Ruelle zeta at zero for nearly hyperbolic 3-manifolds

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Feb 23, 2021
Studying $m_R(0)$: order of vanishing at 0 of the Ruelle zeta function for the geodesic flow on a negatively curved 3-manifold $(\Sigma, g)$

- $g = g_H$ hyperbolic $\implies m_R(0) = 4 - 2b_1(\Sigma)$ \cite{fried1986}

- $g =$ generic perturbation of $g_H$ $\implies m_R(0) = 4 - b_1(\Sigma)$ \cite{cekic2020}

This is in contrast with the case $\dim \Sigma = 2$ where $m_R(0) = b_1(\Sigma) - 2$ for all negatively curved $(\Sigma, g)$ \cite{dzworski2017}

Motivated by Fried’s conjecture ’87 relating the values at 0 of twisted dynamical zeta functions to analytic torsion
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Geodesic and contact flows

- $(\Sigma, g)$ a compact connected oriented Riemannian $n$-dim manifold
- $M = S\Sigma$ the sphere bundle of $(\Sigma, g)$, $\pi_\Sigma : M \to \Sigma$ projection map
- $\alpha(x,v)(\xi) = \langle v, d\pi_\Sigma(x,v)\xi \rangle_g$ canonical 1-form on $M$
- $\alpha$ is a contact form: $d\text{vol}_\alpha := \alpha \wedge (d\alpha)^{n-1}$ is nonvanishing
- Geodesic flow: $\varphi_t = e^{tX} : M \to M$ where $X \in C^\infty(M; TM)$ given by
  \[ \iota_X \alpha = 1, \quad \iota_X d\alpha = 0 \]
- $g$ has negative sectional curvature $\implies \varphi_t$ is Anosov:
  \[ TM = E_0 \oplus E_u \oplus E_s, \quad E_0 = \mathbb{R}X, \]
  \[ \exists C, \theta > 0 : \quad \|d\varphi_{-t}|_{E_u}\|, \|d\varphi_t|_{E_s}\| \leq Ce^{-\theta t}, \quad t \geq 0 \]
- Define $E_u^* := (E_0 \oplus E_u)^\perp$, $E_s^* := (E_0 \oplus E_s)^\perp$ subsets of $T^* M$
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Define the **Ruelle zeta function**

\[ \zeta_R(\lambda) = \prod_\gamma (1 - e^{-\lambda T_\gamma}), \quad \text{Re } \lambda \gg 1 \]

where the product is over all primitive closed geodesics \( \gamma \) of periods \( T_\gamma \)

- The function \( \zeta_R(\lambda) \) continues meromorphically to \( \lambda \in \mathbb{C} \)
  [Giulietti–Liverani–Pollicott ’13, D–Zworski ’16]
  Conjectured by Smale ’67; partial progress by
  Ruelle ’76, Parry–Pollicott ’90, Rugh ’96, Fried ’95

- Define the vanishing order \( m_R(0) \in \mathbb{Z} \):
  \[ \lambda^{-m_R(0)} \zeta_R(\lambda) \text{ holomorphic and nonvanishing at } \lambda = 0 \]

**Question**

Can we describe \( m_R(0) \) in terms of topological invariants of \( \Sigma \), such as the Betti numbers \( b_k(\Sigma) = \dim H^k(\Sigma; \mathbb{R}) \)?
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More general zeta functions $\zeta_\rho(\lambda)$ twisted by a representation $\rho : \pi_1(\Sigma) \to U(m)$; $\zeta_R$ corresponds to the trivial $\rho : \pi_1(\Sigma) \to U(1)$

- $\rho$ is called acyclic if $H^k_\rho(\Sigma; \mathbb{R}) = 0$ for all $k$
- Fried '86 studied the hyperbolic case (curvature $= -1$):

$$m_R(0) = \begin{cases} b_1(\Sigma) - 2, & \text{dim } \Sigma = 2 \\ 4 - 2b_1(\Sigma), & \text{dim } \Sigma = 3 \end{cases}$$

For $\rho$ acyclic, he computed $m_\rho(0) = 0$ and $\zeta_\rho(0) = T_\rho^2$ where $T_\rho$ is the analytic torsion. Fried's conjecture: same formula for $\zeta_\rho(0)$ holds for general locally homogeneous $(\Sigma, g)$

- Fried's conjecture proved for locally symmetric spaces by Shen '16, following Moscovici–Stanton '91, Bismut '11
- All the above use Selberg trace formulas + representation theory
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What happens for general (not locally symmetric) negatively curved $\Sigma$?

- **D–Zworski '17**: $m_R(0) = b_1(\Sigma) - 2$ when $\dim \Sigma = 2$; applies to general contact Anosov flows in dimension 3

- Extended to surfaces with boundary by Hadfield '18, to the nonorientable case by Borns-Weil–Shen '20

- Cekić–Paternain '19: studied $m_R(0)$ for general volume preserving Anosov flows on a 3-manifold $M$ and showed it depends on the properties of the flow, not just on the topology of $M$

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Statement of the result

Theorem 1 [Čekić–D–Küster–Paternain '20]

Let \((\Sigma, g_H)\) be a compact connected oriented hyperbolic 3-manifold. Then:

1. If \(g = g_H\) then \(m_R(0) = 4 - 2b_1(\Sigma)\)

2. If \(g\) is a generic conformal perturbation of \(g_H\) then \(m_R(0) = 4 - b_1(\Sigma)\)

Here generic conformal perturbation is understood as follows:

- there exists an open dense \(\mathcal{O} \subset C^\infty(\Sigma; \mathbb{R})\) such that
  - for any \(a \in \mathcal{O}\) there exists \(\varepsilon > 0\) such that for all \(\tau \in (-\varepsilon, \varepsilon) \setminus \{0\}\)
    - the metric \(g = e^{\tau a}g_H\) has \(m_R(0) = 4 - b_1(\Sigma)\)

