EXERCISES FOR THE MINICOURSE ON
FRACTAL UNCERTAINTY PRINCIPLE
(WITH SOLUTIONS)

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Abstract. These are companion exercises to the minicourse given at the Spring School on Transfer Operators, organized by the Bernoulli Center, Lausanne, in March 2021.

1. Describe all the elements $\gamma \in \text{SL}(2, \mathbb{R})$ such that
$$\gamma(\mathbb{R} \setminus I_2^o) = I_1$$
where $I_1 := [1, 2]$, $I_2 := [-1, 0]$.

Note that these $\gamma$ are all hyperbolic, i.e. $|\text{tr} \gamma| > 2$, which implies that $\gamma$ has two fixed points on $\mathbb{R}$, one attractive and one repulsive. Find these fixed points. Show that any point in $I_1^o$ is the attractive point of some $\gamma$ and similarly for repulsive points and $I_2^o$.

Solution: We need
$$\gamma(-1) = 2, \quad \gamma(0) = 1.$$ 
Writing
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1$$
we get the equations
$$\frac{b - a}{d - c} = 2, \quad \frac{b}{d} = 1.$$ 
Writing out in terms of $a, b$, we get
$$c = \frac{a + b}{2}, \quad d = b,$$
and using the equation $ad - bc = 1$ we get
$$(a - b)b = 2.$$ 
So it makes sense to parametrize by $b \neq 0$, obtaining
$$\gamma = \begin{pmatrix} \frac{b + 2}{b} & b \\ \frac{b + 1}{b} & b \end{pmatrix}, \quad \gamma(x) = 1 + \frac{x}{(b^2 + 1)x + b^2}.$$ 
The fixed point equation is $\gamma(x) = x$, which can be written as the quadratic equation
$$cx^2 + (d - a)x - b = 0.$$
which has solutions
\[ x_\pm = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c} = \frac{1 \pm \sqrt{b^4 + b^2 + 1}}{b^2 + 1}. \]

To see which one is attractive and which one is repulsive, compute
\[ \gamma'(x_\pm) = \frac{1}{(cx_\pm + d)^2} \quad \text{where} \quad cx_\pm + d = \frac{a + b \pm \sqrt{(a + d)^2 - 4}}{2}. \]

We see that \( \gamma'(x_+) < 1 < \gamma'(x_-) \), so \( x_+ \) is the attractive point and \( x_- \) is the repulsive one. From the mapping properties of \( \gamma \), or by direct computation, we see that \( x_+ \in I_1 \) and \( x_- \in I_2 \). Moreover, as \( b \to 0 \) we have
\[ x_+ \to 2, \quad x_- \to 0 \]
and as \( b \to \infty \) we have
\[ x_+ \to 1, \quad x_- \to -1 \]
which gives the last statement.

2. Let \( \Gamma \subset \text{SL}(2, \mathbb{R}) \) be a Schottky group, with generators \( \gamma_1, \ldots, \gamma_r \). Show that it is a free group with these generators, i.e. for any word \( a \in \mathcal{W} \), if \( \gamma_a = I \) then \( a = \emptyset \).

**Solution:** Assume that \( a = a_1 \ldots a_n \) is a nonempty word. Since \( \infty \) is contained in the complement of \( I_{\overline{a_n}} \), we have \( \gamma_{a_n}(\infty) \in I_{a_n} \). Since \( a_n \neq \overline{a_{n-1}} \), \( \gamma_{a_n}(\infty) \) is in the complement of \( I_{\overline{a_{n-1}}} \), thus \( \gamma_{a_{n-1}} \gamma_{a_n}(\infty) \in I_{a_{n-1}} \). Repeating this argument, we get \( \gamma_a(\infty) \in I_{a_1} \). In particular, \( \gamma_a(\infty) \neq \infty \), so \( \gamma_a \) cannot be the identity.

3. This exercise explains why elements of Schottky groups have bounded distortion.

(a) We first discuss the way that a general element \( \gamma \in \text{SL}(2, \mathbb{R}) \) can map an interval to another interval. Assume that \( I, J \subset \mathbb{R} \) are intervals such that \( \gamma(I) = J \). Define the **distortion factor** of \( \gamma \) on \( I \) by
\[ \alpha(\gamma, I) := \log \frac{\gamma^{-1}(\infty) - x_1}{\gamma^{-1}(\infty) - x_0} \in \mathbb{R} \quad \text{where} \quad I = [x_0, x_1]. \]
(If \( \gamma^{-1}(\infty) = \infty \), that is \( \gamma \) is an affine map, then we put \( \alpha(\gamma, I) := 0 \).) Show that \( \gamma \) can be factorized as
\[ \gamma = \gamma_J \gamma_a(\gamma, I) \gamma_I^{-1}, \quad \gamma_I := \begin{pmatrix} e^{\alpha/2} & 0 \\ e^{-\alpha/2} - e^{\alpha/2} & e^{-\alpha/2} \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \]
where \( \gamma_I, \gamma_J \in \text{SL}(2, \mathbb{R}) \) are the affine maps such that \( \gamma_I([0, 1]) = I, \gamma_J([0, 1]) = J \).

