the origin and define for $u$ satisfying (E.5.4),

$$u_j := (I - \text{Op}_h(a_j))u, \quad a_j(x, \xi) := 1 - \psi(|\xi|/j).$$

We have $u_j \in H^{s+1}_h(M)$ since $\sigma_h(I - \text{Op}_h(a_j)) = \psi(|\xi|/j) \in S^{-1}(M)$ and thus $I - \text{Op}_h(a_j) \in \Psi^{-1}_h(M)$.

A direct calculation shows that the symbols $a_j$ are bounded uniformly in the class $S^{1,0}_1(T^*M)$ and converge to 0 in the class $S^{1,0}_1(T^*M)$. Therefore, Lemma E.25 implies that $\chi(u_j - u) \to 0$ in $H^s_h$. Next,

$$(E.5.6) \quad \mathbf{Pu}_j - \mathbf{Pu} = \mathbf{P}\text{Op}_h(a_j)u = \text{Op}_h(a_j)\mathbf{Pu} + [\mathbf{P}, \text{Op}_h(a_j)]u.$$  

Since $\mathbf{Pu} \in H^{s-k+1}_h(M)$, Lemma E.25 implies that $\chi\text{Op}_h(a_j)\mathbf{Pu} \to 0$ in $H^{s-k+1}_h$. Now,

$$[\mathbf{P}, \text{Op}_h(a_j)] = \frac{\hbar}{i} \text{Op}_h(\{\sigma_h(\mathbf{P}), a_j\}) + \hbar^2 R_j, \quad R_j \in \Psi^{k-2}_h(M).$$

Since $a_j$ converges to zero in $S^{1,0}_1(T^*M)$, the remainder $R_j$ satisfies

$$\|\chi R_j\|_{H^s_h \to H^{s-k+1}_h} \to 0 \quad \text{as} \ j \to \infty.$$  

The family of symbols $\{\sigma_h(\mathbf{P}), a_j\}$ is uniformly bounded in $S^{k-1}_1(T^*M)$ and converges to 0 in $S^{k}_1(T^*M)$; by Lemma E.25, we have $\chi[\mathbf{P}, \text{Op}_h(a_j)]u \to 0$ in $H^{s-k+1}_h$. It follows from (E.5.6) that $\chi(\mathbf{Pu}_j - \mathbf{Pu}) \to 0$ in $H^{s-k+1}_h$, finishing the proof of (E.5.5). \hfill \Box

As an immediate corollary of Lemma E.47, we obtain

**Lemma E.48.** Let $\mathbf{P} \in \Psi^k_h(M)$ be properly supported and assume that $u \in H^{s}_h(M), v \in \mathcal{H}^{s-k-1}_\text{comp}(M)$ satisfy

$$(E.5.7) \quad \mathbf{Pu} \in H^{s-k+1}_h(M), \quad \mathbf{P}^*v \in \mathcal{H}^s_\text{comp}(M).$$

Then

$$(E.5.8) \quad \langle \mathbf{Pu}, v \rangle_{L^2} = \langle u, \mathbf{P}^*v \rangle_{L^2}.$$  

**Proof.** Let $\chi \in C_0^\infty(M)$ be equal to 1 on supp $v$. By Lemma E.47, there exist a sequence $u_j \in C^\infty(M)$ such that

$$\|\chi(u_j - u)\|_{H^s} \to 0, \quad \|\chi(\mathbf{Pu}_j - \mathbf{Pu})\|_{H^{s-k+1}_h} \to 0.$$  

Since $u_j$ is smooth and $v$ is compactly supported, we have

$$\langle \mathbf{Pu}_j, v \rangle_{L^2} = \langle u_j, \mathbf{P}^*v \rangle_{L^2},$$

and (E.5.8) follows by taking the limit $j \to \infty$. \hfill \Box
### E. PROPAGATION ESTIMATES

**E.5.1. Propagation of singularities.** The most standard situation when the bound (E.5.3) holds is when the wavefront set of $A$ (defined in §E.2.1) is controlled by the elliptic set of $B$ (defined in §E.2.2) via the Hamiltonian flow of $p = \text{Re} \sigma_h(P)$, provided that $q = -\text{Im} \sigma_h(P)$ has the correct sign:

**THEOREM E.49 (Propagation of singularities).** Let $P \in \Psi^k(M)$ be properly supported, $p, q \in S^k(T^*M)$ be defined in (E.5.2), and

\[
\varphi_t := \exp(t\langle \xi \rangle^{1-k}H_p) : T^*M \to T^*M
\]

be the flow of the field defined in Proposition E.5. Let $A, B, B_1 \in \Psi^0(M)$ be compactly supported and the following **sign condition** hold:

\[
q \geq 0 \quad \text{on } \text{WF}_h(B_1).
\]

Assume finally the following **control condition**: for each $(x, \xi) \in \text{WF}_h(A)$, there exists $T \geq 0$ such that

\[
\varphi_{-T}(x, \xi) \in \text{ell}_h(B); \quad \varphi_t(x, \xi) \in \text{ell}_h(B_1) \quad \text{for all } t \in [-T, 0].
\]

Then for all $s, N$, some $\chi \in C_0^\infty(M)$, and all $u \in C^\infty(M)$, $f := Pu$,

\[
\|Au\|_{H^s_h} \leq C\|Bu\|_{H^s_h} + C h^{-1}\|B_1 f\|_{H^{s-k+1}_h} + O(h^\infty)\|\chi u\|_{H^{\infty}_h}.
\]

**REMARKS.** 1. The operators $A, B, B_1$ do not need to be of order 0. For instance, we could instead take $B \in \Psi^\ell(M)$ in which case the first term on the right-hand side of (E.5.12) is replaced by $C\|Bu\|_{H^{s-\ell}_h}$. This can be deduced from the original (E.5.12) by introducing the operator $\tilde{B} := E_{-\ell}B$ where $E_{-\ell} \in \Psi^\ell(M)$ is elliptic everywhere.

2. The normalization (E.5.9) of the Hamiltonian flow is convenient since it extends to the fiber-radial compactification $\overline{T^*M}$ and thus lets us handle singularities at fiber infinity as well. When $M$ is noncompact, the flow $\varphi_t$ might not be defined for all $t$; it is then implied in (E.5.11) that $\varphi_t(x, \xi)$ exists for $t \in [-T, 0]$.
3. Applying Theorem E.49 to the operator $-\mathbf{P}$, we can reverse the direction of propagation in (E.5.11) provided that we flip the sign condition (E.5.10). (See Figure E.1 and Exercise E.26.) In particular, if $q = 0$, then propagation of singularities applies in both directions. To make the presentation shorter, we state the results for one direction of propagation, but use them in both directions.

4. It follows from Lemma E.47 that (E.5.12) holds for all $u \in H^s_{\text{loc}}(M), f \in H^{s-k+1}_{\text{loc}}(M)$. An even stronger statement is available: if $u$ is merely a distribution and we know that $Bu \in H^s, B_1f \in H^{s-k+1}$, then $Au$ lies in $H^s$. See Exercise E.28.

The first step of the proof of Theorem E.49 is to construct an escape function:

**Lemma E.50.** Assume that (E.5.11) holds for all $(x, \xi) \in \text{WF}_h(A)$ and fix $\beta \geq 0$. Then there exists $g \in C_0^\infty(T^*M)$ such that supp $g \subset \ell h(B_1)$ and

- $g \geq 0$ everywhere;
- $g > 0$ on $\text{WF}_h(A)$;
- $\langle \xi \rangle^{1-k}H_p g \leq -\beta g$ in a neighbourhood of $T^*M \setminus \ell h(B)$.

**Proof.** 1. We first consider the case when $\text{WF}_h(A) = \{(x_0, \xi_0)\}$ consists of a single point. If $(x_0, \xi_0) \in \partial T^*M$, then we embed $T^*M$ in a manifold without boundary and extend the vector field $\langle \xi \rangle^{1-k}H_p$ there. Using (E.5.11), take $T \geq 0$ such that

$$\varphi_{-T}(x_0, \xi_0) \in \ell h(B); \quad \varphi_t(x_0, \xi_0) \in \ell h(B_1) \quad \text{for all } t \in [-T, 0].$$

If $\langle \xi \rangle^{1-k}H_p(x_0, \xi_0) = 0$, then $\varphi_t(x_0, \xi_0) = (x_0, \xi_0)$ for all $t$. This implies that $(x_0, \xi_0) \in \ell h(B)$ and it suffices to take any $g$ such that $g \geq 0$ everywhere, $g(x_0, \xi_0) > 0$, and supp $g \subset \ell h(B)$.

We now assume that $\langle \xi \rangle^{1-k}H_p(x_0, \xi_0) \neq 0$. Then there exist $\delta > 0$ and a hypersurface $\Sigma$ passing through $(x_0, \xi_0)$ such that the map

$$\Phi : (-T - \delta, \delta) \times \Sigma \rightarrow T^*M, \quad \Phi(t, x, \xi) = \varphi_t(x, \xi),$$

is a diffeomorphism onto its image and

$$\Phi((-T - \delta, -T + \delta) \times \Sigma) \subset \ell h(B), \quad \Phi((-T - \delta, \delta) \times \Sigma) \subset \ell h(B_1).$$

Take a function $\psi \in C_0^\infty(-T - \delta, \delta)$ with the following properties:

- $\psi \geq 0$ everywhere;
- $\psi(0) > 0$;
- $\psi' \leq -\beta \psi$ outside of $(-T - \delta/2, -T + \delta/2)$. 


A function $\psi_0$ with these properties for $\beta = 0$ is easy to construct explicitly, see Figure E.2; for general $\beta$, it suffices to put $\psi(t) := e^{-\beta t} \psi_0(t)$. Take also nonnegative $\chi \in C^\infty_0(\Sigma)$ such that $\chi(x_0,\xi_0) > 0$. Then the function

$$g := (\psi \otimes \chi) \circ \Phi^{-1},$$

extended by zero outside of the image of $\Phi$, has the required properties.

2. We now consider the general case. For each $(x_0,\xi_0) \in \text{WF}_h(A)$, let $g(x_0,\xi_0) \in C^\infty_0(T^*M)$ be the function constructed in part 1 of the proof. Then $g(x_0,\xi_0) > 0$ on some open neighbourhood $U(x_0,\xi_0)$ of $(x_0,\xi_0)$. Using compactness of $\text{WF}_h(A)$, we cover it with finitely many sets $U(x_1,\xi_1), \ldots, U(x_m,\xi_m)$. The sum of the corresponding functions

$$g = g(x_1,\xi_1) + \cdots + g(x_m,\xi_m)$$

then has the required properties. \hfill \Box

Armed with Lemma E.50, we now prove Theorem E.49 by means of a positive commutator argument (more pedantically, a negative commutator argument but we will use the standard nomenclature):

Proof of Theorem E.49. 1. Fix a volume form on $M$ and recall the notation

(E.5.13) \hspace{1cm} \text{Re } R := \frac{R + R^*}{2}, \quad \text{Im } R := \frac{R - R^*}{2i}

for operators $R : C^\infty(M) \to C^\infty(M)$.

Fix a constant $\beta > 0$, to be chosen later, and let $g$ be the escape function constructed in Lemma E.50. Then $g$ is a symbol in the class $S^0(T^*M)$. We also fix a metric on $M$ and define $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ using this metric.
Using (E.1.28), take the compactly supported operator
\[ G := \mathrm{Op}_h(\langle \xi \rangle^{s + \frac{1 - k}{2}}) \in \Psi_s^{s + \frac{1 - k}{2}}(M), \quad \text{WF}_h(G) \subset \ell_h(B_1), \]
and the properly supported operator
\[ Y := \mathrm{Op}_h(\langle \xi \rangle^{\frac{k - 1}{2}}) \in \Psi_{\frac{k - 1}{2}}^{\frac{k - 1}{2}}(M). \]
For \( u \in C^\infty(M) \) and \( f = P u \), we write
\[ (E.5.15) \quad \Im \langle (\text{Re} P)u, G^* Gu \rangle = \Re \langle (\Im P)u, G^* Gu \rangle + \Re \langle (\Im P)G^* Gu, u \rangle. \]
We will proceed by bounding from above the terms on the right-hand side.
2. We first write
\[ \Im \langle (\text{Re} P)u, G^* Gu \rangle = \langle (\text{Re} P)u, G^* Gu \rangle - \langle G^* Gu, (\text{Re} P)u \rangle = \langle G^* (\text{Re} P)u, u \rangle - \langle (\text{Re} P)G^* Gu, u \rangle = h(Zu, u), \]
where the symmetric compactly supported operator \( Z \) is a commutator:
\[ (E.5.16) \quad Z = \frac{i}{2h}[\text{Re} P, G^* G] \in \Psi_s^{2s}(M), \quad \text{WF}_h(Z) \subset \ell_h(B_1). \]
The semiclassical principal symbol of \( Z \) is
\[ (E.5.17) \quad \sigma_h(Z) = \frac{1}{2}\{p, \langle \xi \rangle^{2s + 1 - k} g^2 \} = \langle \xi \rangle^{2s}(g \langle \xi \rangle^{1-k} H_p g + \left(s + \frac{1 - k}{2}\right)\langle \xi \rangle^{1-k} H_p \langle \xi \rangle^2 g^2). \]
Choose a constant \( C_1 > 0 \) such that
\[ (E.5.18) \quad \left(s + \frac{1 - k}{2}\right)\langle \xi \rangle^{1-k} H_p \langle \xi \rangle \leq C_1 \quad \text{on supp } g. \]
By Lemma E.50, we have in a neighbourhood of \( T^*M \setminus \ell_h(B) \),
\[ \langle \xi \rangle^{-2s} \sigma_h(Z + (\beta - C_1)(Y G)^*(Y G)) \leq 0. \]
By Proposition E.35 applied to \(- (Z + (\beta - C_1)(Y G)^*(Y G))\), we have
\[ h(Zu, u) \leq (C_1 - \beta)h\|YGu\|^2_{L^2} + Ch\|Bu\|^2_{H^s_h} \]
\[ (E.5.19) \quad + Ch^2\|B_1 u\|^2_{H^{s-1/2}_h} + O(h^\infty)\|\chi u\|^2_{H^{N}_h}. \]
3. We next compute
\[ \Re \langle (\text{Im} P)u, G^* Gu \rangle = \Re \langle G(\text{Im} P)u, Gu \rangle = \langle (\text{Im} P)Gu, Gu \rangle + \Re \langle [G, \text{Im} P]u, Gu \rangle. \]
We have \( \sigma_h(\text{Im } P) = -q \). The sign condition (E.5.10) implies that
\[ \sigma_h(\text{Im } P) \leq 0 \text{ on } \text{ell}_h(B_1) \supset \text{WF}_h(G). \]