- First result on instability of \(m_R(0)\) under metric perturbations
- Our proof of part 1 is different from [Fried '86], using geometric rather than algebraic techniques
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- General idea: “$\zeta(\lambda) = \det(\lambda - P)$” for some operator $P$

- This should be understood as $\partial_\lambda \log \zeta(\lambda) = \text{tr}(\lambda - P)^{-1}$ with the right definition of trace

- Vanishing order of $\zeta$ at 0 = dimension of the space of generalized eigenstates at 0 $\{ u | \exists \ell : P^\ell u = 0 \}$

- One can write the vanishing order $m_R(0)$ of $\zeta_R$ using the dimensions of certain spaces of Pollicott–Ruelle generalized resonant forms $\text{Res}_{0}^{k,\infty}$

- Our strategy is to describe $\text{Res}_{0}^{k,\infty}$ in terms of the de Rham cohomology of $\Sigma$

- In this talk we will focus on the case $k = 1$
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$M = S\Sigma$, $\dim \Sigma = 3$, $X \in C^\infty(M; TM)$ generates the geodesic flow

Our operators: $P_{k,0} = \mathcal{L}_X$ acting on $\Omega^k_0 := \{ \omega \in \bigwedge^k T^*M \mid \iota_X \omega = 0 \}$

For certain anisotropic Sobolev spaces $\mathcal{H}, \mathcal{D}_P$ the operator $P_{k,0} - \lambda : \mathcal{D}_P(M; \Omega^k_0) \to \mathcal{H}(M; \Omega^k_0)$ is Fredholm of index 0

Blank–Keller–Liverani '02, Liverani '04,'05, Baladi '05, Gouëzel–Liverani '06, Baladi–Tsujii '07, Butterley–Liverani '07

We will use the microlocal/scattering theory approach:
Faure–Roy–Sjöstrand '08, Faure–Sjöstrand '11, D–Zworski '16
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- The poles of $(P_{k,0} - \lambda)^{-1}$ are called Pollicott–Ruelle resonances

- Generalized resonant states at $\lambda = 0$:
  $$\text{Res}_{0}^{k,\infty} = \{u \in \mathcal{D}_P(M; \Omega^k_0) \mid \exists \ell : \mathcal{L}_X^\ell u = 0\}$$

- D–Zworski '16, using Hörmander’s propagation of singularities, Melrose’s radial estimates, and Atiyah–Bott–Guillemin trace formula:
  $$m_R(0) = \sum_{k=0}^{4} (-1)^k \dim \text{Res}_{0}^{k,\infty}$$
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Resonance multiplicities

Theorem 1 follows from \( m_R(0) = \sum_{k=0}^{4} (-1)^k \dim \text{Res}^{k,\infty}_0 \) and

Theorem 2 [Cekić–D–Küster–Paternain ’20]

Let \((\Sigma, g_H)\) be a compact connected oriented hyperbolic 3-manifold. Then the dimensions of \(\text{Res}^{k,\infty}_0\) are:

<table>
<thead>
<tr>
<th>(k)</th>
<th>Hyperbolic</th>
<th>Perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(2b_1(\Sigma))</td>
<td>(b_1(\Sigma))</td>
</tr>
<tr>
<td>2</td>
<td>(2b_1(\Sigma) + 2)</td>
<td>(b_1(\Sigma) + 2)</td>
</tr>
<tr>
<td>3</td>
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<tr>
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\((d\alpha)^i \wedge : \text{Res}^{2-j,\infty}_0 \rightarrow \text{Res}^{2+j,\infty}_0\) isomorphisms \(\Rightarrow\) study \(k = 0, 1, 2\)
Resonance multiplicities

Theorem 1 follows from $m_{R}(0) = \sum_{k=0}^{4}(-1)^{k} \dim \text{Res}_{0}^{k, \infty}$ and

**Theorem 2 [Cekić–D–Küster–Paternain ’20]**

Let $(\Sigma, g_{H})$ be a compact connected oriented hyperbolic 3-manifold. Then the dimensions of $\text{Res}_{0}^{k, \infty}$ are:

<table>
<thead>
<tr>
<th>$k$</th>
<th>Hyperbolic</th>
<th>Perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$2b_{1}(\Sigma)$</td>
<td>$b_{1}(\Sigma)$</td>
</tr>
<tr>
<td>2</td>
<td>$2b_{1}(\Sigma) + 2$</td>
<td>$b_{1}(\Sigma) + 2$</td>
</tr>
<tr>
<td>3</td>
<td>$2b_{1}(\Sigma)$</td>
<td>$b_{1}(\Sigma)$</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$$(d\alpha)^i \wedge : \text{Res}_{0}^{2-j, \infty} \rightarrow \text{Res}_{0}^{2+j, \infty}$$ isomorphisms $\Rightarrow$ study $k = 0, 1, 2$
Resonant and coresonant states

- **Generalized resonant states:**
  \[ \text{Res}_{0}^{k,\infty} = \{ u \in \mathcal{D}_{P}(M; \Omega_{0}^{k}) \mid \exists \ell : \mathcal{L}_{X}^{\ell} u = 0 \} \]

- \( \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}) = \{ u \in \mathcal{D}'(M; \Omega_{0}^{k}) \mid \text{WF}(u) \subset E_{u}^{*} \} \)
  defined using wavefront set \( \text{WF}(u) \subset T^{*}M \setminus 0 \)

- **Resonant states:** \( \text{Res}_{0}^{k} = \{ u \in \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}) \mid \mathcal{L}_{X} u = 0 \} \)

- **Coresonant states:** \( \text{Res}_{0*}^{k} = \{ u* \in \mathcal{D}'_{E_{s}^{*}}(M; \Omega_{0}^{k}) \mid \mathcal{L}_{X} u* = 0 \} \)
  \( \text{Res}_{0*}^{k} = \mathcal{J}^{*} \text{Res}_{0}^{k} \) where \( \mathcal{J} : M \to M, \mathcal{J}(x, v) = (x, -v) \)