(b) Show that for each \( R \) there exists \( C \) such that in the notation of part (a)
\[ |\alpha(\gamma, I)| \leq R \quad \Rightarrow \quad C^{-1} \frac{|J|}{|I|} \leq \gamma'(x) \leq C \frac{|J|}{|I|} \quad \text{for all} \quad x \in I. \]
(c) Let $\Gamma$ be a Schottky group generated by $\gamma_1, \ldots, \gamma_r \in \text{SL}(2, \mathbb{R})$. Show that there exists $C_\Gamma$ such that for all nonempty $a = a_1 \ldots a_n \in \mathcal{W}$ we have
\[ C_\Gamma^{-1} |I_a| \leq \gamma'_a(x) \leq C_\Gamma |I_a| \quad \text{for all} \quad x \in I_{a_n}. \]
That is, the derivatives of the map $\gamma_a$ are of comparable size at different points of $I_{a_n}$.

(d) Using the following special case of $\Gamma$-equivariance of the Patterson–Sullivan measure $\mu$:
\[ \mu(I_a) = \int_{I_{a_n}} (\gamma'_a(x))^\delta \, d\mu(x) \]
and the fact that $\mu(I_a) > 0$ for every $a \in \mathcal{A}$, show that for some constant $C_\Gamma$ depending only on $\Gamma$
\[ C_\Gamma^{-1} |I_a| \delta \leq \mu(I_a) \leq C_\Gamma |I_a| \delta. \]
Using this, show that $\Lambda_\Gamma$ is $\delta$-regular up to scale 0 with some constant depending only on $\Gamma$.

**Solution:** See §2 in arXiv:1704.02909.

4. This exercise explains why the transfer operator is of trace class on $\mathcal{H}(D)$. (See for instance Dyatlov–Zworski, *Mathematical Theory of Scattering Resonances*, Appendix B.4, for an introduction to trace class operators.) We consider the following simpler setting: $D \subset \mathbb{C}$ is the unit disk, $\mathcal{H}(D)$ is the space of holomorphic functions in $L^2(D)$ (it is a closed subspace of $L^2$ and thus a Hilbert space), and we consider the operator
\[ L : \mathcal{H}(D) \to \mathcal{H}(D), \quad Lf(z) = f(z/2). \]
Show that $L$ is trace class using one or both of the following methods:

(a) the fact that $\{z^k\}_{k \in \mathbb{N}_0}$ is an orthogonal basis in $\mathcal{H}(D)$;

**Solution:** We have $L(z^k) = 2^{-k}z^k$, so $L$ is self-adjoint on $\mathcal{H}(D)$ and has eigenvalues $2^{-k}$, $k \in \mathbb{N}_0$. The series $\sum_{k=0}^\infty 2^{-k}$ converges, so $L$ is trace class.

(b) the Cauchy integral formula, where $\gamma \subset D$ is a contour surrounding the disk $\{|z| \leq \frac{1}{2}\}$
\[ Lf(z) = \frac{1}{2\pi i} \oint_{\gamma} L_w f(z) \, dw, \quad L_w f(z) = \frac{f(w)}{w - z/2}, \]
together with the fact that each $L_w$ is a rank 1 operator. (This solution easily adapts to the transfer operators that we study, where the key fact is that $\gamma_a(D_b) \not\subset D_a$ when $a \neq b$.)

**Solution:** Each $L_w$ is a rank 1 operator, in fact $L_w = u_w \otimes \delta_w$ where $\delta_w : \mathcal{H}(D) \to \mathbb{C}$ is the delta function at $w$, $\delta_w(f) = f(w)$, and $u_w(z) = \frac{1}{w - z/2} \in \mathcal{H}(D)$. Thus in particular $L_w$ is trace class. Since both $\delta_w$ and $u_w$ depend continuously on $w$ (the first one as a functional on $\mathcal{H}(D)$ with operator norm, the second one as an element of $\mathcal{H}(D)$), $L_w$ depends continuously on $w$ in the Banach space of trace class operators.
on $\mathcal{H}(D)$. So the integral above converges in that Banach space, which shows that $L$ is trace class.