Using a pseudodifferential partition of unity, we write \( \text{Im } P \) as a sum of two operators, one of which has nonpositive principal symbol and the wavefront set of the other one does not intersect \( \text{WF}_h(G) \). Applying Proposition E.23 to the first of these operators, we get
\[
\langle (\text{Im } P)u, Gu \rangle \leq C_2 h \| Gu \|_{H^{k-1/2}_h}^2 + C_2 h \| Gu \|_{L^2}^2 + O(h^\infty) \| u \|_{H^{k-N}_h}^2
\]
(E.5.20)

for some constants \( C_2 \) which are independent of the choice of \( G \); for the last inequality, we used Theorem E.32. Next,
\[
\text{Re} \langle (G, \text{Im } P)u, Gu \rangle = \text{Re} \langle G^*[G, \text{Im } P]u, u \rangle.
\]

The operator \( G^*[G, \text{Im } P] \) lies in \( h\Psi^{2s'}_h(M) \) and
\[
\sigma_h(h^{-1}G^*[G, \text{Im } P]) = ig\{g, q\}
\]
is purely imaginary; therefore, \( \text{Re} (G^*[G, \text{Im } P]) \in h^2\Psi^{2s-1}_h(M) \). Since the wavefront set of this operator is contained in \( \text{ell}_h(B_1) \), we get by Theorem E.32
\[
\langle \text{Re} (G^*[G, \text{Im } P])u, u \rangle \leq C h^2 \| B_1 u \|_{H^{k-1/2}_h}^2 + O(h^\infty) \| u \|_{H^{k-N}_h}^2.
\]
(E.5.21)

Adding (E.5.20) and (E.5.21), we get
\[
\text{Re} \langle (\text{Im } P)u, G^* Gu \rangle \leq C_2 h \| Gu \|_{L^2}^2
\]
(E.5.22)

\[ + C h^2 \| B_1 u \|_{H^{k-1/2}_h}^2 + O(h^\infty) \| u \|_{H^{k-N}_h}^2. \]

4. Adding (E.5.19) and (E.5.22) and using (E.5.15), we arrive to
\[
\text{Im} \langle f, G^* Gu \rangle \leq (C_1 + C_2 - \beta) h \| Gu \|_{L^2}^2 + C h \| Bu \|_{H^{k}_h}^2
\]
\[ + C h^2 \| B_1 u \|_{H^{k-1/2}_h}^2 + O(h^\infty) \| u \|_{H^{k-N}_h}^2. \]

We now put \( \beta := C_1 + C_2 + 1 \) and use the following corollary of the Cauchy–Schwarz inequality and Theorem E.32:
\[
\| f, G^* Gu \| \leq C \| Gf \|_{H^{(1-k)/2}_h} \| Gu \|_{H^{k-1/2}_h}
\]
(E.5.23)
\[ \quad \leq C \| B_1 f \|_{H^{k-1-k+1}_h} \| Gu \|_{L^2} + O(h^\infty) \| u \|_{H^{k-N}_h}^2 \]
to get
\[
\| Gu \|_{L^2}^2 \leq C \| Bu \|_{H^{k}_h}^2 + C h^{-1} \| B_1 f \|_{H^{k-1-k+1}_h} \| Gu \|_{L^2}
\]
(E.5.24)
\[ + C h \| B_1 u \|_{H^{k-1/2}_h}^2 + O(h^\infty) \| u \|_{H^{k-N}_h}^2. \]
This implies the following estimate:
\[
\|Au\|_{H^k_h} \leq C\|Bu\|_{H^k_h} + C h^{-1}\|B_1 f\|_{H^{-k+1}_h} + Ch^{1/2}\|B_1 u\|_{H^{k-1/2}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h},
\]
(E.5.25)
where we have used Theorem E.32 and the fact that WF\(_h\)(A) \(\subset \text{ell}_h(G)\) to bound \(\|Au\|_{H^k_h}\) in terms of \(\|YGu\|_{L^2}\).

5. We finally remove the term \(Ch^{1/2}\|B_1 u\|_{H^{k-1/2}_h}\). For that, we use induction to prove the following estimate for each \(\ell \in \mathbb{N}\):
\[
\|Au\|_{H^k_h} \leq C\|Bu\|_{H^k_h} + C h^{-1}\|B_1 f\|_{H^{-k+1}_h} + Ch^{\ell/2}\|B_1 u\|_{H^{k-\ell/2}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h}. \tag{E.5.26}
\]

For \(\ell = 1\), this is exactly (E.5.25). Now, assuming that (E.5.26) is true for some \(\ell\), we prove it for \(\ell + 1\). Take compactly supported \(B_2 \in \Psi^0_h(M)\) such that the control condition (E.5.11) holds for \((A,B,B)\). By (E.5.26) to \((B_2,B,B)\) and by (E.5.25) to \((A,B,B)\); in particular, WF\(_h\)(\(B_2\)) \(\subset \text{ell}_h(B_1)\). To construct \(B_2\), it suffices to make it microlocalized in a small neighbourhood of the union of segments \(\{\varphi_t(x,\xi) \mid t \in [-T,0]\}\) with \((x,\xi) \in \text{WF}_h(A)\) and \(T\) given by (E.5.11) – see Figure E.1.

Applying (E.5.26) to \((A,B,B)\) and (E.5.25) to \((B_2,B,B)\), we get
\[
\|Au\|_{H^k_h} \leq C\|Bu\|_{H^k_h} + C h^{-1}\|B_2 f\|_{H^{-k+1}_h} + Ch^{\ell/2}\|B_2 u\|_{H^{k-\ell/2}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h},
\]
\[
\|B_2 u\|_{H^{k-\ell/2}_h} \leq C\|Bu\|_{H^{k-\ell/2}_h} + Ch^{-1}\|B_1 f\|_{H^{-k-\ell/2+1}_h} + Ch^{1/2}\|B_1 u\|_{H^{k-\ell/2+1/2}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h}.
\]
Combining these estimates (and using Theorem E.32 to bound \(\|B_2 f\|\) via \(\|B_1 f\|\)) we obtain (E.5.26) for \(\ell + 1\). The fact that (E.5.26) holds for all \(\ell\) immediately implies (E.5.12), finishing the proof. \(\square\)

We now state a basic positive commutator estimate for the case when \(P\) has real principal symbol, which assumes the existence of an escape function which satisfies a sign condition (E.5.27). This estimate will in particular be used in the proofs of radial points estimates in §E.3.2 below.

**Lemma E.51.** Let \(P \in \Psi^k_h(M)\) be properly supported, \(p := \text{Re}\sigma_h(P) \in S^k(T^*M)\), \(A,B,B_1 \in \Psi^0_h(M)\) be compactly supported, and \(g \in C^\infty_0(T^*M)\) is such that

1. \(\text{Im}\sigma_h(P) = 0\) near \(\text{supp } g\);
2. \(g \geq 0\) everywhere, \(\text{WF}_h(A) \subset \{g > 0\}\), and \(\text{supp } g \subset \text{ell}_h(B_1)\);
(3) we have in a neighbourhood of $\mathcal{T}^*M \setminus \text{ell}_h(B)$, for some constant $\delta > 0$ and a fixed Riemannian metric on $M$ in the definition of $(\xi)$,

\begin{equation}
\langle \xi \rangle^{1-k} \left( H_p g + \sigma_h(h^{-1} \text{Im } P) g + \left( s + \frac{1-k}{2} \right) H_p \langle \xi \rangle g \right) \leq -\delta g.
\end{equation}

Then for all $N$, some $\chi \in C^\infty_0(M)$, and all $u \in C^\infty(M)$, $f := Pu$,

\begin{equation}
\| Au \|_{H^k_h} \leq C \| Bu \|_{H^s_h} + Ch^{-1} \| B_1 f \|_{H^{s-1}+1}_h + \mathcal{O}(h^\infty) \| \chi u \|_{H^N_h}.
\end{equation}

**REMARK.** The expression $\langle \xi \rangle^{1-k} \sigma_h(h^{-1} \text{Im } P)$ makes sense near supp $g$ since

$$h^{-1} \text{Im } P := \frac{P - P^*}{2ih}$$

lies in $\Psi^{k-1}_h(M)$ microlocally near WF$_h(B_1)$. Strictly speaking, one should replace $\sigma_h(h^{-1} \text{Im } P)$ by the principal symbol of $X(h^{-1} \text{Im } P)$, where $X \in \Psi^0_h(M)$ satisfies

$$\text{WF}_h(X) \subset \{ \langle \xi \rangle^{-k} \text{Im } \sigma_h(P) = 0 \},$$

$$X = I + \mathcal{O}(h^\infty)$$

microlocally near WF$_h(B_1)$.

**Proof.** Put

$$G := \text{Op}_h(\langle \xi \rangle^{s+\frac{1-k}{2}} g), \quad Y := \text{Op}_h(\langle \xi \rangle^{\frac{k-1}{2}}).$$

Following steps 1–2 in the proof of Theorem E.49, we obtain

$$\text{Im } \langle f, G^* Gu \rangle = h \text{ Re } \langle Z' u, u \rangle,$$

where

$$Z' = Z + h^{-1} G^* G (\text{Im } P) \in \Psi^{2s}_h(M), \quad \text{WF}_h(Z') \subset \text{ell}_h(B_1),$$

$Z \in \Psi^{2s}_h$ is defined in (E.5.16), and $h^{-1} G^* G (\text{Im } P) \in \Psi^{2s}_h$ by assumption (1).

Using (E.5.17), we calculate

$$\langle \xi \rangle^{-2s} \sigma_h(Z') = g \langle \xi \rangle^{1-k} H_p g$$

$$+ \left( \langle \xi \rangle^{1-k} \sigma_h(h^{-1} \text{Im } P) + \left( s + \frac{1-k}{2} \right) \langle \xi \rangle^{1-k} H_p \langle \xi \rangle \right) g^2.$$

By (E.5.27), we see that

$$\langle \xi \rangle^{-2s} \sigma_h(Z' + \delta(YG)^*(YG)) \leq 0$$

in a neighbourhood of $\mathcal{T}^*M \setminus \text{ell}_h(B)$.

By Proposition E.35 applied to $-(Z' + \delta(YG)^*(YG))$, we have

$$h \langle Z' u, u \rangle \leq -\delta h \| Y Gu \|^2_{L^2} + Ch \| Bu \|^2_{H^s_h}$$

$$+ Ch^2 \| B_1 u \|^2_{H^{s-1/2}_h} + \mathcal{O}(h^\infty) \| \chi u \|^2_{H^N_h}.$$
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Arguing similarly to step 4 of the proof of Theorem E.49, we obtain (E.5.28).

□

E.5.2. Radial points estimates. We now show that the control condition (E.5.11) can in some situations be relaxed. In these cases, the positivity in the positive commutator argument comes not from the Hamiltonian derivative of the escape function $H_p g$, but from the other terms in (E.5.27).

More specifically, our estimates will be associated to radial sources/sinks, defined as follows:

**Definition E.52.** (Radial source/sink) Let $\kappa : T^* M \setminus 0 \to \partial T^* M$ be the projection map, see (E.1.8). Take $p \in S^k(T^* M; \mathbb{R})$ and consider the flow

$$\varphi_t := \exp(t \langle \xi \rangle^{1-k} H_p) : T^* M \to T^* M.$$ (E.5.29)

We say that a compact $\varphi_t$-invariant set $L \subset \{\langle \xi \rangle^{-k} p = 0\} \cap \partial T^* M$

is a radial source for $p$, if there exists a neighbourhood $U \subset T^* M$ of $L$ such that uniformly in $(x, \xi) \in U$,

$$\kappa(\varphi_t(x, \xi)) \to L, \quad t \to -\infty;$$ (E.5.30)

$$|\varphi_t(x, \xi)| \geq C^{-1} e^{\theta |t|} |\xi|, \quad t \leq 0,$$ (E.5.31)

for some $C, \theta > 0$. Here $|\cdot|$ denotes a norm on the fibers of $T^* M$.

A radial sink for $p$ is by definition a radial source for $-p$. (See Figure E.4 below.)