- **Pairing:** \( u \in \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}), \ u* \in \mathcal{D}'_{E_{s}^{*}}(M; \Omega_{0}^{4-k}) \mapsto \int_{M} \alpha \wedge u \wedge u* \)

- **Semisimplicity:** \( \text{Res}_{0}^{k,\infty} = \text{Res}_{0}^{k} \), equivalent to the pairing being nondegenerate on \( \text{Res}_{0}^{k} \times \text{Res}_{0*}^{4-k} \)

- The case \( k = 0 \) is simple: \( \text{Res}_{0}^{0,\infty} = \text{Res}_{0}^{0} = \mathbb{R}1 \)
Resonant and coresonant states

- **Generalized resonant states**: 
  \[ \text{Res}^{k,\infty}_0 = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0) | \exists \ell : \mathcal{L}^\ell_X u = 0 \} \]

- **\( \mathcal{D}'_{E^*_u}(M; \Omega^k_0) \)**: 
  \[ \{ u \in \mathcal{D}'(M; \Omega^k_0) | \text{WF}(u) \subset E^*_u \} \]
  defined using wavefront set \( \text{WF}(u) \subset T^*M \setminus 0 \)

- **Resonant states**: 
  \[ \text{Res}^k_0 = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0) | \mathcal{L}_X u = 0 \} \]

- **Coresonant states**: 
  \[ \text{Res}^k_{0^*} = \{ u^* \in \mathcal{D}'_{E^*_s}(M; \Omega^k_0) | \mathcal{L}_X u^* = 0 \} \]
  \[ \text{Res}^k_{0^*} = J^* \text{Res}^k_0 \text{ where } J : M \rightarrow M, J(x, \nu) = (x, -\nu) \]

- **Pairing**: 
  \[ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0), \ u^* \in \mathcal{D}'_{E^*_s}(M; \Omega^{4-k}_0) \mapsto \int_M \alpha \wedge u \wedge u^* \]

- **Semisimplicity**: 
  \[ \text{Res}^{k,\infty}_0 = \text{Res}^k_0, \text{ equivalent to the pairing being nondegenerate on } \text{Res}^k_0 \times \text{Res}^{4-k}_{0^*} \]

- **The case \( k = 0 \)** is simple: 
  \[ \text{Res}^{0,\infty}_0 = \text{Res}^0_0 = \mathbb{R}1 \]
Resonant and coresonant states

- **Generalized resonant states:**
  \[ \text{Res}^{k,\infty}_0 = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0) | \exists \ell : \mathcal{L}^\ell_X u = 0 \} \]

- \[ \mathcal{D}'_{E^*_u}(M; \Omega^k_0) = \{ u \in \mathcal{D}'(M; \Omega^k_0) | \text{WF}(u) \subset E^*_u \} \]
  defined using wavefront set \( \text{WF}(u) \subset T^* M \setminus 0 \)

- **Resonant states:** \( \text{Res}^k_0 = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0) | \mathcal{L}_X u = 0 \} \)

- **Coresonant states:** \( \text{Res}^k_{0*} = \{ u_* \in \mathcal{D}'_{E^*_s}(M; \Omega^k_0) | \mathcal{L}_X u_* = 0 \} \)
  \( \text{Res}^k_{0*} = \mathcal{J}^* \text{Res}^k_0 \) where \( \mathcal{J} : M \to M, \mathcal{J}(x, v) = (x, -v) \)

- **Pairing:** \( u \in \mathcal{D}'_{E^*_u}(M; \Omega^k_0), \ u_* \in \mathcal{D}'_{E^*_s}(M; \Omega^{4-k}_0) \mapsto \int_M \alpha \wedge u \wedge u_* \)

- **Semisimplicity:** \( \text{Res}^{k,\infty}_0 = \text{Res}^k_0 \), equivalent to
  the pairing being nondegenerate on \( \text{Res}^k_0 \times \text{Res}^{4-k}_{0*} \)

- **The case \( k = 0 \) is simple:** \( \text{Res}^{0,\infty}_0 = \text{Res}^0_0 = \mathbb{R}1 \)
Resonant and coresonant states

- **Generalized resonant states:**
  \[ \text{Res}^k_{0,\infty} = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k_0) \mid \exists \ell : \mathcal{L}^\ell_X u = 0 \} \]

- **D’E_u^*(M; \Omega^k_0) = \{ u \in \mathcal{D}'(M; \Omega^k_0) \mid \text{WF}(u) \subset E_u^* \} \]
  defined using wavefront set \( \text{WF}(u) \subset T^*M \setminus 0 \)

- **Resonant states:**
  \[ \text{Res}^k_0 = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k_0) \mid \mathcal{L}^\ell_X u = 0 \} \]

- **Coresonant states:**
  \[ \text{Res}^k_{0*} = \{ u_* \in \mathcal{D}'_{E_s^*}(M; \Omega^k_0) \mid \mathcal{L}^\ell_X u_* = 0 \} \]
  \[ \text{Res}^k_{0*} = \mathcal{J}^* \text{Res}^k_0 \] where \( \mathcal{J} : M \to M, \mathcal{J}(x, v) = (x, -v) \)

- **Pairing:**
  \[ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k_0), \quad u_* \in \mathcal{D}'_{E_s^*}(M; \Omega^{4-k}_0) \quad \mapsto \quad \int_M \alpha \wedge u \wedge u_* \]

- **Semisimplicity:**
  \[ \text{Res}^k_{0,\infty} = \text{Res}^k_0, \text{ equivalent to}\]
  the pairing being nondegenerate on \( \text{Res}^k_0 \times \text{Res}^{4-k}_{0*} \)

- **The case \( k = 0 \) is simple:**
  \[ \text{Res}^{0,\infty}_0 = \text{Res}^0_0 = \mathbb{R}1 \]
Resonant and coresonant states

- Generalized resonant states:
  $$\text{Res}_{0,\infty}^k = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^k) \mid \exists \ell : \mathcal{L}_X^\ell u = 0 \}$$