5. Assume that $\Gamma$ is a Schottky group generated by just two intervals $I_1, I_2$. (The corresponding convex co-compact hyperbolic surface is a hyperbolic cylinder.) Let $x_1 \in I_1, x_2 \in I_2$ be the fixed points of $\gamma_1$ (and thus of $\gamma_2 = \gamma_1^{-1}$). Let $L_s : \mathcal{H}(D) \to \mathcal{H}(D)$ be the transfer operator where $D = D_1 \cup D_2 \subset \mathbb{C}$.

Show that the resonances (i.e. the values $s \in \mathbb{C}$ for which the equation $L_s u = u$ has a nonzero solution $u \in \mathcal{H}(D)$) are given by

$$s = -j + \frac{2\pi i}{\ell} k, \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}, \quad \ell := -\log \gamma_1'(x_1) = -\log \gamma_2'(x_2) > 0.$$ 

(In fact, $\ell$ is the length of the closed geodesic on the cylinder $\Gamma \backslash \mathbb{H}^2$.)

**Hint:** if $L_s u = u$, then let $j$ be the vanishing order of $u$ at $x_1$ and expand the equation at $z = x_1$.

**Solution:** First of all, putting $x := x_1, y := x_2$ in the identity $|\gamma_1(x) - \gamma_1(y)|^2 = \gamma_1'(x)\gamma_1'(y)|x - y|^2$ we get $\gamma_1'(x_1)\gamma_1'(x_2) = 1$. Thus the definition of $\ell$ makes sense.

We have for $u \in \mathcal{H}(D)$

$$L_s u(z) = \begin{cases} (\gamma_1'(z))^s u(\gamma_1(z)), & z \in D_1; \\ (\gamma_2'(z))^s u(\gamma_2(z)), & z \in D_2. \end{cases}$$

The disks $D_1, D_2$ do not interact so we can consider $u$ separately on these two. Let us focus on $D_1$.

Assume that $L_s u = u$ for some $s \in \mathbb{C}$ and $u \in \mathcal{H}(D_1) \setminus \{0\}$. Let $j \in \mathbb{N}_0$ be the vanishing order of $u$ at $z = x_1$. Multiplying $u$ by a constant we may assume that

$$u(z) = (z-x_1)^j + \mathcal{O}(|z-x_1|^{j+1}) \quad \text{as} \quad z \to x_1.$$ 

Expanding the identity $u(z) = L_s u(z)$ at $z = x_1$ and using that

$$\gamma_1(z) - x_1 = e^{-\ell}(z-x_1) + \mathcal{O}(|z-x_1|^2)$$

we get

$$(z-x_1)^j + \mathcal{O}(|z-x_1|^{j+1}) = e^{-\ell(s+j)}(z-x_1)^j + \mathcal{O}(|z-x_1|^{j+1})$$

which implies that $e^{-\ell(s+j)} = 1$ and thus

$$s = -j + \frac{2\pi i}{\ell} k \quad \text{for some} \quad k \in \mathbb{Z}. \quad (0.1)$$

Now, assume that $s$ has the form $(0.1)$ for some $j \in \mathbb{N}_0, k \in \mathbb{Z}$. We construct a nonzero $u \in \mathcal{H}(D)$ such that $L_s u = u$. Let us write

$$\gamma_1'(z) = e^{-\psi(z)}, \quad \gamma_1(z) - x_1 = (z-x_1)e^{-\psi(z)}, \quad z \in D_1$$

where $\psi(z)$ is a function such that $\psi(x_1) = 0$ and $\lim_{z \to x_1} \psi(z) \neq 0$. Then $u(z) = (z-x_1)^j + \mathcal{O}(|z-x_1|^{j+1})$ satisfies $L_s u(z) = u(z)$ for $z \in D_1$. The case $z \in D_2$ is similar.
where \( \varphi, \psi \) are holomorphic and bounded on \( D_1 \) and \( \varphi(x_1) = \psi(x_1) = \ell \). We look for \( u \) in the form
\[
u(z) = (z - x_1)^j e^{v(z)}
\]
where \( v \) is some bounded holomorphic function on \( D_1 \). Then \( L_s u = u \) is equivalent to the following equation for \( v \):
\[
e^{v(z)} = e^{-s\varphi(z) - j\psi(z) + v(\gamma_1(z))}, \quad z \in D_1.
\]
To satisfy the latter it suffices to construct \( v \) such that
\[
v(z) = v(\gamma_1(z)) + \theta(z), \quad z \in D_1
\]
where \( \theta(z) := -s\varphi(z) - j\psi(z) + 2\pi ik \) is holomorphic and bounded on \( D_1 \) and \( \theta(x_1) = 0 \).

Now, to solve (0.2) we put
\[
v(z) := \sum_{n=0}^{\infty} \theta(\gamma_1^n(z)), \quad z \in D_1
\]
where the terms of the series are holomorphic in \( D_1 \) and the series converges uniformly in \( D_1 \) since \( \gamma_1^n(z) \to x_1 \) exponentially fast as \( n \to \infty \).