**Remark.** Note that (E.5.30) and (E.5.31) together imply

$$\varphi_t(x, \xi) \to L \quad as \quad t \to -\infty.$$ (E.5.32)

**Example.** Consider the following operator $P \in \Psi^1_h(\mathbb{R})$:

$$P := x(hD_x) + i\gamma h = \frac{hx}{i} \partial_x + i\gamma h, \quad \gamma \in \mathbb{C}.$$ (E.5.33)

Its principal symbol $p := \sigma_h(P)$ is given by

$$p(x, \xi) = x\xi, \quad H_p = x\partial_x - \xi\partial_\xi.$$  

Then the following set is a radial source for $p$:

$$L := \partial T^* \mathbb{R} \cap \{x = 0\}$$

See Figure E.3 for a phase space picture of the flow $\varphi_t = e^{tH_p}$. We encourage the reader to look at Exercises E.32 and E.35 which explain the definitions and statements below for this example.

To make the subprincipal condition in our estimates invariant, we use the following
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**PROPOSITION E.53.** Let $\varphi_t$ be defined by (E.5.29), $L \subset T^*M$ be a compact $\varphi_t$-invariant set, and $a \in S_h^{k-1}(T^*M; \mathbb{R})$. The following conditions are equivalent:

1. for $T > 0$ large enough, we have

$$\int_0^T (\langle \xi \rangle^{1-k}a) \circ \varphi_t \, dt > 0 \quad \text{on } L;$$

(E.5.34)

2. there exists $b \in S^0(T^*M)$ such that

$$\langle \xi \rangle^{1-k}(a + H_p b) > 0 \quad \text{on } L.$$

(E.5.35)

If either of these conditions holds, we say that $a$ is **eventually positive** on $L$ with respect to $p$. We similarly define the notion of an **eventually negative** symbol.

**REMARKS.** 1. It follows from (E.5.35) that eventual positivity of $a$ does not depend on the choice of the Riemannian metric in the definition of $\langle \xi \rangle$. Moreover, since $L$ is $\varphi_t$-invariant, the direction of propagation in (E.5.34) does not matter.

2. In the case when the vector field $\langle \xi \rangle^{1-k}H_p$ vanishes on $L$, eventual positivity of $a$ is simply equivalent to $a$ being positive on $L$. This more restrictive assumption on $H_p$ is true for the radial sources/sinks arising in the study of asymptotically hyperbolic manifolds in Chapter 5. However, the radial estimates presented here apply to more general settings such as Kerr–de Sitter metrics and Anosov flows.
Proof. 1. Assume that there exists $T > 0$ such that (E.5.34) holds. Put
\[
b := \frac{1}{T} \int_0^T (T - t)(\langle \xi \rangle^{1-k}a) \circ \varphi_t \ dt \in S^0(T^* M),
\]
then integration by parts shows that
\[
\langle \xi \rangle^{1-k} H_p b = \frac{1}{T} \int_0^T (T - t) \partial_t ((\langle \xi \rangle^{1-k}a) \circ \varphi_t) \ dt
\]
\[
= -\langle \xi \rangle^{1-k} a + \frac{1}{T} \int_0^T (\langle \xi \rangle^{1-k}a) \circ \varphi_t \ dt,
\]
therefore
\[
\langle \xi \rangle^{1-k}(a + H_p b) = \frac{1}{T} \int_0^T (\langle \xi \rangle^{1-k}a) \circ \varphi_t \ dt > 0 \quad \text{on } L
\]
and (E.5.35) follows.

2. Now, assume that (E.5.35) holds. Fix $\varepsilon > 0$ such that
\[
a_1 := \langle \xi \rangle^{1-k}(a + H_p b) \geq \varepsilon \quad \text{on } L,
\]
and a constant $C$ such that $|b| \leq C$ on $L$. Since $L$ is $\varphi_t$-invariant, we have
\[
\int_0^T (\langle \xi \rangle^{1-k}a) \circ \varphi_t \ dt = \int_0^T a_1 \circ \varphi_t \ dt - \int_0^T (\langle \xi \rangle^{1-k} H_p b) \circ \varphi_t \ dt
\]
\[
\geq \varepsilon T - b \circ \varphi_T + b
\]
\[
\geq \varepsilon T - 2C
\]
on $L$; for $T$ large enough, this implies (E.5.34). \qed

We are now ready to state the first radial points estimate, which gives a priori estimates near a radial source provided we are in sufficiently high Sobolev regularity:

**THEOREM E.54 (High regularity radial estimate).** Let $P \in \Psi^k_h(M)$ be properly supported, $p := \text{Re} \sigma_h(P) \in S^k(T^* M)$,
\[
L \subset \partial T^* M \cap \{\langle \xi \rangle^{-k} p = 0\}
\]
be a radial source for $P$ (see Definition E.52), and $\text{Im} \sigma_h(P) = 0$ near $L$. Let $s \in \mathbb{R}$ satisfy the following **threshold condition**: the symbol
\[
\sigma_h(h^{-1} \text{Im } P) + \left(s + \frac{1 - k}{2}\right) H_p \langle \xi \rangle^2
\]
is eventually negative on $L$ with respect to $p$ as defined in Proposition E.53.
**Figure E.4.** An illustration of Theorem E.54 for radial sources (left) and Theorem E.56 for radial sinks (right). The horizontal line on the top denotes $\partial T^* M$.

Fix compactly supported $B_1 \in \Psi_0^0(M)$ such that $L \subset \text{ell}_h(B_1)$. Then there exists compactly supported $A \in \Psi_0^0(M)$ such that $L \subset \text{ell}_h(A)$ and for each $N$, some $\chi \in C_0^\infty(M)$, and all $u \in C^\infty(M)$, $f := Pu$, we have

$$(E.5.37) \quad \|Au\|_{H^s_h} \leq C h^{-1} \|B_1 f\|_{H^{s-k+1}_h} + O(h^\infty) \|\chi u\|_{H^{-N}}.$$

**REMARKS.**
1. The condition that $(E.5.36)$ is eventually negative is independent of the choice of the density in the definition of $\text{Im} \ P$ and the metric in the definition of $\langle \xi \rangle$. Moreover, this condition is satisfied for $s > 0$ large enough. See Exercises E.30 and E.31 for details.
2. Using Theorem E.49, we get bounds of the form $(E.5.37)$ for each $A$ such that every backwards trajectory of $\phi_t$ starting at $\text{WF}_h(A)$ converges to $L$.
3. As in the case of Theorem E.49, an application of Lemma E.47 gives $(E.5.37)$ for all $u \in H^s_{\text{loc}}(M)$, $f \in H^{s-k+1}_{\text{loc}}(M)$. An even stronger statement is available which gives a priori regularity of $u$ near $L$ assuming that $u$ lies in any sufficiently high Sobolev class – see Exercise E.33.

Similarly to Theorem E.49, the proof of Theorem E.54 relies on an escape function construction:

**LEMMA E.55.** Assume that $L \subset \partial T^* M$ is a radial source for $p \in S^k(T^* M; \mathbb{R})$, and let $U \subset T^* M$ be some open neighbourhood of $L$. Then there exists a function $\chi \in C_0^\infty(U)$ such that

- $\chi \geq 0$ everywhere;
- $\chi > 0$ on $L$;
- $\langle \xi \rangle^{1-k} H_p \chi \leq 0$ everywhere.

**Proof.** By $(E.5.32)$, we may shrink $U$ so that $\varphi_t(x, \xi) \to L$ as $t \to -\infty$ uniformly in $(x, \xi) \in U$. Take $\psi \in C_0^\infty(U; [0, 1])$ such that $\psi = 1$ near $L$. Then for $T > 0$ large enough, we have

$$(E.5.38) \quad t \geq T, \ (x, \xi) \in \text{supp} \ \psi \quad \implies \quad \psi(\varphi_{-t}(x, \xi)) = 1.$$
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Put
\[ \chi := \int_T^{2T} \psi \circ \varphi_t \, dt. \]
Then \( \chi \geq 0 \) everywhere and, since \( L \) is \( \varphi_t \)-invariant, \( \chi > 0 \) on \( L \). We also see from (E.5.38) that \( \text{supp} \chi \subset U \). It remains to show that
\[ \langle \xi \rangle^{-k} H_p \chi = \psi \circ \varphi_{2T} - \psi \circ \varphi_T \leq 0. \]
Indeed, suppose that \( \psi(\varphi_{2T}(x, \xi)) > \psi(\varphi_T(x, \xi)) \) for some \( (x, \xi) \). Since \( 0 \leq \psi \leq 1 \) everywhere, this implies that \( \psi(\varphi_{2T}(x, \xi)) > 0 \) and \( \psi(\varphi_T(x, \xi)) < 1 \). By (E.5.38), we arrive to a contradiction. \( \square \)

We now give the proof of the radial estimate, using the positive commutator estimate from Lemma E.51:

**Proof of Theorem E.54.** 1. Using Proposition E.53 and the fact that (E.5.36) is eventually negative, choose \( b \in \mathcal{C}^\infty (T^* M) \) such that
\[ (E.5.39) \quad \langle \xi \rangle^{-k} \left( \sigma_h(h^{-1} \text{Im} \mathcal{P}) + \left( s + \frac{1 - k}{2} \right) \frac{H_p \langle \xi \rangle}{\langle \xi \rangle} + H_p b \right) < 0 \quad \text{on } L. \]
Let \( B_2 \in \Psi_0^0(M) \) be compactly supported and satisfy \( L \subset \text{ell}_h(B_2) \); we will fix it in the second step of the proof. Take a neighbourhood \( U \) of \( L \) such that for some constant \( \delta > 0 \),
\[ (E.5.40) \quad U \subset \text{ell}_h(B_2), \]
\[ (E.5.41) \quad \text{Im} \sigma_h(\mathcal{P}) = 0 \quad \text{on } U, \]
\[ (E.5.42) \quad \langle \xi \rangle^{-k} \left( \sigma_h(h^{-1} \text{Im} \mathcal{P}) + \left( s + \frac{1 - k}{2} \right) \frac{H_p \langle \xi \rangle}{\langle \xi \rangle} + H_p b \right) \leq -\delta \quad \text{on } U. \]
Let \( \chi \in \mathcal{C}^\infty_0(U) \) be the function constructed in Lemma E.55 and put
\[ g := e^b \chi \in \mathcal{C}^\infty_0(T^* M). \]
Put \( B := 0 \) and take \( A \in \Psi_0^0(M) \) which is elliptic on \( L \) and satisfies \( \text{WF}_h(A) \subset \{ \chi > 0 \} \). Then the assumptions of Lemma E.51 are satisfied, with \( B_1 \) replaced by \( B_2 \). In particular, (E.5.27) follows from (E.5.42) together with the inequality \( \langle \xi \rangle^{-k} H_p \chi \leq 0 \). The estimate (E.5.28) then gives
\[ (E.5.43) \quad \| Au \|_{H^s_h} \leq C h^{-1} \| B_2 f \|_{H^{-k+1}_h} \]
\[ + C h^{1/2} \| B_2 u \|_{H^{s-1/2}_h} + \mathcal{O}(h^\infty) \| \chi u \|_{H^{s-N}_h}. \]

2. It remains to remove the term \( C h^{1/2} \| B_2 u \|_{H^{s-1/2}_h} \) from (E.5.43). Let \( V \) be a neighbourhood of \( L \) such that
\[ V \subset \text{ell}_h(B_1), \quad \text{Im} \sigma_h(\mathcal{P}) = 0 \quad \text{on } V. \]
We choose $B_2$ elliptic on $L$ and with the following property: for each $(x, \xi) \in \WF_h(B_2)$, the trajectory $\varphi_t(x, \xi)$ of the flow (E.5.29) converges to $L$ as $t \to -\infty$ and lies inside $V$ for all $t \leq 0$. The existence of such operator is guaranteed by (E.5.32).

By Theorem E.49, we have

$$\|B_2 u\|_{H^{-1/2}_h} \leq C \|Au\|_{H^{-1/2}_h} + Ch^{-1} \|B_1 f\|_{H^{s-k+1}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h}.$$  

Combined with (E.5.43), this gives (using Theorem E.32 to bound $\WF$ in terms of $\|u\|_{B^s_{2,2}}$)  

$$\|Au\|_{H^s_h} \leq Ch^{-1} \|B_1 f\|_{H^{s-k+1}_h} + Ch^{1/2} \|Au\|_{H^{-1/2}_h} + O(h^\infty)\|\chi u\|_{H^N_h}.$$  

By Proposition E.20 with $\alpha := (2Ch^{1/2})^{-1}$, $r := s - 1/2$, $s_1 := -N$, $s_2 := s$, and putting the resulting term $\frac{1}{2} \|Au\|_{H^s_h}$ on the left-hand side, we get

$$\|Au\|_{H^s_h} \leq Ch^{-1} \|B_1 f\|_{H^{s-k+1}_h} + Ch^{N+s-1/2} \|Au\|_{H^{-N}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h}.$$  

Since $N$ can be chosen arbitrarily large, this implies (E.5.37). \qed

The second radial points estimate bounds the solution near a radial sink provided that we control it in a punctured neighbourhood of the sink and work in sufficiently low Sobolev regularity:

**THEOREM E.56 (Low regularity radial estimate).** Let $P \in \Psi^k_h(M)$ be properly supported, $p := \Re \sigma_h(P) \in S^k(T^*M)$,

$$L \subset \partial T^*M \cap \{\langle \xi \rangle^{-k} p = 0\}$$

be a radial sink for $P$ (see Definition E.52), and $\Im \sigma_h(P) = 0$ near $L$. Let $s \in \mathbb{R}$ satisfy the following **threshold condition**: the symbol

$$(E.5.44) \quad \sigma_h(h^{-1} \Im P) + \left(s + \frac{1-k}{2}\right) H_k\langle \xi \rangle$$

is eventually negative on $L$ with respect to $p$ as defined in Proposition E.53.