- **\( \mathcal{D}'_{E_u^*}(M; \Omega_0^k) = \{ u \in \mathcal{D}'(M; \Omega_0^k) \mid \text{WF}(u) \subset E_u^* \} \)**
  defined using wavefront set \( \text{WF}(u) \subset T^* M \setminus 0 \)

- Resonant states: \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^k) \mid \mathcal{L}_X u = 0 \} \)

- Coresonant states: \( \text{Res}_{0*}^k = \{ u^* \in \mathcal{D}'_{E_{s_u}^*}(M; \Omega_0^k) \mid \mathcal{L}_X u^* = 0 \} \)
  \( \text{Res}_{0*}^k = \mathcal{J}^* \text{Res}_0^k \) where \( \mathcal{J} : M \to M, \mathcal{J}(x, \nu) = (x, -\nu) \)

- Pairing: \( u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^k), \ u^* \in \mathcal{D}'_{E_{s_u}^*}(M; \Omega_0^{4-k}) \mapsto \int_M \alpha \wedge u \wedge u^* \)

- Semisimplicity: \( \text{Res}_{0,\infty}^k = \text{Res}_0^k \), equivalent to the pairing being nondegenerate on \( \text{Res}_0^k \times \text{Res}_{0*}^{4-k} \)

- The case \( k = 0 \) is simple: \( \text{Res}_{0,\infty}^0 = \text{Res}_0^0 = \mathbb{R}1 \)
Resonant and coresonant states

- **Generalized resonant states:**
  \[ \text{Res}_{0}^{k, \infty} = \{ u \in \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}) | \exists \ell : \mathcal{L}_{\chi}^{\ell} u = 0 \} \]

- **Denoted using wavefront set WF:**
  \[ \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}) = \{ u \in \mathcal{D}'(M; \Omega_{0}^{k}) | \text{WF}(u) \subset E_{u}^{*} \} \]
  defined using \( \text{WF}(u) \subset T^{*}M \setminus 0 \)

- **Resonant states:**
  \[ \text{Res}_{0}^{k} = \{ u \in \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}) | \mathcal{L}_{\chi} u = 0 \} \]

- **Coresonant states:**
  \[ \text{Res}_{0}^{k*} = \mathcal{J}^{*} \text{Res}_{0}^{k} \text{ where } \mathcal{J} : M \rightarrow M, \mathcal{J}(x, v) = (x, -v) \]

- **Pairing:**
  \[ u \in \mathcal{D}'_{E_{u}^{*}}(M; \Omega_{0}^{k}), \ u^{*} \in \mathcal{D}'_{E_{s}^{*}}(M; \Omega_{0}^{4-k}) \rightarrow \int_{M} \alpha \wedge u \wedge u^{*} \]

- **Semisimplicity:**
  \[ \text{Res}_{0}^{k, \infty} = \text{Res}_{0}^{k}, \text{ equivalent to the pairing being nondegenerate on } \text{Res}_{0}^{k} \times \text{Res}_{0}^{4-k} \]

- **The case \( k = 0 \) is simple:**
  \[ \text{Res}_{0}^{0, \infty} = \text{Res}_{0}^{0} = \mathbb{R} \]
Resonant and coresonant states

- **Generalized resonant states:**
  \[ \text{Res}_{0}^{k,\infty} = \{ u \in D'_{E_u^*}(M; \Omega_{0}^k) \mid \exists \ell : L_{\ell}^X u = 0 \} \]

- **D'_{E_u^*}(M; \Omega_{0}^k) = \{ u \in D'(M; \Omega_{0}^k) \mid \text{WF}(u) \subset E_{u}^* \} **
  defined using wavefront set \( \text{WF}(u) \subset T^* M \setminus 0 \)

- **Resonant states:** \( \text{Res}_{0}^{k} = \{ u \in D'_{E_u^*}(M; \Omega_{0}^k) \mid L_{X} u = 0 \} \)

- **Coresonant states:** \( \text{Res}_{0}^{k,\ast} = \{ u_{\ast} \in D'_{E_\ast^s}(M; \Omega_{0}^k) \mid L_{X} u_{\ast} = 0 \} \)
  \( \text{Res}_{0}^{k,\ast} = J^{\ast} \text{Res}_{0}^{k} \) where \( J : M \to M, J(x, \nu) = (x, -\nu) \)

- **Pairing:** \( u \in D'_{E_u^*}(M; \Omega_{0}^k), \ u_{\ast} \in D'_{E_\ast^s}(M; \Omega_{0}^{4-k}) \mapsto \int_M \alpha \wedge u \wedge u_{\ast} \)

- **Semisimplicity:** \( \text{Res}_{0}^{k,\infty} = \text{Res}_{0}^{k} \), equivalent to the pairing being nondegenerate on \( \text{Res}_{0}^{k} \times \text{Res}_{0}^{4-k} \)

- **The case** \( k = 0 \) is simple: \( \text{Res}_{0}^{0,\infty} = \text{Res}_{0}^{0} = \mathbb{R}1 \)
Closed resonant 1-forms

- \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \mathcal{L}_X u = 0 \} \), \( \mathcal{L}_X = d\iota_X + \iota_X d \)

- Closed forms: \( \text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0, \ du = 0 \} \)

- Cohomology map: \( \pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R}), \quad \pi_k(u) = [u]_{H^k} \)

- \( \pi_k \) can be defined because \( \mathcal{D}'_{E_u^*} \) is closed under \( (d\delta + \delta d + 1)^{-1} \):
  
  \[ u \in \mathcal{D}'_{E_u^*}, \ du \in \mathcal{C}\infty \implies u = v + dw \text{ for some } v \in \mathcal{C}\infty, \ w \in \mathcal{D}'_{E_u^*} \]

Lemma: \( \pi_1 \) is an isomorphism

**Injectivity:** if \( u \in \text{Res}_0^1 \) and \( u = df, \ f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R}) \), then \( Xf = \iota_X u = 0 \), so \( f \in \text{Res}_0^0 = \mathbb{R}1 \) and \( u = df = 0 \)

**Surjectivity:** if \( v \in \mathcal{C}\infty(M; \Omega^1) \) and \( dv = 0 \), then \( \int_M (\iota_X v) \ d\text{vol}_\alpha = 0 \), so by the Fredholm property there exists \( f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R}) \) with \( Xf = \iota_X v \).