6. Show the following version of the ‘Patterson–Sullivan’ gap: if \( \Re s > \delta \) then the equation \( L_s u = u \) has no nonzero solution \( u \in \mathcal{H}(D) \). To do this, show that a sufficiently large power \( L^n_s \) is a contracting operator on \( C(I) \) with the supremum norm, by writing out \( L^n_s \) as a sum over words in \( \mathcal{W}^n \) and using the results of Exercise 3.

**Solution:** Put \( \alpha := \Re s > \delta \). Take large \( n \). Then for any \( f \in C(I) \) we have
\[
L^n_s f(x) = \sum_{\substack{a \in \mathcal{W}^n \\text{\scriptsize{a} \rightarrow b}}} (\gamma'_a(x))^a f(\gamma_a(x)), \quad x \in I_b
\]
where \( a \rightarrow b \) means that \( a_n \neq \tilde{b} \) where \( a = a_1 \ldots a_n \).

By Exercise 3(c) we have \( |(\gamma'_a(x))^a| = |\gamma'_a(x)|^\alpha \leq C|I_a|^\alpha \) for \( x \in I_b, a \rightarrow b \). Here \( C \) is a constant independent of \( n \). Therefore
\[
\sup_I |L^n_s f| \leq r_n \sup_I |f|, \quad r_n := C \sum_{a \in \mathcal{W}^n} |I_a|^\alpha.
\]
Now by Exercise 3(d) we have
\[
\sum_{a \in \mathcal{W}^n} |I_a|^\delta \leq C \sum_{a \in \mathcal{W}^n} \mu(I_a) \leq C.
\]
Since \( \alpha > \delta \) and \( \max_{a \in \mathcal{W}^n} |I_a| \to 0 \) as \( n \to \infty \), we get \( r_n \to 0 \) as \( n \to \infty \). Thus for \( n \) large enough, \( L^n_s \) is a contraction on \( C(I) \) with the uniform norm. If \( u \in \mathcal{H}(D) \) and \( L_s u = u \), then it is easy to see that \( f := u|_I \in C(I) \) and \( L^n_s f = f \), which implies that \( u|_I = 0 \) and thus (by analytic continuation for instance) \( u = 0 \).
7. Fix $\delta \in [0, 1]$ and define the $h$-dependent intervals

$$X = Y = [-h^{1-\delta}, h^{1-\delta}].$$

Show that there exists a constant $c > 0$ such that

$$\| 1_{X} \mathcal{F}_h 1_{Y} \|_{L^2(\mathbb{R})} \to L^2(\mathbb{R}) \geq ch^{\max(0, \frac{1}{2} - \delta)}.$$  

(Hint: apply this operator to a dilated cutoff function supported in $Y$.)

**Solution:** Fix $\chi \in C^\infty_c((-1, 1))$ such that $\| \chi \|_{L^2} = 1$ and $\hat{\chi}(0) \neq 0$ and define

$$u(y; h) = h^{\frac{1-\delta}{2}} \chi(h^{\delta-1}y), \quad \| u \|_{L^2} = 1, \quad \text{supp } u \subset Y.$$  

Then

$$\mathcal{F}_h u(x) = \frac{h^{-\delta/2}}{\sqrt{2\pi}} \hat{\chi}(h^{-\delta}x),$$

so we compute

$$\| 1_{X} \mathcal{F}_h 1_{Y} u \|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \| \hat{\chi} \|_{L^2([-h^{1-2\delta}, h^{1-2\delta}])} \geq ch^{\max(0, \frac{1}{2} - \delta)}.$$  

8. Let $Z \subset W$ be a partition, i.e. a finite set of nonempty words such that

$$\Lambda_{\Gamma} = \bigcup_{a \in Z} (\Lambda_{\Gamma} \cap I_a).$$

Let $\mathcal{Z} := \{ \bar{a} \mid a \in Z \}$ where $\bar{a} := a_n \ldots a_1$. Define the transfer operator $\mathcal{L}_{z,s}$ by

$$\mathcal{L}_{z,s} f(z) = \sum_{a \in \mathcal{Z}, \ a \sim b} (\gamma_{a'}(z))^s f(\gamma_{a'}(z)), \quad z \in D_b$$

where for $a = a_1 \ldots a_n$ we put $a' := a_1 \ldots a_{n-1}$ and say $a \sim b$ if $a_n = b$. Assume that $u \in \mathcal{H}(D)$ satisfies $\mathcal{L}_s u = u$. Show that $\mathcal{L}_{z,s} u = u$.

**Solution:** See Lemma 2.4 in arXiv:1704.02909.