Fix compactly supported $B_1 \in \Psi^0_h(M)$ such that $L \subset \ell_h(B_1)$. Then there exist compactly supported $A, B \in \Psi^k_h(M)$ such that $L \subset \ell_h(A)$, $\WF_h(B) \subset \ell_h(B_1) \setminus L$, and for each $N$, some $\chi \in C^0_\infty(M)$, and all $u \in C^\infty(M)$, $f := Pu$, we have

$$(E.5.45) \quad \|Au\|_{H^s_h} \leq C \|Bu\|_{H^s_h} + Ch^{-1} \|B_1 f\|_{H^{s-k+1}_h} + O(h^\infty)\|\chi u\|_{H^{-N}_h}.$$  

**REMARKS.** 1. Similarly to Theorem E.54, the expression (E.5.44) makes sense near $L$ and its eventual negativity is invariant. Moreover, the threshold condition is satisfied for large negative $s$ – see Exercise E.31.
Similarly to Theorems E.49 and E.54, an application of Lemma E.47 gives (E.5.45) for all \( u \in H^s_{\text{loc}}(M), \ f \in H^{s-k+1}_{\text{loc}}(M) \). An even stronger statement is available: if \( u \) is merely a distribution and \( B u \in H^s, \ B_1 f \in H^{s-k+1} \), then \( A u \in H^s \). See Exercise E.34.

**Proof.** As in the proof of Theorem E.54, choose \( b \in C^\infty(T^*M) \) and a neighbourhood \( U \) of \( L \) such that for some constant \( \delta > 0, \)

\[
U \subset \ell_h(B_1),
\]

\[
\text{Im} \sigma_h (P) = 0 \text{ on } U,
\]

\[
\langle \xi \rangle^{1-k} \left( \sigma_h(h^{-1} \text{Im} P) + \left( s + \frac{1-k}{2} \right) \frac{H_p(\xi)}{\langle \xi \rangle} + H_p b \right) \leq -\delta \text{ on } U.
\]

Let \( \chi_1, \chi_2 \in C_0^\infty(U; [0,1]) \) and \( \psi \in C_0^\infty(U \setminus L; [0,1]) \) satisfy

\[
\chi_2 = 1 \text{ near } L, \quad \chi_1 = 1 \text{ near supp } \chi_2, \quad \psi = 1 \text{ near supp } \chi_1 \cap \text{supp}(1-\chi_2).
\]

Put

\[
g := e^b \chi_1, \quad A := \text{Op}_h(\chi_2), \quad B := \text{Op}_h(\psi), \quad B_2 := A + B.
\]

Then the assumptions of Lemma E.51 are satisfied, with \( B_1 \) replaced by \( B_2 \). In particular, the condition (E.5.27) follows from (E.5.48) and the fact that \( \text{supp}(H_p \chi_1) \subset \{ \psi = 1 \} \). The estimate (E.5.28) (using Theorem E.32 to bound \( \| B_2 f \|_{H^{s-k+1}_h} \) in terms of \( \| B_1 f \|_{H^{s-k+1}_h} \)) then gives

\[
\| Au \|_{H^s_h} \leq C \| Bu \|_{H^s_h} + Ch^{-1} \| B_1 f \|_{H^{s-k+1}_h} + Ch^{1/2} \| B_2 u \|_{H^{s-1/2}_h} + O(h^\infty) \| \chi u \|_{H^{-N}_h}.
\]

Recalling that \( B_2 = A + B \), we see that

\[
\| Au \|_{H^s_h} \leq C \| Bu \|_{H^s_h} + Ch^{-1} \| B_1 f \|_{H^{s-k+1}_h} + Ch^{1/2} \| Au \|_{H^{s-1/2}_h} + O(h^\infty) \| \chi u \|_{H^{-N}_h}.
\]

It remains to use Proposition E.20 as in the proof of Theorem E.54. \( \square \)

**E.5.3. Hyperbolic estimates.** We finally study hyperbolic estimates for second order differential operators, used in §5.5.

Throughout this section we assume that \( \overline{M} \) is a compact manifold with boundary \( \partial M, \ t : \overline{M} \to [0, \infty) \) is a boundary defining function (see Definition 5.1), and we are given a product structure (see (5.1.2))

\[
(t, y) : t^{-1}([0,1]) \to [0,1)_t \times (\partial M)_y.
\]
We use this product structure to identify \( \{ t < 1 \} \subset \overline{M} \) with \([0,1) \times \partial M\). (The theorems below apply also to the case when \( M \) is noncompact but has a compact boundary.) Using Definition E.21, we consider the spaces \( \bar{\mathcal{H}}^s(M) \), \( \dot{\mathcal{H}}^s(M) \), and the semiclassical norms \( \| \cdot \|_{\bar{\mathcal{H}}^s_h(M)} \), \( \| \cdot \|_{\dot{\mathcal{H}}^s_h(M)} \) on these spaces.

We assume that \( P \in \text{Diff}^2_h(M) \) is a second order semiclassical differential operator (see §E.1.1) with coefficients which are smooth up to the boundary \( \partial M \). Then

\[
P : \bar{\mathcal{H}}^s(M) \to \bar{\mathcal{H}}^{s-2}(M), \quad P : \dot{\mathcal{H}}^s(M) \to \dot{\mathcal{H}}^{s-2}(M).
\]

The semiclassical principal symbol of \( P \) is a second order polynomial on the fibers

\[
p(t) := \sigma_h(P) \in \text{Poly}^2(T^*M).
\]

We also consider the nonsemiclassical principal symbol \( p_0 \), which is the second order part of \( p \). On \( \{ t < 1 \} \), \( p \) and \( p_0 \) are functions of \( (t,y,\tau,\eta) \) where \( \tau \in \mathbb{R}, \eta \in T^*_y(\partial M) \) are the momentum variables corresponding to \( t,y \) and

\[
p_0(t,y,\tau,\eta) := \frac{1}{2} \langle [\partial^2_{(\tau,\xi)} p](t,y)(\tau,\xi), (\tau,\xi) \rangle.
\]

**DEFINITION E.57.** We say that:

- \( P \) is **hyperbolic** with respect to \( t \) on \( \{ t < 1 \} \), if \( p \) is real-valued and for each \( (t,y,\eta) \in [0,1) \times T^*(\partial M), \eta \neq 0 \), the equation

\[
\text{E.5.51} \quad p_0(t,y,\tau,\eta) = 0, \quad \tau \in \mathbb{R},
\]

has two distinct real solutions \( \tau_\pm \).

- \( P \) is **semiclassically hyperbolic** with respect to \( t \) on \( \{ t < 1 \} \), if it is hyperbolic and for each \( (t,y,\eta) \in [0,1) \times T^*(\partial M) \), the equation

\[
\text{E.5.52} \quad p(t,y,\tau,\eta) = 0, \quad \tau \in \mathbb{R},
\]

has two distinct real solutions \( \tau_\pm \).

**EXAMPLE.** Consider the case of \( M = (0,1) \) and \( P = (hD_t)^2 - (hD_y)^2 + V(y), \ V \in C^\infty((0,1)) \). Then

\[
p(t,y,\tau,\eta) = \tau^2 - \eta^2 + V(y), \quad p_0(t,y,\tau,\eta) = \tau^2 - \eta^2.
\]

For any \( V \), \( P \) is hyperbolic. On the other hand to \( P \) is semiclassically hyperbolic only if \( V < -c < 0 \) on \( (0,1) \). In the definition of semiclassical hyperbolicity we assume that the operator is hyperbolic. That is necessary in view of the example \( P = (hD_t)^2 - 1 \).

We now state the main estimates of this section. We start with the case when \( P \) is hyperbolic and the constants in the estimates depend on \( h \):
THEOREM E.58 (Hyperbolic estimate I). Assume that $P$ is hyperbolic with respect to $t$ on $\{ t < 1 \}$. Take $\chi_1, \chi_2 \in C^\infty(\bar{M})$ satisfying

\begin{equation}
\chi_1 = 1 \text{ near } \{ t \geq 1 \}, \quad \chi_2 = 1 \text{ on } \{ t \leq 1 \}.
\end{equation}

Then for each $u \in \dot{H}^s(M)$ such that $Pu \in \dot{H}^{s-1}(M)$, we have

\begin{equation}
\|(1 - \chi_1)u\|_{\dot{H}^s(M)} \leq C\|\chi_2 Pu\|_{\dot{H}^{s-1}(M)} + C\|\chi_1 \chi_2 u\|_{\dot{H}^s(M)},
\end{equation}

and for each $v \in \dot{H}^s(\bar{M})$ such that $Pv \in \dot{H}^{s-1}(\bar{M})$, we have

\begin{equation}
\|(1 - \chi_1)v\|_{\dot{H}^s(\bar{M})} \leq C\|\chi_2 Pv\|_{\dot{H}^{s-1}(\bar{M})}.
\end{equation}

In both cases the constant $C$ depends on $h$.

REMARK. This somewhat peculiar formulation is useful in proving the Fredholm property of the Vasy operator in §5.6.

The estimates (E.5.54), (E.5.55) are analogous to the well-posedness of the Cauchy problem for hyperbolic equations. Indeed, the norm $\|\chi_1 \chi_2 u\|_{\dot{H}^s}$ on the right-hand side of (E.5.54) controls the behaviour of $u$ near $\{ t = 1 \}$, and the left-hand side of (E.5.54) controls the norm of $u$ outside the support of $\chi_1$. In particular if we choose $\chi_1 \equiv 0$ for $0 \leq t \leq \frac{1}{2}$, it controls the norm of $u$ in that region.

For (E.5.55), the requirement that $v \in \dot{H}^s_{\text{loc}}(\bar{M})$ corresponds to the vanishing of $v, \partial_t v$ on $\{ t = 0 \}$: $v$ is supported in $t \geq 0$.

When $P$ is semiclassically hyperbolic, we can control the constants in (E.5.54), (E.5.55) uniformly as $h \to 0$:

THEOREM E.59 (Hyperbolic estimate II). Assume that $P$ is semiclassically hyperbolic with respect to $t$ on $\{ t < 1 \}$. Take cutoff functions $\chi_1, \chi_2$ satisfying (E.5.53). Then for each $u, v$ as in Theorem E.58,

\begin{equation}
\|(1 - \chi_1)u\|_{\dot{H}^s_h(M)} \leq Ch^{-1}\|\chi_2 Pu\|_{\dot{H}^{s-1}_h(M)} + C\|\chi_1 \chi_2 u\|_{\dot{H}^s(M)},
\end{equation}

\begin{equation}
\|(1 - \chi_1)v\|_{\dot{H}^s_h(\bar{M})} \leq Ch^{-1}\|\chi_2 Pv\|_{\dot{H}^{s-1}_h(\bar{M})}.
\end{equation}

In both cases the constant $C$ is independent of $h$.

We start the proofs of Theorems E.58 and E.59 with an energy estimate, proved by a semiclassical version of the factorization method presented in [H6III, §23.2]. To state it, define for $s \in \mathbb{R}$ and $u \in C^\infty([0, 1) \times \partial M)$ the quantity

\begin{equation}
E_{s,u}(t) = \|u(t)\|_{\dot{H}^s_h(\partial M)} + h\|\partial_t u(t)\|_{\dot{H}^{s-1}_h(\partial M)}, \quad t \in [0, 1).
\end{equation}
LEMMA E.60. Let $0 \leq t_0 \leq t_1 < 1$, $s \in \mathbb{R}$. Then:

1. Assume that $P$ is semiclassically hyperbolic with respect to $t$ in $\{ t < 1 \}$. Suppose that $u \in C^\infty([0,1] \times \partial M)$. Then, in the notation of (E.5.58),
   \[
   (E.5.59) \quad \sup_{0 \leq t \leq t_1} E_{s,u}(t) \leq CE_{s,u}(t_0) + Ch^{-1} \int_{t_0}^{t_1} \| Pu(t) \|_{H_k^{-1}(\partial M)} dt
   \]
   with the constant $C$ independent of $t_0$ and $h$.

2. Assume that $P$ is hyperbolic with respect to $t$ in $\{ t < 1 \}$. Then the estimate (E.5.59) holds, but with $C$ depending on $h$.

Proof. We assume that $P$ is semiclassically hyperbolic, indicating in Step 4 below what changes should be made for the second statement of the lemma. To simplify the formulas below, we henceforth assume that $t_0 = 0$, with the same method applying to the case of general $t_0$.

1. The coefficient of $-h^2 \partial_t^2 = (hD_t)^2$ in $P$ does not vanish in $\{ t < 1 \}$. Multiplying $P$ by a nonvanishing function, we may assume that this coefficient is equal to 1 on $\{ t \leq t_1 \}$. Since $P$ is semiclassically hyperbolic, using the quadratic formula we factorize
   \[
   (E.5.60) \quad p(t,y,\tau,\eta) = (\tau - a_1(t,y,\eta))(\tau - a_2(t,y,\eta)) \quad \text{on} \quad \{ t \leq t_1 \}
   \]
   where the functions $a_j(t,y,\eta)$ are real-valued symbols in $S^1(T^*\partial M)$ smooth in $t \in [0,t_1]$ and satisfying for some $c > 0$
   \[
   (E.5.61) \quad a_2(t,y,\eta) - a_1(t,y,\eta) \geq c\langle \eta \rangle \quad \text{on} \quad \{ t \leq t_1 \}.
   \]
   (The lower bound $\sim \langle \eta \rangle$ follows from standard hyperbolicity which is part of the semiclassical definition.)