Take \( u := v - df \in \text{Res}_0^1 \cap \ker d \), then \( \pi_1(u) = [v]_{H^1} \)
Closed resonant 1-forms

- \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, \ i_X du = 0 \}, \quad \Omega^k = \wedge^k T^* M \)

- Closed forms: \( \text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, \ du = 0 \} \)

- Cohomology map: \( \pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R}), \quad \pi_k(u) = [u]_{H^k} \)

- \( \pi_k \) can be defined because \( \mathcal{D}'_{E^*_u} \) is closed under \( (d\delta + \delta d + 1)^{-1} \):

  \[
  u \in \mathcal{D}'_{E^*_u}, \ du \in C^\infty \implies u = v + dw \text{ for some } v \in C^\infty, \ w \in \mathcal{D}'_{E^*_u}
  \]

**Lemma:** \( \pi_1 \) is an isomorphism

**Injectivity:** if \( u \in \text{Res}_0^1 \) and \( u = df, \ f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \), then \( Xf = \iota_X u = 0 \), so \( f \in \text{Res}_0^0 = \mathbb{R}1 \) and \( u = df = 0 \)

**Surjectivity:** if \( v \in C^\infty(M; \Omega^1) \) and \( dv = 0 \), then \( \int_M (\iota_X v) \, d\text{vol}_\alpha = 0 \), so by the Fredholm property there exists \( f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \) with \( Xf = \iota_X v \).

Take \( u := v - df \in \text{Res}_0^1 \cap \ker d \), then \( \pi_1(u) = [v]_{H^1} \)
Closed resonant 1-forms

- \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, \iota_X du = 0 \}, \quad \Omega^k = \wedge^k T^* M \)

- **Closed forms:** \( \text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, \ du = 0 \} \)

- **Cohomology map:** \( \pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R}), \quad \pi_k(u) = [u]_{H^k} \)

- \( \pi_k \) can be defined because \( \mathcal{D}'_{E^*_u} \) is closed under \((d\delta + \delta d + 1)^{-1}:

  u \in \mathcal{D}'_{E^*_u}, du \in C^\infty \implies u = v + dw \) for some \( v \in C^\infty, w \in \mathcal{D}'_{E^*_u} \)

**Lemma:** \( \pi_1 \) is an isomorphism

**Injectivity:** if \( u \in \text{Res}_0^1 \) and \( u = df, f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \), then \( Xf = \iota_X u = 0 \), so \( f \in \text{Res}_0^0 = \mathbb{R}1 \) and \( u = df = 0 \)

**Surjectivity:** if \( v \in C^\infty(M; \Omega^1) \) and \( dv = 0 \), then \( \int_M (\iota_X v) \ dv_{vol_\alpha} = 0 \), so by the Fredholm property there exists \( f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \) with \( Xf = \iota_X v \). Take \( u := v - df \in \text{Res}_0^1 \cap \ker d \), then \( \pi_1(u) = [v]_{H^1} \)
Closed resonant 1-forms

- \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, \iota_X du = 0 \} \), \( \Omega^k = \wedge^k T^* M \)

- Closed forms: \( \text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^k) \mid \iota_X u = 0, du = 0 \} \)

- Cohomology map: \( \pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R}) \), \( \pi_k(u) = [u]_{H^k} \)

- \( \pi_k \) can be defined because \( \mathcal{D}'_{E^*_u} \) is closed under \( (d\delta + \delta d + 1)^{-1} \):

  \[
  u \in \mathcal{D}'_{E^*_u}, du \in C^\infty \implies u = v + dw \text{ for some } v \in C^\infty, w \in \mathcal{D}'_{E^*_u}.
  \]

Lemma: \( \pi_1 \) is an isomorphism

Injectivity: if \( u \in \text{Res}_0^1 \) and \( u = df \), \( f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \), then

\[ Xf = \iota_X u = 0, \text{ so } f \in \text{Res}_0^0 = \mathbb{R}1 \text{ and } u = df = 0 \]

Surjectivity: if \( v \in C^\infty(M; \Omega^1) \) and \( dv = 0 \), then \( \int_M (\iota_X v) \, d\text{vol}_\alpha = 0 \),

so by the Fredholm property there exists \( f \in \mathcal{D}'_{E^*_u}(M; \mathbb{R}) \) with \( Xf = \iota_X v \).

Take \( u := v - df \in \text{Res}_0^1 \cap \ker d \), then \( \pi_1(u) = [v]_{H^1} \).
Closed resonant 1-forms

- \( \text{Res}_0^k = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0, \iota_X du = 0 \} \), \( \Omega^k = \wedge^k T^* M \)
- Closed forms: \( \text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0, du = 0 \} \)
- Cohomology map: \( \pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R}), \quad \pi_k(u) = [u]_{H^k} \)
- \( \pi_k \) can be defined because \( \mathcal{D}'_{E_u^*} \) is closed under \( (d\delta + \delta d + 1)^{-1} \):
  \[ u \in \mathcal{D}'_{E_u^*}, \; du \in C^\infty \quad \implies \quad u = v + dw \text{ for some } v \in C^\infty, \; w \in \mathcal{D}'_{E_u^*} \]

Lemma: \( \pi_1 \) is an isomorphism

**Injectivity:** if \( u \in \text{Res}_0^1 \) and \( u = df \), \( f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R}) \), then
\[ Xf = \iota_X u = 0, \; \text{so } f \in \text{Res}_0^0 = \mathbb{R}1 \text{ and } u = df = 0 \]

**Surjectivity:** if \( v \in C^\infty(M; \Omega^1) \) and \( dv = 0 \), then \( \int_M (\iota_X v) \, d \text{vol}_\alpha = 0 \), so by the Fredholm property there exists \( f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R}) \) with \( Xf = \iota_X v \).

Take \( u := v - df \in \text{Res}_0^1 \cap \ker d \), then \( \pi_1(u) = [v]_{H^1} \).
Closed resonant 1-forms

- $\text{Res}_0^k = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0, \iota_X du = 0 \}$, $\Omega^k = \wedge^k T^* M$
- Closed forms: $\text{Res}_0^k \cap \ker d = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0, du = 0 \}$
- Cohomology map: $\pi_k : \text{Res}_0^k \cap \ker d \to H^k(M; \mathbb{R})$, $\pi_k(u) = [u]_{H^k}$
- $\pi_k$ can be defined because $\mathcal{D}'_{E_u^*}$ is closed under $(d\delta + \delta d + 1)^{-1}$:
  
  $u \in \mathcal{D}'_{E_u^*}, du \in C^\infty \implies u = v + dw$ for some $v \in C^\infty$, $w \in \mathcal{D}'_{E_u^*}$

Lemma: $\pi_1$ is an isomorphism

**Injectivity:** if $u \in \text{Res}_0^1$ and $u = df$, $f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R})$, then $Xf = \iota_X u = 0$, so $f \in \text{Res}_0^0 = \mathbb{R}1$ and $u = df = 0$

**Surjectivity:** if $v \in C^\infty(M; \Omega^1)$ and $dv = 0$, then $\int_M (\iota_X v) \, d\text{vol}_\alpha = 0$, so by the Fredholm property there exists $f \in \mathcal{D}'_{E_u^*}(M; \mathbb{R})$ with $Xf = \iota_X v$.

Take $u := v - df \in \text{Res}_0^1 \cap \ker d$, then $\pi_1(u) = [v]_{H^1}$
Resonant forms, hyperbolic case

- We know that $\mathcal{C} := \text{Res}^1_0 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \text{Res}^1_0$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega^1_0$
- The $\frac{\pi}{2}$-rotation $\mathcal{I} : E_u^* \to E_u^*$ commutes with $\mathcal{L}_X$ because the flow $\varphi_t = e^{tX}$ is conformal on $E_u^*$: $|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|$, $\xi \in E_u^*(\rho)$
- Thus $\mathcal{I}$ acts on $\text{Res}^1_0 = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^1_0) : \mathcal{L}_X u = 0 \}$
- If $u \in \mathcal{C} \setminus \{0\}$ then $d\mathcal{I}(u) \neq 0$: express $[d\alpha \wedge \mathcal{I}(u)]_{H^3}$ via $\pi_1(u)$
- We show that $\text{Res}^1_0 = \mathcal{C} \oplus \mathcal{I}(\mathcal{C})$ is $2b_1(\Sigma)$-dimensional and semisimplicity holds for $k = 1$, so $\dim \text{Res}^{1,\infty}_0 = 2b_1(\Sigma)$
- We also show that $\text{Res}^2_0 = \text{Res}^2_0 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\text{Res}^1_0$ is $(b_1(\Sigma) + 2)$-dimensional where $\psi$ is an explicit smooth 2-form
- We finally show $\dim \text{Res}^{2,\infty}_0 = 2b_1(\Sigma) + 2$: get $b_1(\Sigma)$ Jordan blocks
Resonant forms, hyperbolic case

- We know that $C := \text{Res}_0^1 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \text{Res}_0^1$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega_0^1$
- The $\frac{\pi}{2}$-rotation $\mathcal{I} : E_u^* \rightarrow E_u^*$ commutes with $\mathcal{L}_X$ because the flow $\varphi_t = e^{tX}$ is conformal on $E_u^*$: $|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|$, $\xi \in E_u^*(\rho)$
- Thus $\mathcal{I}$ acts on $\text{Res}_0^1 = \{ u \in \mathcal{D}_E' (M; \Omega_0^1) | \mathcal{L}_X u = 0 \}$
- If $u \in C \setminus \{0\}$ then $d\mathcal{I}(u) \neq 0$: express $[d\alpha \wedge \mathcal{I}(u)]_{\mathcal{H}_3}$ via $\pi_1(u)$
- We show that $\text{Res}_0^1 = C \oplus \mathcal{I}(C)$ is $2b_1(\Sigma)$-dimensional and semisimplicity holds for $k = 1$, so $\dim \text{Res}_0^{1,\infty} = 2b_1(\Sigma)$
- We also show that $\text{Res}_0^2 = \text{Res}_0^2 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\text{Res}_0^1$ is $(b_1(\Sigma) + 2)$-dimensional where $\psi$ is an explicit smooth 2-form
- We finally show $\dim \text{Res}_0^{2,\infty} = 2b_1(\Sigma) + 2$: get $b_1(\Sigma)$ Jordan blocks
We know that \( C := \text{Res}^1_0 \cap \ker d \) has dimension \( b_1(M) = b_1(\Sigma) \).

We show every \( u \in \text{Res}^1_0 \) is a section of \( E^*_u = (E_0 \oplus E_u) \perp \subset \Omega^1_0 \).

The \( \frac{\pi}{2} \)-rotation \( \mathcal{I} : E^*_u \to E^*_u \) commutes with \( \mathcal{L}_X \) because the flow \( \varphi_t = e^{tX} \) is conformal on \( E^*_u \): \(|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|, \xi \in E^*_u(\rho)\).

Thus \( \mathcal{I} \) acts on \( \text{Res}^1_0 = \{ u \in \mathcal{D}'_{E^*_u}(M; \Omega^1_0) \mid \mathcal{L}_X u = 0 \} \).

If \( u \in C \setminus \{0\} \) then \( d\mathcal{I}(u) \neq 0 \): express \([d\alpha \wedge \mathcal{I}(u)]_{\mathcal{H}3} \) via \( \pi_1(u) \).