   For $k \in \mathbb{R}$, denote by $C^\infty_t \Psi^k_H(\partial M)$ the class of operators in $\Psi^k_H(\partial M)$ depending smoothly on $t \in [0,t_1]$. Using a quantization procedure $\text{Op}_h$ on $\partial M$, consider the operators
   \[
   A_j(t) = \text{Op}_h(a_j(t,\bullet)) \in C^\infty_t \Psi^1_H(\partial M).
   \]
   On $\{ t \leq t_1 \}$ we have
   \[
   (E.5.62) \quad P = (hD_t - A_1(t))(hD_t - A_2(t))
   \]
   \[
   + hC^\infty_t \Psi^0_H(\partial M) hD_t + hC^\infty_t \Psi^1_H(\partial M)
   \]
   and same is true when $A_1(t), A_2(t)$ switch places.

2. We first prove the following estimate for the operators $hD_t - A_j(t)$, valid for any $s \in \mathbb{R}$, $j = 1, 2$, $v \in C^\infty([0,t_1] \times \partial M)$ and $t \in [0,t_1]$:
   \[
   (E.5.63) \quad \|v(t)\|_{H^s_k(\partial M)} \leq C\|v(0)\|_{H^s_k(\partial M)}
   \]
   \[
   + Ch^{-1} \int_{0}^{t} \| (hD_r - A_j(r))v(r) \|_{H^s_k(\partial M)} dr.
   \]
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To show (E.5.63), take an invertible elliptic operator $Y_s \in \Psi_h^s(\partial M)$ and put

$$F(t) := \|Y_s v(t)\|_{L^2(\partial M)}, \quad C^{-1}\|v(t)\|_{H^s_h(\partial M)} \leq F(t) \leq C\|v(t)\|_{H^s_h(\partial M)}.$$  

Then, since $\text{Im}(Y_s^* Y_s A_j(t)) \in hC^\infty \Psi^2_h(\partial M)$,

$$h\partial_t F(t^2) = -2 \text{Im}(Y_s hD_t v(t), Y_s v(t))_{L^2}$$

$$\leq -2 \text{Im}(Y_s (hD_t - A_j(t)) v(t), Y_s v(t))_{L^2} + Ch\|v(t)\|_{H^s_h(\partial M)}^2$$

$$\leq CF(t)\|(hD_t - A_j(t)) v(t)\|_{H^s_h(\partial M)} + ChF(t)^2.$$  

The function $F_\varepsilon(t) := \sqrt{F(t)^2 + \varepsilon}$ is smooth in $t \in [0, t_1]$ for all $\varepsilon > 0$ and

$$\partial_t F_\varepsilon(t) \leq Ch^{-1}\|(hD_t - A_j(t)) v(t)\|_{H^s_h(\partial M)} + CF(t).$$

Integrating and letting $\varepsilon \to 0$, we obtain for all $t \in [0, t_1]$

$$F(t) \leq F(0) + Ch^{-1}\int_0^t \|(hD_r - A_j(r)) v(r)\|_{H^s_h(\partial M)} dr + C\int_0^t F(r) dr.$$  

Grönwall’s inequality (A.5.1) (applied with $B(t) \equiv C$) gives (E.5.63).

3. Assume first that $u \in C^\infty([0, 1], \partial M)$. Applying (E.5.63) to the operator $hD_t - A_{3-j}(t)$ and $v(t) := (hD_t - A_j(t)) u(t)$, $j = 1, 2$, and using (E.5.62), we obtain for all $t \in [0, t_1]$,

$$\|(hD_t - A_j(t)) u(t)\|_{H^{s-1}_h(\partial M)} \leq C E_{s, u}(0)$$

$$+ Ch^{-1}\int_0^t \|Pu(r)\|_{H^{s-1}_h(\partial M)} dr + C\int_0^t E_{s, u}(r) dr.$$  

By (E.5.61), the operator $A_1(t) - A_2(t)$ is elliptic in the class $\Psi^1_h(\partial M)$ and hence, by the elliptic estimate (Theorem E.32), we have for all $t \in [0, t_1]$

$$\|u(t)\|_{H^s_h(\partial M)} \leq C\|(A_1(t) - A_2(t)) u(t)\|_{H^{s-1}_h(\partial M)}$$

$$+ Ch\|u(t)\|_{H^{s-1}_h(\partial M)}.$$  

The first term on the right-hand side is estimated by (E.5.64) and the second one, by $h\|u(0)\|_{H^{s-1}_h(\partial M)}$ and the integral of $h\|\partial_t u(t)\|_{H^{s-1}_h(M)}$; thus

$$\|u(t)\|_{H^s_h(\partial M)} \leq C E_{s, u}(0) + Ch^{-1}\int_0^t \|Pu(r)\|_{H^{s-1}_h(\partial M)} dr$$

$$+ C\int_0^t E_{s, u}(r) dr, \quad t \in [0, t_1].$$  

We also have

$$h\|\partial_t u(t)\|_{H^{s-1}_h(\partial M)} \leq \|(hD_t - A_1(t)) u(t)\|_{H^{s-1}_h(\partial M)} + C\|u(t)\|_{H^s_h(\partial M)}.$$  

where the first term on the right-hand side is estimated by (E.5.64). Combining the last two estimates, we obtain

\[ E_{s,u}(t) \leq CE_{s,u}(0) + C h^{-1} \int_0^t \|Pu(r)\|_{H^{-1}_h(\partial M)} \, dr \]

(E.5.66)

\[ + C \int_0^t E_{s,u}(r) \, dr, \quad t \in [0,t_1]. \]

The estimate (E.5.59) now follows by Grönwall’s inequality (A.5.1).

4. We finally make the weaker assumption that \( P \) is hyperbolic with respect to \( t \) on \( \{ t < 1 \} \) and explain how to obtain (E.5.59) with the constant depending on \( h \).

As before, we may assume that the coefficient of \( (hD_t)^2 \) in \( P \) is equal to 1. The discriminant of the quadratic equation \( p(t,y,\tau,\eta) = 0 \) in \( \tau \) is asymptotic to the discriminant of the equation \( p_0(t,y,\tau,\eta) = 0 \) as \( |\eta| \to \infty \). Since \( P \) is hyperbolic, there exists \( q \in C^\infty_0(T^*\partial M) \) such that \( q \geq 0 \) everywhere and the quadratic equation

\[ p(t,y,\tau,\eta) - q(y,\eta) = 0, \quad \tau \in \mathbb{R} \]

has two distinct solutions for all \( t \in [0,t_1] \) and \( (y,\eta) \in T^*\partial M \).

Let \( Q := Op_h(q) \in \Psi^\text{comp}_h(\partial M) \). Applying Steps 1–3 above to the operator \( P - Q \), we obtain for all \( t \in [0,t_1] \)

\[ E_{s,u}(t) \leq CE_{s,u}(0) + Ch^{-1} \int_0^t \|(P - Q)u(r)\|_{H^{-1}_h(\partial M)} \, dr. \]

Since \( \|Qu(r)\|_{H^{-1}_h(\partial M)} \leq C\|u(r)\|_{H^s_h(\partial M)} \), we have for all \( t \in [0,t_1] \)

\[ E_{s,u}(t) \leq CE_{s,u}(0) + Ch^{-1} \int_0^t \|Pu(r)\|_{H^{-1}_h(\partial M)} \, dr + Ch^{-1} \int_0^t E_{s,u}(r) \, dr. \]

By Grönwall’s inequality (A.5.1) now applied with \( B(t) = C/h \) we obtain (E.5.59) with the constant \( C e^{C/h} \).

The energy estimate (E.5.59) is the standard tool in proving existence and uniqueness of the Cauchy problem. Although that is not part of Theorems E.58 and E.59 it is convenient to have it in place. We again adapt the presentation in [HörIII, §23.2].

**Theorem E.61** (Existence of uniqueness for the Cauchy problem). Suppose that \( P \) is semiclassically hyperbolic in the sense of Definition E.57, \( s \in \mathbb{R} \) and that \( 0 \leq t_0 \leq t_1 < \). Then for any \( \varphi_j \in H^s_h(\partial M), \ j = 0,1 \) and \( f \in L^1((t_0,t_1); H^{-1}_h(\partial M)) \) there exists a unique

\[ u \in C^0([0,1];H^s_h(\partial M)) \cap C^1([0,1];H^{-1}_h(\partial M)), \]

(E.5.67)

\[ Pu \in L^1((0,1);H^{-1}_h(\partial M)), \]
such that $Pu = f$ in $(t_0, t_1) \times \partial M$ and $(hD_t)^j u|_{t=t_0} = \varphi_j$, $j = 0, 1$ and (E.5.59) holds.

Proof. To simplify notation we take $t_0 = 0$ and $h = 1$ – the estimates involving $h$ will follow from (E.5.59).

1. Let $v \in C^\infty_c([0, t_1) \times \partial M)$. Applying (E.5.59) to $P^*$ (which is also semiclassically hyperbolic) with $t$ replaced by $t_1 - t$ and $s$ replaced by $-s + 1$ we obtain the estimate

$$(E.5.68) \quad \|v(t)\|_{H^{-s+1}(\partial M)} + \|D_t v(t)\|_{H^{s-1}(\partial M)} \leq C \int_0^{t_1} \|P^* v(t)\|_{H^{-s}(\partial M)} dt.$$ 

2. Writing $P = D_t^2 + P_1(t, y, D_0)D_t + P_0(t, y, D_y)$ (as in the proof of Lemma E.60) we can assume that the coefficient of $D_t^2$ is identically equal to 1), we define the following anti-linear form

$$\Phi(v) := \int_0^{t_1} \langle f, v \rangle dt - i \langle \varphi_0, h D_t v(0) + P_1^* v(0) \rangle - i \langle \varphi_1, v(0) \rangle.$$ 

Using (E.5.68) we see that (with $H^r := H^r(\partial M)$)

$$(E.5.69) \quad |\Phi(v)| \leq C \left( \int_0^{t_1} \|f(t)\|_{H^{s-1}} dt + \sum_{j=0,1} \|\varphi_j\|_{H^{s-1}} \right) \int_0^{t_1} \|P^* v(t)\|_{H^{-s-1}} dt.$$ 

The Hahn–Banach Theorem then shows that the anti-linear form $P^* v \mapsto \Phi(v)$ extends to a bounded anti-linear form on $L^1((0, t_1); H^{-s})$ and hence there exists $u \in L^\infty((0, t_0); H^s)$ such that

$$(E.5.70) \quad \Phi(v) = \int_0^{t_1} \langle u(t), P^* v(t) \rangle dt,$$

$$\|u\|_{L^\infty((0, t_0); H^s)} \leq C \int_0^{t_1} \|f(t)\|_{H^{s-1}} dt + C \sum_{j=0,1} \|\varphi_j\|_{H^{s-1}}.$$ 

3. By taking $v \in C^\infty_c((0, t_1) \times \partial M)$ we see from (E.5.70) and the definition of $\Phi$ that $Pu = f$ in $(0, t_1) \times \partial M$ and that $u \in L^2((0, t_1); H^s(\partial M))$.