We show that \( \text{Res}^1_0 = C \oplus \mathcal{I}(C) \) is \( 2b_1(\Sigma) \)-dimensional and semisimplicity holds for \( k = 1 \), so \( \dim \text{Res}^1_{0,\infty} = 2b_1(\Sigma) \).

We also show that \( \text{Res}^2_0 = \text{Res}^2_0 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\text{Res}^1_0 \) is \( (b_1(\Sigma) + 2) \)-dimensional where \( \psi \) is an explicit smooth 2-form.

We finally show \( \dim \text{Res}^2_{0,\infty} = 2b_1(\Sigma) + 2 \): get \( b_1(\Sigma) \) Jordan blocks.
Resonant forms, hyperbolic case

- We know that $C := \text{Res}_0^1 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \text{Res}_0^1$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega_0^1$
- The $\frac{\pi}{2}$-rotation $I : E_u^* \to E_u^*$ commutes with $\mathcal{L}_X$ because the flow $\varphi_t = e^{tX}$ is conformal on $E_u^*$: $|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|$, $\xi \in E_u^*(\rho)$
- Thus $I$ acts on $\text{Res}_0^1 = \{u \in \mathcal{D}_{E_u^*}'(M; \Omega_0^1) \mid \mathcal{L}_X u = 0\}$
- If $u \in C \setminus \{0\}$ then $dI(u) \neq 0$: express $[d\alpha \wedge I(u)]_{\mathcal{H}^3}$ via $\pi_1(u)$
- We show that $\text{Res}_0^1 = C \oplus I(C)$ is $2b_1(\Sigma)$-dimensional and semisimplicity holds for $k = 1$, so $\dim \text{Res}_0^1,\infty = 2b_1(\Sigma)$

- We also show that $\text{Res}_0^2 = \text{Res}_0^2 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\text{Res}_0^1$ is $(b_1(\Sigma) + 2)$-dimensional where $\psi$ is an explicit smooth 2-form
- We finally show $\dim \text{Res}_0^2,\infty = 2b_1(\Sigma) + 2$: get $b_1(\Sigma)$ Jordan blocks
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- We know that $\mathcal{C} := \text{Res}_0^1 \cap \ker d$ has dimension $b_1(M) = b_1(\Sigma)$
- We show every $u \in \text{Res}_0^1$ is a section of $E_u^* = (E_0 \oplus E_u)^\perp \subset \Omega^1_0$
- The $\frac{\pi}{2}$-rotation $\mathcal{I} : E_u^* \rightarrow E_u^*$ commutes with $\mathcal{L}_X$ because the flow $\varphi_t = e^{tX}$ is conformal on $E_u^*$: $|d\varphi_t(\rho)^{-T}\xi| = e^t|\xi|$, $\xi \in E_u^*(\rho)$
- Thus $\mathcal{I}$ acts on $\text{Res}_0^1 = \{ u \in \mathcal{D}'_{E_u^*}(M; \Omega^1_0) \mid \mathcal{L}_X u = 0 \}$
- If $u \in \mathcal{C} \setminus \{0\}$ then $d\mathcal{I}(u) \neq 0$: express $[d\alpha \wedge \mathcal{I}(u)]_{H^3}$ via $\pi_1(u)$
- We show that $\text{Res}_0^1 = \mathcal{C} \oplus \mathcal{I}(\mathcal{C})$ is $2b_1(\Sigma)$-dimensional and semisimplicity holds for $k = 1$, so $\dim \text{Res}_0^{1,\infty} = 2b_1(\Sigma)$
- We also show that $\text{Res}_0^2 = \text{Res}_0^2 \cap \ker d = \mathbb{R}d\alpha \oplus \mathbb{R}\psi \oplus d\text{Res}_0^1$ is $(b_1(\Sigma) + 2)$-dimensional where $\psi$ is an explicit smooth 2-form
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- We finally show $\dim \text{Res}_0^{2,\infty} = 2b_1(\Sigma) + 2$: get $b_1(\Sigma)$ Jordan blocks
Resonant forms for perturbations

- Consider now the **perturbed metric** \( g_\tau = e^{\tau a} g_H \), \( a \in \mathcal{C}^\infty(\Sigma; \mathbb{R}) \)
- Define \( \pi_\Sigma : M = S\Sigma \to \Sigma; \ J : M \to M, \ J(x, v) = (x, -v) \)
- We still have \( \dim(\text{Res}_0^1 \cap \ker d) = b_1(\Sigma) \), need to show that all non-closed elements of \( \text{Res}_0^1 \) are moved by the perturbation
- A first variation calculation shows that we need nondegeneracy of
  \[
  du \in d(\text{Res}_0^1), \quad du_* \in d(\text{Res}_0^1) \quad \mapsto \quad \int_M (\pi_*^* a) \alpha \wedge du \wedge du_*
  \]
- Take for simplicity \( b_1(\Sigma) = 1 \), then enough to show
  \[
  u \in \text{Res}_0^1, \quad du \neq 0 \quad \Rightarrow \quad \int_M (\pi_*^* a) \alpha \wedge du \wedge J^*(du) \neq 0
  \]
- That’s true for generic \( a \) as long as \( \pi_*^* (\alpha \wedge du \wedge J^*(du)) \neq 0 \)
  where \( \pi_*^* : \mathcal{D}'(M; \Omega^k) \to \mathcal{D}'(\Sigma; \Omega^{k-2}) \) is the pushforward on forms
Resonant forms for perturbations

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  \[ du \in d(\text{Res}_0^1), \quad du_* \in d(\text{Res}_{0*}^1) \quad \mapsto \quad \int_M (\pi_\Sigma^* a) \alpha \wedge du \wedge du_* \]
- Take for simplicity $b_1(\Sigma) = 1$, then enough to show
  \[ u \in \text{Res}_0^1, \quad du \neq 0 \quad \Rightarrow \quad \int_M (\pi_\Sigma^* a) \alpha \wedge du \wedge J^*(du) \neq 0 \]
- That's true for generic $a$ as long as $\pi_{\Sigma*}(\alpha \wedge du \wedge J^*(du)) \neq 0$
  where $\pi_{\Sigma*} : \mathcal{D}'(M; \Omega^k) \to \mathcal{D}'(\Sigma; \Omega^{k-2})$ is the pushforward on forms
Resonant forms for perturbations