Let us assume first that $f \in C^\infty((0, 1) \times \partial M)$. We claim that for all $j \in \mathbb{N},$

$$(E.5.71) \quad u \in H^{j}((0, t_1); H^{s-j}(\partial M)),$$

that is $D_t^j u \in L^2((0, t_1); H^{s-j}(\partial M))$, $\ell \leq j$. In fact, we already noted that this holds for $j = 0$. So assume that $u \in H^{j}((0, t_1); H^{s-j}(\partial M))$. Then by
differentiating the equation,

\[ D_t \begin{pmatrix} D^j_t u \\ D^{j+1}_t u \end{pmatrix} = \begin{pmatrix} 0 & I \\ P_{j0} & P_{j1} \end{pmatrix} \begin{pmatrix} D^j_t u \\ D^{j+1}_t u \end{pmatrix} + \begin{pmatrix} 0 \\ D^j_t f \end{pmatrix}, \]

where \( P_{j\ell} \in \text{Diff}^{2-\ell}(\partial M) \). Hence, by the inductive hypothesis

\[ D_t^{j+2} u \in L^2((0, t_1); H^{s-j-2}(\partial M)) + \bar{H}^{-1}((0, t_1); H^{s-j-1}(\partial M)), \]

which implies (writing \( \bar{H'} H' := \bar{H}'((0, t_1), H'((\partial M)) \))

\[ D_t^{j+1} u \in (\bar{H}^1 H^{s-j-2} + L^2 H^{s-j-1}) \cap \bar{H}^{-1} H^{s-j} \]

\[ \subset L^2((0, t_1); H^{s-j-1}(\partial M)), \]

proving (E.5.71). The inclusion we used,

\[ H^1(\mathbb{R}; H^{s-j-2}(\partial M)) \cap H^{-1}(\mathbb{R}; H^{s-j}(\partial M)) \subset L^2(\mathbb{R}, H^{s-j-1}(\partial M)), \]

is easily checked by using an orthonormal basis of a Laplacian on \( \partial M \),

\[-\Delta_{\partial M} v_j = \lambda_j^2 v_j \]

and the fact that

\[ \|u\|^2_{H'(\mathbb{R}; H^s(\partial M))} \approx \sum_j (\lambda_j)^2 \int_\mathbb{R} (\tau)^2 |\langle \tilde{u}(\tau, \cdot), \varphi_j \rangle_{L^2(\partial M)}|^2, \]

\[ \tilde{u}(\tau, y) := \int_\mathbb{R} u(t, y)e^{-i\tau \cdot} dt. \]

4. Let us also assume that \( \varphi_j \in C^\infty(\partial M) \). That means that (E.5.70) holds for any \( s \in \mathbb{R} \) and (E.5.71) shows that \( u \in C^\infty([0, t_1] \times \partial M) \). But then we can use the formula for \( \Phi(v) \) in (E.5.70) and integrate by parts:

\[ \Phi(v) = \int_0^{t_1} \langle Pu, v \rangle dt - i\langle u(0), hD_tv(0) + P^*_tv(0) \rangle - i\langle D_tu(0), v(0) \rangle. \]

Since \( v \in C^\infty_c([0, t_1] \times \partial M) \) is arbitrary, a comparison with (E.5.69) shows that

\[ Pu = f, \quad u(0) = \varphi_0, \quad D_tu(0) = \varphi_1, \]

while (E.5.59) gives

\[ \|u\|_{L^\infty((0, t_0); H^s)} + \|D_t u\|_{L^\infty((0, t_0); H^{s-1})} \]

\[ \leq C \int_0^{t_1} \|f(t)\|_{H^{s-1}} dt + C \sum_{j=0,1} \|\varphi_j\|_{H^{s-1}}. \]

5. For \( f \) and \( \varphi_j \)'s in the statement of the theorem choose \( f^\nu \in C^\infty([0, t_1] \times \partial M) \) and \( \varphi_j^\nu \in C^\infty(\partial M) \) such that as \( \nu \to \infty \),

\[ f^\nu \to f \quad \text{in } L^1((0, t_1); H^{s-1}(\partial M)) \quad \text{and} \quad \varphi_j^\nu \to \varphi_j \quad \text{in } H^{s-j}(\partial M). \]

Proceeding as in Step 4 we obtain \( u^\nu \in C^\infty([0, t_1] \times \partial M) \) such that \( Pu^\nu = f^\nu \)

and \( D^j_t u^\nu(0) = \varphi_j^\nu, \quad j = 0, 1 \) and (E.5.74) holds for \( u^\nu \). But this shows that
$u^\nu$ forms a Cauchy sequence in $C([0,t_1]; H^s(\partial M)) \cap C^1([0,t_1]; H^{s-1}(\partial M))$. The is the desired solution $u$.

6. It remains to show uniqueness. Hence assume that $u \in C([0,t_1]; H^s(\partial M)) \cap C^1([0,t_1]; H^{s-1}(\partial M))$ and that $P u = 0$ and $u(0) = D_t u(0) = 0$. The argument in Step 3 of the proof shows that $u \in \dot{H}^k([0,t_1]; H^{s-k}(\partial M)) \cap \dot{H}^{k+1}([0,t_1]; H^{s-k-1}(\partial M))$ for any $k$. For $k$ large enough the proof of Lemma E.60 applies (only finite regularity in $t$ is ultimately needed) and then (E.5.59) shows that $u(t) = 0$ in $H^{s-k}(\partial M)$, $0 \leq t \leq t_1$, which of course means that $u \equiv 0$. \hfill $\square$

Before proving Theorem E.59 we introduce another family of Sobolev spaces: for $t_1 < 1$,

$$H^{s,r}_h := H^{s,r}_h(\mathbb{R} \times \partial M) := \{ u : \| u \|_{H^{s,r}_h} < \infty \},$$

$$\| u \|_{H^{s,r}_h} := \|(I - h^2 \Delta_{\partial M})^{\frac{1}{2}} u \|_{H^s(\mathbb{R} \times \partial M)},$$

(E.5.75)

$$\dot{H}^{s,r}_h := \dot{H}^{s,r}_h([0,t_1] \times \partial M)$$

$$\dot{H}^{s,r}_h = \{ u |_{(-\infty,t_1) \times \partial M} : u \in H^{s,r}_h, \supp u \subset [0,\infty) \},$$

$$\| u \|_{\dot{H}^{s,r}_h} := \inf \| u \|_{H^{s,r}_h},$$

where the infimum is over $U$'s such that $u = U|_{(-\infty,t_1) \times \partial M}$. The following lemma collects the estimates we need:

**Lemma E.62.** For spaces $\dot{H}^{s,r}_h$ defined in (E.5.75) we have

(E.5.76) $$s + r \leq s' + r', \ s \leq s' \implies \| u \|_{\dot{H}^{s,r}_h} \leq \| u \|_{\dot{H}^{s',r'}_h},$$

and for $C$ depending on $s$ only,

(E.5.77) $$\| u \|_{\dot{H}^{s,r}_h} \leq C \left( \| u \|_{\dot{H}_h^{0,r+s}} + \| (hD_t)^2 u \|_{\dot{H}_h^{s-2,r}} \right).$$

Moreover,

(E.5.78) $$f \in \dot{H}^{-1,r}_h \implies \exists f_0, f_1 \in \dot{H}^{s,r-j}_h \ f = f_1 + hD_t f_0,$$

and $\| f \|_{\dot{H}^{-1,r}_h} \leq C \| f \|_{\dot{H}^{s,r}_h}$.

**Proof.** 1. Inequality (E.5.76) follows from the the estimate $\| w \|_{\dot{H}^{s,r}_h} \leq \| w \|_{\dot{H}^{s',r'}_h}$, $w \in H^{s,r}_h(\mathbb{R} \times \partial M)$ and that is immediate from the definition of the norm in (E.5.75).

2. Again we only need to prove this for $w \in H^{s,r}_h(\mathbb{R} \times \partial M)$. If

(E.5.79) $$\mathcal{F}_h w(\tau,y) := (2\pi^2)^{-1/2} \int_{\mathbb{R}} w(t,y)e^{-it\tau/h},$$

then

$$\| w \|_{H^{s,r}_h} \simeq \| (1 + \tau^2 - h^2 \Delta_{\partial M})^{\frac{1}{2}} (I - h^2 \Delta_{\partial M})^{\frac{1}{2}} \mathcal{F}_h w(\tau,\bullet) \|_{L^2(\mathbb{R} \times \partial M)}.$$
Hence, (E.5.77) follows from the fact that for \( \lambda > 1 \) (and \( s \in \mathbb{R} \)),
\[
(\lambda^2 + \lambda)^{s} \lambda^r \leq C \left( (\lambda^{s+r} + \tau^4 (\lambda^2 + \lambda)^{s-2} \lambda^r) \right).
\]
(Divide both sides by \( \lambda^{s+r} \) and put \( \sigma = \tau^2 / \lambda \); then the inequality reads \( (1 + \sigma)^s \leq C(1 + \sigma^2(1 + \sigma)^{s-2}) \) which holds with \( C = 2^{\max(s,1)} \).

3. To obtain (E.5.78) we follow the proof of [HöIII, Theorem B.2.4] and note that
\[
\Lambda_{s,r}w := F^{-1}_h \left( \left( (I - h^2 \Delta_{\partial M})^{\frac{1}{2}} + i\tau \right)^{-s} (I - h^2 \Delta_{\partial M})^{-r} F_h w(\tau, \bullet) \right)
\]
is an isomorphism \( \Lambda_{s,r} : L^2([0, \infty) \times \partial M) \to \tilde{H}^{s,r}_h([0, \infty) \times \partial M) \). That is because the support of the inverse Fourier transform of \( \tau \mapsto (\lambda + i\tau)^{-s} \) is contained in \([0, \infty) \). (This follows from a version of the Schwartz–Paley–Wiener Theorem [HöI, Theorem 7.3.1] since for \( \lambda > 0 \), \( \tau \mapsto (\lambda + i\tau)^{-s} \) is holomorphic and polynomially bounded in \( \text{Im} \tau \leq 0 \) – see [HöI, Exercise 7.1.17].) By using the spectral decomposition of \(-\Delta_{\partial M}\) (see (E.5.73)) we see that on each eigenstate \( \Lambda_{s,r} \) is a convolution in \( t \) by a function supported in \([0, \infty)\), which preserves the support property.

Since
\[
\Lambda_{s-1,r} = ((I - h^2 \Delta_{\partial M})^{\frac{1}{2}} + ihD_t)\Lambda_{s,r},
\]
we obtain the desired decomposition. \( \square \)

**Proof of Theorem E.59.** The operator \( P \) can be extended to \((-\infty, 1) \times \partial M\) and remain hyperbolic there. We use the same notation for that extension.

1. Let us fix \( t_1 \in (0, 1) \). It is useful in the proof to have the following uniqueness statement: if \( u \in \tilde{H}^s(M) \) and \( Pu|_{[0,t_1] \times \partial M} = 0 \) then \( u|_{[0,t_1] \times \partial M} = 0 \). In fact, define \( U \in H^s((-\infty, t_1) \times \partial M) \), \( \text{supp } U \subset [0, t_1) \times \partial M \) by \( U|_{[0,t_1] \times \partial M} = u \). Then \( PU = 0 \) and \( U|_{t<0} = 0 \). Arguing as in Step 3 of the proof of Theorem E.61 we see that for any \( j \), \( U \in \tilde{H}^{s+j}((-\infty, t_1); H^{s-j}(\partial M)) \). But then if \( j \) is large enough we conclude that \( U = 0 \) since we can apply Lemma E.60 on \((-\delta, t_1)\). (That is, with \( t = -\delta \) playing the role of \( t = 0 \) in the lemma).

2. We will prove that for all \( s \) and \( r \), if \( f \in \tilde{H}^{s-1,r} \) then there exists a unique \( u \in \tilde{H}^{s,r}_h \), \( Pu = f \). That \( u \) satisfies
\[
(E.5.80) \quad \|u\|_{\tilde{H}^{s,r}_h} \leq C_h \|Pu\|_{\tilde{H}^{s-1,r}_h}.
\]
We note that (E.5.55) follows from (E.5.80) if we choose \( t_1 \) so that \( t_1|_{\text{supp } \chi_t} > t_1 \) and put \( r = 0 \).

3. We first prove (E.5.80) for \( s \geq 1 \). In that case,
\[
Pu = \tilde{H}^{s-1,r}_h \subset L^1((0, t_1); \tilde{H}^{s+r}_h)
\]
and we can apply Theorem E.61 to see that

\[ u \in C([0, t_1); H^{s+r}(\partial M)) \cap C^1([0, t_1); H^{s+r-1}(\partial M)), \]

(E.5.81)

\[ \|u(t)\|_{H^{s+r}(\partial M)} + \|hD_t u(t)\|_{H^{s+r-1}(\partial M)} \leq \frac{C}{h} \|Pu\|_{\dot{H}^{s-1,r}}, \]

and \( u(0) = D_t u(0) = 0 \). (We can solve the equation for \(-\delta < t < t_0\) and use the support condition on \( u \) to see that \( u \) is the unique solution given in Theorem E.61.)

4. We already know that \( u \in \dot{H}^{0,r+s}_h \) and we claim that

(E.5.82)

\[ \|u\|_{\dot{H}^{\min(j,s),r+s-\min(j,s)}_h} \leq \frac{C}{h} \|Pu\|_{\dot{H}^{s-1,r}_h}, \]

with the case of \( j = 0 \) (recall that \( s \geq 1 \)) covered by (E.5.81). Hence assume that (E.5.82) is valid for \( j \geq 0 \). Then

\[ (hD_t)^2 u = -P_0 u - P_1 hD_t u + Pu, \quad Pu \in \dot{H}^{s-1,r}_h, \]

\[ P_0 u \in \dot{H}^{\min(j,s),r+s-\min(j,s)-2}_h, \quad P_1 hD_t u \in \dot{H}^{\min(j,s)-1,r+s-\min(j,s)-1}_h. \]

This implies that

\[ (hD_t)^2 u \in \dot{H}^{\min(j,s)-1,r+s-\min(j,s)-1}_h + \dot{H}^{s-1,r}_h \subset \dot{H}^{\min(j+1,s)-2,r+s-\min(j+1,s)}_h, \]

with estimates by \( C\|Pu\|_{\dot{H}^{s-1,r}_h}/h \). In fact, if \( j + 1 \leq s \) or \( s \leq j \) this is immediate. If \( j < s < j + 1 \) then \( s - 2 \leq j - 1 \) and the estimate follows from (E.5.76). We now use (E.5.77) to obtain (E.5.82).

By taking \( j \) large enough we see that \((hD_t)^2 u \in \dot{H}^{s-2,r}_h\) and (E.5.77) shows that (E.5.80) holds in the case of \( s \geq 1 \) and for all \( r \in \mathbb{R} \).

5. To handle (E.5.80) for \( s \leq 1 \) we proceed by induction on \( j \) with \( s \in [-j+1, -j+2] \) with the case of \( j = 0 \) established in Step 3. In other words, we can assume that (E.5.80) is valid for \( s \) replaced by \( s + 1 \), in the sense that for all \( f \in \dot{H}^{s,r}_h \) there exists \( u \in \dot{H}^{s+1,r}_h \) such that \( Pu = f \) (with \( r' \in \mathbb{R} \) is arbitrary) and (E.5.80) holds with \( s \) replaced by \( s + 1 \) and \( r \) by \( r' \).