- Consider now the perturbed metric $g_\tau = e^{\tau a} g_H$, $a \in C^\infty(\Sigma; \mathbb{R})$
- Define $\pi_\Sigma : M = S\Sigma \to \Sigma$; $J : M \to M$, $J(x, v) = (x, -v)$
- We still have $\dim(\text{Res}^1_0 \cap \ker d) = b_1(\Sigma)$, need to show that all non-closed elements of $\text{Res}^1_0$ are moved by the perturbation
- A first variation calculation shows that we need nondegeneracy of

$$du \in d(\text{Res}^1_0), \quad du^* \in d(\text{Res}^1_0^*), \quad \mapsto \quad \int_M (\pi^*_\Sigma a) \alpha \wedge du \wedge du^*$$

- Take for simplicity $b_1(\Sigma) = 1$, then enough to show

$$u \in \text{Res}^1_0, \quad du \neq 0 \quad \implies \quad \int_M (\pi^*_\Sigma a) \alpha \wedge du \wedge J^*(du) \neq 0$$

- That's true for generic $a$ as long as $\pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) \neq 0$
  where $\pi_{\Sigma^*} : \mathcal{D}'(M; \Omega^k) \to \mathcal{D}'(\Sigma; \Omega^{k-2})$ is the pushforward on forms
Nontriviality of first variation

- Working only with the hyperbolic metric now
- Given $u \in \text{Res}^1_0$, $du \neq 0$, need $\pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) \neq 0$
- Write $\pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) = F \, d\text{vol}_g$ for some $F \in \mathcal{D}'(\Sigma; \mathbb{R})$
- Difficult to show that $F \neq 0$ because cannot evaluate $F$ at points

**Main identity**

We have $Q_4 F = -\frac{1}{6} \Delta_g |\sigma|_g^2$ where

- $\sigma = \pi_{\Sigma^*}(\alpha \wedge du)$ is a nonzero harmonic 1-form on $\Sigma$
- $Q_4 f(x) = \int_{\mathbb{H}^3} \cosh^{-4} d_{\mathbb{H}^3}(x, y) f(y) \, d\text{vol}_g(y)$ descends to $Q_4 : \mathcal{D}'(\Sigma) \to C^\infty(\Sigma)$ where $\Sigma = \Gamma \backslash \mathbb{H}^3$

- If $F = 0$, then $\Delta_g |\sigma|_g^2 = 0$, so $|\sigma|_g$ is constant, but this is impossible!
Nontriviality of first variation

Working only with the hyperbolic metric now

Given \( u \in \text{Res}_0^1, \ du \neq 0 \), need \( \pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) \neq 0 \)

Write \( \pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) = F \, d\ vol_g \) for some \( F \in D'(\Sigma; \mathbb{R}) \)

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Nontriviality of first variation

- Working only with the hyperbolic metric now
- Given \( u \in \text{Res}_1^0 \), \( du \neq 0 \), need \( \pi_{\Sigma^*}(\alpha \wedge du \wedge J^*(du)) \neq 0 \)
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Two conjectures

**Conjecture 1**

Let \((\Sigma, g)\) be a **generic** negatively curved compact connected oriented 3-manifold. Then:

- semisimplicity holds and \(d(\text{Res}_k^0) = 0\) for all \(k = 0, \ldots, 4\)
- \(\dim \text{Res}_0^0 = 1\), \(\dim \text{Res}_0^1 = b_1(\Sigma)\), \(\dim \text{Res}_0^2 = b_1(\Sigma) + 2\)
- \(m_R(0) = 4 - b_1(\Sigma)\)

The set of \(g\) satisfying Conjecture 1 is open:

\[
\dim \text{Res}_0^1,\infty \leq b_1(\Sigma), \quad \dim \text{Res}_0^2,\infty \leq b_1(\Sigma) + 2 \quad \implies \quad \text{Conjecture 1 holds}
\]

**Conjecture 2**

Let \(\rho : \pi_1(\Sigma) \to U(m)\) be acyclic: \(H^\bullet_\rho(\Sigma; \mathbb{R}) = 0\). Then \(\text{Res}_0^k = 0\) for all \(k\)

Conjecture 2 + DGRS '20 \(\implies\) \(\zeta_\rho(0)\) is locally constant under perturbations of \(\rho, g\), which could lead to a solution of Fried’s conjecture
Two conjectures

Conjecture 1
Let $(\Sigma, g)$ be a generic negatively curved compact connected oriented 3-manifold. Then:

- semisimplicity holds and $d(\text{Res}_0^k) = 0$ for all $k = 0, \ldots, 4$
- $\dim \text{Res}_0^0 = 1$, $\dim \text{Res}_0^1 = b_1(\Sigma)$, $\dim \text{Res}_0^2 = b_1(\Sigma) + 2$
- $m_R(0) = 4 - b_1(\Sigma)$

The set of $g$ satisfying Conjecture 1 is open:
\[
\dim \text{Res}_0^1,\infty \leq b_1(\Sigma), \quad \dim \text{Res}_0^2,\infty \leq b_1(\Sigma) + 2 \quad \Rightarrow \quad \text{Conjecture 1 holds}
\]

Conjecture 2
Let $\rho : \pi_1(\Sigma) \to U(m)$ be acyclic: $H^\bullet(\Sigma; \mathbb{R}) = 0$. Then $\text{Res}_0^k = 0$ for all $k$

Conjecture 2 + DGRS '20 $\quad \Rightarrow \quad \zeta_\rho(0)$ is locally constant under perturbations of $\rho, g$, which could lead to a solution of Fried’s conjecture
Thank you for your attention!