So, assume that \( f = Pu \in \dot{H}^{s-1,r}_h \) and use (E.5.78) to write \( f = f_1 + hD_t f_0, f_j \in \dot{H}^{s,r-j}_h \).

The inductive hypothesis shows that there exist \( w_j \in \dot{H}^{s+1,r-j}_h \) such that

\[ Pu_j = f_j. \]

If \( u_1 := w_1 + hD_t w_0 \) then

\[ Pu_1 - f = g, \quad g := [P, hD_t]w_0 \in \dot{H}^{s,r-1}_h, \]

\[ \|g\|_{\dot{H}^{s,r-1}_h} \leq C h \|w_0\|_{\dot{H}^{s+1,r}_h} \leq C \|f_0\|_{\dot{H}^{s,r}_h} \leq C \|f\|_{\dot{H}^{s-1,r}_h}. \]
We can use the inductive hypothesis again to find \( u_2 \in \dot{H}_h^{s+1,r-1} \subset \dot{H}_h^{s,r} \) such that \( Pu_2 = -g \). Then \( u := u_1 + u_2, u \in \dot{H}_h^{s,r} \) such that \( Pu = f \) and (E.5.80) holds. Step 1 shows that, for \( r = 0 \), the solution is unique and hence it has to be equal to \( u \) in Theorem E.59.

6. We now show how (E.5.56) follows from (E.5.80). Hence, suppose that \( u \in \dot{H}_h^s(M) \) and \( Pu \in \dot{H}_h^{s-1}(M) \). Then for some \( t_2 < 1 \),

\[
Pw = f, \quad w := (1 - \chi_1)u, \quad f := (1 - \chi_1)Pu - [P, \chi_1]u,
\]

\[
t < t_2 \text{ on supp } w \text{ and supp } f,
\]

\[
\|f\|_{\dot{H}_h^{s-1}([0,t_2) \times \partial M)} \leq \|\chi_2 Pu\|_{\dot{H}_h^{s-1}(M)} + h\|\chi_2 u\|_{\dot{H}_h^s}.
\]

We now change the variable \( t \) to \( \tilde{t} = t_2 - t \) which means that in the notation \( f \in \dot{H}_h^{s-1,0}([0,t_2) \tilde{t} \times \partial M) \) and \( w \in \dot{H}_h^{s-1,0}([0,t_2) \tilde{t} \times \partial M) \),

\[
\|f\|_{\dot{H}_h^{s-1,0}} = \|f\|_{\dot{H}_h^{s-1}(M)}, \quad \|w\|_{\dot{H}_h^{s,0}} = \|w\|_{\dot{H}_h^s(M)}.
\]

But then (E.5.56) follows from (E.5.80). \( \square \)

E.6. NOTES

For a fixed \( h \), say \( h = 1 \), the theory presented here is the standard theory of pseudodifferential operators known as microlocal analysis. Good introductions include Alinhac–Gérard [AG07] and Grigis–Sjöstrand [GS94]. A major treatise is Hörmander [HöI]–[HöIV].

The semiclassical theory with various applications is presented in several texts: Robert [Ro05], Helffer [He88] (potential wells, Witten complex), Martinez [Ma02a] (FBI transform), Dimassi–Sjöstrand [DS99] (fine spectral asymptotics), Zworski [Zw12] (broad introduction, semiclassical defect measures), Guillemin–Sternberg [GS13] (functorial approach, Maslov indices), Combescure–Robert [CR12] (coherent states). The reader can consult these works for history of the subject and further references.

Although most of the material reviewed in this appendix can be found in the abovementioned texts, Chapter 5 requires an extension of the theory in which propagation estimates are provided uniformly in the standard microlocal and semiclassical senses. This includes our definition of the semiclassical wave front as subset of \( T^*M \cup \partial T^*M \) coming from [Dy12] and [DZ16a] and radial propagation estimates, also from [DZ16a]. The latter were originally presented in the context of scattering theory by Melrose [Me94] and were first used for scattering on asymptotically hyperbolic manifolds by Vasy [Va13]. See Dyatlov–Guillarmou [DG14] for more general propagation estimates.
E.7. EXERCISES

Section E.1

1. Let $M$ be a manifold and $A : C^\infty(M) \to C^\infty(M)$ a linear operator. For $f \in C^\infty(M)$, let $\text{ad}_f(A) = [f, A]$ be the commutator of $A$ with the multiplication operator by $f$. Let $k \in \mathbb{N}_0$.

(a) Show that $A = A(h) \in \text{Diff}^k_h(M)$ if and only if $\text{ad}_f A \in h \text{Diff}^{k-1}_h(M)$, $h \partial_h A - k A \in \text{Diff}^{k-1}_h(M)$ for each $f \in C^\infty(M)$, with $\text{Diff}^{-1}_h(M) := 0$.

(b) Let $A \in \text{Diff}^k_h(M)$. Show that $\sigma_h(A) = a_0 + a_1 + \cdots + a_k$, where $a_\ell$ is a homogeneous polynomial of degree $\ell$ on the fibers of $T^*M$ satisfying

$$\text{(ad}_f^\ell A)g(x) = \ell! (ih)^\ell a_\ell(x, df(x))g(x) + \mathcal{O}(h^{\ell+1})$$

for all $h$-independent $f, g \in C^\infty(M)$.

2. Let $M$ be a compact manifold, $a \in C^\infty(T^*M)$, and fix a Riemannian metric $g$ on $M$. Show that $a \in S^{0}_{1, 0}(T^*M)$ if and only if

$$\sup_{s>1} \|a_s\|_{C^N((1 \leq |\xi| \leq 2))} < \infty$$

for each $N$, where $a_s(x, \xi) := a(x, s\xi)$.

3. Suppose $a \in S^{k}_{1, 0}(T^*M)$ for some $k > 0$ and $|a| \leq 1$ everywhere. Show that $a \in S^{\varepsilon}_{1, 0}(T^*M)$ for all $\varepsilon > 0$.

4. Construct a symbol $a \in S^{0}_{1, 0}(T^*\mathbb{R})$ and a diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $a(x, \varphi'(x)\xi)$ does not lie in $S^{0}_{1, 0}(T^*\mathbb{R})$.

5. Let $\text{Op}_h^M$ be defined in (E.1.28). Take $a(x) \in C^\infty(M)$ and consider it as a function on $T^*M$. Show that $\text{Op}_h^M(a)$ is the multiplication operator by $a$.

6. This exercise outlines an alternative proof of Proposition E.18.

(a) Use (E.1.37) to reduce to case $k = s = 0$.

(b) Fix constants $N \in \mathbb{N}_0$, $C_0 > \sup |a|$. Use induction on $N$ to construct $b_N(h) \in \overline{S}^0_{1, 0}(T^*\mathbb{R}^n)$, $r_N(h) \in \overline{S}^-_{1, 0}(T^*\mathbb{R}^n)$ such that

$$C_0^2 = \text{Op}_h(b_N)^* \text{Op}_h(a) + \text{Op}_h(b_N^*) \text{Op}_h(b_N) + h^N \text{Op}_h(r_N).$$

(c) Show that for $r \in \overline{S}^-_{1, 0}(T^*\mathbb{R}^n)$ and $N \geq n+1$, the norm $\|\text{Op}_h(r)\|_{L^2 \to L^2}$ is bounded uniformly in $h$. (Hint: use integration by parts in $\xi$ to show that the Schwartz kernel $K_{\text{Op}_h(r)}(x, y)$ is $O(h^{-n}((x-y)/h)^{-R})$ for each $R \in \mathbb{N}_0$. Then apply Schur’s inequality (A.5.3).)

(d) Argue similarly to the proof of Proposition E.24 to show

$$\|\text{Op}_h(a)u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n),$$
where $C$ is independent of $u$ and $h$, and finish the proof.

7. Let $\overline{M}$ be a compact manifold with boundary $\partial M \neq \emptyset$ and interior $M$. Consider the map

$$\hat{H}^s(\overline{M}) \to \hat{H}^s(M), \quad u \mapsto u|_M.$$  

(a) Show that (E.7.1) is one-to-one if and only if $s \geq -\frac{1}{2}$.
(b) Show that the image of (E.7.1) is dense in $\hat{H}^s(M)$ if and only if $s \leq \frac{1}{2}$.
(Hint: use the duality of $\hat{H}^s(M)$ with $\hat{H}^{-s}(\overline{M})$.)
(c) Show that the image of (E.7.1) is closed if and only if $\ldots$.

Section E.2

8. Let $(M,g)$ be a Riemannian manifold and $u \in D'(M)$. Use Theorem E.32 to show that if $\Delta_g u$ is smooth, then so is $u$. (Hint: let $P = -h^2\Delta_g$ and use the fact that $A \in \Psi^0_h(M)$ is smoothing when WF$_h(A) \cap \partial T^* M = \emptyset$.)

Is the result still true if $(M,g)$ is a Lorentzian manifold (see §5.7) and $\Delta_g$ is replaced by the d’Alembert–Beltrami operator?

9. Use Theorem E.32 to show that for each $\chi_0 \in C^\infty_0(\mathbb{R})$, there exists $\chi \in C^\infty_0(\mathbb{R})$ such that

$$\|\chi_0 u\|_{L^2} \leq C\|\chi(hD_x + i)u\|_{L^2} + O(h^\infty)\|\chi u\|_{L^2}$$

for all $u \in C^\infty(\mathbb{R})$. Give a direct proof of (E.7.2) and show that the $O(h^\infty)$ term there cannot be removed.

10. Fix $r \in \mathbb{R}$ and consider the family of operators

$$X_\varepsilon := Op_h((\varepsilon \xi)^{-r}) \in \Psi^r_h(M), \quad \varepsilon \in (0,1),$$

where $Op_h$ is defined in (E.1.28).

(a) Show that $X_\varepsilon = Op_{\varepsilon h}(\xi)^{-r}$.

(b) Using Proposition E.31, show that there exists a family of operators $Y_\varepsilon \in \Psi^r_h(M)$ such that uniformly in $\varepsilon$,

$$X_\varepsilon Y_\varepsilon = I + O(h^\infty)_{q^{-\infty}}, \quad Y_\varepsilon X_\varepsilon = I + O(h^\infty)_{q^{-\infty}}$$

(c) Let $A \in \Psi^k_h(M)$. Show that $X_\varepsilon AY_\varepsilon - A \in h\Psi^{k-1}_h(M)$ has full symbol (defined in (E.1.31)) uniformly bounded in $\varepsilon$ in the class $hS^{k-1}_{1,0}(T^* M)$ and

$$\sigma_h(h^{-1}(X_\varepsilon AY_\varepsilon - A)) = i\hbar \langle \varepsilon \xi \rangle^{r}\{\sigma_h(A), \langle \varepsilon \xi \rangle^{-r}\}.$$  

(Hint: write $X_\varepsilon AY_\varepsilon - A = [X_\varepsilon, A]Y_\varepsilon + O(h^\infty)_{q^{-\infty}}$. To estimate $[X_\varepsilon, A]$, split $A$ into two parts, one of which is microlocalized in $\{|\xi| \leq \varepsilon^{-1}\}$ and the other one is $\varepsilon^{-k}$ times the $Op_{\varepsilon h}$ quantization of some symbol in $S^k_{1,0}$.)
(d) Let $r > 0$ and $u \in H^{s-r}_{\text{comp}}(M)$. Assume that $\|X_\varepsilon u\|_{H^s}$ is bounded uniformly as $\varepsilon \to 0$. Show that $u \in H^{s-r}_{\text{comp}}(M)$. (Hint: reduce to the case of $\mathbb{R}^n$ and use the Monotone Convergence Theorem.)

11. Show that a family of operators $B : C_0^\infty(M) \to \mathcal{D}'(M')$ is $h$-tempered if and only if for each $\chi \in C_0^\infty(M)$, $\chi' \in C_0^\infty(M')$ there exist $C, N$ such that $\|\chi' B \chi\|_{H^N_{h \to H^{-N}_{h}}} \leq C h^{-N}$.

12. Under the assumptions of Proposition E.38, use stationary phase in the $y, \xi$ variables to show that for each $B \in \Psi^k_h(M)$, $Bu$ has the form (E.2.9) modulo a $O(h^{\infty})C_0^\infty$ remainder, with the amplitude

$$b(x, \theta; h) = \sigma_h(B)(x, \partial_x \varphi(x, \theta))a(x, \theta) + O(h).$$

13. Assume that the conditions of Proposition E.38 are satisfied, and $\varphi$ is nondegenerate in the sense that $\partial_{\theta_1} \varphi, \ldots, \partial_{\theta_m} \varphi$ are linearly independent on the set $\{\partial_\theta \varphi = 0\}$. Show that the set

$$\Lambda_\varphi = \{(x, \partial_x \varphi(x, \theta)) \mid \partial_\theta \varphi(x, \theta) = 0\}$$

is an immersed Lagrangian submanifold of $T^*M$.

14. Use Proposition E.39 to show that $(x_0, \xi_0) \notin WF_h(u)$ if and only if there exists a function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a neighbourhood $V$ of $\xi_0$ in the radial compactification of $\mathbb{R}^n$ such that the Fourier transform $\widehat{\chi u}$ satisfies

$$\widehat{\chi u}(\xi/h) = O(h^{\infty}(\xi)^{-\infty}), \quad \xi \in V.$$

15. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and define the standard wavefront set

$$WF(u) \subset T^*\mathbb{R}^n \setminus 0$$

following [HöI, §8.1]: a point $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus 0$ does not lie in $WF(u)$ if and only if there exists a function $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a conic neighbourhood $V$ of $\xi_0$ in $\mathbb{R}^n$ such that $\widehat{\chi u}(\xi) = O((\xi)^{-\infty})$, $\xi \in V$.

(a) Prove that if $u \in \mathcal{D}'(\mathbb{R}^n)$ is independent of $h$, then $WF(u) = WF_h(u) \cap (T^*\mathbb{R}^n \setminus 0)$.

(b) Show that

$$\text{singsupp } u = \pi(WF(u))$$

where $\pi(x, \xi) = x$ and singsupp $u$ is the complement of elements of $\mathbb{R}^n$ having open neighbourhoods $U$ such that $u|_U \in C^\infty(U)$. 

16. Let $M$ be a manifold, $\Sigma \subset M$ be an embedded submanifold, fix volume forms on $M, \Sigma$, and let $a \in C^\infty_0(\Sigma)$. Define the distribution

$$u = a \otimes \delta_\Sigma \in \mathcal{E}'(M), \quad \langle u, \varphi \rangle = \int_\Sigma a\varphi \, dx, \quad \varphi \in C^\infty(M).$$

Show that the standard wavefront set (see (E.7.5)) of $u$ is contained in the conormal bundle

$$(E.7.6) \quad N^*\Sigma = \{(x, \xi) \in T^*M \setminus 0 : x \in \Sigma, \xi \perp T_x\Sigma\}.$$

(Hint: use (E.2.12) for a multiplication operator.)

17. Using the previous exercise, obtain a restriction on $\WF'(\varphi^*)$ (defined similarly to (E.2.6) using the standard wavefront set of the Schwartz kernel), where $\varphi^*$ is the pullback operator by a smooth map $\varphi : M_1 \to M_2$.

18. Let $M$ be a manifold, $\Sigma \subset M$ an embedded submanifold, and assume that $u(h) \in \mathcal{D}'(M)$ is $h$-tempered and satisfies

$$\WF_h(u) \cap (\partial T^*M \cap N^*\Sigma) = \emptyset.$$ 

Assume moreover that $u(h) \in C^\infty(M)$ for each $h$, but with no uniform control in $h$. Show that the restrictions $u(h)|_\Sigma$ are $h$-tempered.

19. Calculate the semiclassical wavefront sets of the following distributions on $\mathbb{R}^n$:

(a) $e^{-\frac{|x|^2}{2h}}$;

(b) $\chi(x/h)$, where $\chi \in \mathcal{S}(\mathbb{R}^n)$;

(c) $e^{-1/h}\delta_0(x)$.

20. For two manifolds $M_1, M_2$, show that the natural map

$$T^*M_1 \times T^*M_2 \to T^*(M_1 \times M_2)$$

extends to continuous maps

$$\overline{T}^*M_1 \times T^*M_2 \to \overline{T}^*(M_1 \times M_2), \quad T^*M_1 \times \overline{T}^*M_2 \to \overline{T}^*(M_1 \times M_2),$$

but not to a continuous map $T^*M_1 \times T^*M_2 \to T^*(M_1 \times M_2)$. (This explains why we do not handle the fiber infinity in Proposition E.41.)

21. Let $A \in \Psi^k_h(M)$. Prove that

$$\WF'_h(A) = \{(x, \xi, x, \xi) : (x, \xi) \in \WF_h(A)\},$$

where the left-hand side uses (E.2.6) and the right-hand side, Definition E.27.

22. Let $B(h) : C^\infty_0(M_2) \to \mathcal{D}'(M_1)$ be $h$-tempered. Show that a point $(x, \xi, y, \eta) \in T^*(M_1 \times M_2)$ does not lie in $\WF'_h(B)$ if and only if there
exists neighbourhoods $U(x, \xi) \subset T^*M_1$, $V(y, \eta) \subset T^*M_2$ such that for each $h$-tempered family of functions $f \in C^\infty_0(M_2)$, we have

$$WF_h(f) \subset V \implies WF_h(Bf) \cap U = \emptyset.$$  

Section E.3

Section E.4

Section E.5

23. Let $M$ be a compact manifold with a fixed smooth density and $X$ be a divergence free vector field. Using Lemma E.47, prove that the operator

$$P := \frac{1}{i} X : \mathcal{D}_P \to L^2(M), \quad \mathcal{D}_P = \{u \in L^2(M) \mid Pu \in L^2(M)\}$$

is self-adjoint on $L^2(M)$.

24. Show that Lemma E.48 does not hold under the assumptions

$$u \in H^s_{\text{loc}}, \quad Pu \in H^{s-k+1+\varepsilon}_{\text{comp}}, \quad v \in H^{-s+k-1-\varepsilon}_{\text{comp}}, \quad P^*v \in H^s_{\text{comp}}$$

for any $\varepsilon > 0$, using the example $P = x \in \Psi^0_h(\mathbb{R})$, $u = (x + i0)^{-1}$, and $v = \delta_0$.

25. Let $P \in \Psi^k_h(M)$ and $p = \sigma_h(P)$ be real-valued. Consider a segment

$$\gamma := \{e^{t(\xi)} \cdot e^{H_P(x_0, \xi_0)} \mid t \in [0, T]\}, \quad (x_0, \xi_0) \in T^*M.$$  

Use Theorem E.49 to prove the following statement: for each $u \in C^\infty(M)$ such that $\gamma \cap WF_h(Pu) = \emptyset$, we have either $\gamma \subset WF_h(u)$ or $\gamma \cap WF_h(u) = \emptyset$.

26. Let $q \in C^\infty(\mathbb{R})$ be a nonnegative function and $\chi_1, \chi_2 \in C^\infty_0(\mathbb{R})$ satisfy $\text{supp} \chi_1 \subset (0, \infty)$ and $\chi_2(0) \neq 0$.

(a) Use Theorem E.49 to show that there exists $\chi \in C^\infty_0(\mathbb{R})$ such that

$$(E.7.7) \quad \|\chi_1 u\|_{L^2} \leq C\|\chi_2 u\|_{L^2} + Ch^{-1}\|\chi(hD_x - iq)u\|_{L^2} + O(h^\infty)\|\chi u\|_{L^2}$$

for all $u \in C^\infty(\mathbb{R})$. Give a direct proof of (E.7.7).

(b) Show that $h^{-1}$ cannot be replaced in (E.7.7) by $h^{-\delta}$ for $\delta < 1$.

(c) Does (E.7.7) still hold when $q$ is allowed to be negative? (Hint: consider $q(x) = x - 1$.)

27. Show that $Ch^{-1}\|B_1f\|_{H^{s-k+1}}$ cannot in general be replaced in (E.5.12) by $Ch^{-1}\|B_1f\|_{H^{s-k+\delta}}$ for $\delta < 1$.

28. This exercise outlines a proof of the following strengthening of Theorem E.49: if $u \in \mathcal{D}'(M)$, $f = Pu$, and $Bu \in H^s$, $B_1f \in H^{s-k+1}$, then $Au \in H^s$ and (E.5.12) holds.
(a) Fix $r > 0$ and let $X_\varepsilon = \text{Op}_h(\langle \varepsilon \xi \rangle^{-r}) \in \Psi_h^{-r}(M)$, $Y_\varepsilon \in \Psi_h(M)$, $\varepsilon > 0$, be the families of operators from Exercise E.10. Put

$$P_\varepsilon := X_\varepsilon Y_\varepsilon \in \Psi^k_h(M).$$

Applying Theorem E.49 to $P_\varepsilon$ and $X_\varepsilon u$ and using Lemma E.47 and Exercise E.10(c), show the following estimate with bounds uniform in $\varepsilon$ [TODO a bit of subtlety here because of the $B_{1\varepsilon}$ condition]:

$$\|X_\varepsilon Au\|_{H^k_h} \leq C\|X_\varepsilon Bu\|_{H^k_h} + Ch^{-1}\|X_\varepsilon B_1 f\|_{H^k_h} + O(h^\infty)\|\chi u\|_{H^{N-1}}$$

for all $u \in H^{s-r}_{\text{loc}}(M)$ such that $f := Pu \in H^{s-r-k+1}_{\text{loc}}(M)$.

(b) Complete the proof by noting that $\chi u \in H^{-N}$ for large enough $N$, fixing $r$ large enough depending on $N$, and using Exercise E.10(d).

29. Show that in Lemma E.51, it is enough to require that (E.5.27) be satisfied on the critical set $\{\langle \xi \rangle^{-k} p = 0\}$, if we additionally assume that $\langle \xi \rangle^{-k}$ has no critical points on $\{\langle \xi \rangle^{-k} p = 0\} \cap \text{WF}_h(B_1)$. (Hint: add a term of the form $h \text{Re}(f, XG u)$ to the left-hand side of (E.5.15), with an appropriately chosen $X \in \Psi^{s-\frac{k+1}{2}}_h$.)

30. Show that if $a$ is eventually positive on $L$ with respect to $p$ (see Proposition E.53) and $a_1 \in S^0(T^*M; \mathbb{R})$, then $a + H_p a_1$ is eventually positive as well. Use this to show that eventual negativity of (E.5.36) and (E.5.44) is independent of:

(a) the metric in the definition of $\langle \xi \rangle$; in fact $\langle \xi \rangle$ may be replaced by any real-valued symbol in $S^1(T^*M)$ which is elliptic on $L$; and

(b) the density on $M$ in the definition of $\text{Im} \ P$. (Hint: use the identity $\psi^{-1} P^* \psi = P^* + \psi^{-1}[P^*, \psi]$ valid for positive $\psi \in C^\infty(M)$.)

31. Use (E.5.31) and (E.5.34) to show that $\langle \xi \rangle^{-1} H_p \langle \xi \rangle$ is eventually negative on a radial source and eventually positive on a radial sink. Deduce that

(a) if $s_1 < s_2$ and (E.5.36) is eventually negative for $s = s_1$, then it is eventually negative for $s = s_2$;

(b) if $s_1 < s_2$ and (E.5.44) is eventually negative for $s = s_2$, then it is eventually negative for $s = s_1$;

(c) (E.5.36) is eventually negative for $s > 0$ large enough;

(d) (E.5.44) is eventually negative for $-s > 0$ large enough.

32. Consider the operator $P := x(hD_x) + i\gamma h$ and its radial source $L = \partial^T \mathbb{R} \cap \{x = 0\}$, see (E.5.33) and Figure E.3.

(a) Show that (E.5.36) is eventually negative on $L$ if and only if $s > \text{Re} \gamma + \frac{1}{2}$. 
(b) Let \( s > \text{Re} \gamma + \frac{1}{2} \). Use Theorems E.49 and E.54 and Lemma E.47 to show that for each compactly supported \( A \in \Psi_0^0(\mathbb{R}) \) such that \( \text{WF}_h(A) \cap \{ \xi = 0 \} = \emptyset \), there exists \( \chi \in C_0^\infty(\mathbb{R}) \) such that for each

\[
u \in H^s_{\text{loc}}(\mathbb{R}), \quad f := Pu \in H^s_{\text{loc}}(\mathbb{R}),
\]

we have the following estimate for all \( N \):

\[
\| Au \|_{H^s} \leq Ch^{-1} \| \chi f \|_{H^s_h} + O(h^\infty) \| \chi u \|_{H^{-N}_h}.
\]

(e) Assume that \( s < \text{Re} \gamma + \frac{1}{2} \). Show that (E.7.8) no longer holds, by taking \( u(x) = x^\gamma / \Gamma(\gamma + 1) \) and using Exercise E.15 and the formulas for the Fourier transform of \( u \), see [HöI, Example 7.1.17]. (Hint: use repeated integration by parts to show that the Fourier transform of \((1 - \chi_0)u\) is rapidly decaying at infinity for \( \chi_0 \in C_0^\infty(\mathbb{R}) \) which is equal to 1 near the origin.)

33. Following the strategy of Exercise E.28, obtain the following strengthening of Theorem E.54: if (E.5.36) is eventually negative for some \( s' < s \), \( u \in \mathcal{D}'(M) \), \( f = Pu \), and \( B_1 u \in H^{s'} \), \( B_1 f \in H^{s-k+1} \), then \( Au \in H^s \) and (E.5.37) holds. (The main difference from Exercise E.28 is to use (E.7.4) and Exercise E.31 to verify that the threshold condition holds on \( H^s_h \) for \( P_\varepsilon \) uniformly in \( \varepsilon \) when \( r := s - s' \).)

34. Following the strategy of Exercise E.28, obtain the following strengthening of Theorem E.56: if \( u \in \mathcal{D}'(M) \), \( f = Pu \), and \( Bu \in H^s \), \( B_1 f \in H^{s-k+1} \), then \( Au \in H^s \) and (E.5.45) holds. (As in Exercise E.33, the new component is the verification of the threshold condition for \( P_\varepsilon \).)

35. Show that the conclusion of Exercise E.34 is false for operator \(-P\), where \( P \) was studied in Exercise E.32, when \( s > \text{Re} \gamma + \frac{1}{2} \). (Hint: consider \( u(x) = x^\gamma / \Gamma(\gamma + 1) \) and use that it is smooth away from zero.)
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